Pliable Index Coding via Conflict-Free Colorings of Hypergraphs

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***Abstract*—In the pliable index coding (PICOD) problem, a server is to serve multiple clients, each of which possesses a unique subset of the complete message set as side information and requests a new message which it does not have. The goal of the server is to do this using as few transmissions as possible. This work presents a hypergraph coloring approach to the PICOD problem. A *conflict-free coloring* of a hypergraph is known from literature as an assignment of colors to its vertices so that each edge of the graph contains one uniquely colored vertex. For a given PICOD problem represented by a hypergraph consisting of messages as vertices and request-sets as edges, we present achievable PICOD schemes using conflict-free colorings of the PICOD hypergraph. Various graph theoretic parameters arising out of such colorings (and some new coloring variants) then give a number of upper bounds on the optimal PICOD length, which we study in this work. Our achievable schemes based on hypergraph coloring include scalar as well as vector linear PICOD schemes. For the scalar case, using the correspondence with conflict-free coloring, we show the existence of an achievable scheme which has length $O(\log^2 \Gamma)$, where Γ refers to a parameter of the hypergraph that captures the maximum 'incidence' number of other edges on any edge. This result improves upon known achievability results in PICOD literature, in some parameter regimes.**

Due to space restrictions, the full version of this paper containing proofs of claims, additional examples and results, is made available in [1].

I. Introduction

The Index Coding problem introduced by Birk and Kol in [2] consists of a system with a server containing m messages and n receivers connected by a broadcast channel. Each receiver has a subset of the messages at the server as *side-information* and demands a particular new message. The goal of the index coding problem is to design a transmission scheme at the server which uses minimum number of transmissions to serve all receivers, also called the length of the index code. The index coding problem is a canonical problem in information theory and has been addressed by a variety of techniques (see [3]–[6] for instance).

A variant of the index coding problem, called pliable index coding (PICOD), was introduced by Brahma and Fragouli in [7]. The pliable index coding problem relaxes the index coding setup, such that each receiver requests *any* message which is not present in its side-information (i.e., any message from its *request-set*). It was shown in [7] that finding the optimal length of a PICOD problem is NP-hard in general. However the existence of a code with length $O(\min\{\log m(1+\log^+(\frac{n}{\log m})), m, n\})$ was proved using a probabilistic argument (where $\log^+(x) = \max\{0, \log(x)\}$). When $m = n^\delta$ for some constant $\delta > 0$, this means that $O(\log^2 n)$ is sufficient. Some algorithms for designing pliable index codes based on greedy and set-cover techniques were also presented and compared in [7]. In [8], a polynomial-time algorithm was presented for general PICOD problems which achieves a length $O(\log^2 n)$. Thus, unlike the index coding problem which has instances for which the required length can be $O(n)$ (for instance, the directed n-cycle problem [3]), much fewer transmissions are sufficient in general for PICOD instances. For several special classes of PICOD problems, distinguished by the structure of the side-information or request-sets of the receivers, achievability and converse results were presented in [9]–[12]. Pliable index coding has also been proposed for efficient data exchange in real-world applications, such as in the data shuffling phase of distributed computing [13].

In this work, we present a graph coloring approach to pliable index coding. A *conflict-free coloring* of a hypergraph is an assignment of labels to its vertices so that each edge of the graph contains at least one vertex which has a label distinct from others. Conflict-free colorings were introduced by Even et al. in [14], motivated by a problem of frequency assignment in wireless communications. Since then, it has been extensively studied in the context of general hypergraphs, hypergraphs induced by neighborhoods in graphs, hypergraphs induced by simple paths in a graph, hypergraphs that naturally arise in geometry, etc. See [15] for a survey on conflict-free colorings. In [2]–[4], the (classical) index coding problem is represented by a directed graph, and vertex colorings of this graph were shown to give index codes. Our present work can be considered a parallel for the PICOD problem where we use *conflict-free* colorings of a hypergraph representing the PICOD problem to obtain pliable index codes.

Any PICOD problem can be equivalently represented using a hypergraph $\mathcal{H}$ with the vertices representing the messages, and the request-sets as hyperedges. We show that conflict-free colorings (and several other new variants) of this hypergraph then give achievability schemes for PICOD. Using this connection, we show that $O(\log^2 \Gamma)$ transmissions are sufficient for any PICOD problem, where Γ is a parameter associated with the intersection between edges of the PICOD hypergraph. This result improves over known achievability results [7], [8] for some parameter ranges.

Our specific contributions and organization of this paper are as follows.

- After briefly reviewing the PICOD problem setup in Section II and conflict-free colorings in Section III, we

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define in Section III-A a generalization of conflict-free coloring called *k-fold conflict-free colorings* that we shall use in this work.

- In Section IV, we show that a k-fold conflict-free coloring of the hypergraph $\mathcal{H}$ which represents the given PICOD problem results in a k-vector pliable index code. Thus, the *conflict-free chromatic number*, i.e. the minimum number of colors in any conflict-free coloring, bounds the optimal PICOD length from above (Theorem 2).
- In Section IV-A, we define the notion of *conflict-free collection of colorings* of hypergraphs, and show that this notion gives a refined upper bound (which we call the *conflict-free covering number*) than the conflict-free chromatic number (Theorem 3). Using such conflict-free collections, we show that $O(\log^2 \Gamma)$ transmissions suffice for scalar PICOD schemes, where Γ refers to the maximum 'incidence' number of other edges on any single edge (Theorem 4). Our proof for Theorem 4 uses a probabilistic argument, but this can be converted into a polynomial time algorithm using known techniques (Remark IV-C).
- We show separation (gaps) between the various parameters presented in this work (Example 1, Lemma 3).

We conclude the paper in Section V with directions for future work. The full version of this paper, available in [1], discusses PICOD schemes arising from 'local' versions of the conflict-free number and the conflict-free covering number (reminiscent of [4] for the original index coding problem). These further improve the achievable lengths of PICOD schemes over the parameters defined in this work.

Notations: Let $[n] \triangleq \{1, \ldots, n\}$ for positive integer n. For sets A, B we denote by $A \backslash B$ the set of elements in A but not in B. We abuse notation to denote $A \backslash \{b\}$ as $A \backslash b$. The set of k-subsets of any set A is given by $\binom{A}{k}$. The span of a set of vectors U is denoted by $span(U)$. The dimension of a subspace W is denoted by $\dim(W)$. Unless mentioned explicitly, all logarithms in the paper are to the base e. $\emptyset$ denotes the empty set.

II. Pliable Index Coding Problem

We briefly review the pliable index coding problem, introduced in [7]. Consider a communication problem defined as follows. There are m messages denoted by $\{x_i : i \in [m]\}$ where x_i lies in some finite alphabet $\mathcal{A}$. These m messages are available at a server. Consider n receivers indexed by $[n]$. Assume that there is a noise-free broadcast channel between the server and the receivers. Each receiver r has some subset of messages available apriori, as *side-information*, which we denote as $\{x_i : i \in S_r\}$. Let the indices of the symbols not available as side-information at receiver r be denoted as $I_r \triangleq [m] \backslash S_r$. We call $\{x_i : i \in I_r\}$ as the *request-set* of receiver r. Each receiver r demands from the source any symbol from its request-set. The messages indexed by $[m]$, the receivers indexed by $[n]$, and the request-sets $\mathfrak{I} \triangleq \{I_r : r \in [n]\}$ together define a $(n, m, \mathfrak{I})$-*pliable index coding problem (PICOD problem).* We assume that $|I_r| \geq 1, \forall r$, as any receiver with $|I_r| = 0$ can be removed from the problem description as it has all the symbols. Consider a hypergraph $\mathcal{H}$ with vertex set $V = [m]$ and edge set $\mathfrak{I} = \{I_r : r \in [n]\}$. Then this hypergraph equivalently captures the PICOD problem.

A *pliable index code* (PIC) consists of a collection of (a) an encoding function at the server which encodes the m messages to an ℓ-length codeword, denoted by $\phi : \mathcal{A}^m \rightarrow \mathcal{A}^\ell$ and (b) decoding functions $\{\psi_r : r \in [n]\}$ where $\psi_r : \mathcal{A}^\ell \times \mathcal{A}^{|S_r|} \rightarrow \mathcal{A}$ denotes the decoding function at receiver r such that

$$\psi_r \left(\phi(\{x_i : i \in [m]\}), \{x_i : i \in S_r\} \right) = x_d, \text{ for some } d \in I_r.$$

The quantity ℓ is called the *length* of the PIC. We are interested to design pliable index codes which have small ℓ.

In this work we assume $\mathcal{A} = \mathbb{F}^k$ for some finite field $\mathbb{F}$ and integer $k \geq 1$. Thus the message x_i is represented as $x_i = (x_{i,1}, \ldots, x_{i,k}) \in \mathbb{F}^k$. We refer to these codes as *k-vector PICs*, while the $k = 1$ case is also called *scalar PIC*. We focus on linear PICs, i.e., one in which the encoding and decoding functions are linear. In that case, the encoder ϕ is represented by a $\ell \times mk$ matrix (denoted by G) such that $\phi(\{x_i : i \in [m]\}) = G\boldsymbol{x}^T$, where $\boldsymbol{x} = (x_{1,1}, \ldots, x_{1,k}, \ldots, x_{m,1}, \ldots, x_{m,k})$. We denote the smallest ℓ such that there is a linear k-vector PIC for the PICOD problem given by the hypergraph $\mathcal{H}$ as $\ell_k^*(\mathcal{H})$.

The following definition and lemma (which is proved in [8]) describe when G can lead to correct decoding at the receivers.

Definition 1. For a $(n, m, \mathfrak{I})$-PICOD problem, a matrix G with mk columns indexed as $G_{i,j} : i \in [m], j \in [k]$, is said to *satisfy* receiver $r \in [n]$, if the following property (P) is satisfied by G

(P) There exists some $d \in I_r$ such that $\dim(span(\{G_{d,j} : j \in [k]\})) = k$ and

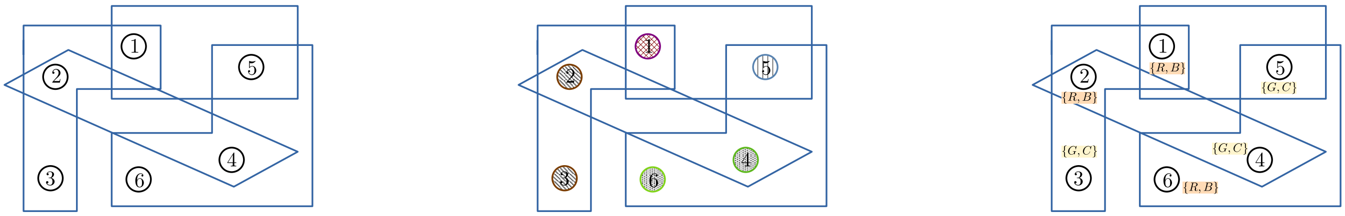
$$span(\{G_{d,j} : j \in [k]\}) \bigcap span\left(\{G_{i,j} : \ \forall i \in I_r \backslash d, \ j \in [k]\}\right) = \{\mathbf{0}\},$$

Lemma 1 ([8] Lemmas 1 and 6)**.** *A matrix G with mk columns is the encoder of a PIC for a $(n, m, \mathfrak{I})$-PICOD problem if and only if the property (P) of Definition 1 is true for each receiver $r \in [n]$.*

Lemma 2 below is useful to prove achievability results for PICOD problems in this work.

Lemma 2. *For a $(n, m, \mathfrak{I})$-PICOD problem, let $\{G^p : p \in [P]\}$ denote a collection of matrices, where G^p is of size $L_p \times mk$, such that for each $r \in [n]$, there exists some matrix G^p which satisfies receiver r. Then the matrix $G = \begin{bmatrix} G^1 \\ \vdots \\ G^P \end{bmatrix}$ of size $(\sum_{p \in [P]} L_p) \times mk$ is the encoder of a PIC for the given PICOD problem.*

Proof: For each $r \in [n]$, there exists some matrix G^p such that Property (P) holds for r (with respect to some $d \in I_r$). By simple linear algebra, we see that the matrix G too must satisfy property (P) for receiver r (with respect to $d \in I_r$), and hence satisfies r. Applying Lemma 1, the proof is complete. ■



(a) (b) (c)

Figure 1: Figure (a) shows a hypergraph $\mathcal{H}$ with 6 vertices and edge set $\mathcal{E} = \{\{1,2,3\},\{1,5\},\{2,4\},\{4,5,6\}\}$. Figure (b) represents a 1-fold conflict-free coloring with 4 colors, with the color classes $\{1\},\{2,3\},\{5\},\{4,6\}$. Figure (c) shows a 2-fold conflict-free coloring using the colors $\{R,G,B,C\}$.

III. Conflict-Free Colorings of Hypergraphs

In this section we review the definition of conflict-free colorings of a hypergraph and discuss some existing results in this regard. Further we also define the more general notion of k-fold conflict-free colorings, hence subsuming the existing notion within the $k=1$ case.

Let $\mathcal{H} = (V,\mathcal{E})$ be a hypergraph. Let $C : V \to [L]$ be a coloring of V, where L is a positive integer. Consider a hyperedge $E \in \mathcal{E}$. We say C is a *conflict-free coloring for the hyperedge* E if there is a vertex $v \in E$ such that $C(v) \neq C(u)$, $\forall u \in E \setminus \{v\}$. That is, in such a coloring, E contains a vertex whose color is distinct from that of every other vertex in E. We say C is a *conflict-free coloring of the hypergraph* $\mathcal{H}$ if C is a conflict-free coloring for every $E \in \mathcal{E}$. The *conflict-free chromatic number* of $\mathcal{H}$, denoted by $\chi_{CF}(\mathcal{H})$, is the minimum L such that there is a conflict-free coloring $C : V \to [L]$ of $\mathcal{H}$. The following theorem on conflict-free coloring on hypergraphs is due to Pach and Tardos [16], which we will use to obtain one of our main results (Theorem 4 and hence Corollary 1) in Section IV-B.

Theorem 1 (Theorem 1.2 in [16])**.** *For any positive integers t and Γ, the conflict-free chromatic number of any hypergraph in which each edge is of size at least $2t-1$ and each edge intersects at most Γ others is $O(t\Gamma^{1/t}\log\Gamma)$. There is a randomized polynomial time algorithm to find such a coloring.*

A. Generalizing conflict-free colorings to k-fold colorings

We now generalize the idea of conflict-free colorings to k-fold conflict-free colorings. To the best of our knowledge this generalized notion is not available in literature.

Definition 2. A k-fold coloring of a hypergraph $\mathcal{H} = (V,\mathcal{E})$ is an assignment of k-sized subsets of $[L]$ to the vertices V, given by $C : V \to \binom{[L]}{k}$. A k-fold coloring C is *conflict-free for edge* $E \in \mathcal{E}$ if there exists some $v \in E$ such that $C(v) \cap C(v') = \emptyset$, for each $v' \in E\backslash v$. A coloring C is a *k-fold conflict-free coloring* for $\mathcal{H}$ if C is a k-fold conflict-free coloring for each edge in $\mathcal{E}$. We define the *k-fold conflict-free chromatic number* of $\mathcal{H}$ as the minimum L such that a k-fold conflict-free coloring of $\mathcal{H}$ exists as defined above, and denote it by $\mathcal{X}_{k,CF}(\mathcal{H})$.

For convenience of definition above, we have used the set $[L]$ to represent the set of labels in the coloring. More generally, we could (and will) use any finite set as the set of labels.

Remark 1. The $k=1$ case of Definition 2 corresponds to the usual conflict-free coloring. Thus $\mathcal{X}_{1,CF}(\mathcal{H}) = \mathcal{X}_{CF}(\mathcal{H})$. ■

Fig. 1 gives an example of 1-fold and 2-fold conflict-free coloring. Clearly, $\mathcal{X}_{k,CF}(\mathcal{H}) \leq k\mathcal{X}_{CF}(\mathcal{H})$ as we can always obtain a k-fold conflict-free coloring from a 1-fold conflict-free coloring by expanding each color into k unique colors. However we show an example here for which this inequality is strict.

Example 1. Consider the hypergraph given by vertex set $V = \{a,\ldots,e\}$ and $\mathcal{E} = \{\{a,c\},\{b,e\}, \ \{b,d\},\{c,e\},\{a,d\}\}$. Consider any 1-fold coloring of this graph. If only two colors were allowed, then it is easy to check that we get a contradiction as we seek to satisfy the conflict-free property. It is also easy to find a conflict-free coloring with 3 colors, for instance, give color 1 to vertices $\{a,b\}$, color 2 to $\{c,d\}$ and color 3 to vertex e. Thus $\mathcal{X}_{1,CF} = 3$.

By similar arguments as above we can show that there cannot be a 2-fold conflict-free coloring with 4 colors. Now consider the following 2-fold coloring with 5 colors denoted by $\{1,\ldots,5\}$. Let set $\{1,2\}$ be assigned to vertex a, $\{2,3\}$ to b, $\{3,4\}$ to c, $\{4,5\}$ to d and $\{5,1\}$ to e. It is easy to check that this is a 2-fold conflict-free coloring. Thus $\mathcal{X}_{2,CF}(\mathcal{H}) = 5 < 6 = 2(\mathcal{X}_{1,CF}(\mathcal{H}))$. ■

IV. Relationship of PIC to Conflict-free Coloring

In this section, we show that a k-fold conflict-free coloring of the hypergraph $\mathcal{H}(V = [m],\mathfrak{I})$ gives a k-vector linear PIC scheme for the PICOD problem given by $\mathcal{H}$. To do this, we define the following matrix associated with a conflict-free coloring of $\mathcal{H}$.

Definition 3 (Indicator Matrix associated to a coloring)**.** Let $C : V \to \binom{[L]}{k}$ denote a k-fold coloring of $\mathcal{H}(V,\mathfrak{I})$. Let $C(i) = \{C_{i,1},\ldots,C_{i,k}\}$ denote the subset assigned to the vertex $i \in [m]$. Consider a standard basis of the L-dimensional vector space over $\mathbb{F}$, denoted by $\{e_1,\ldots,e_L\}$. Now consider the matrix $L \times mk$ matrix G (with columns indexed as $\{G_{i,j} : i \in [m], j \in [k]\}$) constructed as follows.

- For each $i \in [m], j \in [k]$, column $G_{i,j}$ of G is fixed to be $e_{C_{i,j}}$.

We call G as the *indicator matrix associated with the coloring* C.

Using the indicator matrix associated with a conflict-free coloring of $\mathcal{H}$, we shall prove our first bound on $\ell_k^*(\mathcal{H})$.

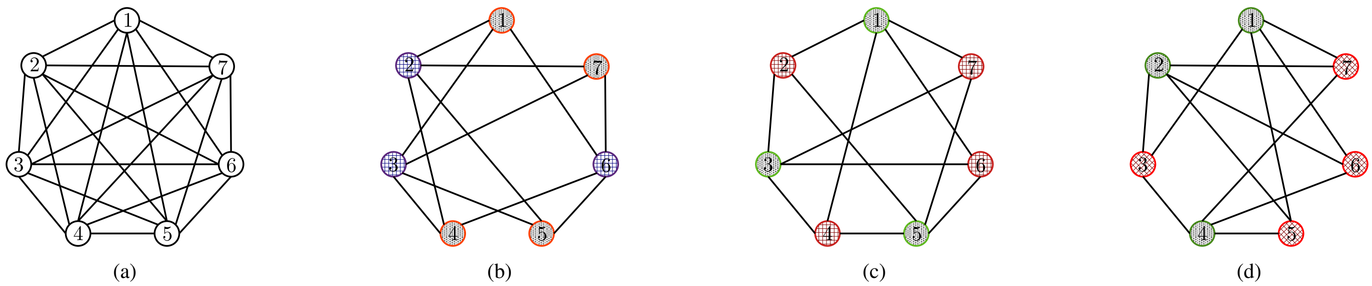


Figure 2: Figure (a) shows the hypergraph K_7 which is the complete graph on 7 vertices. It requires 7 colors for a conflict-free coloring, thus $\mathcal{X}_{CF}(K_7) = 7$. The three figures (b),(c) and (d) depict a collection of 1-fold conflict-free colorings, each figure corresponding to one coloring using 2 colors. Note that only those edges satisfied by the coloring are represented in (b), (c), and (d). In (b), the two color classes are $\{1, 4, 5, 7\}$ and $\{2, 3, 6\}$. In (c), they are $\{2, 4, 6, 7\}$ and $\{1, 3, 5\}$ and in (d) they are $\{1, 2, 4\}$ and $\{3, 5, 6, 7\}$. It can be checked that each edge of K_7 is conflict-free in at least one of these colorings. Thus $\alpha_{CF}(\mathcal{H}) \leq 6 < \mathcal{X}_{CF}(\mathcal{H})$.

Theorem 2. $\ell_k^*(\mathcal{H}) \leq \mathcal{X}_{k,CF}(\mathcal{H})$.

Proof: Let $C : V \to \binom{[L]}{k}$ denote a k-fold conflict-free coloring of $\mathcal{H}$. We first show that there exists a L-length k-vector linear PIC for the problem defined by $\mathcal{H}$. Let G denote the indicator matrix associated with the coloring C as defined in Definition 3. Let $C(i) = \{C_{i,1}, \ldots, C_{i,k}\}$ be the set assigned to vertex i. We show that G satisfies Lemma 1 and hence is a valid encoder for a k-vector linear PIC.

In any conflict-free coloring of $\mathcal{H}$, every edge I_r of $\mathcal{H}$ has a vertex d such that $C(d) \cap C(i) = \emptyset, \forall i \in I_r \setminus d$. Then, clearly, $\{e_{C_{d,j}} : j \in [k]\} \cap \{e_{C_{i,j}} : j \in [k]\} = \emptyset$, for any $i \in I_r \backslash d$. This also means $span(\{e_{C_{d,j}} : j \in [k]\}) \cap span(\{e_{C_{i,j}} : i \in I_r \backslash d, j \in [k]\}) = \{\mathbf{0}\}$, as the vectors $\{e_1, \ldots, e_L\}$ are basis vectors. Further, as $|\{C_{d,j} : j \in [k]\}| = k$, hence $\dim(span(\{e_{C_{d,j}} : j \in [k]\})) = k$. Thus, G satisfies every receiver r and is thus a valid encoder by Lemma 1. By definition of $\mathcal{X}_{k,CF}(\mathcal{H})$, the proof is complete. ■

Example 2. Consider the PICOD problem represented by the hypergraph $\mathcal{H}$ with vertex set $V = \{1, \ldots, 8\}$ and edge set $\mathcal{E} = \{\{1, 2, 4, 6\}, \{1, 2, 3, 5\}, \{2, 3, 4, 7\}, \{1, 3, 4, 8\}, \{2, 5, 6, 7\}, \{1, 5, 6, 8\}, \{3, 5, 7, 8\}, \{4, 6, 7, 8\}\}$. Consider a coloring C which assigns color 1 to vertices $\{1, 2, 3, 4\}$ and color 2 to vertices $\{5, 6, 7, 8\}$. Note that this is a valid (1-fold) conflict-free coloring of $\mathcal{H}$. The indicator matrix associated with this coloring is given by

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

It can be checked that the above matrix satisfies the condition in Lemma 1 for the PICOD problem defined by $\mathcal{H}$. ■

A. Conflict-free coverings and PICOD

In the following discussion, we will define a new parameter called the k-fold conflict-free covering number, which will improve upon the upper bound on the optimal length as given in Theorem 2.

Definition 4 (*Conflict-free collection, conflict-free covering number*)**.** Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Let $\mathfrak{C} = \{C^1, \ldots, C^P\}$ where each $C^p : V \to \binom{[L_p]}{k}$ is a k-fold coloring of the hypergraph $\mathcal{H}$. We say $\mathfrak{C}$ is a *conflict-free collection of k-fold colorings of $\mathcal{H}$*, if for every $E \in \mathcal{E}$, there exists a $C^p \in \mathfrak{C}$ such that C^p is a k-fold conflict-free coloring for E. The quantity $\alpha_{k,CF}(\mathcal{H}) \triangleq \min_{\mathfrak{C}} \sum_{p=1}^{P} L_p$ representing the minimum sum $\sum_{p=1}^{P} L_p$ over all possible collections $\mathfrak{C}$ (over all P) as defined above, is called the *k-fold conflict-free covering number of $\mathcal{H}$*. We denote the number $\alpha_{1,CF}(\mathcal{H})$ simply as $\alpha_{CF}(\mathcal{H})$.

The following observation is easy to show, as any k-fold conflict-free coloring C of $\mathcal{H}$ also gives a conflict-free collection containing just C.

Observation 1. $\alpha_{k,CF}(\mathcal{H}) \leq \chi_{k,CF}(\mathcal{H})$.

Fig. 2 gives an example hypergraph for which $\mathcal{X}_{CF}(\mathcal{H}) > \alpha_{CF}(\mathcal{H})$. In the following theorem, we show that the optimal length of k-vector code for $\mathcal{H}$ is bounded by $\alpha_{k,CF}(\mathcal{H})$, thus improving the bound in Theorem 2.

Theorem 3. $\ell_k^*(\mathcal{H}) \leq \alpha_{k,CF}(\mathcal{H})$.

Proof: Let $\mathfrak{C} = \{C^p : p \in [P]\}$ be a conflict-free collection of k-fold colorings of $\mathcal{H}(V = [m], \mathfrak{I})$, where $C^p : V \to \binom{[L_p]}{k}$. We will first show a PIC for $\mathcal{H}$ with length $\sum_{p \in [P]} L_p$. The proof then follows by definition of $\alpha_{k,CF}(\mathcal{H})$.

Let $G^p : p \in [P]$ denote the indicator matrices as defined in Definition 3 associated with the colorings $C^p : p \in [P]$ respectively. By definition of the conflict-free collection, for each $I_r \in \mathfrak{I}$ we have by arguments similar to the proof of Theorem 2, that there is some G^p which satisfies receiver r. Then, by Lemma 2, the matrix $G = \begin{bmatrix} G^1 \\ \vdots \\ G^P \end{bmatrix}$ is a valid encoder of a k-vector PIC to the given PICOD problem of length $\sum_{p \in [P]} L_p$. This completes the proof. ■

We now discuss the separation between $\alpha_{k,CF}(\mathcal{H})$ and $\mathcal{X}_{k,CF}(\mathcal{H})$ for the $k = 1$ case. This is a generalization of example in Fig. 2.

Lemma 3. *There exist a hypergraph $\mathcal{H}$ with n hyperedges for which $\alpha_{CF}(\mathcal{H}) = \Theta(\log n)$ while $\mathcal{X}_{CF}(\mathcal{H}) = \Theta(\sqrt{n})$.*

Proof: Consider the 2-uniform hypergraph with m vertices and all the 2-sized subsets of $[m]$ as hyperedges. Thus $n = \binom{m}{2}$. It is easy to see that any conflict-free coloring of this graph requires $m = \Theta(\sqrt{n})$ colors.

Let us now turn our attention to $\alpha_{CF}(\mathcal{H})$. Consider a conflict-free collection of P colorings $C^p : p \in [P]$, for some integer P, each with number of colors L_p, such that $\sum_{p \in P} L_p = \alpha_{CF}(\mathcal{H})$. For each $p \in [P]$, let G^p be the indicator matrix associated with the coloring C^p. Consider the $(\alpha_{CF}(\mathcal{H}) \times m)$ binary matrix $G = \begin{bmatrix} G^1 \\ \vdots \\ G^P \end{bmatrix}$. By Theorem 3, this is a valid encoder for a PIC of $\mathcal{H}$. Since every 2-sized subset of $[m]$ is a hyperedge in $\mathcal{H}$, no two columns of G are thus identical. Thus, $\alpha_{CF}(\mathcal{H}) \geq \log_2 m$. In order to prove an upper bound for $\alpha_{CF}(\mathcal{H})$, let $P := \lceil \log_2 m \rceil$. Given any assignment of distinct P-bit binary vectors to the elements of $[m]$, one can construct a conflict-free collection of P colorings of $\mathcal{H}$ given as $C^p : [m] \to \{c_p^0, c_p^1\}$ for $p \in [P]$, where $C^p(j) = c_p^0$ (or $= c_p^1$) if the p-th bit in the binary vector associated with j is 0 (respectively, 1). Thus, $\alpha_{CF}(\mathcal{H}) \leq 2\lceil \log_2 m \rceil$. ■

B. An upper bound on $\alpha_{CF}(\mathcal{H})$ and thus on $\ell_1^*(\mathcal{H})$

In the remainder of this section, we focus on the scalar ($k = 1$) case and prove an upper bound (in Theorem 4) on $\alpha_{CF}(\mathcal{H})$ (and thus on the optimal linear scalar PIC length) based on a readily computable parameter associated with the graph $\mathcal{H}$. Towards that end we make the following observation.

Let $\mathcal{H} = (V, \mathcal{E})$, $\mathcal{H}_1 = (V, \mathcal{E}_1)$, and $\mathcal{H}_2 = (V, \mathcal{E}_2)$ be three hypergraphs defined on the same vertex set V. We say $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, if $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$. We then have the observation which follows by definition of $\alpha_{CF}(\mathcal{H})$.

Observation 2. Let $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$. Then, $\alpha_{CF}(\mathcal{H}) \leq \alpha_{CF}(\mathcal{H}_1) + \alpha_{CF}(\mathcal{H}_2)$.

Theorem 4. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph where every hyperedge intersects with at most Γ other hyperedges, for any $\Gamma > e$. Then, $\alpha_{CF}(\mathcal{H}) = O(\log^2 \Gamma)$.*

Proof. Let $\kappa := 2\log(\Gamma) - 1$. Let $\mathcal{G} = (V, \mathcal{E}_G)$ be a hypergraph defined on the vertex set V with $\mathcal{E}_G = \{E \in \mathcal{E} \ : \ |E| \geq \kappa\}$. From Theorem 1 and Observation 1, we know that $\alpha_{CF}(\mathcal{G}) = O(\log^2 \Gamma)$. Let $P := \lceil \log \kappa \rceil$. For $0 \leq i \leq P$, let $\mathcal{H}_i = (V, \mathcal{E}_i)$, where $\mathcal{E}_i = \{E \in \mathcal{E} \ : \ \frac{k_i}{2} \leq |E| < k_i\}$ and $k_i = \frac{\kappa}{2^i}$. Clearly, $\mathcal{H} = \mathcal{G} \cup \mathcal{H}_0 \cup \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_P$. We shall use the following claim whose proof uses the Lovász Local Lemma [17] and is relegated to the longer version of our paper [1].

Claim 1. $\alpha_{CF}(\mathcal{H}_i) \leq 2(\lceil 5k_i \log \Gamma \rceil)$.

Using Claim 1, we have

$$\begin{aligned}\sum_{i=0}^{P} \alpha_{CF}(\mathcal{H}_i) &\leq 2\sum_{i=0}^{P} \lceil 5k_i \log \Gamma \rceil \leq 10 \log \Gamma \sum_{i=0}^{P} k_i + 2P + 2 \\ &\leq 10 \log \Gamma \sum_{i \geq 0} \frac{\kappa}{2^i} + 2P + 2 \\ &\leq 20\kappa \log \Gamma + 2P + 2 = O(\log^2 \Gamma) \qquad (1)\end{aligned}$$

Now using Observation 2, we have that $\alpha_{CF}(\mathcal{H}) \leq \alpha_{CF}(\mathcal{G}) + \sum_{i=0}^{P} \alpha_{CF}(\mathcal{H}_i)$. Using (1) now, the proof is complete. □

Using Theorem 4 in conjunction with Theorem 3, we have the following achievability result for the PICOD problem. The achievability of lengths m, n are trivial consequences of the problem setup.

Corollary 1. *For any $(n, m, \mathfrak{I})$-PICOD problem, let $\Gamma = \max_{r \in [n]} |\{r' \in [n] \backslash r : I_r \cap I_{r'} \neq \emptyset\}|$. Then there exists a binary linear scalar PIC for the given problem with length $O(\min\{\log^2 \Gamma, m, n\})$. Thus $\ell_1^*(\mathcal{H}) = O(\min\{\log^2 \Gamma, m, n\})$.*

C. Comparison with known achievability results

The original work [7] showed the existence of an achievable scheme with length $O(\min\{\log m(1 + \log^+(\frac{n}{\log m})), m, n\})$ (where $\log^+(x) = \max\{0, \log(x)\}$). For $m = n^\delta$ for some $\delta > 0$, this means the existence of a PIC with length $O(\log^2 n)$ is guaranteed. Our result, Theorem 4, gives an upper bound based on the parameter Γ of the hypergraph. Given the set of vertices V and edges $\mathcal{E}$ of a hypergraph, the parameter Γ can be determined in $O(|V||\mathcal{E}|^2)$ time by a simple algorithm which runs through each edge computing its intersection with all other edges. Further the parameter $\Gamma \leq |\mathcal{E}| - 1 = n - 1$ always, but it could be much smaller in general, as suggested by the below example.

Example 3. Consider the hypergraph $\mathcal{H} = (V, \mathcal{E})$, where $V = [m], \mathcal{E} = \{\{i, i+1, i+2\} : i \in [m-2]\}\}$ for $m \geq 3$. Since every hyperedge overlaps with at most 3 other edges, we have $\Gamma = 3$. The result from [7] suggests the existence of a code of length $O(\log^2 m)$, where by Theorem 4, we have a code of constant length (as m grows). ■

Thus we see that the bound in Theorem 4 could be much smaller than the bound from [7]. A more precise characterization of Γ in terms of $|V|$ and $|\mathcal{E}|$ could offer more insights on this gap between the two upper bounds.

In [8] an achievable scheme was presented for a PICOD problem with n receivers with length $O(\log^2 n)$. The algorithm in [8] had running time polynomial in the problem parameters m, n. our result also yields a polynomial time algorithm. However, the algorithm does not follow immediately from the proof. The main difficulty in getting a deterministic algorithm is the presence of the Local Lemma in the proof. Derandomization of the Local Lemma to provide an constructive algorithm has been studied [18], [19]. Applying Theorem 1.1 (1) in [19], we get a conflict-free coloring of a hypergraph using $O(t\Gamma^{\frac{1+\epsilon}{t}} \log \Gamma)$ colors, where t and Γ are as defined in Theorem 1 and $\epsilon > 0$ is a constant. This suffices to get a deterministic polynomial time coloring algorithm for the hypergraph $\mathcal{G}$ in the proof of Theorem 4 using $O(\log^2 \Gamma)$ colors. In a similar way, one can get polynomial time algorithms for constructing conflict-free collection of colorings for hypergraphs $\mathcal{H}_i$ in the proof such that the total number of colors used across all the colorings in such a collection is $O(k_i \log \Gamma)$.

V. Discussion

Through our conflict-free coloring approach, we have proved the existence of a pliable index code with length relying on the incidence parameter Γ. It would be interesting to give a simpler polynomial-time algorithm for the same. Also, future work could include explicit schemes for k-vector pliable index coding which give non-trivial improvements over simple extensions of scalar index codes.

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