

Representations through a Monoid on the set of Fuzzy Implications

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Abstract

Fuzzy implications are one of the most important fuzzy logic connectives. In this work, we conduct a systematic algebraic study on the set \mathbb{I} of all fuzzy implications. To this end, we propose a binary operation, denoted by \otimes , which makes (\mathbb{I}, \otimes) a non-idempotent monoid. While this operation does not give a group structure, we determine the largest subgroup \mathbb{S} of this monoid and using their representation define a group action of \mathbb{S} that partitions \mathbb{I} into equivalence classes. Based on these equivalence classes, we obtain a hitherto unknown representation of the two main families of fuzzy implications, viz., the f - and g -implications.

Keywords: Semigroup, Monoid, Group action, Fuzzy logic connectives, Fuzzy implications.

MSC 2010: Primary: 20M32 ; Secondary: 03B52.

1. Introduction

Fuzzy implications, along with triangular norms (t-norms, in short) form the two most important fuzzy logic connectives. They are a generalization of the classical implication and conjunction, respectively, to multi-valued logic and play an equally important role in fuzzy logic as their counterparts in classical logic.

Definition 1.1. ([3], Definition 1.1.1) A binary operation I on $[0, 1]$ is called a *fuzzy implication* if

- (i) I is decreasing in the first variable and increasing in the second variable,
- (ii) $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.

The set of all fuzzy implications will be denoted by \mathbb{I} . Table 1 (see also [3]) lists some examples of basic fuzzy implications.

Fuzzy implications have many applications in various fields like fuzzy control, approximate reasoning, decision making, fuzzy logic, etc. Their applicational value has been the *raison d'être* for more than three decades of research on these operations and have made it essential to study various aspects of fuzzy implications in depth. Their analytical properties like continuity, intersections of families of fuzzy implications, relationship between the properties etc., have been studied extensively in a comprehensive manner (see, for instance, the research monograph of Baczyński and Jayaram [3]).

An algebraic study of fuzzy implications can be done along the following lines:

- (i) Let \mathcal{L} denote the underlying set from which fuzzy propositions can assume their truth values. Usually, $\mathcal{L} = [0, 1]$ or at least a poset. Then one imposes some axioms or properties on the fuzzy implication \longrightarrow and studies the logics obtained or the equivalent algebras generated. For instance, see [5, 19, 10, 6, 7].

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Name	Formula
Lukasiewicz	$I_{\mathbf{LK}}(x, y) = \min(1, 1 - x + y)$
Gödel	$I_{\mathbf{GD}}(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$
Reichenbach	$I_{\mathbf{RC}}(x, y) = 1 - x + xy$
Kleene-Dienes	$I_{\mathbf{KD}}(x, y) = \max(1 - x, y)$
Goguen	$I_{\mathbf{GG}}(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if } x > y \end{cases}$
Rescher	$I_{\mathbf{RS}}(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases}$
Yager	$I_{\mathbf{YG}}(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if } x > 0 \text{ or } y > 0 \end{cases}$
Weber	$I_{\mathbf{WB}}(x, y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$
Fodor	$I_{\mathbf{FD}}(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(1 - x, y), & \text{if } x > y \end{cases}$
Smallest FI	$I_{\mathbf{0}}(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \\ 0, & \text{if } x > 0 \text{ and } y < 1 \end{cases}$
Largest FI	$I_{\mathbf{1}}(x, y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } y > 0 \\ 0, & \text{if } x = 1 \text{ and } y = 0 \end{cases}$
Most Strict FI ([21])	$I_{\mathbf{D}}(x, y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0 \end{cases}$

Table 1: Examples of fuzzy implications

- (ii) One can also define a *closed* binary operation \otimes on the set \mathbb{I} and study the algebraic structures obtained on it. For instance, let the operation \otimes be the lattice operation of pointwise meet \wedge or join \vee . From the view point of abstract algebra, we obtain that (\mathbb{I}, \wedge) and (\mathbb{I}, \vee) are commutative, integral, idempotent monoids.

1.1. Motivation for this work

In this work, we take the second of the above two approaches. Note that such a study would have two important ramifications.

- (A) Firstly, since (\mathbb{I}, \otimes) is closed, given two fuzzy implications $I_1, I_2 \in \mathbb{I}$, $I_1 \otimes I_2 \in \mathbb{I}$ and hence gives a way of generating new fuzzy implications from given ones.
- (B) Secondly, if one were able to impose a richer algebraic structure, say (\mathbb{I}, \otimes) forms a group, then one can apply results from group theory to obtain deeper and better perspectives of the different families of fuzzy implications. For instance, it is well known that if a group G is not simple, it has a nontrivial normal subgroup N which partitions G . Now, it is easy to see that to generate the whole of G , when $O(G) < \infty$, it is sufficient to store $O(N) + O(\frac{G}{N})$ elements. Further, since any $g \in G$ is in one of these cosets, we know that $g = n \cdot g'$. If N is a nontrivial normal subgroup with some desirable properties then we have a *unique* decomposition of g into components with known properties.

Thus our motivation for this study is to propose a binary operation \otimes on \mathbb{I} that would give a rich enough algebraic structure to glean newer and better perspectives on fuzzy implications.

1.2. Existing structures on the set of Fuzzy Implications \mathbb{I}

As noted above, a *closed* binary operation \otimes on the set \mathbb{I} allows one to study the algebraic structures obtained on it. For instance, it is well known that, considering fuzzy implications as functions on $[0, 1]^2$ to $[0, 1]$, with the usual point wise lattice operations, not only does one generate new fuzzy implications but also obtain a complete, completely distributive lattice structure on \mathbb{I} (see **Theorem 2.4** below). From the abstract algebraic view, we obtain a commutative, integral monoid [3]. See also the works of Khaledi *et. al.* [11, 12], which can be seen as the study of some special submonoids of \mathbb{I} .

Note, however, that in the above construction, we always obtain idempotent monoids, which also means that given a fuzzy implication one cannot iterate with itself to obtain new fuzzy implications. Looking at fuzzy implications as giving rise to fuzzy relations on $[0, 1]^2$, Baczyński *et. al.*, [1] studied the sup- T composition operation, denoted $\overset{T}{\circ}$, (where T is a t-norm - a commutative, associative, increasing binary operation on $[0, 1]$ with 1 as the unit element) as a possible binary operation on the set \mathbb{I} . For quite a large class of t-norms T (see [8], [3], [9]), they have characterized the subset $\mathbb{I}^\circ \subsetneq \mathbb{I}$ on which this operation is closed. In fact, they have shown that $(\mathbb{I}^\circ, \overset{T}{\circ})$ forms a *non-idempotent* semi group.

It can also be easily seen that a convex combination of two fuzzy implications will also be a fuzzy implication. While considering this as a binary operation shows that \mathbb{I} is a convex set, it is yet to be explored whether it does give rise to any algebra on \mathbb{I} .

For more details on such studies, please refer to [3], Chapter 6.

1.3. Main contributions of this work

From the above, it is clear that algebraic structures that are both non-idempotent and defined on the entire \mathbb{I} are worthy of exploration. Further, a deeper study of their algebraic properties, along the lines suggested in point (B) above, is imperative.

In this work, we propose a binary operation \otimes on \mathbb{I} that makes (\mathbb{I}, \otimes) a non-idempotent monoid. This is the first work in which such a rich structure has been obtained on the entire set of fuzzy implications \mathbb{I} .

Unfortunately, w.r.to the proposed \otimes , we do have right zero elements in \mathbb{I} , thus precluding the possibility of graduating (\mathbb{I}, \otimes) from a monoid to a group, for instance, through the Grothendieck construction. However, there exist many subgroups of (\mathbb{I}, \otimes) . We characterize the largest such subgroup \mathbb{S} and, based on their representation, propose a group action of \mathbb{S} on \mathbb{I} . Clearly, this group action partitions \mathbb{I} into equivalence classes.

In tune with the main motivation of this work, our study shows that many of the existing families turn out to be pieces of this partition and based on this partition we obtain, once again for the first time, representations of the fuzzy implications from the families of f - and g -implications. In particular, we show that every f - and g -implication is a φ -pseudo conjugate (see **Definition 4.7**) of either the Yager I_{YG} , Reichenbach I_{RC} or the Goguen implication I_{GG} .

It should be remarked that the binary operation \otimes proposed in this work, is not unknown. For instance, the work of [21] can be looked at as discussing the idempotent elements of the operation \otimes proposed herein. In fact, the guiding principle there is the verification of the validity of a related classical logic tautology involving two-valued implications, viz., $(p \longrightarrow (p \longrightarrow q)) \models (p \longrightarrow q)$, where p, q are two-valued propositions, suitably extended to the setting of fuzzy logic connectives. However, it should be highlighted that this is the first work to view a generalization of the above tautology as a binary operation \otimes on the set of fuzzy implications \mathbb{I} , explore and study deeply the algebraic structure this operation imposes. Further, this work also shows the utility and desirability of such studies by obtaining hitherto unknown results on the representation of some families of fuzzy implications.

Further, as is stated above, our focus in this work, is as in point (B) above. An analytical study of the proposed operations, i.e., along the lines of point (A) above, whether they really lead to new implications, the properties they preserve, closures of families of fuzzy implications w.r.to these operations, etc., will be taken up in another work. Some partial results from such a study have already been presented in [22] (see also [14]).

1.4. Outline of this work

In Section 2, we begin by proposing a binary operation \otimes on the set \mathbb{I} of all fuzzy implications and show that (\mathbb{I}, \otimes) forms a monoid and discuss the properties preserved under this operation. In Section 3, noting that (\mathbb{I}, \otimes) does not become a group, we characterize the largest subgroup \mathbb{S} of (\mathbb{I}, \otimes) and based on the obtained representation of \mathbb{S} , in Section 4, we define a group action of \mathbb{S} that partitions \mathbb{I} . In Section 5, after introducing the Yager's families of f - and g -implications, we explore and illustrate that the equivalence classes obtained allow us to provide the much needed representations of these two families.

2. Monoid structure on the set of all fuzzy implications

In this section, given two fuzzy implications $I, J \in \mathbb{I}$, we begin by proposing a novel way of obtaining a new fuzzy implication from I, J . To this end, we propose a binary operation \otimes on the set \mathbb{I} of all fuzzy implications and show that (\mathbb{I}, \otimes) forms a monoid. Further, we determine some submonoids of (\mathbb{I}, \otimes) , thus showing that some desirable properties of fuzzy implications are preserved under this operation.

2.1. A Novel Way of Generating Fuzzy Implications

In the literature we can find very few methods of obtaining fuzzy implications from fuzzy implications that also give rise to some algebras on \mathbb{I} , see Chapter 6 in [3]. For an overview of the construction methods, with or without such algebraic underpinnings, please see a recent excellent survey of Massanet and Torrens [18].

Here in the following we recall a method of obtaining a new fuzzy implication from given pair of fuzzy implications that was already proposed in [22].

Definition 2.1 ([22]). For any $I, J \in \mathbb{I}$, we define $I \otimes J: [0, 1]^2 \rightarrow [0, 1]$ as

$$(I \otimes J)(x, y) = I(x, J(x, y)), \quad x, y \in [0, 1]. \quad (1)$$

The following result shows that the function $I \otimes J$ is, indeed, a fuzzy implication.

Theorem 2.2. *The function $I \otimes J$ is a fuzzy implication, i.e., $I \otimes J \in \mathbb{I}$.*

Proof. It is enough to show that $I \otimes J$ satisfies the axioms in Definition 1.1.

- (i) Let $x_1, x_2, y \in [0, 1]$ be such that $x_1 \leq x_2$. Then $J(x_1, y) \geq J(x_2, y)$. Then $(I \otimes J)(x_1, y) = I(x_1, J(x_1, y)) \geq I(x_2, J(x_2, y)) = (I \otimes J)(x_2, y)$ showing that $I \otimes J$ is decreasing in the first variable. Similarly, one can show that $I \otimes J$ is increasing in the second variable.
- (ii) $(I \otimes J)(0, 0) = I(0, J(0, 0)) = I(0, 1) = 1$,
 $(I \otimes J)(1, 1) = I(1, J(1, 1)) = I(1, 1) = 1$,
 $(I \otimes J)(1, 0) = I(1, J(1, 0)) = I(1, 0) = 0$.

□

As noted above the binary operation \otimes , firstly, gives a new way of obtaining fuzzy implications from given ones. An analytical study of \otimes , viz., the properties they preserve, powers of fuzzy implications w.r.to \otimes , their closure w.r.to some well-known families of fuzzy implications have been explored and presented in [22]. Table 2 gives the fuzzy implications obtained with the above operation performed on some of the basic fuzzy implications listed in Table 1.

Secondly, looking at \otimes as a binary operation on \mathbb{I} , one can discuss the algebraic structure imposed by \otimes on \mathbb{I} . In fact, as we show below (\mathbb{I}, \otimes) is actually a monoid.

Theorem 2.3. *(\mathbb{I}, \otimes) forms a monoid, whose identity element is given by*

$$I_{\mathbf{D}}(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ y, & \text{if } x > 0. \end{cases}$$

I	J	$I \circledast J$
$I_{\mathbf{RC}}$	$I_{\mathbf{LK}}$	$\begin{cases} 1, & \text{if } x \leq y \\ 1 - x^2 + xy, & \text{if } x > y \end{cases}$
$I_{\mathbf{GG}}$	$I_{\mathbf{RC}}$	$\begin{cases} 1, & \text{if } x \leq 1 - x + xy \\ \frac{1 - x + xy}{x}, & \text{otherwise} \end{cases}$
$I_{\mathbf{YG}}$	$I_{\mathbf{WB}}$	$I_{\mathbf{WB}}$
$I_{\mathbf{KD}}$	$I_{\mathbf{RS}}$	$\begin{cases} 1, & \text{if } x \leq y \\ 1 - x, & \text{if } x > y \end{cases}$
$I_{\mathbf{FD}}$	$I_{\mathbf{RC}}$	$\begin{cases} 1, & \text{if } x \leq 1 - x + xy \\ 1 - x + xy & \text{otherwise} \end{cases}$
$I_{\mathbf{YG}}$	$I_{\mathbf{GD}}$	$\begin{cases} 1, & \text{if } x \leq y \\ y^x, & \text{if } x > y \end{cases}$
$I_{\mathbf{GD}}$	$I_{\mathbf{LK}}$	$\begin{cases} 1, & \text{if } x \leq \frac{1+y}{2} \\ 1 - x + y, & \text{otherwise} \end{cases}$
$I_{\mathbf{RC}}$	$I_{\mathbf{KD}}$	$\max(1 - x^2, 1 - x + xy)$

Table 2: Composition of some fuzzy implications w.r.to \circledast .

Proof. From Theorem 2.2, it follows that \circledast is a closed binary operation on the set \mathbb{I} . For associativity of \circledast , let $I, J, K \in \mathbb{I}$ and $x, y \in [0, 1]$. Then

$$\begin{aligned}
(I \circledast (J \circledast K))(x, y) &= I(x, (J \circledast K)(x, y)) \\
&= I(x, J(x, K(x, y))) \\
&= (I \circledast J)(x, K(x, y)) \\
&= ((I \circledast J) \circledast K)(x, y)
\end{aligned}$$

showing that \circledast is associative. Further,

$$\begin{aligned}
(I \circledast I_{\mathbf{D}})(x, y) &= I(x, I_{\mathbf{D}}(x, y)) \\
&= \begin{cases} 1, & \text{if } x = 0, \\ I(x, y), & \text{if } x > 0. \end{cases} \\
&= I(x, y)
\end{aligned}$$

and similarly $I_{\mathbf{D}} \circledast I = I$. Thus $I_{\mathbf{D}}$ becomes the identity element in \mathbb{I} . \square

Here we recall the following theorem from [3] to give more richer structure on the set \mathbb{I} .

Theorem 2.4 ([3], Theorem 6.1.1). *The set (\mathbb{I}, \preceq) is a complete, completely distributive lattice with the lattice operations join \vee and meet \wedge defined by*

$$\begin{aligned}
(I \vee J)(x, y) &:= \max(I(x, y), J(x, y)), & x, y \in [0, 1], \\
(I \wedge J)(x, y) &:= \min(I(x, y), J(x, y)), & x, y \in [0, 1],
\end{aligned}$$

where $I, J \in \mathbb{I}$ and \preceq is the usual pointwise ordering on the set of binary functions.

Remark 2.5. From Theorem 2.4, we have $(\mathbb{I}, \preceq, \vee, \wedge)$ is a lattice. In fact, $(\mathbb{I}, \preceq, \vee, \wedge)$ is also a bounded lattice with I_0, I_1 being the least and greatest fuzzy implications. From Theorem 2.3, $(\mathbb{I}, \circledast)$ is a monoid. Together with all these operations \mathbb{I} becomes a lattice ordered monoid as the following lemma illustrates.

Lemma 2.6. *The pentuple $(\mathbb{I}, \otimes, \preceq, \vee, \wedge)$ is a lattice ordered monoid.*

Proof. It is enough to show that \otimes is compatible with lattice operations. Let $I, J, K \in \mathbb{I}$ and $x, y \in [0, 1]$. Then,

$$\begin{aligned} (I \otimes (J \vee K))(x, y) &= I(x, (J \vee K)(x, y)) \\ &= I(x, \max(J(x, y), K(x, y))) \\ &= \max(I(x, J(x, y)), I(x, K(x, y))) \\ &= \max((I \otimes J)(x, y), (I \otimes K)(x, y)) \\ &= ((I \otimes J) \vee (I \otimes K))(x, y) \end{aligned}$$

Thus $I \otimes (J \vee K) = (I \otimes J) \vee (I \otimes K)$. Similarly, one can prove the following:

$$\begin{aligned} (I \vee J) \otimes K &= (I \otimes K) \vee (J \otimes K) , \\ I \otimes (J \wedge K) &= (I \otimes J) \wedge (I \otimes K) , \\ (I \wedge J) \otimes K &= (I \otimes K) \wedge (J \otimes K) . \end{aligned}$$

□

2.2. Basic Properties of Fuzzy Implications

In the following we list a few of the most important properties of fuzzy implications. Note that they are also a generalization of the corresponding properties of the classical implication to multi valued logic.

Definition 2.7 ([3], **Definition 1.3.1**). A fuzzy implication I is said to satisfy

- (i) the left neutrality property (NP) if

$$I(1, y) = y, \quad y \in [0, 1] . \quad (\text{NP})$$

- (ii) the ordering property (OP), if

$$x \leq y \iff I(x, y) = 1 . \quad (\text{OP})$$

- (iii) the identity principle (IP), if

$$I(x, x) = 1, \quad x \in [0, 1] . \quad (\text{IP})$$

- (iv) the exchange principle (EP), if

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1] . \quad (\text{EP})$$

- (v) the law of importation w.r.to a t-norm T , if

$$I(x, I(y, z)) = I(T(x, y), z), \quad x, y, z \in [0, 1] . \quad (\text{LI})$$

Let Φ denote the set of all increasing bijections $\varphi: [0, 1] \rightarrow [0, 1]$. It follows automatically that if $\varphi \in \Phi$ then $\varphi(0) = 0$ and $\varphi(1) = 1$.

Definition 2.8. ([15], Pg. 156) Let $I \in \mathbb{I}$ and $\varphi \in \Phi$. Define $I_\varphi: [0, 1]^2 \rightarrow [0, 1]$ by

$$I_\varphi(x, y) = \varphi^{-1}(I(\varphi(x), \varphi(y))), \quad x, y \in [0, 1].$$

Clearly $I_\varphi \in \mathbb{I}$ for all $\varphi \in \Phi$ and is called the φ -conjugate of I . If $I_\varphi = I$ for all $\varphi \in \Phi$, then I is called *self conjugate* or *invariant*. The set of all *invariant* or *self conjugate* fuzzy implications is denoted by \mathbb{I}_{inv} .

Let us denote by \mathbb{I}_P the subset of \mathbb{I} in which every element satisfies the property P , where $P \in \{(\text{NP}), (\text{EP}), (\text{IP}), (\text{OP})\}$. For instance, $\mathbb{I}_{\text{NP}} = \{I \in \mathbb{I} \mid I \text{ satisfies } (\text{NP})\}$.

Lemma 2.9. *The subsets $\mathbb{I}_{\text{NP}}, \mathbb{I}_{\text{inv}}$ are submonoids of (\mathbb{I}, \otimes) , while \mathbb{I}_{IP} is a subsemigroup of (\mathbb{I}, \otimes) .*

Proof. It is enough to show that $\mathbb{I}_{\text{NP}}, \mathbb{I}_{\text{inv}}, \mathbb{I}_{\text{IP}}$ are closed w.r.to \otimes . Further, note that $I_{\text{D}} \in \mathbb{I}_{\text{inv}}, \mathbb{I}_{\text{NP}}$ and so they also have the identity element. Here we show that \mathbb{I}_{inv} is closed w.r.to \otimes , while the closure of \otimes over $\mathbb{I}_{\text{NP}}, \mathbb{I}_{\text{IP}}$ follows similarly.

Let $I, J \in \mathbb{I}_{\text{inv}}$ and $\varphi \in \Phi$. For any $x, y \in [0, 1]$, we have

$$\begin{aligned} (I \otimes J)_\varphi(x, y) &= \varphi^{-1}((I \otimes J)(\varphi(x), \varphi(y))) \\ &= \varphi^{-1}(I(\varphi(x), J(\varphi(x), \varphi(y)))) \\ &= \varphi^{-1}(I(\varphi(x), \varphi(J_\varphi(x, y)))) \\ &= \varphi^{-1}(I(\varphi(x), \varphi(J(x, y)))) \\ &= I_\varphi(x, J(x, y)) = I(x, J(x, y)) \\ &= (I \otimes J)(x, y). \end{aligned}$$

Thus $(I \otimes J)_\varphi = I \otimes J$, for all $\varphi \in \Phi$. □

The above result shows that \otimes carries over (NP), (IP) and invariance of the original pair of fuzzy implications to the newly generated one. However, (EP) and (OP) are not preserved as can be seen from Example 2.10.

Example 2.10. (i) From Table 1.4 in [3], it is clear that $I = I_{\text{GD}}, J = I_{\text{LK}}$ satisfy (OP). However, $I \otimes J$ does not satisfy (OP) because $(I \otimes J)(0.4, 0.2) = 1$ but $0.4 > 0.2$ (see Table 2 for their explicit formula).

(ii) Again from Table 1.4 in [3], it is clear that $I = I_{\text{RC}}, J = I_{\text{KD}}$ satisfy (EP). However, $(I \otimes J)(0.3, (I \otimes J)(0.8, 0.5)) = 0.91$ where as $(I \otimes J)(0.8, (I \otimes J)(0.3, 0.5)) = 0.928$. Thus $I \otimes J$ does not satisfy (EP) even if I, J satisfy the same.

3. Subgroups of (\mathbb{I}, \otimes) : Characterization and Representation

3.1. (\mathbb{I}, \otimes) is not a group

From Theorem 2.3 we know that (\mathbb{I}, \otimes) is a monoid. However, the following illustrates why the richer group structure is not available on (\mathbb{I}, \otimes) . Take $I_{\mathbf{1}} \in \mathbb{I}$. It is easy to check that $I_{\mathbf{1}}$ is a right zero element of (\mathbb{I}, \otimes) . i.e., $I \otimes I_{\mathbf{1}} = I_{\mathbf{1}}$ for all $I \in \mathbb{I}$. Thus there does not exist any $J \in \mathbb{I}$ such that $J \otimes I_{\mathbf{1}} = I_{\mathbf{D}}$. i.e., the inverse of $I_{\mathbf{1}} \in \mathbb{I}$ w.r.to \otimes does not exist. Thus the algebraic structure (\mathbb{I}, \otimes) is only a monoid and not a group.

Remark 3.1. (i) Note that $I_{\mathbf{1}}$ is not the only right zero element. In fact, every fuzzy implication of the type K^δ given below is a right zero element of (\mathbb{I}, \otimes) , for any $\delta \in]0, 1]$

$$K^\delta(x, y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } (x = 1 \text{ and } y \geq \delta), \\ 0, & \text{otherwise.} \end{cases}$$

(ii) One cannot apply some well-known techniques of obtaining a group, viz., the Grothendieck construction method of obtaining an abelian group from a commutative monoid, due to both the presence of zero elements in (\mathbb{I}, \otimes) and also the absence of commutativity.

3.2. Subgroups of the monoid (\mathbb{I}, \otimes)

Though (\mathbb{I}, \otimes) is not a group, there still exist many subgroups of this monoid. To this end, we determine the invertible elements of (\mathbb{I}, \otimes) . The following example presents an example of a fuzzy implication that is invertible w.r.to \otimes .

Example 3.2. The fuzzy implication defined by

$$I(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ y^3, & \text{if } x > 0, \end{cases}$$

is invertible w.r.to \otimes in \mathbb{I} , because there exists a unique fuzzy implication

$$J(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ y^{\frac{1}{3}}, & \text{if } x > 0, \end{cases}$$

such that $I \otimes J = I_{\mathbf{D}} = J \otimes I$.

The following lemma characterizes the set of all invertible elements of the monoid (\mathbb{I}, \otimes) .

Lemma 3.3. *Let $I \in \mathbb{I}$. Then I is invertible w.r.to \otimes if and only if there exists a unique $J \in \mathbb{I}$ such that for any $x \in (0, 1]$ and $y \in [0, 1]$*

$$I(x, J(x, y)) = y = J(x, I(x, y)) . \quad (2)$$

Proof. (\implies): Let $I \in \mathbb{I}$ be invertible w.r.to \otimes . i.e., there exists a unique $J \in \mathbb{I}$ such that $I \otimes J = I_{\mathbf{D}} = J \otimes I$. In otherwords,

$$I(x, J(x, y)) = I_{\mathbf{D}}(x, y) = J(x, I(x, y)), \quad x, y \in [0, 1].$$

But for $x > 0$, $I_{\mathbf{D}}(x, y) = y$. Thus for $x > 0$, $I(x, J(x, y)) = y = J(x, I(x, y))$.

(\impliedby): Conversely, assume that there exists a unique $J \in \mathbb{I}$ such that for $x > 0$, $I(x, J(x, y)) = y = J(x, I(x, y))$. Since $I, J \in \mathbb{I}$ and $I \otimes J, J \otimes I \in \mathbb{I}$, we have $I(x, J(x, y)) = I_{\mathbf{D}}(x, y) = J(x, I(x, y))$. Thus I is invertible w.r.to \otimes . \square

Lemma 3.4. *The solutions of Eq.(2), for all $x \in (0, 1]$ and $y \in [0, 1]$, are of the form $I(x, y) = \varphi(y)$ and $J(x, y) = \varphi^{-1}(y)$, for some $\varphi \in \Phi$.*

Proof. Let $I, J \in \mathbb{I}$ satisfy (2), i.e., $I(x, J(x, y)) = y = J(x, I(x, y))$, for all $x > 0$ and $y \in [0, 1]$.

Let $x_0 > 0$ be fixed arbitrarily and define two functions $\varphi_{x_0}, \psi_{x_0} : [0, 1] \longrightarrow [0, 1]$ as $\varphi_{x_0}(y) = I(x_0, y)$ and $\psi_{x_0}(y) = J(x_0, y)$. Clearly, both $\varphi_{x_0}, \psi_{x_0}$ are increasing functions on $[0, 1]$.

Then $I(x_0, J(x_0, y)) = \varphi_{x_0}(\psi_{x_0}(y)) = (\varphi_{x_0} \circ \psi_{x_0})(y) = y$ for every $y \in [0, 1]$. Similarly, $J(x_0, I(x_0, y)) = \psi_{x_0}(\varphi_{x_0}(y)) = (\psi_{x_0} \circ \varphi_{x_0})(y) = y$ for every $y \in [0, 1]$. Thus $\varphi_{x_0} = \psi_{x_0}^{-1}$ and φ_{x_0} is a bijection. Hence $\varphi_{x_0} \in \Phi$ for every $x_0 > 0$.

Since x_0 is chosen arbitrarily, $\varphi_x = \psi_x^{-1}$ for all $x > 0$. Thus for $x > 0$, I, J are of the form, $I(x, y) = \varphi_x(y)$ and $J(x, y) = \varphi_x^{-1}(y)$.

Let $0 < x_1 \leq x_2$. Then $I(x_1, y) \geq I(x_2, y)$ implies that $\varphi_{x_1}(y) \geq \varphi_{x_2}(y)$ and $J(x_1, y) \geq J(x_2, y)$ implies that $\varphi_{x_1}^{-1}(y) \geq \varphi_{x_2}^{-1}(y)$ for all $y \in [0, 1]$. Now,

$$\begin{aligned} \varphi_{x_1}^{-1} \geq \varphi_{x_2}^{-1} &\implies \varphi_{x_1} \circ \varphi_{x_1}^{-1} \geq \varphi_{x_1} \circ \varphi_{x_2}^{-1} \\ &\implies \mathbf{id} \geq \varphi_{x_1} \circ \varphi_{x_2}^{-1} \\ &\implies \mathbf{id} \geq \varphi_{x_1} \circ \varphi_{x_2}^{-1} \geq \varphi_{x_2} \circ \varphi_{x_2}^{-1} \\ &\implies \mathbf{id} \geq \varphi_{x_1} \circ \varphi_{x_2}^{-1} \geq \mathbf{id} , \end{aligned}$$

from which it follows $\varphi_{x_1} \circ \varphi_{x_2}^{-1} \equiv \mathbf{id}$, i.e., $\varphi_{x_1}(y) = \varphi_{x_2}(y)$ for all $y \in [0, 1]$. Since x_1, x_2 are arbitrarily chosen $\varphi_{x_1} \equiv \varphi_{x_2} \equiv \varphi$ (say) for all $x_1, x_2 > 0$. Thus $I(x, y) = \varphi(y)$ and $J(x, y) = \varphi^{-1}(y)$, for some $\varphi \in \Phi$. \square

Now we are ready to give the representation of every invertible element of the monoid (\mathbb{I}, \otimes) and thus determine its largest subgroup. From Lemmata 3.3 and 3.4 we have the following result.

Theorem 3.5. $I \in \mathbb{I}$ is invertible w.r.to \otimes if and only if

$$I(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(y), & \text{if } x > 0, \end{cases} \quad (3)$$

where the function $\varphi: [0, 1] \rightarrow [0, 1]$ is an increasing bijection.

Clearly, the largest subgroup of (\mathbb{I}, \otimes) is one that contains all the invertible elements of \mathbb{I} w.r.to \otimes . Let \mathbb{S} be the set of all invertible elements of (\mathbb{I}, \otimes) , i.e., \mathbb{S} is the set of all fuzzy implications of the form (3) for some $\varphi \in \Phi$. In fact, (\mathbb{S}, \otimes) is the largest subgroup contained in (\mathbb{I}, \otimes) .

Example 3.6. Let $\mathbb{S}_R, \mathbb{S}_Q$ be the sets of all fuzzy implications of the form

$$I_r(x, y) = \begin{cases} 1, & \text{if } x = 0 \\ y^r, & \text{if } x > 0 \end{cases}, \text{ for every } r \in \mathbb{R}^{>0},$$

$$I_q(x, y) = \begin{cases} 1, & \text{if } x = 0 \\ y^q, & \text{if } x > 0 \end{cases}, \text{ for every } q \in \mathbb{Q}^{>0},$$

respectively. It is easy to see that $\mathbb{S}_R, \mathbb{S}_Q$ are subgroups of (\mathbb{I}, \otimes) .

Proposition 3.7. Every element of \mathbb{S} satisfies (EP), i.e., $\mathbb{S} \subsetneq \mathbb{I}_{\mathbf{EP}}$.

Proof. Let $I \in \mathbb{S}$. From (3) we have that

$$I(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(y), & \text{if } x > 0, \end{cases}$$

for some $\varphi \in \Phi$. Let $x, y, z \in [0, 1]$. If $x = 0$ or $y = 0$ then we are done. So let $x > 0, y > 0$. Now $I(x, I(y, z)) = I(x, \varphi(z)) = \varphi(\varphi(z))$ and $I(y, I(x, z)) = I(y, \varphi(z)) = \varphi(\varphi(z))$ thus showing that I has (EP). \square

Remark 3.8. (i) Clearly, the inclusion in Proposition 3.7 is strict. For example $I_{\mathbf{RC}} \in \mathbb{I}_{\mathbf{EP}}$ but $I_{\mathbf{RC}} \notin \mathbb{S}$.

(ii) From the discussion following Lemma 2.9, we see that $(\mathbb{I}_{\mathbf{EP}}, \otimes)$ is not closed, while we have obtained a subset \mathbb{S} of $\mathbb{I}_{\mathbf{EP}}$ which is closed w.r.to \otimes .

(iii) Note that \mathbb{S} is not the largest subset of $\mathbb{I}_{\mathbf{EP}}$ that is closed w.r.to \otimes . For instance, if we define \mathbb{U} as the set of all fuzzy implications of the form

$$I(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ \psi(y), & \text{if } x > 0, \end{cases}$$

for some increasing function, not necessarily a *bijection*, $\psi: [0, 1] \rightarrow [0, 1]$ such that $\psi(0) = 0$ and $\psi(1) = 1$, then every element of \mathbb{U} satisfies (EP). Obviously $\mathbb{S} \subsetneq \mathbb{U}$.

(iv) Elements of \mathbb{S} do not satisfy either (OP) or (IP) and the only element satisfying (NP) is the identity $I_{\mathbf{D}}$ of \otimes .

3.3. (\mathbb{S}, \otimes) is isomorphic to (Φ, \circ)

Let \circ denote the usual composition of functions. Then it is well known that (Φ, \circ) is a group. Interestingly, the subgroup (\mathbb{S}, \otimes) is isomorphic to (Φ, \circ) , as the following result illustrates.

Theorem 3.9. *The groups (Φ, \circ) , (\mathbb{S}, \otimes) are isomorphic to each other.*

Proof. Let $h: \Phi \rightarrow \mathbb{S}$ be defined by $h(\varphi) = I$ where

$$I(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(y), & \text{if } x > 0. \end{cases}$$

It is easy to see that the map h is one one and onto. Let $\varphi_1, \varphi_2 \in \Phi$ and $h(\varphi_1) = I_1, h(\varphi_2) = I_2$ where

$$I_i(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi_i(y), & \text{if } x > 0, \end{cases}$$

for $i = 1, 2$. Now

$$\begin{aligned} (h(\varphi_1) \otimes h(\varphi_2))(x, y) &= (I_1 \otimes I_2)(x, y) \\ &= I_1(x, I_2(x, y)) \\ &= \begin{cases} 1, & \text{if } x = 0, \\ \varphi_1(\varphi_2(y)), & \text{if } x > 0, \end{cases} \\ &= h(\varphi_1 \circ \varphi_2)(x, y). \end{aligned}$$

Thus h is an isomorphism. □

4. Partitions obtained from group action of \mathbb{S} on \mathbb{I}

Our stated motivation for this work is to obtain rich algebraic structures on \mathbb{I} that would throw more light on fuzzy implications by providing newer insights and connections between existing families and properties of fuzzy implications. Unfortunately, we find that (\mathbb{I}, \otimes) is not even a group, which precludes further applications of known results. For instance, the theory and results based on normal subgroups cannot be applied to obtain any kind of unique decomposition and hence, some characterization or representation results.

However, in this section, with the help of the largest subgroup \mathbb{S} of the monoid (\mathbb{I}, \otimes) obtained in Section 3.2, we define a group action, which further partitions the set \mathbb{I} . As shown in Section 5, the equivalence classes obtained thus lead us to some hitherto unknown representations of many well-known families of fuzzy implications, thus vindicating both the proposed binary operation \otimes and, in general, the algebraic approach taken in this work.

4.1. Pseudo-conjugacy of Fuzzy Implications

Definition 4.1. ([20], Pg. 488) Let $(G, *)$ be a group and H be a non empty set. A function $\bullet: G \times H \rightarrow H$ is called a group action if, for all $g_1, g_2 \in G$ and $h \in H$, \bullet satisfies the following two conditions:

- (i) $g_1 \bullet (g_2 \bullet h) = (g_1 * g_2) \bullet h$.
- (ii) $e \bullet h = h$ where e is the identity of G .

Definition 4.2. Let $\bullet: \mathbb{S} \times \mathbb{I} \rightarrow \mathbb{I}$ be a map defined by

$$(K, I) \rightarrow K \bullet I = K \otimes I \otimes K^{-1}.$$

Lemma 4.3. *The function \bullet is a group action of \mathbb{S} on \mathbb{I} .*

Proof. (i) Let $K_1, K_2 \in \mathbb{S}$ and $I \in \mathbb{I}$.

$$\begin{aligned} K_1 \bullet (K_2 \bullet I) &= K_1 \otimes (K_2 \bullet I) \otimes K_1^{-1} \\ &= K_1 \otimes K_2 \otimes I \otimes K_2^{-1} \otimes K_1^{-1} \\ &= (K_1 \otimes K_2) \otimes I \otimes (K_1 \otimes K_2)^{-1} \\ &= (K_1 \otimes K_2) \bullet I . \end{aligned}$$

(ii) Similarly, $I_{\mathbf{D}} \bullet I = I_{\mathbf{D}} \otimes I \otimes I_{\mathbf{D}}^{-1} = I$, since $I_{\mathbf{D}}$ is the identity of (\mathbb{I}, \otimes) .

Thus \bullet is group action of \mathbb{S} on \mathbb{I} . □

Definition 4.4. Let $I, J \in \mathbb{I}$. Define $I \sim J \iff J = K \bullet I$ for some $K \in \mathbb{S}$. In other words, $I \sim J \iff J = K \otimes I \otimes K^{-1}$ for some $K \in \mathbb{S}$.

Lemma 4.5. *The relation \sim is an equivalence relation and it partitions the set \mathbb{I} .*

Proof. A direct verification will show that \sim is an equivalence relation. □

Remark 4.6. Let $I \in \mathbb{I}$. Then the equivalence class containing I will be of the form

$$[I] = \{J \in \mathbb{I} \mid J = K \otimes I \otimes K^{-1} \text{ for some } K \in \mathbb{S}\}.$$

Since any $K \in \mathbb{S}$ is of the form

$$K(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(y), & \text{if } x > 0. \end{cases}$$

for some $\varphi \in \Phi$, we have that, if $J \in [I]$, then $J(x, y) = \varphi(I(x, \varphi^{-1}(y)))$ for all $x, y \in [0, 1]$.

Definition 4.7. If $I, J \in \mathbb{I}$ are related as $J(x, y) = \varphi(I(x, \varphi^{-1}(y)))$ for all $x, y \in [0, 1]$ for some $\varphi \in \Phi$, we say that J is a φ -**pseudo conjugate** of I , or alternately and equivalently, I is a φ^{-1} -pseudo conjugate of J .

4.2. Properties preserved by Pseudo-conjugacy

Interestingly, these equivalence classes do preserve some properties of the fuzzy implications as the following lemma illustrates.

Lemma 4.8. *Let $I \in \mathbb{I}$ and $J \in [I]$. Then*

- (i) *I satisfies (LI) w.r.to $T \iff J$ satisfies (LI) w.r.to T .*
- (ii) *I satisfies (EP) $\iff J$ satisfies (EP).*
- (iii) *I satisfies (NP) $\iff J$ satisfies (NP).*
- (iv) *I is continuous $\iff J$ is continuous.*
- (v) *Range of I is trivial \iff Range of J is trivial.*

Proof. In the following we prove only (i) and (ii), as points (iii)–(v) can be proven similarly.

- (i) Let I satisfies (LI) w.r.to a t-norm T and $J \in [I]$. Then $J(x, y) = \varphi(I(x, \varphi^{-1}(y)))$ for some $\varphi \in \Phi$. Now it follows that

$$\begin{aligned} J(x, J(y, z)) &= J(x, \varphi(I(y, \varphi^{-1}(z))) \\ &= \varphi(I(x, I(y, \varphi^{-1}(z)))) \\ &= \varphi(I(T(x, y), \varphi^{-1}(z))) \\ &= J(T(x, y), z) . \end{aligned}$$

Thus J satisfies (LI). The other implication can be proven similarly.

- (ii) Let I satisfy (EP) and $J \in [I]$. Then from Remark 4.6, it follows that J is a φ -pseudo conjugate of I , i.e., $J(x, y) = \varphi(I(x, \varphi^{-1}(y)))$ for some $\varphi \in \Phi$. Now,

$$\begin{aligned}
J(x, J(y, z)) &= J(x, \varphi(I(y, \varphi^{-1}(z)))) \\
&= \varphi(I(x, I(y, \varphi^{-1}(z)))) \\
&= \varphi(I(y, I(x, \varphi^{-1}(z)))) \\
&= J(y, \varphi(I(x, \varphi^{-1}(z)))) \\
&= J(y, J(x, z)) .
\end{aligned}$$

Thus J satisfies (EP). The other implication can be proven similarly. □

The following Lemmata show that unlike fuzzy implications satisfying (NP) or (EP), not all the φ -pseudo conjugates of a fuzzy implication satisfying (IP) (or (OP)) satisfy (IP) (or (OP)). The proofs of the following lemmata are straightforward.

Lemma 4.9. *Let $I \in \mathbb{I}_{\text{IP}}$ and $J \in [I]$, i.e., $J(x, y) = \varphi(I(x, \varphi^{-1}(y)))$ for some $\varphi \in \Phi$. Then J satisfies (IP) only if $\varphi(x) \leq x$ for all $x \in [0, 1]$.*

Lemma 4.10. *Let $I \in \mathbb{I}_{\text{OP}}$ and $J \in [I]$, i.e., $J(x, y) = \varphi(I(x, \varphi^{-1}(y)))$ for some $\varphi \in \Phi$. Then J satisfies (OP) only if $\varphi(x) = x$ for all $x \in [0, 1]$.*

5. Representation results based on the group action of \mathbb{S} on \mathbb{I}

As noted in the Introduction, fuzzy implications have many applications. Hence, there is always a need for generating newer implications with myriad properties that are suitable for a specific task.

The method we have proposed here, based on the binary operation \otimes , obtains a new fuzzy implication from given two fuzzy implications. However, one needs a way of generating fuzzy implications systematically in the first place. There are mainly two ways of generating fuzzy implications in the literature, which give rise to some established families of fuzzy implications, viz.,

- (i) From other fuzzy logic connectives, from whence we obtain, for instance, the families of (S,N)-, R-, QL-implications,
- (ii) From monotone functions, from whence we obtain, for instance, the families of f - and g -implications proposed by Yager [23].

The analytical aspects of the families of (S,N), R, QL-implications obtained from other fuzzy logic connectives, viz., the properties they satisfy, their characterizations, representations and intersections among them have been quite well studied and the results largely established, see [3], Chapters 2 and 4.

Yager [23] proposed two ways of obtaining fuzzy implications from generators of t-norms [13], which are monotone functions, which have come to be known as the families of f - and g -implications. However, in the case of the families of f - and g -implications, the analytical study is far from complete.

For instance, while the properties that these two families satisfy have been known from the time of their introduction, see Yager [23], further tautologies that they satisfy [4], and the intersections among these two families and with the families of (S, N), R-implications [2] and with the families of QL, D-implications [16], were dealt with only a little later. Some characterization results have been proposed only recently by Massanet and Torrens [17]. However, representations of fuzzy implications from these two families is as yet unknown.

In this section, we explore and illustrate that the group action defined in Subsection 4.1 and the equivalence classes obtained from them allow us to provide the much needed representations of these two families.

5.1. f -implications

Yager [23] proposed a method of generating fuzzy implications from unary monotonic functions on $[0,1]$. These monotonic functions are called f -generators. In fact, these f -generators are nothing but additive generators of continuous Archimedean t -norms (see [13], pg. 74).

Definition 5.1 ([3], **Definition 3.1.1** and [23], **Pg. 197**). Let $f : [0, 1] \longrightarrow [0, \infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I : [0, 1]^2 \longrightarrow [0, 1]$ defined by

$$I(x, y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0, 1],$$

with the understanding $0 \cdot \infty = 0$, is called an f -implication and the function f is called an f -generator of I . Here we write $I = I_f$ to emphasize explicitly the relation between I and f .

The family of all f -implications will be denoted by \mathbb{I}_f .

Example 5.2 ([3], **Example 3.1.3**). (i) If we take the f -generator $f_I(x) = -\ln x$, then we obtain the Yager implication $I_{\mathbf{YG}}$ (See, Table 1).

(ii) If we take the f -generator $f_c(x) = 1 - x$, then we obtain the Reichenbach implication $I_{\mathbf{RC}}$ (See, Table 1).

(iii) Let us consider the f -generator $f(x) = \cos(\frac{\pi}{2}x)$, which is a continuous and strictly decreasing trigonometric function such that $f(0) = \cos 0 = 1$ and $f(1) = \cos \frac{\pi}{2} = 0$. Its inverse is given by $f^{-1}(x) = \frac{2}{\pi} \cdot \cos^{-1} x$ and the corresponding f -implication is given by

$$I_f(x, y) = \frac{2}{\pi} \cos^{-1} \left(x \cdot \cos \left(\frac{\pi}{2} y \right) \right), \quad x, y \in [0, 1].$$

(iv) Let us consider the Frank's class of additive generators (see [13], pg. 110) as the f -generators which are given by

$$f^s(x) = -\ln \left(\frac{s^x - 1}{s - 1} \right), \quad s > 0, s \neq 1.$$

Then $f^s(0) = \infty$, its inverse is given by $(f^s)^{-1}(x) = \log_s (1 + (s - 1)e^{-x})$ and the corresponding f -implication, for every s , is given by

$$I_{f^s}(x, y) = \log_s (1 + (s - 1)^{1-x} (s^y - 1)^x), \quad x, y \in [0, 1].$$

(v) If we take the Yager's class of additive generators, viz., $f^\lambda(x) = (1 - x)^\lambda$, where $\lambda \in (0, \infty)$, as the f -generators, then $f^\lambda(0) = 1$, its inverse is given by $(f^\lambda)^{-1}(x) = 1 - x^{\frac{1}{\lambda}}$ and the corresponding f -implication, for every $\lambda \in (0, \infty)$, is given by

$$I_{f^\lambda}(x, y) = 1 - x^{\frac{1}{\lambda}} (1 - y), \quad x, y \in [0, 1].$$

5.2. f -implications - The two subfamilies

As shown by Baczyński and Jayaram [2] if f is an f -generator such that $f(0) < \infty$, then the function $f_1 : [0, 1] \rightarrow [0, 1]$ defined by

$$f_1(x) = \frac{f(x)}{f(0)}, \quad x \in [0, 1], \quad (4)$$

is a well defined f -generator and the f -implications defined from both f and f_1 are identical, i.e., $I_f \equiv I_{f_1}$ and moreover $f_1(0) = 1$. In other words, it is enough to consider only decreasing generators f for which $f(0) = \infty$ or $f(0) = 1$.

Let us denote by

- $\mathbb{I}_{\mathbb{R}, \infty}$ - the family of all f -implications such that $f(0) = \infty$,

- $\mathbb{I}_{\mathbb{F},1}$ – the family of all f - implications such that $f(0) = 1$,
- Clearly, $\mathbb{I}_{\mathbb{F}} = \mathbb{I}_{\mathbb{F},\infty} \cup \mathbb{I}_{\mathbb{F},1}$.

Remark 5.3. Note that for every f - generator f , the function $f \circ \varphi: [0, 1] \rightarrow [0, \infty]$ is strictly decreasing and $(f \circ \varphi)(1) = 0$ for all $\varphi \in \Phi$. Thus $f \circ \varphi$ is also an f -generator for every $\varphi \in \Phi$.

Our first result shows that if I is an f -implication then every φ -pseudo conjugate of I is also an f -implication.

Lemma 5.4. *Let $I \in \mathbb{I}$ and $J \in [I]$. Then $I \in \mathbb{I}_{\mathbb{F}} \iff J \in \mathbb{I}_{\mathbb{F}}$.*

Proof. Let $I \in \mathbb{I}_{\mathbb{F}}$ and $J \in [I]$. Then $I(x, y) = f^{-1}(x \cdot f(y))$ for some generator f . Now,

$$\begin{aligned} J(x, y) &= \varphi(I(x, \varphi^{-1}(y))) \\ &= \varphi(f^{-1}(x \cdot f(\varphi^{-1}(y)))) \\ &= (f \circ \varphi^{-1})^{-1}(x \cdot (f \circ \varphi^{-1})(y)) \\ &= I_{f \circ \varphi^{-1}}(x, y) \end{aligned}$$

Thus J is an f -implication. Analogously one can prove the converse. \square

In fact, the following two results show that Lemma 5.4 can be made even stronger.

Lemma 5.5. *Let $I \in \mathbb{I}$ and $J \in [I]$. Then $I \in \mathbb{I}_{\mathbb{F},\infty} \iff J \in \mathbb{I}_{\mathbb{F},\infty}$.*

Proof. Let I be an f -implication generated by some f -generator f such that $f(0) = \infty$. Let $J \in [I]$. From Lemma 5.4, it follows that $J = I_{f \circ \varphi^{-1}}$ for some $\varphi \in \Phi$. From Remark 5.3, it follows that $f \circ \varphi^{-1}$ is also an f -generator. Moreover $(f \circ \varphi^{-1})(0) = f(\varphi^{-1}(0)) = f(0) = \infty$. Thus $J \in \mathbb{I}_{\mathbb{F},\infty}$. \square

Corollary 5.6. *Let $I \in \mathbb{I}$ and $J \in [I]$. Then $I \in \mathbb{I}_{\mathbb{F},1} \iff J \in \mathbb{I}_{\mathbb{F},1}$.*

5.3. Representation of f -implications

As mentioned earlier, we give the first representation results for the family of f -implications. Our results, based on the group action and the equivalence classes obtained therefrom, show that every f -implication is a φ -pseudo conjugate of either the Yager implication $I_{\mathbf{YG}}$ or the Reichenbach implication $I_{\mathbf{RC}}$. We would like to highlight that this fact is not at all obvious, see for instance, Example 5.2 above.

Theorem 5.7. $\mathbb{I}_{\mathbb{F},\infty} = [I_{\mathbf{YG}}]$.

Proof. We know that $I_{\mathbf{YG}}$ is an f -implication with the generator $f(x) = -\ln x$. Observe that $f(0) = \infty$ and hence $I_{\mathbf{YG}} \in \mathbb{I}_{\mathbb{F},\infty}$. Let $J \in [I_{\mathbf{YG}}]$. From Lemma 5.5, it follows that $J \in \mathbb{I}_{\mathbb{F},\infty}$. Thus $[I_{\mathbf{YG}}] \subseteq \mathbb{I}_{\mathbb{F},\infty}$.

Now, let $I \in \mathbb{I}_{\mathbb{F},\infty}$. i.e., $I = I_f$ for some f -generator f such that $f(0) = \infty$. Take $\varphi(x) = f^{-1}(-\ln x)$. Then $\varphi(0) = 0$ and $\varphi(1) = 1$. Moreover φ is an increasing bijection and hence $\varphi \in \Phi$. Take $f_l(x) = -\ln x$. Then $(f_l \circ \varphi^{-1})(x) = f_l(e^{-f(x)}) = -\ln(e^{-f(x)}) = f(x)$. Thus $I = I_f = I_{f_l \circ \varphi^{-1}}$. This implies that $I \in [I_{\mathbf{YG}}]$ and consequently $\mathbb{I}_{\mathbb{F},\infty} \subseteq [I_{\mathbf{YG}}]$. \square

Theorem 5.8. $\mathbb{I}_{\mathbb{F},1} = [I_{\mathbf{RC}}]$.

Proof. We know that $I_{\mathbf{RC}}$ is an f -implication with the f -generator $f(x) = 1 - x$. Note that $f(0) = 1$ and hence $I_{\mathbf{RC}} \in \mathbb{I}_{\mathbb{F},1}$. Let $J \in [I_{\mathbf{RC}}]$. From Corollary 5.6, it follows that $J \in \mathbb{I}_{\mathbb{F},1}$. Thus $[I_{\mathbf{RC}}] \subseteq \mathbb{I}_{\mathbb{F},1}$.

Now, let $I \in \mathbb{I}_{\mathbb{F},1}$. Then $I = I_f$ for some f -generator f such that $f(0) = 1$. Take $\varphi(x) = 1 - f(x)$. It is clear that $\varphi(0) = 0, \varphi(1) = 1$ and $\varphi(x)$ is increasing bijection on $[0, 1]$. Moreover $I = I_f = I_{f_c \circ \varphi^{-1}}$ where $f_c(x) = 1 - x$. Hence $I \in [I_{\mathbf{RC}}]$. \square

Corollary 5.9. (i) *An $I \in \mathbb{I}_{\mathbb{F},\infty}$ if and only if $I(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ \varphi([\varphi^{-1}(y)]^x), & \text{if } x > 0 \text{ or } y > 0 \end{cases}$, for some*

$\varphi \in \Phi$.

(ii) *An $I \in \mathbb{I}_{\mathbb{F},1}$ if and only if $I(x, y) = \varphi(1 - x + x\varphi^{-1}(y))$, for some $\varphi \in \Phi$.*

(iii) $\mathbb{I}_{\mathbb{F}} = \mathbb{I}_{\mathbb{F},\infty} \cup \mathbb{I}_{\mathbb{F},1} = [I_{\mathbf{YG}}] \cup [I_{\mathbf{RC}}]$.

5.4. g -implications.

In the same paper, Yager [23] proposed another family of fuzzy implications, viz., g -implications from increasing monotone functions on $[0, 1]$. These monotone functions are called g -generators.

Definition 5.10 ([3], **Definition 3.2.1** and [23], **Pg. 201**). Let $g: [0, 1] \rightarrow [0, \infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I: [0, 1]^2 \rightarrow [0, 1]$ defined by

$$I(x, y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \quad x, y \in [0, 1],$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a g -implication, where the function $g^{(-1)}$ is the pseudo inverse of g given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x), & \text{if } x \in [0, g(1)], \\ 1, & \text{if } x \in [g(1), \infty], \end{cases}$$

and the function g is called g -generator of I . Here we write $I = I_g$ to emphasize explicitly the relation between I and g .

The family of all g -implications will be denoted by \mathbb{I}_G .

Example 5.11 ([3], **Example 3.2.4**). Much like the f -generators, the g -generators can be seen as continuous additive generators of continuous Archimedean t -conorms (see [13], page. 79). Once again, the following examples illustrate this idea.

- (i) If we take the g -generator $g_l(x) = -\ln(1-x)$, then we obtain the following fuzzy implication:

$$I_{g_l}(x, y) = I(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ 1 - (1-y)^{\frac{1}{x}}, & \text{if } x \in (0, 1] \text{ or } y \in (0, 1]. \end{cases}$$

- (ii) If we take the g -generator $g_c(x) = x$, then we obtain the Goguen implication $I_{g_c}(x, y) = I_{\mathbf{GG}}$.
(iii) One can easily calculate that for the g -generator $g(x) = -\frac{1}{\ln x}$ we obtain the Yager implication $I_{\mathbf{YG}}$, which is also an f -implication.
(iv) If we take the trigonometric function $g_t(x) = \tan\left(\frac{\pi}{2}x\right)$, which is a continuous function with $g_t(0) = 0$, $g_t(1) = \infty$, as the g -generator, then its inverse is $g_t^{-1}(x) = \frac{2}{\pi} \tan^{-1}(x)$ and we obtain the following g -implication:

$$I_{g_t}(x, y) = \frac{2}{\pi} \tan^{-1}\left(\frac{1}{x} \cdot \tan\left(\frac{\pi}{2}y\right)\right), \quad x, y \in [0, 1].$$

- (v) If we take the Yager's class of additive generators, $g^\lambda(x) = x^\lambda$, where $\lambda \in (0, \infty)$, as the g -generators, then $g^\lambda(1) = 1$ for every λ , its pseudo-inverse is given by $(g^\lambda)^{(-1)}(x) = \min(1, x^{\frac{1}{\lambda}})$ and the g -implication is given by

$$I_{g^\lambda}(x, y) = \min\left(1, \frac{y}{x^{\frac{1}{\lambda}}}\right) = \begin{cases} 1, & \text{if } x^{\frac{1}{\lambda}} \leq y, \\ \frac{y}{x^{\frac{1}{\lambda}}}, & \text{otherwise,} \end{cases} \quad x, y \in [0, 1].$$

- (vi) If we take the Frank's class of additive generators (see [13], page. 110),

$$g^s(x) = -\ln\left(\frac{s^{1-x} - 1}{s - 1}\right), \quad s > 0, s \neq 1,$$

as the g -generators, then for every s , we have $g^s(1) = \infty$,

$$(g^s)^{-1}(x) = 1 - \log_s(1 + (s-1)e^{-s}),$$

and the corresponding g -implication is given by

$$I_{g^s}(x, y) = 1 - \log_s\left(1 + (s-1)\frac{x-1}{x}(s^{1-y} - 1)^{\frac{1}{x}}\right), \quad x, y \in [0, 1].$$

5.5. g -implications-The two subfamilies

Once again, if g is a g -generator such that $g(1) < \infty$, then the function $g_1: [0, 1] \rightarrow [0, 1]$ defined by

$$g_1(x) = \frac{g(x)}{g(1)}, \quad x \in [0, 1], \quad (5)$$

is a well defined g -generator and the g -implications defined from both g and g_1 are identical, i.e., $I_g \equiv I_{g_1}$ and moreover $g_1(1) = 1$. In other words, it is enough to consider only increasing generators for which $g(1) = \infty$ or $g(1) = 1$.

Let us denote by

- $\mathbb{I}_{\mathbb{G}, \infty}$ – the family of all g - implications such that $g(1) = \infty$,
- $\mathbb{I}_{\mathbb{G}, 1}$ – the family of all g - implications such that $g(1) = 1$,
- Clearly, $\mathbb{I}_{\mathbb{G}} = \mathbb{I}_{\mathbb{G}, \infty} \cup \mathbb{I}_{\mathbb{G}, 1}$.

Remark 5.12. Note that for every g - generator g , the function $g \circ \varphi: [0, 1] \rightarrow [0, \infty]$ is strictly increasing and $(g \circ \varphi)(0) = 0$ for all $\varphi \in \Phi$. Thus $g \circ \varphi$ is also a g - generator for every $\varphi \in \Phi$.

Our first result shows that if I is a g -implication then every φ -pseudo conjugate of I is also a g -implication.

Lemma 5.13. *Let $I \in \mathbb{I}$ and $J \in [I]$. Then $I \in \mathbb{I}_{\mathbb{G}} \iff J \in \mathbb{I}_{\mathbb{G}}$.*

Proof. Analogous to Lemma 5.4. □

5.6. Representation of g -implications

Once again, we give the first representation results of the above family of g -implications which show that every g -implication is a φ -pseudo conjugate of either the Yager implication $I_{\mathbf{YG}}$ or the Goguen implication $I_{\mathbf{GG}}$.

We first require the following result from [3] which shows that the set of f -implications generated from f -generators such that $f(0) = \infty$ and the set of g -implications generated from g -generators such that $g(1) = \infty$ are identical.

Proposition 5.14 ([3], Proposition 4.4.1). *The following equalities are true.*

$$\mathbb{I}_{\mathbb{F}, 1} \cap \mathbb{I}_{\mathbb{G}} = \emptyset, \quad (6)$$

$$\mathbb{I}_{\mathbb{F}} \cap \mathbb{I}_{\mathbb{G}, 1} = \emptyset, \quad (7)$$

$$\mathbb{I}_{\mathbb{F}, \infty} = \mathbb{I}_{\mathbb{G}, \infty}. \quad (8)$$

Theorem 5.15. $\mathbb{I}_{\mathbb{G}, \infty} = [I_{\mathbf{YG}}]$.

Proof. Since $\mathbb{I}_{\mathbb{G}, \infty} = \mathbb{I}_{\mathbb{F}, \infty}$ and $\mathbb{I}_{\mathbb{F}, \infty} = [I_{\mathbf{YG}}]$, proof follows directly. □

Theorem 5.16. $\mathbb{I}_{\mathbb{G}, 1} = [I_{\mathbf{GG}}]$.

Proof. We know that $I_{\mathbf{GG}} = I_g$ where $g(x) = x$. Since $g(1) = 1$, clearly $I_{\mathbf{GG}} \in \mathbb{I}_{\mathbb{G}, 1}$ and consequently $[I_{\mathbf{GG}}] \subseteq \mathbb{I}_{\mathbb{G}, 1}$.

Let $I \in \mathbb{I}_{\mathbb{G}, 1}$ i.e., $I = I_g$ for some generator g such that $g(1) = 1$. Take $\varphi(x) = g(x)$. Then $I = I_g = I_{g_1 \circ \varphi^{-1}}$ where $g_1(x) = x$. It follows that $I \in [I_{\mathbf{GG}}]$ and consequently $\mathbb{I}_{\mathbb{G}, 1} \subseteq [I_{\mathbf{GG}}]$. □

Corollary 5.17. (i) *An $I \in \mathbb{I}_{\mathbb{G}, \infty}$ if and only if $I(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ \varphi([\varphi^{-1}(y)]^x), & \text{if } x > 0 \text{ or } y > 0 \end{cases}$, for some $\varphi \in \Phi$.*

(ii) *An $I \in \mathbb{I}_{\mathbb{G}, 1}$ if and only if $I(x, y) = \begin{cases} 1, & \text{if } \varphi(x) \leq y, \\ \varphi\left(\frac{\varphi^{-1}(y)}{\varphi(x)}\right), & \text{if } \varphi(x) > y, \end{cases}$*

for some $\varphi \in \Phi$.

(iii) $\mathbb{I}_{\mathbb{G}} = \mathbb{I}_{\mathbb{G}, \infty} \cup \mathbb{I}_{\mathbb{G}, 1} = [I_{\mathbf{YG}}] \cup [I_{\mathbf{GG}}]$.

6. Concluding Remarks

Our motivation for this study was to propose a binary operation \otimes on the set \mathbb{I} of all fuzzy implications that would give a rich enough algebraic structure to glean newer and better perspectives on fuzzy implications.

The operation \otimes proposed in this work not only gave a novel way of generating newer fuzzy implications from given ones, but also, for the first time, imposed a non-idempotent monoid structure on \mathbb{I} . By defining a suitable group action on \mathbb{I} and the equivalence classes obtained therefrom, we have determined hitherto unknown representation for two of the main families of fuzzy implications, viz., the f - and g -implications.

We believe that the above monoid structure needs to be investigated deeper. Accordingly, earnest efforts are underway along these lines. Already, our investigations on the center of this monoid (\mathbb{I}, \otimes) , have also led us to study some related but interesting semigroup homomorphisms on this monoid. We intend to present these and other results, in a future work.

Acknowledgment

The authors would like to acknowledge CSIR HRDG-INDIA (09/1001/(0008)/2011-EMR-I), DST - INDIA, (SR/FTP/2862/2011-12), respectively, for their financial support.

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