Nonlinear Free Vibration Analysis of Nanobeams with Nonlocal and Surface Effects

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Dedication

Dedicated To

Achan and Amma
(Mr. S. Vijayakumaran Nair and Mrs. P. Sudharmani Amma)

Abstract

Micro electro-mechanical system (MEMS) based sensors and actuators are widely used in almost every field due to many advantages over the conventional devices in terms of stability, accuracy, sensitivity and operating flexibility, etc. Many resonant sensors and actuators are characterized their resonant frequencies and damping. Therefore, it is essential to compute character frequency. As the size of the micro devices reduce to nanoscale, the resonance frequency become dependent of size related factors such as nonlocal effect as well as surface effects. Therefore, in this thesis, we present a detailed theory and derive resonance frequency models theoretically and numerically for the simply-supported, fixed-fixed, and the cantilever nanobeams.

To model the nonlocal effects under axial loading, we first obtain the governing equation by including the nonlocal effects in bending and axial terms. Subsequently, we obtain the analytical models for different resonance frequencies as a function of nonlocal effects. We found that higher modes are more sensitive to the nonlocal parameters. However, the nonlocal model mentioned above is valid for small amplitude oscillation. To improve its range, we modified the governing equation by including additional nonlocal effects in inertia and damping, and also geometric nonlinearity. Subsequently, we found the frequencies as a function of all the important parameters by using the method of multiple scale. To validate the models numerically, we present a linear finite element model to capture nonlocal and surface effects. However, this model does not capture geometric nonlinearity but it can be extended to complex geometries. Finally, we validate our model with each other and also present the limitation of each models.

The models developed in this thesis are applicable to different types of beams which can be widely used in the design of senstive nanoscale sensors and actuators. However, it can be improved by extending it to Timoshenko beams.

Contents

Chapter 1

Introduction

1.1 Motivation

Microelectromechanical systems, MEMS, have been used to develop different types of sensitive dynamic sensors and actuators in many important areas, for example, transportation, communication, automated manufacturing, environmental monitoring, health care, defense systems, and a wide range of consumer products. MEMS are inherently small, thus offering attractive characteristics such as reduced size, weight, and power dissipation and improved speed and precision compared to their macroscopic counterparts. As Integrated Circuit (IC) fabrication technology continues to scale toward deep sub micron and nanometer feature sizes, a variety of nanoelectromechanical systems (NEMS) can be envisioned in the foreseeable future. Nanoscale mechanical devices and systems integrated with nanoelectronics have opened a vast number of new exploratory research areas in science and engineering. To design the dynamic NEMS devices, it is important to understand their vibrational characteristics such as the resonance frequencies and the damping. Although, many of such findings are reported through experiments, the theoretical details are often presented through the underlying assumptions. Among many, one of the basic assumptions which is used to compute the theoretical resonant frequencies is the validation of the theory of elasticity. In this work, we are investigating the combined effects of the surface effect, nonlocal effect with and without geometric nonlinearities which can be modeled numerically [1] as well as analytically [2].

Since the controlled experiments on nanoscale are difficult to perform, the mechanical behaviours of the nanostructures are usually investigated using mathematical simulations such as atomistic, atomistic-continuum mechanics and continuum mechanics approaches. On the other hand, the atomistic and atomistic-continuum mechanics simulation methods consume much time and are computationally expensive for analyzing the system. Therefore, the continuum mechanics based approach is often applied. However, the classical continuum theory cannot predict the small length scale effect and size dependence of material properties at the nanoscale. The length scale effect and the size dependence of material properties are due to the long-range inter-atomic interaction and the energy associated with atoms at free surfaces of the nanostructures, respectively. Such effects can be incorporated separately using the nonlocal as well as surface elasticity theory.

1.2 Literature Survey

In order to include the length scale effects, it has been suggested that nonlocal continuum theory developed by Eringen [3, 4, 5] could be used in the continuum models for accurate prediction of mechanical behaviours of nanostructures [6]. Nonlocal theory of Eringen is based on the assumption that the stress at a material point is considered as a function of the strain field at all the material points in the neighbourhood of the continuum body. The inter-atomic forces and atomic length scales directly come to the constitutive relations as material parameters [3, 4, 5]. In traditional continuum mechanics, the surface free energy is neglected in comparison with the bulk energy because it is associated with only a few layers of atoms near the surface and the ratio of the volume occupied by the surface atoms and the total volume of material of interest is extremely small[7]. As the structural size decreases towards the nanoscale regime, due to the high surface/volume ratio, the surface-to-bulk energy ratio increases. Hence, the surface free energy becomes a significant part of the total elastic potential energy and should be taken into account. Both the experimental observations [8] and theoretical analyses [9] indicate that surface layers differ from their bulk counterparts, their elastic responses are intrinsically size-dependent and consequently, the physical and chemical properties of nanomaterials become size-dependent. Gurtin and Murdoch [10, 11] presented a surface elasticity theory by modeling the surface as a two-dimensional membrane adhering to the underlying bulk material without slipping to account for the effect of surfaces or interfaces on mechanical properties. It has been shown that with correctly chosen surface elastic properties, this surface elasticity theory [12] explains various size-dependent phenomena at the nanoscale and the predictions fit well with atomistic simulations and experimental measurements.

Mahmoud et al. [13] developed a nonlocal finite element model for investigation of bending behavior of Euler-Bernoulli nanobeam, including surface effects. Natural boundary conditions such as the end moments and forces are expressed in terms of nonlocal stresses. Several computational experiments have been carried out to investigate the size dependent behavior due to the nature of nonlocal elasticity and surface effects.

An analytical study on the nonlinear free vibration of functionally graded nano beams with surface effects had been done by Shahrokh et.al [14]. The main goal of this work was to study the surface effects, tension and density, on the nonlinear free vibration of functionally graded nanobeams based on the Euler-Bernoulli beam theory considering the surface equilibrium condition. The Von-Karman geometric nonlinearity is taken into account with the assumption of cubic variation of normal stress through the thickness. The method of multiple scales has been used as an analytical solution for the nonlinear governing equation.

The other works on the functionally graded nano beams were done by Ke et al.[15] and Sharabiani and Yazdi [16]. Ke et al.[15] investigated the nonlinear free vibration of functionally graded nanocomposite beams reinforced by single-walled carbon nanotubes (SWCNTs) based on Timoshenko beam theory and Von-Karman geometric nonlinearity. Sharabiani et al.[16] studied surface effects including surface elasticity and surface tension on nonlinear free vibration of functionally graded nanobeams based on the Euler-Bernoulli beam theory. They did not consider the surface equilibrium condition in derivation of the governing equation.

The surface stress effect on the mechanics of nanostructures has recently been studied. Miller and Shenoy [17] presented that the bending/axial deformation of a nanoscale beam is well depicted by a continuum model that accounts for the surface stress effect. Cuenot et al.[18] experimentally showed that the bending deformation of a one dimensional nanostructure such as nanowires is significantly affected by the surface stress. Lilley [19] have taken into account the surface elasticity-based continuum mechanics model that allows the fundamental insights into the role of surface elasticity on the mechanical properties (i.e.bending deformation) of nanowires. Wang and Li [20] have recently reported that the surface effect significantly determines the elastic properties of nanowires using the density functional theory (DFT) simulations. Yun and Park [21] have provided that, by using multiscale simulations based on surface Cauchy-Born model [22, 23, 24] the bending behaviour of FCC metal nanowires is governed by surface stress forces.

A semi analytical method for the nonlinear vibration of Euler-Bernoullie beams with generalized boundary condition was done by Yan Liu et. al [25]. The method makes use of Linstedt-Poincare perturbation technique to transform the nonlinear governing equations into a linear differential equation system, whose solutions are then sought through the use of differential quadrature approximation in space domain and an analytical series expansion in time domain.

Raffaele et.al [26] proposed an integral nonlocal model to take into account the effect of micro structure on the dynamic response of resonant MEMS devices. They studied that the influence of a finite material length scale manifests itself the damping behavior at nano and micro level. Semi analytical results are computed by averaging only in the transverse direction whereas finite element formulation for longitudinal direction is still under development.

Saeed Abbasion et.al [27] studied the size dependence in free vibration analysis of microscaled Timoshenko beams. They presented a comprehensive model to study the influence of surface elasticity and residual surface tension on the natural frequency of transverse vibrations of micro beams in the presence of rotary inertia and shear deformation effects. It has been concluded that the frequency of vibration of micro and nano beams are size dependent and the results tend to results of classical beam models when the beam length increases.

Li et.al [28] investigated the natural frequency, steady -state resonance and stability for transverse vibrations of nanobeam subjected to variable axial tension as well as compression based on nonlocal elasticity theory. Through the study it has been identified that the instability regions are greatly influenced by nonlocal nanoscale and they become smaller with stronger nonlocal effects .

Behnam Gheshlaghi et.al [29] has done studies on the effect of surface elasticity and tension on the free nonlinear vibrations of nanobeams based on Euler-Bernoulli beam theory in conjunction with von Karman geometric nonlinear model. The results reveal that by increasing the nanobeam dimensions all natural frequencies gradually approach the frequency limit, which entails an expected decrease in the surface effects. More importantly, the surface effects are notably less influential at higher vibration amplitudes. Moreover, inspection of problem phase trajectory plot indicates that the effect of including the surface effects is similar to increasing the magnitude of the velocity component of problem initial conditions .

Murmu et.al [30] studied the dynamic characteristics of a damped viscoelastic nonlocal beams utilizing Kelvin-Voigt and three parameter standard viscoelastic models, velocity dependent external damping and nonlocal Euler-Bernoulli beam theory. They investigated that the external damping parameters have simple effects on the natural frequencies and the dependence with nonlocal parameter is not so strong. Also it showed that nonlocal parameters decrease the sensitivity of the viscoelastic parameter on the damped natural frequencies.

Lei et.al [31] also studied the dynamic behaviour of damped viscoelastic nonlocal beams, but carried out in Timoshenko beam model. The governing equation of motion and the corresponding characteristic equation for the complex frequencies are derived. The theory is applied to a dynamics of a single walled carbon nanotube. It has been identified that the external damping parameter has linear effects on the natural frequencies.

Ru et.al [32] incorporated the size effect of dissipative surface stress on the quality factor of microbeams. The suggested model is an extension of Zener model from bulk materials to the surface in the presence of an initial surface tension. He identified that the amount of surface dissipation depends on the specific values of surface stress parameters and relaxation times.

Singha et.al [33] conducted studies on the nonlinear vibration of laminated skew plates by finite element method. The formulation includes the effects of shear deformation and rotary inertia. The variation of nonlinear frequency ratios with amplitudes is brought out considering different parameters such as skew angle, fiber orientation and boundary condition. It has been investigated that the nonlinear frequency ratio in general increases with increase in thickness and skew angle.

Reza Ansari et.al [34] studied the nonlinear finite element vibration analysis of double walled carbon nanotubes based on Timoshenko beam theory. The finite element is employed to discretize the nonlinear governing equations which are solved to obtain the nonlinear vibration frequencies. The effect of material constant of the surrounding elastic medium and geometric parameters on the vibrational beahaviour are investigated.

Ji Wang et.al [35] formulated the first-order Mindlin plate equations in the finite element method to study the high frequency thickness-shear vibrations of quartz crystal plates. They considered kinematic as well as material nonlinearities in the analysis and obtained frequency response relations. An advancement in the nonlinear finite element analysis is required to study the nonlinearities coming from miniatured crystal resonators.

Nazemnezhad et.al [36] conducted studies on the nonlinear free vibration of nanobeams considering surface effects using Euler Bernoulli beam theory. Accordingly surface density was introduced into the governing equations and it was concluded that the surface density has negligible effect on the variation of the fundamental natural frequency with respect to the length of nanobeam. It was also determined that the effect of surface density is independent of the amplitude ratio.

Wang et.al [37] performed a variational consistent derivation of the governing equations and boundary conditions for the free vibration of beams based on Eringen's nonlocal elasticity theory and the Timoshenko beam theory. Studies proved that the mode shapes are affected by the effects of small length scale, transverse shear deformation and rotary inertia except for the case of simply supported beams. The exact solutions from this model serve as a reference for verifying the numerical vibration solutions obtained from other mathematical models.

Slimani et.al [38] characterized the free non linear vibration behavior of composite beams by using polynomial finite element method with shape functions based on Legendre polynomials or sinusoidal functions. It was observed that asymptotic linearization can be a exact method under the assumption of uni-model excitation because it takes into account of the higher order harmonics. Confrontations of the results also indicated that the simple iterative finite element formulation and the closed form expression gives results close to those of the first order approximation of the asymptotic linearization method.

In the present, we first present analytical models considering the combined effects of non-local, surface, damping and geometric nonlinearity to compute the free vibration frequencies. Subsequently, we present the linear finite element modeling to capture the effect of non-local and surface effects. Finally, we compare the analytical and finite element models with some discussion. In the following section, we present the outline of the work.

1.3 Outline of the thesis

This thesis consists of five chapters. In the first chapter, we have described the motivation behind the selection of this work related to nonlocal and surface effects after presenting a brief introduction to MEMS and NEMS devices. The influence of these parameters in the vibrational characteristic of nanobeams are also mentioned. Towards the end, a review of different works performed by various researchers in this area are presented.

In the second chapter, we investigate the effect of nonlocal effect on the beam with uniform axial load. To do it, we first present the mathematical formulation involving the computation of frequencies. Subsequently, we analyze the nonlocal effect on the frequencies of the simply supported beam, fixed-fixed and cantilever beams, respectively.

In the third chapter, we obtain approximate formula of generalized frequency considering nonlocal effect, surface effect, damping effect, and the geometric nonlinearity, etc., using the method of multiple scales. To do it, we derive the governing equation by taking into account the surface, nonlocal and geometric nonlinear parameters. Subsequently, we apply Galerkin method and, then, the method of multiple scale to solve the governing equations to obtain an approximate expression of the resonance frequencies. Finally, we discuss the results showing the influence of all the parameters on the frequencies.

The fourth chapter deals with the linear finite element modeling of the nonlocal and surface effects in nanobeams. To do it, we apply variation principle on the governing equation containing the nonlocal and surface effects to obtain the weak form of the equation. Subsequently, by approximating the displacement using the Lagrange interpolation function between the nodes, we get the stiffness and mass matrix and then the resonance frequencies corresponding to different modes. Finally, we present the effect of nonlocal and surface effects on the frequencies.

The fifth chapter deals with the validation and comparison of analytical and numerical models. On comparing the results, we found that the proposed model which include the effects of surface roughness, non-local effect, damping and the geometric nonlinearity, show good agreement with the given model from the literature. After comparison, it concluded that the proposed model is more precise and it can depict the vibrational characteristics more effectively.

In the sixth chapter, we have summarized our results corresponding to the analysis. Finally, the conclusion is drawn based on the usefulness of this work. We end the chapter with the future perspectives of the work presented in the thesis.

Chapter 2

Nonlocal effects under axial loading

This chapter presents the theoretical foundation and modeling of nano beams subjected to an axial force under the influence of nonlocal effects. Based on the literature, it is found that the nonlocal effects reduces the effective stiffness of the nanobeam. However, the contradictory behavior in the stiffness can be obtained by changing the axial load [48]. Li et al.[48] modeled the nonlocal effect in simply-supported beam subjected to uniform axial load. Using the variational principle, they first obtain the nonlocal bending moment and subsequently a sixth order partial differential equation. In this chapter, we briefly present the underlying theory to derive the governing equation capturing the nonlocal effects. Based on the exact mode shape, we obtain the solution using the method variable separation. Then, we validate the solution methodology by comparing the frequencies of simply supported beam with the results obtained by Li et al.[48]. Subsequently, we obtain the modal frequencies of cantilever and fixed-fixed beams, respectively.

2.1 Formulation of nonlocal equations of motion

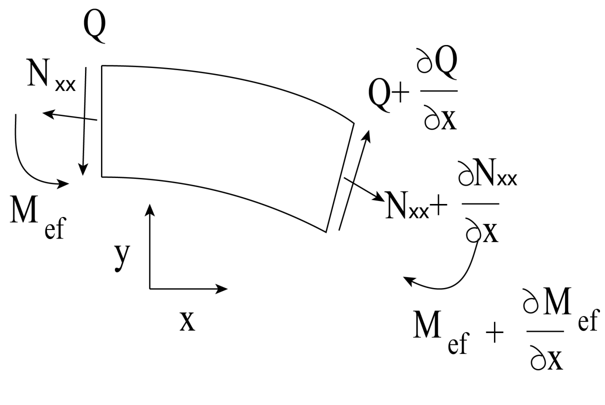

Figure 2.1: Force and moment equilibrium for an element of a nanobeam.

Consider a nanobeam with an initial axial tension N_{xx} and length L. The force equilibrium diagram of an element of the nanobeam is illustrated in Fig. 2.1 in which M_{ef} is the effective bending moment according to the nonlocal elasticity theory, N_{xx} is the internal axial force, Q is the shear force, x the axial coordinate, and y the transverse coordinate. Considering the linear vibration with only small deformation, the dynamic equation of motion for the element can be obtained using the Newton's second law of motion and the moment equilibrium condition as [48]

$$\frac{\partial^2 M_{\mathrm{ef}}}{\partial x^2} + N_{\mathrm{xx}}\frac{\partial^2 w}{\partial x^2} - \rho A\frac{\partial^2 w}{\partial t^2} = 0 \tag{2.1}$$

where, ρ is the material mass density, A is the cross-sectional area of nanobeam, and w is the transverse displacement. M_{ef} is the effective bending moment which is given by

$$M_{\mathrm{ef}} = M - 2\sum_{n=1}^{\infty}\mu^n\frac{\partial^{2n} M}{\partial x^{2n}} \qquad (n = 1, 2, ...) \tag{2.2}$$

where, M is the nonlocal bending moment,

$$M = \mu\frac{\partial^2 M}{\partial x^2} - EI\frac{\partial^2 w}{\partial x^2}, \tag{2.3}$$

and EI is the flexural rigidity and μ is the nonlocal scale parameter which is defined as per Eringen's theory [3, 4] as $\mu = (e_0 \times a)^2$. Two other quantities e_0 and a are a constant dependent on material and an internal characteristic length, respectively, and they capture nonlocal effects.

Neglecting the higher order terms and assuming $n = 1$ in M_{ef} (the most significant nonlocal effect is retained), we obtain the resulting equation by substituting M_{ef} into eqn. (2.1) as

$$\frac{\partial^2 M}{\partial x^2} - 2\mu\frac{\partial^4 M}{\partial x^4} + N_{\mathrm{xx}}\frac{\partial^2 w}{\partial x^2} - \rho A\frac{\partial^2 w}{\partial t^2} = 0. \tag{2.4}$$

From equations (2.2) and (2.3), the governing equation of motion for a nanobeam subjected to an initial axial tension can be derived as

$$\rho A\frac{\partial^2 w}{\partial t^2} - N_{\mathrm{xx}}\frac{\partial^2 w}{\partial x^2} - \mu\left(2EI\frac{\partial^6 w}{\partial x^6} + \rho A\frac{\partial^4 w}{\partial x^2\partial t^2} - N_{\mathrm{xx}}\frac{\partial^4 w}{\partial x^4}\right) = -EI\frac{\partial^4 w}{\partial x^4}. \tag{2.5}$$

The relation between nonlocal bending moment and transverse displacement is also obtained according to eqns. (2.2) and (2.3) as

$$M = \mu\rho A\frac{\partial^2 w}{\partial t^2} + [EI - \mu N_{\mathrm{xx}}]\frac{\partial^2 w}{\partial x^2} + 2\mu EI\frac{\partial^4 w}{\partial x^4}. \tag{2.6}$$

From equations (2.1) and (2.6), we get

$$\begin{aligned} M_{\mathrm{ef}} &= \mu\rho A\frac{\partial^2 w}{\partial t^2} + [EI - \mu N_{\mathrm{xx}}]\frac{\partial^2 w}{\partial x^2} + 2\mu EI\frac{\partial^4 w}{\partial x^4} \\ &-2\mu\left(\mu\rho A\frac{\partial^4 w}{\partial x^2\partial t^2} + [EI - \mu N_{\mathrm{xx}}]\frac{\partial^4 w}{\partial x^4} + 2\mu EI\frac{\partial^6 w}{\partial x^6}\right). \end{aligned} \tag{2.7}$$

Introducing the following dimensionless parameters and variables,

$$\bar{x} = \frac{x}{L}, \qquad \bar{w} = \frac{w}{L}, \qquad \bar{t} = t\sqrt{\frac{EI}{\rho A L^4}}, \qquad \tau = \frac{\sqrt{\mu}}{L}, \qquad T = \frac{N_{\rm xx}L^2}{EI}, \qquad \bar{M} = \frac{ML}{EI}. \tag{2.8}$$

the governing equation can also be written in dimensionless form as follows

$$\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} - T\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} - \tau^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}^2} + T\tau^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} - 2\tau^2 \frac{\partial^6 \bar{w}}{\partial \bar{x}^6} = 0 \tag{2.9}$$

To solve the above equation, we apply the method of variables separation under which the solution of eqn. (2.9) can be assumed as

$$\bar{w}_n(\bar{x}, \bar{t}) = C\phi_n(\bar{x})q_n(\bar{t}), \tag{2.10}$$

where, C is an arbitrary constant, $\phi_n(\bar{x})$ is the vibration mode function, $q_n(\bar{t})$the temporal function with respect to $\bar{t}$ and $n = 1, 2, 3, ...$denotes the mode number. By substituting eqn. (2.10) into eqn. (2.9), one obtains an ordinary differential equation as

$$\frac{d^2 q}{d\bar{t}^2} + \omega_n^2 q = 0 \tag{2.11}$$

$$-\omega_n^2 \phi_n + (\tau^2 \omega_n^2 - T)\frac{d^2 \phi_n}{d\bar{x}^2} + (T\tau^2 + 1)\frac{d^4 \phi_n}{d\bar{x}^4} - 2\tau_2 \frac{d^6 \phi_n}{d\bar{x}^4} = 0, \tag{2.12}$$

where, ω_n is the dimensionless vibration frequency.

2.2 Frequencies of different types of nanobeams

2.2.1 Simply Supported Beam

To illustrate the effects of nonlocal nanoscale τ and dimensionless axial tension T on the vibration behaviour of a nanobeam, a simply supported nanobeam is considered first. The vibrational mode shape function is assumed as

$$\phi_n(\bar{x}) = \sin n\pi\bar{x} \qquad (n = 1, 2, 3, ..), \tag{2.13}$$

which satisfies the boundary conditions as well. The substitution of eqn. (2.13) into eqn. (2.12) and solving for the dimensionless natural frequency ω_n we will get the expression as follows,

$$\omega_n = n\pi\sqrt{\frac{T + (T\tau^2 + 1)n^2\pi^2 + 2\tau^2 n^4 \pi^4}{1 + n^2\pi^2\tau^2}}, \tag{2.14}$$

Taking the first and third mode frequencies as test examples, we present the comparison of frequencies based on nonlocal stress theory and classical vibration theory as shown in Fig. 2.2. Subsequently, the effect of nonlocal scale is observed same as the one present in Li et al. [48]. Apparently, frequencies for a nonlocal nanobeam are significantly higher than the corresponding solutions based on classical vibration.

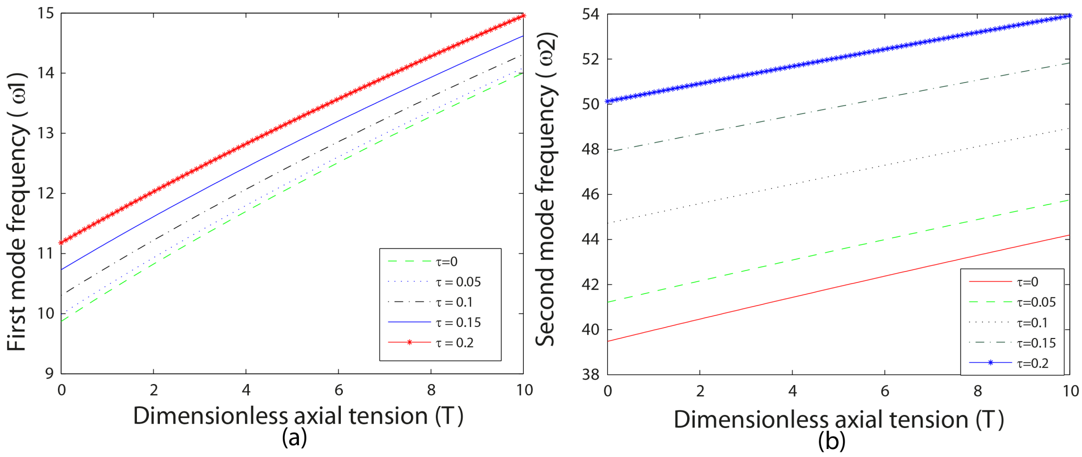

Figure 2.2: Comparison of vibration frequency obtained from nonlocal theory and classical theory for (a) first and (b) second mode of simply supported beam.

Here, to show the nonlocal effect, we vary τ to obtain the variation of the vibration frequencies as illustrated in Figs. 2.3 and 2.4, respectively. From the figures, it is observed that the free vibration frequencies increase with increasing τ. Hence, the stiffness of nanobeam increases for higher τ and the rate of increase is particularly marked for higher vibration modes. A possible explanation is that larger nonlocal nanoscale indicates stronger intermolecular interaction constraints and, thus, higher stiffness.

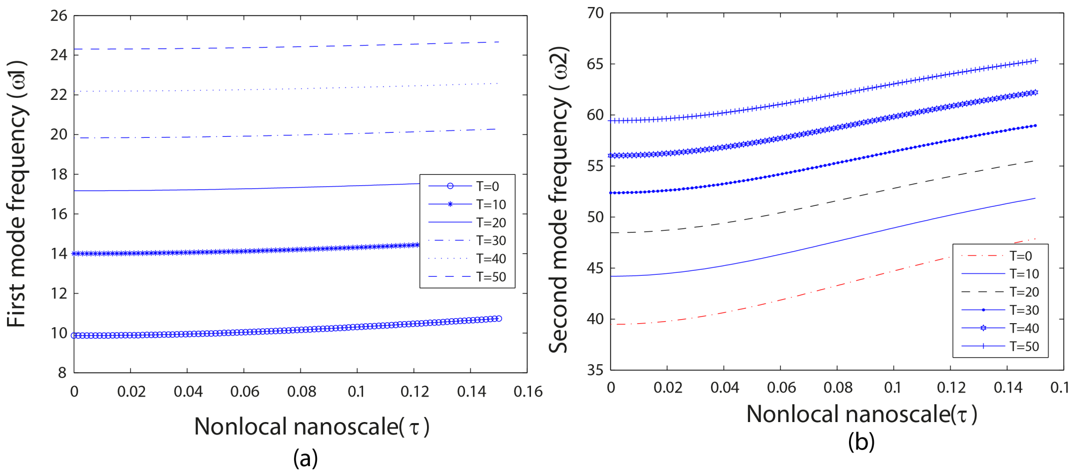

Figure 2.3: Effect of nonlocal nanoscale τ on (a) first and (b) second mode vibration frequency for increasing dimensionless axial tension for simply supported beam.

2.2.2 Cantilever Beam

To illustrate the effects of nonlocal nanoscale τ and dimensionless axial tension T on the vibration behaviour of a cantilever nanobeam, the vibrational mode shape function is assumed as

$$\phi_n(\bar{x}) = (\cosh a_n\bar{x} - \cos a_n\bar{x}) - \sigma_n(\sinh a_n\bar{x} - \sin a_n\bar{x}) \qquad (n = 1, 2, 3, ..), \tag{2.15}$$

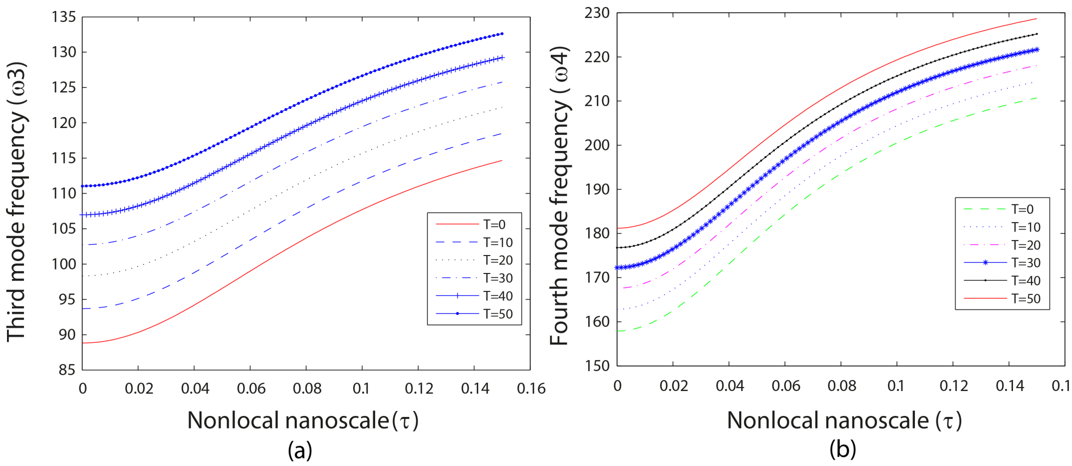

Figure 2.4: Effect of nonlocal nanoscale τ on (a) third and (b) fourth mode vibration frequency for increasing dimensionless axial tension for simply supported beam.

where $\sigma = \frac{\cos a_n L + \cosh a_n L}{\sin a_n L + \sinh a_n L}$ and the value of a_n and σ_n are different for different modes. This is satisfying the boundary conditions as well. The substitution of eqn. (2.15) into eqn. (2.12) yields the dimensionless natural frequency expressions for different modes as written below.

$$\omega_1 = 0.53 \frac{\sqrt{(0.93\tau^2 + 0.285 \times 10^9)(0.123 \times 10^{11} + 0.123 \times 10^{11} T\tau^2 + 0.8069 \times 10^6 \tau^2 + 0.32 \times 10^5 T)}}{(0.93 \times 10^4 \tau^2 + 0.28 \times 10^9)}, \tag{2.16}$$

$$\omega_2 = 0.5 \frac{\sqrt{(-5.31 \times 10^6 \tau^2 + 5.003 \times 10^8)(2.42 \times 10^{11} + 2.42 \times 10^{11} T\tau^2 - 2.66 \times 10^9 \tau^2 - 5.31 \times 10^6 T)}}{(2.65 \times 10^6 \tau^2 - 2.501 \times 10^8)} \tag{2.17}$$

$$\omega_3 = 2.23 \frac{\sqrt{(40070\tau^2 + 1.99 \times 10^6)(1.52 \times 10^9 + 1.52 \times 10^9 T\tau^2 + 6.092 \times 10^7 \tau^2 + 8014T)}}{(40070\tau^2 + 1.99 \times 10^6)} \tag{2.18}$$

$$\omega_4 = 1.7321 \frac{\sqrt{(0.174 \times 10^6 \tau^2 + 0.191 \times 10^6)(0.9182 \times 10^9 + 0.9182 \times 10^9 T\tau^2 + 0.16 \times 10^{10} \tau^2 + 0.58 \times 10^5 T)}}{(.174 \times 10^6 \tau^2 + .191 \times 10^6)}, \tag{2.19}$$

The effect of τ on the vibration frequencies is illustrated in Fig. 2.6. From the figure, it is observed that the free vibration frequencies increase with increasing τ. Hence, the stiffness of nanobeam increases for higher τ and the rate of increase is particularly marked for higher vibration modes.

2.2.3 Fixed-Fixed Beam

To illustrate the effects of nonlocal nanoscale τ and dimensionless axial tension T on the vibration behaviour of a fixed-fixed nanobeam, the vibrational mode shape function is assumed as

$$\phi_n(\bar{x}) = (\cosh a_n\bar{x} - \cos a_n\bar{x}) - \sigma_n(\sinh a_n\bar{x} - \sin a_n\bar{x}) \qquad (n = 1, 2, 3, ..), \tag{2.20}$$

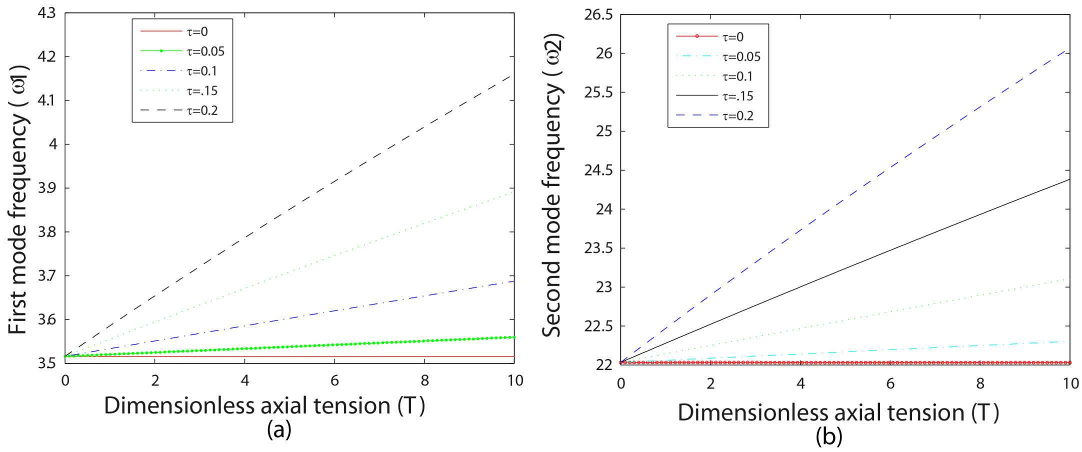

Figure 2.5: Comparison of vibration frequency obtained from nonlocal theory and classical theory for first and fourth mode of Cantilever beam.

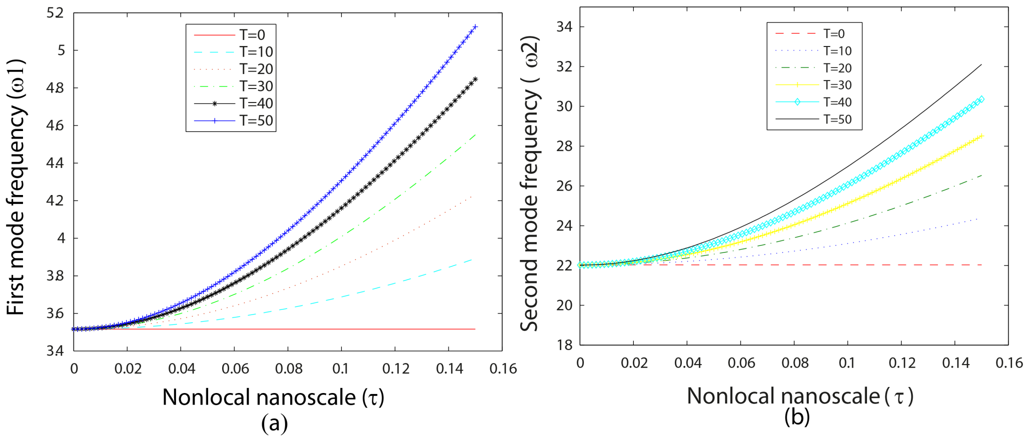

Figure 2.6: Effect of nonlocal nanoscale τ on (a) first and (b) second mode vibration frequency for increasing dimensionless axial tension for cantilever beam.

where $\sigma_n = \frac{\cos a_n L + \cosh a_n L}{\sin a_n L + \sinh a_n L}$ and satisfies the boundary conditions as well. The values of σ_n and a_n are different for different modes. The substitution of (2.20) into (2.12) yields the dimensionless first mode natural frequency as

$$\omega_1 = 3.16\frac{\sqrt{(2.72\times10^{10}\tau^2+1.58\times10^9)(7.9\times10^{10}+7.9\times10^{10}T\tau^2+2.72\times10^{12}\tau^2+2.71\times10^9T)}}{(2.71\times10^{10}\tau^2+1.58\times10^9)}, \tag{2.21}$$

and for the second mode we will get the expression for frequency as

$$\omega_2 = 2.24\frac{\sqrt{(6.76\times10^5\tau^2+11691)(8.872\times10^6+8.87\times10^6T\tau^2+1.02\times10^9\tau^2+1.353\times10^5T)}}{(6.76\times10^5\tau^2+11691)}, \tag{2.22}$$

The effect of τ on the vibration frequencies is illustrated in Fig. 2.8. From the figure, it is observed that the free vibration frequencies increase with increasing τ. Hence, the stiffness of nanobeam increases for higher τ and the rate of increase is particularly marked for higher vibration modes.

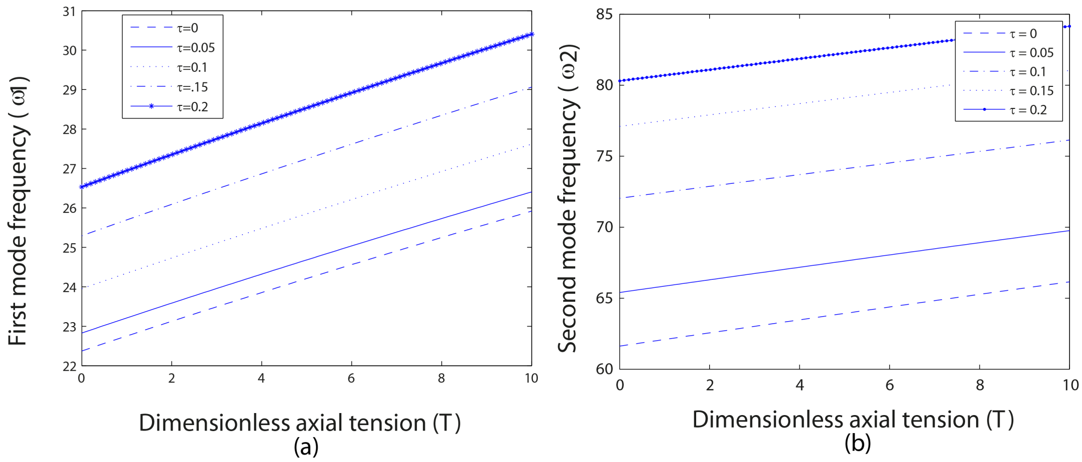

Figure 2.7: Comparison of vibration frequency obtained from nonlocal theory and classical theory for (a) first and (b) second mode of fixed-fixed beam.

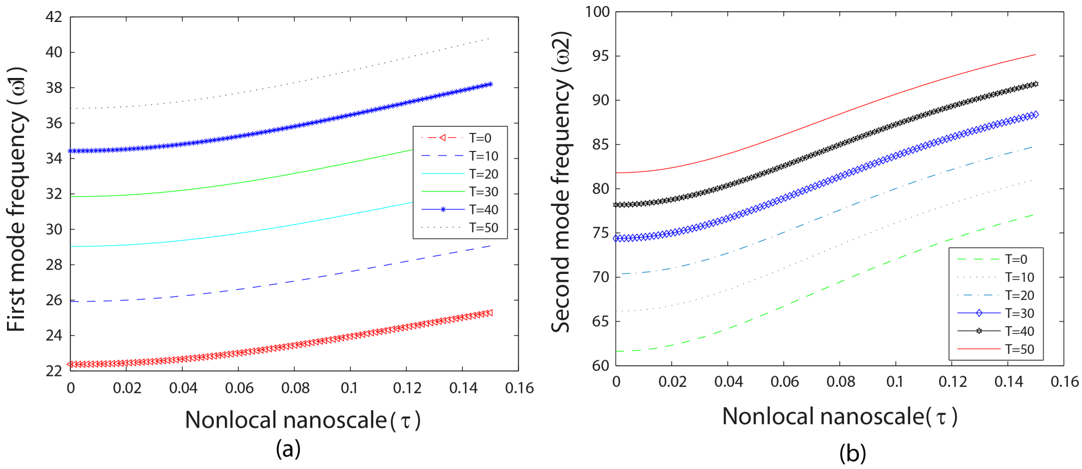

Figure 2.8: Effect of nonlocal nanoscale τ on (a) first and (b) second mode vibration frequency for increasing dimensionless axial tension for fixed-fixed beam.

2.3 Results and Discussion

In this chapter, we obtain the analytical expression of modal frequencies of simply supported, cantilever and the fixed-fixed beam, respectively. On analyzing the variation of frequencies with dimensionless axial tensions for different beams, it is evident that the nonlocal nanoscale is playing a major role in the vibrational characteristics of nanobeams. It is also obsered that the nonlocal effects become more significant in the higher modes. Such kind of behaviour is found to be consistent for the beam with different boundary conditions. When the nonlocal nonlocal parameter value has been put to zero, it merges with the classical results.

While analyzing the frequency vs nonlocal nanoscale parameters for various axial tension values, we can see common changes happening for all types nanobeams considered in this chapter. In this case, we vary tension values ranging from 0 to 50, in steps of 10 and the nonlocal parameter from 0 to 0.16. As the tension value increases the frequency increases; and for the higher modes, the curves are more crowded towards the zero tension plot. Thus, we can summarize that as the tension value increases, for higher modes the frequency change is less when compared with the lower modes of nanobeams irrespective of end conditions.

Chapter 3

Combined effects of nonlocal and surface effects

In this chapter, we obtain the modified governing equation of the nanobeam based on the Euler-Bernoulli beam theory (EBT) to include the influence of surface effects, nonlocal effects, damping and geometric nonlinearity. The classical theories of Gurtin and Murdoch [10] is utilized to formulate the surface effects. For nonlocal effect, the theories proposed by Eringen [3] is utilized. The geometric nonlinearity is included in the Green strain tensor subjected to Von-Karman assumptions. We also include nonlocal effect in damping [30].

3.1 Mathematical formulation based on Euler beam theory

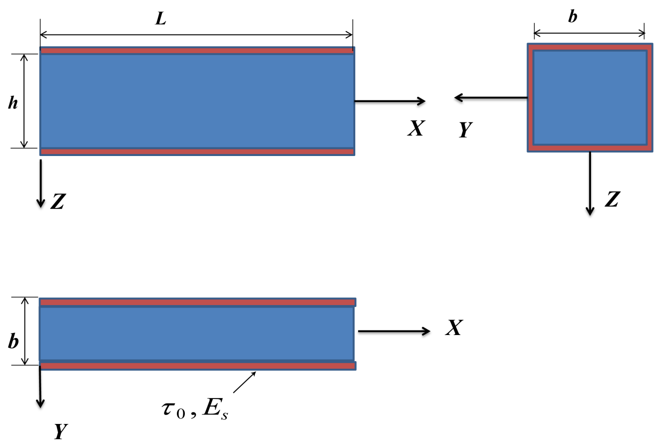

Figure 3.1: Different views of a nanobeam with surface layer (shown in different shade on the outer surface). Here, τ_0 and E_s are residual surface tension and elastic surface modulus.

Consider a nanobeam of length L, uniform width b and thickness h as shown in Fig. 3.1. A cartesian coordinate system (x, z) is used to label the material points of the nanobeam in the unstressed reference configuration.

The displacements $\overline{u}$ (in the x-direction), $\overline{w}$ (in the z-direction) can be approximated as,

$$\overline{u}(x,z,t) = u(x,t) - z\frac{\partial w}{\partial x}, \qquad \overline{w}(x,z,t) = w(x,t) \tag{3.1}$$

where, u and w are the axial and transverse displacement components of a material point on the mid-plane of the beam. Taking the nonzero Green's strain tensor components under the assumptions of Von-Karman, the strains can be written in terms of displacement as,

$$\epsilon_{\mathrm{xx}} = \epsilon^0_{\mathrm{xx}} - z\frac{\partial^2 w}{\partial x^2}, \qquad \epsilon^0_{\mathrm{xx}} = \frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^2 \tag{3.2}$$

To capture the surface effect, the classical constitutive relation of the surface boundaries ($y = \pm b/2, z = \pm h/2$) as given by Gurtin and Murdoch [10, 11] and also the classical constitutive relations for the internal material of the beam ($-b/2 < y < b/2, -h/2 < z < h/2$) can be expressed as

$$\sigma_s = \tau_0 + E_s\varepsilon_{xx}\,, \qquad \sigma_{\mathrm{xx}} = E\varepsilon_{\mathrm{xx}} \tag{3.3}$$

where, τ_0 and E_s are the residual surface tension in the axial direction and the surface elastic modulus, respectively. E and G are Young's modulus, shear modulus of the internal material of the beam, respectively.

The stress resultants also containing the surface effects can be utilized to find the effective axial force and the moment as

$$N_{\mathrm{xx}} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{\mathrm{xx}} b dz + \oint \sigma^s ds = (EA)_s\varepsilon^0_{\mathrm{xx}} + 2\tau_0(b+h), \tag{3.4}$$

$$M_{\mathrm{xx}} = \int_{\frac{-h}{2}}^{\frac{h}{2}} z\sigma_{\mathrm{xx}} b dz + \oint z\sigma^s ds = -(EI)_s\left[-\frac{\partial^2 w}{\partial x^2}\right] \tag{3.5}$$

where, $(EA)_s$ and $(EI)_s$ are the effective in-plane and flexural rigidities, respectively, and which can be written as follow,

$$(EA)_s = EA + 2E_s(b+h) \qquad \text{and} \qquad \mathrm{(EI)_s = E(\frac{bh^3}{12}) + E_s(\frac{h^3}{6} + \frac{bh^2}{2})}. \tag{3.6}$$

In order to include the small scale effect, it has been suggested that nonlocal continuum theory developed by Eringen [3, 4, 5] could be used in the continuum models for accurate prediction of mechanical behaviors of nanostructures. Eringen's theory states that stress at a reference point is a function of the strain field at neighbourhood point in the body. Here the size effects are taken into account by the integration of a scale parameter into classical continuum models.The internal characteristic length can be considered as a material parameter by the following differential constitutive relation

$$(1-\mu\frac{\partial^2}{\partial x^2})\sigma_{ij}^{nl}=\sigma_{ij}^{l} \qquad \text{and} \qquad \mu=(\mathrm{e}_0\times \mathrm{a})^2 \tag{3.7}$$

where, μ is the scale parameter and e_0 is the material constant and a is a parameter depending upon the internal characteristic length ; σ_{ij}^{nl} and σ_{ij}^{l} are the non-local and local stress tensor components, respectively.

Using the strain-displacement relation from equation (3.2), stress-strain relations given in equation (3.3), nonlocal constitutive relations from equation (3.7) and stress resultant definitions as per equations (4.3) and (3.5), one can express the stress resultants in terms of the displacement components as

$$\left(1-\mu\frac{\partial^2}{\partial x^2}\right)N_{\mathrm{xx}}^{nl}=(EA)_s\epsilon_{\mathrm{xx}}^{0}+2\tau_0(b+h) \tag{3.8}$$

$$M_{\mathrm{xx}}^{nl}-\mu\frac{\partial^2 M_{\mathrm{xx}}^{nl}}{\partial x^2}=-(EI)_s\left[\frac{\partial^2 w}{\partial x^2}\right]. \tag{3.9}$$

The variational form of the nonlinear equations of motion which include the nonlocal and surface energy effects for EBT can be written as:

$$\int_{t1}^{t2}\int_0^L\left(m_0\frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}+M_{\mathrm{xx}}^{nl}\frac{\partial^2\delta w}{\partial x^2}-N_{\mathrm{xx}}^{nl}\delta\varepsilon_{xx}^{0}\right)dxdt=0 \tag{3.10}$$

where t_1 and t_2 are two arbitrary times , $m_0=\int_{\frac{-h}{2}}^{\frac{h}{2}}\rho b dz$.

The in-plane equation of motion is given by

$$\frac{\partial N_{\mathrm{xx}}^{nl}}{\partial x}=0. \tag{3.11}$$

Using equations (3.8) and (3.11), the expression of local in-plane resultant force N_{xx}^{l} and its partial differentiation w.r.t. x can be written in terms of displacement components (u,w) as

$$N_{\mathrm{xx}}=(EA)_s[\frac{\partial u}{\partial x}+\frac{1}{2}(\frac{\partial w}{\partial x})^2]+2\tau_0(b+h) \tag{3.12}$$

$$\frac{d(EA)_s}{dx}\frac{\partial u}{\partial x}+(EA)_s\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\frac{d(EA)_s}{dx}(\frac{\partial w}{\partial x})^2+(EA)_s\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+2(b+h)\frac{\partial \tau_0}{\partial x}+2\tau_0\frac{d(b+h)}{dx}=0. \tag{3.13}$$

Similarly, the nonlinear transverse equations of motion based on EBT from equation

$$\frac{\partial^2 M_{\mathrm{xx}}^{nl}}{\partial x^2}+\frac{\partial}{\partial x}(N_{\mathrm{xx}}^{nl}\frac{\partial w}{\partial x})-m_0\frac{\partial^2 w}{\partial t^2}=0. \tag{3.14}$$

Using equation (3.9), the final equation can be written as

$$\frac{\partial^2}{\partial x^2}\left[(EI)_s\frac{\partial^2 w}{\partial x^2}\right]-\left(1-\mu\frac{\partial^2}{\partial x^2}\right)\left(N_{\mathrm{xx}}^{nl}\frac{\partial^2 w}{\partial x^2}\right)+\left(1-\mu\frac{\partial^2}{\partial x^2}\right)\left(m_0\frac{\partial^2 w}{\partial t^2}\right)=0 \tag{3.15}$$

The above equation include geometric nonlinearity and the nonlocal (small scale) in the axial tension and inertial terms. It also includes the surface effects. However, we rewrite this equation to include nonlocal effect in other important terms also to do higher order analysis.

3.2 Modified governing equation

In this section, we are modifying the nonlinear transverse equation of motion based on Euler-Bernoulli beam theory (EBT) which is given by eqn. (3.15). In the previous governing equation, the only two terms, axial and inertial, are shown to be under the influence of nonlocal effect. Here, we are modifying the equation with a nonlocal damping as well as bending term by pre-multiplying the correponding terms with $(1 - \mu\frac{\partial^2}{\partial x^2})$. The final equation can be written as

$$\left(1-\mu\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2}{\partial x^2}\left[(EI)_s\frac{\partial^2 w}{\partial x^2}\right]-\left(1-\mu\frac{\partial^2}{\partial x^2}\right)N^{nl}_{\mathrm{xx}}\frac{\partial^2 w}{\partial x^2}+\left(1-\mu\frac{\partial^2}{\partial x^2}\right)C\frac{\partial w}{\partial t}+\left(1-\mu\frac{\partial^2}{\partial x^2}\right)\rho A\frac{\partial^2 w}{\partial t^2}=0 \tag{3.16}$$

where, $(EI)_s$ is the surface induced flexural rigidity and N^{nl}_{xx} is the resultant force which can be expressed as

$$N^{nl}_{\mathrm{xx}} = N_o + \frac{1}{2}(\frac{\partial w}{\partial x})^2(EA)_s + 2\tau_0(b+h) \tag{3.17}$$

and N_0 is the initial force.

Upon substitution of eqn. (3.17) into eqn. (3.16) and further simplification, we will get the modified equation as

$$(EI)_s\frac{\partial^4 w}{\partial x^4} - \mu(EI)_s\frac{\partial^6 w}{\partial x^6} + \mu N_o\frac{\partial^4 w}{\partial x^4} + \frac{1}{2}\mu(EA)_s(\frac{\partial w}{\partial x})^2\frac{\partial^4 w}{\partial x^4} + 2\mu\tau_0(b+h)\frac{\partial^4 w}{\partial x^4} - N_o\frac{\partial^2 w}{\partial x^2} - \frac{1}{2}(EA)_s(\frac{\partial w}{\partial x})^2\frac{\partial^2 w}{\partial x^2} - 2\tau_0(b+h)\frac{\partial^2 w}{\partial x^2} + C\frac{\partial w}{\partial t} - \mu C\frac{\partial^3 w}{\partial x^2\partial t} + \rho A\frac{\partial^2 w}{\partial t^2} - \mu\rho A\frac{\partial^4 w}{\partial x^2\partial t^2} = 0. \tag{3.18}$$

3.2.1 Nondimensionalization

To apply the perturbation technique, we first non-dimensionalize the equation with the following non-dimensional parameters, $\bar{x} = \frac{x}{L},\ \bar{w} = \frac{w}{L},\ \bar{t} = t\sqrt{\frac{(EI)_s}{\rho AL^4}},\ C = \zeta\bar{t}$.

The final form of equations can be written as

$$Q_1\frac{\partial^4\bar{w}}{\partial\bar{x}^4} - F\frac{\partial^6\bar{w}}{\partial\bar{x}^6} + Q_3(\frac{\partial\bar{w}}{\partial\bar{x}})^2\frac{\partial^4\bar{w}}{\partial\bar{x}^4} - Q_2\frac{\partial^2\bar{w}}{\partial\bar{x}^2} - Q_4(\frac{\partial\bar{w}}{\partial\bar{x}})^2\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + \zeta\frac{\partial\bar{w}}{\partial\bar{t}} - F\zeta\frac{\partial^3\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2} + \frac{\partial^2\bar{w}}{\partial\bar{t}^2} - F\frac{\partial^4\bar{w}}{\partial\bar{x}^2\partial\bar{t}^2} = 0 \tag{3.19}$$

where, $Q_1 = 1 + \frac{\mu N_o}{(EI)_s} + \frac{2\mu\tau_0(b+h)}{(EI)_s}$, $Q_2 = \frac{L^2 N_o}{(EI)_s} + \frac{2\tau_0 L^2(b+h)}{(EI)_s}$,$F = \frac{\mu}{L^2}$, $Q_3 = \frac{\mu(EA)_s}{2(EI)_s}$ and $Q_4 = \frac{(EA)_s}{2(EI)_s}$.

To solve the equation, we assume the solution based on the single mode approximation and also assume the form of the transverse displacement as $\bar{w}(x,t) = q(t).\phi(x)$. Substituting $\bar{w}(x,t)$ into the governing equation and then applying Galerkin method, the equation reduces to the following form,

$$Q_1\phi\phi_{xxxx}q - F\phi\phi_{xxxxxx}q + Q_3\phi(\phi_x)^2q^3\phi_{xxxx} - Q_2\phi q\phi_{xx} - Q_4\phi q^3\phi_{xx}(\phi_x)^2 + \zeta\dot{q}\phi^2 - F\zeta\phi\dot{q}\phi_{xx} + \ddot{q}\phi^2 - F\ddot{q}\phi_{xx}\phi = 0. \tag{3.20}$$

Integrating the above equation from 0 to 1 and then simplifying it, we get

$$\ddot{q} + B_1\dot{q} + B_2q^3 + B_3q = 0 \tag{3.21}$$

where, B_1, B_2 and B_3 are the coefficients depending upon the mode shapes and end conditions of nanobeams. Expresions of all these quantities are mentioned in the later section for different beams.

3.2.2 Solution methodology using the method of multiple scales

We apply the method of multiple scales (MMS), one of the perturbation method, to solve the ordinary differential equation (3.21). To obtain the weak nonlinear effect, terms associated with damping and nonlinear stiffness are rescaled with ϵ. The rescaled form of the governing equation can be written as

$$\ddot{q} + \epsilon B_1\dot{q} + \epsilon B_2q^3 + B_3q = 0. \tag{3.22}$$

Assuming the form of the solution as

$$q(T_0, T_1) = q_0(T_0, T_1) + \epsilon q_1(T_0, T_1) \tag{3.23}$$

where, $T_0 = t$, the fast time scale and $T_1 = \epsilon t$, the slow time scale, $\dot{q} = (D_0 + \epsilon D_1)q$ and $\ddot{q} = (D_0^2 + 2\epsilon D_0 D_1 + \bigcirc(\epsilon^2))q$, $\frac{\partial}{\partial T_i} = D_i$ and subsequently substituting it in the governing equation, we write the following equations by comparing the coefficients of the same powers of ϵ

$$\bigcirc(\epsilon^0) : D_0^2q_0 + B_3q_0 = 0, \tag{3.24}$$

$$\bigcirc(\epsilon^1) : D_0^2q_1 + B_3q_1 = -2D_0D_1q_0 - B_1D_0q_0 - B_2q_0^3, . \tag{3.25}$$

The eqn. (3.24) is a standard harmonic equation having a solution of the form

$$q_0 = a_0(\cos(T_0 + \beta_0)) \tag{3.26}$$

where, $a_0 = a_0(T_1)$ and $\beta_0 = \beta_0(T_1)$. After substituting q_0 into eqn. (3.25), and then expanding some of the terms, we get

$$D_0^2q_1 + B_3q_1 = 2\sin(T_0 + \beta_0)\frac{da_0}{dT_1} + 2a_0\cos(T_0 + \beta_0)\frac{d\beta_0}{dT_1} - \frac{3}{4}a_0^3B_2\cos(T_0 + \beta_0) - \frac{1}{4}B_2a_0^3\cos(3T_0 + 3\beta_0) + B_1a_0\sin(T_0 + \beta_0). \tag{3.27}$$

Upon collecting the coefficients of $\cos(T_0 + \beta_0)$ and $\sin(T_0 + \beta_0)$, we obtain following the equations, Coefficients of $\cos(T_0 + \beta_0)$:

$$2a_0\frac{d\beta_0}{dT_1} - \frac{3}{4}a_0^3B_2 = 0 \tag{3.28}$$

& Coefficients of $\sin(T_0 + \beta_0)$:

$$2\frac{da_0}{dT_1} + a_0B_1 = 0 \tag{3.29}$$

From the above two equations, we can get the expression for β_0 and $\frac{da_0}{dT_1}$ as follow

$$\beta_0(T_1) = \frac{3}{8}B_2a_0^2T_1 + \beta_0 \tag{3.30}$$

$$\frac{da_0}{dT_1} = -\frac{B_1a_0}{2}. \tag{3.31}$$

On solving these two equations simultaneously using Runge-Kutte 4^{th} order method, we can get the values of a_0 and B_2.

On eliminating the secular terms, we can write the expression for $D_0^2q_1 + B_3q_1$ as

$$D_0^2q_1 + B_3q_1 = -\frac{B_2}{4}a_0^3(cos(3T_0 + 3\beta_0 + \frac{9}{8}B_2a_0^2T_1)). \tag{3.32}$$

Assuming the solution for q_1 as $q_1 = C1\cos(3T_0 + 3\beta_0)$ and after substituting it in the above equation, we get the value of $C1$ as $C1 = -\frac{B_2a_0^3}{4(B_3-9)}$. The expression of q_1 becomes

$$q_1 = -\frac{B_2a_0^3}{4(B_3 - 9)}\cos(3(1 + \frac{3}{8}B_2a_0^2\epsilon)t + 3\beta_0). \tag{3.33}$$

Substituting the expression for q_1 and q_0 in eqn. (3.23), we get the final form of solution as,

$$q = a_0\cos((1 + \frac{3}{8}B_2a_0^2\epsilon)t + \beta_0) + \epsilon\frac{B_2a_0^3}{4(9 - B_3)}\cos(3(1 + \frac{3}{8}B_2a_0^2\epsilon)t + 3\beta_0). \tag{3.34}$$

Comparing eqn. (3.34) with the stanadard form of solution

$$q = a_0\cos(\tilde{\omega}t + \beta_0) + \epsilon\frac{B_2a_0^3}{4(9 - B_3)}\cos(3\tilde{\omega}t + 3\beta_0), \tag{3.35}$$

the modified expression for frequency can be obtained as

$$\tilde{\omega} = 1 + \frac{3}{8}B_2a_0^2\epsilon. \tag{3.36}$$

Here, a_0 is the function of B_1, B_2 is a function of nonlocal and surface parameters, thus, the modified frequency $\tilde{\omega}$ is a function damping, nonlocal, surface as well as nonlinear stiffness parameters. In the following section, we apply this solution to different beams. The material properties and geometrical parameters taken for this analysis are given in Table 3.1.

Table 3.1: Parameters

Parameters	Value
h	$100nm$
b	$300nm$
E	$70GPa$
E_s	$5.1882N/m$
τ_0	$0.9108N/m$
ρ	$2400kg/m^3$
N_0	$2.1\mu N$

3.3 Results and Discussion

3.3.1 Simply Supported Beam

Mode-1

For a simply supported beam, in order to find out the first mode frequency, we proceed as follow. Substituting the mode shape equation $\phi(x) = \sin(\pi x)$ in eqn. (3.20), we get the following form of the equation.

$$\ddot{q}[\frac{1}{2} + \frac{\pi^2 F}{2}] + \dot{q}[\zeta(\frac{1}{2} + \frac{\pi^2 F}{2})] + q^3[\frac{\pi^6 Q_3}{8} + \frac{\pi^4 Q_4}{8}] + q[\frac{Q_1\pi^4}{2} + \frac{F\pi^6}{2} + \frac{Q_2\pi^2}{2} - \frac{F\pi^4}{2}] = 0. \quad (3.37)$$

On comparing eqn. (3.37) with eqn. (3.21), we get the following expressions for B_1,B_2 and B_3

$$B_1 = \zeta, \quad (3.38)$$

$$B_2 = \frac{\pi^4(Q_4 + \pi^2 Q_3)}{4(1 + \pi^2 F)}, \quad (3.39)$$

$$B_3 = \frac{Q_1\pi^4 + F\pi^6 + Q_2\pi^2 - F\pi^4}{1 + \pi^2 F}. \quad (3.40)$$

To describe the dependence of damping, ζ and nonlocal parameters μ, we show the frequency variation with the different values ζ and μ in the Fig. 3.2.

Figures 3.2(a) and (b) show the variation of frequencies with non-zero combination of ζ and μ. Figure 3.2(a) shows that the frequency reduces as μ increases. Later it becomes independent of nonlocal parameter. Figure 3.2(b) also shows the variation of frequency with damping. It shows that it reduces with damping and then become zero. Similar kind of variations are discussed for other modes as well in the following sections.

Mode-3

To find the third mode frequency, we take the mode shape as $\phi(x) = \sin(3\pi x)$ and then substitute it in eqn. (3.20) to find the following frequency equation.

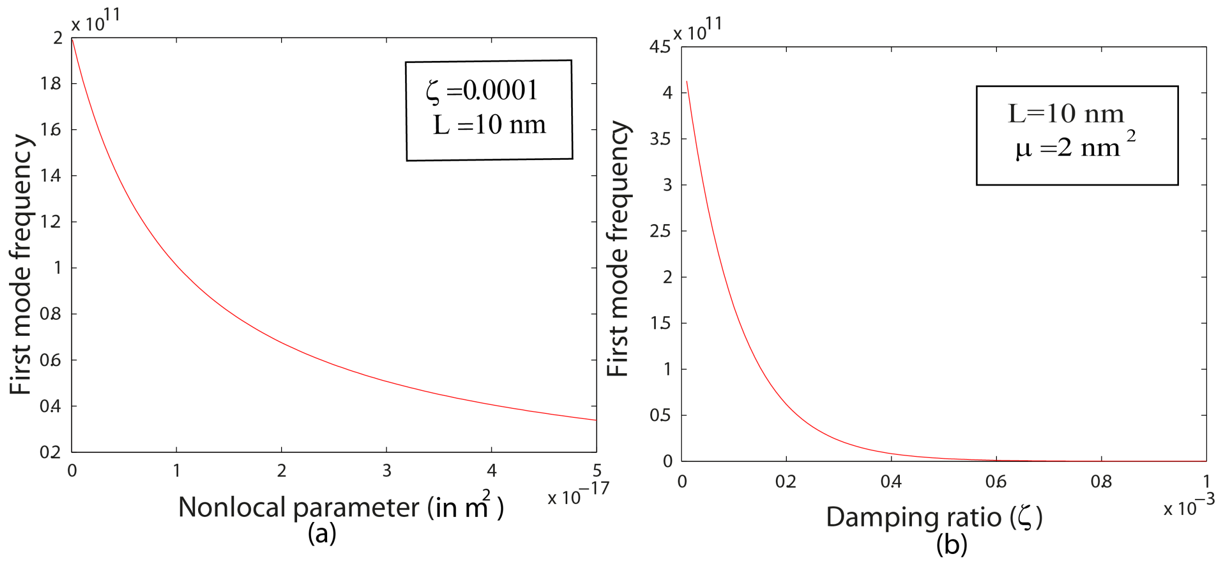

Figure 3.2: Plots showing the variation of first mode frequency of simply supported beam against (a) the nonlocal parameter (μ) for constant damping ratio (ζ) and length of beam (b) damping ratio (ζ) for a constant length and μ value.

$$\ddot{q}[\frac{1}{2} + \frac{9\pi^2 F}{2}] + \dot{q}[\zeta(\frac{1}{2} + \frac{9\pi^2 F}{2})] + q^3[\frac{729\pi^6 Q_3}{8} + \frac{81\pi^4 Q_4}{8}] \tag{3.41}$$

$$+q[\frac{81 Q_1 \pi^4}{2} + \frac{729 F \pi^6}{2} + \frac{9 Q_2 \pi^2}{2} - \frac{81 F \pi^4}{2}] = 0 \tag{3.42}$$

Comparing eqn. (3.42) with eqn. (3.21), we get the expressions for B_1,B_2 and B_3

$$B_1 = \zeta, \tag{3.43}$$

$$B_2 = \frac{\pi^4(81Q_4 + 729\pi^2 Q_3)}{4(1 + 9\pi^2 F)}, \tag{3.44}$$

$$B_3 = \frac{81Q_1\pi^4 + 729F\pi^6 + 9Q_2\pi^2 - 81F\pi^4}{1 + 9\pi^2 F}. \tag{3.45}$$

Figure 3.3 show the similar variation of frequencies with non-zero combination of ζ and μ.

Mode-5

The fifth mode frequency of a simply supported beam can be analyzed as follows. Substituting the mode shape equation $\phi(x) = \sin(5\pi x)$ in eqn. (3.20) we get the ordinary differential equation as given below.

$$\ddot{q}[\frac{1}{2} + \frac{25\pi^2 F}{2}] + \dot{q}[\zeta(\frac{1}{2} + \frac{25\pi^2 F}{2})] + q^3[\frac{15625\pi^6 Q_3}{8} + \frac{625\pi^4 Q_4}{8}] \tag{3.46}$$

$$+q[\frac{625 Q_1 \pi^4}{2} + \frac{15625 F \pi^6}{2} + \frac{25 Q_2 \pi^2}{2} - \frac{625 F \pi^4}{2}] = 0 \tag{3.47}$$

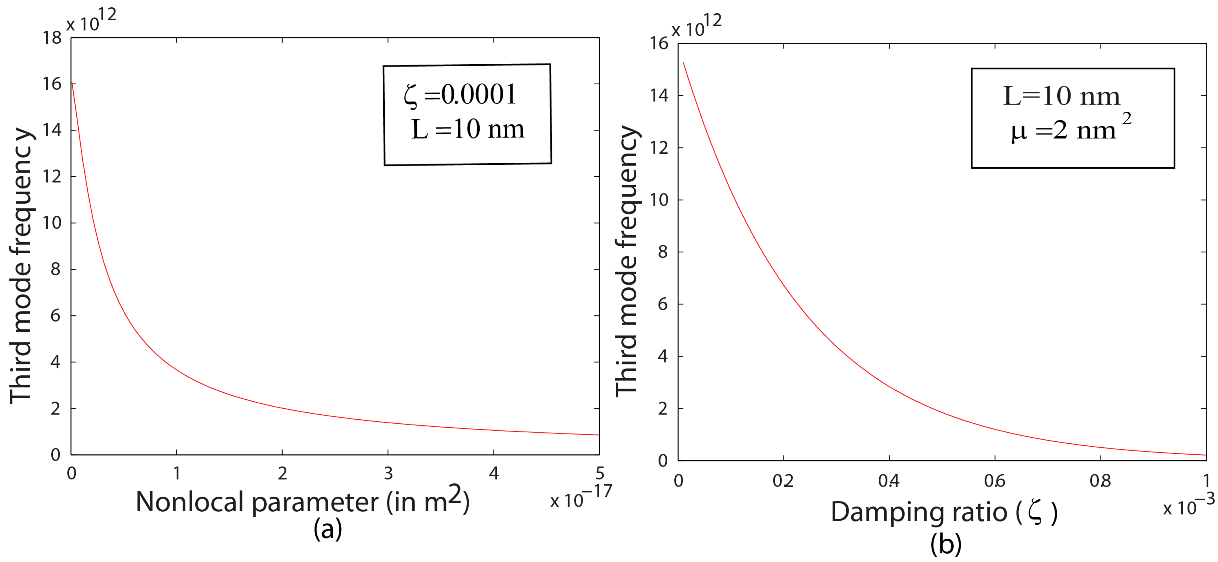

Figure 3.3: Plots showing the variation of third mode frequency of simply supported beam against (a) the nonlocal parameter (μ) for constant damping ratio (ζ) and length of beam (b) damping ratio (ζ) for a constant length and μ value.

Comparing eqn. (3.47) with eqn. (3.21) we obtain the expressions for B_1,B_2 and B_3

$$B_1 = \zeta, \tag{3.48}$$

$$B_2 = \frac{\pi^4(625Q_4 + 15625\pi^2 Q_3)}{4(1+25\pi^2 F)}, \tag{3.49}$$

$$B_3 = \frac{625Q_1\pi^4 + 15625F\pi^6 + 25Q_2\pi^2 - 625F\pi^4}{1+25\pi^2 F}. \tag{3.50}$$

The variation in fifth mode frequency with the non-zero combination of μ and ζ are shown in fig. 3.4

3.3.2 Fixed-Fixed Beam

Mode-1

For a fixed-fixed beam to get the first mode frequency we substitute the mode shape equation $\phi(x) = (\cosh a_1 x - \cos a_1 x) - \sigma_1(\sinh a_1 x - \sin a_1 x)$ in eqn. (3.20) where $\sigma_1 = .9825$ and $a_1 = 4.73$ for first mode. Upon substitution we arrive at the following differential equation.

$$\ddot{q}[12.3018F + 1.0035] + \dot{q}[\zeta(1.0035 + 12.3018F)] + q^3[4707.33Q_3 + 74.126Q_4] \tag{3.51}$$

$$+q[5657.08F + 12.3018Q_2 + 500.564Q_1] = 0 \tag{3.52}$$

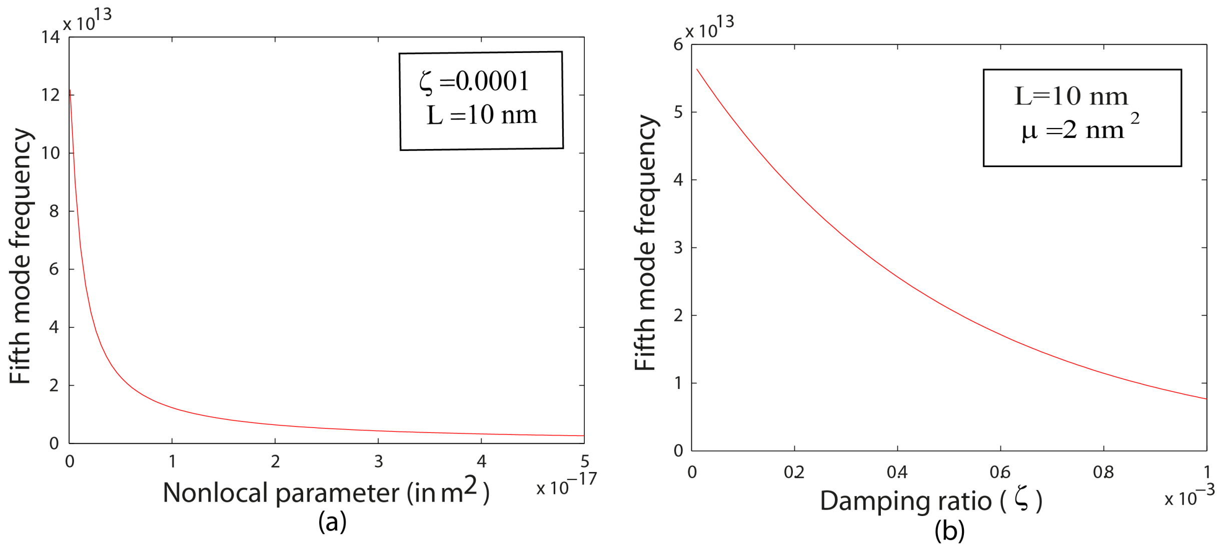

Figure 3.4: Plots showing the variation of fifth mode frequency of simply supported beam against (a) the nonlocal parameter (μ) for constant damping ratio (ζ) and length of beam (b) damping ratio (ζ) for a constant length and μ value.

Comparing eqn. (3.52) with eqn. (3.21)we get the expressions for B_1,B_2 and B_3

$$B_1 = \zeta, \tag{3.53}$$

$$B_2 = \frac{4707.33Q_3 + 74.126Q_4}{12.3018F + 1.0035}, \tag{3.54}$$

$$B_3 = \frac{5657.08F + 12.3018Q_2 + 500.564Q_1}{12.3018F + 1.0035}. \tag{3.55}$$

To describe the dependence of damping, ζ and nonlocal parameters μ, we show the frequency variation with the different values ζ and μ in the Fig. 3.5. Figures 3.5 (a) and (b) show the variation of frequencies with non-zero combination of ζ and μ. Figure 3.5(a) shows that the frequency reduces as μ increases. Later it becomes independent of nonlocal parameter. Figure 3.5 (b) also shows the variation of frequency with damping. It shows that it reduces with damping and then become zero. Similar kind of variations are discussed for other modes as well in the following sections.

Mode2

To get the second mode frequency we substitute the mode shape equation $\phi(x) = (\cosh a_2 x - \cos a_2 x) - \sigma_2(\sinh a_2 x - \sin a_2 x)$ in eqn. (3.20) where $\sigma_2 = 1.000777$ and $a_2 = 7.85$ for second mode. Upon substitution we get the frequency equation.

$$\ddot{q}[46.034F + 1.0035] + \dot{q}[\zeta(1.0035 + 46.034F)] + q^3[1.08 \times 10^5 Q_3 + 1181.0871Q_4] \\ + q[1.71 \times 10^5 F + 46.034Q_2 + 3798.68Q_1] = 0 \tag{3.56}$$

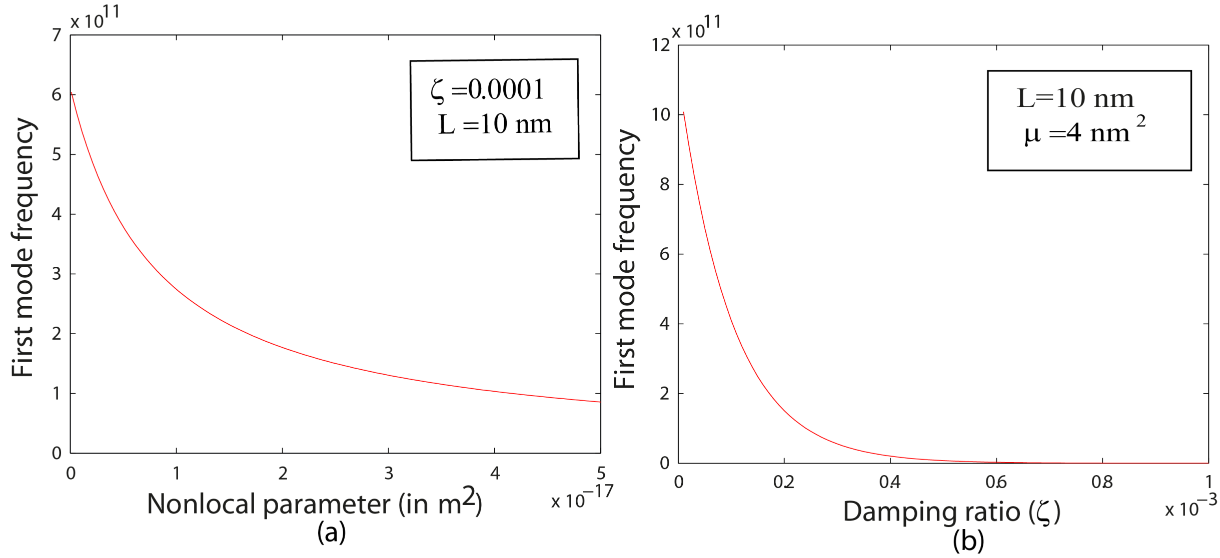

Figure 3.5: Plots showing the variation of first mode frequency of fixed-fixed beam against (a) the nonlocal parameter (μ) for constant damping ratio (ζ) and length of beam (b) damping ratio (ζ) for a constant length and μ value.

Comparing eqn. (3.56) with eqn. (3.21)we will get the expressions for B_1,B_2 and B_3

$$B_1 = \zeta, \tag{3.57}$$
$$B_2 = \frac{1.08 \times 10^5 Q_3 + 1181.0871 Q_4}{46.034F + 1.0035}, \tag{3.58}$$
$$B_3 = \frac{1.71 \times 10^5 F + 46.034 Q_2 + 3798.68 Q_1}{46.034F + 1.0035}. \tag{3.59}$$

Figure 3.6 show the similar kind of variation for non-zero combination of μ and ζ.

Mode-4

To obtain the fourth mode frequency response of fixed-fixed beam we substitute the mode shape equation $\phi(x) = (\cosh a_4 x - \cos a_4 x) - \sigma_4(\sinh a_4 x - \sin a_4 x)$ in eqn. (3.20) where $\sigma_4 = 1$ and $a_4 = 14.1372$ for fourth mode. Upon substitution we get the equation as given

$$\ddot{q}[178.65F + .96463] + \dot{q}[\zeta(178.65F + .96463)] + q^3[3.90694 \times 10^6 Q_3 + 17853.042 Q_4] \tag{3.60}$$
$$+q[7.097 \times 10^6 F + 178.65 Q_2 + 38531.45 Q_1] = 0$$

Comparing eqn. (3.60) with eqn. (3.21)we get the expressions for B_1,B_2 and B_3

$$B_1 = \zeta, \tag{3.61}$$
$$B_2 = \frac{3.90694 \times 10^6 Q_3 + 17853.042 Q_4}{178.65F + .96463}, \tag{3.62}$$
$$B_3 = \frac{7.097 \times 10^6 F + 178.65 Q_2 + 38531.45 Q_1}{178.65F + .96463}. \tag{3.63}$$

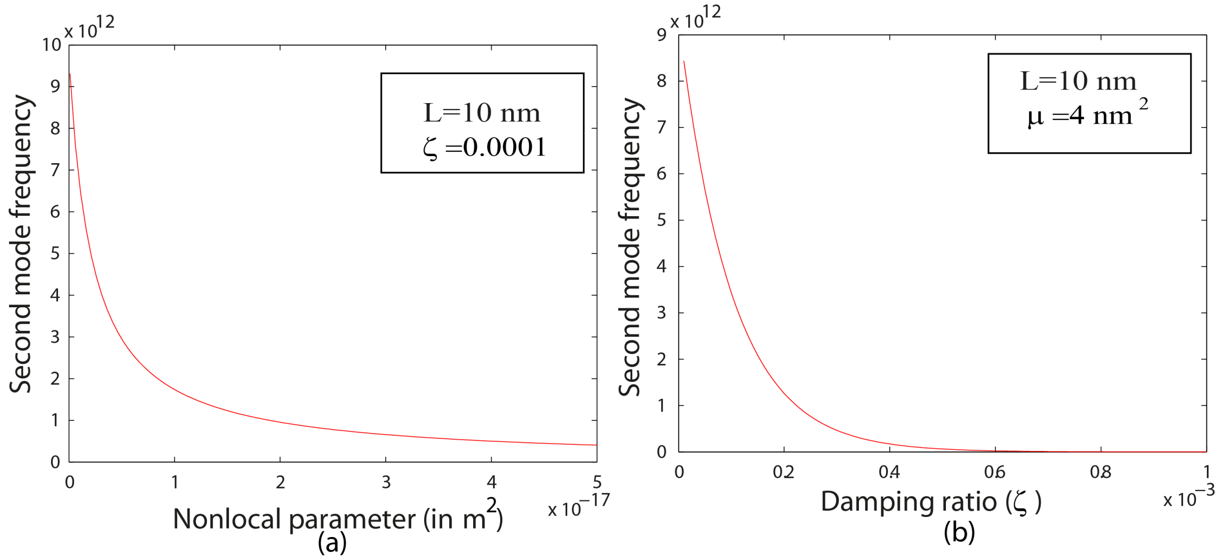

Figure 3.6: Plots showing the variation of second mode frequency of fixed-fixed beam against (a) the nonlocal parameter (μ) for constant damping ratio (ζ) and length of beam (b) damping ratio (ζ) for a constant length and μ value.

The similar variation in frequency for non-zero combination of μ and ζ are shown in fig. 3.7

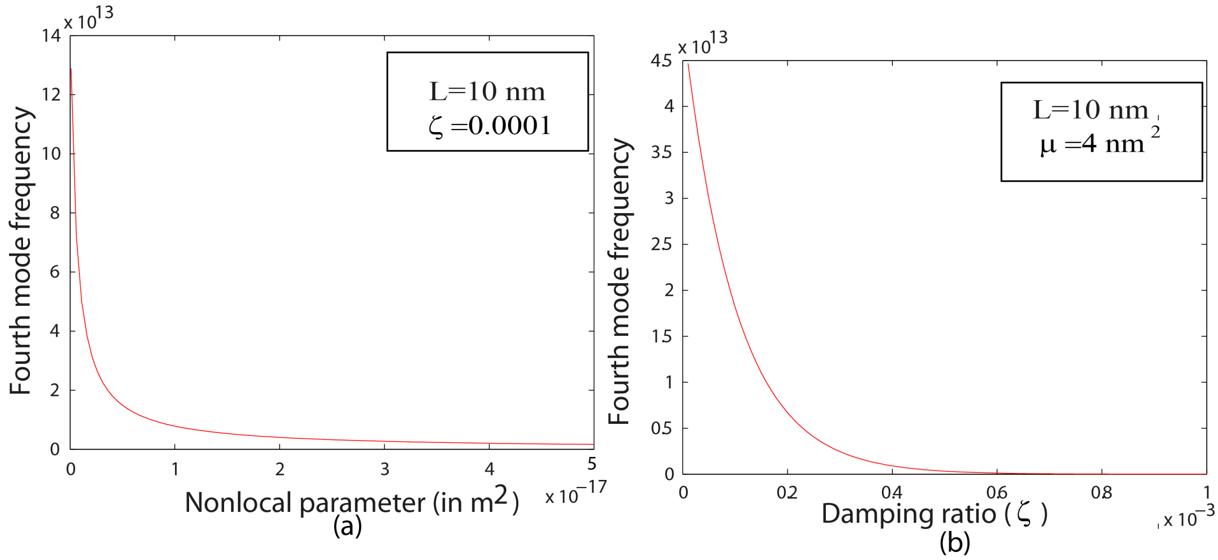

Figure 3.7: Plots showing the variation of fourth mode frequency of fixed-fixed beam against (a) the nonlocal parameter (μ) for constant damping ratio (ζ) and length of beam (b) damping ratio (ζ) for a constant length and μ value.

3.3.3 Cantilever Beam

Mode-3

For a cantilever beam in order to find out the third mode frequency we substitute the mode shape equation $\phi(x) = (\cosh a_3 x - \cos a_3 x) - \sigma_3(\sinh a_3 x - \sin a_3 x)$ in eqn. (3.20) where $\sigma_3 = .999225$ and $a_3 = 7.8548$ for third mode. Subsequently we obtain the ordinary differential equation as given below.

$$\ddot{q}[45.91F + .999] + \dot{q}[\zeta(45.91F + .999)] + q^3[2.515 \times 10^5 Q_3 + 1170.35Q_4] \quad (3.64)$$
$$+q[1.71 \times 10^5 F + 45.91Q_2 + 3806.376Q_1] = 0$$

Comparing eqn. (3.64) with eqn. (3.21) we obtain the expressions for B_1,B_2 and B_3.

$$B_1 = \zeta, \quad (3.65)$$
$$B_2 = \frac{2.515 \times 10^5 Q_3 + 1170.35Q_4}{45.91F + .999}, \quad (3.66)$$
$$B_3 = \frac{1.71 \times 10^5 F + 45.91Q_2 + 3806.376Q_1}{45.91F + .999}. \quad (3.67)$$

To describe the dependence of damping, ζ and nonlocal parameters μ, we show the frequency variation with the different values ζ and μ in the Fig. 3.8.

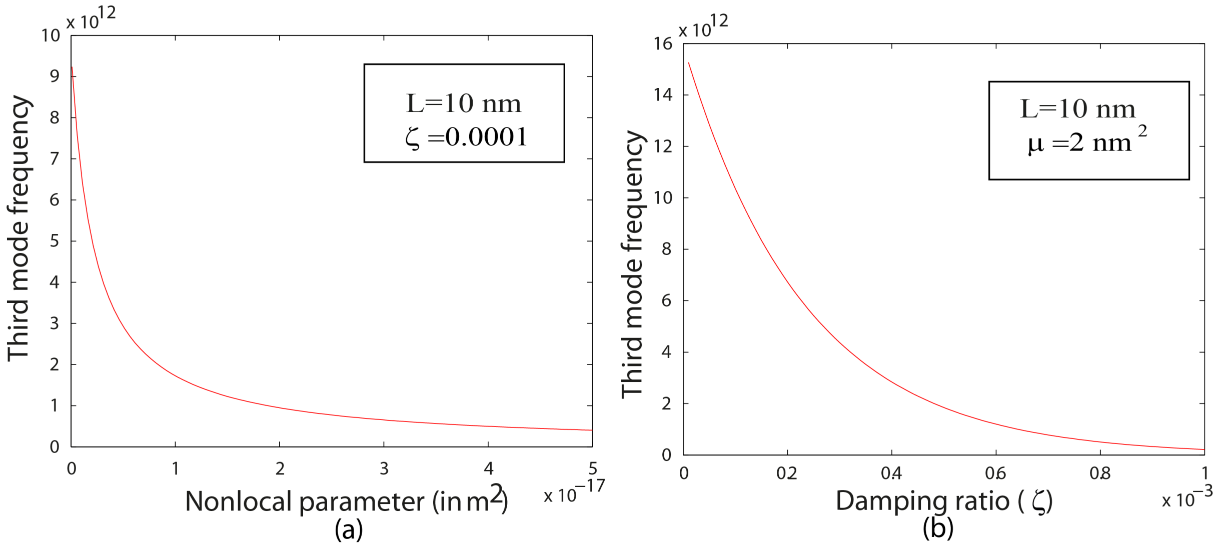

Figure 3.8: Plots showing the variation of third mode frequency of cantilever beam against (a) the nonlocal parameter (μ) for constant damping ratio (ζ) and length of beam (b) damping ratio (ζ) for a constant length and μ value.

Figures 3.8 (a) and (b) show the variation of frequencies with non-zero combination of ζ and μ. Figure 3.8(a) shows that the frequency reduces as μ increases. Later it becomes independent of nonlocal parameter. Figure 3.8 (b) also shows the variation of frequency with damping. It shows that it reduces with damping and then become zero. Similar kind of variations are discussed for other modes as well in the following sections.

Mode-4

To find the fourth mode frequency we take the mode shape equation $\phi(x) = (\cosh a_4 x - \cos a_4 x) - \sigma_4(\sinh a_4 x - \sin a_4 x)$ and then substitute in eqn. (3.20). The value of $\sigma_4 = 1.000033$ and $a_4 = 10.9586$ for fourth mode. Upon substitution we get the frequency equation.

$$\ddot{q}[98.75F + .9891] + \dot{q}[\zeta(98.75F + .9891)] + q^3[1.46 \times 10^6 Q_3 + 5333.18Q_4] \\ +q[1.41 \times 10^6 F + 98.75Q_2 + 14264.62Q_1] = 0 \tag{3.68}$$

Comparing eqn. (3.68) with eqn. (3.21)we obtain the expressions for B_1,B_2 and B_3

$$B_1 = \zeta, \tag{3.69}$$

$$B_2 = \frac{1.46 \times 10^6 Q_3 + 5333.18Q_4}{98.75F + .9891}, \tag{3.70}$$

$$B_3 = \frac{1.41 \times 10^6 F + 98.75Q_2 + 14264.62Q_1}{98.75F + .9891}. \tag{3.71}$$

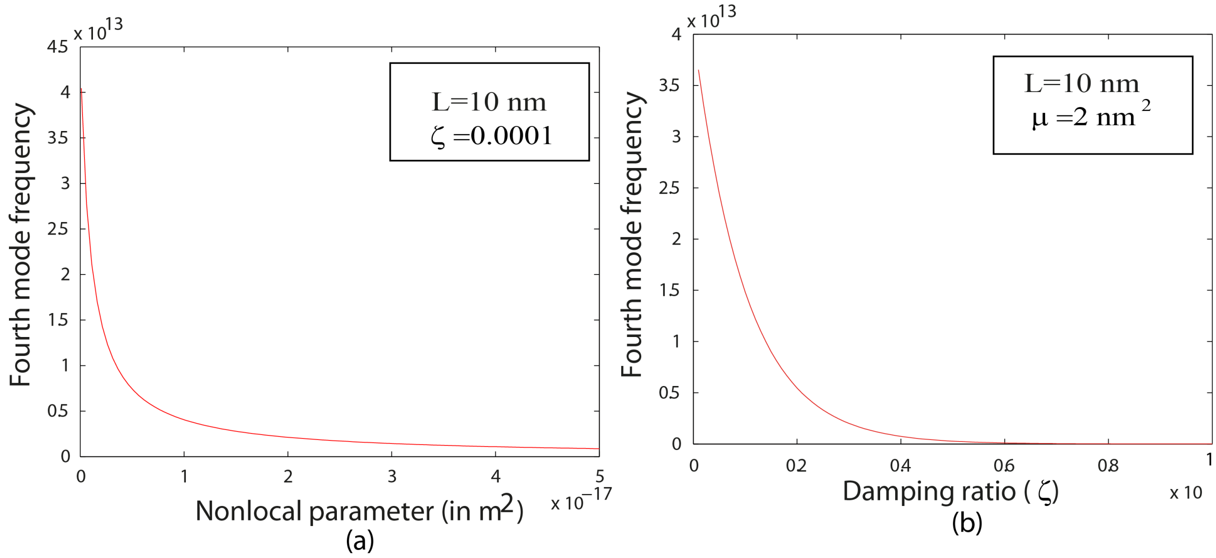

Figure 3.9: Plots showing the variation of fourth mode frequency of cantilever beam against (a) the nonlocal parameter (μ) for constant damping ratio (ζ) and length of beam (b) damping ratio (ζ) for a constant length and μ value.

Figure 3.9 show similar variation of frequencies for non-zero combination of ζ and μ.

Mode-5

The fifth mode frequency of a cantilever beam can be analyzed as follows. We take the mode shape equation $\phi(x) = (\cosh a_5 x - \cos a_5 x) - \sigma_5(\sinh a_5 x - \sin a_5 x)$ and then putting it in eqn. (3.20), where $\sigma_5 = 1$ and $a_5 = 14.1372$ for fifth mode, to get the frequency equation.

$$\ddot{q}[178.65F + .96463] + \dot{q}[\zeta(178.65F + .96463)] + q^3[3.906 \times 10^6 Q_3 + 17853.041 Q_4] \\ +q[7.01 \times 10^6 F + 178.45 Q_2 + 38531.45 Q_1] = 0 \tag{3.72}$$

Comparing eqn. (3.72) with eqn. (3.21)we get the expressions for B_1,B_2 and B_3

$$B_1 = \zeta, \tag{3.73}$$

$$B_2 = \frac{3.906 \times 10^6 Q_3 + 17853.041 Q_4}{178.65F + .96463}, \tag{3.74}$$

$$B_3 = \frac{7.01 \times 10^6 F + 178.45 Q_2 + 38531.45 Q_1}{178.65F + .96463}. \tag{3.75}$$

The variation in frequency with non-zero combination of μ and ζ are shown in fig. 3.10

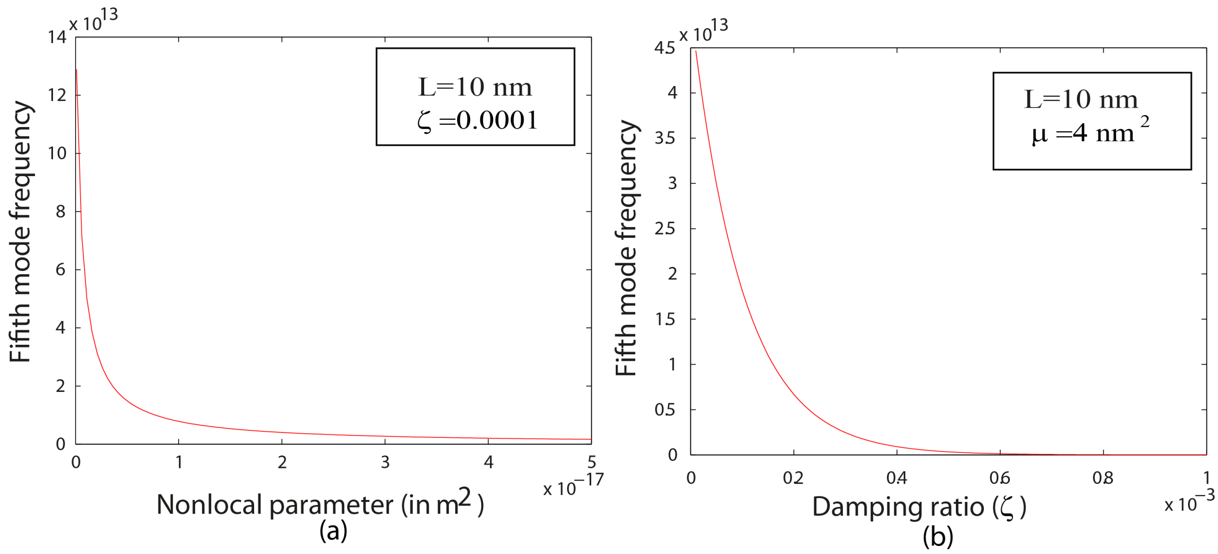

Figure 3.10: Plots showing the variation of fifth mode frequency of cantilever beam against (a) the nonlocal parameter (μ) for constant damping ratio (ζ) and length of beam (b) damping ratio (ζ) for a constant length and μ value.

3.3.4 Summary

In this chapter, we obtained modified governing equation by including the nonlocal parameters in inertial, stiffness, axial as well as damping terms. Subsequently, we obtain the frequency equation for different modes of the simply supported, fixed-fixed, and cantilever beams. Finally, we obtained the variation of different modal frequencies with non-local parameter and damping. In all the modes, we found that the frequency reduces as the nonlocal as well damping increases. We also found that the frequency become independent of nonlocal parameter after certain cut-off value.

Chapter 4

Linear finite element analysis of nonlocal and surface effects

In this chapter, we apply finite element method to obtain the modal frequencies of Euler-Bernoulli nanobeam with nonlocal and surface effects. Under this formulation, the geometric non-linearity as well as damping are neglected for the sake of simplicity. The nonlocal as well as surface effects are modeled in a similar ways as described in the previous chapters.

4.1 Displacement, Strains, Stress and Moments

The displacements $\overline{u}$ (in the x-direction) and $\overline{w}$ (in the z-direction) can be approximated as,

$$\overline{u}(x,z,t) = u(x,t) - z\frac{\partial w}{\partial x}, \qquad \overline{w}(x,z,t) = w(x,t) \tag{4.1}$$

where, u and w are the axial and transverse displacement at the mid plane of the beam. As per Euler-Bernoulli beam theory the nonzero strains can be written as

$$\epsilon_{\mathrm{xx}} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}, \tag{4.2}$$

The stress resultant and moments can also be defined as,

$$N_{\mathrm{xx}} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{\mathrm{xx}} b dz + \oint \sigma_s ds = (EA)_s \frac{\partial u}{\partial x} + 2\tau_0(b+h), \tag{4.3}$$

$$M_{\mathrm{xx}} = \int_{\frac{-h}{2}}^{\frac{h}{2}} z\sigma_{\mathrm{xx}} b dz + \oint z\sigma_s ds = -(EI)_s \left[\frac{\partial^2 w}{\partial x^2}\right] \tag{4.4}$$

where, $(EA)_s$ and $(EI)_s$ are the effective in-plane and flexural rigidities, respectively, which can be written as

$$(EA)_s = EA + 2E_s(b+h) \qquad \text{and} \qquad \mathrm{(EI)_s} = \mathrm{E}(\frac{\mathrm{bh^3}}{12}) + \mathrm{E_s}(\frac{\mathrm{h^3}}{6} + \frac{\mathrm{bh^2}}{2}). \tag{4.5}$$

In order to get the variational form of the equations, we are using the Hamilton's principle. It states that for a particular time interval, say t_1 and t_2, the time integral of the hamiltonian attains extremum value. Applying the variation, we can write the equation as

$$\int_{t_1}^{t_2} (\delta U - \delta T) dt = 0. \tag{4.6}$$

where, δU is the virtual strain energy and δT is the virtual kinetic energy.
The expressions for δU and δT can be written as

$$\delta U = b \int \int_{\frac{-h}{2}}^{\frac{h}{2}} \sigma_{\text{xx}} \delta \epsilon_{\text{xx}} dz dx = b \int \left(N_{\text{xx}} \delta \frac{\partial u}{\partial x} - M_{\text{xx}} \delta \frac{\partial^2 w}{\partial x^2} \right) dx \tag{4.7}$$

$$\begin{aligned} \delta T = b \int \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho \left[\left(\frac{\partial u}{\partial t} - z \frac{\partial^2 w}{\partial t \partial x} \right) \left(\frac{\partial \delta u}{\partial t} - z \frac{\partial^2 \delta w}{\partial t \partial x} \right) + \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} \right] \\ = b \int \left[I_0 \left(\frac{\partial u}{\partial t} \frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} \right) - I_1 \left(\frac{\partial u}{\partial t} \frac{\partial^2 \delta w}{\partial t \partial x} + \frac{\partial^2 w}{\partial t \partial x} \frac{\partial \delta u}{\partial t} \right) + I_2 \frac{\partial^2 w}{\partial t \partial x} \frac{\partial^2 \delta w}{\partial t \partial x} \right] \end{aligned} \tag{4.8}$$

By substituting eqn. (4.7) and eqn. (4.8) in eqn. (4.6) we obtain ,

$$\begin{aligned} b \int \int \left(N_{\text{xx}} \delta \frac{\partial u}{\partial x} - M_{\text{xx}} \delta \frac{\partial^2 w}{\partial x^2} \right) dx + b \int \int I_0 \left(\frac{\partial u}{\partial t} \frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} \right) dx - \\ b \int \int I_1 \left(\frac{\partial u}{\partial t} \frac{\partial^2 \delta w}{\partial t \partial x} + \frac{\partial^2 w}{\partial t \partial x} \frac{\partial \delta u}{\partial t} \right) dx + b \int \int I_2 \frac{\partial^2 w}{\partial t \partial x} \frac{\partial^2 \delta w}{\partial t \partial x} dx = 0 \end{aligned} \tag{4.9}$$

The Euler-Lagrange equations utilizing the above equations can be arrived as

$$\frac{\partial N_{\text{xx}}}{\partial x} = I_0 \frac{\partial^2 u}{\partial t^2} - I_1 \frac{\partial^3 w}{\partial x \partial t^2} \tag{4.10}$$

$$\frac{\partial^2 M_{\text{xx}}}{\partial x^2} = I_0 \frac{\partial^2 w}{\partial t^2} + I_1 \frac{\partial^3 u}{\partial x \partial t^2} - I_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} \tag{4.11}$$

Using Eringen's theory for nonlocal effect, we can write the following equation,

$$N_{\text{xx}} - \mu \frac{\partial^2 N_{\text{xx}}}{\partial x^2} = (EA)_s \frac{\partial u}{\partial x} + 2\tau_0 (b + h). \tag{4.12}$$

By utilizing eqns. (4.10) and (4.11), we finally arrive at the following expressions for N_{xx} and M_{xx}:

$$N_{\text{xx}} = \mu \left[I_0 \frac{\partial^3 u}{\partial x \partial t^2} - I_1 \frac{\partial^4 w}{\partial x^2 \partial t^2} \right] + (EA)_s \frac{\partial u}{\partial x} + 2\tau_0 (b + h) \tag{4.13}$$

$$M_{xx} = \mu \left[I_0 \frac{\partial^2 w}{\partial t^2} + I_1 \frac{\partial^3 u}{\partial x \partial t^2} - I_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} \right] - (EI)_s \left[\frac{\partial^2 w}{\partial x^2} \right] \tag{4.14}$$

where, $I_0 = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho dz$, $I_1 = \int_{\frac{-h}{2}}^{\frac{h}{2}} z \rho dz$ and $I_2 = \int_{\frac{-h}{2}}^{\frac{h}{2}} z^2 dz$

4.2 Finite Element Formulation

To obtain the finite element formulation of the beam, we substitute eqns. (4.13) and (4.14) into eqn. (4.7) and (4.8) and then find δU as write eqn. (4.6) as

$$\delta U = b\int \left[(EA)_s \frac{\partial u}{\partial x}\frac{\partial(\delta u)}{\partial x} + 2\tau_0(b+h)\frac{\partial(\delta u)}{\partial x} + \mu I_0 \frac{\partial^3 u}{\partial x \partial t^2}\frac{\partial(\delta u)}{\partial x} - \mu I_1 \frac{\partial^4 w}{\partial x^2 \partial t^2}\frac{\partial(\delta u)}{\partial x} - \mu I_0 \frac{\partial^2 w}{\partial t^2}\frac{\partial^2(\delta w)}{\partial x^2}\right] dx$$
$$+ b\int \left[-\mu I_1 \frac{\partial^3 u}{\partial x \partial t^2}\frac{\partial^2(\delta w)}{\partial x^2} + \mu I_2 \frac{\partial^4 w}{\partial x^2 \partial t^2}\frac{\partial^2(\delta w)}{\partial x^2} + (EI)_s \frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\delta w)}{\partial x^2}\right] dx. \tag{4.15}$$

Finally, the variational form of the equation from eqn. (4.6)

$$b\int_0^T\int_0^L \left[(EA)_s \frac{\partial u}{\partial x}\frac{\partial(\delta u)}{\partial x} + 2\tau_0(b+h)\frac{\partial(\delta u)}{\partial x} + \mu I_0 \frac{\partial^3 u}{\partial x \partial t^2}\frac{\partial(\delta u)}{\partial x} - \mu I_1 \frac{\partial^4 w}{\partial x^2 \partial t^2}\frac{\partial(\delta u)}{\partial x} - \mu I_0 \frac{\partial^2 w}{\partial t^2}\frac{\partial^2(\delta w)}{\partial x^2}\right] dxdt$$
$$+ b\int_0^T\int_0^L \left[-\mu I_1 \frac{\partial^3 u}{\partial x \partial t^2}\frac{\partial^2(\delta w)}{\partial x^2} + \mu I_2 \frac{\partial^4 w}{\partial x^2 \partial t^2}\frac{\partial^2(\delta w)}{\partial x^2} + (EI)_s \frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\delta w)}{\partial x^2} - I_0 \frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} - I_0 \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}\right] dxdt$$
$$+ b\int_0^T\int_0^L \left[I_1 \frac{\partial u}{\partial t}\frac{\partial^2 \delta w}{\partial t \partial x} + I_1 \frac{\partial^2 w}{\partial t \partial x}\frac{\partial \delta u}{\partial t} - I_2 \frac{\partial^2 w}{\partial t \partial x}\frac{\partial^2 \delta w}{\partial t \partial x}\right] dxdt = 0. \tag{4.16}$$

To obtain the finite element model, we divide the region with two node elements with three degrees of freedom per node. Three degrees of freedom contain axial displacement, transverse displacement and angular displacement. Here, we are using Lagrange interpolation function to interpolate axial displacement and Hermite function functions to interpolate transverse and angular displacements. Assuming the axial and transverse displacement in terms of nodal degrees of freedom and interpolation functions, we get $u = N_u U$ and $w = N_w U$, where, $N_u = \begin{bmatrix} N_1 & 0 & 0 & N_4 & 0 & 0 \end{bmatrix}$,$\mathrm{N}_w = \begin{bmatrix} 0 & N_2 & N_3 & 0 & N_5 & N_6 \end{bmatrix}. Here,$ $\mathrm{N}_1 = 1 - \frac{x}{l}$ and $N_4 = \frac{x}{l}$ are the Lagrange interpolation functions. $N_2 = \frac{l^3 - 3lx^2 - 2x^3}{l^3}$; $N_3 = \frac{l^2 x - 2lx^2 + x^3}{l^2}$; $N_5 = \frac{3lx^2 - 2x^3}{l^3}$ and $N_6 = \frac{x^3 - lx^2}{l^2}$ are the Hermite functions.

After substituting the shape functions in eqn. 4.16, we get the following form of the variational equation

$$b\int_0^T\int_0^L \left[(EA)_s \left(\frac{\partial N_u}{\partial x}\right)^2 U\delta U + 2\tau_0(b+h)\frac{\partial N_u}{\partial x}\delta U + \mu I_0 \left(\frac{\partial N_u}{\partial x}\right)^2 \frac{\partial^2 U}{\partial t^2}\delta U - \mu I_1 \frac{\partial^2 N_w}{\partial x^2}\frac{\partial N_u}{\partial x}\frac{\partial^2 U}{\partial t^2}\delta U\right] dxdt$$
$$+ b\int_0^T\int_0^L \left[-\mu I_0 N_w \frac{\partial^2 N_w}{\partial x^2}\frac{\partial^2 U}{\partial t^2}\delta U - \mu I_1 \frac{\partial N_u}{\partial x}\frac{\partial^2 N_w}{\partial x^2}\frac{\partial^2 U}{\partial t^2}\delta U + \mu I_2 \left(\frac{\partial^2 N_w}{\partial x^2}\right)^2 \frac{\partial^2 U}{\partial t^2}\delta U + (EI)_s \left(\frac{\partial^2 N_w}{\partial x^2}\right)^2 U\delta U\right] dxdt$$
$$+ b\int_0^T\int_0^L \left[-I_0 (N_u)^2 \frac{\partial^2 U}{\partial t^2}\delta U - I_0 (N_w)^2 \frac{\partial^2 U}{\partial t^2}\delta U + I_1 N_w \frac{\partial N_u}{\partial x}\frac{\partial^2 U}{\partial t^2}\delta U\right] dxdt$$
$$+ b\int_0^T\int_0^L \left[I_1 \frac{\partial N_w}{\partial x}\frac{\partial^2 U}{\partial t^2}\delta U - I_2 \left(\frac{\partial N_w}{\partial x}\right)^2 \frac{\partial^2 U}{\partial t^2}\delta U\right] dxdt = 0. \tag{4.17}$$

After eliminating the variational operator and doing rearrangement, we get

$$
\begin{aligned}
b\int_0^L \left[\mu I_0\left(\frac{\partial N_u}{\partial x}\right)^2 + \mu I_2\left(\frac{\partial^2 N_w}{\partial x^2}\right)^2 - 2\mu I_1 \frac{\partial^2 N_w}{\partial x^2}\frac{\partial N_u}{\partial x} - \mu I_0 N_w \frac{\partial^2 N_w}{\partial x^2}\right]\ddot{U} \\
+b\int_0^L \left[-I_2\left(\frac{\partial N_w}{\partial x}\right)^2 - I_0 N_w^2 - I_0 N_u^2 + I_1 N_w \frac{\partial N_u}{\partial x} + I_1 \frac{\partial N_w}{\partial x}\right]\ddot{U} \\
+b\int_0^L \left[(EA)_s\left(\frac{\partial N_u}{\partial x}\right)^2 + (EI)_s\left(\frac{\partial^2 N_w}{\partial x^2}\right)^2\right] U = 0.
\end{aligned}
\tag{4.18}
$$

On integrating the eqn. (4.18), we can write it in the following form

$$[M]\ddot{U} + [K][U] = 0 \tag{4.19}$$

where, $[M] = [M_l] + [\mu M_{nl}]$ which is the sum of local and nonlocal masses. Assuming $U = Ue^{i\omega t}$, we get the eigen equation in the following form

$$[K][U] = \omega^2 [M][U], \tag{4.20}$$

where, ω is the modal frequencies.

4.3 Results and Discussion

The frequency variation of simply supported beam is demonstrated using the linear finite element model developed in this chapter. For doing the analysis, the following dimensions and properties are used. $L = 10nm$, $E = 160GPa$, $A = 1nm^2$, $E_s = 5.1882N/m$, $\tau_0 = .9108N/m$, $\rho = 2400Kg/m^3$. We solve eqn. (4.20) to find the frequencies and mode-shapes for the first three modes of the beam.

First we do the convergence study in which we divide the beam into 2, 4, 6, 8, 10, and 12 elements. Figure 4.1 shows the variation of the first mode frequency with the number of elements. We found that the converged solution is obtained for 10 elements. Therefore, all our analysis is done with 10 elements. To do the analysis, we vary the frequencies with the nonlocal and surface

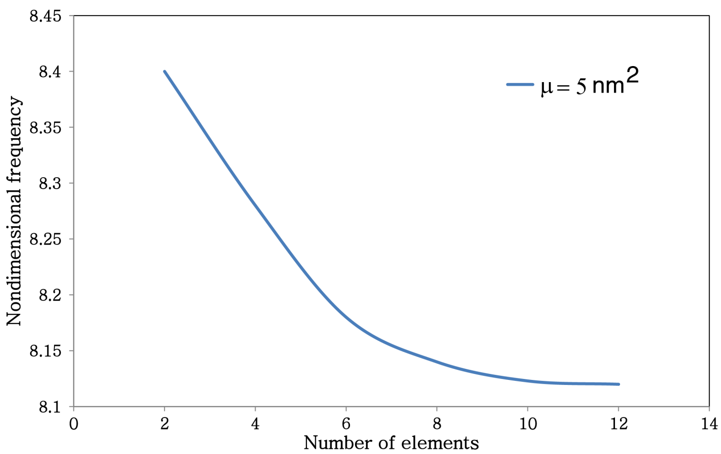

Figure 4.1: The variation of frequency with the number of elements for a given μ value.

parameters. Figure 4.2 shows the variation of frequency with nonlocal parameters for different slenderness ratios (i.e., L/H) 10, 20 and 100, respectively.

For all the three modes, the frequencies decrease with the nonlocal parameter, μ.

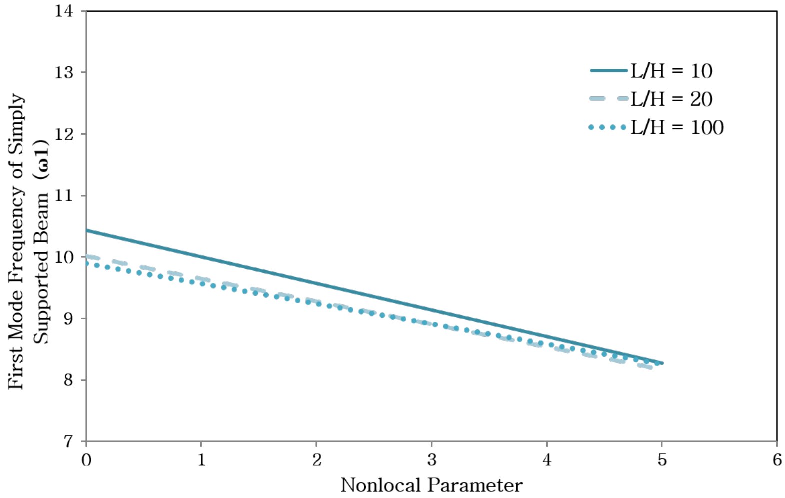

Figure 4.2: Effect of nonlocal parameter on first mode vibration frequency of simply supported beam for different slenderness ratios.

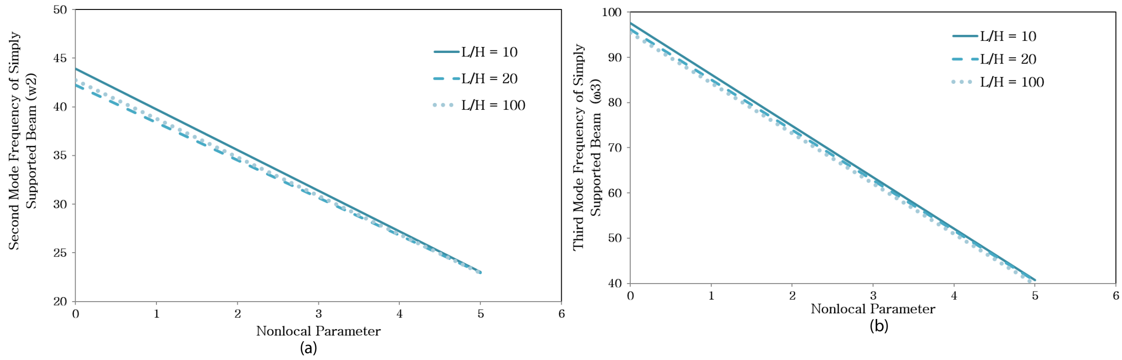

Figure 4.3: Effect of nonlocal parameter on (a) second and (b) third mode vibration frequency of simply supported beam for different slenderness ratios.

Chapter 5

Comparison of proposed model with other models

In this chapter, we first validate all the models and then compare their limitations over different values of nonlocal parameters. For the validation and comparison of results, we take simply supported beam and find different modal frequencies using different models.

The first model (nonlocal model), which is described in chapter 2, captures only nonlocal effect with axial and bending terms without considering geometric nonlinearity and surface effects. Therefore, it is limited to small amplitude vibration problem. The second model (proposed model), which we are proposing in chapter 3, contain not only nonlocal effects in inertial, bending, axial and damping, but also captures geometric nonlinearity and surface effects. In chapter 3, we have present linear finite element formulation (FEM model) with local and surface effects with geometric nonlinearity.

5.1 Validation and Comparison of models

For validation and comparison, we take simply supported beam with the geometric and mechanical properties as mentioned in Table 5.1.To obtain the solution, we are using MAPLE and MATLAB.

Table 5.1: Parameters

Parameters	Value
L	$1\mu m$
B	$300nm$
H	$100nm$
E	$160GPa$
E_s	$5.1882N/m$
τ_0	$.9108N/m$
ρ	$2400kg/m^3$
N	$.8\mu N$
ζ	0.00001

5.1.1 Validation

Figure 5.1 shows comparison of the variation of first mode frequency with nonlocal parameters obtained from FEM model, nonlocal model, and proposed model, respectively, under ideal condition. Under this condition, geometric nonlinearity is neglected, non-local effect is just limited to bending and axial terms. It shows that all the models behave similar to each other. Thus, all the model validate with respect to each other. It is also found that the variation is similar to the results present by Li et al. [48]. We also present the comparison of third modal frequency (Fig. In 5.2) to validate the models. On comparing, we find that all our models work similar to each other. Hence, they are validated under the common operating condition.

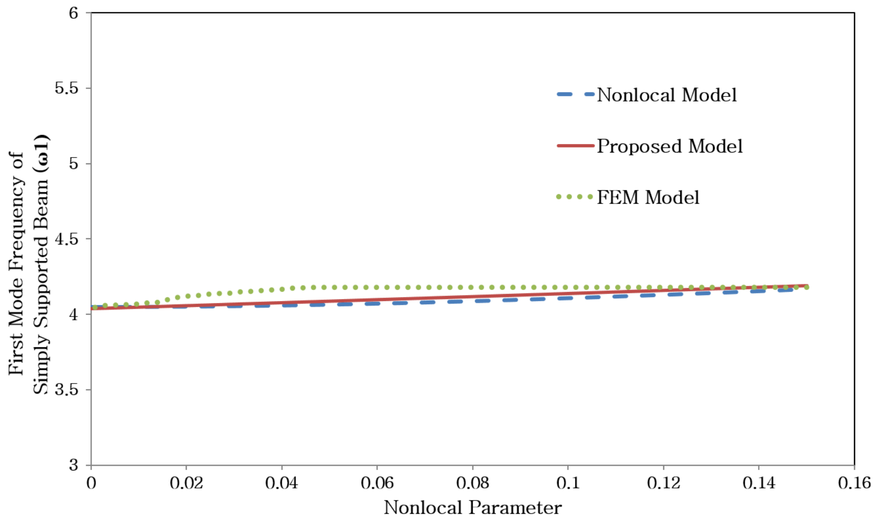

Figure 5.1: First mode frequency validation of simply supported beam depicted in all models.

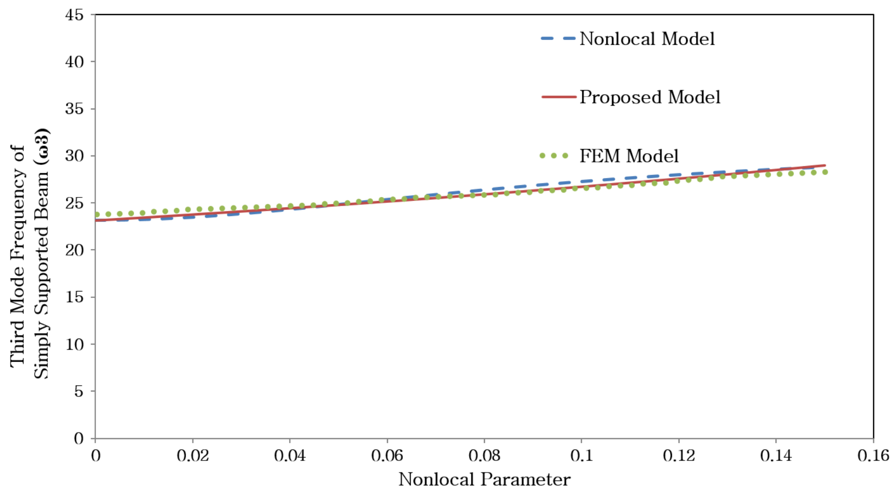

Figure 5.2: Third mode frequency variation for SS beam depicted in all models.

5.1.2 Comparison

In this section, we compare the results from all the models for non-ideal cases. Figure 5.3 shows the variation of FEM and nonlocal models when the nonlocal and surface effects are considered. It clearly shows the difference in the results as nonlocal model does not consider surface effects. Figure 5.4 shows the comparison of nonlocal and proposed model without damping effect. It also shows the difference in the results as the proposed model also considers geometric nonlinearity due to axial elongation.

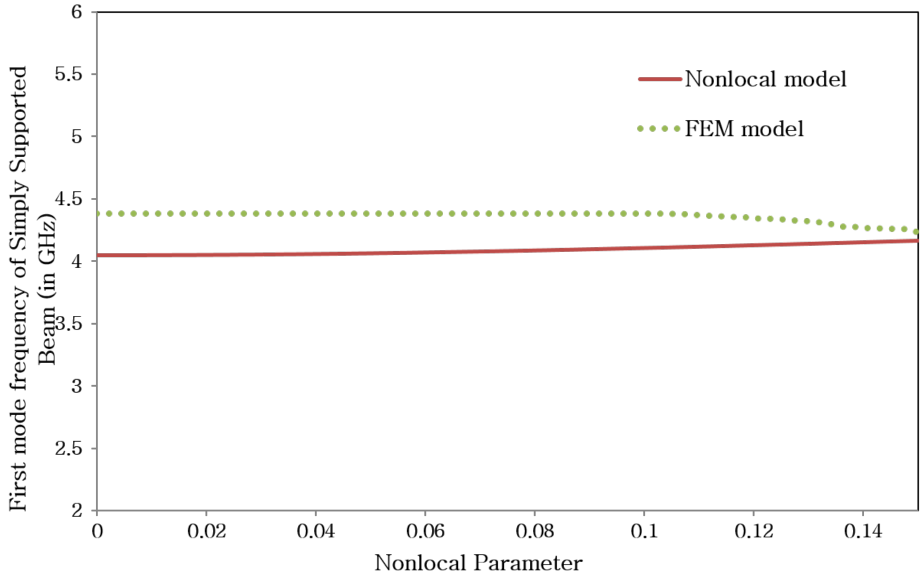

Figure 5.3: Comparison of FEM model with Nonlocal model for SS beam mode1.

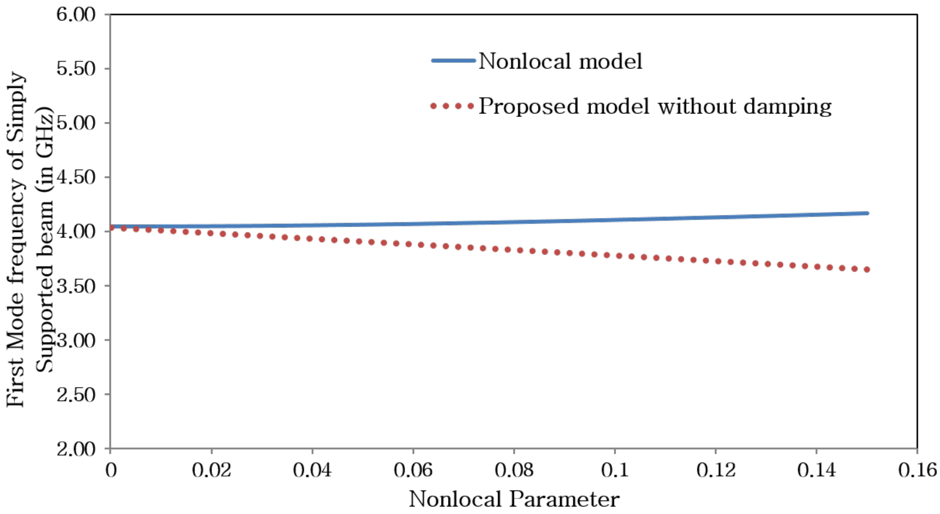

Figure 5.4: Comparison of Nonlocal model with proposed model without damping for SS beam mode1 .

Chapter 6

Conclusions and Future work

In this thesis, we first developed analytical models simply-supported, fixed-fixed, and cantilever beam by considering the nonlocal effects in bending and axial term. Using this model, we studied the influence of axial tensions under different nonlocal parameters for fundamental and higher modes. We found that higher modes are more sensitive to the nonlocal parameters. However, we also found that the model developed in chapter 2 is valid for small amplitude oscillation. To improve the modeling, we modified the governing equation by including additional nonlocal effects in inertia and damping, and also geometric nonlinearity. Subsequently, we found the frequencies of simply supported beam, fixed-fixed beam and cantilever and studied their variation with nonlocal parameters and damping. Finally, we present linear finite element model to capture nonlocal and surface effects which can be used to model the structure with complex geometry.

All the models presented in the thesis are based on Euler-Bernoulli beams. Hence, they can be extended to Timosenko beams. Generalized nonlinear FEM model can also be developed for generalized object. Moreover, the experimental validation of the models can also be done to find the limitations.

Appendix

Here, we present the local/element stiffness and mass matrices which are obtained from FEM formulation in chapter 4.

$$K = \begin{bmatrix} (EA)_s/L & 0 & 0 & -(EA)_s/L & 0 & 0 \\ 0 & 12(EI)_s/L^3 & 6(EI)_s/L^2 & 0 & -12(EI)_s/L^3 & 6(EI)_s/L^2 \\ 0 & 6(EI)_s/L^2 & 4(EI)_s/L & 0 & -6(EI)_s/L^2 & 2(EI)_s/L \\ -(EA)_s/L & 0 & 0 & (EA)_s/L & 0 & 0 \\ 0 & -12(EI)_s/L^3 & -6(EI)_s/L^2 & 0 & 12(EI)_s/L^3 & -6(EI)_s/L^2 \\ 0 & 6(EI)_s/L^2 & 2(EI)_s/L & 0 & -6(EI)_s/L^2 & 4(EI)_s/L \end{bmatrix}$$

$$\mathrm{M}_1 = \rho hb \begin{bmatrix} L/3 & 0 & 0 & L/6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ L/6 & 0 & 0 & L/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \mathrm{M}_2 = \mu \begin{bmatrix} \rho A/L & 0 & 0 & -\rho A/L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\rho A/L & 0 & 0 & \rho A/L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{M}_3 = \tfrac{\rho AL}{420} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -4L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -4L^2 & 0 & -22L & 4L^2 \end{bmatrix}; \mathrm{M}_4 = -\tfrac{A\rho h^2}{12} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6/5L & 1/10 & 0 & -6/5L & 1/10 \\ 0 & 1/10 & 2L/15 & 0 & -1/10 & -L/30 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6/5L & -1/10 & 0 & 6/5L & -1/10 \\ 0 & 1/10 & -L/30 & 0 & -1/10 & 2L/15 \end{bmatrix}$$

$$\mathrm{M}_5 = \tfrac{\mu A\rho h^2}{12} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12/L^3 & 6/L^2 & 0 & -12/L^3 & 6/L^2 \\ 0 & 6/L^2 & 4/L & 0 & -6/L^2 & 2/L \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12/L^3 & -6/L^2 & 0 & 12/L^3 & -6/L^2 \\ 0 & 6/L^2 & 2/L & 0 & -6/L^2 & 4/L \end{bmatrix}$$

$$\mathrm{M}_6 = -\mu\rho A \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6/5L & -1/10 & 0 & 6/5L & -1/10 \\ 0 & -1/10 & -2L/15 & 0 & 1/10 & L/30 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6/5L & 1/10 & 0 & -6/5L & 1/10 \\ 0 & -1/10 & L/30 & 0 & 1/10 & -2L/15 \end{bmatrix}$$

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