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Fuzzy implications: alpha migrativity and generalised laws of importation

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ABSTRACT

In this work, we discuss the law of α-migrativity as applied to fuzzy implication functions in a meaningful way. A generalisation of this law leads us to Pexider-type functional equations connected with the law of importation, viz., the generalised law of importation $I(C(x,\alpha),y)=I(x,J(\alpha,y))$ (GLI) and the generalised cross-law of importation $I(C(x,\alpha),y)=J(x,I(\alpha,y))$ (CLI), where C is a generalised conjunction. In this article we investigate only (GLI). We begin by showing that the satisfaction of law of importation by the pairs (C, I) and/or (C, J) does not necessarily lead to the satisfaction of (GLI). Hence, we study the conditions under which these three laws are related.

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1. Introduction

1.1. On associativity-type functional equations

This paper deals with t-norms, t-conorms, fuzzy negations and fuzzy implications, see, e.g. [2,12]. Let us consider the following associativity-type functional equation involving the functions F_i: $[0, 1]^2 \to [0, 1]$ for $i = 1, 2, 3, 4$:

$$F_1(F_2(x,\alpha),y)=F_3(x,F_4(\alpha,y)),\quad \alpha,x,y\in[0,1]. \tag{1}$$

studied by Aczél et al. [1], see also [13]. If in (1) all $F_i = F$, then it reduces to the well-known associativity of F. If we consider each of the F_i's to be either a fuzzy conjunction or a fuzzy implication, then we obtain variants of different well-known functional equations considered in the literature devoted to fuzzy logic. For instance, if $F_1 = F_3 = T$ is a t-norm and $F_2 = F_4 = T_{\mathbf{P}}$, the product t-norm given by $T_{\mathbf{P}}(x,y)=xy$, then we obtain the migrativity functional equation of a t-norm (see [7,18]):

$$T(x\alpha,y)=T(x,\alpha y), \tag{2}$$

written usually in the form $T(\alpha x, y) = T(x, \alpha y)$. Note that if (2) is satisfied for a fixed $\alpha\in(0, 1)$ and for all $x, y\in[0, 1]$, then T is said to be α-migrative, while if it is true for all $\alpha\in(0, 1)$ and all $x, y\in[0, 1]$, then T is said to be migrative. In fact, for

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arbitrary t-norms T_1, T_2, if $F_1 = F_3 = T_1$, and $F_2 = F_4 = T_2$, then the above migrativity functional Eq. (2) can be generalised as follows:

$$T_1(T_2(x, \alpha), y) = T_1(x, T_2(\alpha, y)), \tag{3}$$

and it is called the generalised migrativity equation [10]. Since t-norms are symmetric, we can write (3) equivalently as $T_1(T_2(\alpha, x), y) = T_1(x, T_2(\alpha, y))$. The related generalised cross-migrativity equation [8] is defined as follows:

$$T_1(T_2(x, \alpha), y) = T_2(x, T_1(\alpha, y)), \tag{4}$$

or, since all t-norms are symmetric, as $T_1(T_2(\alpha, x), y) = T_2(x, T_1(\alpha, y))$. Similarly, as above, if (3) and (4) hold only for a fixed $\alpha \in (0, 1)$, then we say the pair of t-norms (T_1, T_2) is α-migrative and α-cross migrative, respectively.

If $F_1 = F_3 = F_4 = I$, a fuzzy implication and $F_2 = T$, a t-norm, then we obtain the well-known law of importation (see [2]):

$$I(T(x, \alpha), y) = I(x, I(\alpha, y)). \tag{5}$$

Of course (5) for t-norms can be also written as follows: $I(T(\alpha, x), y) = I(x, I(\alpha, y))$.

1.2. Motivation for this work

The above generalisations of the migrativity Eqs. (2)–(4) are obtained by considering different t-norms T_i in (2) and have been well studied. Similarly, the migrativity equations have been studied by weakening the functions T_i involved from the class of t-norms to more general conjunctive-type aggregation operators.

Borrowing the terminology from Bustince et. al. [4] we can say that a binary aggregation function A is α-B-migrative, with $\alpha \in [0, 1]$ w.r.t. an aggregation function B if the identity

$$A(B(x, \alpha), y) = A(x, B(\alpha, y))$$

holds for any $x, y \in [0, 1]$. Note that neither A nor B are assumed to be symmetric (like it is for t-norms), so, as authors write, it is important to keep the "correct" order of the arguments. Bustince et al. [5] discuss the migrativity of uninorms and nullnorms, while Mas et al. [15] discuss the migrativity of uninorms and nullnorms over t-norms and t-conorms and, further, Qiao and Hu [20] studied it for overlap and grouping functions[1]

However, in the case of the law of importation, the only generalisations of (5) are those where the t-norm T has been weakened to more generalised conjunctions, see for instance, [17]. Investigations of (5) to the following generalised versions have not been done so far:

$$I_1(T(x, \alpha), y) = I_1(x, I_2(\alpha, y)), \tag{6}$$

$$I_1(T(x, \alpha), y) = I_2(x, I_1(\alpha, y)). \tag{7}$$

It is well-known that the migrativity equation, as it stands, cannot be satisfied by a t-conorm (see [9]). A meaningful way of defining migrativity for a t-conorm is yet to be proposed. Similar is the case with the migrativity equation for a fuzzy implication.

In this work, we begin by proposing a definition of α-migrativity for fuzzy implications. Showing that it relates to a special case of (6) and (7) above, we go on to investigate the solutions of a further generalised (6), where the t-norm is replaced by a generalised fuzzy conjunction C leaving the investigations of (7) to a future work. Please note that this article is a revised and extended version of the conference paper [3].

Our current exposition is largely theoretical and aims to establish some connections between the laws of alpha migrativity and the laws of importations. However, we believe that this connection will be useful in results and applications wherein the law of importation plays an important role, as is already illustrated in the following works [11,14,16].

2. The laws of migrativity and (6) — A motivation for study

In this paper we use the following definition of fuzzy implication.

Definition 2.1 (cf. Baczyński and Jayaram [2]). A function I: $[0, 1]^2 \to [0, 1]$ is called a *fuzzy implication* if

1. I is non-increasing in the first variable,
2. I is non-decreasing in the second variable,
3. $I(0, 0) = 1 = I(1, 1)$ and $I(1, 0) = 0$.

[1] Note that there exist other generalisations of the migrativity equation, see for instance Ouyang [19] or Durante and Ricci [6]. However, these do not fall in the general framework of associativity-type functional equation.

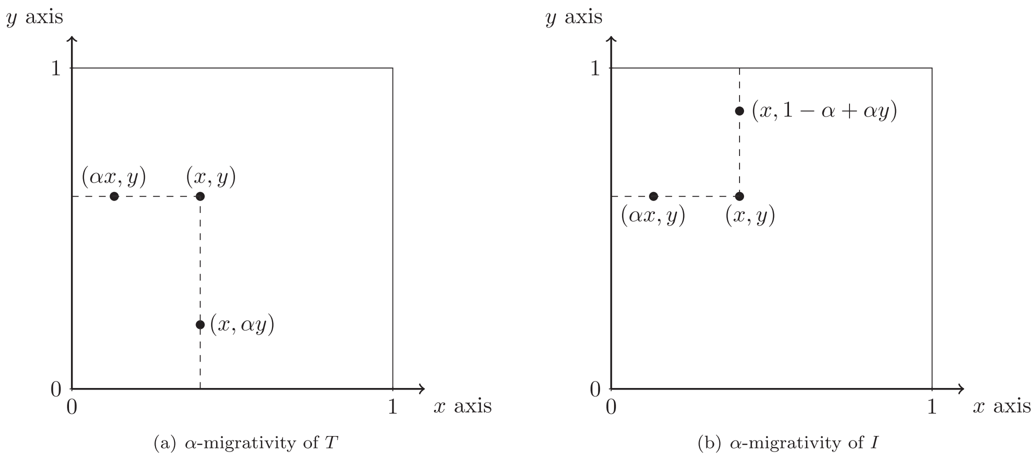

Fig. 1. A graphical interpretation of α-migrativity of t-norms and fuzzy implications.

We denote the set of all fuzzy implications by $\mathcal{FI}$.

In principle, (2) states that the output of the t-norm T should be invariant to scaling by α on either of the arguments. Note that while (2) implies commutativity

$$\begin{aligned} T(x,y) &= T(1\cdot x, y) = T(1, x\cdot y) \\ &= T(1, y\cdot x) = T(1\cdot y, x) = T(y,x), \quad x,y\in[0,1], \end{aligned}$$

the converse is not true, which can be seen from the fact that not all t-norms are migrative.

Note that for a t-norm T, the left and bottom boundaries are the constant zero function, i.e., $T(0,y)=0=T(x,0)$. Let $\alpha\in(0,1)$ and consider an arbitrary point (x, y) in the unit square. Then the α-migrativity of T states that if we move along the points joining (x, y) and $(0, y)$ or (x, y) and $(x, 0)$ proportionately, then their T values are equal. Taking a cue from the above interpretation of the α-migrativity and noting that I is bimonotonic, we propose the following definition of α-migrativity for fuzzy implications in a similar spirit (see Fig. 1).

Definition 2.2. Let $\alpha\in(0,1)$ be fixed. A fuzzy implication I is said to be α-migrative, if it satisfies

$$I(x\alpha, y) = I(x, 1-\alpha+\alpha y), \quad x,y\in[0,1]. \tag{8}$$

If I is α-migrative for every $\alpha\in(0,1)$, then I is said to be migrative.

Note that when $\alpha=0$ or $\alpha=1$, then every I is α-migrative. It can be easily seen that the only neutral fuzzy implication (i.e., when $I(1,y)=y$, for all $y\in[0,1]$) that satisfies (8) for all $\alpha\in(0,1)$ is the Reichenbach implication $I_{\mathbf{RC}}(x,y)=1-x+xy$. Writing (8) as follows

$$I(T_{\mathbf{P}}(x,\alpha), y) = I(x, I_{\mathbf{RC}}(\alpha, y)), \quad x,y\in[0,1], \tag{9}$$

we note the following, which in fact, leads us to the main motivations for the study of (6) and its further generalisations.

1. When $I=I_{\mathbf{RC}}$ in (9), then (9) reduces to the identity as the pair $(T_{\mathbf{P}}, I_{\mathbf{RC}})$ satisfies (5), see Remark 3.5(i).
2. One can immediately see that (8) is a special case of (6) where $I_1=I$ with the pair $(C=T_{\mathbf{P}}, I_2=I_{\mathbf{RC}})$ being fixed.
3. Finally, it is well known that given a t-norm T and a strong negation N one can obtain a fuzzy implication I, usually called the (S,N)-implication, as follows:

   $$I(x,y) = N(T(x, N(y))), \quad x,y\in[0,1].$$

   Let us apply the above to a t-norm T that is migrative. We obtain the following identity:

   $$\begin{aligned} I(x\alpha, y) &= N(T(x\alpha, N(y))) \\ &= N(T(x, \alpha N(y))) \\ &= N(T(x, N(N(\alpha N(y))))) \\ &= I(x, N(\alpha N(y))). \end{aligned} \tag{10}$$

   Taking the standard strong negation $N(x)=1-x$ the above identity reduces to (9). Further, we can write

   $$N(\alpha N(y)) = N(T_{\mathbf{P}}(\alpha, N(y))) = J(\alpha, y).$$

   Clearly, J is the (S,N)-implication obtained from the pair $(T_{\mathbf{P}}, N)$. Substituting J in (10) and generalising $T_{\mathbf{P}}$ to a conjunction C leads us to a further generalisation of (6) as will be discussed below.

3. The generalised law of importation (11)

We begin with some preliminaries.

Definition 3.1. A fuzzy conjunction C is a monotone extension of the boolean conjunction from {0, 1} to [0,1], i.e., C: $[0, 1]^2 \to [0, 1]$ is a binary operation such that

1. $C(x, 0) = 0 = C(0, x)$ for all $x \in [0, 1]$,
2. $C(1, 1) = 1$,
3. C is non-decreasing in each of the variables.

We denote the set of all fuzzy conjunctions by $\mathcal{C}$.

Definition 3.2 (see [2]). For an $I \in \mathcal{FI}$, the partial function N_I: $[0, 1] \to [0, 1]$ defined as $N_I(x) := I(x, 0)$ is called the natural negation obtained from I and it is a fuzzy negation.

Definition 3.3. Let $C \in \mathcal{C}$ and $I, J \in \mathcal{FI}$. The triple (C, I, J) is said to satisfy the generalised law of importation if it satisfies

$$I(C(x, \alpha), y) = I(x, J(\alpha, y)), \quad \alpha, x, y \in [0, 1]. \tag{11}$$

The first natural question to answer is the following:

Are there distinct fuzzy implications $I, J \in \mathcal{FI}$ such that the above equations are valid for some $C \in \mathcal{C}$?

Let us denote by I_0 the least and by I_1 the greatest fuzzy implications (see [2, Proposition 1.1.7]) which are given as follows:

$$I_0(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad I_1(x, y) = \begin{cases} 0, & \text{if } (x, y) = (1, 0), \\ 1, & \text{otherwise.} \end{cases}$$

For I_1 (I_0) we have some sufficient conditions on C and J so that the triplet (C, I_1, J) ((C, I_0, J)) satisfies (11).

Remark 3.4. Let $C \in \mathcal{C}$ and $J \in \mathcal{FI}$.

1. If C is an operation such that $C(x, y) = 1 \Longleftrightarrow x = y = 1$ and J is a function such that $J(x, y) = 0 \Longleftrightarrow x = 1$ and $y = 0$, then the triple (C, I_1, J) satisfies (11).
   To see this, note that it is clear that LHS / RHS of (11) with $I = I_1$ can either be 0 or 1. Now,
   $$\begin{aligned} \text{LHS of } (11) = 0 &\Longleftrightarrow I_1(C(x, \alpha), y) = 0 \Longleftrightarrow x = \alpha = 1 \text{ and } y = 0, \\ \text{RHS of } (11) = 0 &\Longleftrightarrow I_1(x, J(\alpha, y)) = 0 \Longleftrightarrow x = 1 \text{ and } J(\alpha, y) = 0 \Longleftrightarrow x = \alpha = 1 \text{ and } y = 0. \end{aligned}$$
2. It can be similarly shown that if C is an operation such that $C(x, y) = 0 \Longleftrightarrow x = 0$ or $y = 0$, i.e., C has no zero divisors, and J is a function such that $J(x, y) = 1 \Longleftrightarrow x = 0$ or $y = 1$, then the triple (C, I_0, J) satisfies (11).

Note that if $I = J$ and $C = T$ is a t-norm, then (11) reduces to (5). If only $I = J$ is considered, then from (11) for the triple (I, I, C) a weaker form of the law of importation for I, dealing with the couple (I, C), is obtained. To avoid new notations, we will call also this new law (i.e., $I(C(x, \alpha), y) = I(x, I(\alpha, y))$ for all $x, y, \alpha \in [0, 1]$) as the importation law, with notation (5). In the following remark, we only mention a few basic results and refer the readers to the following works for more details [17].

Remark 3.5.

1. Note that there exist pairs (C, J) that satisfy (5). For instance, let $C \in \mathcal{C}$ be associative and let $f \in (0, 1]$ be the left-neutral element of C. Then, for a strong negation N, if we define $J(x, y) = N(C(x, N(y)))$, then $J \in \mathcal{FI}$ and f is also the left-neutral element of J. It can be easily verified that the pair (C, J) satisfies (5). In fact,
   $$\begin{aligned} \text{LHS of } (5) &= J(C(x, \alpha), y) = N(C(C(x, \alpha), N(y))) \\ &= N(C(x, N(N(C(\alpha, N(y)))))) \\ &= J(x, N(C(\alpha, N(y)))) \\ &= J(x, J(\alpha, y)) = \text{RHS of } (5). \end{aligned}$$
   For instance, the following pairs $(T_{\mathbf{M}}, I_{\mathbf{KD}})$ and $(T_{\mathbf{P}}, I_{\mathbf{RC}})$, where
   $$T_{\mathbf{M}}(x, y) = \min(x, y), \quad I_{\mathbf{KD}}(x, y) = \max(1 - x, y),$$
   can be obtained as above and they satisfy (5).
2. Note also that not all pairs (C, J) that satisfy (5) need have N_J to be strong. For instance, consider the associative conjunction C given below and the corresponding J obtained from the strong negation $N(x) = 1 - x$:
   $$C(x, y) = \begin{cases} y, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad J(x, y) = \begin{cases} y, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Table 1
The satisfaction of (5) by either/both the pairs (C, J) and (C, I) is neither necessary nor sufficient for the triple (C, I, J) to satisfy (11).

C	J	I	(C, J) has (LI)	(C, I) has (LI)	(GLI)
C_4	I_4	J_4	✓	✓	✓
$C_\wedge^y$	$I_{0.5}$	$J_{0.5}$	×	✓	✓
C_5^1	$I_{\mathbf{D}}$	I_1	✓	×	✓
C_5^1	I_6^3	I_1	×	×	✓
C_6	I_0	I_1	✓	✓	×
C_5^1	I_1	$I_{\mathbf{D}}$	×	✓	×
C_6	$I_{\mathbf{D}}$	J_4	✓	×	×
C_6	$I_{\mathbf{LK}}$	$I_{\mathbf{RC}}$	×	×	×

Clearly, every $f \in (0, 1]$ is a left-neutral element of both C and J but N_J is not strong. It can be easily verified that the pair (C, J) satisfies (5). We can obtain further pairs (C, J) that satisfy (5), e.g. let us consider C_2 and its residual J_2 given below:

$$C_2(x,y) = \begin{cases} y, & x = 1, \\ 0, & \text{otherwise}, \end{cases} \qquad J_2(x,y) = \begin{cases} y, & x = 1, \\ 1, & \text{otherwise}. \end{cases}$$

Clearly, the pair (C_2, J_2) satisfies (5) but

$$N_{J_2}(x) = \begin{cases} 1, & x < 1, \\ 0, & x = 1, \end{cases}$$

is the greatest fuzzy negation and is not strong.

Remark 3.6. Note also that the satisfaction of (5) by either/both the pairs (C, J) and (C, I) is neither necessary nor sufficient for the triple (C, I, J) to satisfy (11) as can be seen from Table 1. In this table the following functions are used:

$$C_4(x,y) = \begin{cases} 0, & y = 0, \\ x, & \text{otherwise}, \end{cases} \qquad I_4(x,y) = \begin{cases} 1 - x^2, & y < 1, \\ 1, & y = 1, \end{cases}$$

$$C_\wedge^y(x,y) = \begin{cases} \min(x,y), & \min(x,y) \le 0.5, \\ y, & \text{otherwise}, \end{cases} \qquad J_4(x,y) = \begin{cases} 1, & y = 1, \\ 1 - x, & \text{otherwise}, \end{cases}$$

$$C_5^1(x,y) = \begin{cases} 0, & y = 0, \\ x, & \text{otherwise}, \end{cases} \qquad I_{0.5}(x,y) = \begin{cases} 1, & x \le 0.5 \text{ or } y = 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$C_6(x,y) = \begin{cases} 1, & (x,y) = (1,1), \\ \frac{xy}{2}, & \text{otherwise}. \end{cases} \qquad J_{0.5}(x,y) = \begin{cases} 1, & x \le 0.5, \\ y, & \text{otherwise}, \end{cases}$$

$$I_{\mathbf{D}}(x,y) = \begin{cases} 1, & x = 0 \text{ or } y = 1, \\ y, & \text{otherwise}, \end{cases} \qquad I_6^3(x,y) = \begin{cases} y, & x = 1, \\ 0, & x > 0 \text{ and } y = 0, \\ 1, & \text{otherwise}, \end{cases}$$

$$I_{\mathbf{LK}}(x,y) = \min(1, 1 - x + y), \quad \text{the Łukasiewicz implication.}$$

In this work, we discuss the relationships that exist between the satisfaction of (5) by (C, J), (C, I) and the satisfaction of (11) by the triple (C, I, J).

3.1. Solutions of (11) - some necessary conditions

We begin by presenting some necessary conditions on the triple (C, I, J) to satisfy (11). Towards this end we require the following properties of an I.

Definition 3.7 (see [2])**.** An $I \in \mathcal{FI}$ is said to satisfy

1. the left neutrality property if
$$I(1,y) = y, \quad y \in [0,1], \tag{12}$$
2. the law of contraposition (or in other words, the contrapositive symmetry) with respect to fuzzy negation N, if
$$I(x,y) = I(N(y), N(x)), \quad x,y \in [0,1]. \tag{13}$$

By $\mathcal{H}$ we denote the following family of functions, which are also sometimes referred to as unary aggregation functions:

$$\mathcal{H} = \{h : [0,1] \to [0,1] \mid h \text{ is non-decreasing with } h(0) = 0, h(1) = 1\}.$$

Lemma 3.8. *Let the triple (C, I, J) satisfy (11) and let $e \in (0, 1]$ be the left-neutral element of C.*

1. *There exists an $h \in \mathcal{H}$ such that $I(x, y) = h(J(x, y))$.*
2. *$N_I(x) = 0$ whenever $x \in [e, 1]$.*
3. *If e is also the left-neutral element of I, then $I = J$.*

Proof.

1. Substituting $x = e$ we obtain from (11),

$$I(C(e,\alpha), y) = I(\alpha, y) = I(e, J(\alpha, y)), \quad \alpha, x, y \in [0,1]. \tag{14}$$

Let $h(u) = I(e, u)$. Now, if $x, y, u, v \in [0, 1]$ such that $J(x, y) = J(u, v)$, then

$$I(x, y) = I(C(e, x), y) = I(e, J(x, y)) = I(e, J(u, v)) = I(C(e, u), v) = I(u, v).$$

Since I is well-defined so is h and hence $I(x, y) = h(J(x, y))$ from (14). Now,

$$0 = I(1, 0) = h(J(1, 0)) = h(0),$$
$$h(1) = I(e, 1) = 1,$$

and since I is non-decreasing in the second variable, h is non-decreasing. Thus $h \in \mathcal{H}$.

2. Further, $h(0) = I(e, 0) = 0$, and by the non-increasingness of I in the first variable, we have that $N_I(x) = 0$ whenever $x \in [e, 1]$. Hence N_I cannot be a strict negation.
3. Clearly, if e is also the left-neutral element of I, then for any $\alpha, y \in [0, 1]$, we have

$$I(\alpha, y) = I(C(e, \alpha), y) = I(e, J(\alpha, y)) = J(\alpha, y).$$

□

It is easy to see that if $e = 1$ and if I has the neutrality property (12), then $h = id_{[0,1]}$ and hence $I = J$.

3.2. Investigation of (11) when the pair (C, J) satisfies (5)

We begin with a sufficient condition for the satisfaction of (11).

Theorem 3.9. *Let $C \in \mathcal{C}$ and $I, J \in \mathcal{FI}$ and consider the following three statements:*

1. *The triple (C, I, J) satisfies (11).*
2. *The pair (C, J) satisfies (5).*
3. *There exists an $h \in \mathcal{H}$ such that $I = h(J)$.*

Further, let us consider the following two properties on C and h:

(a) *There exists $e \in (0, 1]$ such that $C(e, x) = x$ for all $x \in [0, 1]$, i.e., e is the left-neutral element of C.*
(b) *h is one-to-one.*

Then the following implications are valid:

(1) *If (a) is true, then (i) $\Longrightarrow$ (iii).*
(2) *Without any further assumption, (ii) and (iii) $\Longrightarrow$ (i).*
(3) *If (b) is true, then (iii) and (i) $\Longrightarrow$ (ii).*

Proof.

(1) Let $e \in (0, 1]$ be the left-neutral element of C. The implication (i) $\Longrightarrow$ (iii) follows from Lemma 3.8.
(2) Let (ii) and (iii) be true. To show that (11) is valid note that

$$\begin{aligned} \text{LHS of } (11) &= I(C(x,\alpha), y) = h(J(C(x,\alpha), y)) \\ &= h(J(x, J(\alpha, y))) \\ &= I(x, J(\alpha, y)) = \text{RHS of } (11). \end{aligned}$$

(3) From (iii) and (i) we have that $h(J(C(x,\alpha), y)) = h(J(x, J(\alpha, y)))$. Now, if h is an injection, then clearly, $J(C(x,\alpha), y) = J(x, J(\alpha, y))$ and the pair (C, J) satisfies (5).

□

Remark 3.10. We now discuss the necessity of the different conditions employed in Theorem 3.9.

1. Let us consider another pair (C_5, J_5), where

$$C_5(x,y) = \begin{cases} 0, & y < 0.5, \\ x, & y \geq 0.5, \end{cases} \quad J_5(x,y) = \begin{cases} 1-x, & y \leq 0.5, \\ 1, & y > 0.5. \end{cases}$$

As is shown below, this pair (C_5, J_5) satisfies (5),

$$\begin{aligned} \textit{LHS of } (5) &= J_5(C_5(x,\alpha), y) = J_5\left(\begin{cases} 0, & \alpha < 0.5, \\ x, & \alpha \geq 0.5 \end{cases}, y\right) \\ &= \begin{cases} 1, & y \leq 0.5, \alpha < 0.5, x \in [0,1], \\ 1-x, & y \leq 0.5, \alpha \geq 0.5, x \in [0,1], \\ 1, & y > 0.5, \alpha, x \in [0,1], \end{cases} \\ \text{RHS of } (5) &= J_5(x, J_5(\alpha, y)) = J_5\left(x, \begin{cases} 1-\alpha, & y \leq 0.5, \\ 1, & y > 0.5, \end{cases}\right) \\ &= \begin{cases} 1-x, & y \leq 0.5, 1-\alpha \leq 0.5, x \in [0,1], \\ 1, & y \leq 0.5, 1-\alpha > 0.5, x \in [0,1], \\ 1, & y > 0.5, \alpha, x \in [0,1], \end{cases} \quad = \text{LHS of } (5). \end{aligned}$$

Note that $N_J(x) = 1 - x$ is a strong negation but C_5 does not have any left-neutral element and hence the assumption (a) of Theorem 3.9 is not valid. Interestingly, if an $I \in \mathcal{FI}$ is such that the triple (C_5, I, J_5) satisfies (11), then there does exist an $h \in \mathcal{H}$ such that $I = h(J_5)$. To see this, let us consider such an I. Now, if $y = 0$, then from (11) for the triple (C_5, I, J_5) we have

$$\begin{aligned} \text{LHS of } (11) &= I(C_5(x,\alpha), 0) = N_I[C_5(x,\alpha)], \\ \text{RHS of } (11) &= I(x, J_5(\alpha, 0)) = I(x, 1-\alpha). \end{aligned}$$

From the above, we obtain
- if $\alpha < 0.5$, then $I(x, 1-\alpha) = 1$,
- if $\alpha \geq 0.5$, then $I(x, 1-\alpha) = N_I(x) = I(x, 0)$.

Thus, I is independent of y for any $y \in [0, 0.5]$ and we can represent I as follows

$$I(x,y) = \begin{cases} 1, & y > 0.5, \\ k(x), & y \leq 0.5, \end{cases}$$

where $k(x) = N_I(x) = I(x, 0)$. Clearly, k: $[0,1] \to [0,1]$ is a decreasing function such that $k(0) = 1$ and $k(1) = 0$. Now, let us define $h(x) = k(1-x)$. It is obvious that $h \in \mathcal{H}$. It can be shown that $I = h(J_5)$.

2. In Theorem 3.9, as the proof demonstrates, the assumption (ii) is not necessary, i.e., even if the pair (C, J) does not satisfy (5), there can exist an $h \in \mathcal{H}$ such that the triple (C, I, J) satisfies (11) with $I = h(J)$. In fact, even if both of (a) and (ii) are not true, we can still have that the triple (C, I, J) satisfies (11) with $I = h(J)$. To see this consider the following triple of functions $(C_5^1, I_1^0, J = I_6^3)$ (see Remark 3.6), where

$$I_1^0(x,y) = \begin{cases} 0, & x > 0 \text{ and } y = 0, \\ 1, & \text{otherwise}. \end{cases}$$

Clearly, C_5^1 has no left-neutral element. Also, it can be easily verified that the pair $(C_5^1, J = I_6^3)$ does not satisfy (5). For instance, if $\alpha \in (0,1)$, then

$$I_6^3(C_5^1(1,\alpha), \alpha) = I_6^3(1,\alpha) = \alpha \neq 1 = I_6^3(1, I_6^3(\alpha,\alpha)) = I_6^3(1,1).$$

However, for

$$h(u) = \begin{cases} 0, & u = 0, \\ 1, & \text{otherwise}, \end{cases} \tag{15}$$

we have $I_1^0 = h(J)$ and the triple $(C_5^1, I_1^0, J = I_6^3)$ satisfies (11), as shown below:

$$\begin{aligned} \text{LHS of } (11) &= I_1^0(C_5^1(x,\alpha), y) = I_1^0\left(\begin{cases} 0, & \alpha = 0, \\ x, & \alpha \neq 0, \end{cases} y\right) \\ &= \begin{cases} 1, & \alpha = 0, x, y \in [0,1], \\ I_1^0(x,y), & \alpha \neq 0, x, y \in [0,1], \end{cases} = \begin{cases} 0, & \alpha, x \in (0,1], y = 0, \\ 1, & \text{otherwise}, \end{cases} \\ \text{RHS of } (11) &= I_1^0(x, I_6^3(\alpha, y)) = I_1^0\left(x, \begin{cases} y, & \alpha = 1, \\ 0, & \alpha \in (0,1] \text{ and } y = 0, \\ 1, & \text{otherwise}, \end{cases}\right) \end{aligned}$$

$$= \begin{cases} I_1^0(x,y), & \alpha = 1, x, y \in [0,1], \\ I_1^0(x,0), & \alpha \in (0,1], y = 0, x \in [0,1], \\ I_1^0(x,1), & \text{otherwise}, \end{cases} = \begin{cases} 1, & \alpha = 1, x = 0, y \in [0,1], \\ 0, & \alpha = 1, x \in (0,1], y = 0, \\ 1, & \alpha = 1, x \in (0,1], y \in (0,1], \\ 1, & \alpha \in (0,1], y = 0, x = 0, \\ 0, & \alpha \in (0,1], y = 0, x \in (0,1], \\ 1, & \text{otherwise}, \end{cases}$$

$$= \begin{cases} 0, & \alpha, x \in (0,1], y = 0, \\ 1, & \text{otherwise}, \end{cases} = \text{LHS of (11)}.$$

3. Similarly the injectivity of h is not necessary for (iii) and (i) to imply (ii). For instance, let us consider the pair $I_{\mathbf{D}}, I_1 \in \mathcal{FI}$. Once again, for the function h in (15) we have that $I_1 = h(I_{\mathbf{D}})$. Clearly, h is not an injection. The triple $(C_5^1, I_1, J = I_{\mathbf{D}})$ satisfies (11) as can be seen below:

$$\text{LHS of (11)} = I_1(C_5^1(x,\alpha), y) = I_1\left(\begin{cases} 0, & \alpha = 0, \\ x, & \alpha \neq 0, \end{cases} y\right) = \begin{cases} 0, & \alpha \neq 0, x = 1, y = 0, \\ 1, & \text{otherwise}, \end{cases}$$

$$\text{RHS of (11)} = I_1(x, I_{\mathbf{D}}(\alpha, y)) = \begin{cases} 0, & x = 1 \text{ and } I_{\mathbf{D}}(\alpha, y) = 0, \\ 1, & \text{otherwise}, \end{cases}$$

$$= \begin{cases} 0, & \alpha \neq 0, x = 1, y = 0, \\ 1, & \text{otherwise}, \end{cases} = \text{LHS of (11)}.$$

Interestingly, even when h is not one-to-one, the pair $(C_5^1, I_{\mathbf{D}})$ satisfies (5), as shown below:

$$\text{LHS of (5)} = I_{\mathbf{D}}(C_5^1(x,\alpha), y) = I_{\mathbf{D}}\left(\begin{cases} 0, & \alpha = 0, \\ x, & \alpha \neq 0, \end{cases} y\right)$$

$$= \begin{cases} 1, & \alpha = 0, x, y \in [0,1], \\ I_{\mathbf{D}}(x,y), & \alpha \neq 0, x, y \in [0,1], \end{cases} = \begin{cases} 1, & \alpha = 0, x, y \in [0,1], \\ 1, & \alpha \neq 0, x = 0, y \in [0,1], \\ y, & \alpha \neq 0, x \neq 0, y \in [0,1], \end{cases}$$

$$\text{RHS of (5)} = I_{\mathbf{D}}(x, I_{\mathbf{D}}(\alpha, y)) = I_{\mathbf{D}}\left(x, \begin{cases} 1, & \alpha = 0, \\ y, & \alpha > 0, \end{cases}\right)$$

$$= \begin{cases} 1, & \alpha = 0, x, y \in [0,1], \\ 1, & \alpha \neq 0, x = 0, y \in [0,1], \\ y, & \alpha \neq 0, x \neq 0, y \in [0,1], \end{cases} = \text{LHS of (5)}.$$

3.3. *Investigations of (11) when the pair (C, I) satisfies (5)*

In this section, we investigate along similar lines as was done in the previous section, but with the assumption that the pair (C, I) satisfies (5). We begin by presenting the following result.

Theorem 3.11. *Let $C \in \mathcal{C}$ and $I, J \in \mathcal{FI}$ and consider the following statements:*

1. *The triple (C, I, J) satisfies (11).*
2. *The pair (C, I) satisfies (5).*
3. *$I = J$.*

Then the following implications are valid:

(1) *If I has a left-neutral element, then (i) and (ii) $\Longrightarrow$ (iii).*
(2) *Without any further assumption, we have two implications: (ii) and (iii) $\Longrightarrow$ (i); (iii) and (i) $\Longrightarrow$ (ii).*

Proof.

(1) Clearly, (i) and (ii) imply that $I(x, I(\alpha, y)) = I(x, J(\alpha, y))$ for all $x, \alpha, y \in [0, 1]$. Let I have a left-neutral element, i.e., there exists $e \in (0, 1]$ such that $I(e, y) = y$ for all $y \in [0, 1]$. Then, letting $x = e$ in the above equation, gives us that $I = J$ on $[0, 1]^2$.
(2) The implications (ii) and (iii) $\Longrightarrow$ (i) and (iii) and (i) $\Longrightarrow$ (ii) are trivially true.

□

Remark 3.12. It can be shown that the presence of a left-neutral element of I is not necessary, but only sufficient. To see this consider the following pair $(C_0^y, I_{\mathbf{RS}})$, where

$$C_0^y(x,y) = \begin{cases} 0, & x = 0 \text{ or } y = 0, \\ y, & \text{otherwise}, \end{cases} \quad I_{\mathbf{RS}}(x,y) = \begin{cases} 1, & x \leq y, \\ 0, & x > y. \end{cases}$$

The pair $(C_0^y, I_{\mathbf{RS}})$ satisfies (5) as can be seen from the following:

$$\text{LHS of (5)} = I_{\mathbf{RS}}(C_0^y(x,\alpha), y) = \begin{cases} 1, & C_0^y(x,\alpha) \le y, \\ 0, & C_0^y(x,\alpha) > y, \end{cases} = \begin{cases} 1, & \alpha \le y, \\ 0, & \alpha > y, \end{cases} = I_{\mathbf{RS}}(\alpha, y).$$

$$\text{RHS of (5)} = I_{\mathbf{RS}}(x, I_{\mathbf{RS}}(\alpha, y)) = \begin{cases} 1, & x \le I_{\mathbf{RS}}(\alpha,y), \\ 0, & x > I_{\mathbf{RS}}(\alpha,y), \end{cases}$$

$$= \begin{cases} 1, & x = 0 \text{ or } \alpha \le y, \\ 0, & x > 0 \text{ and } \alpha > y, \end{cases} = \begin{cases} 1, & \alpha \le y, \\ 0, & \alpha > y, \end{cases} = I_{\mathbf{RS}}(\alpha, y) = \text{LHS of (5)}.$$

Let us now assume that the triple $(C_0^y, I_{\mathbf{RS}}, J)$ satisfies (11). We will show that $J = I_{\mathbf{RS}}$.

- Let $\alpha,\ y \in (0, 1)$ such that $\alpha \le y$ and $J(\alpha, y) = \beta$. If $x = 1$, then

$$\text{LHS of (11)} = I_{\mathbf{RS}}(C_0^y(x,\alpha), y) = I_{\mathbf{RS}}(C_0^y(1,\alpha), y) = I_{\mathbf{RS}}(\alpha, y) = 1,$$
$$\text{RHS of (11)} = I_{\mathbf{RS}}(x, J(\alpha, y)) = I_{\mathbf{RS}}(1, J(\alpha, y)) = I_{\mathbf{RS}}(1, \beta).$$

  Since $I_{\mathbf{RS}}(1, \beta) = 1 \iff \beta = 1$ we have that $J(\alpha, y) = 1$ if $\alpha \le y$.
- Let $\alpha,\ y \in (0, 1)$ such that $\alpha > y$ and $J(\alpha, y) = \beta$. We claim that $\beta = 0$. If not, let $x = \beta > 0$. Then

$$\text{LHS of (11)} = I_{\mathbf{RS}}(C_0^y(x,\alpha), y) = I_{\mathbf{RS}}(C_0^y(\beta,\alpha), y) = I_{\mathbf{RS}}(\alpha, y) = 0,$$
$$\text{while, RHS of (11)} = I_{\mathbf{RS}}(x, J(\alpha, y)) = I_{\mathbf{RS}}(\beta, J(\alpha, y)) = I_{\mathbf{RS}}(\beta, \beta) = 1.$$

  Thus $J(\alpha, y) = 0$, when $\alpha > y$ and hence $J = I_{\mathbf{RS}}$.

4. Laws of migrativity: Some comments and concluding remarks

In this work, we have proposed and investigated a meaningful generalisation of the laws of migrativity to fuzzy implications, viz., (11) and (16), which lead us to two generalised versions of the law of importation (5). One of these two laws, viz., (11) has been investigated in depth in the context of the relationships between the satisfaction of (5) by the underlying fuzzy implication and (11). Some interesting perspectives on how these can also be obtained when using t-norms that are themselves migrative have also been presented.

In Section 2 we discussed how the α-migrativity (8) of fuzzy implications is related to the generalised law of importation (11). This was also the case when an α-migrative t-norm was used to obtain the corresponding (S,N)-implication. Interestingly, studying the α-migrativity of an R-implication w.r.t. an α-migrative t-norm T (for a given $\alpha \in (0, 1]$) leads us to the cross law of importation as shown below. Firstly, for each $x,\ y \in [0, 1]$ we have

$$I_T(x\alpha, y) = I_T(\alpha x, y) = \sup\{t \in [0,1] \mid T(\alpha x, t) \le y\} = \sup\{t \in [0,1] \mid T(x, \alpha t) \le y\}.$$

Since, as t varies over [0,1], αt varies over $[0, \alpha]$. Thus substituting $u = \alpha t$ we have

$$\begin{aligned} I_T(\alpha x, y) &= \sup\left\{\frac{u}{\alpha} \in [0,1] \mid T(x,u) \le y\right\} \\ &= \frac{1}{\alpha}\sup\{u \in [0,\alpha] \mid T(x,u) \le y\} \\ &= \min\left(1, \frac{1}{\alpha}\sup\{v \in [0,1] \mid T(x,v) \le y\}\right) \\ &= \min\left(1, \frac{I_T(x,y)}{\alpha}\right) \\ &= I_{\mathbf{GG}}(\alpha, I_T(x,y)). \end{aligned}$$

Of course, if $\alpha = 0$, then $I_T(\alpha x, y) = I_T(0, y) = 1$ and $I_{\mathbf{GG}}(\alpha, I_T(x, y)) = I_{\mathbf{GG}}(0, I_T(x, y)) = 1$, thus, in general, we obtain the following cross-law of importation, viz.,

$$I(C(x,\alpha), y) = J(x, I(\alpha, y)), \quad x, y, \alpha \in [0,1]. \tag{16}$$

Along similar lines, one can define the law of α-migrativity of t-conorms as follows.

Definition 4.1. Let $\alpha \in (0, 1)$ be fixed. A t-conorm S is said to be α-migrative, if it satisfies

$$S(1 - \alpha + \alpha x, y) = S(x, 1 - \alpha + \alpha y), \quad x, y \in [0,1]. \tag{17}$$

If S is α-migrative for every $\alpha \in (0, 1)$, then S is said to be migrative.

We intend to investigate the above functional equations in a future work.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Michał Baczyński: Formal analysis, Data curation, Writing - review & editing. **Balasubramaniam Jayaram:** Formal analysis, Data curation, Writing - review & editing. **Radko Mesiar:** Formal analysis, Data curation, Writing - review & editing.

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References

[1] J. Aczél, V. Belousov, M. Hosszú, Generalized associativity and bisymmetry on quasigroups, Acta Mathematica Academiae Scientiarum Hungaricae 11 (1960) 127–136.
[2] M. Baczyński, B. Jayaram, Fuzzy Implications, Vol. 231 of Studies in Fuzziness and Soft Computing, Springer-Verlag, Berlin Heidelberg, 2008.
[3] M. Baczyński, B. Jayaram, R. Mesiar, On alpha-migrativity of Fuzzy Implications and the Generalised Laws of Importation, in: SCIS&ISIS2018 in conjunction with ISWS2018 - Joint 10th International Conference on Soft Computing and Intelligent Systems and 19th International Symposium on Advanced Intelligent Systems, Toyama, Japan, December 5–8, IEEE, 2018, pp. 581–586.
[4] H. Bustince, B. De Baets, J. Fernández, R. Mesiar, J. Montero, A generalization of the migrativity property of aggregation functions, Inf Sci 191 (2012) 76–85.
[5] H. Bustince, J. Montero, R. Mesiar, Migrativity of aggregation functions, Fuzzy Sets Syst. 160 (6) (2009) 766–777.
[6] F. Durante, R.G. Ricci, Supermigrativity of aggregation functions, Fuzzy Sets Syst. 335 (2018) 55–66.
[7] F. Durante, P. Sarkoci, A note on the convex combinations of triangular norms, Fuzzy Sets Syst. 159 (2008) 77–80.
[8] J. Fodor, E.P. Klement, R. Mesiar, Cross-migrative triangular norms, Int. J. Intell. Syst. 27 (2012) 411–428.
[9] J. Fodor, I. Rudas, On continuous triangular norms that are migrative, Fuzzy Sets Syst. 158 (2007) 1692–1697.
[10] J. Fodor, I. Rudas, An extension of the migrative property for triangular norms, Fuzzy Sets Syst. 168 (2011) 70–80.
[11] B. Jayaram, On the law of importation $(x \wedge y) \longrightarrow z \equiv (x \longrightarrow (y \longrightarrow z))$ in fuzzy logic, IEEE Trans. Fuzzy Syst. 16 (2008) 130–144.
[12] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Vol. 8 of Trends in Logic, Kluwer Academic Publishers, Dordrecht, 2000.
[13] G. Maksa, Quasisums and generalized associativity, Aequationes Mathematicae 69 (2005) 6–27.
[14] S. Mandal, B. Jayaram, Bandler-Kohout subproduct with Yager's classes of fuzzy implications, IEEE Trans. Fuzzy Syst. 22 (2014) 469–482.
[15] M. Mas, M. Monserrat, D. Ruiz-Aguilera, J. Torrens, Migrative uninorms and nullnorms over t-norms and t-conorms, Fuzzy Sets Syst. 261 (2015) 20–32.
[16] M. Mas, M. Monserrat, J. Torrens, A characterization of (U,N), RU, QL and D-implications derived from uninorms satisfying the law of importation, Fuzzy Sets Syst. 161 (2010) 1369–1387.
[17] S. Massanet, J. Torrens, The law of importation versus the exchange principle on fuzzy implications, Fuzzy Sets Syst. 168 (2011) 47–69.
[18] R. Mesiar, V. Novák, Open problems from the 2nd International Conference on Fuzzy Sets Theory and Its Applications, Fuzzy Sets Syst. 81 (1996) 185–190.
[19] Y. Ouyang, Generalizing the migrativity of continuous t-norms, Fuzzy Sets Syst. 211 (2013) 73–83.
[20] J. Qiao, B.Q. Hu, On generalized migrativity property for overlap functions, Fuzzy Sets Syst. 357 (2019) 91–116.