# The ⊛-composition - A Novel Generating Method of Fuzzy Implications: An Algebraic Study

NAGESWARA RAO VEMURI

A Thesis Submitted to

Indian Institute of Technology Hyderabad

In Partial Fulfillment of the Requirements for

The Degree of Doctor of Philosophy

भारतीय प्रौद्योगिकी संस्थान हैदराबाद

Indian Institute of Technology Hyderabad

Department of Mathematics

September 2014

# Declaration

I declare that this written submission represents my ideas in my own words, and where ideas or words of others have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

V. Nageswara Rao

(Signature)

(NAGESWARA RAO VEMURI)

MA1CP001

(Roll No.)

## Approval Sheet

This Thesis entitled "The $\circledast$-composition - A Novel Generating Method of Fuzzy Implications: An Algebraic Study" by NAGESWARA RAO VEMURI is approved for the degree of Doctor of Philosophy from IIT Hyderabad

19.09.2014.

(Prof. Akhil Ranjan Roy) Examiner
Dept. of Mathematics
IIT Kharagpur

19/9/14

(Dr. Challa Subrahmanya Sastry) Examiner
Dept. Mathematics
IIT Hyderabad

(Dr. Balasubramaniam Jayaram ) Adviser
Dept. of Mathematics
IIT Hyderabad

19/9/14

(Dr. M.V.Panduranga Rao) Chairman
Dept. of Computer Science Engineering
IIT Hyderabad

# Acknowledgements

It is my privilege to have worked with Dr. Balasubramaniam Jayaram, my guide and my inspiration, who introduced me to the interesting field of fuzzy logic. It has been an honor to be his first Ph.D student. I express my best regards to him for teaching me many aspects of academics as well as life. I would like to express my sincere gratitude towards him for his continuous support and evaluation of my work.

Besides my guide, I am very much grateful to Dr. C.S.Sastry, Head, Dept. of Mathematics, not only for providing the necessary facilities but also the sufficient encouragement for me to carry out this research. Special thanks are also due to Prof. P.V.Subramanyam for giving me the opportunity to participate in many Symposia, Seminars and Workshops at IIT Madras.

I am highly obliged to Prof. R.Tandon (SMS, UoH) and Dr. M.V.Panduranga Rao (CSE, IITH), Dr. G.Ramesh, Dr. P.A.L.Narayana and Dr. D.V.Naidu (Dept.Math, IITH) for being on my Doctoral Committee and assessing my work constantly. I would like to acknowledge Dr. A Prabhakar for the many simulating conversations we have had.

I am thankful to Prof. R.Mesiar for inviting me to visit Dept. of Mathematics, STU, Slovakia in 2014 and giving the opportunity to discuss with his departmental colleagues.

I would like to thank A. Sai Hareesh (Research Scholar, SSSIHL) for the fruitful discussions we had when he visited Dept.of Mathematics, IITH in 2012.

I am in debt to acknowledge all my teachers who have guided and supported me throughout my education.

I am very fortunate to have worked with my many colleagues and friends Sayantan Mandal, Ramu Naidu, Santhosh, Deepika, Ganesh, Anji and Pradip in several departmental activities. The other juniors and M.Sc students have also extended their help to me. My best wishes to all of them.

I am delighted to thank my friends Jagga, Pankaj, Karthik, Raja, Srinivas, Sudharshan, Kashfuddoja, Rajesh, Prasad and Aravind whose companionship has left with me an infinite number of memories of my stay in the hostel.

I would like to thank also Anupama, Vandana, Upender, Ashok, Pattabhi, Teja, Sudhakar, Ravi Gujjula, Durga Rao, Srinivas, Bhaskar, Mahender, Chiru and Venkat for encouraging me throughout my Ph.D.

I would also like to thank my family friends Bhaskar, Raju, Malli, Showry, Naga Raju and Suresh for making my stay at home enjoyable.

I express my sincere affection towards my brothers and sisters, Pavani, Sarojini, Babu Rao, Sharon, Ratna, Sowmya and John, whose love has kept me happy and cheerful.

Words cannot express my feeling for my parents, Sri. V.K.Raju and Smt. Ruthamma. They have been my love, courage and strength since my childhood. Special thanks to my grand parents and all the relatives.

The financial support given by CSIR HRDG (INDIA) and facilities provided by IIT Hyderabad to carry out this research work are also greatly acknowledged.

# Dedication

To my parents for their love, support and care.

# Abstract

Fuzzy implications are one of the two most important fuzzy logic connectives, the other being t-norms. They are a generalisation of the classical implication from two-valued logic to the multi-valued setting.
A binary operation $I$ on $[0,1]$ is called a *fuzzy implication* if

(i) $I$ is decreasing in the first variable,

(ii) $I$ is increasing in the second variable,

(iii) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

The set of all fuzzy implications defined on $[0,1]$ is denoted by $\mathbb{I}$.

Fuzzy implications have many applications in fields like *fuzzy control, approximate reasoning, decision making, multivalued logic, fuzzy image processing*, etc. Their applicational value necessitates new ways of generating fuzzy implications that are fit for a specific task. The generating methods of fuzzy implications can be broadly categorised as in the following:

**(M1):** From binary functions on $[0,1]$, typically other fuzzy logic connectives, viz., $(S,N)$-, $R$-, $QL$-implications,

**(M2):** From unary functions on [0,1], typically monotonic functions, for instance, Yager's $f$-, $g$-implications, or from fuzzy negations,

**(M3):** From existing fuzzy implications.

**Motivation for this thesis:**

Among the above generating methods, the third one, namely the generating method **(M3)**, has an interesting fallout. This generating method not only generates new fuzzy implications from fuzzy implications but also, often, gives rise to algebraic structures on the set $\mathbb{I}$. All the existing generating methods of fuzzy implications from fuzzy implications involve either other fuzzy logic connective(s) or parameter(s). Moreover the richest algebraic structure that is available so far is a semigroup, that too, on a subset of $\mathbb{I}$, under some assumptions. In this study our objectives are the following:

**(Obj 1):** Propose new generating method(s) of fuzzy implications from fuzzy implications without using other fuzzy logic connectives or parameters.

**(Obj 2):** Ensure that the generating method(s) that we propose would give richer algebraic structures on $\mathbb{I}$ that would allow us to glean newer and better perspectives of fuzzy implications.

**The research work carried out in this thesis:**

The contents of this thesis can be subdivided into the following three parts.

**Part-I : The $\circledast$-composition-A novel generating method and a monoid structure.**

(i) For any $I, J \in \mathbb{I}$, we propose their $\circledast$-composition $I \circledast J$ in the following manner:

$$(I \circledast J)(x, y) = I(x, J(x, y)), \qquad x, y \in [0, 1].$$

Then we show that $I \circledast J$ is indeed a fuzzy implication, i.e., $I \circledast J \in \mathbb{I}$. Note that this clearly achieves our first objective **(Obj 1)**, since we do not use any other fuzzy logic connectives or parameters.

(ii) Looking at the $\circledast$-composition as a binary operation on $\mathbb{I}$, we show that $(\mathbb{I}, \circledast)$ is, indeed, a non-idempotent monoid, the richest algebraic structure on whole of $\mathbb{I}$ known so far, thus achieving partially, our second objective, **(Obj 2)**.

Since the $\circledast$-composition on $\mathbb{I}$ can be looked in two different ways, viz, a generating method of fuzzy implications and a binary operation on the set $\mathbb{I}$, we explore the $\circledast$-composition on $\mathbb{I}$ along these two aspects in Part II and Part III, respectively.

**Part-II : The $\circledast$-composition w.r.to properties, functional equations, families and powers.**

(i) We study the $\circledast$-closures of fuzzy implications with respect to some desirable properties of fuzzy implications and some functional equations involving fuzzy implications.

(ii) We study the effect of $\circledast$-composition on fuzzy implications that are obtained from the remaining two generating methods, viz., **(M1)** and **(M2)**.

(iii) Given an $I \in \mathbb{I}$, one could always compose $I$ with itself to obtain $I \circledast I \in \mathbb{I}$. Using the associativity of $\circledast$, we define the $n^{\text{th}}$ powers of $I$ and study the convergence of the powers of fuzzy implications in the limiting case.

(iv) Further, we investigate whether the self-composition of fuzzy implications w.r.to $\circledast$ leads to newer fuzzy implications. In the case where the self-composition of fuzzy implications generates new fuzzy implications, we examine the powers of of fuzzy implications w.r.to the desirable properties and functional equations.

**Part-III: Algebraic aspects of $\mathbb{I}$ w.r.to the $\circledast$-composition.**

(i) From Part I, we know that the generative method $\circledast$ that we have proposed makes $\mathbb{I}$ a non-idempotent monoid. Unfortunately, due to the presence of zero elements, we note that $(\mathbb{I}, \circledast)$ is not a group and can not be made a group by some well known techniques like Grothendieck construction due to the lack of commutativity. However, we characterise the largest subgroup $\mathbb{S}$ of $\mathbb{I}$ and obtain its representation.

(ii) Based on the representation of $\mathbb{S}$, we define group actions of $\mathbb{S}$ on $\mathbb{I}$ and show that these group actions lead to the following:

   (a) Representations of Yager's families of fuzzy implications in terms of the following three basic fuzzy implications, namely, the Yager implication, the Reichenbach implication and Goguen implication.

(b) An algebraic connotation of some conjugacy classes of fuzzy implications that were proposed earlier.

(iii) We know that any monoid is injectively homomorphic to the set of all right translations defined on it. Since $(\mathbb{I}, \circledast)$ is a monoid, we determine the set of all right translation homomorphisms on $(\mathbb{I}, \circledast)$ which helps us to characterise few important subsets like center and the set of right zero elements of $(\mathbb{I}, \circledast)$.

**Highlights of the work contained in this thesis:**

- Proposed a novel generating method of fuzzy implications without using any other fuzzy logic connective(s), parameters or transformations.
- Obtained the richest algebraic structure known so far, on the whole of $\mathbb{I}$, namely, a non-idempotent monoid.
- For the first time, showed that the Yager's families of fuzzy implications can be seen as pseudo-conjugates of three basic fuzzy implications.
- Conjugacy transformations, that were proposed earlier from a purely analytical perspective, were shown to have clear algebraic connotations.

**Publications from this thesis work:**

**Journal Publications**

1. **N. R. Vemuri**., B. Jayaram.: Representations through a monoid on the set of fuzzy implications. Fuzzy Sets and Systems 247 (2014) 51-67.
2. **N. R. Vemuri**., B. Jayaram.: The $\circledast$-composition of fuzzy implications: Closures with respect to Properties, Powers and Families. **Accepted for publication** in *Fuzzy Sets and Systems*. doi:10.1016/j.fss.2014.10.004.
3. **N. R. Vemuri**., B. Jayaram.: Homomorphisms on the Monoid of Fuzzy Implications and the Iterative Functional Equation $I(x, I(x, y)) = I(x, y)$. **Revised version submitted** to *Information Sciences*.
4. **N. R. Vemuri**., B. Jayaram.: Bijective transformations of of Fuzzy Implications- An algebraic perspective. **Submitted** to *Kybernetika*.

**Conference Publications**

1. **N. R. Vemuri**., B. Jayaram.: Fuzzy Implications: Novel generating process and the consequent algebras. S. Greco *et al.* (Eds.): IPMU, Part II, CCIS 298, pp. 358-367. Springer-Verlag. Heidelberg 2012.
2. **N. R. Vemuri**., B. Jayaram.: Homomorphisms on the monoid $(\mathbb{I}, \circledast)$ of fuzzy implications. in: The 2013 IEEE International Conference on Fuzzy Systems FUZZ-IEEE 2013, Hyderabad, 7-10, July 2013. doi.10.1109/FUZZ-IEEE.2013.6622436.

3. **N. R. Vemuri**., B. Jayaram.: Homomorphisms on the monoid of fuzzy implications $(\mathbb{I}, \circledast)$ - A complete characterization, in: Maji *et al.* (Eds.), PReMI 2013, Springer. pp. 563-568.

4. **N. R. Vemuri**., B. Jayaram.: Conjugacy relation via the group action on the monoid of fuzzy implications. in : Klement *et al.*(Eds). International conference on Fuzzy Sets : Theory and Applications, FSTA 2014. pp. 112. January 26-31.

# Contents

# Part I

# Fuzzy Implications and the $\circledast$-composition

# Chapter 1

# Fuzzy Implications and Generating Methods : A Brief Review

As complexity rises, precise statements lose meaning
and meaningful statements lose precision
– Lotfi A. Zadeh (1921-)

Fuzzy implications, along with triangular norms (t-norms, in short) form the two most important fuzzy logic connectives. They are a generalisation of the classical implication and conjunction, respectively, to multivalued logic and play an equally important role in fuzzy logic as their counterparts in classical logic. The truth table of classical implication is given in Table 1.1.

| $p$ | $q$ | $p \Longrightarrow q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

Table 1.1: Truth Table of Classical Implication Operator

In this chapter we begin with recalling the definition of fuzzy implications and present few examples of fuzzy implications. Then we briefly review the different generating methods of fuzzy implications in Section 1.2. In Section 1.3, we recall the existing generative methods of fuzzy implications from fuzzy implications and the algebraic structures on the set of all fuzzy implications thus obtained from them. Based on the discussions in these sections, we derive the main motivation behind this thesis in Section 1.4.

## 1.1 Fuzzy Implications

In the literature one can find many equivalent definitions of fuzzy implications. In the following we recall its definition from [8] (see also [47, 31]).

**Definition 1.1.1** ([8], Definition 1.1.1). *A function* $I\colon [0,1]^2 \longrightarrow [0,1]$ *is called a* fuzzy implication *if it satisfies, for all* $x, x_1, x_2, y, y_1, y_2 \in [0,1]$, *the following conditions:*

$$\text{if } x_1 \le x_2, \text{ then } I(x_1, y) \ge I(x_2, y)\ , \text{ i.e., } I(\,\cdot\,, y) \text{ is decreasing}\ , \tag{I1}$$

$$\text{if } y_1 \le y_2, \text{ then } I(x, y_1) \le I(x, y_2)\ , \text{ i.e., } I(x, \,\cdot\,) \text{ is increasing}\ , \tag{I2}$$

$$I(0,0) = 1\ , I(1,1) = 1\ , I(1,0) = 0\ . \tag{I3}$$

From Definition 1.1.1, it is clear that a fuzzy implication, when restricted to $\{0,1\}$, coincides with the classical implication. The set of all fuzzy implications will be denoted by $\mathbb{I}$. Table 1.2 (see also [8]) lists some examples of basic fuzzy implications.

| Name | Formula |
|---|---|
| Łukasiewicz | $I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$ |
| Gödel | $I_{\mathbf{GD}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{if } x > y \end{cases}$ |
| Reichenbach | $I_{\mathbf{RC}}(x,y) = 1 - x + xy$ |
| Kleene-Dienes | $I_{\mathbf{KD}}(x,y) = \max(1-x, y)$ |
| Goguen | $I_{\mathbf{GG}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \frac{y}{x}, & \text{if } x > y \end{cases}$ |
| Rescher | $I_{\mathbf{RS}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ 0, & \text{if } x > y \end{cases}$ |
| Yager | $I_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if } x > 0 \text{ or } y > 0 \end{cases}$ |
| Weber | $I_{\mathbf{WB}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$ |
| Fodor | $I_{\mathbf{FD}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \max(1-x, y), & \text{if } x > y \end{cases}$ |
| Smallest FI | $I_{\mathbf{0}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } \ y = 1 \\ 0, & \text{if } x > 0 \text{ and } \ y < 1 \end{cases}$ |
| Largest FI | $I_{\mathbf{1}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } \ y > 0 \\ 0, & \text{if } x = 1 \text{ and } \ y = 0 \end{cases}$ |
| Most Strict | $I_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0 \end{cases}$ |

Table 1.2: Examples of fuzzy implications

It is clear that Definition 1.1.1 of fuzzy implication is only a particular generalisation of classical implication from $\{0,1\}$ to the multivalued setting, namely, the unit interval $[0,1]$. Note that Definition 1.1.1 can also be generalised to posets, lattices, chains that are either finite, infinite, bounded or even unbounded (see, [3, 14, 24, 25, 51, 49] ). Further, it can also be generalised to fuzzy sets, interval valued fuzzy sets, type-2 fuzzy sets, etc., which lead to different generalisations of classical implications (see, [2, 13, 16, 26, 45, 50, 52, 23]). Moreover, it should be emphasized that such generalisations are also useful in various contexts and it is worthwhile to study them in a comprehensive manner.

However, in this thesis, we restrict ourselves to the study of fuzzy implications that are defined only on the unit interval $[0, 1]$ (i.e., those that are defined in Definition 1.1.1).

Fuzzy implications play an important role in approximate reasoning, fuzzy control, decision theory, control theory, expert systems, fuzzy mathematical morphology, fuzzy image processing, etc. - see for example [15, 19, 39, 40, 80, 81, 83, 84] or the recent monograph exclusively devoted to fuzzy implications [8]. Hence, there is always a need for a panoply of fuzzy implications satisfying different properties that make them valuable in the appropriate context. The different generation methods of fuzzy implications can be broadly classified into the following three categories, viz,

**(M1)** From binary functions on $[0, 1]$, typically other fuzzy logic connectives, viz., $(S, N)$-, $R$-, $QL$-implications (see [8]),

**(M2)** From unary functions on [0,1], typically monotonic functions, for instance, Yager's $f$-, $g$-implications (see [83]), or from fuzzy negations [11, 38, 58, 72],

**(M3)** From fuzzy implications (see [6, 10, 27, 28, 30, 37, 62]).

## 1.2 Fuzzy Implications - Some Generating Methods

The first generating method mentioned above was one of the earliest approaches taken to obtain fuzzy implications. Herein, a fuzzy implication is obtained from given fuzzy logic connective(s). For instance, given a t-conorm $S$ and a fuzzy negation $N$, one obtains $(S, N)$-implications, which are a generalisation of the material implication in the classical logic. Similarly, from only a t-norm $T$ one can obtain a fuzzy implication as its residuation - once again this is a direct generalisation of the two-valued implication in intuitionistic logic. For definitions of these families, please see Definitions 5.1.1 and 5.2.1 in Chapter 5. The properties, characterisations, representations, intersections between these families of fuzzy implications and their generalisations have been studied by many authors in various contexts and for an up-to-date analysis of these families of fuzzy implications, please see, Chapters 2 - 5 of [8] or the survey paper [9].

A second method for obtaining fuzzy implications was firstly proposed by Yager [83], see also [79]. Two methods of generating fuzzy implications by using monotone functions defined from $[0, 1]$ to $[0, \infty]$ were presented by Yager. These have now come to be known as the $f$- and $g$-generated implications. Their role in approximate reasoning was studied in [83] (also see [55] for more details). Characterisation results of Yager's classes of fuzzy implications were obtained only recently in [61]. This approach has also been successfully applied to more general monotone functions, see for instance [82], and also by a combination of unary monotone functions and binary fuzzy logic connectives, see for instance, [1, 21, 41, 59, 73].

While the above two methods help us in creating fuzzy implications from other unary or binary functions, in general, a third method exists which generates fuzzy implications from given fuzzy implications. In the literature, one finds many generating methods along this approach. Once again, we could divide such approaches as either generative or constructive.

(i) By *generative* methods, we refer to those works which propose a closed form formula for obtaining new fuzzy implications from given ones, often with the help of other fuzzy logic connectives. Interestingly, these methods not only generate fuzzy implications but also, often, impose algebraic structures on the set $\mathbb{I}$. These algebraic structures can present a different perspective of fuzzy implications.

The earliest such works were due to the group of Drewniak, see for instance, [6, 27, 28]. In these works, it was shown that the proposed operations, viewed as binary operations on the set $\mathbb{I}$, give rise to some lattice and group theoretic structures, see, Section 1.3 for more details.

There exist works that have proposed other generative methods but whose algebraic underpinnings are either non-existent or as yet unknown - see, for instance, [10, 30, 72] or the works of Hliněná *et. al.* [37, 38].

(ii) By *constructive* methods, we refer to those methods that somehow depend on the underlying geometry to construct a fuzzy implication from a pair of fuzzy implications, often by specifying the values over different sub-regions of $[0,1]^2$. For instance, the threshold and vertical threshold generation methods of Massanet and Torrens [56, 60, 62, 63] fall under this category. Note that these methods do not always have any algebraic connotations. For more on these constructions, we refer the readers to the recent excellent survey of Massanet and Torrens [64].

## 1.3 Fuzzy Implications from Fuzzy Implications : Existing Generative Methods

As noted earlier, we are interested in generative methods for obtaining new fuzzy implications from existing ones. In the literature only a few such generative methods are known. In this section, we begin by giving a brief review of the existing methods and the algebraic structures they produce on $\mathbb{I}$. For more details, please see, Chapter 6 of [8].

### 1.3.1 Lattice of Fuzzy Implications

The lattice operations of *meet* and *join* were the first to be employed towards generating new fuzzy implications. Bandler and Kohout [12] obtained fuzzy implications by taking the meet and join of a given fuzzy implication I and its reciprocal $I_N(x,y) = I(N(y), N(x))$, where $N$ is a strong negation (for the definition of a strong negation, see, Definition 3.3.6). This method has been discussed extensively under the topic of contrapositivisation of fuzzy implications, see Fodor [30], Balasubramaniam [10].

In general, given $I, J \in \mathbb{I}$, we consider the following 'meet' and 'join' operations:

$$(I \vee J)(x,y) = \max(I(x,y), J(x,y)), \qquad x,y \in [0,1], \qquad I,J \in \mathbb{I}\,, \tag{Latt-Max}$$

$$(I \wedge J)(x,y) = \min(I(x,y), J(x,y)), \qquad x,y \in [0,1], \qquad I,J \in \mathbb{I}\,. \tag{Latt-Min}$$

**Theorem 1.3.1** ([8], Theorem 6.1.1)**.** *The family* $(\mathbb{I}, \leq)$ *is a complete, completely distributive lattice with the lattice operations* (Latt-Max) *and* (Latt-Min)*, where* $\leq$ *is the usual pointwise order on the set of binary functions.*

In fact, one can use any aggregation operator $A$ (see, Definition 1.1, [34]) instead of $\min$ or $\max$ in (Latt-Max) and (Latt-Min), respectively, to obtain a new fuzzy implication as follows:

$$(I \overset{A}{\circ} J)(x,y) = A(I(x,y), J(x,y)), \qquad x, y \in [0,1], \qquad I, J \in \mathbb{I}\ . \tag{1.1}$$

Please refer to the recent work of Reiser *et al.* [69] for the conditions on $A$ for $(I \overset{A}{\circ} J)$ to be a fuzzy implication and the properties the obtained fuzzy implications possess. However, whether they lead to any algebraic or order-theoretic structures on $\mathbb{I}$ with an arbitrary aggregation $A$ is yet to be explored. For more details and other similar works, please refer to [18, 20, 67, 68].

### 1.3.2 Convex Combinations of Fuzzy Implications

We know that fuzzy implications are basically binary functions on $[0,1]$. Thus one can define convex combinations of fuzzy implications in the usual manner.

**Definition 1.3.2.** *Convex combination of two fuzzy implications $I, J \in \mathbb{I}$ is defined as*

$$K(x,y) = \lambda I(x,y) + (1-\lambda) J(x,y), \qquad x, y \in [0,1],\ \lambda \in [0,1]\ .$$

**Theorem 1.3.3** ([8], Theorem 6.2.2)**.** *Convex combination of any two fuzzy implications is also a fuzzy implication. Thus the set $\mathbb{I}$ of all fuzzy implications is a convex set.*

### 1.3.3 Conjugacy Classes of Fuzzy Implications

Let $\Phi$ denote the set of all increasing bijections on $[0,1]$. Note that if $([0,1], *)$ and $([0,1], \diamond)$ are two ordered groupoids, then $\varphi(x * y) = \varphi(x) \diamond \varphi(y)$ is a groupoid homomorphism for any $\varphi \in \Phi$. Conversely, given a binary groupoid operation, one could obtain new groupoid operations from the above as follows: $x * y = \varphi^{-1}\left(\varphi(x) \diamond \varphi(y)\right)$.

Viewing a fuzzy implication as a groupoid on $[0,1]$, Baczyński and Drewniak [5, 70] obtained further fuzzy implications from given ones as above.

**Definition 1.3.4** ([5])**.** *For any $\varphi \in \Phi$ and $I \in \mathbb{I}$, we define the $\varphi$-conjugate of $I$ by,*

$$I_\varphi(x,y) = \varphi^{-1}\left(I(\varphi(x), \varphi(y))\right), \qquad x, y \in [0,1]\ .$$

**Theorem 1.3.5** ([8], Theorem 6.3.1)**.** *If $I \in \mathbb{I}$ then for every $\varphi \in \Phi$, the function $I_\varphi \in \mathbb{I}$.*

**Definition 1.3.6** (cf. [27], [8])**.** *A fuzzy implication $I$ is called* self-conjugate *or* invariant *if $I_\varphi = I$, for all $\varphi \in \Phi$.*

Let $\mathbb{I}_{\mathbf{inv}}$ denote the set of all invariant fuzzy implications.

**Theorem 1.3.7** (cf. [27], [8], Theorem 6.3.8)**.** *$\mathbb{I}_{\mathbf{inv}}$ is a distributive lattice.*

### 1.3.4 Compositions of Fuzzy Implications

**Definition 1.3.8** ([48], Definition 3.1)**.** *A binary operation $T(S)\colon [0,1]^2 \longrightarrow [0,1]$ is called a* t-norm (t-conorm)*, if it is increasing in both the variables, commutative, associative and has $1(0)$ as the neutral element.*

| Name | Formula |
|---|---|
| minimum | $T_{\mathbf{M}}(x,y) = \min(x,y)$ |
| algebraic product | $T_{\mathbf{P}}(x,y) = xy$ |
| Łukasiewicz | $T_{\mathbf{LK}}(x,y) = \max(x+y-1,0)$ |
| drastic product | $T_{\mathbf{D}}(x,y) = \begin{cases} 0, & \text{if } x,y \in [0,1) \\ \min(x,y), & \text{otherwise} \end{cases}$ |
| nilpotent minimum | $T_{\mathbf{nM}}(x,y) = \begin{cases} 0, & \text{if } x+y \leq 1 \\ \min(x,y), & \text{otherwise} \end{cases}$ |

Table 1.3: Basic t-norms

| Name | Formula |
|---|---|
| maximum | $S_{\mathbf{M}}(x,y) = \max(x,y)$ |
| probabilistic sum | $S_{\mathbf{P}}(x,y) = x+y-xy$ |
| Łukasiewicz | $S_{\mathbf{LK}}(x,y) = \min(x+y,1)$ |
| drastic sum | $S_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x,y \in (0,1] \\ \max(x,y), & \text{otherwise} \end{cases}$ |
| nilpotent maximum | $S_{\mathbf{nM}}(x,y) = \begin{cases} 1, & \text{if } x+y \geq 1 \\ \max(x,y), & \text{otherwise} \end{cases}$ |

Table 1.4: Basic t-conorms

In the infix notation, usually a $T(S)$ is denoted by $*(\oplus)$. Tables 1.3 and 1.4 list a few of the t-norms and t-conorms respectively that are considered basic in the literature, which will also be useful in the sequel.

Note that any binary function $F : [0,1]^2 \longrightarrow [0,1]$ can be treated as a binary fuzzy relation on $[0,1]$. Once again, treating a fuzzy implication $I$ as a fuzzy relation, Baczyński and Drewniak [6] employed relational composition operators to obtain new fuzzy implications.

**Definition 1.3.9** (cf. [6], [8], Definition 6.4.1). *Let $I, J \in \mathbb{I}$ and $*$ be a t-norm. Then* sup–$*$ *composition of $I, J$ is given as follows:*

$$(I \overset{*}{\circ} J)(x,y) = \sup_{t\in[0,1]} (I(x,t) * J(t,y)), \qquad x,y \in [0,1]\,. \tag{COMP}$$

**Theorem 1.3.10** (cf. [6],[8], Theorem 6.4.4). *Let $*$ be a t-norm. If $I, J \in \mathbb{I}$, then $(I \overset{*}{\circ} J) \in \mathbb{I} \Longleftrightarrow (I \overset{*}{\circ} J)(1,0) = 0$.*

**Theorem 1.3.11** ([8], Theorem 6.4.12). *If $*$ is a left continuous (l.c.) t-norm, then $\overset{*}{\circ}$ is associative. Thus $(\mathbb{I}, \overset{*}{\circ})$ is a semigroup.*

From Theorem 1.3.10, we note that (COMP) gives a semigroup only on a subset of $\mathbb{I}$. For a further generalisation of this generative method, see Drewniak and Sobera [29]. However, no newer algebraic structures are known.

## 1.4 Motivation for the Proposed Work

In Section 1.3, we have recalled the existing generative methods of fuzzy implications from fuzzy implications and the algebraic structures of $\mathbb{I}$ thus obtained from them. Moreover, from these generating methods the following two facts emerge:

(i) All the existing generative methods of fuzzy implications involve either other fuzzy logic connective(s), parameter(s) or transformations.

(ii) The richest algebraic structure that is available so far is a semigroup, that too, on a subset of $\mathbb{I}$, under some assumptions.

Based on the above two observations we derive our motivation for this research work. Specifically, our objectives for this thesis are the following:

**(Obj 1) :** Propose new generative method(s) of fuzzy implications from fuzzy implications without using other fuzzy logic connectives or parameters.

**(Obj 2) :** Ensure that the generative method(s) that we propose would give richer algebraic structures on $\mathbb{I}$ that would allow us to glean newer and better perspectives of fuzzy implications.

# Chapter 2

# The $\circledast$-composition of Fuzzy Implications

*I have created a new universe from nothing.*
– Janos Bolyai (1802-1860)

From our stated motivation in Section 1.4, it is clear that we are interested in novel generative methods of fuzzy implications from fuzzy implications that would impose richer algebraic structures on $\mathbb{I}$. In this chapter we accomplish the same. In Section 2.1, given any two fuzzy implications we propose a novel composition called the $\circledast$-composition and show that it is also a fuzzy implication. We show that the $\circledast$-composition, when looked at as a binary operation on $\mathbb{I}$, makes the whole of $\mathbb{I}$, a non-idempotent monoid. Towards the end, we give the outline of the thesis in Section 2.3.

## 2.1 The $\circledast$-composition on $\mathbb{I}$ : A Novel Generative Method

In the following given two fuzzy implications $I, J$ we propose the $\circledast$- composition $I \circledast J$ of $I, J$ and show that $I \circledast J$ is indeed a fuzzy implication.

**Definition 2.1.1.** *Given $I, J \in \mathbb{I}$, define $I \circledast J\colon [0,1]^2 \longrightarrow [0,1]$ as*

$$(I \circledast J)(x,y) = I(x, J(x,y)), \qquad x, y \in [0,1]\ .$$

**Theorem 2.1.2.** *The function $I \circledast J$ is a fuzzy implication, i.e., $I \circledast J \in \mathbb{I}$.*

*Proof.* Let $I, J \in \mathbb{I}$ and $x_1, x_2, y \in [0,1]$.

(i) Let $x_1 \leq x_2$. Then $J(x_1, y) \geq J(x_2, y)$. Also we have, $I(x_1, J(x_1, y)) \geq I(x_2, J(x_2, y))$. Thus $(I \circledast J)(x_1, y) \geq (I \circledast J)(x_2, y)$. Similarly, one can show that $I \circledast J$ is increasing in the second variable.

(ii) $(I \circledast J)(0,0) = I(0, J(0,0)) = I(0,1) = 1.$
$(I \circledast J)(1,1) = I(1, J(1,1)) = I(1,1) = 1.$
$(I \circledast J)(1,0) = I(1, J(1,0)) = I(1,0) = 0.$

Thus $I \circledast J$ is a fuzzy implication. $\square$

Note that from Theorem 2.1.2, $\circledast$ is closed on the set $\mathbb{I}$, i.e., it does indeed generate fuzzy implications from given pair of fuzzy implications. Moreover this generating method involves neither other fuzzy logic connectives nor parameters, thus clearly achieving our first objective (**Obj 1**).

Table 2.1 shows some new fuzzy implications obtained from some of the basic fuzzy implications listed in Table 1.2 via the $\circledast$-composition defined in Definition 2.1.1. From Table 2.1 one can note that $\circledast$ indeed generates newer fuzzy implications from fuzzy implications.

| $I$ | $J$ | $I \circledast J$ |
|---|---|---|
| $I_{\mathbf{RC}}$ | $I_{\mathbf{LK}}$ | $\begin{cases} 1, & \text{if } x \leq y \\ 1 - x^2 + xy, & \text{if } x > y \end{cases}$ |
| $I_{\mathbf{GG}}$ | $I_{\mathbf{RC}}$ | $\begin{cases} 1, & \text{if } x \leq 1 - x + xy \\ \dfrac{1 - x + xy}{x}, & \text{otherwise} \end{cases}$ |
| $I_{\mathbf{KD}}$ | $I_{\mathbf{RS}}$ | $\begin{cases} 1, & \text{if } x \leq y \\ 1 - x, & \text{if } x > y \end{cases}$ |
| $I_{\mathbf{RC}}$ | $I_{\mathbf{KD}}$ | $\max(1 - x^2, 1 - x + xy)$ |
| $I_{\mathbf{FD}}$ | $I_{\mathbf{RC}}$ | $\begin{cases} 1, & \text{if } x \leq 1 - x + xy \\ 1 - x + xy & \text{otherwise} \end{cases}$ |
| $I_{\mathbf{YG}}$ | $I_{\mathbf{GD}}$ | $\begin{cases} 1, & \text{if } x \leq y \\ y^x, & \text{if } x > y \end{cases}$ |
| $I_{\mathbf{GD}}$ | $I_{\mathbf{LK}}$ | $\begin{cases} 1, & \text{if } x \leq \frac{1+y}{2} \\ 1 - x + y, & \text{otherwise} \end{cases}$ |

Table 2.1: Compositions of some fuzzy implications w.r.to $\circledast$.

## 2.2 The $\circledast$-composition on $\mathbb{I}$ : A Monoid Structure

In Theorem 2.1.2, we have proved that for $I, J \in \mathbb{I}$, the function $I \circledast J \in \mathbb{I}$. In other words, we have shown that $\circledast$ is indeed a binary operation on the set $\mathbb{I}$. In the following we show that $\circledast$ makes $\mathbb{I}$ a monoid. Note that this is the richest algebraic structure obtained so far on the set $\mathbb{I}$ without any assumptions.

**Theorem 2.2.1.** $(\mathbb{I}, \circledast)$ *forms a monoid, whose identity element is given by*

$$I_{\mathbf{D}}(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ y, & \text{if } x > 0\ . \end{cases}$$

*Proof.* From Theorem 2.1.2, it follows that $\circledast$ is a binary operation on the set $\mathbb{I}$. To see the associativity of $\circledast$, let $I, J, K \in \mathbb{I}$ and $x, y \in [0,1]$. Then

$$\begin{aligned}(I \circledast (J \circledast K))(x,y) &= I(x, (J \circledast K)(x,y)) \\ &= I(x, J(x, K(x,y))) \\ &= (I \circledast J)(x, K(x,y)) \\ &= ((I \circledast J) \circledast K)(x,y),\end{aligned}$$

showing that $\circledast$ is associative. Further,

$$\begin{aligned}(I \circledast I_{\mathbf{D}})(x,y) &= I(x, I_{\mathbf{D}}(x,y)) \\ &= \begin{cases} 1, & \text{if } x = 0, \\ I(x,y), & \text{if } x > 0, \end{cases} \\ &= I(x,y),\end{aligned}$$

and similarly $I_{\mathbf{D}} \circledast I = I$. Thus $I_{\mathbf{D}}$ becomes the identity element in $\mathbb{I}$. $\square$

**Remark 2.2.2.** *Take $I(x,y) = I_{\mathbf{RC}}(x,y) = 1-x+xy$. Then it is easy to see that $(I \circledast I)(x,y) = 1-x^2+x^2y$ is not same as $I_{\mathbf{RC}}$. Thus $\circledast$ is not idempotent in $\mathbb{I}$ and consequently, $(\mathbb{I}, \circledast)$ is a non-idempotent monoid.*

## 2.3 Outline of the rest of the Thesis

In this chapter, we have proposed a novel generative method of fuzzy implications from fuzzy implications, namely, the $\circledast$-composition and presented some of the new fuzzy implications thus obtained. Looking at the $\circledast$-composition as a binary operation on $\mathbb{I}$, we have shown that the set $\mathbb{I}$ becomes a non-idempotent monoid. Since the $\circledast$-composition can be looked in two different ways, viz, a generative method of fuzzy implications and a binary operation on the set $\mathbb{I}$, we explore the $\circledast$-composition on $\mathbb{I}$ along these two aspects in Part II and Part III of this thesis, respectively.

Part II, consisting of the Chapters 3-5, investigates the behavior of the $\circledast$-composition as a generative method of fuzzy implications from fuzzy implications. In particular, we study the $\circledast$-closures of fuzzy implications w.r.to some desirable basic properties, functional equations and families of fuzzy implications. Further, we study the powers of fuzzy implications obtained from the self composition of fuzzy implications w.r.to the $\circledast$-composition. Specifically, we investigate the following:

(i) If $I, J \in \mathbb{I}$ satisfy a property (functional equation or belong to a certain family), we investigate whether $I \circledast J$ also satisfies the same. If not, we try to find the fuzzy implications $I, J$ such that $I \circledast J$ preserves the same property (functional equation or belongs to a certain family).

(ii) If $I \in \mathbb{I}$ satisfies a property (functional equation or belong to a certain family), we investigate whether all the powers of $I$ w.r.to $\circledast$ also satisfy the same. If not, we try to find the fuzzy implications $I$ such that all the powers also satisfy the same property (functional equation or belongs to a certain family).

Part III, consisting of the Chapters 6-8, investigates the algebraic aspects of the monoid $(\mathbb{I}, \circledast)$. Our investigations show the following :

(i) Though $(\mathbb{I}, \circledast)$ is a monoid, we show that it can not be made a group due to the presence of zero elements.

(ii) Since the set of all invertible elements of any monoid forms a subgroup, we characterise the set of all such elements of $(\mathbb{I}, \circledast)$ and give their representations.

(iii) Based on the representations obtained, we propose group actions on $\mathbb{I}$ which lead to newer and better perspectives of some families of fuzzy implications.

(iv) We know that any monoid is injectively homomorphic to its set of all right translations. Since $(\mathbb{I}, \circledast)$ is a monoid, we determine the set of all right translation homomorphisms on $(\mathbb{I}, \circledast)$ which helps us to characterise few important sub algebras like center, set of right zero elements, etc., of $(\mathbb{I}, \circledast)$.

# Part II

# The $\circledast$-composition : As a Generative Method of Fuzzy Implications

# Chapter 3

# The $\circledast$-composition : Closures w.r.to Properties and Functional Equations

*Take what you need; act as you must, and*
*you will obtain that for which you wish!*
– Rene Descartes (1596-1650)

In Chapter 2, we have proposed a novel generative method of fuzzy implications from fuzzy implications. However, the applicability of any generative method of fuzzy implications lies mostly on the preservation of some desirable basic properties and functional equations of the original fuzzy implications. Thus it is essential to investigate the preservation of properties and functional equations of fuzzy implications w.r.to any generating method of fuzzy implications. In this chapter, we attempt to investigate the $\circledast$-closures of fuzzy implications w.r.to properties and functional equations. In particular, we would like to do the following:

*If $I, J \in \mathbb{I}$ satisfy a property (functional equation), we investigate whether $I \circledast J$ also satisfies the same. If not, we try to find those fuzzy implications $I, J$ satisfying that property (functional equation) such that $I \circledast J$ also satisfies the same property (functional equation).*

Towards this end, in Section 3.1, we recall the most desirable basic properties of fuzzy implications and give a comparative analysis of these properties w.r.to the existing generative methods of fuzzy implications discussed in Section 1.3. In Section 3.2, we investigate the closures of $\circledast$-composition w.r.to these basic properties. The $\circledast$-closures of fuzzy implications are also investigated for fuzzy implications satisfying two important functional equations in Section 3.3.

## 3.1 Basic Properties of Fuzzy Implications

In the literature, one finds some properties of fuzzy implications that play a key role in characterising different fuzzy implications and in various applications of fuzzy implications. Note that they are also a natural generalisation of the corresponding properties of the classical implication to multivalued logic. In the following, we recall a few of the most important properties of fuzzy implications (see [8, 66, 75]).

**Definition 3.1.1** (cf. [8], Definition 1.3.1). • *A fuzzy implication $I$ is said to satisfy* *(i) the left neutrality property* (NP) *if*

$$I(1, y) = y, \qquad y \in [0, 1] . \tag{NP}$$

*(ii) the ordering property* (OP)*, if*

$$x \le y \Longleftrightarrow I(x, y) = 1, \qquad x, y \in [0, 1]. \tag{OP}$$

*(iii) the identity principle* (IP)*, if*

$$I(x, x) = 1, \qquad x \in [0, 1] . \tag{IP}$$

*(iv) the exchange principle* (EP)*, if*

$$I(x, I(y, z)) = I(y, I(x, z)), \qquad x, y, z \in [0, 1] . \tag{EP}$$

- *A fuzzy implication $I$ is said to be continuous if it is continuous in both the variables.*

Let $\mathbb{I}_{\mathbf{NP}}$ denote the set of fuzzy implications satisfying (NP). Similarly, let the subsets $\mathbb{I}_{\mathbf{IP}}, \mathbb{I}_{\mathbf{OP}}, \mathbb{I}_{\mathbf{EP}}$ denote the set of fuzzy implications satisfying (IP), (OP) and (EP), respectively. Recall that $\mathbb{I}_{\mathbf{inv}}$ denotes the set of all fuzzy implications satisfying self conjugacy.

### 3.1.1 Existing Generative Methods and Preservation of Basic Properties

In Section 1.3, we have reviewed the existing generative methods of fuzzy implications from fuzzy implications. In the following table we present a comparative analysis of the existing generative methods w.r.to the basic properties. For the relevant results and their proofs please see, for instance, [4, 6, 8, 29].

| Property | $I$ | $J$ | $I \vee J$ | $I \wedge J$ | Convex Combination | $I \overset{*}{\circ} J$ |
|---|---|---|---|---|---|---|
| IP | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| OP | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| NP | ✓ | ✓ | ✓ | ✓ | ✓ | × |
| EP | ✓ | ✓ | × | × | × | × |
| Self conjugacy | ✓ | ✓ | ✓ | ✓ | × | × |
| Continuity | ✓ | ✓ | ✓ | ✓ | ✓ | × |

Table 3.1: Existing generative methods w.r.to different properties.

In the following we investigate the $\circledast$- preservation of the most basic properties of fuzzy implications and some functional equations involving fuzzy implications, as was done in the case of other existing generative methods of fuzzy implications.

## 3.2 The $\circledast$-composition w.r.to the Basic Properties

In this section, as mentioned before, we do the following: Given $I, J \in \mathbb{I}$ satisfying a certain property **P**, we now investigate whether $I \circledast J$ also satisfies the same property or not. If not, then we attempt to characterise $I, J$ such that $I \circledast J$ also satisfies the same property. Towards this end we have the following result.

**Theorem 3.2.1.** *Let $I, J \in \mathbb{I}$ and $\varphi \in \Phi$. Then $(I \circledast J)_\varphi = I_\varphi \circledast J_\varphi$.*

*Proof.* Let $I, J \in \mathbb{I}$ and $\varphi \in \Phi$. Let $x, y \in [0,1]$.

$$\begin{aligned}(I \circledast J)_\varphi(x,y) &= \varphi^{-1}((I \circledast J)(\varphi(x), \varphi(y)))\\ &= \varphi^{-1}(I(\varphi(x), J(\varphi(x), \varphi(y))))\\ &= \varphi^{-1}(I(\varphi(x), \varphi(J_\varphi(x,y))))\\ &= I_\varphi(x, J_\varphi(x,y))\\ &= (I_\varphi \circledast J_\varphi)(x,y).\end{aligned}$$

Thus we have proved that $(I \circledast J)_\varphi = I_\varphi \circledast J_\varphi$. □

**Lemma 3.2.2.** *If $I, J \in \mathbb{I}$ satisfy* (NP) ((IP), *self conjugacy, continuity) then $I \circledast J$ satisfies the same property.*

*Proof.* A direct verification provides the proof. □

### 3.2.1 The $\circledast$-composition w.r.to the Ordering Property (OP)

While the composition $\circledast$ preserves (NP), (IP), self-conjugacy and continuity, this is not true with either the ordering property (OP) or the exchange principle (EP), as is made clear from the following remark.

**Remark 3.2.3.** *(i) From Table 1.2, it is clear that both $I = I_{\mathbf{GD}}, J = I_{\mathbf{LK}}$ satisfy* (OP)*. However, $I \circledast J$ does not satisfy* (OP) *because $(I \circledast J)(0.4, 0.2) = 1$ but $0.4 > 0.2$ (see Table 2.1 for its explicit formula).*

*(ii) However, note that in the above example, $J \circledast I$ satisfies* (OP)*, since $J \circledast I = J = I_{\mathbf{LK}}$, satisfies* (OP)*. In fact, it is easy to check that $I \circledast I_{\mathbf{GD}} = I$ for all $I \in \mathbb{I}_{\mathbf{OP}}$.*

*(iii) Let us consider another example where $I \circledast J$ is neither $I$ nor $J$. To this end, let $I = I_{\mathbf{GG}}$, the Goguen implication, and*

$$J(x,y) = \begin{cases} 1, & \text{if } x \le y,\\ y^2, & \text{if } x > y.\end{cases}$$

*Now, both $I, J$ satisfy* (OP) *and so also their $\circledast$ composition given by*

$$(I \circledast J)(x,y) = \begin{cases} 1, & \text{if } x \le y,\\ \dfrac{y^2}{x}, & \text{if } x > y.\end{cases}$$

*(iv) It is also interesting to note that even when not both of $I, J$ satisfy* (OP)*, one can have that $I \circledast J$ satisfies* (OP)*. To see this, let $I = I_{\mathbf{GG}}$, the Goguen implication which satisfies* (OP)*, and $I = I_{\mathbf{RC}}$, the Reichenbach implication which does not satisfy* (OP)*. Now, $I \circledast J = I_{\mathbf{LK}}$, which does satisfy* (OP)*.*

(v) *Let $I = I_{\mathbf{RC}}$ and $J = I_{\mathbf{KD}}$. From Table 1.2, it is clear that both $I, J$ do not satisfy* (OP). *Moreover, their $\circledast$ composition $I \circledast J$ also does not satisfy* (OP) *because $0.3 < 0.5$ but $(I \circledast J)(0.3, 0.5) = 0.91 < 1$.*

The following result characterises all fuzzy implications $I, J \in \mathbb{I}_{\mathbf{OP}}$ such that $I \circledast J \in \mathbb{I}_{\mathbf{OP}}$.

**Theorem 3.2.4.** *Let $I, J \in \mathbb{I}$ satisfy* (OP). *Then the following statements are equivalent:*

(i) *$I \circledast J$ satisfies* (OP).

(ii) *$J$ satisfies the following for $x, y \in [0, 1]$ :*

$$x > J(x, y), \text{ whenever } x > y \; . \tag{3.1}$$

(iii) *$J(x, y) \leq y$, whenever $x > y$.*

*Proof.* Let $I, J \in \mathbb{I}$ satisfy (OP).

(i) $\Longrightarrow$ (ii) : Let $I \circledast J$ satisfy (OP). Then $(I \circledast J)(x, y) = 1 \Longleftrightarrow x \leq y$,

$$i.e., I(x, J(x, y)) = 1 \Longleftrightarrow x \leq y \; ,$$

$$i.e., x \leq J(x, y) \Longleftrightarrow x \leq y \; ,$$

which implies $x > J(x, y)$ for all $x > y$.

(ii) $\Longrightarrow$ (iii) : Let $J$ satisfy (3.1). If $x > y$, then there exists $\varepsilon > 0$, arbitrarily small, such that $x > y + \varepsilon > y$. Now, from the antitonicity of $J$ in the first variable and (3.1), we have $J(x, y) \leq J(y + \varepsilon, y) < y + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we see that $J(x, y) \leq y$ for all $x > y$.

(iii) $\Longrightarrow$ (i) : Let $J$ satisfy $J(x, y) \leq y$ for all $x > y$.

- Let $x \leq y$. Then, since $J$ satisfies (OP), $J(x, y) = 1$ and consequently, $(I \circledast J)(x, y) = I(x, J(x, y)) = 1$.
- Let $x > y$. Then we have $J(x, y) \leq y < x$. From (OP) of $I$, it follows that $I(x, J(x, y)) < 1$.

In other words, we have $x > y \Longleftrightarrow (I \circledast J)(x, y) < 1$ and hence $I \circledast J$ satisfies (OP). □

In the following example we present a family of fuzzy implications satisfying (OP) which satisfies the conditions in Theorem 3.2.4.

**Example 3.2.5.** (i) *Let us denote by $\mathbb{I}^{\circ}_{\varphi} \subset \mathbb{I}_{\mathbf{OP}}$ such that every $I \in \mathbb{I}^{\circ}_{\varphi}$ is of the following form:*

$$I(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ \varphi(y), & \text{if } x > y, \end{cases}$$

*where $\varphi \in \Phi$ and $\varphi(y) \leq y$ for all $y \in [0, 1]$. Clearly, every $I \in \mathbb{I}^{\circ}_{\varphi}$ satisfies* (3.1) *and hence, if $I, J \in \mathbb{I}^{\circ}_{\varphi}$ then $I \circledast J$ satisfies* (OP). *In fact, $I \circledast J \in \mathbb{I}^{\circ}_{\varphi}$.*

*(ii) However, $\mathbb{I}_{\varphi}^{\circ}$ does not contain all fuzzy implications satisfying* (3.1)*. To see this consider the following fuzzy implication which does satisfy* (3.1) *but does not belong to $\mathbb{I}_{\varphi}^{\circ}$:*

$$J(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \min(y, 1 - \frac{x}{2}), & \text{if } x > y \,. \end{cases}$$

### 3.2.2 The $\circledast$-composition w.r.to the Exchange Principle (EP)

Among the basic properties of fuzzy implications, the exchange principle (EP) is the most important. Along with the ordering property (OP), it implies many other properties. For instance, the following result from [8] shows that (OP) and (EP) are sufficient to make an arbitrary binary function on $[0,1]$ into almost a fuzzy implication with all the desirable properties.

**Lemma 3.2.6** ([8], Lemma 1.3.4)**.** *If a function $I\colon [0,1]^2 \longrightarrow [0,1]$ satisfies* (EP) *and* (OP)*, then $I$ satisfies* (I1), (I3), (NP) *and* (IP).

Once again, as in the case of (OP), the following remark shows that $\circledast$ does not always preserve (EP).

**Remark 3.2.7.** *(i) From Table 1.4 in [8], one notes that both the fuzzy implications $I = I_{\mathbf{RC}}, J = I_{\mathbf{KD}}$ satisfy* (EP)*. Table 2.1 gives the formula for $I_{\mathbf{RC}} \circledast I_{\mathbf{KD}}$. However, $(I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(0.3, (I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(0.8, 0.5)) = 0.91$, while $(I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(0.8, (I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(0.3, 0.5)) = 0.928$. Thus $I_{\mathbf{RC}} \circledast I_{\mathbf{KD}}$ does not satisfy* (EP) *even if $I$ and $J$ satisfy* (EP)*.*

*(ii) Once again, as in the case of* (OP)*, observe that for the same $I, J$ above their composition $J \circledast I$ satisfies* (EP)*, since $I_{\mathbf{KD}} \circledast I_{\mathbf{RC}} = I_{\mathbf{RC}}$.*

*(iii) Let $I = I_{\mathbf{WB}}$ and $J = I_{\mathbf{SQ}}$ where $I_{\mathbf{SQ}}$ is given by*

$$I_{\mathbf{SQ}}(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \{(0,0) \text{ or } (1,1)\}, \\ \max(y^x, (1-x)^{\sqrt{1-y}}), & \text{otherwise}. \end{cases}$$

*From Tables 1.4 and 1.5, it follows respectively, that $I_{\mathbf{WB}}$ satisfies* (EP) *where as $I_{\mathbf{SQ}}$ does not satisfy* (EP)*. However, it easy to check that the composition $I \circledast J$ is equal to $I$ and hence satisfies* (EP)*.*

*(iv) Let $I(x,y) = \max(1-x, y^2)$ and $J(x,y) = (1-x+xy)^{\frac{1}{2}}$. It is easy to check that these two functions $I, J$ are fuzzy implications. Moreover, one can always prove that $I, J$ do not satisfy* (EP) *always, for example,*

$$I(0.2, I(0.3, 0.4)) = 0.8 \neq 0.7 = I(0.3, I(0.2, 0.4)),$$
$$J(0.2, J(0.7, 0)) = 0.9537 \neq 0.923 J(0.7, J(0.2, 0)).$$

*However,*

$$
\begin{aligned}
(I \circledast J)(x,y) &= I(x, J(x,y)) \\
&= I(x, (1-x+xy)^{\frac{1}{2}}) \\
&= \max(1-x, ((1-x+xy)^{\frac{1}{2}})^2) \\
&= \max(1-x, 1-x+xy) \\
&= 1-x+xy = I_{\mathbf{RC}}(x,y),
\end{aligned}
$$

*which satisfies* (EP) *from Table 1.4 in [8].*

From the above, we see that the $\circledast$-composition does not always preserve (EP). In the following we define a property of a pair of fuzzy implications $I, J$ which turns out to be a sufficient condition for the preservation of (EP) by $\circledast$. In fact, as we will see later, this property plays an important role in the sequel.

**Definition 3.2.8.** *A pair* $(I, J)$ *of fuzzy implications is said to be* mutually exchangeable *if*

$$
I(x, J(y,z)) = J(y, I(x,z)), \qquad x, y, z \in [0,1]. \tag{ME}
$$

**Remark 3.2.9.** *(i) In the context of aggregating fuzzy implications, Reiser.et.al. proposed the generalised exchange property (GEP) of two fuzzy implications* $I, J$ *in [69] as follows.*

$$
I(x, J(y,z)) = I(y, J(x,z)), \qquad x, y, z \in [0,1].
$$

*Note that Definition 3.2.8 is different from (GEP). However, when* $I = J \in \mathbb{I}$ *both* (ME) *and the (GEP) of [69] reduce to the usual* (EP) *of* $I$*.*

*(ii) If* $I, J$ *are mutually exchangeable, then* $I \circledast J = J \circledast I$*, i.e.,* $\circledast$ *is commutative on* $I, J$*. To see this, let* $x = y$ *in* (ME)*, which then becomes* $I(x, J(x,z)) = J(x, I(x,z))$*. i.e.,* $(I \circledast J)(x,z) = (J \circledast I)(x,z)$*, for all* $x, z \in [0,1]$*.*

**Example 3.2.10.** *Let* $\epsilon, \delta \in [0,1]$*. Let us consider the following two fuzzy implications:*

$$
I(x,y) = \begin{cases} 1, & \text{if } x \le \epsilon, \\ y^2, & \text{if } x > \epsilon, \end{cases} \quad \text{and } J(x,y) = \begin{cases} 1, & \text{if } x \le \delta, \\ y^3, & \text{if } x > \delta. \end{cases}
$$

*It is easy to check that the pair* $(I, J)$ *satisfies* (ME).

Now, we are ready to give a sufficient condition on the pair $I, J$ of fuzzy implications satisfying (EP) such that their $\circledast$-composition $I \circledast J$ also satisfies (EP).

**Theorem 3.2.11.** *Let* $I, J \in \mathbb{I}$ *satisfy* (EP) *and be mutually exchangeable, i.e., satisfy* (ME)*. Then* $I \circledast J$ *satisfies* (EP).

*Proof.* Let $I, J \in \mathbb{I}_{\mathbf{EP}}$ satisfy (ME) and $x, y, z \in [0,1]$. Then

$$\begin{aligned}
(I \circledast J)(x, (I \circledast J)(y,z)) &= I(x, J(x, (I \circledast J)(y,z))) \\
&= I(x, J(x, I(y, J(y,z)))) \\
&= I(x, I(y, J(x, J(y,z)))) && \lceil\because (I,J) \text{ satisfies (ME)} \\
&= I(y, I(x, J(y, J(x,z)))) && \lceil\because I, J \text{ satisfy (EP)} \\
&= I(y, J(y, I(x, J(x,z)))) && \lceil\because (I,J) \text{ satisfies (ME)} \\
&= (I \circledast J)(y, (I \circledast J)(x,z)) \ .
\end{aligned}$$

Thus $I \circledast J$ also satisfies (EP). $\square$

**Remark 3.2.12.** *The condition that $I, J \in \mathbb{I}_{\mathbf{EP}}$ satisfy* (ME) *for $I \circledast J$ to satisfy* (EP) *is only sufficient but not necessary. To see this, let*

$$I(x,y) = \begin{cases} 1, & \text{if } x \le 0.3, \\ y^2, & \text{if } x > 0.3, \end{cases} \quad \text{and } J(x,y) = \begin{cases} 1, & \text{if } x \le 0.5, \\ \sin(\frac{\pi y}{2}), & \text{if } x > 0.5. \end{cases}$$

*Now, $I \circledast J$ is given by the following formula*

$$(I \circledast J)(x,y) = \begin{cases} 1, & \text{if } x \le 0.5, \\ \sin^2(\frac{\pi y}{2}), & \text{if } x > 0.5 \ . \end{cases}$$

*It is easy to check that $I, J, I \circledast J \in \mathbb{I}_{\mathbf{EP}}$. However, if $x = 0.6, y = 0.7, z = 0.8$, then $I(x, J(y,z)) = 0.9045$ and $J(y, I(x,z)) = 0.8443$. Thus $I, J$ fail to satisfy* (ME).

## 3.3 The $\circledast$-composition : Closures w.r.to Functional Equations

The study of functional equations involving fuzzy implications has attracted much attention not only due to their theoretical aesthetics but also due to their applicational value. In this section, we present two of the most important functional equations involving fuzzy implications, viz., the law of importation (LI) and contraposition w.r.to a strong negation $N$, CP($N$), and study the following question: *If a given pair of fuzzy implications $I, J \in \mathbb{I}$ satisfies one of these functional equations, does $I \circledast J$ also satisfy the same functional equation?*

Note that the choice of these functional equations were based not only on their importance in the literature but also their relevance to the subsequent analysis in this thesis.

### 3.3.1 The Law of Importation w.r.to a t-norm $T$

The law of importation (LI) has been shown to play a major role in the computational efficiency of fuzzy relational inference mechanisms that employ fuzzy implications to relate antecedents and consequents, see for instance, [55, 81].

**Definition 3.3.1** ([8], Definition 1.5.1). *A fuzzy implication $I$ is said to satisfy the law of importation* (LI) *w.r.to a t-norm $T$, if*

$$I(x, I(y,z)) = I(T(x,y), z), \qquad x, y, z \in [0,1]. \tag{LI}$$

In the literature, one finds many weaker versions of the law of importation (LI) where the t-norm $T$ is generalised to a commutative conjunctor, for instance, see the version presented in Massanet and Torrens [57]. However, here in this work we deal only with the classical version of (LI), i.e., where the conjunctor is a t-norm $T$. Note that any $I$ that satisfies (LI) automatically satisfies (EP) too, while the converse is not true, see for instance, Remark 7.3.1 in [8] and [57].

**Remark 3.3.2.** *Note that even if $I, J \in \mathbb{I}$ satisfy* (LI) *w.r.to the same t-norm $T$, $I \circledast J$ may not satisfy* (LI) *w.r.to any t-norm or may satisfy* (LI) *w.r.to same t-norm or even a different t-norm $T'$.*

*(i) Let $I = I_{\mathbf{RC}}, J = I_{\mathbf{YG}}$. It follows from Table 7.1 in [8], that both $I, J$ satisfy* (LI) *w.r.to the product t-norm $T_{\mathbf{P}}(x,y) = xy$. However, $I \circledast J$ given by*

$$(I \circledast J)(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ 1 - x + xy^x, & \text{if } x > 0 \text{ or } y > 0, \end{cases}$$

*does not satisfy* (EP) *since*

$$(I \circledast J)(0.2, (I \circledast J)(0.3, 0.4)) = 0.9487 \neq 0.8752 = (I \circledast J)(0.3, (I \circledast J)(0.2, 0.4)).$$

*From the necessary conditions of* (EP) *for* (LI) *(see, Remark 7.3.1 in [8]), it follows that $I \circledast J$ does not satisfy* (LI) *w.r.to any t-norm $T$.*

*(ii) Consider the fuzzy implication $I_{(n)}(x,y) = 1 - x^n + x^n y$, for some arbitrary but fixed $n \in \mathbb{N}$. Then*

$$I_{(n)}(T_{\mathbf{P}}(x,y), z) = I_{(n)}(xy, z) = 1 - x^n y^n + x^n y^n z \text{ , and}$$
$$I_{(n)}(x, I_{(n)}(y,z)) = I_{(n)}(x, 1 - y^n + y^n z) = 1 - x^n + x^n(1 - y^n + y^n z) = 1 - x^n y^n + x^n y^n z \text{ .}$$

*Thus $I_{(n)}$ satisfies* (LI) *w.r.to the t-norm $T_{\mathbf{P}}$ for any finite $n \in \mathbb{N}$.*

*Now, let $I(x,y) = I_{(1)}(x,y) = I_{\mathbf{RC}}(x,y) = 1 - x + xy$, and $J(x,y) = I_{(2)}(x,y) = 1 - x^2 + x^2 y$. From above, it follows that $I, J$ both satisfy* (LI) *w.r.to $T_{\mathbf{P}}$. Now $(I \circledast J)(x,y) = 1 - x^3 + x^3 y = I_{(3)}(x,y)$, which also satisfies* (LI) *w.r.to $T_{\mathbf{P}}$.*

*(iii) Let $I = I_{\mathbf{RC}}, J = I_{\mathbf{GG}}$. It follows from Table 7.1 in [8] that $I, J$ satisfy* (LI) *w.r.to $T = T_{\mathbf{P}}$. Now, $I_{\mathbf{RC}} \circledast I_{\mathbf{GG}} = I_{\mathbf{LK}}$. From Theorem 7.3.5 in [8], it follows that $I_{\mathbf{LK}}$ satisfies* (LI) *w.r.to only the Lukasiewicz t-norm $T_{\mathbf{LK}}(x,y) = \max(0, x+y-1)$, which means that $I_{\mathbf{RC}} \circledast I_{\mathbf{GG}}$ does not satisfy* (LI) *w.r.to the product t-norm $T_{\mathbf{P}}$ but with a different t-norm $T_{\mathbf{LK}}$.*

*(iv) Finally, let us consider $I, J \in \mathbb{I}$ defined as*

$$I(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ ,} \\ \sin(\frac{\pi y}{2}), & \text{if } x = 1 \text{ ,} \end{cases} \quad \text{and } J(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ ,} \\ y^3, & \text{if } x = 1 \text{ .} \end{cases}$$

*Then it is easy to check that all of $I, J$ and $I \circledast J$ (as given below) satisfy* (LI) *w.r.to any t-norm $T$:*

$$(I \circledast J)(x, y) = \begin{cases} 1, & \textit{if } x < 1 \ , \\ \sin(\frac{\pi y^3}{2}), & \textit{if } x = 1 \ . \end{cases}$$

The following result contains a sufficient condition on the implications $I, J$ satisfying (LI) w.r.to the same t-norm $T$ under which their $\circledast$ composition $I \circledast J$ also satisfies (LI) w.r.to the same $T$. Once again, we see that (ME) plays an important role.

**Lemma 3.3.3.** *Let $I, J \in \mathbb{I}$ satisfy* (LI) *w.r.to a t-norm $T$. If $I, J$ satisfy* (ME) *then $I \circledast J$ satisfies* (LI) *w.r.to the same t-norm $T$.*

*Proof.* Let $I, J \in \mathbb{I}$ satisfy (LI) w.r.to a t-norm $T$ and satisfy (ME).

$$\begin{aligned}(I \circledast J)(T(x, y), z) &= I(T(x, y), J(T(x, y), z)) = I(x, I(y, J(T(x, y), z))) \\ &= I(x, I(y, J(x, J(y, z)))) = I(x, J(x, I(y, J(y, z)))) \qquad \lceil \because \text{ using (ME)} \\ &= I(x, J(x, (I \circledast J)(y, z))) \\ &= (I \circledast J)(x, (I \circledast J)(y, z)).\end{aligned}$$

Thus $I \circledast J$ satisfies (LI) w.r.to the same t-norm $T$. $\square$

**Remark 3.3.4.** *Note that, in Lemma 3.3.3,* (ME) *is only sufficient and not necessary. To see this, let $I, J \in \mathbb{I}$ be as given in Remark 3.3.2(iv). We know that $I \circledast J$ satisfies* (LI) *w.r.to any $T$. However, $I$ and $J$ are not mutually exchangeable. To see this, we note that*

$$I(x, J(y, z)) = \begin{cases} 1, & \textit{if } x < 1 \textit{ or } y < 1 \ , \\ \sin(\frac{\pi z^3}{2}), & \textit{if } x = 1 \textit{ and } \ y = 1 \ , \end{cases}$$

*and*

$$J(y, I(x, z)) = \begin{cases} 1, & \textit{if } x < 1 \textit{ or } y < 1 \ , \\ \sin^3(\frac{\pi z}{2}), & \textit{if } x = 1 \textit{ and } \ y = 1 \ , \end{cases}$$

*which are not identically the same for all $x, y, z \in [0, 1]$. To see this, let $z = \frac{1}{2}$ and $x, y \in (0, 1)$ be arbitrary for instance.*

The following result gives a condition on $I, J$ such that (ME) also becomes necessary for (LI).

**Theorem 3.3.5.** *Let $I, J \in \mathbb{I}$ satisfy* (LI) *w.r.to a t-norm $T$. Further, for all $x \in (0, 1]$, let $I(x, \cdot)$ be one-one and $J(x, \cdot)$ be both one-one and onto, i.e., $J(x, \cdot)$ is an increasing bijection on $[0, 1]$. Then the following statements are equivalent:*

*(i)* *$I, J$ satisfy* (ME).

*(ii)* *$I \circledast J$ satisfies* (LI) *w.r.to $T$.*

*Proof.* (i) $\Longrightarrow$ (ii) : Follows from Lemma 3.3.3.

(ii) $\Longrightarrow$ (i) : Let $I \circledast J$ satisfy (LI) w.r.to same $T$. Then, for any $x, y \in (0,1)$,

$$\begin{aligned}
(I \circledast J)(T(x,y),z) &= (I \circledast J)(x,(I \circledast J)(y,z)) \\
&\Longrightarrow I(x, I(y, J(x, J(y,z)))) = I(x, J(x, I(y, J(y,z)))) \\
&\Longrightarrow I(y, J(x, J(y,z)) = J(x, I(y, J(y,z))) \,. \qquad [\because I(x,\cdot) \text{ is one-one}
\end{aligned}$$

Since $J(x,\cdot)$ is a bijection on $[0,1]$, for every $t \in [0,1]$ and any $y \in [0,1]$ there exists a unique $z \in [0,1]$ such that $t = J(y,z)$. Hence, we have $I(y, J(x,t)) = J(x, I(y,t))$ for all $x, y, t \in [0,1]$, i.e., $I, J$ are mutually exchangeable. □

### 3.3.2 Contrapositive Symmetry w.r.to a Fuzzy Negation $N$

Before discussing the contrapositive symmetry of fuzzy implications, we introduce fuzzy negations. Fuzzy negations, are once again, a generalisation of classical negation from binary logic to multi-valued logic.

**Definition 3.3.6** ([48], Definition 11.3)**.** *A function $N : [0,1] \to [0,1]$ is called a* fuzzy negation *if $N(0) = 1, N(1) = 0$ and $N$ is decreasing. A fuzzy negation $N$ is called*

*(i)* strict *if, in addition, $N$ is strictly decreasing and is continuous,*

*(ii)* strong *if it is an involution, i.e., $N(N(x)) = x$, for all $x \in [0,1]$.*

Table 3.2 lists the basic fuzzy negations, which will also be useful in the sequel.

| Name | Formula |
|---|---|
| Standard | $N_{\mathbf{C}}(x) = 1 - x$ |
| Least | $N_{\mathbf{D_1}}(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases}$ |
| Greatest | $N_{\mathbf{D_2}}(x) = \begin{cases} 1, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \end{cases}$ |

Table 3.2: Basic fuzzy negations

Interestingly, given a fuzzy implication $I$, one can always obtain a fuzzy negation from it as follows.

**Definition 3.3.7** ([8], Definition 1.4.14)**.** *Let $I \in \mathbb{I}$ be any fuzzy implication. The function $N_I : [0,1] \longrightarrow [0,1]$ defined by $N_I(x) = I(x,0)$ is a fuzzy negation and is called the* natural negation *of $I$.*

Contrapositive symmetry of implications is a tautology in classical logic. Contrapositive symmetry of fuzzy implications w.r.to an involutive or a strong negation plays an equally important role in fuzzy logic as its classical counterpart - especially in t-norm based multivalued logics, see for instance, [32, 33, 42, 43, 54, 44].

Once again, many generalisations and weaker versions of the law of contraposition are considered in the literature, see for instance, [8], Section 1.5. However, here we consider only the classical law of contraposition where the involved negation is strong.

**Definition 3.3.8** ([8], Definition 7.3). *An implication $I$ is said to satisfy the contrapositive symmetry w.r.to a fuzzy negation $N$ if*

$$I(x, y) = I(N(y), N(x)), \qquad x, y \in [0, 1] \,. \tag{CP}$$

*In such a case, we often write that I satisfies CP(N).*

Just as (EP) and (LI) are closely related, so are (CP) and (NP). In fact, as the following result shows if a neutral fuzzy implication satisfies (CP) w.r.to some fuzzy negation $N$, then it does so only with its natural negation which should be necessarily strong.

**Lemma 3.3.9** ([8], Lemma 1.5.4). *Let $I \in \mathbb{I}_{\mathbf{NP}}$ satisfy* (CP) *w.r.to a fuzzy negation $N$. Then $N_I = N$ and $N_I$ is strong.*

However, it should be emphasized, that even if a fuzzy implication $I$ does not satisfy (NP), it still can satisfy (CP) with some fuzzy, even strong, negation, see [8], Example 1.5.10.

**Remark 3.3.10.** *Once again, note that even if $I, J$ satisfy* (CP) *w.r.to a fuzzy negation $N$, $I \circledast J$ may not satisfy* (CP) *w.r.to any fuzzy negation $N$ or may satisfy* (CP) *w.r.to the same fuzzy negation $N$.*

*(i) Let $I = I_{\mathbf{RC}}$ and $J = I_{\mathbf{KD}}$. It is easy to check that $I_{\mathbf{RC}}$ and $I_{\mathbf{KD}}$ both satisfy CP($N_{\mathbf{C}}$), i.e.,* (CP) *w.r.to the classical strong negation $N_{\mathbf{C}}(x) = 1 - x$. Now, from the definition of $\circledast$, it follows that, $(I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(x, y) = \max(1 - x^2, 1 - x + xy)$ which has* (NP). *From Lemma 3.3.9 above, we see that since $I_{\mathbf{RC}} \circledast I_{\mathbf{KD}}$ has* (NP), *if it satisfies* (CP) *w.r.to some fuzzy negation $N$, then its natural negation should be strong. However, we see that $N_{I_{\mathbf{RC}} \circledast I_{\mathbf{KD}}}(x) = (I_{\mathbf{RC}} \circledast I_{\mathbf{KD}})(x, 0) = \max(1 - x^2, 1 - x) = 1 - x^2$, which is only strict but not strong. Hence, $I_{\mathbf{RC}} \circledast I_{\mathbf{KD}}$ does not satisfy* (CP) *w.r.to* any *fuzzy negation $N$.*

*(ii) Interestingly, if $I = I_{\mathbf{KD}}$ and $J = I_{\mathbf{RC}}$, $I \circledast J = I_{\mathbf{KD}} \circledast I_{\mathbf{RC}} = I_{\mathbf{RC}}$, which satisfies* (CP) *w.r.to the same negation $N$.*

In the rest of this section, we only consider $I, J \in \mathbb{I}_{\mathbf{NP}}$. If such a pair also satisfies (CP) w.r.to the same fuzzy negation $N$, then the following result gives a necessary condition for $I \circledast J$ to satisfy (CP).

**Theorem 3.3.11.** *Let $I, J \in \mathbb{I}_{\mathbf{NP}}$ satisfy* (CP) *w.r.to a fuzzy negation $N$. If $I \circledast J$ also satisfies CP($N$) then $I(x, N(x)) = N(x)$ for all $x \in [0, 1]$.*

*Proof.* Firstly, from Lemma 3.2.2, we see that $I \circledast J \in \mathbb{I}_{\mathbf{NP}}$ and from Lemma 3.3.9 that $N_J = N$. Further, since $I \circledast J$ satisfies (CP) w.r.to $N$, once again from Lemma 3.3.9 we have that $N_{I \circledast J}(x) = I(x, J(x, 0)) = N(x)$ or equivalently, $I(x, N(x)) = N(x)$. □

In the following we show that if the considered pair $I, J \in \mathbb{I}_{\mathbf{NP}}$ also possesses other desirable properties like (EP) or (OP), then one obtains much stronger results. The following result is helpful in the characterisation results given below. The family of $(S, N)$-implications will be dealt with presently in Section 5.1 of Chapter 5.

**Theorem 3.3.12** ([17], Theorem 5). *Let $I$ be an $(S, N)$-implication with an appropriate t-conorm $S$ and a strong negation $N$. Then $I(x, N(x)) = N(x)$ if and only if $S = \max$.*

**Theorem 3.3.13.** *Let $I, J \in \mathbb{I}_{\mathbf{NP}} \bigcap \mathbb{I}_{\mathbf{EP}}$ satisfy* (CP) *w.r.to a fuzzy negation $N$. Then the following statements are equivalent:*

*(i)* $I \circledast J$ *satisfies CP(*$N$*).*

*(ii)* $I(x, y) = \max(N(x), y) = J(x, y)$*.*

*(iii)* $J \circledast I$ *satisfies CP(*$N$*).*

*Proof.* (i) $\Longrightarrow$ (ii): Since $I$ has (NP) and CP(N), we know from Lemma 3.3.9 that $N = N_I$ is strong. Further, since $I$ satisfies (EP), by the characterisation result for $(S, N)$- implications, viz., Theorem 5.1.3, we see that $I$ is an $(S, N)$- implication. Now, since $I \circledast J$ satisfies CP(N) with a strong negation $N$, from Theorem 3.3.11 we have that $I(x, N(x) = N(x)$ for all $x \in [0, 1]$ and from Theorem 3.3.12 above we have that $I(x, y) = \max(N(x), y)$.

Similarly, one can show that $J$ is also an $(S, N)$- implication with a possibly different t-conorm $S$ other than $\max$, but with the same negation $N$, i.e., $J(x, y) = S(N(x), y)$. Since the t-conorm $\max$ is distributive over any $S$, we have

$$(I \circledast J)(x, y) = \max(N(x), S(N(x), y)) = S(N(x), \max(N(x), y)) \ ,$$

$$(I \circledast J)(N(y), N(x)) = \max(y, S(N(x), y)) = S(y, \max(N(x), y)) \ .$$

Letting $y = 0$ in the above equations and using the fact that $I \circledast J$ satisfies CP($N$), we obtain that $S(N(x), N(x)) = N(x)$ and hence $S = \max$, since $N$ is involutive and hence is onto on $[0, 1]$.

(ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i) are straight-forward now. $\square$

**Theorem 3.3.14.** *Let* $I, J \in \mathbb{I}_{\mathbf{NP}} \bigcap \mathbb{I}_{\mathbf{OP}}$ *satisfy* (CP) *w.r.to the same fuzzy negation* $N$*. Then* $I \circledast J$ *does not satisfy* (CP) *w.r.to any negation* $N$*.*

*Proof.* Suppose $I \circledast J$ satisfies (CP) w.r.to fuzzy negation $N$. Then from Theorem 3.3.11, we see that $I(x, N(x)) = N(x)$ for all $x \in [0, 1]$. In particular, if we take $x = e \in (0, 1)$, the equilibrium point of $N$, i.e., $N(e) = e$, we have that $I(e, N(e)) = I(e, e) = N(e) = e < 1$, a contradiction to the fact that $I$ satisfies (OP). $\square$

## 3.4 Conclusions

| Property | $I$ | $J$ | $I \circledast J$ | Remark |
|---|---|---|---|---|
| IP | ✓ | ✓ | ✓ | |
| OP | ✓ | ✓ | × | Theorem 3.2.4 |
| NP | ✓ | ✓ | ✓ | |
| EP | ✓ | ✓ | × | Theorem 3.2.11 |
| Self conjugacy | ✓ | ✓ | ✓ | |
| Continuity | ✓ | ✓ | ✓ | |
| CP(N) | ✓ | ✓ | × | Theorem 3.3.13 |
| LI | ✓ | ✓ | × | Theorem 3.3.5 |

Table 3.3: Closure of $\circledast$ w.r.to different properties and functional equations.

In this chapter, we have investigated whether the $\circledast$-composition preserves some of the desirable basic properties of fuzzy implications and two important functional equations involving fuzzy implications. Specifically, we have shown that the $\circledast$-composition preserves properties like (NP), (IP), continuity and self conjugacy but fails to preserve (OP) and (EP). Since the $\circledast$-composition does not preserve (OP) and (EP) we have attempted to characterise the largest subsets of $\mathbb{I}_{\mathbf{OP}}, \mathbb{I}_{\mathbf{EP}}$, respectively, such that the $\circledast$-composition is closed on both of them. We have also studied the behavior of $\circledast$-composition w.r.to two functional equations involving fuzzy implications. In this chapter, we have also proposed a new concept of *mutual exchangeability* (ME), a generalisation of (EP) to a pair of fuzzy implications and showed that it plays an important role in the preservation of properties like (EP), (LI). The summary of the main results obtained in this chapter is given in Table 3.3.

# Chapter 4

# Self Composition of Fuzzy Implications w..r.to $\circledast$

*For the things we have to learn before we can do them,*
*we learn by doing them.*
– Aristotle (384 BC – 322 BC)

In Chapter 2, we have shown that if $I, J \in \mathbb{I}$ then $I \circledast J \in \mathbb{I}$. From this it follows that, if $J = I$ then $I \circledast I$ is also an implication on $[0,1]$. Since the binary operation $\circledast$ is associative in $\mathbb{I}$ (see Theorem 2.2.1), one can generate fuzzy implications from a single fuzzy implication.

In this chapter, we begin with the definition of powers of fuzzy implications and investigate their behavior in the limiting case. In Section 4.2 we examine if a fuzzy implication $I$ satisfies a desirable property whether all the powers $I_{\circledast}^{[n]}$ of $I$ satisfy the same or not. In Section 4.3, we study the closures of the powers of $I$ w.r.to functional equations that $I$ satisfies.

## 4.1 Self Composition w.r.to $\circledast - I_{\circledast}^{[n]}$

Since $\circledast$ is an associative binary operation on $\mathbb{I}$, one can define the self composition of fuzzy implications w.r.to $\circledast$.

**Definition 4.1.1.** *Let $I \in \mathbb{I}$. For any $n \in \mathbb{N}$, we define the $n$-th power of $I$ w.r.to the binary operation $\circledast$ as follows: For $n = 1$,*

$$I_{\circledast}^{[n]} = I,$$

*and for $n \geq 2$,*

$$I_{\circledast}^{[n]}(x,y) = I\left(x, I_{\circledast}^{[n-1]}(x,y)\right) = I_{\circledast}^{[n-1]}\left(x, I(x,y)\right)\,, \qquad x, y \in [0,1]\,. \tag{4.1}$$

Note that if $I \in \mathbb{I}$ then $I_{\circledast}^{[n]} \in \mathbb{I}$ for all $n \in \mathbb{N}$. While $I_{\circledast}^{[n]} \in \mathbb{I}$, the following posers still remain :

(i) Whether for every $I$, the powers of $I$ will be different from $I$.

(ii) If so, will they continue to generate implications different from them forever.

| Implication(I) | Order $\mathcal{O}(I)$ | $\lim_{n\to\infty} I_{\circledast}^{[n]}$ |
|---|---|---|
| $I_{\mathbf{RC}}$ | $\infty$ | $I_{\mathbf{WB}}$ |
| $I_{\mathbf{KD}}$ | 1 | $I_{\mathbf{KD}}$ |
| $I_{\mathbf{FD}}$ | 2 | $I_{\mathbf{FD}}^2$ |
| $I_{\mathbf{GD}}$ | 1 | $I_{\mathbf{GD}}$ |
| $I_{\mathbf{GG}}$ | $\infty$ | $I_{\mathbf{WB}}$ |
| $I_{\mathbf{RS}}$ | 1 | $I_{\mathbf{RS}}$ |
| $I_{\mathbf{LK}}$ | $\infty$ | $I_{\mathbf{WB}}$ |
| $I_{\mathbf{WB}}$ | $\infty$ | $I_{\mathbf{WB}}$ |
| $I_{\mathbf{YG}}$ | $\infty$ | $I_{\mathbf{WB}}$ |
| $I_{\mathbf{1}}$ | 1 | $I_{\mathbf{1}}$ |
| $I_{\mathbf{0}}$ | 1 | $I_{\mathbf{0}}$ |

Table 4.1: Powers of the basic fuzzy implications w.r.to $\circledast$ and their orders.

Towards discussing these questions, we propose the following characteristic of a fuzzy implication $I$.

### 4.1.1 Order of a Fuzzy Implication $I$ w.r.to $\circledast$ - $\mathcal{O}(I)$

**Definition 4.1.2.** *An $I \in \mathbb{I}$ is said to be of* fixed point order $n \in \mathbb{N}$ *w.r.to the $\circledast$-composition if $n$ is the smallest integer such that $I_{\circledast}^{[n]} = I_{\circledast}^{[n+1]}$. We denote it by $\mathcal{O}(I)$ and refer to it just as the* order *of an $I$.*

Note that for every $I \in \mathbb{I}$ its order is either finite or infinite. Table 4.1 tabulates the orders and the limiting case behavior of the basic fuzzy implications listed in Table 1.2.

Note that $I_{\mathbf{FD}}^2$ in Table 4.1 is obtained as

$$(I_{\mathbf{FD}} \circledast I_{\mathbf{FD}})(x,y) = (I_{\mathbf{FD}})_{\circledast}^{[2]}(x,y) = \begin{cases} 1, & \text{if } x \leq y \text{ or } x \in [0, 0.5] \ , \\ \max(1-x, y), & \text{otherwise} \ . \end{cases}$$

If $\mathcal{O}(I) = 1$ then $I_{\circledast}^{[2]} = I$ and we do not obtain any new implications from $I \in \mathbb{I}$. In algebraic terms, such $I \in \mathbb{I}$ form the set of idempotents in the monoid $(\mathbb{I}, \circledast)$. Note that the characterisation of all such idempotent elements is a non-trivial task. Some partial results are already available in [77, 78]. In the case when $I \in \mathbb{I}$ comes from a specific family of fuzzy implications, say $(S,N)$- or $R$-implications, this functional equation has been dealt with by Shi *et al.* [72]. Since our motivation in this work is to obtain new fuzzy implications from given ones, in the sequel we only study the case when $\mathcal{O}(I) > 1$.

### 4.1.2 Convergence of Powers of Fuzzy Implications $I_{\circledast}^{[n]}$

We observe from Table 4.1, that most of the basic fuzzy implications do converge to $I_{\mathbf{WB}}$ w.r.to the $\circledast$-composition in the limiting case. The following result explores the context in which this is true for any general fuzzy implication. Note the important role played by the law of importation (Definition 3.3.1).

**Theorem 4.1.3.** *Let $I \in \mathbb{I}$ satisfy* (LI) *w.r.to a t-norm $T$.*

*(i) Then* $I_{\circledast}^{[n]}(x,y) = I\left(x_T^{[n]}, y\right)$*, where* $x_T^{[n]} = T\left(x, x_T^{[n-1]}\right)$ *and* $x_T^{[1]} = x$ *for any* $x \in [0,1]$ *and* $n \in \mathbb{N}$.

*(ii) Further, let* $T$ *be Archimedean, i.e., for any* $x, y \in (0,1)$ *there exists an* $n \in \mathbb{N}$ *such that* $x_T^{[n]} < y$.
*Then* $\lim_{n\to\infty} I_{\circledast}^{[n]}(x,y) = \begin{cases} 1, & \text{if } x < 1\ , \\ I(x,y), & \text{if } x = 1\ . \end{cases}$

*(iii) If, in addition,* $I \in \mathbb{I}_{\mathbf{NP}}$ *then* $\lim_{n\to\infty} I_{\circledast}^{[n]} = I_{\mathbf{WB}}$.

*Proof.* (i) Let $x, y \in [0,1]$. Then $I_{\circledast}^{[2]}(x,y) = I(x, I(x,y)) = I(T(x,x),y)$, since $I$ satisfies (LI). Thus $I_{\circledast}^{[2]}(x,y) = I(x_T^{[2]}, y)$. By induction, we obtain $I_{\circledast}^{[n]}(x,y) = I(x_T^{[n]}, y)$.

(ii) Let $\epsilon > 0$ and $x < 1$. Since $T$ is Archimedean, for any $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that $x_T^{[m]} < \epsilon$. Thus, for any $y \in [0,1]$, $I_{\circledast}^{[n]}(x,y) = I(x_T^{[n]}, y) \longrightarrow I(0,y)$ as $n \longrightarrow \infty$ and $\lim_{n\to\infty} I_{\circledast}^{[n]}(x,y) = 1$.

If $x = 1$, then $I_{\circledast}^{[2]}(1,y) = I(1, I(1,y)) = I(T(1,1),y) = I(1,y)$ and in general, $I_{\circledast}^{[n]}(1,y) = I(1,y)$.

(iii) Follows from (ii) and the fact that $I(1,y) = y$.

□

## 4.2 Closure of $I_{\circledast}^{[n]}$ w.r.to the Basic Properties

In this section, given an $I \in \mathbb{I}$ satisfying a particular property **P**, we investigate whether all the powers $I_{\circledast}^{[n]}$ of $I$ satisfy the same property or not, along the lines of Section 3.2. From Lemma 3.2.2 in Chapter 3, we see that if $I \in \mathbb{I}$ satisfies any one of the properties (NP), (IP), self-conjugacy and continuity then $I_{\circledast}^{[n]}$ satisfies the same for all $n \in \mathbb{N}$. Hence it is enough to investigate the preservation of (OP) and (EP).

Before doing this, in the following we prove that if $I \in \mathbb{I}$ satisfies (EP) then the pair $(I, I_{\circledast}^{[n]})$ satisfies (ME) for any $n \in \mathbb{N}$.

**Lemma 4.2.1.** *If* $I \in \mathbb{I}_{\mathbf{EP}}$ *then the pair* $(I, I_{\circledast}^{[n]})$ *satisfies* (ME) *for all* $n \in \mathbb{N}$*. i.e.,*

$$I(x, I_{\circledast}^{[n]}(y,z)) = I_{\circledast}^{[n]}(y, I(x,z)), \qquad \textit{for all } x,y,z \in [0,1]. \tag{4.2}$$

*Proof.* We prove this by using mathematical induction on $n$. For, $n = 1$, $I$ satisfies (4.2), from the (EP) of $I$. Assume that $I$ satisfies (4.2) for $n = k-1$. Now, for $n = k$,

$$\begin{aligned} I(x, I_{\circledast}^{[k]}(y,z)) &= I(x, I(y, I_{\circledast}^{[k-1]}(y,z))) \\ &= I(x, I_{\circledast}^{[k-1]}(y, I(y,z))) \\ &= I_{\circledast}^{[k-1]}(y, I(x, I(y,z))) \\ &= I_{\circledast}^{[k-1]}(y, I(y, I(x,z))) \\ &= I_{\circledast}^{[k]}(y, I(x,z)), for\ all\ x,y,z \in [0,1]. \end{aligned}$$

Thus the pair $(I, I_{\circledast}^{[n]})$ satisfies (ME) for all $n \in \mathbb{N}$. □

**Theorem 4.2.2.** *If $I \in \mathbb{I}_{\mathbf{EP}}$ then $I_{\circledast}^{[n]}$ satisfies* (EP) *for all $n \in \mathbb{N}$.*

*Proof.* Follows from Lemma 4.2.1 and Theorem 3.2.11. $\square$

Note that a similar result as Theorem 4.2.2 is not true for fuzzy implications satisfying (OP). The following result establishes that if both $I, I_{\circledast}^{[2]}$ satisfy (OP) then $I_{\circledast}^{[n]}$ satisfies (OP) for every $n \in \mathbb{N}$.

**Theorem 4.2.3.** *Let $I \in \mathbb{I}_{\mathbf{OP}}$. The following statements are equivalent:*

*(i)* $I_{\circledast}^{[2]}$ *satisfies* (OP), *i.e.,* $I_{\circledast}^{[2]} \in \mathbb{I}_{\mathbf{OP}}$.

*(ii)* $x > I(x, y)$, *whenever* $x > y$.

*(iii)* $I_{\circledast}^{[n]}$ *satisfies* (OP) *for all* $n \in \mathbb{N}$.

*Proof.* (i) $\Longrightarrow$ (ii) : This follows from Theorem 3.2.4 with $J = I$.

(ii) $\Longrightarrow$ (iii) : Let $x > I(x, y)$, whenever $x > y$. We prove that $I_{\circledast}^{[n]}$ satisfies (OP) for $n$. We do this by using mathematical induction on $n$. Since $I_{\circledast}^{[n]}$ satisfies (OP) for $n = 1$, assume that $I_{\circledast}^{[n]}$ satisfies (OP) for $n = k - 1$. Now we show that $I_{\circledast}^{[n]}$ also has (OP) for $n = k$.

- Let $x \leq y$. Then $I(x, y) = 1$. Hence $I_{\circledast}^{[k]}(x, y) = I_{\circledast}^{[k-1]}(x, I(x, y)) = 1$.
- Let $x > y$. Then from our assumption, we have $x > I(x, y)$ . Now from (OP) of $I_{\circledast}^{[k-1]}$, it follows that $I_{\circledast}^{[k]}(x, y) = I_{\circledast}^{[k-1]}(x, I(x, y)) < 1$ and hence $I_{\circledast}^{[n]}$ satisfies (OP) for all $n \in \mathbb{N}$.

(iii) $\Longrightarrow$ (i) : Follows trivially for $n = 2$. $\square$

**Corollary 4.2.4.** *Let $I \in \mathbb{I}_{\mathbf{OP}}$. If $I_{\circledast}^{[m]}$ satisfies* (OP) *for some $m \in \mathbb{N}$ then $I_{\circledast}^{[n]}$ satisfies* (OP) *for all $n \in \mathbb{N}$.*

## 4.3 Closures of $I_{\circledast}^{[n]}$ w.r.to Functional Equations

In Section 4.2, we have shown that the $\circledast$-composition preserves most of the properties of fuzzy implications w.r.to powers of fuzzy implications. Now, in this subsection, we do the same but for functional equations (LI) and CP(N) satisfied by fuzzy implications.

**Theorem 4.3.1.** *If $I \in \mathbb{I}$ satisfies* (LI) *w.r.to a t-norm $T$, then $I_{\circledast}^{[n]}$ also satisfies* (LI) *w.r.to $T$.*

*Proof.* We prove this also by using mathematical induction on $n$. For $n = 1$, $I_{\circledast}^{[n]}$ satisfies (LI). Assume that $I_{\circledast}^{[n]}$ satisfies (LI) w.r.to the same t-norm $T$ for $n = k - 1$, i.e.,

$$I_{\circledast}^{[k-1]}(T(x, y), z) = I_{\circledast}^{[k-1]}(x, I_{\circledast}^{[k-1]}(y, z))$$

for all $x, y, z \in [0, 1]$. From Theorem 4.1.3(i) recall that if $I$ satisfies (LI) w.r.to a t-norm $T$ then

$I_{\circledast}^{[k]}(x,y) = I(x_T^{[k]}, y)$ for all $x, y \in [0,1]$. Now, for any $x, y, z \in [0,1]$,

$$\begin{aligned}
I_{\circledast}^{[k]}(T(x,y),z) &= I\left((T(x,y))_T^{[k]}, z\right) && \lceil \because \text{From Theorem 4.1.3} \\
&= I\left(T(T(x,y),(T(x,y))_T^{[k-1]}), z\right) && \lceil \because \text{From Definition of } x_T^{[n]} \\
&= I\left(T(x,y), I((T(x,y))_T^{[k-1]}, z)\right) && \lceil \because \text{(LI) of } I \\
&= I\left(T(x,y), I_{\circledast}^{[k-1]}(T(x,y),z)\right) && \lceil \because \text{(LI) of } I_{\circledast}^{[k-1]} \\
&= I\left(x, I(y, I_{\circledast}^{[k-1]}(T(x,y),z)\right) && \lceil \because \text{(LI) of } I \\
&= I\left(x, I(y, I_{\circledast}^{[k-1]}(x, I_{\circledast}^{[k-1]}(y,z)))\right) && \lceil \because \text{(LI) of } I_{\circledast}^{[k-1]} \\
&= I\left(x, I_{\circledast}^{[k-1]}(x, I(y, I_{\circledast}^{[k-1]}(y,z)))\right) && \lceil \because\ (I, I_{\circledast}^{[k-1]}) \text{ satisfies (ME)} \\
&= I_{\circledast}^{[k]}(x, I_{\circledast}^{[k]}(y,z)) \ .
\end{aligned}$$

Thus $I_{\circledast}^{[n]}$ satisfies (LI) for all $n \in \mathbb{N}$. □

**Remark 4.3.2.** *If $I \in \mathbb{I}$ satisfies* (CP) *w.r.to some strong negation $N$, then $I_{\circledast}^{[n]}$ may satisfy CP(N) or may not satisfy CP w.r.to any $N$.*

*(i) Let $I(x,y) = \max(N(x), y)$, for some strong negation $N$. Then clearly $I$ satisfies* (CP) *w.r.to $N$ and so does $I_{\circledast}^{[n]}$ for every $n \in \mathbb{N}$.*

*(ii) If we let $I = I_{\mathbf{RC}}$, then $I$ satisfies* (CP) *w.r.to $N_{\mathbf{C}}(x) = 1 - x$. Since $I_{\mathbf{RC}}$ satisfies both* (NP) *and* (EP)*, from Theorem 3.3.13 we see that, for $I_{\circledast}^{[2]}$ to satisfy CP($N_{\mathbf{C}}$), $I$ should be expressible as $I(x,y) = \max(N(x), y)$ for some negation $N$. Clearly, this is not true, since $I_{\mathbf{RC}}$ cannot be expressed as $\max(N(x), y)$ for any negation $N$. Note that this also means that $I_{\circledast}^{[2]}$ does not (cannot) satisfy* (CP) *w.r.to any negation $N$.*

The following result gives a necessary condition on an $I$ satisfying CP(N) such that all the powers of $I_{\circledast}^{[n]}$ of $I$ also satisfy CP(N).

**Lemma 4.3.3.** *Let $I \in \mathbb{I}$ satisfy* (NP)*. If $I$ satisfies CP(N) then $N_{I_{\circledast}^{[n]}}$ is strong for all $n \in \mathbb{N}$.*

*Proof.* If $I \in \mathbb{I}_{\mathbf{NP}}$ satisfies (CP) then from Lemma 3.3.9 we have that $N_I = N$ should be strong. Further, if $I_{\circledast}^{[2]}$ satisfies CP($N$), then from Theorem 3.3.11, we know that $I(x, N(x)) = N(x)$ for all $x \in [0,1]$. Now,

$$N_{I_{\circledast}^{[2]}}(x) = I(x, I(x,0)) = I(x, N_I(x)) = I(x, N(x)) = N(x) \ ,$$

and hence is strong. That $N_{I_{\circledast}^{[n]}}$ is strong for all $n \in \mathbb{N}$ follows by mathematical induction. □

**Lemma 4.3.4.** *Let $I \in \mathbb{I}$ satisfy* (NP) *and* (EP)*. If $I$ satisfies CP(N) then so does $I_{\circledast}^{[n]}$ for all $n \in \mathbb{N}$.*

*Proof.* From Theorem 3.3.13 with $J = I$, we see that $I_{\circledast}^{[2]}$ satisfies CP($N$) if and only if $I(x,y) = \max(N(x), y)$, which clearly satisfies CP($N$). Further, note that $I_{\circledast}^{[n]} = I$, in this case, and hence satisfies CP($N$) for all $n \in \mathbb{N}$. □

The following result follows trivially from Theorem 3.3.14.

**Lemma 4.3.5.** *Let $I \in \mathbb{I}$ satisfy* (NP) *and* (OP)*. Then $I_{\circledast}^{[n]}$ does not satisfy* (CP) *for any $n \in \mathbb{N}$.*

## 4.4 Conclusions

In this chapter, we have observed that one can, very often, generate new fuzzy implications from a single fuzzy implication via the $\circledast$-composition. Further, from the associativity of $\circledast$-composition, we have defined the $n$-th powers of fuzzy implications, w.r.to the $\circledast$-composition, in a natural way and consequently, investigated the nature of powers of fuzzy implications in the limiting case. Given an $I \in \mathbb{I}$ satisfying a desirable property or a functional equation, we have examined whether all the $n$-th powers $I_{\circledast}^{[n]}$ of $I$ satisfy the same. If not, given a property or a functional equation, we have characterised all fuzzy implications such that they and all their $n$-th powers also satisfy the corresponding property or the functional equation.

# Chapter 5

# The $\circledast$-composition : Closures w.r.to Families

*The family is the test of freedom;*
*because the family is the only thing that*
*the free man makes for himself and by himself.*
– G. K. Chesterton (1874 – 1936)

As noted in Chapter 1, the earliest method of generating fuzzy implications was from binary functions on $[0, 1]$, typically other fuzzy logic connectives, like t-norms and t-conorms, often with the help of other connectives like fuzzy negations. These gave rise to two of the most important and well studied families, viz., $(S, N)$- and $R$- implications. For a comprehensive study of these two families, please refer to [8, 9].

While the above generation process generalised classical implications to the setting of fuzzy logic, it was Yager [83] who first proposed a method of obtaining fuzzy implications from unary functions on $[0, 1]$. He proposed two new families of fuzzy implications, viz., $f$- and $g$- implications. For a comprehensive study of these two families, please refer to [7, 8]. These four families have received a lot of interest and importance from the research community due to their use in both theoretical considerations and practical applications.

Note that among the above four families, $(S, N)$- and $R$-implications can be thought of as representative examples of the first category of generation processes listed in Section 1.2, while the families of $f$- and $g$-implications are representatives of the second category. We now study the effect of the proposed $\circledast$ operation, an example from the third category of the generation processes, on the above 4 families - which are representatives of the first and second categories of the mentioned generation processes.

Once again note that, while the families of $f$- and $g$- implications have been completely characterised, see [61], the families of $(S, N)$- and $R$-implications, though two of the oldest, are yet to be characterised completely. In this work, we deal only with those sub-families of $(S, N)$- and $R$-implications for which characterisation results are available.

In this chapter we study the closure of the binary operation $\circledast$ on the above families of fuzzy implications. More explicitly, we investigate the solutions to the following questions:

- Firstly, if $I, J \in \mathbb{I}$ belong to a certain family of fuzzy implications, then does $I \circledast J$ also belong to the same family? If not, what are the conditions on the underlying operations that ensure that $I \circledast J$ also belongs to the same family?

- Secondly, we investigate the effect on member of each of these families under self-composition w.r.to the $\circledast$ operation, or equivalently the powers of fuzzy implications from these families.

We investigate the solutions of the above two problems in Sections 5.1 - 5.4 for fuzzy implications coming from the families of $(S, N)$-, $R$-, $f$-, $g$- implications, respectively.

## 5.1 The $\circledast$-composition : Closures w.r.to $(S, N)$- implications

One of the first generalisations of a classical implication to the setting of fuzzy logic, in fact, multi-valued logic is based on the classical material implication $p \Longrightarrow q \equiv \neg p \vee q$. The family of $(S, N)$-implications were obtained by substituting a fuzzy negation $N$ for '$\neg$' and a t-conorm $S$ for the join / maximum operation '$\vee$' in the preceeding formula and hence the nomenclature.

### 5.1.1 $(S, N)$- implications

**Definition 5.1.1** ([8], Definition 2.4.1). *A function $I : [0, 1]^2 \longrightarrow [0, 1]$ is called an $(S, N)$-* implication *if there exist a t-conorm $S$ and a fuzzy negation $N$ such that*

$$I(x, y) = S(N(x), y), \qquad x, y \in [0, 1]. \tag{5.1}$$

If $I$ is an $(S, N)$- implication then we will often denote it by $I_{S,N}$. The family of all $(S, N)$-implications will be denoted by $\mathbb{I}_{\mathbb{S},\mathbb{N}}$. Table 5.1 gives some of the basic $(S, N)$- implications. For detailed formulae, please see Tables 1.2 and 1.4 for their formulae.

| $S$ | $N$ | $(S, N)$- implication $I_{S,N}$ |
|---|---|---|
| $S_{\mathbf{M}}$ | $N_{\mathbf{C}}$ | $I_{\mathbf{KD}}$ |
| $S_{\mathbf{P}}$ | $N_{\mathbf{C}}$ | $I_{\mathbf{RC}}$ |
| $S_{\mathbf{LK}}$ | $N_{\mathbf{C}}$ | $I_{\mathbf{LK}}$ |
| $S_{\mathbf{nM}}$ | $N_{\mathbf{C}}$ | $I_{\mathbf{FD}}$ |
| any $S$ | $N_{\mathbf{D1}}$ | $I_{\mathbf{D}}$ |
| any $S$ | $N_{\mathbf{D2}}$ | $I_{\mathbf{WB}}$ |

Table 5.1: Examples of basic $(S, N)$- implications.

Here in the following we recall a few of the desirable properties that an $(S, N)$-implication possesses.

**Proposition 5.1.2** (cf. [8], Proposition 2.4.3). *If $I_{S,N}$ is an $(S, N)$- implication, then*

*(i)* $I_{S,N}$ *satisfies* (NP), (EP).

*(ii)* $N_{I_{S,N}} = N$.

*(iii) If $N$ is strong, then $I_{S,N}$ satisfies CP($N$).*

In the literature, the only available characterisations for $(S,N)$- implications are those that are obtained from continuous negations.

**Theorem 5.1.3** ([8], Theorem 2.4.10)**.** *For a function $I\colon [0,1]^2 \longrightarrow [0,1]$ the following statements are equivalent:*

*(i) $I$ is an $(S,N)$- implication with a continuous fuzzy negation $N$.*

*(ii) $I$ satisfies* (I1), (EP) *and $N_I$ is a continuous fuzzy negation.*

*Moreover, the representation of $(S,N)$- implication* (5.1) *is unique in this case.*

Let us denote some special subsets of $\mathbb{I}_{\mathbb{S},\mathbb{N}}$ by the following:

- $\mathbb{I}_{\mathbb{S},\mathbb{N}_\mathbb{C}}$ - $(S,N)$- implications obtained from continuous negations.
- $\mathbb{I}_{\mathbb{S}_\mathbb{C},\mathbb{N}_\mathbb{C}}$ - $(S,N)$- implications obtained from continuous t-conorms and continuous negations.

### 5.1.2 Closure of $\mathbb{I}_{\mathbb{S},\mathbb{N}_\mathbb{C}}$ w.r.to the $\circledast$-composition

In general, the $\circledast$-composition of two $(S,N)$- implications need not be an $(S,N)$- implication. To see this, let $I,J\in\mathbb{I}_{\mathbb{S},\mathbb{N}}$. Then from Proposition 5.1.2, it follows that $I,J$ satisfy (EP). However, from Remark 3.2.7, it follows that $I\circledast J$ need not satisfy (EP) which implies that $I\circledast J$ need not again be an $(S,N)$- implication. Thus the composition of two $(S,N)$- implications need not be an $(S,N)$- implication.

**Remark 5.1.4.** *(i) On the one hand, if we let $I(x,y)=S_{\mathbf{P}}(1-x,y)$, $J(x,y)=S_{\mathbf{P}}(1-x^2,y)$, which are $(S,N)$- implications, then $(I\circledast J)(x,y)=S_{\mathbf{P}}(1-x^3,y)$, is an $(S,N)$- implication.*

*(ii) On the other hand, if $I=I_{\mathbf{RC}}$, $J=I_{\mathbf{KD}}$ both of which are $(S,N)$- implications (see Table 5.1), then their composition $I_{\mathbf{RC}}\circledast I_{\mathbf{KD}}$, as given in Table 2.1, is not an $(S,N)$- implication since it does not satisfy* (EP)*. To see this, let $x=0.3, y=0.8$ and $z=0.5$. Then*

$$(I\circledast J)(0.3,(I\circledast J)(0.8,0.5)) = 0.91\ ,$$

$$\textit{where as}, (I\circledast J)(0.8,(I\circledast J)(0.3,0.5)) = 0.928\ .$$

*Thus $I\circledast J$ does not satisfy* (EP) *and hence $I\circledast J$ does not become an $(S,N)$- implication.*

The following result gives a sufficient condition for $I\circledast J\in\mathbb{I}_{\mathbb{S},\mathbb{N}}$.

**Theorem 5.1.5.** *Let $I(x,y)=S(N_1(x),y)$ and $J(x,y)=S(N_2(x),y)$ be two $(S,N)$- implications. Then $I\circledast J$ is also an $(S,N)$- implication.*

*Proof.* Let $I(x,y)=S(N_1(x),y)$, $J(x,y)=S(N_2(x),y)$ be two $(S,N)$- implications. Now,

$$\begin{aligned}(I\circledast J)(x,y) &= I(x,J(x,y))\\ &= I(x,S(N_2(x),y))\\ &= S(N_1(x),S(N_2(x),y))\\ &= S(S(N_1(x),N_2(x)),y).\end{aligned}$$

Since for every $t$-conorm $S$ and fuzzy negations $N_1, N_2$, the function $N'(x) = S(N_1(x), N_2(x))$ is again a fuzzy negation, we get $(I \circledast J)(x,y) = S(N'(x), y)$ is an $(S,N)$- implication with $S$ being the t-conorm and $N'(x) = S(N_1(x), N_2(x))$ the fuzzy negation. □

**Remark 5.1.6.** *Note that the converse of Theorem 5.1.5 is not true. For example, we know that $I_{\mathbf{WB}}(x,y) = S(N_{\mathbf{D_2}}(x), y)$ for any t-conorm $S$. Let $I(x,y) = S_1(N(x), y)$ be an $(S,N)$- implication where $S_1$ is a t-conorm different from $S$. Then $I \circledast I_{\mathbf{WB}} = I_{\mathbf{WB}} = I_{\mathbf{WB}} \circledast I$, is an $(S,N)$- implication but $S \neq S_1$.*

Let us consider $I, J \in \mathbb{I}_{\mathbb{S},\mathbb{N}_{\mathbb{C}}}$, i.e., $I(x,y) = S_1(N_1(x), y)$ and $J(x,y) = S_2(N_2(x), y)$, where $S_1, S_2$ are t-conorms and $N_1, N_2$ are two continuous negations. Clearly, $I \circledast J$ satisfies (I1). Note, however that, even when we consider only $(S,N)$- implications obtained from continuous negations, we have that the natural negation $N_{I \circledast J}$ of $I \circledast J$ given by

$$N_{I \circledast J}(x) = (I \circledast J)(x, 0) = I(x, J(x, 0)), \qquad x \in [0,1] ,$$

may not be continuous. For instance, when $I(x,y) = S_{\mathbf{nM}}(1-x, y) = I_{\mathbf{FD}}(x,y)$ and $J(x,y) = S_{\mathbf{P}}(1-x, y) = I_{\mathbf{RC}}(x,y) = 1 - x + xy$, we obtain

$$N_{I \circledast J}(x) = \begin{cases} 1, & \text{if } x \leq \frac{1}{2}, \\ 1 - x, & \text{if } x > \frac{1}{2}. \end{cases}$$

Clearly, $N_{I \circledast J}$ is not continuous at $x = \frac{1}{2}$.

As is already shown above, $I \circledast J$ may not preserve (EP). Clearly, if the pair $(I, J)$ satisfies (ME) then from Theorem 3.2.11 we know that $I \circledast J$ will satisfy (EP). While, this is neither sufficient nor necessary to ensure $I \circledast J \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$, the following result shows that this is equivalent to the condition $S_1 = S_2$ of Theorem 5.1.5 when $I, J \in \mathbb{I}_{\mathbb{S},\mathbb{N}_{\mathbb{C}}}$.

**Theorem 5.1.7.** *Let $I(x,y) = S_1(N_1(x), y), J(x,y) = S_2(N_2(x), y)$ be two $(S,N)$-implications such that $N_1, N_2$ are continuous negations. Then the following statements are equivalent:*

*(i)* $S_1 = S_2$.

*(ii)* $I, J$ *satisfy* (ME).

*Proof.* (i) $\Longrightarrow$ (ii). Let $S_1 = S_2 = S$. Then we have $I(x,y) = S(N_1(x), y), J(x,y) = S(N_2(x), y)$. Now,

$$\text{L.H.S of (ME)} = I(x, J(y,z)) = S(N_1(x), S(N_2(y), z)) = S(S(N_1(x), N_2(y)), z) \text{ , and}$$

$$\text{R.H.S of (ME)} = J(y, I(x,z)) = S(N_2(y), S(N_1(x), z)) = S(S(N_1(x), N_2(y)), z) \text{ .}$$

Thus $I, J$ satisfy (ME).

(ii) $\Longrightarrow$ (i). Let $I, J$ satisfy (ME), i.e., $I(x, J(y,z)) = J(y, I(x,z))$ for all $x, y, z \in [0,1]$. This implies that

$$S_1(N_1(x), S_2(N_2(y), z)) = S_2(N_2(y, S_1(N_1(x), z)))$$

for all $x, y, z \in [0,1]$. Letting $z = 0$, we obtain, for any $x, y \in [0,1]$,

$$S_1(N_1(x), N_2(y)) = S_2(N_1(x), N_2(y)) \text{ .}$$

Since the ranges of $N_1, N_2$ are equal to $[0,1]$, we get, $S_1(a,b) = S_2(a,b)$ for all $a, b \in [0,1]$. $\square$

Finally, if we restrict the underlying t-conorm $S$ also to be continuous, i.e., if we consider $\mathbb{I}_{\mathbb{S}_\mathbb{C},\mathbb{N}_\mathbb{C}} \subsetneq \mathbb{I}_{\mathbb{S},\mathbb{N}}$, then the following result is immediate:

**Corollary 5.1.8.** *Let $I, J \in \mathbb{I}_{\mathbb{S}_\mathbb{C},\mathbb{N}_\mathbb{C}}$. If $I, J$ satisfy* (ME) *then $I \circledast J \in \mathbb{I}_{\mathbb{S}_\mathbb{C},\mathbb{N}_\mathbb{C}}$.*

### 5.1.3 Powers of $(S,N)$- implications w.r.to the $\circledast$-composition

Note that if $I \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$, then it satisfies (EP) and hence from Theorem 4.2.2 it follows that $I_\circledast^{[n]}$ satisfies (EP) for all $n \in \mathbb{N}$.

From Theorem 5.1.5 we have the following:

**Lemma 5.1.9.** *Let $I \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$. Then $I_\circledast^{[n]}$ is also an $(S,N)$- implication.*

While $I_\circledast^{[n]} \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$ for every $n \in \mathbb{N}$, it is interesting to study those $(S,N)$- implications that give rise to newer $(S,N)$- implications. Alternately, it is enough to study $I \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$ such that $\mathcal{O}(I) = 1$, i.e., those $I \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$ that satisfy the following equation (5.2).

$$I(x, I(x,y)) = I(x,y), \qquad x, y \in [0,1]. \tag{5.2}$$

This was already dealt with by Shi *et al.* [74].

**Theorem 5.1.10** ([74], Theorem 10)**.** *An $I \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$ satisfies* (5.2) *if and only if the range of $N$ is a subset of the idempotent elements of $S$.*

**Corollary 5.1.11** ([74], Corollary 2 )**.** *Let $I \in \mathbb{I}_{\mathbb{S},\mathbb{N}_\mathbb{C}}$, i.e., $I(x,y) = S(N(x), y)$ for some continuous negation $N$. Then $I$ satisfies* (5.2) *if and only if $S = S_{\mathbf{M}} = \max$.*

The above results indicate that every $(S,N)$- implication (where $N$ is continuous) other than those derived from the t-conorm *maximum* will give rise to newer $(S,N)$- implications on self-composition with $\circledast$.

Finally, we only note that if $I \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$ then $\mathcal{O}(I) = m > 1$ does not mean that $m = \infty$. For instance, the Fodor implication $I_{\mathbf{FD}} \in \mathbb{I}_{\mathbb{S},\mathbb{N}}$ but $\mathcal{O}(I_{\mathbf{FD}}) = 2$ (see, Table 4.1).

## 5.2 The $\circledast$-composition : Closures w.r.to R- implications

A second family of fuzzy implications obtained under the first category of the generation processes listed in Section 1.2, is the family of residual implications. This is a generalisation of the implication in the classical intuitionistic logic to the setting of fuzzy logic. Once again, for more details on this family regarding their properties, intersections with other families, etc., see for instance, [8].

### 5.2.1 R- implications

**Definition 5.2.1** ([8], Definition 2.5.1)**.** *A function $I : [0,1]^2 \longrightarrow [0,1]$ is called an* R- implication *if there exists a t-norm $T$ such that*

$$I(x,y) = \sup\{t \in [0,1] | T(x,t) \le y\}, \qquad x, y \in [0,1].$$

If $I$ is an $R$-implication generated from a t-norm $T$, then it is denoted by $I_T$. The family of all $R$-implications will be denoted by $\mathbb{I}_{\mathbb{T}}$.

| t-norm $T$ | R-implication $I_T$ |
|---|---|
| $T_{\mathbf{M}}$ | $I_{\mathbf{GD}}$ |
| $T_{\mathbf{P}}$ | $I_{\mathbf{GG}}$ |
| $T_{\mathbf{LK}}$ | $I_{\mathbf{LK}}$ |
| $T_{\mathbf{D}}$ | $I_{\mathbf{WB}}$ |
| $T_{\mathbf{nM}}$ | $I_{\mathbf{FD}}$ |

Table 5.2: Examples of basic R-implications. For detailed formulae, please see Tables 1.2 and 1.3.

**Theorem 5.2.2** ([8], Theorem 2.5.4)**.** *If $T$ is a t-norm, then $I_T$ satisfies* (NP) *and* (IP).

As in the case of $(S, N)$-implications, the characterisation of $R$-implications is available only for $R$-implications those are obtained from *left-continuous* t-norms.

**Theorem 5.2.3** ([8], Theorem 2.5.17)**.** *For a function $I\colon [0,1]^2 \longrightarrow [0,1]$ the following statements are equivalent:*

*(i) $I$ is an R-implication generated from a left-continuous t-norm.*

*(ii) $I$ satisfies* (I2), (EP), (OP) *and it is right-continuous with respect to the second variable.*

*Moreover, the representation of an R-implication, up to a left-continuous t-norm, is unique.*

The set of all R-implications generated from left-continuous t-norms will be denoted by $\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$. Then from Theorem 5.2.3 it follows that $\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}} \subsetneq \mathbb{I}_{\mathbb{T}} \cap \mathbb{I}_{\mathbf{OP}}$. Once again, we focus on the problem of, if $I, J \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$ then does $I \circledast J \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$?

### 5.2.2 Closure of $\mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$ w.r.to the $\circledast$-composition

The $\circledast$-composition of two $R$-implications obtained from left-continuous t-norms need not be an $R$-implication obtained from a left-continuous t-norm.

**Remark 5.2.4.** *(i) On the one hand, let $I = I_{\mathbf{GD}}, J = I_{\mathbf{LK}}$. Clearly $I, J \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$ (see Table 5.2). However, from Remark 3.2.3 we know that $I_{\mathbf{GD}} \circledast I_{\mathbf{LK}}$ does not satisfy* (OP) *and hence from the characterisation Theorem 5.2.3 we see that $I_{\mathbf{GD}} \circledast I_{\mathbf{LK}} \notin \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$.*

*(ii) On the other hand, consider the Goguen and Gödel implications $I_{\mathbf{GG}}, I_{\mathbf{GD}}$ which are two $R$-implications generated from left-continuous t-norms $T_{\mathbf{P}}, T_{\mathbf{M}}$, respectively. Then $I_{\mathbf{GG}} \circledast I_{\mathbf{GD}} = I_{\mathbf{GG}} \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$.*

Even though our focus on the problem is for R-implications obtained from *left-continuous* t-norms, in the following we do investigate the solutions of this problem for more generalised classes of R-implications than those obtained from *left-continuous* t-norms. Towards this, the following result shows that the $\circledast$-composition of two $R$-implications, not necessarily obtained from left-continuous t-norms, is an R-implication only if one of them is the Gödel implication $I_{\mathbf{GD}}$.

**Lemma 5.2.5.** *Let $I, J \in \mathbb{I}_{\mathbb{T}} \cap \mathbb{I}_{\mathbf{OP}}$. Then the following statements are equivalent:*

*(i) $I \circledast J \in \mathbb{I}_{\mathbb{T}} \cap \mathbb{I}_{\mathbf{OP}}$.*

*(ii)* $J = I_{\mathbf{GD}}$.

*(iii)* $I \circledast J = I$.

*Proof.* (i) $\Longrightarrow$ (ii). Let $I \circledast J \in \mathbb{I}_{\mathbb{T}} \cap \mathbb{I}_{\mathbf{OP}}$. Then from Theorem 3.2.4, it follows that $y \geq J(x, y)$ for all $x > y$. Since $J$ is an $R$-implication it follows that $J(x, y) \geq y$ for all $x, y \in [0, 1]$. Thus $J(x, y) = y$ for all $x > y$ and hence $J = I_{\mathbf{GD}}$.

(ii) $\Longrightarrow$ (iii). This follows from a direct verification.

(iii) $\Longrightarrow$ (i). This follows from the hypothesis.

□

For more details on when an $I \in \mathbb{I}_{\mathbb{T}}$ satisfies (OP), please refer to [9], Proposition 5.8.

### 5.2.3 Powers of R-implications w.r.to the $\circledast$-composition

From Lemma 5.2.5 the following result is obvious:

**Lemma 5.2.6.** *Let $I \in \mathbb{I}_{\mathbb{T}_{\mathbb{LC}}}$. Then the following statements are equivalent:*

*(i)* $I_{\circledast}^{[n]} \in \mathbb{I}_{\mathbb{T}}$ *for all* $n \in \mathbb{N}$.

*(ii)* $I = I_{\mathbf{GD}}$.

## 5.3 The $\circledast$-composition : Closures w.r.to $f$- implications

While the previous sections dealt with fuzzy implications obtained from other fuzzy logic connectives, which were, in essence, generalisations of classical implications to the setting of fuzzy logic, in the following sections we deal with the Yager's families of fuzzy implications, which were proposed by Yager in [83]. In fact, a specific example which can be thought of as a precursor to the generalisation of Yager was firstly suggested by Villar and Sanz-Bobi in [79] and the first such example they obtained was the Yager implication $I_{\mathbf{YG}}$. However, it was Yager who made use of the additive generators of Archimedean t-norms and t-conorms as unary operators on $[0, 1]$ to propose two new families of fuzzy implications, viz., $f$- and $g$- implications .

Towards this end, in this section, we investigate the behavior of $\circledast$-composition on Yager's class of $f$- implications.

### 5.3.1 $f$- implications

In this subsection we present the definitions, some relevant properties and characterisations of $f$-implications and proceed along the lines similar to that of Sections 5.1 and 5.2, exploring their closures w.r.to the $\circledast$-composition.

**Definition 5.3.1** ([8], Definition 3.1.1). *Let $f : [0, 1] \longrightarrow [0, \infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I : [0, 1]^2 \longrightarrow [0, 1]$ defined by*

$$I(x, y) = f^{-1}(x \cdot f(y)) \ , \qquad x, y \in [0, 1] \ , \tag{5.3}$$

*with the understanding* $0 \cdot \infty = 0$*, is called an* f-implication*. If $I$ is an $f$-implication then it is denoted by $I_f$. The family of all $f$-implications will be denoted by $\mathbb{I}_{\mathbb{F}}$.*

In the following, we list out some important but relevant results that give the properties and characterisations of the family of $f$-implications.

**Theorem 5.3.2** ([8], Theorem 3.17)**.** *If $f$ is an $f$-generator then $I_f$ satisfies* (NP) *and* (EP).

As shown by Baczyński and Jayaram [7] if $f$ is an $f$-generator such that $f(0) < \infty$, then the function $f_1 \colon [0,1] \longrightarrow [0,1]$ defined by

$$f_1(x) = \frac{f(x)}{f(0)}\,, \qquad x \in [0,1]\,, \tag{5.4}$$

is a well defined $f$-generator and the $f$-implications defined from both $f$ and $f_1$ are identical, i.e., $I_f \equiv I_{f_1}$ and moreover $f_1(0) = 1$. In other words, it is enough to consider only decreasing generators $f$ for which $f(0) = \infty$ or $f(0) = 1$.

Let us denote by

- $\mathbb{I}_{\mathbb{F},\infty}$ – the family of all $f$- implications such that $f(0) = \infty$,
- $\mathbb{I}_{\mathbb{F},1}$ – the family of all $f$- implications such that $f(0) = 1$,
- Clearly, $\mathbb{I}_{\mathbb{F}} = \mathbb{I}_{\mathbb{F},\infty} \cup \mathbb{I}_{\mathbb{F},1}$.

Now, in the following we recall the characterisations of $f$-implications belonging to $\mathbb{I}_{\mathbb{F},\infty}$ and $\mathbb{I}_{\mathbb{F},1}$.

**Theorem 5.3.3** (cf. [61], Theorem 6)**.** *Let $I \colon [0,1] \longrightarrow [0,1]$ be a binary function. Then the following statements are equivalent:*

*(i) $I$ is an $f$-implication with $f(0) < \infty$, i.e., $I \in \mathbb{I}_{\mathbb{F},1}$.*

*(ii) $I$ satisfies* (LI) *w.r.to $T_{\mathbf{P}}$ and $N_I$ is strict negation.*

*Moreover, the $f$- generator is unique upto a positive multiplicative constant and is given by $f(x) = N_I^{-1}(x)$.*

**Theorem 5.3.4** (cf. [61], Theorem 12)**.** *Let $I \colon [0,1]^2 \longrightarrow [0,1]$ be a binary function. Then the following statements are equivalent:*

*(i) $I$ is an $f$-implication with $f(0) = \infty$, i.e., $I \in \mathbb{I}_{\mathbb{F},\infty}$.*

*(ii) $I$ satisfies* (LI) *w.r.to $T_{\mathbf{P}}$, $I$ is continuous except $(0,0)$ and $I(x,y) = 1 \Longleftrightarrow x = 0$ or $y = 1$.*

### 5.3.2 Closure of $\mathbb{I}_{\mathbb{F}}$ w.r.to the $\circledast$-composition

Let $I, J \in \mathbb{I}_{\mathbb{F}}$. From Theorem 5.3.2, it follows that both $I, J$ satisfy (EP). However, from Remark 3.2.7, we know that $I \circledast J$- need not satisfy (EP). Thus the $\circledast$ composition of two $f$-implications need not be an $f$-implication.

**Remark 5.3.5.** *(i) Consider the fuzzy implications $I = I_{\mathbf{YG}} \in \mathbb{I}_{\mathbb{F},\infty}$ and $J = I_{\mathbf{RC}} \in \mathbb{I}_{\mathbb{F},1}$. Then their composition $I \circledast J = I_{\mathbf{YG}} \circledast I_{\mathbf{RC}}$ is as given in Remark 3.3.2(i). Once again, from the same remark, we see that $I_{\mathbf{YG}} \circledast I_{\mathbf{RC}}$ does not satisfy* (EP) *and hence from Remark 7.3.1 in [8] cannot satisfy* (LI) *w.r.to any t-norm $T$. Clearly, now, $I_{\mathbf{YG}} \circledast I_{\mathbf{RC}} \notin \mathbb{I}_{\mathbb{F}}$.*

*(ii) Let $I(x,y) = I_{\mathbf{RC}}(x,y) = 1 - x + xy$, $J(x,y) = 1 - x^2 + x^2 y$. Then $N_J(x) = 1 - x^2$, a strict negation. Moreover, from Remark 3.3.2(iii), $J$ satisfies* (LI) *w.r.to $T_{\mathbf{P}}$. Finally from Theorem 5.3.3, it follows that $J \in \mathbb{I}_{\mathbb{F}}$. Once again, by the above arguments the composition $I \circledast J$ which is given by $(I \circledast J)(x,y) = 1 - x^3 + x^3 y$, also belongs to $\mathbb{I}_{\mathbb{F}}$.*

To begin with, the following results show that, if the $\circledast$-composition of two $f$-implications $I_{f_1}, I_{f_2}$ is again an $f$-implication, then either both $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$ or both $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$.

**Lemma 5.3.6.** *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F}}$ be such that $I_{f_1} \circledast I_{f_2} = I_h \in \mathbb{I}_{\mathbb{F}}$, for some $f$-generators $f_1, f_2, h$. If $f_1(0) < \infty$ and $f_2(0) < \infty$ then $h(0) < \infty$.*

*Proof.* Let $I_{f_1}, I_{f_2}$ be two $f$-implications such that $I_{f_1} \circledast I_{f_2} = I_h$ is an $f$-implication. Then $f_1, f_2, h$ satisfy the following equation.

$$f_1^{-1}\left(x \cdot f_1 \circ f_2^{-1}(x \cdot f_2(y))\right) = h^{-1}(x \cdot h(y)), \qquad x, y \in [0,1]. \tag{5.5}$$

Let $x > 0$ and $y = 0$. Then

$$\begin{aligned} f_2^{-1}(x \cdot f_2(0)) > 0 &\Longrightarrow f_1(f_2^{-1}(x \cdot f_2(0))) < \infty \\ &\Longrightarrow x \cdot f_1 \circ f_2^{-1}(x \cdot f_2(0)) < \infty \\ &\Longrightarrow f_1^{-1}\left(x \cdot f_1 \circ f_2^{-1}(x \cdot f_2(0))\right) > 0 \text{ , i.e., L.H.S. of (5.5)} > 0 \text{ .} \end{aligned}$$

Thus we have R.H.S. of (5.5) is also greater than $0$ or equivalently, $h^{-1}(x \cdot h(0)) > 0$. Now, if $h(0) = \infty$ then $x \cdot h(0) = \infty$ and hence $h^{-1}(x \cdot h(0)) = 0$, a contradiction. Thus $h(0) < \infty$. This completes the proof. □

**Theorem 5.3.7.** *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F}}$ be such that $I_{f_1} \circledast I_{f_2} = I_h \in \mathbb{I}_{\mathbb{F}}$, for some $f$-generators $f_1, f_2, h$. Then the following statements are equivalent:*

*(i) $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$.*

*(ii) $I_h \in \mathbb{I}_{\mathbb{F},\infty}$.*

*Proof.* Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F}}$ be such that $I_{f_1} \circledast I_{f_2} = I_h$ is an $f$-implication, for some $f$-generators $f_1, f_2, h$.

(i) $\Longrightarrow$ (ii). Let $f_1(0) = \infty = f_2(0)$. We prove that $h(0) = \infty$. Now, $I_{f_1} \circledast I_{f_2} = I_h$ is the expression given in (5.5). Once again, let $x > 0$ and $y = 0$ in (5.5). Then

$$\begin{aligned} \text{L.H.S. of (5.5)} &= f_1^{-1}(x \cdot f_1 \circ f_2^{-1}(x \cdot f_2(0)) \\ &= f_1^{-1}(x \cdot f_1 \circ f_2^{-1}(x \cdot \infty)) \\ &= f_1^{-1}(x \cdot f_1 \circ f_2^{-1}(\infty)) \\ &= f_1^{-1}(x \cdot f_1(0)) \\ &= f_1^{-1}(x \cdot \infty) = f_1^{-1}(\infty) = 0. \end{aligned}$$

Now R.H.S. of (5.5) $= h^{-1}(x \cdot h(0)) = 0$ implies that $x \cdot h(0) = h(0)$. Since $x > 0$, either $h(0) = 0$ or $h(0) = \infty$. Now, from monotonicity of $h$, it follows that $h(0) = \infty$.

(ii) $\Longrightarrow$ (i). Let $h(0) = \infty$. We prove that $f_1(0) = \infty = f_2(0)$. Now, R.H.S. of (5.5) = $h^{-1}(x \cdot h(0)) = 0$ and hence L.H.S. of (5.5) = $f_1^{-1}(x \cdot f_1 f_2^{-1}(x \cdot f_2(0))) = 0$. i.e., $x \cdot f_1 f_2^{-1}(x \cdot f_2(0)) = f_1(0)$.

Suppose that $f_1(0) < \infty$. Once again, we have the following implications:

$$f_1 \circ f_2^{-1}(x \cdot f_2(0)) < \infty \Longrightarrow f_2^{-1}(x \cdot f_2(0)) > 0 \Longrightarrow f_2(0) < \infty .$$

However, from Lemma 5.3.6, we know that if $f_1(0) < \infty$ and $f_2(0) < \infty$ then $h(0) < \infty$, a contradiction to the fact that $h(0) = \infty$.

□

Similar to Theorem 5.3.7, we have the following result:

**Corollary 5.3.8.** *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F}}$ be such that $I_{f_1} \circledast I_{f_2} = I_h \in \mathbb{I}_{\mathbb{F}}$, for some $f$-generators $f_1, f_2, h$. Then the following statements are equivalent:*

*(i)* $I_{f_1}$ *or* $I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$.

*(ii)* $I_h \in \mathbb{I}_{\mathbb{F},1}$.

*Proof.* On the one hand, if $I_h \in \mathbb{I}_{\mathbb{F},1}$ then the fact that one of $I_{f_1}$ or $I_{f_2}$ should belong to $\mathbb{I}_{\mathbb{F},1}$ follows from the contrapositive of Theorem 5.3.7.

On the other hand, if one of $I_{f_1}$ or $I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$ but $I_{f_1} \circledast I_{f_2} = I_h \in \mathbb{I}_{\mathbb{F}}$, then once again it is clear from Theorem 5.3.7 that $I_h$ cannot be in $\mathbb{I}_{\mathbb{F},\infty}$ and hence is in $\mathbb{I}_{\mathbb{F}} \setminus \mathbb{I}_{\mathbb{F},\infty} = \mathbb{I}_{\mathbb{F},1}$. □

Note that what the above results show is, if two $f$-implications compose to give an $f$-implication, then where their composition would fall. However, it is not true that the composition of any arbitrary pair of $f$-implications from $\mathbb{I}_{\mathbb{F},\infty}$ or $\mathbb{I}_{\mathbb{F},1}$ will again be an $f$-implication, as the following example shows.

**Example 5.3.9.** *(i) Let $I_{f_1}(x, y) = I_{\mathbf{YG}}(x, y)$ and $I_{f_2}(x, y) = \log_2(1 + (2^y - 1)^x)$ whose $f$-generator is $f_2(x) = -\ln(2^x - 1)$. From Example 3.1.3 (i) and (iv) in [8], it follows that both $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$. Now, $I_{f_1} \circledast I_{f_2}$ is given by*

$$(I_{f_1} \circledast I_{f_2})(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 , \\ \log_2(1 + (2^{y^x} - 1)^x), & \text{if } x > 0 \text{ or } y > 0 . \end{cases}$$

*It is easy to check that*

$$(I_{f_1} \circledast I_{f_2})(0.3, (I_{f_1} \circledast I_{f_2})(0.5, 0.8)) = 0.2761 ,$$

*while*

$$(I_{f_1} \circledast I_{f_2})(0.5, (I_{f_1} \circledast I_{f_2})(0.3, 0.8)) = 0.2242 .$$

*This implies that $I_{f_1} \circledast I_{f_2}$ does not satisfy* (EP) *and hence $I_{f_1} \circledast I_{f_2} \notin \mathbb{I}_{\mathbb{F}}$.*

*(ii) Let $f_1(x) = 1 - x^2$ and for $e = \dfrac{\sqrt{5}-1}{2}$,*

$$f_2(x) = \begin{cases} 1 + \frac{x(e-1)}{e}, & \text{if } x \le e\ , \\ e + \frac{(x-e)e}{e-1}, & \text{if } x \ge e\ . \end{cases}$$

*Clearly, both $f_1, f_2$ are decreasing functions with $f_1(0) = f_2(0) = 1$ and $f_1(1) = f_2(1) = 0$ ( in fact, both $f_1, f_2$ are fuzzy negations), and hence can be used as $f$-generators to obtain $f$-implications, $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$ using* (5.3)*, where $f_1^{-1}(x) = \sqrt{1-x}$ for $x \in [0,1]$ and $f_2^{-1}(x) = f_2(x)$ for all $x \in [0,1]$. Now,*

$$(I_{f_1} \circledast I_{f_2})(x,y) = f_1^{-1}\left(x \cdot f_1 \circ f_2(x \cdot f_2(y))\right)\ , x, y \in [0,1]\ .$$

*Once again, it is easy to check that*

$$(I_{f_1} \circledast I_{f_2})(0.6, (I_{f_1} \circledast I_{f_2})(0.7, 0)) = 0.8904\ ,$$

$$\textit{while, } (I_{f_1} \circledast I_{f_2})(0.7, (I_{f_1} \circledast I_{f_2})(0.6, 0)) = 0.9036\ .$$

*This implies that $I_{f_1} \circledast I_{f_2}$ does not satisfy* (EP) *and hence $I_{f_1} \circledast I_{f_2} \notin \mathbb{I}_{\mathbb{F}}$.*

In the following, we investigate the conditions under which the composition of two $f$- implications will be an $f$-implication. Towards this end, we present a few small but important results.

**Lemma 5.3.10** ([8], Lemma 3.1.8)**.** *Let $I_f$ be an $f$- implication. Then $I_f(\cdot, y)$ is one-one for all $y \in (0,1)$.*

**Lemma 5.3.11.** *Let $I_f$ be an $f$- implication and let $x > 0$ . Then*

*(i) $I_f(x, \cdot)$ is one-one.*

*(ii) Further, if $f(0) = \infty$ then $I_f(x, \cdot)$ is an increasing bijection on $[0,1]$.*

*Proof.* Let $I = I_f$ be an $f$- implication and let $x > 0$.

(i) Let $y_1, y_2 \in [0,1]$ and $I_f(x, y_1) = I_f(x, y_2)$. Then

$$\begin{aligned} f^{-1}(x \cdot f(y_1)) = f^{-1}(x \cdot f(y_2)) &\Longrightarrow x \cdot f(y_1) = x \cdot f(y_2) \\ &\Longrightarrow f(y_1) = f(y_2) \\ &\Longrightarrow y_1 = y_2\ . \end{aligned}$$

Thus $I_f(x, \cdot)$ is one-one for all $x \in (0,1]$.

(ii) Let $f(0) = \infty$. Now, $I_f(x, 0) = f^{-1}(x \cdot f(0)) = f^{-1}(x \cdot \infty) = f^{-1}(\infty) = 0$ and $I_f(x, 1) = 1$.

Since $f$ is continuous, the range of $I_f(x, \cdot)$ is the entire $[0,1]$ and hence $I_f(x, \cdot)$ is an increasing bijection on $[0,1]$. □

**Remark 5.3.12.** *In Lemma 5.3.11(ii), if $f(0) < \infty$ then $I_f(x, \cdot)$ need not be a bijection on $[0,1]$ for all $x > 0$. For example, let $f(x) = 1 - x$, which is the $f$- generator of the Reichenbach implication $I_{\mathbf{RC}} \in \mathbb{I}_{\mathbb{F},1}$. Clearly, $f(0) = 1 < \infty$. When $x = 0.2$, $I_f(0.2, y) = 0.8 + 0.2y$ for all $y \in [0,1]$. Here the range of $I_f(0.2, \cdot)$ is equal to $[0.8, 1]$.*

**Theorem 5.3.13.** *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$. Then the following statements are equivalent:*

*(i)* $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$.

*(ii)* $I_{f_1} \circledast I_{f_2}$ *satisfies* (LI) *w.r.to* $T_{\mathbf{P}}$.

*(iii)* $I_{f_1}, I_{f_2}$ *are mutually exchangeable, i.e., the pair* $(I_{f_1}, I_{f_2})$ *satisfies* (ME).

*Proof.* (i) $\implies$ (ii) : Let $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$. Then from the characterisation result, Theorem 5.3.4, of $f$-implications with $f(0) = \infty$, it follows that $I_{f_1} \circledast I_{f_2}$ satisfies (LI) w.r.to $T_{\mathbf{P}}$.

(ii) $\implies$ (iii) : Now assume that $I_{f_1} \circledast I_{f_2}$ satisfies (LI) w.r.to $T_{\mathbf{P}}$. From Lemma 5.3.11(i), it follows that $I_{f_1}(x, \cdot)$ is one-one and from Lemma 5.3.11(ii) that $I_{f_2}(x, \cdot)$ is a bijection on $[0, 1]$. Now, from Theorem 3.3.5, it follows immediately that $I_{f_1}, I_{f_2}$ are mutually exchangeable.

(iii) $\implies$ (i) : Let $I_{f_1}, I_{f_2}$ be mutually exchangeable. Then from Theorem 3.3.5, $I_{f_1} \circledast I_{f_2}$ satisfies (LI) w.r.to $T_{\mathbf{P}}$. Since $I_{f_1}, I_{f_2}$ are continuous except at $(0, 0)$, $I_{f_1} \circledast I_{f_2}$is also continuous except at $(0, 0)$. Moreover

$$\begin{aligned}
(I_{f_1} \circledast I_{f_2})(x, y) = 1 &\iff I_{f_1}(x, I_{f_2}(x, y)) = 1 \\
&\iff x = 0 \text{ or } I_{f_2}(x, y) = 1 \\
&\iff x = 0 \text{ or } x = 0 \text{ or } y = 1 \\
&\iff x = 0 \text{ or } y = 1\ .
\end{aligned}$$

Now, from Theorem 5.3.4 we see that $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},\infty}$. □

In the case $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$, then we have only some sufficient conditions as the following results show.

**Lemma 5.3.14.** *If* $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$*, then* $N_{I_{f_1} \circledast I_{f_2}}$ *is a strict negation.*

*Proof.* Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$. Then from Theorem 5.3.3 we know that $N_{I_{f_1}}, N_{I_{f_2}}$ are strict negations. Now,

$$(N_{I_{f_1} \circledast I_{f_2}})(x) = (I_{f_1} \circledast I_{f_2})(x, 0) = f_1^{-1}(x \cdot f_1 f_2^{-1}(x \cdot f_2(0))) = f_1^{-1}(x \cdot f_1(N_{I_{f_2}}(x))).$$

Clearly, $N_{I_{f_1} \circledast I_{f_2}}$ being the composition of continuous functions is continuous. To show that it is a strict negation, it suffices to show that it is strictly decreasing. From the antitonicity of $f_1, f_2$ we have the following implications:

$$\begin{aligned}
x_1 < x_2 &\implies N_{I_{f_2}}(x_1) > N_{I_{f_2}}(x_2) \\
&\implies f_1(N_{I_{f_2}}(x_1)) < f_1(N_{I_{f_2}}(x_2)) \\
&\implies x_1 \cdot f_1(N_{I_{f_2}}(x_1)) < x_1 \cdot f_1(N_{I_{f_2}}(x_2)) < x_2 \cdot f_1(N_{I_{f_2}}(x_2)) \\
&\implies f_1^{-1}(x_1 \cdot f_1(N_{I_{f_2}}(x_1))) > f_1^{-1}(x_2 \cdot f_1(N_{I_{f_2}}(x_2))) \\
&\text{i.e., } (N_{I_{f_1} \circledast I_{f_2}})(x_1) > (N_{I_{f_1} \circledast I_{f_2}})(x_2)\ .
\end{aligned}$$

This completes the proof. □

**Theorem 5.3.15.** *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$. If $I_{f_1}, I_{f_2}$ are mutually exchangeable then $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$.*

*Proof.* Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$. Then from Theorem 5.3.3, $I_{f_1}, I_{f_2}$ satisfy (LI) w.r.to $T_{\mathbf{P}}$ and $N_{I_{f_1}}, N_{I_{f_2}}$ are strict negations. Now from Lemma 3.3.3, if $I_{f_1}, I_{f_2}$ are mutually exchangeable then $I_{f_1} \circledast I_{f_2}$ satisfies (LI) w.r.to $T_{\mathbf{P}}$. Moreover from Lemma 5.3.14, it follows directly that $N_{I_{f_1} \circledast I_{f_2}}$ is a strict negation. Again from Theorem 5.3.3 it follows that $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$. $\square$

Note, however, it is not clear whether the converse of Lemma 5.3.15 is true, i.e., whether the mutual exchangeability of $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$ is also necessary for $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$ and hence we have only the following result, the proof of which follows from Theorem 5.3.3 and Lemma 5.3.15.

**Corollary 5.3.16.** *Let $I_{f_1}, I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$. Let us consider the following statements:*

*(i)* $I_{f_1} \circledast I_{f_2} \in \mathbb{I}_{\mathbb{F},1}$.

*(ii)* $I_{f_1} \circledast I_{f_2}$ *satisfies* (LI) *w.r.to* $T_{\mathbf{P}}$.

*(iii)* $I_{f_1}, I_{f_2}$ *are mutually exchangeable.*

*Then, the following implications are true: (i) $\Longleftrightarrow$ (ii) and (iii) $\Longrightarrow$ (i).*

### 5.3.3 Powers of $f$-implication w.r.to the $\circledast$-composition

**Theorem 5.3.17.** *If $I_f \in \mathbb{I}_{\mathbb{F},\infty}$ then $(I_f)^{[n]}_{\circledast} \in \mathbb{I}_{\mathbb{F},\infty}$ for all $n \in \mathbb{N}$.*

*Proof.* Let $I_f \in \mathbb{I}_{\mathbb{F},\infty}$. The proof is done by using mathematical induction on $n$.

**Base Step :** Note that $(I_f)^{[2]}_{\circledast}(x,y) = (I_f \circledast I_f)(x,y) = I_f(x, I_f(x,y))$, since $I_f$ satisfies (EP), the (repeated) pair $(I_f, I_f)$ satisfies (ME) and from Theorem 5.3.13(iii) we see that $I_f \circledast I_f \in \mathbb{I}_{\mathbb{F},\infty}$.

**Induction Step :** Now, let us assume that $(I_f)^{[k-1]}_{\circledast} \in \mathbb{I}_{\mathbb{F},\infty}$. Since $(I_f)^{[k-1]}_{\circledast} \in \mathbb{I}_{\mathbb{F},\infty}$, $(I_f)^{[k-1]}_{\circledast}$ satisfies (LI) w.r.to $T_{\mathbf{P}}$ and is continuous except at $(0,0)$ and $(I_f)^{[k-1]}_{\circledast}(x,y) = 1 \Longleftrightarrow x = 0$ or $y = 1$. Since $I_f, (I_f)^{[k-1]}_{\circledast}$ satisfy (EP) from Theorem 5.3.2 and then from Lemma 4.2.1, we see that $I_f, (I_f)^{[k-1]}_{\circledast}$ satisfy (ME). Now, from Theorem 5.3.13, it follows that $(I_f)^{[k]}_{\circledast} = I_f \circledast (I_f)^{[k-1]}_{\circledast} \in \mathbb{I}_{\mathbb{F},\infty}$. $\square$

The proof of the following result is similar to that of Theorem 5.3.17.

**Theorem 5.3.18.** *If $I_f \in \mathbb{I}_{\mathbb{F},1}$ then $(I_f)^{[n]}_{\circledast} \in \mathbb{I}_{\mathbb{F},1}$ for all $n \in \mathbb{N}$.*

**Corollary 5.3.19.** *If $I_f \in \mathbb{I}_{\mathbb{F}}$ then $(I_f)^{[n]}_{\circledast} \in \mathbb{I}_{\mathbb{F}}$ for all $n \in \mathbb{N}$.*

**Lemma 5.3.20.** *Let $I = I_f$ be an $f$-implication. Then $I^{[n]}_{\circledast}(x,y) = I(x^n, y)$ for all $x, y \in [0,1], n \in \mathbb{N}$.*

*Proof.* Let $I = I_f$ be an $f$-implication. Since we know $I$ satisfies (LI) w.r.to the product t-norm $T_{\mathbf{P}}(x,y) = xy$, the result follows immediately from Theorem 4.1.3. $\square$

**Lemma 5.3.21.** *Let $I \in \mathbb{I}_{\mathbb{F}}$. Then $\mathcal{O}(I) = \infty$.*

*Proof.* Suppose for some $m \in \mathbb{N}$, $I^{[m]}_{\circledast} = I^{[m+1]}_{\circledast}$. Let $x, y \in (0,1)$ be arbitrarily chosen. Then

$$
\begin{aligned}
I^{[m]}_{\circledast}(x,y) &= I^{[m+1]}_{\circledast}(x,y) \\
&\Longrightarrow I_f(x^m, y) = I_f(x^{m+1}, y) \\
&\Longrightarrow f^{-1}\left(x^m \cdot f(y)\right) = f^{-1}(x^{m+1} \cdot f(y)) \\
&\Longrightarrow x^m \cdot f(y) = x^{m+1} \cdot f(y) \\
&\Longrightarrow x = 0 \text{ or } y = 1 \text{ or } x = 1\ ,
\end{aligned}
$$

which is a contradiction. Thus $\mathcal{O}(I) = \infty$. $\square$

**Corollary 5.3.22.** *No $f$-implication satisfies the idempotent equation* (5.2).

## 5.4 The $\circledast$-composition : Closures w.r.to $g$- implications

In this section, we discuss the closure of the $\circledast$-composition w.r.to the second family of fuzzy implications proposed by Yager, viz., the $g$-implications. The results and proofs in this section largely mirror those that were given in the earlier section that dealt with $f$-implications (Section 5.3) and hence only a sketch of the proof is given wherever necessary.

### 5.4.1 $g$- implications

**Definition 5.4.1** ([8], Definition 3.2.1)**.** *Let $g : [0,1] \longrightarrow [0,\infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I : [0,1]^2 \longrightarrow [0,1]$ defined by*

$$I(x,y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \qquad x, y \in [0,1] ,$$

*with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a* g- generated implication, *where the function $g^{(-1)}$ is the pseudo inverse of $g$ given by*

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x), & \text{if } x \in [0,\ g(1)] , \\ 1, & \text{if } x \in [g(1),\ \infty] . \end{cases}$$

The family of all $g$-generated implications is denoted by $\mathbb{I}_{\mathbb{G}}$. Once again, it can be shown that it is sufficient to consider two types of $g$-generators, viz., those with $g(1) = \infty$ and $g(1) = 1$. Let us denote by

- $\mathbb{I}_{\mathbb{G},\infty}$ – the family of all $g$-generated implications such that $g(1) = \infty$.
- $\mathbb{I}_{\mathbb{G},1}$ – the family of all $g$-generated implications such that $g(1) < \infty$.

In the following we list out some relevant properties and characterisation results for $g$-implications.

**Theorem 5.4.2** ([8], Theorem 3.2.8)**.** *Let $g$ be a $g$-generator.*

*(i)* *$I_g$ satisfies* (NP) *and* (EP).

*(ii)* *$I_g$ is continuous except at the point $(0,0)$.*

**Theorem 5.4.3** (cf. [61], Theorem 14)**.** *Let $I \colon [0,1] \longrightarrow [0,1]$ be a binary function. Then the following statements are equivalent:*

*(i)* *$I$ is a $g$-implication with $g(1) = \infty$.*

*(ii)* *$I$ satisfies* (LI) *w.r.to $T_{\mathbf{P}}$, $I$ is continuous except at $(0,0)$ and $I(x,y) = 1 \Longleftrightarrow x = 0$ or $y = 1$.*

**Theorem 5.4.4** (cf. [61], Theorem 17)**.** *Let $I \colon [0,1] \longrightarrow [0,1]$ be a binary function. Then the following statements are equivalent:*

*(i)* *$I$ is a $g$-implication with $g(1) < \infty$.*

*(ii)* *$I$ satisfies* (LI) *w.r.to $T_{\mathbf{P}}$ and there exists a continuous strictly increasing function $t\colon [0,1] \longrightarrow [0,1]$ with $t(0) = 0$ and $t(1) = 1$ such that $I(x,y) = 1 \Longleftrightarrow y \geq t(x)$.*

*Moreover, the $f$- generator is unique upto a positive multiplicative constant and it is given by $f(x) = N_I^{-1}(x)$.*

We recall the following result from [8] which shows that the set of $f$-implications generated from $f$-generators such that $f(0) = \infty$ and the set of $g$-implications generated from $g$-generators such that $g(1) = \infty$ are identical.

**Proposition 5.4.5** ([8], Proposition 4.4.1)**.** *The following equalities are true:*

$$\begin{aligned} \mathbb{I}_{\mathbb{F},1} \cap \mathbb{I}_{\mathbb{G}} &= \emptyset \ , \\ \mathbb{I}_{\mathbb{F}} \cap \mathbb{I}_{\mathbb{G},1} &= \emptyset \ , \\ \mathbb{I}_{\mathbb{F},\infty} &= \mathbb{I}_{\mathbb{G},\infty} \ . \end{aligned}$$

### 5.4.2 $g$- implications and the $\circledast$-composition

**Lemma 5.4.6.** *Let $I_{g_1}, I_{g_2} \in \mathbb{I}_{\mathbb{G}}$ be such that $I_{g_1} \circledast I_{g_2} = I_h \in \mathbb{I}_{\mathbb{G}}$. Then $I_{g_1}, I_{g_2} \in \mathbb{I}_{\mathbb{G},\infty} \Longleftrightarrow I_h \in \mathbb{I}_{\mathbb{G},\infty}$.*

*Proof.* Proof follows from Proposition 5.4.5 and Theorem 5.3.7. □

**Corollary 5.4.7.** *Let $I_{g_1}, I_{g_2} \in \mathbb{I}_{\mathbb{G}}$ be such that $I_{g_1} \circledast I_{g_2} = I_h \in \mathbb{I}_{\mathbb{G}}$. Then $I_{g_1}$ or $I_{g_2} \in \mathbb{I}_{\mathbb{G},1} \Longleftrightarrow I_h \in \mathbb{I}_{\mathbb{G},1}$.*

**Theorem 5.4.8.** *Let $I_{g_1}, I_{g_2} \in \mathbb{I}_{\mathbb{G},\infty}$. Then the following statements are equivalent:*

*(i)* $I_{g_1} \circledast I_{g_2} \in \mathbb{I}_{\mathbb{G},\infty}$.

*(ii)* *$I_{g_1} \circledast I_{g_2}$ satisfies* (LI) *w.r.to $T_{\mathbf{P}}$.*

*(iii)* *$I_{g_1}, I_{g_2}$ are mutually exchangeable.*

*Proof.* Proof follows from Theorems 5.4.5 and 5.3.13. □

**Theorem 5.4.9.** *Let $I_{g_1}, I_{g_2} \in \mathbb{I}_{\mathbb{G},1}$. If $I_{g_1}, I_{g_2}$ are mutually exchangeable then $I_{g_1} \circledast I_{g_2} \in \mathbb{I}_{\mathbb{G},1}$.*

*Proof.* Proof is similar to that of Theorem 5.3.15. □

### 5.4.3 Powers of $g$- implications w.r.to the $\circledast$-composition.

**Theorem 5.4.10.** *If $I_g \in \mathbb{I}_{\mathbb{G},\infty}$ then $(I_g)_{\circledast}^{[n]} \in \mathbb{I}_{\mathbb{G},\infty}$ for all $n \in \mathbb{N}$.*

*Proof.* Proof follows from Theorem 5.3.17 and Proposition 5.4.5. □

**Theorem 5.4.11.** *If $I_g \in \mathbb{I}_{\mathbb{G},1}$ then $(I_g)_{\circledast}^{[n]} \in \mathbb{I}_{\mathbb{G},1}$ for all $n \in \mathbb{N}$.*

*Proof.* Proof is similar to that of Theorem 5.4.10. □

**Corollary 5.4.12.** *If $I_g \in \mathbb{I}_{\mathbb{G}}$ then $(I_g)_{\circledast}^{[n]} \in \mathbb{I}_{\mathbb{G}}$ for all $n \in \mathbb{N}$.*

**Lemma 5.4.13.** *Let $I = I_g$ be a $g$-implication. Then $I_{\circledast}^{[n]}(x,y) = I(x^n, y)$ for all $x, y \in [0,1], n \in \mathbb{N}$.*

*Proof.* Proof is similar to that of Lemma 5.3.20. $\square$

**Lemma 5.4.14.** *Let $I \in \mathbb{I}_{\mathbb{G}}$. Then $\mathcal{O}(I) = \infty$.*

*Proof.* Proof is similar to that of Lemma 5.3.21. $\square$

**Corollary 5.4.15.** *No g-implication satisfies the idempotent equation* (5.2).

## 5.5 Conclusions

In this chapter we have investigated the effect of the $\circledast$-composition on the fuzzy implications that are obtained from the other two existing generating methods of fuzzy implications **(M1)** and **(M2)** as discussed in Section 1.1. In particular, we have considered the following questions: Firstly, if $I, J$ belong to a certain family of fuzzy implications does $I \circledast J$ also belong to the same family? Secondly, do all the powers of a fuzzy implication coming from a certain family belong to the same family or not?
In the investigations of above questions, our study shows the following:

- In general, $\mathbb{I}_{\mathbb{S},\mathbb{N}}$ is not closed w.r.to the $\circledast$-composition and hence we have determined some sufficient and / or necessary conditions on some subsets of $\mathbb{I}_{\mathbb{S},\mathbb{N}}$ such that $\circledast$ is closed in them. However, we have shown that all the powers of an $(S, N)$-implication w.r.to the $\circledast$-composition are again $(S, N)$-implications.

- In the case of $R$-implications obtained from *left-continuous* t-norms, we have shown that the $\circledast$-composition of two such implications belongs to the same family only if one of them is the Gödel implication $I_{\mathbf{GD}}$. Unlike in the case of $(S, N)$-implications, the only $R$-implication obtained from *left-continuous* t-norm such that all the powers are again $R$-implications is the Gödel implication $I_{\mathbf{GD}}$.

- It is interesting to see that the *mutual exchangeability* (ME) of fuzzy implications becomes both necessary and sufficient condition in the preservation of Yager's classes of fuzzy implications w.r.to the $\circledast$-composition. Further we have proven that the Yager's classes of fuzzy implications are closed w.r.to the self-composition of $\circledast$.

# Part III

# The $\circledast$-composition : As a Binary Operation on $\mathbb{I}$

# Chapter 6

# Algebraic Structures of $(\mathbb{I}, \circledast)$

*Algebra is, properly speaking, the Analysis of equations.*
– Joseph Alfred Serret (1819-1885)

In Chapter 2, we have shown that the $\circledast$-composition when looked at as a binary operation on $\mathbb{I}$, makes it a non-idempotent monoid (see Theorem 2.2.1). In this chapter we explore further algebraic aspects of the set of fuzzy implications w.r.to the binary operation $\circledast$. Towards this end, in Section 6.1, when equipped with lattice operations, we show that $(\mathbb{I}, \circledast)$ becomes a lattice ordered monoid and also we show that this monoid can not be made a group. Following this we characterise the largest subgroup $\mathbb{S}$ contained in $(\mathbb{I}, \circledast)$ and give their representations in Section 6.2. Isomorphic classes of $\mathbb{S}$ are obtained in Section 6.3.

## 6.1 $(\mathbb{I}, \circledast, \preceq, \vee, \wedge)$ : A Lattice Ordered Monoid

Here we recall definitions of $\vee$ and $\wedge$ from Section 1.3.1.

**Theorem 6.1.1** ([8], Theorem 6.1.1)**.** *The set $(\mathbb{I}, \preceq)$ is a complete, completely distributive lattice with the lattice operations join $\vee$* (Latt-Max) *and meet $\wedge$* (Latt-Min).

**Remark 6.1.2.** *From Theorem 6.1.1, we have $(\mathbb{I}, \preceq, \vee, \wedge)$ is a lattice. In fact, $(\mathbb{I}, \preceq, \vee, \wedge)$ is also a bounded lattice with $I_{\mathbf{0}}, I_{\mathbf{1}}$ being the least and greatest fuzzy implications. From Theorem 2.2.1, we know that $(\mathbb{I}, \circledast)$ is a monoid. Together with all these operations $\mathbb{I}$ becomes a lattice ordered monoid as the following lemma illustrates.*

**Lemma 6.1.3.** *The pentuple $(\mathbb{I}, \circledast, \preceq, \vee, \wedge)$ is a lattice ordered monoid where $\vee, \wedge$ are as in* (Latt-Max) *and* (Latt-Min).

*Proof.* It is enough to show that the binary operation $\circledast$ is compatible with the lattice operations.

Let $I, J, K \in \mathbb{I}$ and $x, y \in [0,1]$. Then,

$$\begin{aligned}
(I \circledast (J \vee K))(x,y) &= I(x, (J \vee K)(x,y)) \\
&= I(x, \max(J(x,y), K(x,y))) \\
&= \max(I(x, J(x,y)), I(x, K(x,y))) \\
&= \max((I \circledast J)(x,y), (I \circledast K)(x,y)) \\
&= ((I \circledast J) \vee (I \circledast K))(x,y).
\end{aligned}$$

Thus $I \circledast (J \vee K) = (I \circledast J) \vee (I \circledast K)$. Similarly, one can prove the following:

$$\begin{aligned}
(I \vee J) \circledast K &= (I \circledast K) \vee (J \circledast K)\ , \\
I \circledast (J \wedge K) &= (I \circledast J) \wedge (I \circledast K)\ , \\
(I \wedge J) \circledast K &= (I \circledast K) \wedge (J \circledast K)\ .
\end{aligned}$$

Thus $(\mathbb{I}, \circledast, \preceq, \vee, \wedge)$ is a lattice ordered monoid. □

### 6.1.1 $(\mathbb{I}, \circledast)$ Is Not a Group

From Theorem 2.2.1 we know that $(\mathbb{I}, \circledast)$ is a monoid. However, the following illustrates why the richer group structure is not available on $(\mathbb{I}, \circledast)$. Take $I_{\mathbf{1}} \in \mathbb{I}$. It is easy to check that $I_{\mathbf{1}}$ is a right zero element of $(\mathbb{I}, \circledast)$, i.e., $I \circledast I_{\mathbf{1}} = I_{\mathbf{1}}$ for all $I \in \mathbb{I}$. Thus there does not exist any $J \in \mathbb{I}$ such that $J \circledast I_{\mathbf{1}} = I_{\mathbf{D}}$, i.e., the inverse of $I_{\mathbf{1}} \in \mathbb{I}$ w.r.to $\circledast$ does not exist. Thus the algebraic structure $(\mathbb{I}, \circledast)$ is only a monoid and not a group.

**Remark 6.1.4.** *(i) Note that $I_{\mathbf{1}}$ is not the only right zero element. In fact, every fuzzy implication of the type $K^{\delta}$ given below is a right zero element of $(\mathbb{I}, \circledast)$ , for any $\delta \in (0,1]$*

$$K^{\delta}(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } (x = 1 \text{ and } \ y \geq \delta), \\ 0, & \text{if } x = 1 \text{ and } y > \delta. \end{cases}$$

*For a proof of this argument, see Lemma 8.3.3.*

*(ii) One cannot apply some well-known techniques of obtaining a group from a monoid, viz., the Grothendieck construction method of obtaining an abelian group from a commutative monoid, due to both the presence of zero elements in $(\mathbb{I}, \circledast)$ and also the absence of commutativity.*

## 6.2 Subgroups of $(\mathbb{I}, \circledast)$

From Section 6.1.1, it is clear that $(\mathbb{I}, \circledast)$ is only a monoid but not a group. Though $(\mathbb{I}, \circledast)$ is not a group, there still exist many subgroups of this monoid. For instance, see the following example.

**Example 6.2.1.** *Let $\mathbb{S}_R, \mathbb{S}_Q$ be the sets of all fuzzy implications of the form*

$$I_r(x,y) = \begin{cases} 1, & \text{if } x = 0 \\ y^r, & \text{if } x > 0 \end{cases}\ , \text{ for every } r \in \mathbb{R}^{>0}\ ,$$

$$I_q(x,y) = \begin{cases} 1, & \textit{if } x = 0 \\ y^q, & \textit{if } x > 0 \end{cases}, \textit{ for every } q \in \mathbb{Q}^{>0} ,$$

*respectively. It is easy to see that* $\mathbb{S}_R, \mathbb{S}_Q$ *are subgroups of* $(\mathbb{I}, \circledast)$.

Towards the characterisation of subgroups of $(\mathbb{I}, \circledast)$, we determine the invertible elements of $(\mathbb{I}, \circledast)$. In, other words, we characterise the set of all invertible fuzzy implications w.r.to the operation $\circledast$ and investigate their representations.

### 6.2.1 Characterisation of Invertible Elements

In this subsection we characterise the set of all invertible elements of $(\mathbb{I}, \circledast)$. Towards this end, we have the following the lemma.

**Lemma 6.2.2.** *Let* $I \in \mathbb{I}$. *Then the following statements are equivalent:*

*(i)* $I$ *is invertible w.r.to* $\circledast$.

*(ii) There exists a unique* $J \in \mathbb{I}$ *such that for any* $x \in (0,1]$ *and* $y \in [0,1]$,

$$I(x, J(x,y)) = y = J(x, I(x,y)) . \tag{6.1}$$

*Proof.* (i) $\Longrightarrow$ (ii). Let $I \in \mathbb{I}$ be invertible w.r.to $\circledast$, i.e., there exists a unique $J \in \mathbb{I}$ such that $I \circledast J = I_{\mathbf{D}} = J \circledast I$. In other words,

$$I(x, J(x,y)) = I_{\mathbf{D}}(x,y) = J(x, I(x,y)), \qquad x, y \in [0,1].$$

But for $x > 0, I_{\mathbf{D}}(x,y) = y$. Thus for $x > 0$, $I(x, J(x,y)) = y = J(x, I(x,y))$.

(ii) $\Longrightarrow$ (i). Conversely, assume that there exists a unique $J \in \mathbb{I}$ such that for $x > 0$, $I(x, J(x,y)) = y = J(x, I(x,y))$. Since $I, J \in \mathbb{I}$ and $I \circledast J, J \circledast I \in \mathbb{I}$, we have $I(x, J(x,y)) = I_{\mathbf{D}}(x,y) = J(x, I(x,y))$. Since $I, J \in \mathbb{I}$, it follows that $I(0,y) = 1 = J(0,y)$ which is also equal to $I_{\mathbf{D}}(0,y)$. Thus $I$ is invertible w.r.to $\circledast$.

□

### 6.2.2 Representation of Invertible Elements

Though we have obtained the characterisations of invertible elements of $(\mathbb{I}, \circledast)$, it is worthful only if we have the representations of those elements. In this subsection we attempt to find the representations of invertible elements of $(\mathbb{I}, \circledast)$. Note that in order to get their representations we need to solve the functional equation (6.1). Recall that $\Phi$ is the set of all increasing bijections on $[0,1]$.

**Lemma 6.2.3.** *The solutions of* (6.1)*, for all* $x \in (0,1]$ *and* $y \in [0,1]$*, are of the form* $I(x,y) = \varphi(y)$ *and* $J(x,y) = \varphi^{-1}(y)$*, for some* $\varphi \in \Phi$*. r*

*Proof.* Let $I, J \in \mathbb{I}$ satisfy (6.1), i.e., $I(x, J(x,y)) = y = J(x, I(x,y))$, for all $x > 0$ and $y \in [0,1]$.

Let $x_0 > 0$ be fixed arbitrarily and define two functions $\varphi_{x_0}, \psi_{x_0} \colon [0,1] \longrightarrow [0,1]$ as $\varphi_{x_0}(y) = I(x_0, y)$ and $\psi_{x_0}(y) = J(x_0, y)$. Clearly, both $\varphi_{x_0}, \psi_{x_0}$ are increasing functions on $[0,1]$.

Then $I(x_0, J(x_0, y)) = \varphi_{x_0}(\psi_{x_0}(y)) = (\varphi_{x_0} \circ \psi_{x_0})(y) = y$ for every $y \in [0,1]$. Similarly, $J(x_0, I(x_0, y)) = \psi_{x_0}(\varphi_{x_0}(y)) = (\psi_{x_0} \circ \varphi_{x_0})(y) = y$ for every $y \in [0,1]$. Thus $\varphi_{x_0} = \psi_{x_0}^{-1}$ and $\varphi_{x_0}$ is a bijection. Hence $\varphi_{x_0} \in \Phi$ for every $x_0 > 0$.

Since $x_0$ is chosen arbitrarily, $\varphi_x = \psi_x^{-1}$ for all $x > 0$. Thus for $x > 0$, $I, J$ are of the form, $I(x,y) = \varphi_x(y)$ and $J(x,y) = \varphi_x^{-1}(y)$.

Let $0 < x_1 \leq x_2$. Then $I(x_1, y) \geq I(x_2, y)$ implies that $\varphi_{x_1}(y) \geq \varphi_{x_2}(y)$ and $J(x_1, y) \geq J(x_2, y)$ implies that $\varphi_{x_1}^{-1}(y) \geq \varphi_{x_2}^{-1}(y)$ for all $y \in [0,1]$. Now,

$$\begin{aligned}
\varphi_{x_1}^{-1} \geq \varphi_{x_2}^{-1} &\implies \varphi_{x_1} \circ \varphi_{x_1}^{-1} \geq \varphi_{x_1} \circ \varphi_{x_2}^{-1} \\
&\implies \mathbf{id} \geq \varphi_{x_1} \circ \varphi_{x_2}^{-1} \\
&\implies \mathbf{id} \geq \varphi_{x_1} \circ \varphi_{x_2}^{-1} \geq \varphi_{x_2} \circ \varphi_{x_2}^{-1} \\
&\implies \mathbf{id} \geq \varphi_{x_1} \circ \varphi_{x_2}^{-1} \geq \mathbf{id} \,,
\end{aligned}$$

from which it follows $\varphi_{x_1} \circ \varphi_{x_2}^{-1} \equiv \mathbf{id}$, i.e., $\varphi_{x_1}(y) = \varphi_{x_2}(y)$ for all $y \in [0,1]$. Since $x_1, x_2$ are arbitrarily chosen $\varphi_{x_1} \equiv \varphi_{x_2} \equiv \varphi$ (say) for all $x_1, x_2 > 0$. Thus $I(x,y) = \varphi(y)$ and $J(x,y) = \varphi^{-1}(y)$, for some $\varphi \in \Phi$. □

Now we are ready to give the representation of every invertible element of the monoid $(\mathbb{I}, \circledast)$ and thus determine its largest subgroup. From Lemmata 6.2.2 and 6.2.3 we have the following result.

**Theorem 6.2.4.** *An $I \in \mathbb{I}$ is invertible w.r.to $\circledast$ if and only if*

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \,, \\ \varphi(y), & \text{if } x > 0 \,, \end{cases} \tag{6.2}$$

*where the function $\varphi \colon [0,1] \longrightarrow [0,1]$ is an increasing bijection. Moreover, in this case, the inverse $J$ of the fuzzy implication $I$ is given by*

$$J(x,y) = \begin{cases} 1, & \text{if } x = 0 \,, \\ \varphi^{-1}(y), & \text{if } x > 0 \,. \end{cases}$$

The following presents an example of a fuzzy implication that is invertible w.r.to $\circledast$ and its inverse.

**Example 6.2.5.** *The fuzzy implication defined by*

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ y^3, & \text{if } x > 0, \end{cases}$$

*is invertible w.r.to $\circledast$ in $\mathbb{I}$, because there exists a unique fuzzy implication*

$$J(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ y^{\frac{1}{3}}, & \text{if } x > 0, \end{cases}$$

*such that* $I \circledast J = I_{\mathbf{D}} = J \circledast I$.

Clearly, the largest subgroup of $(\mathbb{I}, \circledast)$ is one that contains all the invertible elements of $\mathbb{I}$ w.r.to $\circledast$. Let $\mathbb{S}$ be the set of all invertible elements of $(\mathbb{I}, \circledast)$, i.e., $\mathbb{S}$ is the set of all fuzzy implications of the form (6.2) for some $\varphi \in \Phi$.

**Proposition 6.2.6.** *Every element of* $\mathbb{S}$ *satisfies* (EP)*, i.e.,* $\mathbb{S} \subsetneq \mathbb{I}_{\mathbf{EP}}$.

*Proof.* Let $I \in \mathbb{S}$. From (6.2) we have that

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(y), & \text{if } x > 0, \end{cases}$$

for some $\varphi \in \Phi$. Let $x, y, z \in [0,1]$. If $x = 0$ or $y = 0$ then we are done. So, let $x > 0, y > 0$. Now, $I(x, I(y,z)) = I(x, \varphi(z)) = \varphi(\varphi(z))$ and $I(y, I(x,z)) = I(y, \varphi(z)) = \varphi(\varphi(z))$ thus showing that $I$ has (EP). □

**Remark 6.2.7.** *(i) Clearly, the inclusion in Proposition 6.2.6 is strict. For example* $I_{\mathbf{RC}} \in \mathbb{I}_{\mathbf{EP}}$ *but* $I_{\mathbf{RC}} \notin \mathbb{S}$.

*(ii) From the discussion in Section 3.2.2, we see that* $(\mathbb{I}_{\mathbf{EP}}, \circledast)$ *is not closed, while we have obtained a subset* $\mathbb{S}$ *of* $\mathbb{I}_{\mathbf{EP}}$ *which is closed w.r.to* $\circledast$.

*(iii) Note that* $\mathbb{S}$ *is not the largest subset of* $\mathbb{I}_{\mathbf{EP}}$ *that is closed w.r.to* $\circledast$. *For instance, if we define* $\mathbb{U}$ *as the set of all fuzzy implications of the form*

$$I(x,y) = \begin{cases} 1, & \textit{if } x = 0, \\ \psi(y), & \textit{if } x > 0, \end{cases}$$

*for some increasing function, not necessarily a* bijection, $\psi \colon [0,1] \to [0,1]$ *such that* $\psi(0) = 0$ *and* $\psi(1) = 1$*, then every element of* $\mathbb{U}$ *satisfies* (EP)*. Obviously* $\mathbb{S} \subsetneq \mathbb{U}$.

*(iv) Elements of* $\mathbb{S}$ *do not satisfy either* (OP) *or* (IP) *and the only element satisfying* (NP) *is the identity* $I_{\mathbf{D}}$ *of* $(\mathbb{I}, \circledast)$.

## 6.3 Isomorphic Classes of $(\mathbb{S}, \circledast)$

Let $\circ$ denote the usual composition of functions. Then it is well known that $(\Phi, \circ)$ is a group. Interestingly, the subgroup $(\mathbb{S}, \circledast)$ is isomorphic to $(\Phi, \circ)$, as the following result illustrates.

**Theorem 6.3.1.** *The groups* $(\Phi, \circ)$, $(\mathbb{S}, \circledast)$ *are isomorphic to each other.*

*Proof.* Let $h \colon \Phi \longrightarrow \mathbb{S}$ be defined by $h(\varphi) = I$ where

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(y), & \text{if } x > 0. \end{cases}$$

It is easy to see that the map $h$ is one-one and onto. Let $\varphi_1, \varphi_2 \in \Phi$ and $h(\varphi_1) = I_1, h(\varphi_2) = I_2$ where

$$I_i(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi_i(y), & \text{if } x > 0, \end{cases}$$

for $i = 1, 2$. Now

$$\begin{aligned}(h(\varphi_1) \circledast h(\varphi_2))(x,y) &= (I_1 \circledast I_2)(x,y) \\ &= I_1(x, I_2(x,y)) \\ &= \begin{cases} 1, & \text{if } x = 0, \\ \varphi_1(\varphi_2(y)), & \text{if } x > 0, \end{cases} \\ &= h(\varphi_1 \circ \varphi_2)(x,y).\end{aligned}$$

Thus $h$ is an isomorphism. □

## 6.4 Conclusions

In this chapter, we have shown that the algebraic structure $(\mathbb{I}, \circledast)$, when equipped with lattice operations, becomes a lattice ordered monoid. Moreover, we have shown that because of the presence of zero elements $(\mathbb{I}, \circledast)$ does not become a group. Further, we have also shown that the presence of zero elements and absence of commutativity preclude the possibility of graduating $(\mathbb{I}, \circledast)$ from a monoid to a group. Noting that there exist non-trivial subgroups of $(\mathbb{I}, \circledast)$ we set out to find the largest subgroup of $(\mathbb{I}, \circledast)$. Since the set of all invertible elements of a monoid forms the largest subgroup, we have characterised the set of all invertible elements of $(\mathbb{I}, \circledast)$ which is denoted by $\mathbb{S}$ and determined their representations. Further we have shown that $\mathbb{S}$ is isomorphic to the group $\Phi$ of all increasing bijections, the composition of functions being the binary operation.

# Chapter 7

# Group Actions on $\mathbb{I}$ and Conjugacy Classes

*Mathematics is the art of giving the same name to different things.*
–Jules Henri Poincare

We recall from Chapter 1 that one of our objectives for this thesis is to obtain a rich algebraic structure on $\mathbb{I}$ that would throw more light on fuzzy implications by providing newer insights and connections between existing families and properties of fuzzy implications. Unfortunately, we have found that $(\mathbb{I}, \circledast)$ is only a monoid but not a group, which precludes further applications of known results. For instance, the theory and results based on normal subgroups cannot be applied to obtain any kind of unique decomposition and hence, some characterisation or representation results.

We know that the set of all invertible elements of a monoid forms a subgroup. In Section 6.2, we have characterised all such elements of the monoid $(\mathbb{I}, \circledast)$, whose set is denoted by $\mathbb{S}$, and obtained their representations. In this chapter we investigate the group actions of $\mathbb{S}$ on $\mathbb{I}$.

In Section 7.1, we begin by recalling the conjugacy classes of fuzzy implications that have been proposed in the literature. In Section 7.2, we discuss the action of a group on a non-empty set. With the help of the largest subgroup $\mathbb{S}$ of $(\mathbb{I}, \circledast)$, we propose three different group actions on the set $\mathbb{I}$. In Section 7.3, we obtain, with the help of the group action we propose here, some hitherto unknown and simpler representations of Yager's families of fuzzy implications, namely, $f$-, $g$- implications. With the help of the group action proposed in Section 7.4, we show that the equivalence classes obtained through this group action are nothing but the conjugacy classes proposed by Baczyński and Drewniak in [5]. Finally in Section 7.5, we propose yet another group action of $\mathbb{S}$ on $\mathbb{I}$ and show that the equivalence classes that we obtain are exactly the conjugacy classes proposed by Jayaram and Mesiar in [41], in the context of generating special implications from special implications.

## 7.1 Existing Conjugacy Classes of Fuzzy Implications

In this section we recall two conjugacy classes of fuzzy implications proposed by Baczyński and Drewniak [5] and Jayaram and Mesiar [41].

### 7.1.1 Conjugacy Classes Proposed by Baczyński and Drewniak

In Section 1.3.3, we have recalled the conjugacy classes of fuzzy implications proposed by Baczyński and Drewniak in [5]. For a given $I \in \mathbb{I}$ and $\varphi \in \Phi$, they defined $I_\varphi$ as follows:

$$I_\varphi(x,y) = \varphi^{-1}\left(I(\varphi(x),\varphi(y))\right), \qquad x,y \in [0,1].$$

Clearly, $I_\varphi \in \mathbb{I}$ for all $I \in \mathbb{I}$ and $\varphi \in \Phi$. In the literature, $I_\varphi$ is called a $\varphi$-conjugate of $I$ and if $I_\varphi = I$ for all $\varphi \in \Phi$ then $I$ is a self conjugate or invariant (see Section 1.3.3). Moreover, defining a relation $\sim_{\mathcal{B}}$ on $\mathbb{I}$ as $I \sim_{\mathcal{B}} J \iff J = I_\varphi$ for some $\varphi \in \Phi$, they showed that $\sim_{\mathcal{B}}$ is an equivalence relation on $\mathbb{I}$ in which the equivalence classes of $I \in \mathbb{I}$ are given by

$$\begin{aligned}[I]_{\sim_{\mathcal{B}}} &= \{J \in \mathbb{I} | I \sim_{\mathcal{B}} J\} \\ &= \{J \in \mathbb{I} | J = I_\varphi \text{ for some } \varphi \in \Phi\} \\ &= \{J \in \mathbb{I} | J(x,y) = \varphi^{-1}\left(I(\varphi(x),\varphi(y))\right) \text{ for some } \varphi \in \Phi\} \\ &= \{J(x,y) = \varphi^{-1}\left(I(\varphi(x),\varphi(y))\right) \text{ for some } \varphi \in \Phi\}\,. \end{aligned} \tag{7.1}$$

In the following we note that the above equivalence relation preserves some of the most desirable basic properties of fuzzy implications as well as is closed w.r.to some families of fuzzy implications.

**Proposition 7.1.1** ([8], Proposition 1.3.6)**.** *Let $\varphi \in \Phi$. If $I \in \mathbb{I}$ satisfies* (NP) *(*(IP), (OP), (EP)*), then $I_\varphi$ also satisfies* (NP) *(*(IP), (OP), (EP)*).*

**Theorem 7.1.2** ([8], Theorem 2.4.5)**.** *If $I_{S,N}$ is an (S,N)-implication, then the $\Phi$-conjugate of $I_{S,N}$ is also an (S,N)-implication generated from the $\Phi$-conjugate t-conorm of $S$ and the $\Phi$-conjugate fuzzy negation of $N$, i.e., if $\varphi \in \Phi$, then*

$$(I_{S,N})_\varphi = I_{S_\varphi,N_\varphi}\,.$$

**Proposition 7.1.3** ([8], Proposition 2.5.10)**.** *If $I_T$ is an R-implication, then the $\Phi$-conjugate of $I_T$ is also an R-implication generated from the $\Phi$-conjugate t-norm of $T$, i.e., if $\varphi \in \Phi$, then*

$$(I_T)_\varphi = I_{T_\varphi}\,.$$

### 7.1.2 Conjugacy Classes Proposed by Jayaram and Mesiar

In [41], Jayaram and Mesiar studied a new class of fuzzy implications, namely, special fuzzy implications, which were defined as follows.

**Definition 7.1.4** (cf.[36], [41], Definition 1.1)**.** *A fuzzy implication $I$ is said to be special, if for any $\epsilon > 0$ and for all $x, y \in [0,1]$ such that $x+\epsilon, y+\epsilon \in [0,1]$ the following condition is fulfilled:*

$$I(x,y) \le I(x+\epsilon, y+\epsilon), \tag{SP}$$

In particular they characterised the set of all fuzzy implications that are also special. In the context of generating special fuzzy implications from special fuzzy implications they proposed the following transformation.

**Definition 7.1.5** ([41], Definition 9.7). *Let $\varphi \in \Phi$ and $I \in \mathbb{I}$. Define the following function:*

$$\varphi(I)(x,y) = \varphi(I(x,y)), \qquad x,y \in [0,1].$$

Further they showed that the transformation defined in Definition 7.1.5 preserves all the axioms of fuzzy implications including (SP) (see, Proposition 9.8 in [41]). Moreover, one can define $\sim_{\mathcal{J}}$ on $\mathbb{I}$ as follows:

$$I \sim_{\mathcal{J}} J \Longleftrightarrow J = \varphi(I)\ , \tag{7.2}$$

for some $\varphi \in \Phi$. Then it is not difficult to see that $\sim_{\mathcal{J}}$ is, in fact, an equivalence relation on $\mathbb{I}$ and partitions it into equivalence classes which are given by

$$\begin{aligned}[I]_{\sim_{\mathcal{J}}} &= \{J \in \mathbb{I} | I \sim_{\mathcal{J}} J\} \\ &= \{J \in \mathbb{I} | J = \varphi(I) \text{ for some } \varphi \in \Phi\} \\ &= \{J \in \mathbb{I} | J(x,y) = \varphi(I(x,y)) \text{ for some } \varphi \in \Phi\}\ .\end{aligned}$$

The following result shows one of the important properties that equivalence classes obtained as above have.

**Proposition 7.1.6** ([41], Definition 9.8). *$I \in \mathbb{I}$ is a special fuzzy implication $\Longleftrightarrow$ $\varphi(I)$ is a special fuzzy implication.*

In Sections 7.1.1 and 7.1.2 we have recalled the conjugacy classes proposed earlier by Baczyński and Drewniak, Jayaram and Mesiar, respectively, in different contexts. However, one should notice that there does not exist, in the literature, any algebraic perspectives for either of the discussed conjugacy classes so far. In our investigations we have found that these conjugacy classes are exactly the equivalence classes obtained via some group actions of $\mathbb{S}$ on $\mathbb{I}$. Towards this end, in Section 7.2, we recall the notion of action of a group on a non-empty set and the equivalence classes obtained thereof to make the above claims more explicit in Sections 7.4 and 7.5.

## 7.2 Group Action on a Set and Consequent Partition

We know that every equivalence relation on a non-empty set gives rise to a partition of the set into equivalence classes and vice versa. Unfortunately, it is always not easy to define a nontrivial relation on a set which also becomes an equivalence relation on it. Further given a partition it is not easy to find the underlying equivalence relation. However, one can achieve this with the help of group actions acting on the set under consideration.

In the following we briefly recall the basic concepts of a group action and how one always can obtain a partition of a set into equivalence classes via the equivalence relation defined with the help of the group action. More details can be found in any book dealing with the theory of groups, for example, [71].

**Definition 7.2.1** ([71], Pg. 488). *Let $(S,*)$ be a group and $T$ be a non empty set. A function $\bullet\colon S\times T \longrightarrow T$ is called a group action if, for all $s_1, s_2 \in S$ and $t \in T$, $\bullet$ satisfies the following two conditions:*

*(i)* $s_1 \bullet (s_2 \bullet t) = (s_1 * s_2) \bullet t$.

*(ii)* $e \bullet t = t$ *where* $e$ *is the identity of* $S$.

**Remark 7.2.2.** *Let* $(S, *)$ *be a group and* $T$ *be a non empty set. Let* $\bullet$ *be a group action of* $S$ *on* $T$*. Define* $t_1 \sim t_2 \in T$ *if and only if* $t_1 = s \bullet t_2$ *for some* $s \in S$*. Then it is easy to see that the relation* $\sim$ *is an equivalence relation on* $T$ *and it partitions* $T$ *into equivalence classes. The equivalence class containing* $t \in T$ *is given by the set* $\{u \in T | t \sim u\}$.

From the discussion above, it is clear that every group action on a set induces a partition on the considered set and hence different group actions will lead to different partitions of the set. Here, in the following sections, we propose three group actions on the set $\mathbb{I}$ and find the equivalence classes obtained thereof. In fact, we show that one of the group actions that we propose leads to hitherto unknown representations of Yager's classes of fuzzy implications while the other two give algebraic connotations of conjugacy classes discussed in Sections 7.1.1 and 7.1.2, respectively.

## 7.3 Pseudo-conjugacy Classes of Fuzzy Implications

In this section, we propose a group action of $\mathbb{S}$ on $\mathbb{I}$ and investigate the equivalence classes induced from it. Further, we show that these equivalence classes preserve most of the basic properties of fuzzy implications. Moreover, we find that the equivalence classes of fuzzy implications coming from Yager's families (i.e., both $f$-, $g$- implications). Further we show firstly that the family of Yager's $f$- implications consists of only two equivalence classes whose representatives are $I_{\mathbf{RC}}$ and $I_{\mathbf{YG}}$ implications and secondly that the family of Yager's $g$- implications also consists of two equivalence classes with $I_{\mathbf{GG}}$ and $I_{\mathbf{YG}}$ being the representative elements.

### 7.3.1 Group action and Pseudo-conjugacy

In the following we define a group action of $\mathbb{S}$ on $\mathbb{I}$.

**Definition 7.3.1.** *Let* $\bullet : \mathbb{S} \times \mathbb{I} \longrightarrow \mathbb{I}$ *be a map defined by*

$$(K, I) \longrightarrow K \bullet I = K \circledast I \circledast K^{-1}.$$

**Lemma 7.3.2.** *The function* $\bullet$ *is a group action of* $\mathbb{S}$ *on* $\mathbb{I}$.

*Proof.* (i) Let $K_1, K_2 \in \mathbb{S}$ and $I \in \mathbb{I}$.

$$\begin{aligned} K_1 \bullet (K_2 \bullet I) &= K_1 \circledast (K_2 \bullet I) \circledast K_1^{-1} \\ &= K_1 \circledast K_2 \circledast I \circledast K_2^{-1} \circledast K_1^{-1} \\ &= (K_1 \circledast K_2) \circledast I \circledast (K_1 \circledast K_2)^{-1} \\ &= (K_1 \circledast K_2) \bullet I \ . \end{aligned}$$

(ii) Similarly, $I_{\mathbf{D}} \bullet I = I_{\mathbf{D}} \circledast I \circledast I_{\mathbf{D}}^{-1} = I$, since $I_{\mathbf{D}}$ is the identity of $(\mathbb{I}, \circledast)$.
Thus $\bullet$ is group action of $\mathbb{S}$ on $\mathbb{I}$. □

Recall from Remark 7.2.2 that, every group action leads to an equivalence relation on the set under consideration. In the following we define a relation on $\mathbb{I}$ using the group action defined in Definition 7.3.1 which leads to an equivalence relation on $\mathbb{I}$ .

**Definition 7.3.3.** *Let $I, J \in \mathbb{I}$. Define $I \overset{\bullet}{\sim} J \iff J = K \bullet I$ for some $K \in \mathbb{S}$. In other words, $I \overset{\bullet}{\sim} J \iff J = K \circledast I \circledast K^{-1}$ for some $K \in \mathbb{S}$.*

**Remark 7.3.4.** *Let $I \in \mathbb{I}$. Then the equivalence class containing $I$ will be of the form*

$$[I]_{\underset{\sim}{\bullet}} = \{J \in \mathbb{I} | J = K \circledast I \circledast K^{-1} for\ some\ K \in \mathbb{S}\}.$$

*Since any $K \in \mathbb{S}$ is of the form*

$$K(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(y), & \text{if } x > 0, \end{cases}$$

*for some $\varphi \in \Phi$, we have that, if $J \in [I]_{\underset{\sim}{\bullet}}$, then $J(x,y) = \varphi(I(x, \varphi^{-1}(y)))$ for all $x, y \in [0,1]$.*

**Definition 7.3.5.** *If $I, J \in \mathbb{I}$ are related as $J(x,y) = \varphi(I(x, \varphi^{-1}(y)))$ for all $x, y \in [0,1]$ for some $\varphi \in \Phi$, then we say that $J$ is a $\varphi$-**pseudo conjugate** of $I$, or alternately and equivalently, $I$ is a $\varphi^{-1}$-pseudo conjugate of $J$.*

### 7.3.2 $\varphi$- pseudo conjugates of Fuzzy Implications and Basic Properties

Interestingly, $\varphi$- pseudo conjugates do preserve some properties of the fuzzy implications as the following lemma illustrates.

**Lemma 7.3.6.** *Let $I \in \mathbb{I}$ and $J \in [I]_{\underset{\sim}{\bullet}}$. Then*

*(i) $I$ satisfies* (LI) *w.r.to $T \iff J$ satisfies* (LI) *w.r.to $T$.*

*(ii) $I$ satisfies* (EP) $\iff$ *$J$ satisfies* (EP).

*(iii) $I$ satisfies* (NP) $\iff$ *$J$ satisfies* (NP).

*(iv) $I$ is continuous $\iff$ $J$ is continuous.*

*(v) Range of $I$ is trivial $\iff$ Range of $J$ is trivial.*

*Proof.* In the following we prove only (i) and (ii), as points (iii)–(v) can be proven similarly.

(i) Let $I$ satisfy (LI) w.r.to a t-norm $T$ and $J \in [I]_{\underset{\sim}{\bullet}}$. Then from Remark 7.3.4, it follows that $J$ is a $\varphi$-pseudo conjugate of $I$. i.e., $J(x,y) = \varphi(I(x, \varphi^{-1}(y)))$ for some $\varphi \in \Phi$. Now it follows that

$$\begin{aligned} J(x, J(y,z)) &= J(x, \varphi(I(y, \varphi^{-1}(z))) \\ &= \varphi(I(x, I(y, \varphi^{-1}(z)))) \\ &= \varphi(I(T(x,y), \varphi^{-1}(z)))) \\ &= J(T(x,y), z)) \ . \end{aligned}$$

Thus $J$ satisfies (LI). The converse can be proven similarly.

(ii) Let $I$ satisfy (EP) and $J \in [I]_{\underset{\sim}{\bullet}}$. Then $J(x, y) = \varphi(I(x, \varphi^{-1}(y)))$ for some $\varphi \in \Phi$. Now,

$$\begin{aligned} J(x, J(y, z)) &= J(x, \varphi(I(y, \varphi^{-1}(z)))) \\ &= \varphi(I(x, I(y, \varphi^{-1}(z)))) \\ &= \varphi(I(y, I(x, \varphi^{-1}(z)))), \qquad [\because\ I \text{ has (EP)} \\ &= J(y, \varphi(I(x, \varphi^{-1}(z)))) \\ &= J(y, J(x, z)) \end{aligned}$$

Thus $J$ satisfies (EP). The converse can be proven similarly. $\square$

The following lemmata show that unlike fuzzy implications satisfying (NP) or (EP), not all the $\varphi$-pseudo conjugates of a fuzzy implication satisfying (IP) (or (OP)) satisfy (IP) (or (OP)).

**Lemma 7.3.7.** *Let $I \in \mathbb{I}_{\mathbf{IP}}$ and $J \in [I]_{\underset{\sim}{\bullet}}$, i.e., $J(x, y) = \varphi(I(x, \varphi^{-1}(y)))$ for some $\varphi \in \Phi$. If $\varphi(x) \leq x$ for all $x \in [0, 1]$ then $J$ satisfies* (IP).

*Proof.* Let $I \in \mathbb{I}_{\mathbf{IP}}$. i.e., $I(x, x) = 1$ for all $x \in [0, 1]$. Let $x, y \in [0, 1]$ be such that $x \leq y$. Since every $I \in \mathbb{I}$ is increasing in the second variable, it follows that $1 = I(x, x) \leq I(x, y)$. Thus if $I$ has (IP) then $I(x, y) = 1$, whenever $x \leq y$. Let $\varphi(x) \leq x$ for all $x \in [0, 1]$, i.e., $x \leq \varphi^{-1}(x)$ for all $x \in [0, 1]$. Then $I(x, \varphi^{-1}(x)) = 1$ and hence $\varphi(I(x, \varphi^{-1}(x))) = 1$, i.e., $J$ satisfies (IP). $\square$

**Remark 7.3.8.** *The converse of Lemma 7.3.7 is not true. To see this let $I \in \mathbb{I}$ be defined as*

$$I(x, y) = \begin{cases} 1, & \text{if } x < 1, \\ y^2, & \text{if } x = 1 \text{ and } y \leq \frac{1}{2}, \\ y, & \text{otherwise} \end{cases}$$

*It is clear that $I$ satisfies* (IP). *If $J \in [I]_{\underset{\sim}{\bullet}}$ then $J(x, y) = \varphi(I(x, \varphi^{-1}(y)))$ for some $\varphi \in \Phi$. Take $\varphi(x) = \sin(\frac{\pi x}{2})$. Note that $\varphi(x) \geq x$ for all $x \in [0, 1]$. However, for this choice of $\varphi$, $J$ has the following form.*

$$J(x, y) = \begin{cases} 1, & \text{if } x < 1, \\ \sin\left(\dfrac{2}{\pi}(\sin^{-1}(y))^2\right), & \text{if } x = 1 \text{ and } y \leq \frac{1}{2}, \\ y, & \text{otherwise.} \end{cases}$$

*From the expression of $J$, one can easily conclude that $J$ has* (IP).

**Lemma 7.3.9.** *Let $I \in \mathbb{I}_{\mathbf{OP}}$ and $J \in [I]_{\underset{\sim}{\bullet}}$, i.e., $J(x, y) = \varphi(I(x, \varphi^{-1}(y)))$ for some $\varphi \in \Phi$. Then the following statements are equivalent:*

*(i) $J$ satisfies* (OP).

*(ii) $\varphi(x) = x$ for all $x \in [0, 1]$.*

*Proof.* (i) $\Longrightarrow$ (ii). Let $J$ satisfy (OP). Then

$$\begin{aligned} x \leq y &\Longleftrightarrow J(x,y) = 1 \\ &\Longleftrightarrow \varphi(I(x, \varphi^{-1}(y))) = 1 \\ &\Longleftrightarrow I(x, \varphi^{-1}(y)) = 1 \\ &\Longleftrightarrow x \leq \varphi^{-1}(y) \\ &\Longleftrightarrow \varphi(x) \leq y. \end{aligned}$$

Thus we have proved that $x \leq y \Longleftrightarrow \varphi(x) \leq y$ which implies that $\varphi(x) = x$ for all $x \in [0,1]$.

(ii) $\Longrightarrow$ (i). Follows trivially.

□

### 7.3.3 An Alternative Representation of $f$- implications

In this section, we focus on the family of $f$-implications introduced earlier, please see Section 5.3.1 for more details. Based on the group action proposed in Definition 7.3.1 and the equivalence classes obtained therefrom in Remark 7.3.4, we show that every $f$-implication is a $\varphi$-pseudo conjugate of either the Yager implication $I_{\mathbf{YG}}$ or the Reichenbach implication $I_{\mathbf{RC}}$. We would like to highlight that this fact is not at all obvious, see for instance, Example 7.3.10(iii), (iv) and (v) below.

**Example 7.3.10** ([8], Example 3.1.3)**.** *(i) If we take the $f$-generator $f_l(x) = -\ln x$, then we obtain the Yager implication $I_{\mathbf{YG}}$ (see, Table 1.2).*

*(ii) If we take the $f$-generator $f_c(x) = 1 - x$, then we obtain the Reichenbach implication $I_{\mathbf{RC}}$ (see, Table 1.2).*

*(iii) Let us consider the $f$-generator $f(x) = \cos(\frac{\pi}{2}x)$, which is a continuous and strictly decreasing trigonometric function such that $f(0) = \cos 0 = 1$ and $f(1) = \cos\frac{\pi}{2} = 0$. Its inverse is given by $f^{-1}(x) = \frac{2}{\pi} \cdot \cos^{-1} x$ and the corresponding $f$- implication is given by*

$$I_f(x,y) = \frac{2}{\pi}\cos^{-1}\left(x \cdot \cos\left(\frac{\pi}{2}y\right)\right) , \qquad x,y \in [0,1] .$$

*(iv) Let us consider the Frank's class of additive generators of t-norms* (see [48], pg. 110) *as the $f$-generators which are given by*

$$f^s(x) = -\ln\left(\frac{s^x - 1}{s-1}\right) , \qquad s > 0 \, , s \neq 1 \, .$$

*Then $f^s(0) = \infty$, its inverse is given by $(f^s)^{-1}(x) = \log_s\left(1 + (s-1)e^{-x}\right)$ and the corresponding $f$- implication, for every $s$, is given by*

$$I_{f^s}(x,y) = \log_s\left(1 + (s-1)^{1-x}(s^y - 1)^x\right) , \qquad x,y \in [0,1] .$$

*(v) If we take the Yager's class of additive generators, viz., $f^\lambda(x) = (1-x)^\lambda$, where $\lambda \in (0,\infty)$, as the $f$-generators, then $f^\lambda(0) = 1$, its inverse is given by $(f^\lambda)^{-1}(x) = 1 - x^{\frac{1}{\lambda}}$ and the corresponding $f$-*

*implication, for every $\lambda \in (0, \infty)$, is given by*

$$I_{f^\lambda}(x, y) = 1 - x^{\frac{1}{\lambda}}(1 - y)\ , \qquad x, y \in [0, 1]\ .$$

**Remark 7.3.11.** *Let $f$ be an $f$- generator. If any $\varphi \in \Phi$ then the function $f \circ \varphi \colon [0,1] \longrightarrow [0, \infty]$ is strictly decreasing and $(f \circ \varphi)(1) = 0$. Thus $f \circ \varphi$ is also an $f$-generator for every $\varphi \in \Phi$.*

Our first result shows that if $I$ is an $f$-implication then every $\varphi$-pseudo conjugate of $I$ is also an $f$-implication.

**Lemma 7.3.12.** *Let $I \in \mathbb{I}$ and $J \in [I]_{\underset{\sim}{\bullet}}$. Then $I \in \mathbb{I}_{\mathbb{F}} \iff J \in \mathbb{I}_{\mathbb{F}}$.*

*Proof.* Let $I \in \mathbb{I}_{\mathbb{F}}$ and $J \in [I]_{\underset{\sim}{\bullet}}$. Then $I(x, y) = f^{-1}(x \cdot f(y))$ for some generator $f$. Now,

$$\begin{aligned} J(x, y) &= \varphi(I(x, \varphi^{-1}(y))) \\ &= \varphi(f^{-1}(x \cdot f(\varphi^{-1}(y)))) \\ &= (f \circ \varphi^{-1})^{-1}(x \cdot (f \circ \varphi^{-1})(y)) \\ &= I_{f \circ \varphi^{-1}}(x, y). \end{aligned}$$

Thus $J$ is an $f$-implication. Analogously one can prove the converse. □

In fact, the following two results show that Lemma 7.3.12 can be made even stronger.

**Lemma 7.3.13.** *Let $I \in \mathbb{I}$ and $J \in [I]_{\underset{\sim}{\bullet}}$. Then $I \in \mathbb{I}_{\mathbb{F},\infty} \iff J \in \mathbb{I}_{\mathbb{F},\infty}$.*

*Proof.* Let $I$ be an $f$-implication generated by some $f$-generator $f$ such that $f(0) = \infty$. Let $J \in [I]_{\underset{\sim}{\bullet}}$. From Lemma 7.3.12, it follows that $J = I_{f \circ \varphi^{-1}}$ for some $\varphi \in \Phi$. From Remark 7.3.11, it follows that $f \circ \varphi^{-1}$ is also an $f$-generator. Moreover $(f \circ \varphi^{-1})(0) = f(\varphi^{-1}(0)) = f(0) = \infty$. Thus $J \in \mathbb{I}_{\mathbb{F},\infty}$. □

**Corollary 7.3.14.** *Let $I \in \mathbb{I}$ and $J \in [I]_{\underset{\sim}{\bullet}}$. Then $I \in \mathbb{I}_{\mathbb{F},1} \iff J \in \mathbb{I}_{\mathbb{F},1}$.*

**Theorem 7.3.15.** $\mathbb{I}_{\mathbb{F},\infty} = [I_{\mathbf{YG}}]_{\underset{\sim}{\bullet}}$.

*Proof.* We know that $I_{\mathbf{YG}}$ is an $f$- implication with the generator $f(x) = -\ln x$ (see, Example 7.3.10(i)). Observe that $f(0) = \infty$ and hence $I_{\mathbf{YG}} \in \mathbb{I}_{\mathbb{F},\infty}$. Let $J \in [I_{\mathbf{YG}}]_{\underset{\sim}{\bullet}}$. From Lemma 7.3.13, it follows that $J \in \mathbb{I}_{\mathbb{F},\infty}$. Thus $[I_{\mathbf{YG}}]_{\underset{\sim}{\bullet}} \subseteq \mathbb{I}_{\mathbb{F},\infty}$.

Now, let $I \in \mathbb{I}_{\mathbb{F},\infty}$, i.e., $I = I_f$ for some $f$-generator $f$ such that $f(0) = \infty$. Take $\varphi(x) = f^{-1}(-\ln x)$. Then $\varphi(0) = 0$ and $\varphi(1) = 1$. Moreover $\varphi$ is an increasing bijection and hence $\varphi \in \Phi$. Take $f_l(x) = -\ln x$. Then $(f_l \circ \varphi^{-1})(x) = f_l(e^{-f(x)}) = -\ln(e^{-f(x)}) = f(x)$. Thus $I = I_f = I_{f_l \circ \varphi^{-1}}$. This implies that $I \in [I_{\mathbf{YG}}]_{\underset{\sim}{\bullet}}$ and consequently $\mathbb{I}_{\mathbb{F},\infty} \subseteq [I_{\mathbf{YG}}]_{\underset{\sim}{\bullet}}$. □

**Theorem 7.3.16.** $\mathbb{I}_{\mathbb{F},1} = [I_{\mathbf{RC}}]_{\underset{\sim}{\bullet}}$.

*Proof.* We know that $I_{\mathbf{RC}}$ is an $f$- implication with the $f$- generator $f(x) = 1 - x$ (see, Example 7.3.10(ii)). Note that $f(0) = 1$ and hence $I_{\mathbf{RC}} \in \mathbb{I}_{\mathbb{F},1}$. Let $J \in [I_{\mathbf{RC}}]_{\underset{\sim}{\bullet}}$. From Corollary 7.3.14, it follows that $J \in \mathbb{I}_{\mathbb{F},1}$. Thus $[I_{\mathbf{RC}}]_{\underset{\sim}{\bullet}} \subseteq \mathbb{I}_{\mathbb{F},1}$.

Now, let $I \in \mathbb{I}_{\mathbb{F},1}$. Then $I = I_f$ for some $f$- generator $f$ such that $f(0) = 1$. Take $\varphi(x) = 1 - f(x)$. It is clear that $\varphi(0) = 0, \varphi(1) = 1$ and $\varphi(x)$ is increasing bijection on $[0, 1]$. Moreover $I = I_f = I_{f_c \circ \varphi^{-1}}$ where $f_c(x) = 1 - x$. Hence $I \in [I_{\mathbf{RC}}]_{\underset{\sim}{\bullet}}$. □

**Corollary 7.3.17.** *(i) An* $I \in \mathbb{I}_{\mathbb{F},\infty}$ *if and only if for some* $\varphi \in \Phi$*,*

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ \varphi\left(\left[\varphi^{-1}(y)\right]^x\right), & \text{if } x > 0 \text{ or } y > 0. \end{cases}$$

*(ii) An* $I \in \mathbb{I}_{\mathbb{F},1}$ *if and only if for some* $\varphi \in \Phi$*,* $I(x,y) = \varphi\left(1 - x + x\varphi^{-1}(y)\right)$.

*(iii)* $\mathbb{I}_{\mathbb{F}} = \mathbb{I}_{\mathbb{F},\infty} \cup \mathbb{I}_{\mathbb{F},1} = [I_{\mathbf{YG}}]_{\underset{\sim}{\bullet}} \cup [I_{\mathbf{RC}}]_{\underset{\sim}{\bullet}}$.

The above results, as it often happens in mathematics, merge aesthetics with utility; through the group action we have proposed, we have shown that every $f$-implication is a pseudo-conjugate of either the Yager implication $I_{\mathbf{YG}}$ or the Reichenbach implication $I_{\mathbf{RC}}$, a fact that is not at all apparent from Example 7.3.10.

### 7.3.4 An Alternative Representation of $g$- implications

From Chapter 5, we recall that the class of $g$-implications is another class of Yager's implications (see, Definition 5.4.1). In this subsection we examine the effect of the group action that we have proposed on the family of $g$-implications and investigate the equivalence classes for different subsets of $g$-implications, viz., $\mathbb{I}_{\mathbb{G},\infty}$ and $\mathbb{I}_{\mathbb{G},1}$ . Based on our results, once again for the first time, we show that every $g$-implication is a pseudo-conjugate of either the Yager implication $I_{\mathbf{YG}}$ or the Goguen implication $I_{\mathbf{GG}}$.

**Example 7.3.18** ([8], Example 3.2.4). *Much like the $f$-generators, the $g$-generators can be seen as continuous additive generators of continuous Archimedean t-conorms* (see Chapter 4, [48]). *Once again, the following examples illustrate this idea.*

*(i) If we take the g-generator* $g_l(x) = -\ln(1-x)$*, then we obtain the following fuzzy implication:*

$$I_{g_l}(x,y) = I(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \ , \\ 1 - (1-y)^{\frac{1}{x}}, & x \in (0,1] \text{ or } y \in (0,1] \ . \end{cases}$$

*(ii) If we take the g-generator* $g_c(x) = x$*, then we obtain the Goguen implication* $I_{g_c} = I_{\mathbf{GG}}$.

*(iii) One can easily calculate that for the g-generator* $g(x) = -\dfrac{1}{\ln x}$ *we obtain the Yager implication* $I_{\mathbf{YG}}$*, which is also an $f$-implication.*

*(iv) If we take the trigonometric function* $g_{\mathbf{t}}(x) = \tan\left(\frac{\pi}{2}x\right)$*, which is a continuous function with* $g_{\mathbf{t}}(0) = 0, g_{\mathbf{t}}(1) = \infty$*, as the g-generator, then its inverse is* $g_{\mathbf{t}}^{-1}(x) = \frac{2}{\pi}\tan^{-1}(x)$ *and we obtain the following g- implication:*

$$I_{g_{\mathbf{t}}}(x,y) = \frac{2}{\pi}\tan^{-1}\left(\frac{1}{x}\cdot\tan\left(\frac{\pi}{2}y\right)\right) \ , \qquad x, y \in [0,1] \ .$$

*(v) If we take the Yager's class of additive generators,* $g^{\lambda}(x) = x^{\lambda}$*, where* $\lambda \in (0,\infty)$*, as the g-generators, then* $g^{\lambda}(1) = 1$ *for every* $\lambda$*, its pseudo-inverse is given by* $(g^{\lambda})^{(-1)}(x) = \min(1, x^{\frac{1}{\lambda}})$ *and the g-*

*implication is given by*

$$I_{g^\lambda}(x,y) = \min\left(1, \frac{y}{x^{\frac{1}{\lambda}}}\right) = \begin{cases} 1, & \textit{if } x^{\frac{1}{\lambda}} \le y \ , \\ \dfrac{y}{x^{\frac{1}{\lambda}}}, & \textit{if } x^{\frac{1}{\lambda}} > y \ , \end{cases} \qquad x, y \in [0,1] \ .$$

*(vi) If we take the Frank's class of additive generators of t-conorms* (see Remark 4.8, [48]),

$$g^s(x) = -\ln\left(\frac{s^{1-x}-1}{s-1}\right) , \qquad s > 0 \ , s \neq 1 \ ,$$

*as the $g$-generators, then for every $s$, we have $g^s(1) = \infty$,*

$$(g^s)^{-1}(x) = 1 - \log_s\left(1 + (s-1)e^{-s}\right)$$

*and the corresponding $g$- implication is given by*

$$I_{g^s}(x,y) = 1 - \log_s\left(1 + (s-1)^{\frac{x-1}{x}}(s^{1-y}-1)^{\frac{1}{x}}\right) \ , \qquad x, y \in [0,1] \ .$$

**Remark 7.3.19.** *Note that for every $g$- generator $g$, the function $g \circ \varphi \colon [0,1] \longrightarrow [0,\infty]$ is strictly increasing and $(g \circ \varphi)(0) = 0$ for all $\varphi \in \Phi$. Thus $g \circ \varphi$ is also a $g$- generator for every $\varphi \in \Phi$.*

Our first result shows that if $I$ is a $g$-implication then every $\varphi$-pseudo conjugate of $I$ is also a $g$-implication.

**Lemma 7.3.20.** *Let $I \in \mathbb{I}$ and $J \in [I]_{\underset{\sim}{\bullet}}$. Then $I \in \mathbb{I}_{\mathbb{G}} \iff J \in \mathbb{I}_{\mathbb{G}}$.*

Once again, the following results show that every $g$-implication is a $\varphi$-pseudo conjugate of either the Yager implication $I_{\mathbf{YG}}$ or the Goguen implication $I_{\mathbf{GG}}$.

**Theorem 7.3.21.** $\mathbb{I}_{\mathbb{G},\infty} = [I_{\mathbf{YG}}]_{\underset{\sim}{\bullet}}$.

*Proof.* From Proposition 5.4.5 and Theorem 7.3.15, the proof follows directly. □

**Theorem 7.3.22.** $\mathbb{I}_{\mathbb{G},1} = [I_{\mathbf{GG}}]_{\underset{\sim}{\bullet}}$.

*Proof.* We know that $I_{\mathbf{GG}} = I_g$ where $g(x) = x$. Since $g(1) = 1$, clearly $I_{\mathbf{GG}} \in \mathbb{I}_{\mathbb{G},1}$ and consequently $[I_{\mathbf{GG}}]_{\underset{\sim}{\bullet}} \subseteq \mathbb{I}_{\mathbb{G},1}$.

Let $I \in \mathbb{I}_{\mathbb{G},1}$ i.e., $I = I_g$ for some generator $g$ such that $g(1) = 1$. Take $\varphi(x) = g(x)$. Then $I = I_g = I_{g_1 \circ \varphi^{-1}}$ where $g_1(x) = x$. It follows that $I \in [I_{\mathbf{GG}}]_{\underset{\sim}{\bullet}}$ and consequently $\mathbb{I}_{\mathbb{G},1} \subseteq [I_{\mathbf{GG}}]_{\underset{\sim}{\bullet}}$. □

**Corollary 7.3.23.** *(i) An $I \in \mathbb{I}_{\mathbb{G},\infty}$ if and only if for some $\varphi \in \Phi$,*

$$I(x,y) = \begin{cases} 1, & \textit{if } x = 0 \textit{ and } y = 0 \\ \varphi\left(\left[\varphi^{-1}(y)\right]^x\right), & \textit{if } x > 0 \textit{ or } y > 0. \end{cases}$$

.

*(ii) An $I \in \mathbb{I}_{\mathbb{G},1}$ if and only if, for some $\varphi \in \Phi$, $I(x,y) = \begin{cases} 1, & \textit{if } \varphi(x) \le y, \\ \varphi\left(\frac{\varphi^{-1}(y)}{x}\right), & \textit{if } \varphi(x) > y. \end{cases}$*

*(iii)* $\mathbb{I}_{\mathbb{G}} = \mathbb{I}_{\mathbb{G},\infty} \cup \mathbb{I}_{\mathbb{G},1} = [I_{\mathbf{YG}}]_{\underset{\sim}{\bullet}} \cup [I_{\mathbf{GG}}]_{\underset{\sim}{\bullet}}$.

Once again, the above results show that the Yager implication $I_{\mathbf{YG}}$ and the Goguen implication $I_{\mathbf{GG}}$ act as seeds from which the entire family of $g$-implications can be obtained.

## 7.4 Algebraic Connotation of Baczyński and Drewniak Conjugacy Classes

In Section 7.1.1, we have recalled the conjugacy classes of fuzzy implications proposed by Baczyński and Drewniak. However, it should be noticed that there exists no algebraic connotation of these conjugacy classes so far. In this section we attempt to give an algebraic interpretation to these conjugacy classes and show that these conjugacy classes are exactly the equivalence classes of fuzzy implications obtained from a group action of $\mathbb{S}$ on the set $\mathbb{I}$.

Towards this, we first propose yet another new generating method of fuzzy implications from fuzzy implications and show that this method imposes a semigroup structure on the set $\mathbb{I}$.

**Definition 7.4.1.** *Let $I, J \in \mathbb{I}$. Define $I\Delta J \colon [0,1]^2 \longrightarrow [0,1]$ as follows:*

$$(I\Delta J)(x,y) = I(J(1,x), J(x,y)), \qquad x,y \in [0,1]. \tag{7.3}$$

Observe that Definitions 2.1.1 and 7.4.1 are not identically same on $\mathbb{I}$.

**Theorem 7.4.2.** *The function $I\Delta J$ is a fuzzy implciation. i.e., $I\Delta J \in \mathbb{I}$.*

*Proof.* Let $I, J \in \mathbb{I}$ and $x_1, x_2, y \in [0,1]$.

(i) Let $x_1 \le x_2$. Then $J(x_1, y) \ge J(x_2, y)$ and $J(1, x_1) \le J(1, x_2)$.

$$\begin{aligned}(I\Delta J)(x_1,y) &= I(J(1,x_1), J(x_1,y))\\ &\ge I(J(1,x_1), J(x_2,y))\\ &\ge I(J(1,x_2), J(x_2,y))\\ &\ge (I\Delta J)(x_2,y).\end{aligned}$$

Thus $I\Delta$ is decreasing in the first variable. Similarly, one can show that $I\Delta J$ is increasing in the second variable.

(ii) $(I\Delta J)(0,0) = I(J(1,0), J(0,0)) = I(0,1) = 1$.
$(I\Delta J)(1,1) = I(J(1,1), J(1,1)) = I(1,1) = 1$.
$(I\Delta J)(1,0) = I(J(1,1), J(1,0)) = I(1,0) = 0$.

Thus $I\Delta J$ is a fuzzy implication. □

From Theorem 7.4.2, it follows that $I\Delta J \in \mathbb{I}$ for all $I, J \in \mathbb{I}$. Algebraically speaking, $\Delta$ becomes a binary operation on the set $\mathbb{I}$. In fact, in the following, we show that $\Delta$ is associative in $\mathbb{I}$, thus making $(\mathbb{I}, \Delta)$ a semigroup.

**Theorem 7.4.3.** $(\mathbb{I}, \Delta)$ *is a semigroup.*

*Proof.* From Theorem 7.4.2, it is enough to show that $\Delta$ is associative in $\mathbb{I}$. To show this, let $I, J, K \in \mathbb{I}$ and $x, y \in [0,1]$. Then

$$\begin{aligned}(I\Delta(J\Delta K))(x,y) &= I((J\Delta K)(1,x),(J\Delta K)(x,y))\\ &= I(J(K(1,1),K(1,x)),J(K(1,x),K(x,y)))\\ &= I(J(1,K(1,x)),J(K(1,x),K(x,y))),\\ \text{and, } ((I\Delta J)\Delta K)(x,y) &= (I\Delta J)(K(1,x),K(x,y))\\ &= I(J(1,K(1,x)),J(K(1,x),K(x,y))).\end{aligned}$$

Thus $\Delta$ is associative in $\mathbb{I}$ and $(\mathbb{I},\Delta)$ forms a semigroup. □

Unlike $(\mathbb{I},\circledast)$, the semigroup $(\mathbb{I},\Delta)$ is not a monoid. However, in the following, we show that the binary operations $\circledast$ and $\Delta$ are identically the same on $\mathbb{S}$, the set of all invertible elements of $(\mathbb{I},\circledast)$.

**Lemma 7.4.4.** *Let $I, J \in \mathbb{S}$. Then $I \circledast J = I\Delta J$.*

*Proof.* Let $I, J \in \mathbb{S}$ and

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0\ ,\\ \varphi(y), & \text{if } x > 0\ ,\end{cases}$$

and

$$J(x,y) = \begin{cases} 1, & \text{if } x = 0\ ,\\ \psi(y), & \text{if } x > 0\ ,\end{cases}$$

for some $\varphi, \psi \in \Phi$. Now,

$$\begin{aligned}(I\Delta J)(x,y) &= I(J(1,x),J(x,y))\\ &= I(\psi(x),J(x,y))\\ &= \begin{cases} 1, & \text{if } x = 0,\\ \varphi(\psi(y)), & \text{if } x > 0,\end{cases}\end{aligned}$$

and,

$$\begin{aligned}(I \circledast J)(x,y) &= I(x,J(x,y))\\ &= \begin{cases} 1, & \text{if } x = 0,\\ \varphi(\psi(y)), & \text{if } x > 0.\end{cases}\end{aligned}$$

Thus $\circledast$ and $\Delta$ are equal on $\mathbb{S}$. □

**Remark 7.4.5.** *From Lemma 7.4.4, one may suspect that the binary operations $\circledast$ and $\Delta$ are identically same on $\mathbb{I}$, but this is not true. To see this, let $I(x,y) = I_{\mathbf{RC}}(x,y) = 1 - x + xy$ and $J(x,y) = \max(1-x, y^2)$. Then it follows that*

$$(I \circledast J)(x,y) = \max(1-x^2, 1-x+xy^2)\ ,$$

*while,*

$$(I\Delta J)(x,y) = \max(1-x^3, 1-x^2+x^2y^2)\ ,$$

*for all* $x, y \in [0,1]$. *With* $x = 0.5, y = 0$, *we see that*

$$(I \circledast J)(x,y) = 0.75 \neq 0.87 = (I\Delta J)(x,y)\ .$$

From Lemma 7.4.4, the following remark is straightforward.

**Lemma 7.4.6.** *For all* $I \in \mathbb{I}, K \in \mathbb{S}$, $K \circledast (I\Delta K^{-1}) = (K \circledast I)\Delta K^{-1}$.

*Proof.* Let $I \in \mathbb{I}, K \in \mathbb{S}$. Then from Theorem 6.2.4, $K$ is given by (6.2), i.e.,

$$K(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(y), & \text{if } x > 0, \end{cases}$$

for some $\varphi \in \Phi$. Then $K^{-1}$ will be given by

$$K^{-1}(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi^{-1}(y), & \text{if } x > 0. \end{cases}$$

**Case (i)**: Let $x = 0$. Then $(K \circledast (I\Delta K^{-1}))(0,y) = 1 = ((K \circledast I)\Delta K^{-1})(0,y)$.
**Case (ii)**: Let $x > 0$. Then

$$\begin{aligned} (K \circledast (I\Delta K^{-1}))(x,y) &= K(x, (I\Delta K^{-1})(x,y)) \\ &= K(x, I(K^{-1}(1,x), K^{-1}(x,y))) \\ &= \varphi(I(\varphi^{-1}(x), \varphi^{-1}(y))) \\ \text{and, } ((K \circledast I)\Delta K^{-1})(x,y) &= (K \circledast I)(K^{-1}(1,x), K^{-1}(x,y)) \\ &= K(K^{-1}(1,x), I(K^{-1}(1,x), K^{-1}(x,y))) \\ &= K(\varphi(x), I(\varphi^{-1}(x), \varphi^{-1}(y))) \\ &= \varphi(I(\varphi^{-1}(x), \varphi^{-1}(y))). \end{aligned}$$

Thus in all cases, $K \circledast (I\Delta K^{-1}) = (K \circledast I)\Delta K^{-1}$. □

We now define yet another group action of $\mathbb{S}$ on $\mathbb{I}$ and study the equivalence classes obtained from it.

**Lemma 7.4.7.** *Let* $\sqcap\colon \mathbb{S} \times \mathbb{I} \longrightarrow \mathbb{I}$ *be defined by*

$$K \sqcap I = K \circledast I\Delta K^{-1}, \qquad K \in \mathbb{S},\ I \in \mathbb{I}. \tag{7.4}$$

*The operation* $\sqcap$*, defined as in* (7.4)*, is a group action of* $\mathbb{S}$ *on* $\mathbb{I}$.

*Proof.* (i) Let $K_1, K_2 \in \mathbb{S}$ and $I \in \mathbb{I}$. Then

$$\begin{aligned}
K_1 \sqcap (K_2 \sqcap I) &= K_1 \circledast (K_2 \sqcap I)\Delta K_1^{-1} \\
&= K_1 \circledast (K_2 \circledast I\Delta K_2^{-1})\Delta K_1^{-1} \\
&= K_1 \circledast K_2 \circledast I\Delta (K_1 \Delta K_2)^{-1}, && \lceil\because \text{ by Lemma 7.4.6.} \\
&= K_1 \circledast K_2 \circledast I\Delta (K_1 \circledast K_2)^{-1}, && \lceil\because \text{ by Lemma 7.4.4.} \\
&= (K_1 \circledast K_2) \sqcap I.
\end{aligned}$$

(ii) $I_{\mathbf{D}} \sqcap I = I_{\mathbf{D}} \circledast I\Delta I_{\mathbf{D}}^{-1} = I$ for all $I \in \mathbb{I}$.
Thus $\sqcap$ is a group action of $\mathbb{S}$ on $\mathbb{I}$. □

**Definition 7.4.8.** *Define $\sim_{\sqcap}$ on $\mathbb{I}$ by $I \sim_{\sqcap} J \iff J = K \circledast I\Delta K^{-1}$ for some $K \in \mathbb{S}$.*

It is easy to verify that $\sim_{\sqcap}$ is an equivalence relation (Lemma 7.4.6 is useful).

**Theorem 7.4.9.** *The conjugacy classes of fuzzy implications proposed by Baczyński* et.al.*, viz.,* (7.1)*, are the equivalence classes of fuzzy implications w.r.to the equivalence relation $\sim_{\sqcap}$, i.e., for any $I \in \mathbb{I}$, we have that $[I]_{\sim_{\mathcal{B}}} = [I]_{\sim_{\sqcap}}$.*

*Proof.* Let $I, J \in \mathbb{I}$ be such that $I \sim_{\sqcap} J$. Then $J = K \circledast I\Delta K^{-1}$ for some $K \in \mathbb{S}$. Let $K \in \mathbb{S}$ be of the form given in (6.2) for some $\varphi^{-1} \in \Phi$. Now,

$$\begin{aligned}
J(x, y) &= (K \circledast I\Delta K^{-1})(x, y) \\
&= K(x, (I\Delta K^{-1})(x, y)) \\
&= K(x, I(K^{-1}(1, x), K^{-1}(x, y))) \\
&= K(x, I(\varphi(x), K^{-1}(x, y))) \\
&= \begin{cases} 1, & \text{if } x = 0 \\ \varphi^{-1}(I(\varphi(x), \varphi(y))), & \text{if } x > 0 \end{cases} \\
&= \varphi^{-1}(I(\varphi(x), \varphi(y))) = I_\varphi(x, y)\ .
\end{aligned}$$

□

From Theorem 7.4.9, we see that the conjugacy classes proposed by Baczyński and Drewniak can also be obtained by a group action of $\mathbb{S}$ on $\mathbb{I}$.

## 7.5 Algebraic Connotation of Jayaram and Mesiar Conjugacy Classes

Here in this section we propose yet another group action of $\mathbb{S}$ on $\mathbb{I}$ and show that the equivalence classes obtained through them are exactly the conjugacy classes proposed by Jayaram and Mesiar [41], viz., (7.2), in the context of special fuzzy implications.

Towards this end, we have the following definition.

**Definition 7.5.1.** *Let $\sqcup \colon \mathbb{S} \times \mathbb{I} \longrightarrow \mathbb{I}$ be defined by $K \sqcup I = K \circledast I, \qquad K \in \mathbb{S},\ I \in \mathbb{I}$.*

**Lemma 7.5.2.** *$\sqcup$ is a group action of $\mathbb{S}$ on $\mathbb{I}$.*

*Proof.* (i) Let $K_1, K_2 \in \mathbb{S}$ and $I \in \mathbb{I}$. Then

$$\begin{aligned} K_1 \sqcup (K_2 \sqcup I) &= K_1 \circledast (K_2 \sqcup I) \\ &= K_1 \circledast K_2 \circledast I \\ &= (K_1 \circledast K_2) \sqcup I. \end{aligned}$$

(ii) $I_{\mathbf{D}} \sqcup I = I_{\mathbf{D}} \circledast I = I$ for all $I \in \mathbb{I}$.
Thus $\sqcup$ is a group action of $\mathbb{S}$ on $\mathbb{I}$. □

**Definition 7.5.3.** *Define $\sim_{\sqcup}$ on $\mathbb{I}$ by $I \sim_{\sqcup} J \Longleftrightarrow J = K \circledast I$ for some $K \in \mathbb{S}$.*

It is easy to verify that $\sim_{\sqcup}$ is an equivalence relation.

**Lemma 7.5.4.** *The equivalence classes of fuzzy implications as given in Definition 7.5.3 are exactly the conjugacy classes proposed by Jayaram and Mesiar, viz.,* (7.2)*, i.e., for any $I \in \mathbb{I}$, we have that $[I]_{\sim_{\mathcal{J}}} = [I]_{\sim_{\sqcup}}$.*

*Proof.* Let $I \in \mathbb{I}$. Then

$$\begin{aligned} [I]_{\sim_{\sqcup}} &= \{J \in \mathbb{I} | J \sim_{\sqcup} I\} \\ &= \{J \in \mathbb{I} | J = K \sqcup I \text{ for some } K \in \mathbb{S}\} \\ &= \{J \in \mathbb{I} | J = K \circledast I \text{ for some } K \in \mathbb{S}\} \\ &= \{J \in \mathbb{I} | J(x,y) = K(x, I(x,y)) \text{ for some } K \in \mathbb{S}\} \\ &= \{J \in \mathbb{I} | J(x,y) = \varphi(I(x,y)) \text{ for some } \varphi \in \Phi\} \qquad [\because \text{ Representation of } K \in \mathbb{S} \text{ from (6.2)} \\ &= \{\varphi(I(x,y)) \mid \varphi \in \Phi\}\ . \end{aligned}$$

□

## 7.6 Conclusions

In this chapter, we have proposed three different group actions of $\mathbb{S}$ on $\mathbb{I}$ and investigated the equivalence classes of $\mathbb{I}$ obtained from them. Based on the partitions obtained from one of the group actions, we have obtained the representations of Yager's families of fuzzy implications in terms of the Reichenbach implication $I_{\mathbf{RC}}$, the Yager implication $I_{\mathbf{YG}}$ and Goguen implication $I_{\mathbf{GG}}$. Finally, we have also shown that the equivalence classes obtained from the other two group actions, viz., $\sqcap, \sqcup$, are nothing but the conjugacy classes proposed earlier in different contexts thus providing an algebraic connotation to these definitions.

# Chapter 8

# Right Translation (Semigroup) Homomorphisms on $(\mathbb{I}, \circledast)$

*Structures are the weapons of a mathematician.*
– Nicolas Bourbaki.

One of our stated motivations is to obtain a richer algebraic structure on the set $\mathbb{I}$ of all fuzzy implications. For instance, if $(\mathbb{I}, \circledast)$ were to form a group, then one can apply results from group theory to obtain deeper and better perspectives of the different families of fuzzy implications. For instance, it is well known that if a group $G$ is not simple, it has a nontrivial normal subgroup $N$ which partitions $G$. Now, it is easy to see that to generate the whole of $G$, when $O(G) < \infty$, it is sufficient to pick $O(N) + O(\frac{G}{N})$ elements. Further, since any $g \in G$ is in one of these cosets, $g$ can be expressed as $g = n \cdot g'$, for some $n \in N$ and a $g'$ which is the representative element of the (same) coset. If $N$ is a nontrivial normal subgroup with some desirable properties then we have a *unique* decomposition of $g$ into components with known properties.

As was made clear in the earlier chapters, $(\mathbb{I}, \circledast)$ does not form a group. In Chapter 7, our exploration took the path of proposing group actions and investigating their equivalence classes. These equivalence classes were able to throw new light on existing concepts or families of fuzzy implications. In this chapter, we further explore the algebraic aspects by investigating some special subsets of the monoid $(\mathbb{I}, \circledast)$.

From the Cayley's theorem for monoids, we know that any monoid $(\mathbb{M}, \otimes)$ is isomorphic to the set of all right translations $\mathcal{R} \subsetneq \mathbb{M}^{\mathbb{M}}$, where $\mathcal{R} = \{g_a : \mathbb{M} \to \mathbb{M} | g_a(x) = x \otimes a, \text{ for a fixed } a \in \mathbb{M}\}$.

In this chapter, we study the right translation semigroup homomorphisms on the monoid $(\mathbb{I}, \circledast)$, i.e., those right translations that also become semigroup homomorphisms on $(\mathbb{I}, \circledast)$. Our study shows that three sub-classes of fuzzy implications, denoted by $\mathbb{A}, \mathbb{K}_\varepsilon, \mathbb{K}^\varepsilon \subsetneq \mathbb{I}$, give rise to right translation semigroup homomorphisms on $\mathbb{I}$.

This study has two interesting fallouts:

(i) One of the above sub-classes, viz., $\mathbb{A}$ turns out to be precisely the set of right zero elements, while another, namely $\mathbb{K}_\varepsilon$, characterises the set of all commuting elements in $\mathbb{I}$, i.e., the center $\mathcal{Z}_{\mathbb{I}}$ of the monoid $(\mathbb{I}, \circledast)$. The study also shows the important role played by the considered form of homomorphisms in determining $\mathcal{Z}_{\mathbb{I}}$.

(ii) Further, the set $\mathbb{A}$ of right zero elements of $\mathbb{I}$ w.r.to $\circledast$, also forms a two-sided ideal of the monoid $(\mathbb{I}, \circledast)$ and thus gives rise to the possibility of defining Rees semigroups of $(\mathbb{I}, \circledast)$.

We begin this chapter, by recalling the notion of right translations on a monoid and introduce the right translations $g_K$ on $(\mathbb{I}, \circledast)$ in Section 8.1 and show that not all of them become semigroup homomorphisms (denoted *s.g.h*, henceforth) on $(\mathbb{I}, \circledast)$. Hence, in Section 8.2 we undertake this study and obtain a few necessary conditions on $K \in \mathbb{I}$ such that $g_K$ is an *s.g.h.* In Section 8.3, we investigate the trivial range fuzzy implications $K$ such that $g_K$ is an *s.g.h* and based on their representations show that they form the set of all right zero elements of the monoid $(\mathbb{I}, \circledast)$. In Section 8.4, we show that in the case of nontrivial range fuzzy implications $K$ such that $g_K$ is an s.g.h, the vertical section $K(1, y)$ must be either identity or zero on $[0, 1)$, i.e., either, $K(1, y) = y$ for all $y \in [0, 1]$ or $K(1, y) = 0$ for all $y \in [0, 1)$. In Section 8.5, we characterise the set of all $K \in \mathbb{I}$ satisfying (NP), i.e., $K(1, y) = y$ for all $y \in [0, 1]$ and obtain the representations of elements of the same. Based on the obtained results, we show that this set is exactly the set of all commuting elements of $(\mathbb{I}, \circledast)$, viz., the center of the monoid. The representations of fuzzy implications $K$ satisfying $K(1, y) = 0$ for all $y \in [0, 1)$ is discussed in Section 8.6.

## 8.1 Right Translations on the Monoid $(\mathbb{I}, \circledast)$

Translations are one of the important transformations that can be defined on a semigroup. In [22], Clifford introduced the notion of translations in the context of extension of semigroups, and later on, their role has been studied in different contexts, for more details, please see, [53].

### 8.1.1 Left and Right Translations on a general monoid

In this subsesction we review the important concepts related to translations, inner translations and their role in the embedding of semigroups.

**Definition 8.1.1** ([53], Chapter 10, Definition 7.2). *A transformation $\psi$ of a semigroup $(U, \cdot)$ is called a left (right) translation if for any elements $x$ and $y$ of $U$,*

$$\psi(x \cdot y) = \psi(x) \cdot y \qquad (\psi(x \cdot y) = x \cdot \psi(y))$$

For every $a \in U$, the functions $\phi_a(x) = a \cdot x, \psi_a(x) = x \cdot a$ are left, right translations of $U$, respectively, and are called the inner left and inner right translations induced by $a$.

**Theorem 8.1.2** ([53], Chapter 10, Theorem 7.5). *In a semigroup $U$, every left (right) translation is an inner right translation if and only if $U$ has left (right) identity.*

**Theorem 8.1.3** ([53], Chapter 10, Theorem 7.7). *If a semigroup $U$ has an identity then every translation, both left and right, is inner.*

The following is the Cayley's theorem for semigroups.

**Theorem 8.1.4** ([46], Theorem 2.34). *For every semigroup $S$ there exists a set $X$ and an injective map $\phi : S \to X^X$ which is a morphism of semigroups from $S$ to $X^X$.*

The above result states that every semigroup $S$ is isomorphic to a subsemigroup of the semigroup of all transformations on an appropriate set $X$. In fact, from the usual proofs of this result, it can be seen that $S$ is isomorphic to the set of all right translations defined over $S$.

### 8.1.2 Right Translations on the monoid $(\mathbb{I}, \circledast)$

In the following, we introduce the right translations on the monoid $(\mathbb{I}, \circledast)$ and show that they are lattice homomorphisms.

**Definition 8.1.5.** *For a fixed $K \in \mathbb{I}$, define $g_K : (\mathbb{I}, \circledast) \longrightarrow (\mathbb{I}, \circledast)$ by*

$$g_K(I) = I \circledast K, \qquad I \in \mathbb{I}.$$

Since $\mathbb{I}$ is a lattice ordered monoid (see Lemma 6.1.3), we have the following result.

**Proposition 8.1.6.** *For every $K \in \mathbb{I}$, the map $g_K$ is a lattice homomorphism.*

*Proof.* Let $K \in \mathbb{I}$. Let $I, J \in \mathbb{I}$ and $x, y \in [0, 1]$. Then,

$$\begin{aligned} g_K(I \vee J)(x, y) &= ((I \vee J) \circledast K)(x, y) \\ &= (I \vee J)(x, K(x, y)) \\ &= \max(I(x, K(x, y)), J(x, K(x, y))) \\ &= \max((I \circledast K)(x, y), (J \circledast K)(x, y)) \\ &= (g_K(I) \vee g_K(J))(x, y). \end{aligned}$$

Similarly, one can prove that

$$g_K(I \wedge J) = g_K(I) \wedge g_K(J), \qquad I, J \in \mathbb{I}.$$

Thus $g_K$ is a lattice homomorphism. □

We have proved that $g_K$'s are lattice homomorphisms for every $K \in \mathbb{I}$. However, for every $K \in \mathbb{I}$, the function $g_K$ need not be a semigroup homomorphism (*s.g.h*) on $\mathbb{I}$ as the following example illustrates.

**Example 8.1.7.** *For instance, when $K(x, y) = I_{\mathbf{LK}}(x, y) = \min(1, 1-x+y)$, the Łukasiewicsz implication, the map $g_K$ is not an* s.g.h. *To see this, let $I(x, y) = I_{\mathbf{KD}}(x, y) = \max(1-x, y)$ and $J(x, y) = I_{\mathbf{RC}}(x, y) = 1 - x + xy$ and $x = 0.4, y = 0.2$, we observe that*

$$g_{I_{\mathbf{LK}}}(I \circledast J)(0.4, 0.2) = 0.92 \neq 1 = \left(g_{I_{\mathbf{LK}}}(I) \circledast g_{I_{\mathbf{LK}}}(J)\right)(0.4, 0.2) .$$

## 8.2 Necessary conditions on $K \in \mathbb{I}$ such that $g_K$ is an *s.g.h.*

Since $g_K$ is not an *s.g.h* for every $K \in \mathbb{I}$, we investigate to characterise and, if possible, determine those fuzzy implications $K$ for which $g_K$ becomes an *s.g.h.*

In the following, we investigate some conditions that $K$ should satisfy for $g_K$ to be an *s.g.h.*

**Proposition 8.2.1.** *Let $K \in \mathbb{I}$ be arbitrarily fixed. Then the following statements are equivalent:*

*(i)* $g_K$ *is an* s.g.h.

*(ii)* $J \circledast K = K \circledast J \circledast K$ *for all* $J \in \mathbb{I}$.

*Proof.* (i) $\Longrightarrow$ (ii) : Let $K \in \mathbb{I}$ and $g_K$ be an *s.g.h.* Then for all $I, J \in \mathbb{I}$, $g_K(I \circledast J) = g_K(I) \circledast g_K(J)$ will imply $I \circledast J \circledast K = I \circledast K \circledast J \circledast K$. If we take $I = I_{\mathbf{D}}$, the identity in $(\mathbb{I}, \circledast)$, it follows that $J \circledast K = K \circledast J \circledast K$ for all $J \in \mathbb{I}$.

(ii) $\Longrightarrow$ (i) : Let $K \in \mathbb{I}$ be such that $J \circledast K = K \circledast J \circledast K$ for all $J \in \mathbb{I}$. This directly implies that $I \circledast J \circledast K = I \circledast K \circledast J \circledast K$ for every $I \in \mathbb{I}$, since every $\circledast$ is a well-defined function on $\mathbb{I}$. Thus $g_K$ is an *s.g.h.*

□

As a consequence of Proposition 8.2.1, we have the following result.

**Lemma 8.2.2.** *Let $K \in \mathbb{I}$ be such that $g_K$ is an* s.g.h. *Then $K^2 = K$.*

*Proof.* From Proposition 8.2.1, it follows that $J \circledast K = K \circledast J \circledast K$ for all $J \in \mathbb{I}$. But when $J = I_{\mathbf{D}}$, the identity of $(\mathbb{I}, \circledast)$, we have that $I_{\mathbf{D}} \circledast K = K \circledast I_{\mathbf{D}} \circledast K$, or equivalently, $K = K \circledast K$. □

**Remark 8.2.3.** *The converse of the previous lemma need not be true always. For example take $K = I_{\mathbf{GD}}$. Clearly $K \circledast K = K$. However, $g_K$ does not always to be an s.g.h. To see this, let, $K = I_{\mathbf{GD}}$, which is such that $K \circledast K = K$ (see, Theorem 11, [74]). Take $J \in \mathbb{I}$ defined by*

$$I_\beta(x, y) = \begin{cases} 1, & \textit{if } x = 0 \textit{ or } y = 1 \,, \\ 0, & \textit{if } x = 1 \textit{ and } y = 0 \,, \\ \beta, & \textit{otherwise.} \end{cases} \tag{8.1}$$

*Now, let $\beta = 0.6, x = 0.4$ and $y = 0.2$. Then*

$$\begin{aligned} (I_\beta \circledast I_{\mathbf{GD}})(0.4, 0.2) &= I_\beta(0.4, I_{\mathbf{GD}}(0.4, 0.2)) \\ &= I_\beta(0.4, 0.2) = 0.6 \\ \textit{while, } (I_{\mathbf{GD}} \circledast I_\beta \circledast I_{\mathbf{GD}})(0.4, 0.2) &= I_{\mathbf{GD}}(0.4, I_\beta(0.4, I_{\mathbf{GD}}(0.4, 0.2))) \\ &= I_{\mathbf{GD}}(0.4, I_\beta(0.4, 0.2)) = I_{\mathbf{GD}}(0.4, 0.6) = 1. \end{aligned}$$

*From Proposition 8.2.1, it follows that $g_K$ is not an* s.g.h.

The above two results convey the necessary conditions that $K$ should satisfy for $g_K$ to become an *s.g.h.* In our quest for determining $K \in \mathbb{I}$ such that $g_K$ becomes an *s.g.h.*, we divide our analysis into two parts, viz., finding $K \in \mathbb{I}$ when the range of $K$ is trivial and the range of $K$ is nontrivial.

## 8.3 $K \in \mathbb{I}$ with trivial range such that $g_K$ is an *s.g.h.*

In this section, we determine completely the fuzzy implications $K \in \mathbb{I}$ whose range is trivial, i.e., $K(x, y) \in \{0, 1\}$ for all $x, y \in [0, 1]$, and for whom the map $g_K$ is an *s.g.h.*

**Theorem 8.3.1.** *Let $K \in \mathbb{I}$ be such that the range of $K$ is trivial. Then the following statements are equivalent:*

*(i)* $g_K$ *is an s.g.h.*

*(ii)* $K = K^\delta$ *for some* $\delta \in (0,1]$, *where*

$$K^\delta(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } (x = 1 \text{ and } y \geq \delta)\ , \\ 0, & \text{if } x = 1 \text{ and } y < \delta\ . \end{cases}$$

*Proof.* (i) $\Longrightarrow$ (ii). Let $g_K$ be an *s.g.h.*

**Claim:** $K(x,y) = 1$, for all $x \in [0,1)$ and for all $y \in [0,1]$.
**Proof of the claim :**

- If $x = 0$, it is trivial that $K(x,y) = 1$ for all $y \in [0,1]$.
- Let $0 < x < 1$. Suppose that for some $y_0 \in [0,1)$, $K(x,y_0) < 1$, i.e., $K(x,y_0) = 0$. Since $g_K$ is an *s.g.h*, it follows that $J \circledast K = K \circledast J \circledast K$ for all $J \in \mathbb{I}$. Now,

$$\begin{aligned}(J \circledast K)(x,y_0) &= J(x, K(x,y_0)) \\ &= J(x,0), \\ (K \circledast J \circledast K)(x,y_0) &= K(x, J(x, K(x,y_0))) \\ &= K(x, J(x,0)).\end{aligned}$$

  Since the range of $K$ is trivial, $J(x,0) \subseteq \{0,1\}$ for all $J \in \mathbb{I}$. This gives a contradiction if we take a $J \in \mathbb{I}$ such that $J(x,0) \notin \{0,1\}$.

Thus $K(x,y) = 1$, for all $x < 1$.
Now for $x = 1, y \in [0,1]$, we have either $K(x,y) = 0$ or $K(x,y) = 1$. Let us define

$$\delta = \sup\{y \in [0,1] | K(1,y) = 0\}.$$

Let us take $K \in \mathbb{I}$ such that $K(1,y)$ is right continuous. Then for $y \geq \delta$, $K(1,y) = 1$ and for $y < \delta$, $K(1,y) = 0$. Thus $K = K^\delta$.

(ii) $\Longrightarrow$ (i). It can be verified easily.

□

**Remark 8.3.2.** *Note that in the proof of Theorem 8.3.1 we have chosen $K^\delta$ such that it is right-continuous in the second variable, when $x = 1$. However, if we choose $K^\delta$ such that it is left-continuous in the second variable at $x = 1$, i.e., $K^\delta(1,y) = 1$ when $y > \delta$ and $K^\delta(1,y) = 0$ when $y \leq \delta$, it can be easily verified that $g_{K^\delta}$ is still an* s.g.h. *This particular choice was made to conform to the tradition in the literature of requiring right-continuity in the second variable, as in the case of implications from which the deresiduum is constructed.*

Interestingly, as the following result shows, the set of all fuzzy implications of the form $K^\delta$ where $\delta \in (0,1]$, i.e., $\{K^\delta | \delta \in (0,1]\}$ is precisely the set of all right zero elements of $\mathbb{I}$ w.r.to $\circledast$.

Before doing so, recall that $I_{\mathbf{0}}, I_{\mathbf{1}} \in \mathbb{I}$ are defined as follows:

$$I_{\mathbf{0}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } \ y = 1, \\ 0, & \text{if } x > 0 \text{ and } \ y < 1, \end{cases} \quad \text{and } I_{\mathbf{1}}(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } \ y > 0, \\ 0, & \text{if } x = 1 \text{ and } \ y = 0. \end{cases} \tag{8.2}$$

**Lemma 8.3.3.** *Let $\mathbb{A} \subset \mathbb{I}$ be the set of all right zero elements of $\circledast$. Then $\mathbb{A} = \{K^{\delta} | \delta \in (0,1]\}$.*

*Proof.*

- If $K = K^{\delta}$ for some $\delta \in (0,1]$, then it is easy to see that $I \circledast K = K$ for all $I \in \mathbb{I}$. Hence $\mathbb{A} \supseteq \mathbb{K}^{\delta}$.

- Let for some $K \in \mathbb{I}$, $I \circledast K = K$ for all $I \in \mathbb{I}$.
  **Claim:** $K(x,y) = \{0,1\}$ for all $x, y \in [0,1]$, i.e., the range of $K$ is trivial.
  *Proof of the claim:* Clearly, if $x = 0$ or $y = 1$ then $K(x,y) = 1 \in \{0,1\}$. Suppose for some $x_0 \in (0,1], y_0 \in [0,1)$ that $\alpha = K(x_0, y_0) \notin \{0,1\}$. Now,

$$\begin{aligned} (I_{\mathbf{0}} \circledast K)(x_0, y_0) &= I_{\mathbf{0}}(x_0, K(x_0, y_0)) \\ &= I_{\mathbf{0}}(x_0, \alpha) = 0 \\ &\neq \alpha = K(x_0, y_0), \end{aligned}$$

  contradicting $I \circledast K = K$ for all $I \in \mathbb{I}$. Thus the range of $K$ is trivial.
  **Claim:** $K(x,y) = 1$, for all $x \in [0,1)$ and for all $y \in [0,1]$.
  *Proof of the claim:* If $x = 0$, then it is trivial. So, let $0 < x < 1$ be fixed arbitrarily. Suppose for $y_0 < 1$, that $K(x, y_0) < 1$. Since the range of $K$ is trivial, $K(x, y_0) = 0$. Now,

$$\begin{aligned} (I_{\mathbf{1}} \circledast K)(x, y_0) &= I_{\mathbf{1}}(x, K(x, y_0)) \\ &= I_{\mathbf{1}}(x, 0) = 1 \\ &\neq 0 = K(x, y_0), \end{aligned}$$

  contradicts the fact that $I \circledast K = K$ for all $I \in \mathbb{I}$. Now define $\delta = \sup\{t | K(1,t) = 0\}$. This implies that $K(1,y) = 0$ for all $y < \delta$ and $K(1,y) = 1$ for all $y > \delta$, because the range of $K$ is trivial. Once again since we are interested in $K \in \mathbb{I}$ right continuous in the second variable, we take that $K(1,\delta) = 1$. Thus $K = K^{\delta}$ for some $\delta \in (0,1]$.

□

**Proposition 8.3.4.** *The monoid $(\mathbb{I}, \circledast)$ does not have left zero elements.*

*Proof.* Let $\mathcal{L}$ denote the set of all left-zero elements of the monoid $(\mathbb{I}, \circledast)$. We claim that $\mathcal{L} = \emptyset$.

On the contrary, let $I, J \in \mathcal{L}$ be two left-zero elements. Then $I \circledast K = I$ and $J \circledast K = K$ for all $K \in \mathbb{I}$. Now, consider a right zero element $K' \in \mathbb{A}$. Then it follows that $I = I \circledast K' = K'$ and $J = J \circledast K' = K'$. This shows that $I = J$ and hence $\mathcal{L}$ is utmost a singleton set. Let $L \in \mathcal{L}$.

Now, let $K_1, K_2 \in \mathbb{A}$ be two distinct right zero elements of $(\mathbb{I}, \circledast)$. Then we have $L = L \circledast K_1 = K_1$ and $L = L \circledast K_2 = K_2$, which leads to a contradiction, since $K_1 \neq K_2$. Thus $\mathcal{L} = \emptyset$ and $(\mathbb{I}, \circledast)$ has no left zero elements. □

**Corollary 8.3.5.** *The monoid $(\mathbb{I}, \circledast)$ has no two-sided zero elements.*

While $(\mathbb{I}, \circledast)$ has no two-sided zero elements, it is interesting to note the following. Clearly, every $K^\delta \in \mathbb{A}$ is a right zero element, i.e., $I \circledast K^\delta = K^\delta$, for any $I \in \mathbb{I}$. If we consider the following composition, $K^\delta \circledast I$ for any $I \in \mathbb{I}$, we obtain that $K^\delta \circledast I = K^\mu \in \mathbb{A}$ for some $\mu \in (0,1]$. In other words, the set $\mathbb{A}$ when composed with $\mathbb{I}$ subsumes it both from the left and the right. In fact, as we show below, the set $\mathbb{A}$ forms a two-sided ideal of the monoid $(\mathbb{I}, \circledast)$.

Recall that a nonempty subset $A$ of a semigroup $S$ is called a two-sided ideal if $AS \subseteq A$ and $SA \subseteq A$.

**Lemma 8.3.6.** *The set $\mathbb{A}$ of all right zero elements of $(\mathbb{I}, \circledast)$ forms a two-sided ideal.i.e., $\mathbb{I}\mathbb{A} = \mathbb{A} = \mathbb{A}\mathbb{I}$.*

*Proof.* From Lemma 8.3.3, it follows that $\mathbb{I}\mathbb{A} = \mathbb{A}$. Now it remains to show that $\mathbb{A}\mathbb{I} = \mathbb{A}$. Before proceeding to show this, for a given $\delta \in (0,1]$ and an $I \in \mathbb{I}$, let us define

$$\delta_I = \inf\{y \in [0,1] | I(1,y) \geq \delta\} . \tag{8.3}$$

Note that $1 \in \{y \in [0,1] | I(1,y) \geq \delta\}$ and hence $\delta_I \in (0,1]$ and is well defined.

Now, let $I \in \mathbb{I}$ and $K \in \mathbb{A}$. Since $K \in \mathbb{A}$, from Lemma 8.3.3 we have that $K = K^\delta$ for some $\delta \in (0,1]$ (see, Theorem 8.3.1, for the definition of $K^\delta$). Now,

$$\begin{aligned}
(K \circledast I)(x,y) &= K^\delta(x, I(x,y)) \\
&= \begin{cases} 1, & \text{if } x < 1 \text{ or } (x = 1 \ \&\ I(1,y) \geq \delta) , \\ 0, & \text{if } x = 1 \ \&\ I(1,y) < \delta , \end{cases} \\
&= \begin{cases} 1, & \text{if } x < 1 \text{ or } (x = 1 \ \&\ y \geq \delta_I) , \\ 0, & \text{if } x = 1 \ \&\ y < \delta_I , \end{cases} \\
&= K^{\delta_I}(x,y) ,
\end{aligned}$$

where $\delta_I$ is as defined in (8.3) above. Since, for every $I \in \mathbb{I}$, there exists a $\delta_I \in (0,1]$, such that $K \circledast I = K^{\delta_I} \in \mathbb{A}$, we see that $\mathbb{A}\mathbb{I} \subseteq \mathbb{A}$. The other inclusion $\mathbb{A} \subseteq \mathbb{A}\mathbb{I}$ follows directly, as the identity $I_{\mathbf{D}} \in \mathbb{I}$. Thus $\mathbb{A}\mathbb{I} = \mathbb{A}$ and $\mathbb{A}$ is a two-sided ideal.

□

## 8.4 $K \in \mathbb{I}$ with nontrivial range such that $g_K$ is an *s.g.h.*

In Section 8.3, we have characterised and found the trivial range fuzzy implications $K$ such that $g_K$ is an *s.g.h.* Further, we have shown that these fuzzy implications form the set of all right zero elements of the monoid $(\mathbb{I}, \circledast)$. In this section, we determine the nontrivial range fuzzy implications $K$ such that $g_K$ is an *s.g.h.* Towards this end, the following result shows that the range of such fuzzy implications $K$ should be the entire $[0,1]$ interval.

**Lemma 8.4.1.** *If the range of $K \in \mathbb{I}$ is nontrivial and $g_K$ is an* s.g.h. *then the range of $K$ is equal to $[0,1]$.*

*Proof.* Let the range of $K \in \mathbb{I}$ be nontrivial and $g_K$ be an *s.g.h.* Since the range of $K$ is nontrivial, there exists $\alpha \in (0,1)$ such that $K(x_0, y_0) = \alpha$ for some $x_0 \in (0,1]$ and $y_0 \in [0,1)$. Let $I_\beta \in \mathbb{I}$ be as defined in Eq.(8.1). Then, $(I_\beta \circledast K)(x_0, y_0) = I_\beta(x_0, K(x_0, y_0)) = I_\beta(x_0, \alpha) = \beta$. Since $g_K$ is an *s.g.h*,

$(K \circledast I_\beta \circledast K)(x_0, y_0) = \beta$, i.e., $\beta$ is in the range of $K$. Since $\beta \in (0,1)$ is chosen arbitrarily, the range of $K$ contains every point of $[0,1]$. Thus the range of $K$ is $[0,1]$. $\square$

While, the above result discusses the range of the fuzzy implication $K$ such that $g_K$ is an *s.g.h* the following result shows that the natural negation (see Definition 3.3.7) of such a $K$ is of trivial range.

**Lemma 8.4.2.** *Let $K \in \mathbb{I}$ be such that $g_K$ is an* s.g.h. *Then the range of $K(\cdot, 0) = \{0,1\}$.*

*Proof.* Let $K \in \mathbb{I}$ be such that $g_K$ is an *s.g.h.* For some $x_0 \in (0,1)$, let $K(x_0, 0) = \alpha$. Consider $I_{\mathbf{0}} \in \mathbb{I}$. Then

$$
\begin{aligned}
(K \circledast I_{\mathbf{0}} \circledast K)(x_0, 0) &= K(x_0, I_{\mathbf{0}}(x_0, K(x_0, 0))) \\
&= K(x_0, I_{\mathbf{0}}(x_0, \alpha)) = K(x_0, 0), \\
\text{while,} \qquad (I_{\mathbf{0}} \circledast K)(x_0, 0) &= I_{\mathbf{0}}(x_0, K(x_0, 0)) \\
&= I_{\mathbf{0}}(x_0, \alpha) = 0.
\end{aligned}
$$

Since $g_K$ is an *s.g.h.* from Proposition 8.2.1, it follows that $K(x_0, 0) = 0 = \alpha$. Note that $x_0 \in (0,1)$ is chosen arbitrarily. Hence $K(x, 0) = \{0,1\}$ for all $x \in [0,1]$. $\square$

Note that the above result characterises the horizontal section $K(\cdot, 0)$ and is trivially true for $K$ whose range is trivial. Now, before characterising the nontrivial range fuzzy implications $K$ such that $g_K$ is an *s.g.h.*, we characterise the vertical section $K(1, \cdot)$ of $K$ which helps us in getting the representations of $K$. Towards this, we propose the following definition.

**Definition 8.4.3.** *Let $K \in \mathbb{I}$. Define the following two real numbers:*

$$\epsilon_0 = \sup\{t \in [0,1] | K(1,t) = 0\}, \tag{8.4}$$

$$\epsilon_1 = \inf\{t \in [0,1] | K(1,t) = 1\}. \tag{8.5}$$

**Remark 8.4.4.** *(i) Let $\epsilon_0, \epsilon_1$ be two real numbers as defined in Definition 8.4.3. For every $K \in \mathbb{I}$, since $K(1,0) = 0$ and $K(1,1) = 1$, the real numbers $\epsilon_0, \epsilon_1$ in the equations* (8.4), (8.5) *are well defined and exist in general.*

*(ii) More importantly, $0 \le \epsilon_0 \le \epsilon_1 \le 1$.*

*(iii) Since $\epsilon_0 \le \epsilon_1$, if $\epsilon_0 = 1$ then $\epsilon_1 = 1$.*

**Proposition 8.4.5.** *Let the range of $K \in \mathbb{I}$ be nontrivial and $g_K$ be an* s.g.h. *Let $\epsilon_0, \epsilon_1 \in [0,1]$ be defined as in Definition 8.4.3. Then the vertical section $K(1, .)$ has the following form:*

$$
K(1,y) = \begin{cases}
0, & \textit{if } y \in [0, \epsilon_0)\ , \\
0 \textbf{\textit{ or }} \epsilon_0, & \textit{if } y = \epsilon_0\ , \\
y, & \textit{if } y \in (\epsilon_0, \epsilon_1)\ , \\
\epsilon_1 \textbf{\textit{ or }} 1, & \textit{if } y = \epsilon_1\ , \\
1, & \textit{if } y \in (\epsilon_1, 1]\ .
\end{cases} \tag{8.6}
$$

*Proof.* Let $K \in \mathbb{I}$ be such that the range of $K$ is nontrivial and $g_K$ is an *s.g.h.*

Further, since, $g_K$ is an *s.g.h*, we see that for all $J \in \mathbb{I}$ the following equality should hold for all $y \in [0,1]$:

$$(J \circledast K)(1,y) = (K \circledast J \circledast K)(1,y).$$
$$\textit{i.e.,}\ J(1,K(1,y)) = K(1,J(1,K(1,y)))\ . \tag{8.7}$$

(i) From the definition of $\epsilon_0, \epsilon_1$ above, it is clear that $K(1,y) = 0$ whenever $0 \leq y < \epsilon_0$ and $K(1,y) = 1$, whenever $\epsilon_1 < y \leq 1$.

(ii) Let $\epsilon_0 < y < \epsilon_1$. We claim that $K(1,y) = y$. If not, let there be a $y_0 \in [0,1)$ such that $K(1,y_0) = y' \neq y_0$. Let us choose a $J \in \mathbb{I}$ such that $J(1,y') = y_0$. Note that such a $J$ is always possible, for instance, $J = I_\beta$ of (8.1) with $\beta = y_0$. Then, we have

$$\begin{aligned}\textbf{LHS}\ \text{of (8.7)} &= J(1,K(1,y_0)) = J(1,y') = y_0,\\ \textbf{RHS}\ \text{of (8.7)} &= K(1,J(1,K(1,y_0)))\\ &= K(1,J(1,y')) = K(1,y_0) = y',\end{aligned}$$

from whence we obtain that $g_K$ is not an *s.g.h.*, a contradiction. Thus $K(1,y) = y$ whenever $\epsilon_0 < y < \epsilon_1$.

(iii) Note that since $\epsilon_0, \epsilon_1$ are only the infimum and supremum of these sets, which are intervals due to the monotonicity of $K$ in the second variable, they may not belong to these intervals themselves. In other words, $K(1,\epsilon_0) \geq 0$ and $K(1,\epsilon_1) \leq 1$.

   (a) Clearly, if $\epsilon_0 = \max\{t \in [0,1] | K(1,t) = 0\}$, then $K(1,\epsilon_0) = 0$.

   (b) However, if $\epsilon_0 \notin \{t \in [0,1] | K(1,t) = 0\}$ then clearly $0 < K(1,\epsilon_0) = \delta$. We claim that $\delta = \epsilon_0$. On the contrary, let $\delta \neq \epsilon$, then, once again, one can choose a $J \in \mathbb{I}$ such that $J(1,\delta) = \epsilon_0$. Then, we have

$$\begin{aligned}\textbf{LHS}\ \text{of (8.7)} &= J(1,K(1,\epsilon_0)) = J(1,\delta) = \epsilon_0\ ,\\ \textbf{RHS}\ \text{of (8.7)} &= K(1,J(1,K(1,\epsilon_0)))\\ &= K(1,J(1,\delta)) = K(1,\epsilon_0) = \delta\ ,\end{aligned}$$

   from whence we obtain that $g_K$ is not an *s.g.h.*, a contradiction. Thus $K(1,\epsilon_0) = \epsilon_0$.

   (c) A similar proof as above shows that if $\epsilon_1 \in \{t \in [0,1] | K(1,t) = 1\}$ then $K(1,\epsilon_1) = 1$, while if $\epsilon_1 \notin \{t \in [0,1] | K(1,t) = 1\}$ then $K(1,\epsilon_1) = \epsilon_1$.

□

In Proposition 8.4.5, even though we are able to characterise the vertical sections $K(1,\cdot)$, it is not clear what values $K(1,\cdot)$ could assume. Now, we investigate all the possible values of $\epsilon_0, \epsilon_1$ in the case range of $K$ is **nontrivial** and $g_K$ is an *s.g.h*.

**Theorem 8.4.6.** *Let $K \in \mathbb{I}$ be such that the range of $K$ is nontrivial and $g_K$ is an* s.g.h *and let $\epsilon_0, \epsilon_1$ be defined as in Definition 8.4.3. Then* *(i)* $\epsilon_1 \neq 0$.

*(ii) If* $\epsilon_0 = 0$, *then* $\epsilon_1 = 1$, *in which case* $K(1, y) = y$ *for all* $y \in [0, 1]$.

*(iii) If* $0 < \epsilon_0 < 1$, *then* $\epsilon_0 \neq \epsilon_1$.

*(iv) If* $\epsilon_0 > 0$, *then* $\epsilon_0 = 1$, *in which case* $K(1, y) = 0$ *for all* $y > 0$.

*Proof.* (i) Let $\epsilon_1 = 0$. This implies that $K(1, y) = 1$ for all $y > 0$. Again it follows from the monotonicity of $I$ in the first variable that $K(x, y) = 1$ for all $x$ and all $y > 0$. Now, from Lemma 8.4.2, it follows that the range of the negation of fuzzy implication $K$ is trivial, i.e., $K(x, 0) \in \{0, 1\}$. So, the range of $K$ becomes $\{0, 1\}$, a contradiction to the fact the range of $K$ is nontrivial. Thus $\epsilon_1 \neq 0$.

(ii) Let $\epsilon_0 = 0$ and suppose that $\epsilon_1 < 1$. Then from (i), it follows that $0 < \epsilon_1 < 1$. So choose a $\delta > 0$ such that $0 < \epsilon_1 + \delta < 1$. Let $0 < y_1 < \epsilon_1$. This implies that $0 < K(1, y_1) = \alpha < 1$. Now, choose a $J \in \mathbb{I}$ such that $J(1, K(1, y_1)) = J(1, \alpha) = \epsilon_1 + \delta$. However, $K(1, J(1, K(1, y_1))) = K(1, \epsilon_1 + \delta) = 1$, which contradicts $g_K$ being an *s.g.h*. Thus $\epsilon_1 = 1$.

(iii) Let $0 < \epsilon_0 < 1$. Suppose that $\epsilon_0 = \epsilon_1$. Then $K(1, \cdot)$ will be of the form

$$K(1, y) = \begin{cases} 1, & \text{if } y \geq \epsilon_0, \\ 0, & \text{if } y < \epsilon_0. \end{cases} \tag{8.8}$$

This implies that $K(x, y) = 1$ for all $x \in [0, 1], y \geq \epsilon_0$. Now we prove that $K(x, y) = 1$ for all $x \in [0, 1), y \in [0, \epsilon_0)$. On the contrary suppose that $\alpha = K(x_0, y_0) < 1$ for some $x_0 \in (0, 1),\ y_0 \in [0, \epsilon_0)$. Since $0 < \epsilon_0 < 1$, choose a $\delta > 0$ such that $0 < \epsilon_0 + \delta < 1$. Now choose a $J \in \mathbb{I}$ such that $J(x_0, K(x_0, y_0)) = J(x_0, \alpha) = \epsilon_0 + \delta \neq 1$. Now, $K(x_0, J(x_0, K(x_0, y_0))) = K(x_0, J(x_0, \alpha)) = K(x_0, \epsilon_0 + \delta) = 1$ a contradiction to the fact that $g_K$ is an *s.g.h*. Thus $K(x_0, y_0) = 1$ for all $x_0 \in [0, 1)$ and $y_0 \in [0, \epsilon_0)$ and $K(x, y) = 1$ for all $x < 1$. Finally from the Eq.(8.8) it follows that the range of $K$ is trivial, a contradicition. Thus $\epsilon_0 \neq \epsilon_1$.

(iv) Let $\epsilon_0 > 0$. Suppose $\epsilon_0 < 1$, i.e., $0 < \epsilon_0 < 1$. Now from (iii) it follows that $\epsilon_0 \neq \epsilon_1$. Let $y_1 \in (\epsilon_0, \epsilon_1)$. Then the Eq.(8.6) implies that $K(1, y_1) = y_1$. Choose a $J \in \mathbb{I}$ be such that $J(1, y_1) = \frac{\epsilon_0}{2}$. Then

$$\begin{aligned} (J \circledast K)(1, y_1) &= J(1, K(1, y_1)) = J(1, y_1) = \frac{\epsilon_0}{2}, \\ \text{and } (K \circledast J \circledast K)(1, y_1) &= K(1, J(1, K(1, y_1))) \\ &= K(1, J(1, y_1)) = K\left(1, \frac{\epsilon_0}{2}\right) = 0 \end{aligned}$$

a contradiction to the fact that $g_K$ is an *s.g.h*. Thus $\epsilon_0 = 1$.

□

From Proposition 8.4.5 and Remark 8.4.4, Corollary 8.4.7 gives the possible values of $\epsilon_0, \epsilon_1$ of nontrivial $K \in \mathbb{I}$ in the case $g_K$ is an *s.g.h*.

**Corollary 8.4.7.** *Let* $K \in \mathbb{I}$ *be such that the range of* $K$ *is nontrivial and* $g_K$ *is an* s.g.h *and let* $\epsilon_0$ *and* $\epsilon_1$ *be defined as in* (8.4) *and* (8.5), *respectively. Then* *(i)* *If* $\epsilon_0 = 0$ *then* $\epsilon_1 = 1$.

*(ii)* *If* $\epsilon_0 > 0$ *then* $\epsilon_0 = 1$.

**Corollary 8.4.8.** *Let the range of* $K$ *be nontrivial and* $g_K$ *be an* s.g.h. *Then one of the following conditions holds:*

*(i)* $K(1, y) = y$, *for all* $y \in [0, 1]$.

*(ii)* $K(1, y) = 0$, *for all* $y \in [0, 1)$.

From the above results, it is clear that if $K$ is a nontrivial range implication such that $g_K$ is an *s.g.h* then $K$ has either (NP) or $K(1, y) = 0$ for all $y \in [0, 1)$. We analyse each of these two cases in Sections 8.5 and 8.6.

## 8.5 $K \in \mathbb{I}_{\mathbf{NP}}$ such that $g_K$ is an s.g.h

To get the representation of $K$ satisfying (NP), we need to take the help of two important algebraic concepts, namely, center and set of idempotent elements. Let us denote the center and the set of all idempotent elements of the monoid $(\mathbb{I}, \circledast)$, respectively, as follows:

$$\mathcal{Z}_{\mathbb{I}} = \{I \in \mathbb{I} | I \circledast J = J \circledast I, \forall J \in \mathbb{I}\} ,$$
$$\mathcal{I}_{\mathbb{I}} = \{I \in \mathbb{I} \mid I \circledast I = I\} .$$

**Remark 8.5.1.** $(\mathcal{Z}_{\mathbb{I}}, \circledast)$ *is a commutative submonoid of* $(\mathbb{I}, \circledast)$.

The following lemma which plays an important role when dealing with the fuzzy implications $K$ having (NP) gives a relation between the sets $\mathcal{Z}_{\mathbb{I}}$ and $\mathcal{I}_{\mathbb{I}}$ in $(\mathbb{I}, \circledast)$.

**Lemma 8.5.2.** *The center* $\mathcal{Z}_{\mathbb{I}}$ *of the monoid* $(\mathbb{I}, \circledast)$ *is contained in the set* $\mathcal{I}_{\mathbb{I}}$, *i.e.,* $\mathcal{Z}_{\mathbb{I}} \subset \mathcal{I}_{\mathbb{I}}$.

*Proof.* Let $K \in \mathcal{Z}_{\mathbb{I}}$. We need to show that $K^2 = K$,

$$i.e., K(x, K(x, y)) = K(x, y), \qquad x, y \in [0, 1].$$

Suppose for some $x_0 \in (0, 1], y_0 \in [0, 1)$ that

$$\alpha = K(x_0, K(x_0, y_0)) \neq K(x_0, y_0) = \beta.$$

Thus $K(x_0, \beta) = \alpha$.
**Claim:** $\beta \notin \{0, 1\}$.

*Proof of the claim:* Let $\beta = 0$, i.e., $K(x_0, y_0) = 0$ and hence $K(x_0, K(x_0, y_0)) = K(x_0, 0) = \alpha \neq \beta = 0$. Then

$$(I_{\mathbf{0}} \circledast K)(x_0, y_0) = I_{\mathbf{0}}(x_0, K(x_0, y_0)) = I_{\mathbf{0}}(x_0, 0) = 0,$$

$$(K \circledast I_{\mathbf{0}})(x_0, y_0) = K(x_0, I_{\mathbf{0}}(x_0, y_0)) = K(x_0, 0) = \alpha \neq 0.$$

Thus $I_{\mathbf{0}} \circledast K \neq K \circledast I_{\mathbf{0}}$, contradicting the fact $K \in \mathcal{Z}_{\mathbb{I}}$. Thus $\beta \neq 0$.
Let $\beta = 1$, i.e., $K(x_0, y_0) = 1$. Then it implies that $K(x_0, K(x_0, y_0)) = 1$, contradicting our assumption $K(x_0, y_0) \neq K(x_0, K(x_0, y_0))$. Thus $\beta \neq 1$.
**Claim:** $\alpha \neq 1$.
*Proof of the claim:* Let $\alpha = 1$, i.e., $K(x_0, K(x_0, y_0)) = \alpha = 1$. We have already proven that $\beta \neq 0, 1$. Now define $I_\beta \in \mathbb{I}$ as in (8.1).

Now, $I_\beta(x_0, K(x_0, y_0)) = \beta$ and $K(x_0, I_\beta(x_0, y_0)) = K(x_0, \beta) = \alpha = 1$. Thus

$$I_\beta(x_0, K(x_0, y_0)) \neq K(x_0, I_\beta(x_0, y_0)),$$

a contradiction to the fact that $K \in \mathcal{Z}_{\mathbb{I}}$. Thus $\alpha \neq 1$.

Now, $I_\beta(x_0, K(x_0, \beta)) = I_\beta(x_0, \alpha) = \beta$ and $K(x_0, I_\beta(x_0, \beta)) = K(x_0, \beta) = \alpha$. Thus

$$I_\beta(x_0, K(x_0, \beta)) \neq K(x_0, I_\beta(x_0, \beta)),$$

a contradiction to the fact that $K \in \mathcal{Z}_{\mathbb{I}}$. Thus $K \in \mathcal{I}_{\mathbb{I}}$ and hence $\mathcal{Z}_{\mathbb{I}} \subset \mathcal{I}_{\mathbb{I}}$. $\square$

**Remark 8.5.3.** *In Lemma 8.5.2, the inclusion is strict. To see this, let us take $I_{\mathbf{0}}, I_{\mathbf{1}} \in \mathbb{I}$ as given in* (8.2). *Then it is strightforward to see that $I_{\mathbf{1}} \circledast I_{\mathbf{1}} = I_{\mathbf{1}}$, i.e., $I_{\mathbf{1}} \in \mathcal{I}_{\mathbb{I}}$. However, at $x = 1, y = 0.4$ we observe that*

$$(I_{\mathbf{1}} \circledast I_{\mathbf{0}})(1, 0.4) = I_{\mathbf{1}}(1, I_{\mathbf{0}}(1, 0.4)) = I_{\mathbf{1}}(1, 0) = 0,$$

$$(I_{\mathbf{0}} \circledast I_{\mathbf{1}})(1, 0.4) = I_{\mathbf{0}}(1, I_{\mathbf{1}}(1, 0.4)) = I_{\mathbf{0}}(1, 1) = 1.$$

*which implies that $I_{\mathbf{1}} \notin \mathcal{Z}_{\mathbb{I}}$. Similarly, one can observe that $I_{\mathbf{GD}}, I_{\mathbf{0}} \in \mathcal{I}_{\mathbb{I}}$ but $I_{\mathbf{GD}}, I_{\mathbf{0}} \notin \mathcal{Z}_{\mathbb{I}}$.*

Based on Lemma 8.5.2, we have a first partial characterisation of $K$ such that $g_K$ is an *s.g.h.*

**Lemma 8.5.4.** *If $K \in \mathcal{Z}_{\mathbb{I}}$ then $g_K$ is an* s.g.h.

*Proof.* Let $K \in \mathcal{Z}_{\mathbb{I}}$. Then, from Lemma 8.5.2, it follows that $K \in \mathcal{I}_{\mathbb{I}}$. Let $I, J \in \mathbb{I}$. Now,

$$\begin{aligned} g_K(I) \circledast g_K(J) &= (I \circledast K) \circledast (J \circledast K) \\ &= (I \circledast K) \circledast (K \circledast J) \\ &= I \circledast (K \circledast K) \circledast J \\ &= I \circledast (K \circledast J) = I \circledast (J \circledast K) \\ &= (I \circledast J) \circledast K = g_K(I \circledast J). \end{aligned}$$

Thus $g_K$ is an *s.g.h.* $\square$

In fact, as we show in the following the converse of Lemma 8.5.4 is also true for neutral implications, i.e., for those $K \in \mathbb{I}$ that satisfy (NP). Before proving this fact, we need the following result which gives a complete characterisation of all $K \in \mathbb{I}$ for which $g_K$ is an *s.g.h.*

**Lemma 8.5.5.** *If $K \in \mathcal{Z}_{\mathbb{I}}$, then the range of $K$ is nontrivial.*

*Proof.* Let $K \in \mathcal{Z}_{\mathbb{I}}$. Suppose that the range of $K$ is trivial. Since $K \in \mathcal{Z}_{\mathbb{I}}$, from Lemma 8.5.4 it follows that $g_K$ is an *s.g.h.* Again from Theorem 8.3.1, it follows that $K = K^\delta$ for some $\delta \in (0, 1]$. Here we claim that $\delta \neq 1$. If $\delta = 1$, then $K = K^{\delta}$ will be of the form

$$K(x,y) = \begin{cases} 1, & \text{if } x < 1 \text{ or } \ y = 1, \\ 0, & \text{if } x = 1 \text{ and } y \neq 1. \end{cases}$$

Now it is easy to see that $(I_{\mathbf{1}} \circledast K)(1, 0.2) = 0$ where as $(K \circledast I_{\mathbf{1}})(1, 0.2) = 1$, proving that $K \notin \mathcal{Z}_{\mathbb{I}}$, a contradiction to the fact $K \in \mathcal{Z}_{\mathbb{I}}$. Thus $\delta \neq 1$. Now, it is easy to find two real numbers $\delta^{'}, \delta^{''} \in (0,1]$ such that $\delta^{''} < \delta < \delta^{'}$. Let $I = I_{\beta}$ as defined in (8.1) with $\beta = \delta^{''}$. Then

$$\begin{aligned} (I \circledast K^{\delta})(1, \delta^{'}) &= I(1, K^{\delta}(1, \delta^{'})) = 1, \\ \text{while, } (K^{\delta} \circledast I)(1, \delta^{'}) &= K^{\delta}(1, I(1, \delta^{'})) = K^{\delta}(1, \delta^{''}) = 0, \end{aligned}$$

contradicting that $K \in \mathbb{Z}_{\mathbb{I}}$. Thus the range of $K$ is nontrivial.

□

**Proposition 8.5.6.** *If $K \in \mathcal{Z}_{\mathbb{I}}$, then $K$ has* (NP)*.*

*Proof.* Let $K \in \mathcal{Z}_{\mathbb{I}}$. From Lemma 8.5.4, it follows that $g_K$ is an *s.g.h* and also from Lemma 8.5.5, it follows that range of $K$ is nontrivial. To prove that $K$ has (NP), from Proposition 8.4.5 it suffices to show that $K(1, y) \neq 0$ or $1$ for any $y \in (0, 1)$.

On the contrary, let $K(1, y_0) = 0$ for some $y_0 \in (0, 1)$. Then, on the one hand,

$$\begin{aligned} (I_{\mathbf{1}} \circledast K)(1, y_0) &= I_{\mathbf{1}}(1, K(1, y_0)) \\ &= I_{\mathbf{1}}(1, 0) = 0 \ , \end{aligned}$$

and on the other hand,

$$\begin{aligned} (K \circledast I_{\mathbf{1}})(1, y_0) &= K(1, I_{\mathbf{1}}(1, y_0)) \\ &= K(1, 1) = 1 \ , \end{aligned}$$

which contradicts the fact $K \in \mathcal{Z}_{\mathbb{I}}$. Thus for any $y_0 \in (0, 1)$, $K(1, y_0) \neq 0$.

Similarly, by taking $I_{\mathbf{0}}$ instead of $I_{\mathbf{1}}$, above we can show that for any $y_0 \in \ (0, 1)$, $K(1, y_0) \neq 1$. From Proposition 8.4.5, we see that this is equivalent to stating $\epsilon_0 = 0$ and $\epsilon_1 = 1$ and hence it follows that $K$ must have (NP). □

We define below a special class of fuzzy implications satisfying (NP).

**Definition 8.5.7.** *For $\epsilon \in [0, 1)$ define*

$$K_{\epsilon}(x,y) = \begin{cases} 1, & \text{if } x \leq \epsilon \ , \\ y, & \text{if } x > \epsilon, \end{cases} \tag{8.9}$$

*and for $\epsilon = 1$, $K_{\epsilon} = I_{\mathbf{WB}}$ where*

$$I_{\mathbf{WB}}(x,y) = \begin{cases} 1, & \text{if } x < 1, \\ y, & \text{if } x = 1. \end{cases} \tag{8.10}$$

Note that $K_\epsilon \in \mathbb{I}$, for all $\epsilon \in [0,1]$ and $\sup K_\epsilon = I_{\mathbf{WB}}$. For notational convenience, we denote the set of all such $K_\epsilon$ fuzzy implications by

$$\mathbb{K}_\epsilon = \{I \in \mathbb{I} | I = K_\epsilon \ for\ some\ \epsilon \in [0,1]\}.$$

The following result lists a few properties of fuzzy implications from the set $\mathbb{K}_\epsilon$.

**Proposition 8.5.8.** *The following properties hold true.*

*(i)* $\epsilon_1 < \epsilon_2 \Longrightarrow K_{\epsilon_1} \leq K_{\epsilon_2}$

*(ii)* $K_{\epsilon_1} \circledast K_{\epsilon_2} = K_{\max(\epsilon_1,\epsilon_2)} = K_{\epsilon_2} \circledast K_{\epsilon_1}$

*(iii)* $\epsilon_1 < \epsilon_2 \Longrightarrow g_{K_{\epsilon_1}}(K_{\epsilon_2}) = g_{K_{\epsilon_2}}(K_{\epsilon_1}) = K_{\epsilon_2}$

*(iv)* $g_{K_\epsilon}(I) = g_I(K_\epsilon),\ for\ all\ I \in \mathbb{I}$

*(v)* $\epsilon_1 < \epsilon_2 \Longrightarrow g_{K_{\epsilon_2}}(\mathbb{I}) \subset g_{K_{\epsilon_1}}(\mathbb{I})$.

**Proposition 8.5.9.** $(\mathbb{K}_\epsilon, \circledast)$ *is a commutative submonoid of* $(\mathbb{I}, \circledast)$.

In the following we present some results relating to the sets $\mathbb{K}_\epsilon$, $\mathcal{Z}_\mathbb{I}$ and $\mathcal{I}_\mathbb{I}$.

**Lemma 8.5.10.** $\mathbb{K}_\epsilon \subseteq \mathcal{Z}_\mathbb{I} \subset \mathcal{I}_\mathbb{I}$.

*Proof.* In Lemma 8.5.2, we proved that $\mathcal{Z}_\mathbb{I} \subset \mathcal{I}_\mathbb{I}$. So here it is enough to show that $\mathbb{K}_\epsilon \subseteq \mathcal{Z}_\mathbb{I}$. Now if $I \in \mathbb{K}_\epsilon$ then

$$I(x,y) = \begin{cases} 1, & \text{if } x \leq \epsilon \ , \\ y, & \text{if } x > \epsilon \end{cases}$$

for some $\epsilon \in [0,1)$ or $I = I_{\mathbf{WB}}$. For any $J \in \mathbb{I}$ and $I \in \mathbb{K}_\epsilon$, we have

$$(I \circledast J)(x,y) = (J \circledast I)(x,y) = \begin{cases} 1, & \text{if } x \leq \epsilon \ , \\ J(x,y), & \text{if } x > \epsilon, \end{cases}$$

showing that $I \in \mathcal{Z}_\mathbb{I}$. If $I = I_{\mathbf{WB}}$, then

$$(I \circledast J)(x,y) = (J \circledast I)(x,y) = \begin{cases} 1, & \text{if } x < 1 \ , \\ J(1,y), & \text{if } x = 1, \end{cases}$$

for all $J \in \mathbb{I}$. Thus $I_{\mathbf{WB}} \in \mathcal{Z}_\mathbb{I}$. □

In fact, the opposite inclusion ( i.e., $\mathcal{Z}_\mathbb{I} \subseteq \mathbb{K}_\epsilon$) is also true, a fact that we prove in Lemma 8.5.12. Now, we are ready to give a complete characterisation and representation of $K \in \mathbb{I}$ satisfying (NP) for which $g_K$ will be an *s.g.h.*

**Theorem 8.5.11.** *Let* $K \in \mathbb{I}$ *satisfy* (NP). *The following statements are equivalent:*

*(i)* $g_K$ *is an* s.g.h.

*(ii)* $K \in \mathbb{K}_\epsilon$.

*Proof.* (i) $\implies$ (ii): Let $g_K$ be an *s.g.h* for some $K \in \mathbb{I}$. Since $K$ has (NP) the range of $K$ is $[0,1]$. Let $\alpha < 1$ be chosen arbitrarily. Then there exists some $x_0 \in (0,1], y_0 \in [0,1)$, such that $K(x_0, y_0) = \alpha < 1$. We keep $K$ fixed, vary $J$ and investigate the equivalence $J \circledast K = K \circledast J \circledast K$.

When $J = I_{\mathbf{0}}$, we have

$$\begin{aligned}(J \circledast K)(x_0, y_0) &= I_{\mathbf{0}}(x_0, K(x_0, y_0)) \\ &= I_{\mathbf{0}}(x_0, \alpha) = 0\ , \\ (K \circledast J \circledast K)(x_0, y_0) &= K(x_0, I_{\mathbf{0}}(x_0, K(x_0, y_0))) \\ &= K(x_0, 0).\end{aligned}$$

Since $g_K$ is an *s.g.h.*, $K(x_0, 0) = 0$. Hence, if $K(x_0, y_0) = \alpha < 1$, then $K(x_0, 0) = 0$. Now,

$$\begin{aligned}(J \circledast K)(x_0, 0) &= J(x_0, K(x_0, 0)) \\ &= J(x_0, 0), \\ \text{and } (K \circledast J \circledast K)(x_0, 0) &= K(x_0, J(x_0, K(x_0, 0))) \\ &= K(x_0, J(x_0, 0)).\end{aligned}$$

Now let us, once again, choose $J$ as in (8.1) with $\beta = y_0$. Thus we have $J(x_0, 0) = y_0$ and hence

$$\begin{aligned}y_0 &= J(x_0, 0) = K(x_0, J(x_0, 0)) \\ &= K(x_0, y_0) = \alpha. \\ &\implies \alpha = y_0\ .\end{aligned}$$

Since $\alpha$ is chosen arbitrarily, we have

$$K(x_0, y) = y, \qquad y \in [0,1]. \tag{8.11}$$

Let $x^* = \inf\{x | K(x, y) = y, \text{ for all } y\} \geq 0$. Note that the infimum exists because $K$ has (NP), i.e., $1 \in \{x | K(x, y) = y, \text{ for all } y\} \neq \emptyset$.

**Claim:** $K(s, y) = 1$, for any $s \in [0, x^*)$ and for all $y \in [0,1]$.

*Proof of the claim:* On the contrary, let us suppose that $1 > K(s, y_0) = y_1 > y_0$ for some $y_0, y_1 \in [0,1)$ and $s \in [0, x^*)$. Now,

$$\begin{aligned}J(s, K(s, y_0)) &= J(s, y_1)\ , \\ \text{and}, K(s, J(s, K(s, y_0))) &= K(s, J(s, y_1))\ .\end{aligned}$$

Once again, choosing a $J$ as in (8.1) with $\beta = y_0$, we have

$$J(s, y_1) = y_0 \text{ and } K(s, J(s, y_1)) = K(s, y_0) = y_1 \ ,$$
$$\implies J(s, K(s, y_0)) \neq K(s, J(s, K(s, y_0))),$$

i.e., $g_K$ is not an *s.g.h.*, a contradiction. Thus $K(s, y) = 1$, for all $s \in [0, x^*)$.

Now the question is what value should one assign to $K(x^*, y)$. Since it is customary to assume left-continuity of fuzzy implications in the first variable, we let $K(x^*, y) = 1$. Note that letting $K(x^*, y) = y$ also gives a $K$ such that $g_K$ is an *s.g.h.*

From the above claim and (8.11) we see that every $K$ is of the form (8.9) for some $\epsilon \in [0, 1)$ or $K = I_{\mathbf{WB}}$.

(ii) $\implies$ (i) : Follows from Lemmata 8.5.10 and 8.5.4.

□

**Lemma 8.5.12.** *Let $K \in \mathcal{Z}_{\mathbb{I}}$. Then $K \in \mathbb{K}_\epsilon$, i.e., $\mathcal{Z}_{\mathbb{I}} \subseteq \mathbb{K}_\epsilon$.*

*Proof.* Let $K \in \mathcal{Z}_{\mathbb{I}}$. From Lemma 8.5.4 it follows that $g_K$ is an *s.g.h* and also from Lemma 8.5.5 it follows that range of $K$ is nontrivial. Further, from Proposition 8.5.6 we know that $K$ has (NP). Again from Theorem 8.5.11 it follows that $K \in \mathbb{K}_\epsilon$. □

**Corollary 8.5.13.** $\mathcal{Z}_{\mathbb{I}} = \mathbb{K}_\epsilon$.

*Proof.* In Lemma 8.5.10 we proved that $\mathcal{Z}_{\mathbb{I}} \supseteq \mathbb{K}_\epsilon$. From Lemma 8.5.12 it follows that $\mathcal{Z}_{\mathbb{I}} \subseteq \mathbb{K}_\epsilon$. □

**Remark 8.5.14.** *From Corollary 8.5.13, it follows that with the help of the* s.g.h. $g_K$*, we have found out the center $\mathcal{Z}_{\mathbb{I}}$ of the monoid $\mathbb{I}$.*

From Corollary 8.5.13, it follows that the center $\mathcal{Z}_{\mathbb{I}}$ is nothing but the set of all fuzzy implications of the form $K_\epsilon$ for some $\epsilon \in [0, 1)$ or $I_{\mathbf{WB}}$. Thus we have found out fuzzy implications $K$ satisfying the functional equation $I \circledast K = K \circledast I$ for all $I \in \mathbb{I}$. For the characterisation of some well-known families of fuzzy implications $I$ satisfying $I \circledast I = I (i.e., I \in \mathcal{I}_{\mathbb{I}})$ of $(\mathbb{I}, \circledast)$, please see [74].

Recall from Corollary 8.4.8 that if the range of $K \in \mathbb{I}$ is nontrivial and $g_K$ is an *s.g.h.* then either, $K(1, y) = y$ for all $y \in [0, 1]$ or $K(1, y) = 0$ for all $y \in [0, 1)$. In Section 8.5, we have characterised and found representations of fuzzy implications $K$ such that $g_K$ is an *s.g.h.* in the case of $K(1, y) = y$ for all $y \in [0, 1]$, i.e., $K$ has (NP). Now it remains to characterise the nontrivial range non-neutral implications $K$ such that $g_K$ is an *s.g.h.* and give their presentations. We take up this task in the following section.

## 8.6 $K \in \mathbb{I} \setminus \mathbb{I}_{\mathbf{NP}}$ such that $g_K$ is an *s.g.h.*

We begin this section by defining the following class of fuzzy implications.

**Definition 8.6.1.** *For $\epsilon \in [0,1)$, define*

$$K^{\epsilon}(x,y) = \begin{cases} 1, & \text{if } x \le \epsilon\ , \\ y, & \text{if } \epsilon < x < 1\ , \\ 0, & \text{if } x = 1\ \&\ y \neq 1 \end{cases} \tag{8.12}$$

For notational convenience, we denote the set of all such $K^{\epsilon}$ fuzzy implications by

$$\mathbb{K}^{\epsilon} = \{I \in \mathbb{I} | I = K^{\epsilon}\ for\ some\ \epsilon \in [0,1[\}.$$

**Theorem 8.6.2.** *Let $K \in \mathbb{I}$ such that $K(1,y) = 0$ for all $y \neq 1$. The following statements are equivalent:*

*(i)* $g_K$ *is an* s.g.h.

*(ii)* $K \in \mathbb{K}^{\epsilon}$.

*Proof.* Let $K \in \mathbb{I}$ such that $K(1,y) = 0$ for all $y \neq 1$.

(i) $\Longrightarrow$ (ii): Let $g_K$ be an *s.g.h* for some $K \in \mathbb{I}$. Since the range of $K$ is nontrival, from Lemma 8.4.1, the range of $K$ is whole of $[0,1]$. Let $0 < \alpha < 1$ be chosen arbitrarily. Then there exist some $x_0 \in (0,1), y_0 \in [0,1)$, such that $0 < K(x_0,y_0) = \alpha < 1$. We keep $K$ fixed, vary $J \in \mathbb{I}$ and investigate the equivalence $J \circledast K = K \circledast J \circledast K$.

When $J = I_{\mathbf{0}}$, we have

$$(J \circledast K)(x_0,y_0) = I_{\mathbf{0}}(x_0, K(x_0,y_0)) = I_{\mathbf{0}}(x_0,\alpha) = 0\ ,$$
$$(K \circledast J \circledast K)(x_0,y_0) = K(x_0, I_{\mathbf{0}}(x_0, K(x_0,y_0))) = K(x_0,0).$$

Since $g_K$ is an *s.g.h.*, $K(x_0,0) = 0$. Hence, if $K(x_0,y_0) = \alpha < 1$, then $K(x_0,0) = 0$. Now, for any $J \in \mathbb{I}$, we have

$$(J \circledast K)(x_0,0) = J(x_0, K(x_0,0)) = J(x_0,0),$$
$$\text{and } (K \circledast J \circledast K)(x_0,0) = K(x_0, J(x_0, K(x_0,0))) = K(x_0, J(x_0,0)).$$

Now let us, once again, choose $J \in \mathbb{I}$ such that $J(x_0,0) = y_0$. Then

$$y_0 = J(x_0,0) = K(x_0, J(x_0,0)) = K(x_0,y_0) = \alpha,$$

which implies that $\alpha = y_0$. Since $\alpha$ is chosen arbitrarily, we have

$$K(x_0,y) = y, \qquad y \in [0,1]. \tag{8.13}$$

Let $x^* = \inf\{x | K(x,y) = y,\ \text{for all}\ \ y\} \ge 0$. Note that the infimum exists because $x_0$ satisfies (8.13).

**Claim:** $K(s,y) = 1$, for any $s \in [0,x^*)$ and for all $y \in [0,1]$.

*Proof of the claim:* On the contrary, let us suppose that $1 > K(s,y_0) = y_1 > y_0$ for some $y_0, y_1$.

Now,

$$J(s, K(s, y_0)) = J(s, y_1) ,$$
$$K(s, J(s, K(s, y_0))) = K(s, J(s, y_1)) .$$

Once again, choosing a $J \in \mathbb{I}$ such that $J(s, y_1) = y_0$, we get

$$J(s, y_1) = y_0 \text{ and } K(s, J(s, y_1)) = K(s, y_0) = y_1 ,$$
$$\implies J(s, K(s, y_0)) \neq K(s, J(s, K(s, y_0))),$$

i.e., $g_K$ is not an *s.g.h.*, a contradiction. Thus $K(s, y) = 1$, for all $s \in [0, x^*)$.

Now the question is what value should one assign to $K(x^*, y)$. Since it is customary to assume left-continuity of fuzzy implications in the first variable, we let $K(x^*, y) = 1$. Note that letting $K(x^*, y) = y$ also gives a $K$ such that $g_K$ is a homomorphism.

From the above claim and (8.13) we see that every $K$ is of the form (8.12) for some $\epsilon \in [0, 1)$.

(ii) $\implies$ (i): This follows easily. Let $K = K^\epsilon$ for some $\epsilon \in [0, 1)$.i.e.,

$$K(x, y) = K^\epsilon(x, y) = \begin{cases} 1, & \text{if } x \leq \epsilon , \\ y, & \text{if } \epsilon < x < 1 , \\ 0, & \text{if } x = 1 \ \& \ y \neq 1. \end{cases}$$

Let $J \in \mathbb{I}$.

$$\begin{aligned} \text{Now, } (J \circledast K)(x, y) &= J(x, K(x, y)) \\ &= \begin{cases} 1, & \text{if } x \leq \epsilon , \\ J(x, y), & \text{if } \epsilon < x < 1 , \\ 0, & \text{if } x = 1 \ \& \ y \neq 1, \end{cases} \\ \text{and, } (K \circledast J \circledast K)(x, y) &= K(x, J(x, K(x, y))) \\ &= \begin{cases} 1, & \text{if } x \leq \epsilon , \\ J(x, y), & \text{if } \epsilon < x < 1 , \\ 0, & \text{if } x = 1 \ \& \ y \neq 1, \end{cases} \end{aligned}$$

which proves that $g_K$ is an *s.g.h.*

□

**Corollary 8.6.3.** *Let $K \in \mathbb{I}$. Then following statements are equivalent:*

*(i)* $g_K$ *is an s.g.h.*

*(ii)* $K \in \mathbb{A} \cup \mathbb{K}_\epsilon \cup \mathbb{K}^\epsilon$.

## 8.7 Concluding Remarks

In this chapter, we have considered the right translations $g_K$ defined on the monoid $(\mathbb{I}, \circledast)$ and shown that they are lattice homomorphisms of $(\mathbb{I}, \preceq, \circledast, \vee, \wedge)$. Since every right translation is not a semigroup homomorphism (*s.g.h.*), we have characterised and found the representations of $K \in \mathbb{I}$ such that the right translations $g_K$ are *s.g.h.* Based on our results we have shown that the set of all trivial range fuzzy implications $K$ for which $g_K$ become *s.g.h* forms the set of right zero elements, as well as a two-sided ideal. Further, our analysis in the case of nontrivial range fuzzy implications $K$ satisfying (NP) have enabled us, not only to determine the center of the monoid $(\mathbb{I}, \circledast)$, but also to obtain a clear representation of its elements.

# Chapter 9

# Conclusions and Future Work

*It is always wise to look ahead,*
*but difficult to look further than you can see.*
– Winston Churchill (1874 – 1965)

In this thesis, we have proposed and studied the following generative method of fuzzy implications:

Given $I, J \in \mathbb{I}$, $I \circledast J \colon [0,1]^2 \longrightarrow [0,1]$ is defined as

$$(I \circledast J)(x,y) = I(x, J(x,y)), \qquad x, y \in [0,1] \ .$$

It was shown that $\circledast$ generates fuzzy implications from fuzzy implications and also $(\mathbb{I}, \circledast)$ forms a non-idempotent monoid. Further, we have explored the $\circledast$-composition on $\mathbb{I}$ in two different ways, viz, as a generating method of fuzzy implications and as a binary operation on the set $\mathbb{I}$.

Firstly, we have studied the $\circledast$-composition as a generating method of fuzzy implications. Though we have shown that $\circledast$-composition preserves most of the properties of fuzzy implications and functional equations involving fuzzy implications, there are some questions that still remain unanswered, for instance, see the following:

**Problem 9.0.1.** *Find the representations of the largest subset $\mathbb{I}^{\circ} \subset \mathbb{I}_{\mathbf{OP}}$ such that every $I \in \mathbb{I}^{\circ}$ satisfies* (3.1).

We have also proposed a new concept of *mutual exchangeability* (ME), a generalisation of (EP), to a pair $(I, J)$ of fuzzy implications and shown that (ME) plays a central role in the $\circledast$-preservation of (EP), (LI) and families of fuzzy implications. Hence it is worth while to investigate the following question:

**Problem 9.0.2.** *Characterise the fuzzy implications $I, J$ such that the pair $(I, J)$ satisfies* (ME).

Secondly, we have investigated the algebraic aspects of the set $\mathbb{I}$ considering $\circledast$ as a binary operation on $\mathbb{I}$. Though $(\mathbb{I}, \circledast)$ forms only a monoid, we have characterised the largest subgroup contained in it and based on its representations we have hitherto unknown representations of Yager's families of fuzzy implications. However, a similar study has to be carried out for the other families of fuzzy implications, namely, $(S,N)$-, $R$-implications, which leads us to the following questions:

**Problem 9.0.3.** *Does there exist any seed members for different families of fuzzy implications, namely, $(S,N)$-, $R$-, $QL$-, $(U,N)$-implications, as in the case of Yager's families of fuzzy implications, from which one could generate all the other members of the same family?*

We have also defined yet another generative method of fuzzy implications, denoted by, $\Delta$ (see Definition 7.4.1) and employed it in obtaining the algebraic connotations of some conjugacy classes that were proposed earlier. Though we have shown that $\Delta$ makes $\mathbb{I}$ a semigroup, further explorations are to be done along the lines of this thesis.

**Problem 9.0.4.** *Investigate the algebraic and analytic aspects of the semigroup ($\mathbb{I}, \Delta$).*

In fact, a similar analysis needs to be carried out also for the other generative methods proposed in [76]. For instance, one can consider the operation $\diamond$ proposed in [76] as

$$(I \diamond J)(x,y) = J(I(y,x), J(x,y)), \qquad x,y \in [0,1] \ ,$$

which forms an *implication* structure on $\mathbb{I}$ along with the operation $\odot_T$ defined on $\mathbb{I}$ in the same work [76]. Our preliminary analysis seems to indicate that $(\mathbb{I}, \odot_T, \diamond, \leq, I_{\mathbf{0}}, I_{\mathbf{1}})$ forms a bounded residuated monoid. We intend to explore this further.

Finally, we have shown that the set of all trivial range fuzzy implications $K$ such that $g_K$ is an *s.g.h.* forms the set $\mathbb{A}$ of all right zero elements as well as a two sided ideal. In fact, it can be easily seen that if $\mathbb{B} \subset \mathbb{I}$ is any other ideal in $(\mathbb{I}, \circledast)$, then $\mathbb{A} \subseteq \mathbb{B}$, i.e., $\mathbb{A}$ is contained in the intersection of all the ideals of the monoid $(\mathbb{I}, \circledast)$. Further, since every two sided ideal leads to a Rees semigroup, the following questions become interesting to investigate:

**Problem 9.0.5.** *(i) Is the set $\mathbb{A}$ the only ideal of the monoid $(\mathbb{I}, \circledast)$?*

*(ii) If yes, determine the Rees semigroup of fuzzy implications obtained by the two sided ideal $\mathbb{A}$.*

# References


[1] I. Aguiló, M. Carbonell, J. Suñer, and J. Torrens, "Dual representable aggregation functions and their derived S-implications," in *Computational Intelligence for Knowledge-Based Systems Design*, ser. Lecture Notes in Computer Science, E. Hüllermeier, R. Kruse, and F. Hoffmann, Eds. Springer Berlin Heidelberg, 2012, vol. 6178, pp. 408–417.

[2] C. Alcalde, A. Burusco, and R. Fuentes-González, "A constructive method for the definition of interval-valued fuzzy implication operators," *Fuzzy Sets and Systems*, vol. 153, no. 2, pp. 211 – 227, 2005.

[3] K. T. Atanassov, "Remarks on the conjuctions, disjunctions and implications of the intuitionistic fuzzy logic," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 09, no. 01, pp. 55–65, 2001.

[4] M. Baczyński, G. Beliakov, H. Bustince, and A. Pradera, *Advances in Fuzzy Implication Functions*, ser. Studies in Fuzziness and Soft Computing. Springer London, Limited, 2013, vol. 300.

[5] M. Baczyński and J. Drewniak, "Conjugacy classes of fuzzy implications," in *Computational Intelligence*, ser. Lecture Notes in Computer Science, B. Reusch, Ed. Springer Berlin Heidelberg, 1999, vol. 1625, pp. 287–298.

[6] M. Baczyński, J. Drewniak, and J. Sobera, "Semigroups of fuzzy implications," *Tatra Mount Math Publishers*, vol. 21, pp. 61–71, 2001.

[7] M. Baczyński and B. Jayaram, "Yager's classes of fuzzy implications: Some properties and intersections," *Kybernetika*, vol. 43, pp. 157 – 182, 2007.

[8] ——, *Fuzzy Implications*, ser. Studies in Fuzziness and Soft Computing. Berlin Heidelberg: Springer-Verlag, 2008, vol. 231.

[9] ——, "(S,N)- and R-implications: A state-of-the-art survey," *Fuzzy Sets and Systems*, vol. 159, no. 14, pp. 1836 – 1859, 2008, formal Methods for Fuzzy Mathematics, Approximation and Reasoning, Part I.

[10] J. Balasubramaniam, "Contrapositive symmetrisation of fuzzy implications - Revisited," *Fuzzy Sets and Systems*, vol. 157, no. 17, pp. 2291–2310, 2006.

[11] ——, "Yager's new class of implications $J_f$ and some classical tautologies," *Information Sciences*, vol. 177, no. 3, pp. 930–946, 2007.

[12] W. Bandler and L. J. Kohout, "Semantics of implication operators and fuzzy relational products," *International Journal of Man-Machine Studies*, vol. 12, no. 1, pp. 89 – 116, 1980.

[13] B. Bedregal, G. Dimuro, R. Santiago, and R. Reiser, "On interval fuzzy S-implications," *Information Sciences*, vol. 180, no. 8, pp. 1373 – 1389, 2010.

[14] B. Bedregal, G. Beliakov, H. Bustince, J. Fernández, A. Pradera, and R. Reiser, "(S,N)-implications on bounded lattices," in *Advances in Fuzzy Implication Functions*, ser. Studies in Fuzziness and Soft Computing, M. Baczyński, G. Beliakov, H. Bustince Sola, and A. Pradera, Eds. Springer Berlin Heidelberg, 2013, vol. 300, pp. 101–124.

[15] I. Bloch, "Duality vs. adjunction for fuzzy mathematical morphology and general form of fuzzy erosions and dilations," *Fuzzy Sets and Systems*, vol. 160, no. 13, pp. 1858–1867, 2009.

[16] H. Bustince, E. Barrenechea, and V. Mohedano, "Intutionistic fuzzy implication operators An expression and main properties," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 12, no. 03, pp. 387–406, 2004.

[17] H. Bustince, P. Burillo, and F. Soria, "Automorphisms, negations and implication operators," *Fuzzy Sets and Systems*, vol. 134, no. 2, pp. 209 – 229, 2003.

[18] H. Bustince, J. Fernández, R. Mesiar, and T. Calvo, *Aggregation Functions in Theory and in Practise: Proceedings of the 7th International Summer School on Aggregation Operators at the Public University of Navarra, Pamplona, Spain, July 16-20, 2013*, ser. Advances in Intelligent Systems and Computing. Springer, 2013.

[19] H. Bustince, J. Fernández, J. Sanz, M. Baczynski, and R. Mesiar, "Construction of strong equality index from implication operators," *Fuzzy Sets and Systems*, vol. 211, pp. 15–33, 2013.

[20] T. Calvo, J. Martín, and G. Mayor, "Aggregation of implication functions," in *Proceedings of the 8th conference of the European Society for Fuzzy Logic and Technology, EUSFLAT-13, Milano, Italy, September 11-13, 2013*, J. Montero, G. Pasi, and D. Ciucci, Eds. Atlantis Press, 2013, pp. 569–574.

[21] M. Carbonell and J. Torrens, "Continuous R-implications generated from representable aggregation functions," *Fuzzy Sets and Systems*, vol. 161, no. 17, pp. 2276 – 2289, 2010.

[22] A. Clifford, "Extensions of semigroups," *Transactions of American Mathematical Society*, vol. 68, pp. 165–173, 1950.

[23] C. Cornelis, G. Deschrijver, and E. E. Kerre, "Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: Construction, classification, application," *International Journal of Approximate Reasoning*, vol. 35, no. 1, pp. 55 – 95, 2004.

[24] G. Deschrijver and E. E. Kerre, "Implicators based on binary aggregation operators in interval-valued fuzzy set theory," *Fuzzy Sets and Systems*, vol. 153, no. 2, pp. 229 – 248, 2005.

[25] ——, "Smets-magrez axioms for R-implicators in interval-valued and intuitionistic fuzzy set theory," *International Journal of Uncertainty and Fuzziness in Knowledge-Based Systems*, vol. 13, no. 4, pp. 453–464, 2005.

[26] Z. P. Dienes, "On an implication function in many-valued systems of logic," *The Journal of Symbolic Logic*, vol. 14, no. 2, pp. 95–97, 1949.

[27] J. Drewniak, "Invariant fuzzy implications," *Soft Computing*, vol. 10, no. 10, pp. 979–980, 2006.

[28] J. Drewniak and J. Sobera, "Compositions of invariant fuzzy implications." *Soft Computing*, vol. 10, no. 6, pp. 514–520, 2006.

[29] ——, "Compositions of fuzzy implications," in *Advances in Fuzzy Implication Functions*, ser. Studies in Fuzziness and Soft Computing, M. Baczyński, G. Beliakov, H. Bustince Sola, and A. Pradera, Eds. Springer, Berlin Heidelberg, 2013, vol. 300, pp. 155–176.

[30] J. C. Fodor, "Contrapositive symmetry of fuzzy implications," *Fuzzy Sets and Systems*, vol. 69, no. 2, pp. 141 – 156, 1995.

[31] J. C. Fodor and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*. Dordrecht: Kluwer Academic, 1994.

[32] S. Gottwald, "Axiomatizations of t-norm based logics - A survey," *Soft Computing.*, vol. 4, no. 2, pp. 63–67, 2000.

[33] S. Gottwald and S. Jenei, "A new axiomatization for involutive monoidal t-norm-based logic," *Fuzzy Sets and Systems*, vol. 124, no. 3, pp. 303–307, 2001.

[34] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap, *Aggregation Functions*, 1st ed., ser. Encyclopedia of Mathematics and its Applications. New York, NY, USA: Cambridge University Press, 2009.

[35] S. Greco, B. Bouchon-Meunier, G. Coletti, M. Fedrizzi, B. Matarazzo, and R. R. Yager, Eds., *Advances in Computational Intelligence - 14th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU 2012, Catania, Italy, July 9-13, 2012. Proceedings, Part II*, ser. Communications in Computer and Information Science, vol. 298. Springer, 2012.

[36] P. Hajek and L. Kohout, "Fuzzy implications and generalized quantifiers," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 4, no. 3, pp. 225–233, 1996.

[37] D. Hliněná, M. Kalina, and P. Král, "Generated implications revisited," in *Advances in Computational Intelligence*, ser. Communications in Computer and Information Science, S. Greco, B. Bouchon-Meunier, G. Coletti, M. Fedrizzi, B. Matarazzo, and R. Yager, Eds. Springer Berlin Heidelberg, 2012, vol. 298, pp. 345–354.

[38] ——, "Implications generated by triples of monotone functions," in *Aggregation Functions in Theory and in Practise*, ser. Advances in Intelligent Systems and Computing, H. Bustince, J. Fernández, R. Mesiar, and T. Calvo, Eds. Springer Berlin Heidelberg, 2013, vol. 228, pp. 441–452.

[39] B. Jayaram, "On the law of importation in fuzzy logic," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 1, pp. 130–144, 2008.

[40] ——, “Rule reduction for efficient inferencing in similarity based reasoning,” *International Journal of Approximate Reasoning*, vol. 48, no. 1, pp. 156–173, 2008.

[41] B. Jayaram and R. Mesiar, “On special fuzzy implications,” *Fuzzy Sets and Systems*, vol. 160, no. 14, pp. 2063 – 2085, 2009.

[42] S. Jenei, “Structure of left-continuous triangular norms with strong induced negations.(III): Construction and decomposition,” *Fuzzy Sets and Systems*, vol. 128, no. 2, pp. 197–208, 2002.

[43] S. Jenei and F. Montagna, “A proof of standard completeness for Esteva and Godo’s Logic MTL,” *Studia Logica*, vol. 70, no. 2, pp. 183–192, 2002.

[44] ——, “Strongly involutive uninorm algebras,” *Journal of Logic and Computation*, vol. 23, no. 3, pp. 707–726, 2013.

[45] F. Karacál, “On the direct decomposability of strong negations and S-implication operators on product lattices,” *Information Sciences*, vol. 176, no. 20, pp. 3011 – 3025, 2006.

[46] M. Kilp, U. Knauer, and A. Mikhalev, *Monoids, Acts and Categories: With Applications to Wreath Products and Graphs. A Handbook for Students and Researchers*, ser. De Gruyter Expositions in Mathematics. De Gruyter, 2000.

[47] L. Kitainik, *Fuzzy Decision Procedures with Binary Relations: Towards a Unified Theory*, ser. Etudes Presentees a la Commission Internationale Pour L’Hist. New York: Kluwer Academic Publishers, 1993.

[48] E. P. Klement, R. Mesiar, and E. Pap, *Triangular Norms*, ser. Trends in Logic. Dordrecht: Kluwer Academic Publishers, 2000, vol. 8.

[49] D. Li and Y. Li, “Algebraic structures of interval-valued (S, N)- fuzzy implications,” *International Journal of Approximate Reasoning*, vol. 53, no. 6, pp. 892 – 900, 2012.

[50] L. Lin and Z. Q. Xia, “Intuitionistic fuzzy implication operators: Expressions and properties,” *Journal of Applied Mathematics and Computing*, vol. 22, no. 3, pp. 325–338, 2006.

[51] H. W. Liu and G. J. Wang, “A note on implicators based on binary aggregation operators in interval-valued fuzzy set theory,” *Fuzzy Sets and Systems*, vol. 157, no. 24, pp. 3231 – 3236, 2006.

[52] H. W. Liu and P. J. Xue, “T-seminorms and implications on a complete lattice,” in *Quantitative Logic and Soft Computing 2010*, ser. Advances in Intelligent and Soft Computing, B.-Y. Cao, G.-J. Wang, S.-L. Chen, and S.-Z. Guo, Eds. Springer Berlin Heidelberg, 2010, vol. 82, pp. 215–225.

[53] E. Ljapin, *Semigroups*. Rhode Island: American Mathematical Society, 1974.

[54] K. Maes, B. D. Baets, and J. C. Fodor, “The unique role of the Lukasiewicz-triplet in the theory of fuzzified normal forms,” *Fuzzy Sets and Systems*, vol. 153, no. 2, pp. 161–179, 2005.

[55] S. Mandal and B. Jayaram, “Bandler-Kohout subproduct with Yager’s classes of fuzzy implications,” *IEEE Transactions on Fuzzy Systems*, 2013. [Online]. Available: http://dx.doi.org/10.1109/TFUZZ.2013.2260551.

[56] S. Massanet, G. Mayor, and J. Torrens, "Constructing non-functionally expressible fuzzy implications," in *Proceedings of the 8th conference of the European Society for Fuzzy Logic and Technology, EUSFLAT-13, Milano, Italy, September 11-13, 2013*, J. Montero, G. Pasi, and D. Ciucci, Eds. Atlantis Press, 2013, pp. 193–199.

[57] S. Massanet and J. Torrens, "The law of importation versus the exchange principle on fuzzy implications," *Fuzzy Sets and Systems*, vol. 168, no. 1, pp. 47–69, 2011.

[58] ——, "On a new class of fuzzy implications: h-implications and generalizations," *Information Sciences*, vol. 181, no. 11, pp. 2111 – 2127, 2011.

[59] ——, "On a generalization of Yager's implications," in *Advances in Computational Intelligence*, ser. Communications in Computer and Information Science, S. Greco, B. Bouchon-Meunier, G. Coletti, M. Fedrizzi, B. Matarazzo, and R. Yager, Eds. Springer Berlin Heidelberg, 2012, vol. 298, pp. 315–324.

[60] ——, "On some properties of threshold generated implications," *Fuzzy Sets and Systems*, vol. 205, pp. 30–49, 2012.

[61] ——, "On the characterization of Yager's implications," *Information Sciences.*, vol. 201, pp. 1–18, 2012.

[62] ——, "Threshold generation method of construction of a new implication from two given ones," *Fuzzy Sets and Systems*, vol. 205, pp. 50–75, 2012.

[63] ——, "On the vertical threshold generation method of fuzzy implication and its properties," *Fuzzy Sets and Systems*, vol. 226, pp. 32–52, 2013.

[64] ——, "An overview of construction methods of fuzzy implications," in *Advances in Fuzzy Implication Functions*, ser. Studies in Fuzziness and Soft Computing, M. Baczyński, G. Beliakov, H. Bustince Sola, and A. Pradera, Eds. Springer, Berlin Heidelberg, 2013, vol. 300, pp. 1–30.

[65] J. Montero, G. Pasi, and D. Ciucci, Eds., *Proceedings of the 8th conference of the European Society for Fuzzy Logic and Technology, EUSFLAT-13, Milano, Italy, September 11-13, 2013*. Atlantis Press, 2013.

[66] K. W. Oh and W. Bandler, "Properties of fuzzy implication operators," *International Journal of Approximate Reasoning*, vol. 1, no. 3, pp. 273 – 285, 1987.

[67] Y. Ouyang, "On fuzzy implications determined by aggregation operators," *Information Sciences*, vol. 193, pp. 153 – 162, 2012.

[68] A. Pradera, "A review of the relationships between aggregation, implication and negation functions," in *Aggregation Functions in Theory and in Practise*, ser. Advances in Intelligent Systems and Computing, H. Bustince, J. Fernández, R. Mesiar, and T. Calvo, Eds. Springer Berlin Heidelberg, 2013, vol. 228, pp. 31–36.

[69] R. H. Reiser, B. Bedregal, and M. Baczyński, "Aggregating fuzzy implications," *Information Sciences*, vol. 253, pp. 126 – 146, 2013.

[70] B. Reusch, Ed., *Computational Intelligence, Theory and Applications, International Conference, 6th Fuzzy Days, Dortmund, Germany, May 25-28, 1999, Proceedings*, ser. Lecture Notes in Computer Science, vol. 1625. Springer, 1999.

[71] J. J. Rotman, *A First Course in Abstract Algebra: With Applications*. New Jersey: Pearson Prentice Hall, 2006.

[72] Y. Shi, B. Gasse, and E. E. Kerre, "Fuzzy implications: Classification and a new class," in *Advances in Fuzzy Implication Functions*, ser. Studies in Fuzziness and Soft Computing, M. Baczyński, G. Beliakov, H. Bustince Sola, and A. Pradera, Eds. Springer Berlin Heidelberg, 2013, vol. 300, pp. 31–51.

[73] Y. Shi, B. Gasse, D. Ruan, and E. Kerre, "On a new class of implications in fuzzy logic," in *Information Processing and Management of Uncertainty in Knowledge-Based Systems. Theory and Methods*, ser. Communications in Computer and Information Science, E. Hüllermeier, R. Kruse, and F. Hoffmann, Eds. Springer Berlin Heidelberg, 2012, vol. 80, pp. 525–534.

[74] Y. Shi, D. Ruan, and E. E. Kerre, "On the characterizations of fuzzy implications satisfying $I(x, y) = I(x, I(x, y))$," *Information Sciences.*, vol. 177, no. 14, pp. 2954–2970, 2007.

[75] P. Smets and P. Magrez, "Implication in fuzzy logic," *International Journal of Approximate Reasoning*, vol. 1, no. 4, pp. 327 – 347, 1987.

[76] N. R. Vemuri and B. Jayaram, "Fuzzy implications: Novel generation process and the consequent algebras," in *Advances on Computational Intelligence - 14th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU 2012, Catania, Italy, July 9-13, 2012. Proceedings, Part II*, ser. Communications in Computer and Information Science, S. Greco, B. Bouchon-Meunier, G. Coletti, M. Fedrizzi, B. Matarazzo, and R. R. Yager, Eds., vol. 298. Springer, 2012, pp. 365–374.

[77] ——, "Homomorphisms on the monoid of fuzzy implications," in *The 2013 International Conference on Fuzzy Systems (FUZZ-IEEE 2013)*. IEEE, 2013, pp. 1–8.

[78] ——, "Homomorphisms on the monoid of fuzzy implications $(\mathbb{I}, \circledast)$ - A complete characterization," in *5th International Conference on Pattern Recognition and Machine Intelligence (PReMI 2013)*, ser. Lecture Notes in Computer Science, P. Maji, A. Ghosh, M. N. Murty, K. Ghosh, and S. K. Pal, Eds., vol. 8251. Springer, 2013, pp. 563–568.

[79] J. Villar and M. Sanz-Bobi, "Semantic analysis of fuzzy models, application to Yager models," in *Advances in Fuzzy Systems and Evolutionary Computation*, ser. Artificial intelligence series, N. Mastorakis, Ed. WSES Press, 2001, pp. 82–87.

[80] M. Štěpnička and B. DeBaets, "Implication-based models of monotone fuzzy rule bases," *Fuzzy Sets and Systems*, vol. 232, pp. 134–155, 2013.

[81] M. Štěpnička and B. Jayaram, "On the suitability of the Bandler-Kohout subproduct as an inference mechanism," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 2, pp. 285–298, 2010.

[82] A. Xie and H. Liu, "A generalization of Yager's f-generated implications," *International Journal of Approximate Reasoning*, vol. 54, no. 1, pp. 35 – 46, 2013.

[83] R. R. Yager, “On some new classes of implication operators and their role in approximate reasoning,” *Information Sciences.*, vol. 167, no. 1-4, pp. 193–216, 2004.

[84] L. A. Zadeh, “Outline of a new approach to the analysis of complex systems and decision processes,” *IEEE Transactions on Systems, Man and Cybernetics*, vol. SMC-3, no. 1, pp. 28–44, 1973.