Homomorphisms on the Monoid of Fuzzy Implications $(\mathbb{I}, \circledast)$ - A Complete Characterization

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Abstract. In [4], we had proposed a novel generating methods of fuzzy implications and investigated algebraic structures on the set of all fuzzy implications, which is denoted by $\mathbb{I}$. Again in [5], we had defined a particular function g_K on the monoid $(\mathbb{I}, \circledast)$ (See Def. 16) and characterised the function K for which g_K is a semigroup homomorphism (s.g.h) in two special cases, i.e., K is with trivial range and $K(1, y) = y$ for all $y \in [0, 1]$(neutrality property). In this work we characterise the nontrivial range non neutral implications K such that g_K is an s.g.h. and also present their representations.

Keywords: Fuzzy implication, neutrality property, homomorphism.

1 Introduction

Fuzzy implications are a generalisation of classical implication to the fuzzy logic. It has many applications in the areas like approxiamte reasoning, control theory, decision making, fuzzy logic and so on. Its definition is as follows:

Definition 1. *[1] A binary function I on $[0, 1]$ is called a fuzzy implication if*

(i) I is decreasing in the first variable and increasing in the second variable,
(ii) $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.

Let the set of all fuzzy implications be denoted by $\mathbb{I}$. For more on fuzzy implications, please see [1].

Among the generating methods of fuzzy implications from fuzzy implications proposed in [4], the following method gives not only new fuzzy implications but also a rich algebraic structure namely, monoid, on the set $\mathbb{I}$ of all fuzzy implications. In this connection we recall few results from [4].

Definition 2. *([4]) For $I, J \in \mathbb{I}$, define $I \circledast J$ as*

$$(I \circledast J)(x, y) = I(x, J(x, y)), \qquad x, y \in [0, 1].$$

Theorem 1. *([4]) $I \circledast J$ is an implication on $[0, 1]$, i.e., $I \circledast J \in \mathbb{I}$.*

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Theorem 2. *([4])* $(\mathbb{I}, \circledast)$ *forms a monoid, whose identity element is given by*

$$I_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ y, & \text{if } x > 0. \end{cases}$$

Definition 3. *([3], [6]) Let* $(S, \star), (T, \odot)$ *be two algebraic structures. A map* $f\colon (S,\star) \to (T, \odot)$ *is called a homomorphism if* $f(a \star b) = f(a) \odot f(b)$, *for all* a, $b \in S$.

Since homomorphisms play an important role in Algebra, the authors in [5] defined a particular function on $(\mathbb{I}, \circledast)$ by fixing a fuzzy implication in the following way.

Definition 4. *([5]) For any fixed* $K \in \mathbb{I}$, *define* $g_K\colon (\mathbb{I}, \circledast) \to (\mathbb{I}, \circledast)$ *by*

$$g_K(I) = I \circledast K, \qquad I \in \mathbb{I}.$$

In [5], it was proved that for every $K \in \mathbb{I}$, the map g_K is a lattice homomorphism. However g_K is not a semigroup homomorphism (s.g.h.) for every $K \in \mathbb{I}$(See Example 28 in [5]). Thus it has become necessary to characterise and represent $K \in \mathbb{I}$ such that g_K is an s.g.h. In the same work [5], we have investigated the class of trivial range implications and nontrivial range neutral implications K such that g_K is an s.g.h.

However, the class of nontrivial range non neutral implications K such that g_K is an s.g.h is not known. In this work, we give complete characterisation and representation results for this class and complete the study of such s.g.h g_K.

2 Semigroup Homomorphisms on $(\mathbb{I}, \circledast)$.

2.1 Trivial Range Implication K

The characterisation and representation of fuzzy implications K such that g_K, defined as in Definition 4, is an s.g.h was completely obtained in the case of $K(x,y) \in \{0,1\}$, for all $x, y \in [0,1]$, please see [5] for more details. We present below the main result of this case.

Theorem 3. *([5]) Let* $K \in \mathbb{I}$ *be such that the range of* K *is trivial. Then the following statements are equivalent:*

(i) g_K *is an* s.g.h.

(ii) $K = K^\delta$ *for some* $\delta \in]0,1]$ *where*

$$K^\delta(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } (x = 1 \text{ and } y \geq \delta)\ , \\ 0, & \text{otherwise}\ . \end{cases}$$

2.2 Nontrivial Range Implication K

In the case of nontrivial range, characterisation and representation of K such that g_K is an s.g.h. was done only for implications K satisfying $K(1,y)=y$ for all $y \in [0,1]$. However, the class of nontrivial implications for which $K(1,y) \neq y$ for some $y \in (0,1)$ has to be found out such that the function g_K is an s.g.h. Before doing this we investigate the range of K such that g_K is an s.g.h.

Lemma 1. *If the range of K is nontrivial and g_K is an s.g.h then the range of K is equal to $[0,1]$.*

In order to obtain the representation of nontrivial K such that g_K is an s.g.h, we characterised the vertical section $K(1,\cdot)$ of K in [5]. For a quick reference we present this result in the following.

Proposition 1. *([5]) Let the range of $K \in \mathbb{I}$ be nontrivial and g_K be an s.g.h. Then there exist $\alpha, \beta \in [0,1]$ such that the vertical section $K(1,.)$ has the following form:*

$$K(1,y) = \begin{cases} 0, & \text{if } y \in [0,\alpha)\ , \\ 0 \textbf{ or } \alpha, & \text{if } y = \alpha\ , \\ y, & \text{if } y \in (\alpha,\beta)\ , \\ \beta \textbf{ or } 1, & \text{if } y = \beta\ , \\ 1, & \text{if } y \in (\beta,1]\ . \end{cases} \tag{1}$$

The following definition helps us in getting representation of K.

Definition 5. *Let $K \in \mathbb{I}$ and g_K is an s.g.h. Define two real numbers ϵ_0, ϵ_1 in the following way:*

$$\epsilon_0 = \sup\{t \in [0,1] | K(1,t) = 0\},$$
$$\epsilon_1 = \inf\{t \in [0,1] | K(1,t) = 1\}\ .$$

For every $K \in \mathbb{I}$, since $K(1,0)=0$ and $K(1,1)=1$, the real numbers ϵ_0, ϵ_1 in Definition 5 are well defined and exist in general. In the following we investigate the possible values of ϵ_0, ϵ_1 for $K \in \mathbb{I}$ such that range of K is nontrivial.

Lemma 2. $\epsilon_1 \neq 0$.

Proof. Let $\epsilon_1 = 0$. This implies that $K(1,y)=1$ for all $y>0$. It follows from the monotonicity of I in the first variable that $K(x,y)=1$ for all $x \in [0,1]$ and $y>0$. So the range of K becomes $\{0,1\}$, a contradiction to the fact the range of K is nontrivial. Thus $\epsilon_1 \neq 0$.

Lemma 3. *If $\epsilon_0 = 0$, then $\epsilon_1 = 1$. This implies that $K(1,y)=y$ for all $y \in [0,1]$.*

Proof. Let $\epsilon_0 = 0$ and suppose that $\epsilon_1 < 1$. Then from Lemma 2, it follows that $0 < \epsilon_1 < 1$. So choose a $\delta > 0$ such that $0 < \epsilon_1 + \delta < 1$. Let $0 < y_1 < \epsilon_1$. This implies that $0 < K(1,y_1) = \alpha < 1$. Now, choose a $J \in \mathbb{I}$ such that $J(1,K(1,y_1)) = J(1,\alpha) = \epsilon_1 + \delta$. But $K(1,J(1,K(1,y_1))) = K(1,\epsilon_1+\delta) = 1$, which contradicts g_K is an s.g.h. Thus $\epsilon_1 = 1$.

Lemma 4. *If $0 < \epsilon_0 < 1$, then $\epsilon_0 \neq \epsilon_1$.*

Proof. Let $0 < \epsilon_0 < 1$. Suppose that $\epsilon_0 = \epsilon_1$. Then $K(1, \cdot)$ will be of the form

$$K(1, y) = \begin{cases} 1, & \text{if } y \geq \epsilon_0 \\ 0, & \text{if } y < \epsilon_0 \end{cases} \tag{2}$$

This implies that $K(x, y) = 1$ for all $x \in [0, 1], y \geq \epsilon_0$. Now we prove that $K(x, y) = 1$ for all $x \in [0, 1[, y \in [0, \epsilon_0[$. On the contrary suppose that $\alpha = K(x_0, y_0) < 1$ for some $x_0 \in]0, 1[,\ y_0 \in [0, \epsilon_0[$. Since $0 < \epsilon_0 < 1$, choose a $\delta > 0$ such that $0 < \epsilon_0 + \delta < 1$. Now choose a $J \in \mathbb{I}$ such that $J(x_0, K(x_0, y_0)) = J(x_0, \alpha) = \epsilon_0 + \delta \neq 1$. Now, $K(x_0, J(x_0, K(x_0, y_0))) = K(x_0, J(x_0, \alpha)) = K(x_0, \epsilon_0 + \delta) = 1$ a contradiction to the fact that g_K is an s.g.h. Thus $K(x_0, y_0) = 1$ for all $x_0 \in [0, 1[$ and $y_0 \in [0, \epsilon_0[$ and $K(x, y) = 1$ for all $x < 1$. Finally from (2) it follows that the range of K is trivial, a contradicition. Thus $\epsilon_0 \neq \epsilon_1$.

Lemma 5. *If $\epsilon_0 > 0$, then $\epsilon_0 = 1$. This implies that $K(1, y) = 0$ for all $y > 0$.*

Proof. Let $\epsilon_0 > 0$. Suppose $\epsilon_0 < 1$, i.e., $0 < \epsilon_0 < 1$. Now Lemma 4 implies that $\epsilon_0 \neq \epsilon_1$. Now let $y_1 \in]\epsilon_0, \epsilon_1[$. Then $K(1, y_1) = y_1$ by (1) of Proposition 1. Choose a $J \in \mathbb{I}$ be such that $J(1, y_1) = \frac{\epsilon_0}{2}$. Then $(J \circledast K)(1, y_1) = J(1, K(1, y_1)) = J(1, y_1) = \frac{\epsilon_0}{2}$ and $(K \circledast J \circledast K)(1, y_1) = K(1, J(1, K(1, y_1))) = K(1, J(1, y_1)) = K(1, \frac{\epsilon_0}{2}) = 0$ a contradiction to g_K is an s.g.h. Thus $\epsilon_0 = 1$.

From Lemmas 3, 5 it follows that if K is a non trivial range implication such that g_K is an s.g.h then K is such that $K(1, y) = y$ for all $y \in [0, 1]$ or $K(1, y) = 0$ for all $y \neq 1$. Once again here we recall that the characterisation of nontrivial K such that g_K is an s.g.h was completely done in the case K satisfies $K(1, y) = y$ for all $y \in [0, 1]$. Here in the following we present main result of this issue. For more results, please see [5].

Definition 6. *([5]) For $\epsilon \in [0, 1[$ define*

$$K_\epsilon(x, y) = \begin{cases} 1, & \text{if } x \leq \epsilon, \\ y, & \text{if } x > \epsilon, \end{cases}$$

Note that $K_\epsilon \in \mathbb{I}$, for all $\epsilon \in [0, 1]$ and $\sup K_\epsilon = I_{\mathbf{WB}}$ where

$$I_{\mathbf{WB}}(x, y) = \begin{cases} 1, & \text{if } x < 1, \\ y, & \text{if } x = 1. \end{cases} \tag{3}$$

For notational convenience, we denote the set of all such K_ϵ implications by

$$\mathbb{K}_\epsilon^+ = \{I \in \mathbb{I} | I = K_\epsilon \text{ for some } \epsilon \in [0, 1[\} \cup I_{\mathbf{WB}}.$$

Theorem 4. *([5]) Let $K \in \mathbb{I}$ be satisfy $K(1, y) = y$ for all $y \in [0, 1]$. Then the following statements are equivalent:*

(i) g_K *is an* s.g.h.
(ii) $K \in \mathbb{K}_\epsilon^+$.

Now it remains to characterise the nontrivial range non neutral implications K such that g_K is an s.g.h. Now we will take up this in the following subsection.

2.3 Characterisation and Representation of K Satisfying $K(1, y) = 0$ for All $y < 1$ Such That g_k Is an s.g.h.

Definition 7. *For* $\epsilon \in [0, 1[$ *define*

$$K^{\epsilon}(x, y) = \begin{cases} 1, & \text{if } x \le \epsilon \ , \\ y, & \text{if } \epsilon < x < 1 \ , \\ 0, & \text{if } x = 1 \ \& \ y \neq 1 \ . \end{cases} \tag{4}$$

For notational convenience, we denote the set of all such K^{ϵ} implications by

$$\mathbb{K}^{\epsilon} = \{I \in \mathbb{I} | I = K^{\epsilon} \ for \ some \ \epsilon \in [0, 1[\}.$$

Theorem 5. *Let* $K \in \mathbb{I}$ *be such that* $K(1, y) = 0$ *for all* $y \neq 1$*. Then* g_K *is an* s.g.h $\Longleftrightarrow K \in \mathbb{K}^{\epsilon}$ *for some* $\epsilon \in [0, 1[$.

Proof. Let $K \in \mathbb{I}$ such that $K(1, y) = 0$ for all $y \neq 1$.
$(\Longrightarrow)$: Let g_K be an *s.g.h* for some $K \in \mathbb{I}$. Since the range of K is non-trival, from Lemma 1, it follows that the range of K is $[0, 1]$. Let $0 < \alpha < 1$ be chosen arbitrarily. Then there exist some $x_0 \in]0, 1[, y_0 \in [0, 1[$, such that $0 < K(x_0, y_0) = \alpha < 1$. We keep K fixed, vary J and investigate the equivalence $J \circledast K = K \circledast J \circledast K$.

When $J = I_{\mathbf{0}}$, we have

$$(J \circledast K)(x_0, y_0) = I_{\mathbf{0}}(x_0, K(x_0, y_0)) = I_{\mathbf{0}}(x_0, \alpha) = 0 \ ,$$
$$(K \circledast J \circledast K)(x_0, y_0) = K(x_0, I_{\mathbf{0}}(x_0, K(x_0, y_0))) = K(x_0, 0).$$

Since g_K is an *s.g.h.*, $K(x_0, 0) = 0$. Hence, if $K(x_0, y_0) = \alpha < 1$, then $K(x_0, 0) = 0$. Now, for any $J \in \mathbb{I}$, we have

$$(J \circledast K)(x_0, 0) = J(x_0, K(x_0, 0)) = J(x_0, 0)$$
$$\text{and } (K \circledast J \circledast K)(x_0, 0) = K(x_0, J(x_0, K(x_0, 0))) = K(x_0, J(x_0, 0)).$$

Now let us, once again, choose $J \in \mathbb{I}$ such that $J(x_0, 0) = y_0$. Then

$$y_0 = J(x_0, 0) = K(x_0, J(x_0, 0)) = K(x_0, y_0) = \alpha.$$

Since α is chosen arbitrarily, we have

$$K(x_0, y) = y, \qquad y \in [0, 1]. \tag{5}$$

Let $x^* = \inf\{x | K(x, y) = y, \text{ for all } \ y\} \ge 0$. Note that the infimum exists because x_0 satisfies (5).

Claim: $K(s, y) = 1$, for any $s \in [0, x^*[$ and for all $y \in [0, 1]$.

Proof of the claim: On the contrary, let us suppose that $1 > K(s, y_0) = y_1 > y_0$ for some y_0, y_1. Now,

$$J(s, K(s, y_0)) = J(s, y_1) \ ,$$
$$K(s, J(s, K(s, y_0))) = K(s, J(s, y_1)) \ .$$

Once again, choosing a $J \in \mathbb{I}$ such that $J(s, y_1) = y_0$, we get

$$J(s, y_1) = y_0 \text{ and } K(s, J(s, y_1)) = K(s, y_0) = y_1 \ ,$$
$$\Longrightarrow J(s, K(s, y_0)) \neq K(s, J(s, K(s, y_0))),$$

i.e., g_K is not an *s.g.h.*, a contradiction. Thus $K(s, y) = 1$, for all $s \in [0, x^*[$.

Now the question is what value should one assign to $K(x^*, y)$. To allow for the possibility that $x^* = 0$ and since it is customary to assume left-continuity of fuzzy implications in the first variable, we let $K(x^*, y) = 1$. Note that letting $K(x^*, y) = y$ also gives a K such that g_K is a homomorphism.

From the above claim and (5) we see that every K is of the form (4) for some $\epsilon \in [0, 1[$.

($\Longrightarrow$): This follows easily.

3 Conclusions

The implications K for which the map g_K defined as in Definition 4 is an s.g.h were characterised and their representations were given in the case where K has trivial range. Further in the nontrivial range case the same was done for K satisfying $K(1, y) = y$ for all $y \in [0, 1]$. In this paper we showed that in the second case there are only two classes of nontrivial range $K \in \mathbb{I}$ for which g_K is an s.g.h. Also we characterised and found the representation of the nontrivial non neutral implications K for which g_K is an s.g.h thus completing the characterisations and representations of $K \in \mathbb{I}$ for which g_K is an s.g.h in all the cases.

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