Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 323 (2017) 94–102
FUZZY sets and systems
www.elsevier.com/locate/fss

Short Communication

T-subnorms with strong associated negation: Some properties

Balasubramaniam Jayaram

Department of Mathematics, Indian Institute of Technology Hyderabad, Hyderabad, 502 285, India

Received 21 September 2016; received in revised form 7 December 2016; accepted 22 January 2017
Available online 25 January 2017

Abstract

In this work we investigate t-subnorms M that have strong associated negation. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeanness, conditional cancellativity, left-continuity, nilpotent elements, etc. In particular, we show that under this setting many of these properties are equivalent. Our investigations lead us to two open problems which are also presented.

Keywords: T-norms; T-subnorms; Archimedeanness; Conditional cancellativity; Left-continuity; R-implications; Residual implications

1. Introduction

The theory of triangular norms and triangular subnorms has been well studied and its applications well-established. Many algebraic and analytical properties of these operations, viz., Archimedeanness, conditional cancellativity, left-continuity, etc., have been studied and their inter-relationships shown (see for instance, [6]).

Yet another way of categorizing t-subnorms is as follows: Given a t-subnorm M, one can obtain its associated negation n_M (see Definitions 2.2 and 2.4 below). Note that n_M is usually not a fuzzy negation, i.e., $n_M(1) \geq 0$. However, we can broadly consider two sub-classes of t-subnorms based on whether their associated negation n_M is strong or not.

In this work, we study the class of t-subnorms whose associated negation n_M is strong. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate some particular classes of these and study the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeanness, conditional cancellativity, left-continuity, etc. In particular, we show that under this setting many of these properties are equivalent. Our investigations have led us to two open problems, which are also presented.

E-mail address: jbala@iith.ac.in.

http://dx.doi.org/10.1016/j.fss.2017.01.012

2. Preliminaries

To make this short note self-contained, we present some important definitions and properties, which can be found in [6,1].

Definition 2.1. A fuzzy negation is a function $N: [0,1] \to [0,1]$ that is non-increasing and such that $N(1) = 0$ and $N(0) = 1$. Further, it is said to be strong or involutive, if $N \circ N = id_{[0,1]}$.

Definition 2.2. A t-subnorm is a function $M: [0,1]^2 \to [0,1]$ such that it is monotonic non-decreasing, associative, commutative and $M(x, y) \le \min(x, y)$ for all $x, y \in [0,1]$, i.e., 1 need not be the neutral element.

Definition 2.3. Let M be a t-subnorm.

(i) If 1 is the neutral element of M, then it becomes a t-norm. We denote a t-norm by T in the sequel.
(ii) M is said to satisfy the Conditional Cancellation Law if, for any $x, y, z \in (0,1]$,

$$M(x, y) = M(x, z) > 0 \text{ implies } y = z\,. \tag{CCL}$$

Alternately, (CCL) implies that on the positive domain of M, i.e., on the set $\{(x, y) \in (0,1]^2 \mid M(x, y) > 0\}$, M is strictly increasing.
(iii) M is said to be *Archimedean*, if for all $x, y \in (0,1)$ there exists an $n \in \mathbb{N}$ such that $x_M^{[n]} < y$.
(iv) An element $x \in (0,1)$ is a *nilpotent* element of M if there exists an $n \in \mathbb{N}$ such that $x_M^{[n]} = 0$.
(v) A t-norm T is said to be *nilpotent*, if it is continuous and if each $x \in (0,1)$ is a nilpotent element of T.

Definition 2.4. Let M be any t-subnorm and $x, y \in [0,1]$.

- The R-implication I_M of M is given by

$$I_M(x, y) = \sup\{t \in [0,1] \mid M(x, t) \le y\}\,. \tag{1}$$

- The associated negation n_M of M is given by

$$n_M(x) = \sup\{t \in [0,1] \mid M(x, t) = 0\}. \tag{2}$$

A brief note on nomenclature is perhaps warranted here. Firstly, the R-implication I_M will be termed a *residual* implication only if the underlying t-subnorm M is left-continuous.

Secondly, while n_M is clearly a non-increasing function and $n_M(0) = 1$, note that it need not be a fuzzy negation, since $n_M(1)$ can be greater than 0. Hence, only in the case n_M is a fuzzy negation we call n_M the *natural negation* of M in this work. However, many results hold even if $n_M(1) > 0$, see for instance [3,9], and hence to preserve this generality in such situations we term n_M as the *associated negation*.

For instance, the following result is true even when $n_M(1) > 0$.

Proposition 2.5 *(cf. [1], Proposition 2.3.4). Let M be any t-subnorm and n_M its associated negation. Then we have the following:*

(i) $M(x, y) = 0 \Longrightarrow y \le n_M(x)$.
(ii) $y < n_M(x) \Longrightarrow M(x, y) = 0$.
(iii) *If M is left-continuous then $y = n_M(x) \Longrightarrow M(x, y) = 0$, i.e., the reverse implication of (i) also holds.*

Proposition 2.6. *Let M be any t-subnorm with n_M being a natural negation with e as its fixed point, i.e., $n_M(e) = e$. Then*

(i) *Every $x \in (0, e)$ is a nilpotent element; in fact, $x_M^{[2]} = 0$ for all $x \in [0, e)$.*
(ii) *In addition, if M is either conditionally cancellative or left-continuous, then e is also a nilpotent element.*

Proof. (i) By definition,

$$n_M(e) = \sup\{t \in [0,1] \mid M(e,t) = 0\} = e,$$

implies that $M(e, e^-) = 0$, from whence we get $M(x,x) \le M(e,e^-) = 0$ for all $x \in [0, e)$. In other words, $x_M^{[2]} = 0$ for all $x \in [0, e)$.

(ii) Let M be conditionally cancellative. If $e_M^{[2]} = 0$ then clearly e is a nilpotent element. If not, then we have $M(e,e) = x < M(1,e) \le e$ and from (ii) above we have $M(x,x) = 0$. Now,

$$e_M^{[4]} = M(M(e,e), M(e,e)) = M(x,x) = 0.$$

If M is left-continuous, then $n_M(e) = \max\{t \in [0,1] \mid M(e,t) = 0\} = e$, i.e., $e \in \{t \in [0,1] \mid M(e,t) = 0\}$ and hence $M(e,e) = 0$, i.e., e is also a nilpotent element. □

Remark 2.7.

(i) In the case n_M is a strong natural negation we can show that if M is conditionally cancellative then every $x \in (0,1)$ is also a nilpotent element, see Remark 5.9(ii).

(ii) Note that without any further assumptions, the set of nilpotent elements need not be the whole of $(0,1)$. For instance, for the nilpotent minimum t-norm

$$T_{\mathbf{nM}}(x,y) = \begin{cases} 0, & \text{if } x + y \le 1, \\ \min(x,y), & \text{otherwise}, \end{cases} \quad x, y \in [0,1],$$

which is left-continuous but not conditionally cancellative, its set of nilpotent elements is $(0, .5]$, while its set of zero divisors is $(0,1)$.

However, Theorem 6.1 gives an equivalence condition for the whole of $(0,1)$ to be the set of nilpotent elements under a suitable condition on n_M.

3. T-subnorms with strong associated negation = t-norms

There are works showing that some classes of t-subnorms M whose associated negations n_M are involutive do become t-norms. Jenei [4], also see [5], showed it for the class of left-continuous M, while Jayaram [2] did the same for conditionally cancellative M. The main result of this section shows that the above results are true in general, i.e., any t-subnorm with a strong natural negation is a t-norm.

The following result was firstly proven by Jenei in [4]. However, we give a very simple proof of this result without resorting to the rotation-invariance property.

Theorem 3.1 *(Jenei, [4], Theorem 3). If M is a left-continuous t-subnorm with n_M being strong, then M is a t-norm.*

Proof. Firstly, note that if M is a left-continuous t-subnorm, then its residual implication satisfies the exchange principle, i.e.,

$$I_M(x, I_M(y,z)) = I_M(y, I_M(x,z)).$$

It follows from the fact that the neutral element of M does not play any role in the proof, see, for instance the proof given for Theorem 2.5.7 in [1].

If n_M is strong, then for every $y \in [0,1]$ there exists $y' \in [0,1]$ such that $n_M(y) = y'$. Now,

$$I_M(1, y') = I_M(1, I_M(y, 0)) = I_M(y, I_M(1, 0)) = I_M(y, 0) = y'.$$

Thus, for all $y' \in [0,1]$,

$$I_M(1, y') = \max\{t \mid M(1,t) \le y'\} = y' \Longrightarrow M(1, y') = y'. \quad \square$$

Theorem 3.2 *(Jayaram [2], Theorem 4.4). Let M be any conditionally cancellative t-subnorm. If n_M is a strong natural negation then M is a t-norm.*

Now, we prove the main result of this section which shows that the above results are true in general.

Theorem 3.3. *Let M be any t-subnorm with n_M being a strong natural negation. M is a t-norm.*

Proof. Note, firstly, that since $n_M(x) = \sup\{t \in [0,1] \mid M(x,t) = 0\}$, is a strong negation, we have that $n_M(z) = 1 \Longleftrightarrow z = 0$ and $n_M(z) = 0 \Longleftrightarrow z = 1$. Equivalently, $M(1,z) = 0 \Longleftrightarrow z = 0$.

On the contrary, let us assume that $M(1,x) = x' \lneq x$ for some $x \in (0,1]$. Since n_M is strong, the following are true:

(i) $n_M(x') > n_M(x)$,
(ii) if $p > n_M(x)$ then $M(x,p) > 0$,
(iii) there exists a $y \in (0,1)$ such that $n_M(x') > y > n_M(x)$ and $M(y,x) = q > 0$ while $M(y,x') = 0$.

Now, by associativity we have

$$\left.\begin{aligned} M(y, M(x,1)) &= M(y,x') = 0 \\ M(M(y,x),1) &= M(q,1) \end{aligned}\right\} \Longrightarrow M(q,1) = 0,$$

a contradiction. Thus $M(1,x) = x$ for **all** $x \in [0,1]$ and hence we have the result. □

In the following sections, we deal with t-subnorms whose associated negations are strong, or equivalently t-norms whose associated negations are strong. We discuss the inter-relationships between the different algebraic and analytical properties for this subclass of t-norms; in particular, Archimedeanness, Conditional Cancellativity, (Left-)continuity and Nilpotence that are relevant to our context. We begin with listing out some established results and go on to present some new ones.

4. Continuity and nilpotence

Let T be a t-norm and n_T a strong negation. The following result, whose proof is straight-forward, shows the equivalence between continuity and nilpotence:

Theorem 4.1 *(Klement et al. [6]). Let T be a t-norm with n_T being strong. Then the following are equivalent:*

(i) T is continuous.
(ii) T is a nilpotent t-norm.

Further, we know that every nilpotent t-norm is both Archimedean and Conditionally cancellative, since every nilpotent t-norm is isomorphic to the Łukasiewicz t-norm and the Archimedeanness and Conditionally cancellativity of T are preserved under isomorphism, see [6], Examples 2.14(iv) and 2.15(v). Trivially, every nilpotent t-norm is also left-continuous.

5. Conditional cancellativity, left continuity and nilpotence

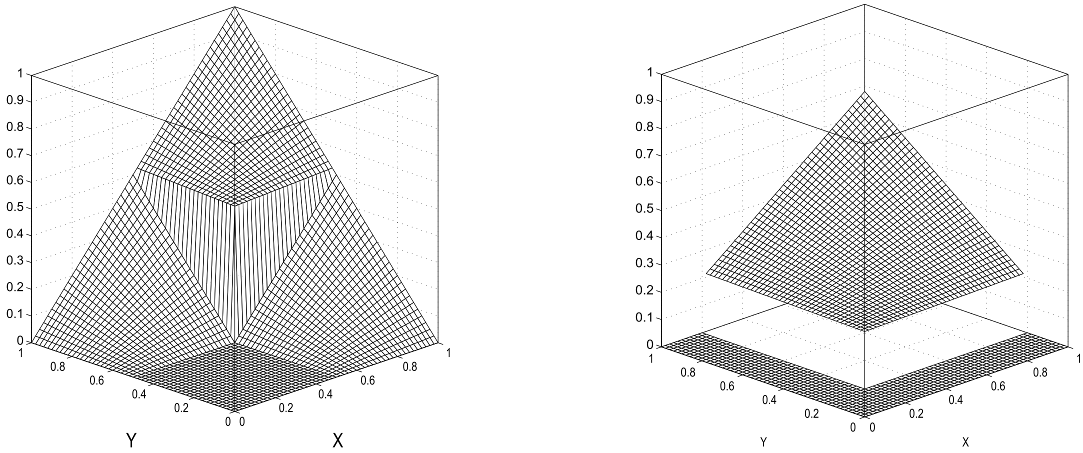
Recently, in Jayaram [2], the following problem of U. Höhle, given in Klement et al. [7] has been solved. Further it was shown that it characterizes the set of all conditionally cancellative t-subnorms.

Problem 5.1 *(U. Höhle, [7], Problem 11).* Characterize all left-continuous t-norms T which satisfy

$$I_T(x, T(x,y)) = \max(n_T(x), y), \quad x, y \in [0,1], \tag{3}$$

where I_T, n_T are as given in (1) and (2) with $M = T$.

Theorem 5.2 *(cf. Jayaram [2], Theorem 3.1). Let M be any t-subnorm, not necessarily left-continuous. Then the following are equivalent:*



(a) $T_{\mathbf{OY}}$ (b) $M_{\mathbf{P_f}}$

Fig. 1. A t-norm and a t-subnorm that are conditionally cancellative.

(i) *The pair* (I_M, M) *satisfies* (3).
(ii) M *is a Conditionally Cancellative t-subnorm.*

Remark 5.3. The following statements follow from Theorem 5.2 with $M = T$, a t-norm:

(i) If a (right) continuous T satisfies (3) along with its R-implication then T is necessarily Archimedean, see [6], Proposition 2.15(ii).
(ii) However, if a left-continuous T satisfies (3) along with its residual implication then T need not be Archimedean and hence not continuous. An example is Hajék's t-norm or the following t-norm $T_{\mathbf{OY}}$ of Ouyang et al. [11], Example 3.4, which is a (CCL) t-norm (and hence a t-subnorm too) that is left-continuous but not continuous at $(0.5, 0.5)$ and hence is not Archimedean (see Fig. 1(a)):

$$T_{\mathbf{OY}}(x, y) = \begin{cases} 2(x-0.5)(y-0.5)+0.5, & \text{if } (x, y) \in (0.5, 1]^2 \\ 2y(x-0.5), & \text{if } (x, y) \in (0.5, 1] \times [0, 0.5] \\ 2x(y-0.5), & \text{if } (x, y) \in [0, 0.5] \times (0.5, 1] \\ 0, & \text{otherwise} \end{cases}.$$

Theorem 5.4 *(Jenei, [4], Theorem 2). Let T be a left-continuous t-norm with n_T being strong. Then the following are equivalent:*

(i) T *is a conditionally cancellative t-norm.*
(ii) T *is a nilpotent t-norm.*

In fact, for a conditionally cancellative t-subnorm M we can give an equivalent condition for it to be left-continuous.

Theorem 5.5. *Let M be a* (CCL) *t-subnorm. Then the following are equivalent:*

(i) $M(x, n_M(x)) = 0$*,* $x \in [0, 1]$.
(ii) M *is left-continuous.*

Proof. (i) $\Longrightarrow$ (ii): Let $M(x, n_M(x)) = 0$, for all $x \in [0, 1]$. On the contrary, let us assume that M is not left-continuous. Then there exist $x_0 \in (0, 1]$, $y_0 \in (0, 1]$ and an increasing sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \in [0, 1)$, such that $\lim_{n\to\infty} x_n = x_0$, but

$$\lim_{n\to\infty} M(x_n, y_0) = M(x_0^-, y_0) = z' < z_0 = M(x_0, y_0).$$

Observe that

$$I_M(y_0, z') = \sup\{t \in [0,1] \mid M(y_0, t) \le z'\} = x_0, \tag{4}$$

since from the monotonicity of M we have $M(y_0, x_n) \le z'$ for every $n \in \mathbb{N}$ and $M(y_0, x_0) = z_0 > z'$. Since M is (CCL), we have from (3)

$$I_M(y_0, z') = I_M(y_0, M(y_0, x_0^-)) = \max(n(y_0), x_0^-).$$

Now, we have two cases. On the one hand, if $I_M(y_0, z') = x_0^- \lneq x_0$, then it is a contradiction to (4). On the other hand, if $I_M(y_0, z') = n(y_0)$, then this implies that $n(y_0) = x_0$ from (4) and hence

$$M(x_0, y_0) = M(n(y_0), y_0) = z_0 = 0,$$

by the hypothesis and hence there does not exist any $z' < z_0$ and hence M is left-continuous.

(ii) $\Longrightarrow$ (i): Follows from Proposition 2.5(iii). □

In other words, Theorem 5.5 states that for a (CCL) t-subnorm M, the only points at which M may not be left-continuous is the boundary of the zero region $Z_M = \{(x, y) \in [0,1]^2 | M(x, y) = 0\}$ which does not contain the origin.

Remark 5.6. In Theorem 5.5 we do not need n_M to be a negation, i.e., $n_M(1) \ge 0$. Consider the following t-subnorm $M_{\mathbf{P_f}}$ (cf. Example 3.15 of [6], see Fig. 1(b)),

$$M_{\mathbf{P_f}} = \begin{cases} 0.2 + \dfrac{3(x-0.2)(y-0.2)}{4}, & \text{if } (x, y) \in (0.2, 1]^2 \\ 0, & \text{otherwise} \end{cases}$$

which is a left-continuous (CCL) t-subnorm but $n_{M_{\mathbf{P_f}}}$ is not a negation since $n_{M_{\mathbf{P_f}}}(1) = 0.2$.

Theorem 5.7. *Let M be a (CCL) t-subnorm whose n_M is strong. Then M is left-continuous.*

Proof. If possible, let $M(x_0, n(x_0)) = p > 0$ for some $x_0 \in (0, 1)$. Since M is (CCL), we have $M(1^-, x_0) < x_0$ and hence by associativity we have

$$M(1^-, M(x_0, n(x_0))) = M(1^-, p)$$
$$M(M(1^-, x_0), n(x_0)) = 0$$

from whence it follows $M(1^-, p) = 0$, i.e., $n(p) = 1$, a contradiction to the fact that n_M is strong. Thus $p = 0$ and the result follows from Theorem 5.5. □

Theorem 5.8. *Let M be a t-subnorm such that n_M is strong. Then the following are equivalent:*

(i) *M is conditionally cancellative.*
(ii) *M is a nilpotent t-norm.*

Proof. If M satisfies (CCL) then M is left-continuous, from Theorem 5.7 and now, using Theorem 5.4 we have the result. □

Remark 5.9.

(i) The nilpotent minimum t-norm $T_{\mathbf{nM}}$ is an example of a t-subnorm M whose n_M is involutive and M satisfies (LEM) with n_M but is not conditionally cancellative and hence is not a nilpotent t-norm.
(ii) In the case n_M is a strong natural negation, from Theorem 5.7 we see that conditionally cancellativity implies left-continuity and from Theorem 5.8 that every $x \in (0, 1)$ is a nilpotent element.

6. Archimedeanness, left continuity and nilpotence

We begin with a result that shows that if n_M is strong, then the Archimedeanness is equivalent to every element $x \in (0, 1)$ being nilpotent. However, unless M is also left-continuous, M is not a nilpotent t-norm.

Theorem 6.1. *Let M be any t-subnorm such that n_M is not completely vanishing, i.e., there exists $z \in (0, 1)$ such that $n_M(z) > 0$. The following are equivalent:*

(i) *Every $x \in (0, 1)$ is a nilpotent element.*
(ii) *M is Archimedean.*

Proof. (i) $\Longrightarrow$ (ii): Follows from Proposition 2.15 (iv) in [6].
(ii) $\Longrightarrow$ (i): Let M be any Archimedean t-subnorm such that n_M is not completely vanishing, i.e., there exists $z \in (0, 1)$ such that $n_M(z) > 0$. By Proposition 2.5(ii) we see that for any $0 < z' < n_M(z)$ we have $M(z', z) = 0$. For any $x \in [0, 1)$, by the Archimedeanness of M, there exists an $n, p \in \mathbb{N}$ such that $x_M^{[n]} < z'$ and $x_M^{[p]} < z$ from whence we have that

$$x_M^{[n+p]} = M\left(x_M^{[n]}, x_M^{[p]}\right) \leq M(z', z) = 0. \quad \square$$

Corollary 6.2. *Let M be any t-subnorm such that n_M is a strong negation. Then the following are equivalent:*

(i) *Every $x \in (0, 1)$ is a nilpotent element.*
(ii) *M is Archimedean.*

The following result is due to Kolesárová [8]:

Theorem 6.3. *Let T be any Archimedean t-norm. Then the following are equivalent:*

(i) *T is left-continuous.*
(ii) *T is continuous.*

Corollary 6.4. *A left-continuous Archimedean t-subnorm M whose n_M is involutive is a nilpotent t-norm.*

Proof. From Theorem 3.1 we see that M is a left-continuous t-norm. From Theorem 6.3, since M is Archimedean it is continuous. Also by Theorem 6.1, we have that every $x \in (0, 1)$ is a nilpotent element. Thus T is nilpotent, i.e., isomorphic to $T_{\mathbf{LK}}(x, y) = \max(0, x + y - 1)$. $\square$

Remark 6.5.

(i) Note that there exist left-continuous Archimedean t-subnorms M that are not continuous and hence their n_M is not involutive. For instance, consider the t-subnorm

$$M(x, y) = \begin{cases} x + y - 1, & \text{if } x + y > \frac{3}{2}, \\ 0, & \text{otherwise }, \end{cases} \quad x, y \in [0, 1].$$

(ii) The nilpotent minimum t-norm $T_{\mathbf{nM}}$ is an example of a left-continuous t-subnorm M whose n_M is involutive but is not Archimedean and hence is not a nilpotent t-norm.
(iii) However, it is not clear whether there exists any non-nilpotent Archimedean t-subnorm M whose n_M is involutive. Clearly such t-(sub)norms are not left-continuous.

Problem 1. Does there exist any non-nilpotent Archimedean t-subnorm M whose n_M is involutive. In other words, is an Archimedean t-subnorm M whose n_M is involutive necessarily left-continuous?

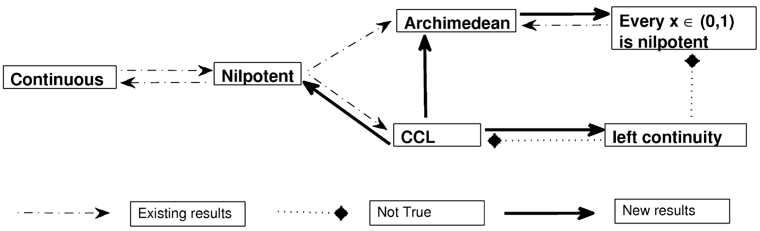


Fig. 2. A summary of the results available so far when n_T is strong.

7. Archimedeanness and conditional cancellativity

In general, there does not exist any inter-relationships between Archimedeanness and conditional cancellativity, as the following examples show.

Example 7.1.

(i) The Ouyang t-norm $T_{\mathbf{OY}}$ is an example of a t-(sub)norm which is not Archimedean but is both left-continuous and conditionally cancellative.

(ii) The following t-norm is neither Archimedean nor left-continuous but is conditionally cancellative:

$$T(x, y) = \begin{cases} 0, & \text{if } xy \leq \frac{1}{2} \,\&\, \max(x, y) < 1 \\ xy, & \text{if } xy > \frac{1}{2} \\ \min(x, y), & \text{otherwise} \end{cases} .$$

(iii) The following t-subnorm is Archimedean and continuous, but not conditionally cancellative:

$$M(x, y) = \max(0, \min(x + y - 1, x - a, y - a, 1 - 2a)),$$

where $a \in (0, 0.5)$. For instance, with $a = 0.25$ we have $M(0.75, 0.75) = M(0.75, 0.8) = 0.5$.

(iv) The nilpotent minimum $T_{\mathbf{nM}}$, whose n_M is strong, is neither Archimedean nor conditionally cancellative, but is left-continuous.

(v) The Lukasiewicz t-norm $T_{\mathbf{LK}}(x, y) = \max(0, x + y - 1)$ is both Archimedean and conditionally cancellative. Further, $n_{T_{\mathbf{LK}}}$ is strong.

In fact, in the case when n_M is strong we have the following partial implication.

Lemma 7.2. *Let M be any t-subnorm whose n_M is strong. If M is conditionally cancellative then M is Archimedean.*

Proof. From Theorem 5.8, we have that if M is conditionally cancellative then M is a nilpotent t-norm from whence it follows that M is Archimedean. □

Problem 2. Does there exist any Archimedean t-subnorm M whose n_M is involutive but is not conditionally cancellative? In other words, is an Archimedean t-subnorm M whose n_M is involutive necessarily conditionally cancellative?

In fact, from Theorem 3.3, it can be easily seen that the above two problems are an alternate formulation of Problem 2.1 in [10].

8. Concluding remarks

In this work, we have shown that t-subnorms whose associated negations are strong are necessarily t-norms. Further, we have studied the inter-relationships between some algebraic and analytical properties of such t-(sub)norms. Fig. 2 gives a pictorial summary of the results that exist so far. Our study has also opened up two interesting open problems.

Acknowledgement

The author would like to acknowledge the project CSIR HRDG-INDIA (25 (0235)/14/EMR II) for financial support.

References

[1] M. Baczyński, B. Jayaram, Fuzzy Implications, Stud. Fuzziness Soft Comput., vol. 231, Springer, Berlin, 2008.
[2] B. Jayaram, Solution to an open problem – a characterization of conditionally cancellative t-subnorms, Aequ. Math. 84 (2012) 235–244.
[3] B. Jayaram, R. Mesiar, On special fuzzy implications, Fuzzy Sets Syst. 160 (2009) 2063–2085.
[4] S. Jenei, Continuity of left-continuous triangular norms with strong induced negations and their boundary condition, Fuzzy Sets Syst. 124 (2001) 35–41.
[5] S. Jenei, On the structure of rotation-invariant semigroups, Arch. Math. Log. 42 (2003) 489–514.
[6] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer, Dordrecht, 2000.
[7] E.P. Klement, R. Mesiar, E. Pap, Problems on triangular norms and related operators, Fuzzy Sets Syst. 145 (2004) 471–479.
[8] A. Kolesárová, A note on Archimedean triangular norms, BUSEFAL 80 (1999) 57–60.
[9] R. Mesiar, A. Mesiarová, Continuous triangular subnorms, Fuzzy Sets Syst. 142 (2004) 75–83.
[10] R. Mesiar, A. Stupňanová, Open problems from the 12th International Conference on Fuzzy Set Theory and Its Applications, Fuzzy Sets Syst. 261 (2015) 112–123.
[11] Y. Ouyang, J. Fang, J. Li, A conditionally cancellative left-continuous t-norm is not necessarily continuous, Fuzzy Sets Syst. 157 (2006) 2328–2332.