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Intersections between some families of (U,N)- and RU-implications

Michał Baczyński[a], Balasubramaniam Jayaram[b]

[a] *Institute of Mathematics, University of Silesia, 40-007 Katowice, ul. Bankowa 14, Poland*
[b] *Department of Mathematics, Indian Institute of Technology Hyderabad, Yeddumailaram - 502 205, India*

Abstract

(U,N)-implications and RU-implications are the generalizations of (S,N)- and R-implications to the framework of uninorms, where the t-norms and t-conorms are replaced by appropriate uninorms. In this work, we present the intersections that exist between (U,N)-implications and the different families of RU-implications obtainable from the well-established families of uninorms.

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1. Introduction

Fuzzy implications, which are a generalization of the classical two-valued implications to the multi-valued setting, play an important role in many applications, viz., Approximate Reasoning, Fuzzy Control, Fuzzy Image Processing, etc. (see [12, 22, 4]). Hence, it is beneficial to have a repertoire of fuzzy implications at one's disposal. Towards this end, many families of fuzzy implications have been proposed, often as a generalization of the classical logic formulae by suitably substituting classical logic operations with their fuzzy logic counterparts. For instance, (S,N)-implications generalize the material implication from the classical logic with a t-conorm instead of the disjunction, while R-implications obtained from a t-norm generalize the intuitionistic (residual) logic implication to the framework of fuzzy logic, whereas QL-implications are the fuzzy counterparts of quantum logic implication.

Each of these families possesses many different properties. For the interrelationships between main axioms of fuzzy implications see the recent article by Shi et al. [26]. The suitability of a particular family or families of fuzzy implications to a given application under consideration largely depends on the properties that the fuzzy implications in them possess. It is in this context that the investigation into overlaps that exist among the families of fuzzy implications assumes significance. Clearly, fuzzy implications that belong to the intersection of two or more families possess all the properties of the corresponding families, thus making them suitable for more applications.

Many works dealing with investigations into intersections between different families of fuzzy implications are known in the literature. Intersections between (S,N)- and R-implications was firstly done by Dubois and Prade [12], see also the works of Fodor [14, 15]. Recently, in [3], a complete characterization of the intersection of the above two families was given. Following this, the intersections between QL-implications and the above two families of fuzzy implications has also been characterized, see [6]. Earlier, the authors, in [2], had also investigated the intersections between the family of f- and g-generated fuzzy implications proposed by Yager.

(U,N)-implications are a generalization of (S,N)-implications, where a t-conorm S is replaced by a uninorm U. A similar generalization of R-implications from the setting of t-norms to the setting of uninorms,

Email addresses: `michal.baczynski@us.edu.pl` (Michał Baczyński), `jbala@iith.ac.in` (Balasubramaniam Jayaram)

referred to here as RU-implications, has been done by De Baets and Fodor [9]. Ruiz and Torrens have investigated, quite extensively, fuzzy implications generated from uninorms [24] and their distributivity [23, 25].

Recently, some characterizations of (U,N)-implications were given by the authors in [5]. However, a similar characterization for RU-implications is yet to be done. Still, many properties and results relating to RU-implications obtained from the main classes of uninorms have been investigated and established. Based on these results, in this work we investigate the intersections that exist between the above two families of fuzzy implications obtained from uninorms. We obtain precise and almost complete overlaps that exist among these families.

Our article has been divided in several parts. After introducing the necessary preliminaries on the basic fuzzy logic operations, viz., fuzzy implications and uninorms in Sections 2 and 3, we present the definition, properties and characterization results - where available - of the families of (U,N)- and RU-implications in Sections 4 and 5, respectively. Section 6 contains the main results of this work - investigation of the intersections that exist between the above two families of fuzzy implications obtained from uninorms, presenting complete and precise overlaps that exist among these families.

2. Fuzzy implications and negations

In this work the following equivalent definition introduced by Fodor and Roubens [17, Definition 1.15] (see also [4]) is used.

Definition 2.1. A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication, if it satisfies, for all $x, y, z \in [0,1]$, the following conditions:

$$\text{if } x \leq y, \text{ then } I(x,z) \geq I(y,z), \tag{I1}$$

$$\text{if } y \leq z, \text{ then } I(x,y) \leq I(x,z), \tag{I2}$$

$$I(0,0) = 1, \tag{I3}$$

$$I(1,1) = 1, \tag{I4}$$

$$I(1,0) = 0. \tag{I5}$$

A fuzzy implication I is said to satisfy the exchange principle, if

$$I(x, I(y,z)) = I(y, I(x,z)), \qquad \text{for all } x, y, z \in [0,1]. \tag{EP}$$

Definition 2.2 (Klement et al. [19, Definition 11.3])**.** A decreasing function $N\colon [0,1] \to [0,1]$ is called a fuzzy negation, if $N(0) = 1$, $N(1) = 0$. Further, a fuzzy negation N is called

(i) strict, if it is strictly decreasing and continuous;
(ii) strong, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

Definition 2.3. Let $I\colon [0,1]^2 \to [0,1]$ be a fuzzy implication and $\alpha \in [0,1[$. If the function $N_I^\alpha\colon [0,1] \to [0,1]$ given by

$$N_I^\alpha(x) = I(x, \alpha), \qquad \text{for all } x \in [0,1],$$

is a fuzzy negation, then it is called the natural negation of I with respect to α.

It should be noted that for any fuzzy implication I we have (I5), so for $\alpha = 0$ we have the natural negation $N_I = N_I^0$ of I (see [1]). Also α should be less than 1 for fuzzy implications, since $I(1,1) = 1$ by (I4).

3. Uninorms

Definition 3.1 (see [27, 18]). An associative, commutative and increasing operation $U\colon [0,1]^2 \to [0,1]$ is called a uninorm, if there exists $e \in [0,1]$, called the neutral element of U, such that

$$U(e,x) = U(x,e) = x, \qquad \text{for all } x \in [0,1].$$

Remark 3.2 (cf. Fodor et al. [18]). (i) If $e = 0$, then U is a t-conorm and if $e = 1$, then U is a t-norm.
(ii) The neutral element e corresponding to a uninorm U is unique.
(iii) For any uninorm U we have $U(0,1) \in \{0,1\}$. A uninorm U such that $U(0,1) = 0$ is called conjunctive and if $U(0,1) = 1$, then it is called disjunctive.
(iv) The structure of a uninorm U with the neutral element $e \in]0,1[$ is always the following. It is like a t-norm on the square $[0,e]^2$, like a t-conorm on the square $[e,1]^2$ and it takes values between the minimum and the maximum in the other cases.

There are several different classes of uninorms introduced in the literature. We only mention relevant details and results, which will be useful in the sequel, connected with the three main classes of uninorms.

3.1. The classes of $\mathcal{U}_{\mathbf{Min}}$ and $\mathcal{U}_{\mathbf{Max}}$

Uninorms verifying that both functions $U(\,\cdot\,,0)$ and $U(\,\cdot\,,1)$ are continuous except at the point e were characterized by Fodor et al. [18], as follows.

Theorem 3.3. *Let $e \in]0,1[$. For a function $U\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) *U is a conjunctive uninorm with the neutral element e, such that the function $x \mapsto U(x,1)$ is continuous for all $x \in [0,e[$.*
(ii) *There exist a t-norm T and a t-conorm S such that*

$$U(x,y) = \begin{cases} e \cdot T\left(\frac{x}{e}, \frac{y}{e}\right), & \text{if } x,y \in [0,e], \\ e + (1-e) \cdot S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & \text{if } x,y \in [e,1], \\ \min(x,y), & \text{otherwise}, \end{cases} \qquad \text{for all } x,y \in [0,1]. \tag{1}$$

Theorem 3.4. *Let $e \in]0,1[$. For a function $U\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) *U is a disjunctive uninorm with the neutral element e, such that the function $x \mapsto U(x,0)$ is continuous for all $x \in]e,1]$.*
(ii) *There exist a t-norm T and a t-conorm S such that*

$$U(x,y) = \begin{cases} e \cdot T\left(\frac{x}{e}, \frac{y}{e}\right), & \text{if } x,y \in [0,e], \\ e + (1-e) \cdot S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & \text{if } x,y \in [e,1], \\ \max(x,y), & \text{otherwise}, \end{cases} \qquad \text{for all } x,y \in [0,1]. \tag{2}$$

The class of uninorms of the form (1) is denoted by $\mathcal{U}_{\mathbf{Min}}$, while the class of uninorms of the form (2) is denoted by $\mathcal{U}_{\mathbf{Max}}$. Note that, even if a t-norm T, a t-conorm S and $e \in]0,1[$ are fixed, a uninorm is not uniquely defined – it can be conjunctive or disjunctive. If U is a conjunctive (disjunctive) uninorm, then we will write $U^{\mathbf{c}}_{T,S,e}$ ($U^{\mathbf{d}}_{T,S,e}$, respectively).

3.2. Idempotent uninorms

A uninorm U such that $U(x, x) = x$ for all $x \in [0, 1]$ is said to be an idempotent uninorm. The class of all idempotent uninorms will be denoted by $\mathcal{U}_{\mathbf{Idem}}$. Martín et al. [20] have characterized all idempotent uninorms, which subsumes the results of Czogała and Drewniak [7] and De Baets [8], who first characterized the class of left-continuous and right-continuous idempotent uninorms.

Theorem 3.5 (Martín et al. [20])**.** *Let $e \in [0, 1]$. For a function $U\colon [0, 1]^2 \to [0, 1]$ the following statements are equivalent:*

(i) *U is an idempotent uninorm with the neutral element e.*
(ii) *There exists a decreasing function $g\colon [0, 1] \to [0, 1]$ with a fixed point e, i.e., $g(e) = e$, satisfying*

$$\begin{aligned} &g(x) = 0, \qquad \text{for all } x \in]g(0), 1],\\ &g(x) = 1, \qquad \text{for all } x \in [0, g(1)[,\\ &\inf\{y \mid g(y) = g(x)\} \leq g(g(x)) \leq \sup\{y \mid g(y) = g(x)\}, \qquad \text{for all } x \in [0, 1], \end{aligned}$$

such that U has the following form

$$U(x, y) = \begin{cases} \min(x, y), & \text{if } (y < g(x)) \text{ or } (y = g(x) \text{ and } x < g(g(x))),\\ \max(x, y), & \text{if } (y > g(x)) \text{ or } (y = g(x) \text{ and } x > g(g(x))),\\ \max(x, y) & \\ \quad \text{or} & \text{if } y = g(x) \text{ and } x = g(g(x)),\\ \min(x, y), & \end{cases} \qquad \text{for all } x, y \in [0, 1].$$

and U is commutative on the set $\{(x, y) \mid y = g(x) \text{ and } x = g(g(x))\}$.

Example 3.6. Let $e \in]0, 1[$ be fixed and let us consider the functions $g_c, g_d\colon [0, 1] \to [0, 1]$ as defined below:

$$g_c(x) = \begin{cases} 1, & \text{if } x < e,\\ e, & \text{if } x \geq e, \end{cases} \qquad g_d(x) = \begin{cases} e, & \text{if } x \leq e,\\ 0, & \text{if } x > e. \end{cases}$$

Then the corresponding idempotent uninorms generated by them are the first kind of uninorms considered by Yager and Rybalov [27]:

$$U^{\mathbf{c},e}_{\mathbf{YR}}(x, y) = \begin{cases} \max(x, y), & \text{if } x, y \in [e, 1],\\ \min(x, y), & \text{otherwise,} \end{cases} \qquad U^{\mathbf{d},e}_{\mathbf{YR}}(x, y) = \begin{cases} \min(x, y), & \text{if } x, y \in [0, e],\\ \max(x, y), & \text{otherwise.} \end{cases}$$

3.3. Representable uninorms

Analogous to the representation theorems for continuous Archimedean t-norms and t-conorms, Fodor et al. [18] have proven the following result.

Theorem 3.7 (Fodor et al. [18])**.** *Let $e \in]0, 1[$. For a function $U\colon [0, 1]^2 \to [0, 1]$ the following statements are equivalent:*

(i) *U is a strictly increasing and continuous uninorm on $]0, 1[^2$ with the neutral element e such that U is self-dual, except in points $(0, 1)$ and $(1, 0)$, with respect to a strong negation N with the fixed point e, i.e.,*

$$U(x, y) = N(U(N(x), N(y))), \qquad \text{for all } x, y \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\}.$$

(ii) *U has a continuous additive generator, i.e., there exists a continuous and strictly increasing function $h\colon [0,1] \to [-\infty, \infty]$, such that $h(0) = -\infty$, $h(e) = 0$ and $h(1) = \infty$, which is uniquely determined up to a positive multiplicative constant, such that*

$$U(x,y) = \begin{cases} 0, & \text{if } (x,y) \in \{(0,1),(1,0)\}, \\ h^{-1}(h(x)+h(y)), & \text{otherwise}, \end{cases} \qquad \text{for all } x,y \in [0,1],$$

or

$$U(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \{(0,1),(1,0)\}, \\ h^{-1}(h(x)+h(y)), & \text{otherwise}, \end{cases} \qquad \text{for all } x,y \in [0,1].$$

Uninorms that can be represented as in Theorem 3.7 are called representable uninorms and this class will be denoted by $\mathcal{U}_{\mathbf{Rep}}$. It should be noted that these operations appeared first in [11].

Remark 3.8. (i) Note that once the additive generator h is fixed, by its strictness e is also unique and hence h generates a unique (up to the constant value on the set $\{(0,1),(1,0)\}$) representable uninorm. If a conjunctive and representable uninorm is generated by h, then we will denote it by $U_h^{\mathbf{c}}$. Similarly, if a disjunctive and representable uninorm is generated by h, then we will denote it by $U_h^{\mathbf{d}}$.

(ii) It is interesting to note that every representable uninorm U_h (conjunctive or disjunctive) gives rise to a natural negation, obtained as

$$N_{U_h}(x) = h^{-1}(-h(x)), \qquad \text{for all } x \in [0,1], \tag{3}$$

which is a strong negation (see Definition 2.2). Also, U_h is self-dual with respect to its natural negation.

Example 3.9. For the additive generator $h_1(x) = \ln\left(\frac{x}{1-x}\right)$, we get the following disjunctive and representable uninorm (in this case $e = \frac{1}{2}$):

$$U_{h_1}^{\mathbf{d}}(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \{(0,1),(1,0)\}, \\ \dfrac{xy}{(1-x)(1-y)+xy}, & \text{otherwise}, \end{cases} \qquad \text{for all } x,y \in [0,1].$$

For other examples of representable uninorms see [18].

3.4. Intersections between the above classes of uninorms

Remark 3.10. (i) For a representable uninorm U, we have $U(x,1) = 1$ for all $x > 0$ and $U(x,0) = 0$ for all $x < 1$. Hence, if U is conjunctive, then the function $x \mapsto U(x,1)$ is not continuous at $x = 0$, while if U is disjunctive, then the function $x \mapsto U(x,0)$ is not continuous at $x = 1$. Therefore no representable uninorm belongs to either $\mathcal{U}_{\mathbf{Min}}$ or $\mathcal{U}_{\mathbf{Max}}$.

(ii) For a representable uninorm generated from h we have

$$U(x,x) = h^{-1}(h(x)+h(x)) = h^{-1}(2h(x)) \neq x$$

whenever $x \in]0,1[\setminus\{e\}$. Therefore no representable uninorm is idempotent.

(iii) From the representation results of $\mathcal{U}_{\mathbf{Min}}$ and $\mathcal{U}_{\mathbf{Max}}$ of Fodor et al. [18] (see the uninorms $U_{\mathbf{YR}}^{\mathbf{c},e}, U_{\mathbf{YR}}^{\mathbf{d},e}$) one can see that an equivalent condition for an idempotent uninorm to belong to $\mathcal{U}_{\mathbf{Min}}$ or $\mathcal{U}_{\mathbf{Max}}$ is that its associated function g should have either the representation g_c or g_d given in Example 3.6. Let us denote these sub-classes of idempotent uninorms by

$$\mathcal{U}_{I,G_c} = \{U \in \mathcal{U}_{\mathbf{Idem}} \mid g = g_c \text{ and } e \in]0,1[\},$$
$$\mathcal{U}_{I,G_d} = \{U \in \mathcal{U}_{\mathbf{Idem}} \mid g = g_d \text{ and } e \in]0,1[\}.$$

From the above discussion it is clear that the following relationships exist among the above families of uninorms:

$$\mathcal{U}_{\mathbf{Min}} \cap \mathcal{U}_{\mathbf{Rep}} = \mathcal{U}_{\mathbf{Max}} \cap \mathcal{U}_{\mathbf{Rep}} = \mathcal{U}_{\mathbf{Idem}} \cap \mathcal{U}_{\mathbf{Rep}} = \emptyset,$$
$$\mathcal{U}_{\mathbf{Min}} \cap \mathcal{U}_{\mathbf{Idem}} = \mathcal{U}_{I,G_c},$$
$$\mathcal{U}_{\mathbf{Max}} \cap \mathcal{U}_{\mathbf{Idem}} = \mathcal{U}_{I,G_d}.$$

4. (U,N)-operations and (U,N)-implications

A natural generalization of (S,N)-implications (see [1]) in the uninorm framework is to consider a uninorm in the place of a t-conorm.

Definition 4.1. A function $I\colon [0,1]^2 \to [0,1]$ is called a (U,N)-operation, if there exist a uninorm U and a fuzzy negation N such that

$$I_{U,N}(x,y) = U(N(x),y), \qquad \text{for all } x,y \in [0,1]. \tag{4}$$

If I is a (U,N)-operation generated from a uninorm U and a negation N, then we will denote it by $I_{U,N}$.

Proposition 4.2 (Baczyński and Jayaram [5, Proposition 5.2])**.** *If $I_{U,N}$ is a (U,N)-operation obtained from a uninorm U with $e \in]0,1[$ as its neutral element, then*

(i) $I_{U,N}$ *satisfies* (I1), (I2), (I5) *and* (EP),
(ii) $N^e_{I_{U,N}} = N$.

If $e \in]0,1[$, then not for every uninorm U the (U,N)-operation is a fuzzy implication. Next result characterizes these (U,N)-operations, which satisfy (I3) and (I4).

Theorem 4.3 (cf. De Baets and Fodor [9, p. 98])**.** *For a uninorm U with the neutral element $e \in]0,1[$ the following statements are equivalent:*

(i) *The (U,N)-operation $I_{U,N}$ is a fuzzy implication.*
(ii) *U is a disjunctive uninorm, i.e., $U(0,1) = U(1,0) = 1$.*

Following the terminology used by Mas et al. [21] for the QL-implications, only if the (U,N)-operation $I_{U,N}$ is a fuzzy implication we use the term (U,N)-implication.

Theorem 4.4 (Baczyński and Jayaram [1, Theorem 6.4])**.** *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) *I is a (U,N)-implication generated from some uninorm U with the neutral element $e \in]0,1[$ and some continuous fuzzy negation N.*
(ii) *I satisfies* (I1), (I3), (EP) *and the function N^e_I is a continuous negation for some $e \in]0,1[$.*

Moreover, the representation (4) *of (U,N)-implication is unique in this case.*

Example 4.5. In the following, we give examples of (U,N)-implications obtained using the classical strong negation $N_{\mathbf{C}}(x) = 1-x$ for all $x \in [0,1]$, and for different uninorms. Note that $I_{\mathbf{KD}}$ is the Kleene-Dienes implication given by $I_{\mathbf{KD}}(x,y) = \max(1-x,y)$, for all $x,y \in [0,1]$.

(i) Let us consider the disjunctive uninorm $U_{\mathbf{LK}}$ from the class $\mathcal{U}_{\mathbf{Max}}$ generated by the triple $(T_{\mathbf{LK}}, S_{\mathbf{LK}}, 0.5)$, where $T_{\mathbf{LK}}$ denotes the Łukasiewicz t-norm $T_{\mathbf{LK}}(x,y) = \max(x+y-1,0)$ and $S_{\mathbf{LK}}$ denotes the Łukasiewicz t-conorm $S_{\mathbf{LK}}(x,y) = \min(x+y,1)$, for all $x,y \in [0,1]$. Then

$$I_{U_{\mathbf{LK}},N_{\mathbf{C}}}(x,y) = \begin{cases} \max(y-x+0.5,0), & \text{if } \max(1-x,y) \le 0.5, \\ \min(y-x+0.5,1), & \text{if } \min(1-x,y) > 0.5, \\ I_{\mathbf{KD}}(x,y), & \text{otherwise}, \end{cases} \qquad \text{for all } x,y \in [0,1].$$

(ii) Let us consider the disjunctive uninorm $U_{\mathbf{M}}$ from the class $\mathcal{U}_{\mathbf{Max}}$ generated from the triple $(T_{\mathbf{M}}, S_{\mathbf{M}}, 0.5)$, where $T_{\mathbf{M}}$ denotes the minimum t-norm $T_{\mathbf{M}}(x,y) = \min(x,y)$ and $S_{\mathbf{M}}$ denotes the maximum t-conorm $S_{\mathbf{M}}(x,y) = \max(x,y)$, for all $x,y \in [0,1]$. Observe, that $U_{\mathbf{M}}$ is also an idempotent uninorm. Then

$$I_{U_{\mathbf{M}},N_{\mathbf{C}}}(x,y) = \begin{cases} \min(1-x,y), & \text{if } \max(1-x,y) \le 0.5, \\ I_{\mathbf{KD}}(x,y), & \text{otherwise}, \end{cases} \qquad \text{for all } x,y \in [0,1].$$

(iii) Let us consider the disjunctive representable uninorm $U^{\mathbf{d}}_{h_1}$ from Example 3.9. Then

$$I_{U^{\mathbf{d}}_{h_1},N_{\mathbf{C}}}(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \{(0,0),(1,1)\}, \\ \dfrac{(1-x)y}{x+y-2xy}, & \text{otherwise}, \end{cases} \qquad \text{for all } x,y \in [0,1].$$

5. RU-Implications

Analogous to the definition of R-implications from t-norms (see [3]), one can also define residual operations from uninorms.

Definition 5.1. A function $I\colon [0,1]^2 \to [0,1]$ is called an RU-operation, if there exists a uninorm U such that

$$I(x,y) = \sup\{t \in [0,1] \mid U(x,t) \le y\}, \qquad \text{for all } x,y \in [0,1]. \tag{5}$$

If I is an RU-operation generated from a uninorm U, then we will often denote it by I_U.

Proposition 5.2 (see [9]). *If U is a uninorm with the neutral element $e \in]0,1[$, then I_U satisfies* (I1), (I2), (I4), (I5). *Moreover, $I_U(e,y) = y$ for all $y \in [0,1]$.*

Next result characterize these RU-operations, which satisfy (I3).

Proposition 5.3 (cf. [9, Proposition 7]). *For a uninorm U with neutral element $e \in]0,1[$ the following statements are equivalent:*

(i) *I_U is a fuzzy implication.*
(ii) *$U(0,y) = 0$ for all $y \in [0,1[$.*

Here again only if the RU-operation I_U is a fuzzy implication we use the term RU-implication. The block structure of a uninorm U precludes any further investigations on the basic properties of RU-implications unless the class to which U belongs is known.

5.1. *RU-Implications from uninorms in the class $\mathcal{U}_{\mathbf{Min}}$*

Firstly let us consider a uninorm U in the class $\mathcal{U}_{\mathbf{Max}}$. Observe that from Proposition 5.3 we get that the RU-operation generated from U is not a fuzzy implication. Therefore in this subsection we consider only (conjunctive) uninorms in $\mathcal{U}_{\mathbf{Min}}$.

Theorem 5.4 (cf. De Baets and Fodor [9, Theorem 6]). *If $U^{\mathbf{c}}_{T,S,e} \in \mathcal{U}_{\mathbf{Min}}$, then the RU-implication obtained from U is given by*

$$I_{U^{\mathbf{c}}_{T,S,e}}(x,y) = \begin{cases} e \cdot I_T\left(\frac{x}{e}, \frac{y}{e}\right), & \text{if } x,y \in [0,e[\text{ and } x > y, \\ e + (1-e) \cdot I_S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & \text{if } x,y \in [e,1] \text{ and } x \le y, \\ e, & \text{if } x,y \in [e,1] \text{ and } x > y, \\ I_{\mathbf{GD}}(x,y), & \text{otherwise}, \end{cases} \qquad \text{for all } x,y \in [0,1],$$

where $I_S(x,y) = \sup\{t \in [0,1] \mid S(x,t) \le y\}$, i.e., it is the residual of the t-conorm S, obtained from (5) *by employing S instead of the uninorm U and $I_{\mathbf{GD}}$ is the Gödel implication defined by the formula*

$$I_{\mathbf{GD}}(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ y, & \text{if } x > y, \end{cases} \qquad \text{for all } x,y \in [0,1].$$

Example 5.5. (i) Let us consider the conjunctive uninorm $U_{\mathbf{LK}} = (T_{\mathbf{LK}}, S_{\mathbf{LK}}, 0.5) \in \mathcal{U}_{\mathbf{Min}}$. Then

$$I_{U_{\mathbf{LK}}}(x,y) = \begin{cases} 0.5 + y - x, & \text{if } (x,y \in [0,0.5[\text{ and } y < x) \text{ or } (x,y \in [0.5,1] \text{ and } y > x), \\ 0.5, & \text{if } x,y \in [0.5,1] \text{ and } y \le x, \\ I_{\mathbf{GD}}(x,y), & \text{otherwise}, \end{cases} \qquad \text{for all } x,y \in [0,1].$$

(ii) Let us consider the conjunctive uninorm $U_{\mathbf{M}} = (T_{\mathbf{M}}, S_{\mathbf{M}}, 0.5) \in \mathcal{U}_{\mathbf{Min}}$. Then

$$I_{U_{\mathbf{M}}}(x, y) = \begin{cases} y, & \text{if } x, y \in [0.5, 1] \text{ and } y > x, \\ 0.5, & \text{if } x, y \in [0.5, 1] \text{ and } y \leq x, \\ I_{\mathbf{GD}}(x, y), & \text{otherwise}, \end{cases} \qquad \text{for all } x, y \in [0, 1].$$

Remark 5.6. If $U \in \mathcal{U}_{\mathbf{Min}}$ with the neutral element $e \in]0, 1[$, then the natural negation of I_U with respect to e is the function

$$N^e_{I_U}(x) = \begin{cases} 1, & \text{if } x \in [0, e[, \\ e, & \text{if } x \in [e, 1], \end{cases}$$

which is not a fuzzy negation. In fact, from the formula of I_U given in Theorem 5.4, it can be seen that, for any $\alpha \in]0, 1[$, the natural negation of I_U with respect to α is not a fuzzy negation, since $N^\alpha_{I_U}(1) = \alpha$ if $\alpha \in]0, e[$ and $N^\alpha_{I_U}(1) = e$ if $\alpha \in [e, 1[$.

5.2. RU-Implications from idempotent uninorms

Firstly we cite the following result on RU-implications generated from idempotent uninorms.

Proposition 5.7. *For a uninorm $U \in \mathcal{U}_{\mathbf{Idem}}$ which has the generator g the following statements are equivalent:*

(i) *I_U is a fuzzy implication.*
(ii) *$g(0) = 1$.*

The following result of Ruiz and Torrens [24] gives the general structure of an I_U obtained from such idempotent uninorms, which subsumes an earlier result by De Baets and Fodor [9].

Theorem 5.8 (Ruiz and Torrens [24, Theorem 4])**.** *If $U \in \mathcal{U}_{\mathbf{Idem}}$ has the generator g such that $g(0) = 1$, then the RU-implication obtained from U is given by*

$$I_U(x, y) = \begin{cases} \max(g(x), y), & \text{if } x \leq y, \\ \min(g(x), y), & \text{if } x > y, \end{cases} \qquad \text{for all } x, y \in [0, 1]. \tag{6}$$

Example 5.9. (i) Let us consider the idempotent uninorm $U^{\mathbf{c},e}_{\mathbf{YR}} \in \mathcal{U}_{\mathbf{Min}}$, with the associated function g_c given in Remark 3.10(ii). Then its RU-implication is given by

$$I_{U^{\mathbf{c},e}_{\mathbf{YR}}}(x, y) = \begin{cases} y, & \text{if } (y < x \text{ and } y \leq e) \text{ or } (y \geq x \text{ and } x \geq e), \\ e, & \text{if } y < x \text{ and } y > e, \\ 1, & \text{otherwise}, \end{cases} \qquad \text{for all } x, y \in [0, 1].$$

(ii) Let us consider the right-continuous idempotent uninorm generated by the classical negation $N_{\mathbf{C}}(x) = 1 - x$, which is obviously not in $\mathcal{U}_{\mathbf{Min}}$, given by

$$U_{N_{\mathbf{C}}}(x, y) = \begin{cases} \min(x, y), & \text{if } y < 1 - x, \\ \max(x, y), & \text{if } y \geq 1 - x, \end{cases} \qquad \text{for all } x, y \in [0, 1].$$

Then its RU-implication is given by

$$I_{U_{N_{\mathbf{C}}}}(x, y) = \begin{cases} \min(1 - x, y), & \text{if } y < x, \\ \max(1 - x, y), & \text{if } y \geq x, \end{cases} \qquad \text{for all } x, y \in [0, 1].$$

(iii) Let us consider the right-continuous idempotent uninorm generated by the strict negation $N_{\mathbf{K}}(x) = 1 - x^2$, which is again not in $\mathcal{U}_{\mathbf{Min}}$, given by

$$U_{N_{\mathbf{K}}}(x,y) = \begin{cases} \min(x,y), & \text{if } y < N_{\mathbf{K}}(x), \\ \max(x,y), & \text{if } y \geq N_{\mathbf{K}}(x), \end{cases} \qquad \text{for all } x,y \in [0,1].$$

Then its RU-implication is given by

$$I_{U_{N_{\mathbf{K}}}}(x,y) = \begin{cases} \min(N_{\mathbf{K}}(x),y), & \text{if } y < x, \\ \max(N_{\mathbf{K}}(x),y), & \text{if } y \geq x, \end{cases} \qquad \text{for all } x,y \in [0,1].$$

5.3. RU-Implications from representable uninorms

Since every representable uninorm satisfies condition (ii) in Proposition 5.3, every RU-operation generated from a representable uninorm is a fuzzy implication. In this case we have the following representation of the RU-implications.

Theorem 5.10 (De Baets and Fodor [9, Theorem 7])**.** *If $U_h \in \mathcal{U}_{\mathbf{Rep}}$, then the RU-implication obtained from U_h is given by*

$$I_{U_h}(x,y) = \begin{cases} 1, & \text{if } (x,y) \in \{(0,0),(1,1)\}, \\ h^{-1}(h(y) - h(x)), & \text{otherwise}, \end{cases} \qquad \text{for all } x,y \in [0,1]. \tag{7}$$

Example 5.11. Let us consider the disjunctive representable uninorm $U^{\mathbf{d}}_{h_1}$ given in Example 3.9. Its RU-implication given in Theorem 5.10 is also the (U,N)-implication $I_{U^{\mathbf{d}}_{h_1},N_{\mathbf{C}}}$ given as in Example 4.5(iii).

Proposition 5.12 (cf. De Baets and Fodor [9])**.** *If $U_h \in \mathcal{U}_{\mathbf{Rep}}$ with the neutral element $e \in]0,1[$, then*

(i) *$I_{U_h}(x,x) = e$ for all $x \in]0,1[$,*
(ii) *the function $N^e_{I_{U_h}}(x) = N_{U_h}(x) = h^{-1}(-h(x))$ is defined for all $x \in [0,1]$ and is a strong negation,*
(iii) *I_{U_h} satisfies the law of contraposition with respect to N_{U_h}, i.e., $I_{U_h}(x,y) = I_{U_h}(N_{U_h}(y), N_{U_h}(x))$, for all $x,y \in [0,1]$.*

Proof. Let U_h be a representable uninorm with the additive generator h and the neutral element $e \in]0,1[$.

(i) From (7), we see that $I_{U_h}(x,x) = h^{-1}(h(x) - h(x)) = h^{-1}(0) = e$, for all $x \in]0,1[$. However, note that $I_{U_h}(0,0) = I_{U_h}(1,1) = 1$.
(ii) The natural negation of I_{U_h} with respect to e is

$$N^e_{I_{U_h}}(x) = I_{U_h}(x,e) = h^{-1}(h(e) - h(x)) = h^{-1}(-h(x)) = N_{U_h}(x), \qquad \text{for all } x \in [0,1],$$

by Remark 3.8(ii), and is a strong negation.
(iii) If $(x,y) \in \{(0,0),(1,1)\}$, then the contrapositivity is obvious from the boundary conditions. If $(x,y) \in [0,1]^2 \setminus \{(0,0),(1,1)\}$, then from the previous point we have

$$\begin{aligned} I_{U_h}(N_{U_h}(y), N_{U_h}(x)) &= h^{-1}\left(h\left(h^{-1}(-h(x))\right) - h\left(h^{-1}(-h(y))\right)\right) \\ &= h^{-1}(-h(x) + h(y)) = I_{U_h}(x,y). \end{aligned}$$

□

6. Intersection between (U,N)- and RU-implications

In the previous sections we have discussed two families of fuzzy implications derived from uninorms, viz., (U,N)-implications and RU-implications. In the case of RU-implications from uninorms, we have specifically considered uninorms U from the three main families of $\mathcal{U}_{\mathbf{Min}}$,$\mathcal{U}_{\mathbf{Rep}}$ and $\mathcal{U}_{\mathbf{Idem}}$. In this section we discuss the intersections that exist among these families of fuzzy implications. Towards this end, we introduce the following notations to denote these families of fuzzy implications:

- $\mathbb{I}_{\mathbb{U},\mathbb{N}}$ – the family of all (U,N)-implications;
- $\mathbb{I}_{\mathbb{U},\mathbb{N}_{\mathbb{C}}}$ – the family of all (U,N)-implications obtained from continuous negations;
- $\mathbb{I}_{\mathbb{U}_{\mathbf{M}}}$ – the family of all RU-implications generated from uninorms in $\mathcal{U}_{\mathbf{Min}}$;
- $\mathbb{I}_{\mathbb{U}_{\mathbf{I}}}$ – the family of all RU-implications generated from uninorms in $\mathcal{U}_{\mathbf{Idem}}$;
- $\mathbb{I}_{\mathbb{U}_{\mathbf{R}}}$ – the family of all RU-implications generated from uninorms in $\mathcal{U}_{\mathbf{Rep}}$.

Needless to state, in the case $e = 0$ we have $\mathbb{I}_{\mathbb{U},\mathbb{N}}$ is the set of all (S,N)-implications, while if $e = 1$ we have $\mathbb{I}_{\mathbb{U}}$ is the set of all R-implications obtained from t-norms. Hence, in the sequel, we consider only uninorms with neutral elements in $]0, 1[$.

6.1. Intersection between $\mathbb{I}_{\mathbb{U},\mathbb{N}}$ and $\mathbb{I}_{\mathbb{U}_{\mathbf{M}}}$

Because of Proposition 4.2(ii) and Remark 5.6 we get

$$\mathbb{I}_{\mathbb{U},\mathbb{N}} \cap \mathbb{I}_{\mathbb{U}_{\mathbf{M}}} = \emptyset.$$

6.2. Intersection between $\mathbb{I}_{\mathbb{U},\mathbb{N}}$ and $\mathbb{I}_{\mathbb{U}_{\mathbf{R}}}$

Proposition 6.1. *Let U_h be a representable uninorm with the additive generator h. Then the RU-implication I_{U_h} is also a (U,N)-implication obtained from the disjunctive representable uninorm $U_h^{\mathbf{d}}$ given by the formula*

$$U_h^{\mathbf{d}}(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \{(0, 1), (1, 0)\}, \\ U_h(x, y), & \text{otherwise}, \end{cases} \qquad \text{for all } x, y \in [0, 1],$$

and its natural negation N_{U_h}, i.e., $I_{U_h} = I_{U_h^{\mathbf{d}},N_{U_h}}$.

Proof. Observe that by Remark 3.8(ii) we get

$$\begin{aligned} I_{U_h^{\mathbf{d}},N_{U_h}}(x, y) = U_h^{\mathbf{d}}(N_{U_h}(x), y) &= \begin{cases} 1, & \text{if } (N_{U_h}(x), y) \in \{(0, 1), (1, 0)\} \\ h^{-1}(h(N_{U_h}(x)) + h(y)), & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } (x, y) \in \{(0, 0), (1, 1)\} \\ h^{-1}(h(h^{-1}(-h(x))) + h(y)), & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & \text{if } (x, y) \in \{(0, 0), (1, 1)\} \\ h^{-1}(-h(x) + h(y)), & \text{otherwise} \end{cases} \\ &= I_{U_h}(x, y), \end{aligned}$$

for all $x, y \in [0, 1]$. From the uniqueness of the representation of (U,N)-implications generated from continuous negations we get the claim. □

Let us denote by

- $\mathbb{I}_{\mathbb{U}^{\mathbf{d}}_{\mathbf{R}},\mathbb{N}_{\mathbb{U}_{\mathbf{R}}}}$ – the family of all (U,N)-implications obtained from disjunctive representable uninorms and their strong natural negations;

The above result can be summarized as follows (see Remark 6.12):

$$\mathbb{I}_{\mathbb{U},\mathbb{N}} \cap \mathbb{I}_{\mathbb{U}_{\mathbf{R}}} = \mathbb{I}_{\mathbb{U}^{\mathbf{d}}_{\mathbf{R}},\mathbb{N}_{\mathbb{U}_{\mathbf{R}}}},$$
$$\mathbb{I}_{\mathbb{U}_{\mathbf{R}}} = \mathbb{I}_{\mathbb{U}^{\mathbf{d}}_{\mathbf{R}},\mathbb{N}_{\mathbb{U}_{\mathbf{R}}}} \subsetneq \mathbb{I}_{\mathbb{U},\mathbb{N}_{\mathbb{C}}} \subsetneq \mathbb{I}_{\mathbb{U},\mathbb{N}}.$$

Remark 6.2. We know that an R-implication I_T obtained from a continuous t-norm is also an (S,N)-implication - in fact, with a strong N - if and only if the t-norm T is nilpotent. The above results seem to suggest that representable uninorms are generalizations of nilpotent t-norms and t-conorms, whereas their definition indicates that they are, in fact, obtained from generators of strict t-norms and t-conorms.

6.3. Intersection between $\mathbb{I}_{\mathbb{U},\mathbb{N}}$ and $\mathbb{I}_{\mathbb{U}_{\mathbf{I}}}$

Ruiz and Torrens [24] have investigated the conditions under which the RU-implication from an idempotent uninorm is also a (U,N)-implication obtained from a strong N. In fact, it can be shown (see Proposition 6.8) that the strongness of N need not be assumed and is consequential of the continuity.

Definition 6.3. A function $g\colon [0,1] \to [0,1]$ is called

(i) sub-involutive, if $g(g(x)) \le x$ for all $x \in [0,1]$,
(ii) super-involutive, if $g(g(x)) \ge x$ for all $x \in [0,1]$.

Lemma 6.4. *Let $N\colon [0,1] \to [0,1]$ be a continuous negation. If, in addition, N is either sub-involutive or super-involutive, then N is involutive i.e., it is a strong negation.*

Proof. We give the proof only for the case when N is sub-involutive. Since a fuzzy negation N is continuous on $[0,1]$ it is onto. Now, for any $x \in [0,1]$ there exists a $y \in [0,1]$ such that $x = N(y)$. Consequently, we get

$$x = N(y) \Longrightarrow N(x) = N(N(y)) \le y \Longrightarrow N(N(x)) \ge N(y) = x,$$

i.e., $N(N(x)) \ge x$. Since N is sub-involutive, we also have that $N(N(x)) \le x$, whence N is involutive. □

Lemma 6.5. *Let U_I be an idempotent uninorm such that $g(0) = 1$. If the R-implication I_{U_I} is also a (U,N)-implication generated from some uninorm U with the neutral element $e \in]0,1[$ and some fuzzy negation N, then $N = g$.*

Proof. From Proposition 4.2(ii) the negation of I_{U_I} with respect to e is a fuzzy negation N. Hence we get

$$N(x) = N^e_{I_{U,N}}(x) = N^e_{I_{U_I}}(x) = I_{U_I}(x,e) = \begin{cases} \max(g(x),e), & \text{if } x \le e \\ \min(g(x),e), & \text{if } x > e \end{cases}, \qquad \text{for all } x \in [0,1].$$

But $g(e) = e$ and g is decreasing, so $N = g$. □

The above results imply that we should consider only idempotent uninorms generated from fuzzy negations. The other necessary condition for a (U,N)-implication is the exchange principle. Characterization of RU-implications generated from idempotent uninorms that satisfy (EP) has been obtained by Ruiz and Torrens [24].

Theorem 6.6 ([24, Theorem 5])**.** *If $U \in \mathcal{U}_{\mathbf{Idem}}$ has the generator g such that $g(0) = 1$, then the following statements are equivalent:*

(i) *RU-implication I_U satisfies* (EP).
(ii) *The following property is satisfied:*

$$\text{if } g(g(x)) < x \text{ for some } x \in [0,1], \text{ then } x > e \text{ and } g(x) = e. \tag{8}$$

Corollary 6.7. *Let $U \in \mathcal{U}_{\mathbf{Idem}}$ have the generator g such that $g(0) = 1$. If g is super-involutive, then the RU-implication I_U satisfies* (EP).

Proposition 6.8. *Let N be a continuous negation and U be an idempotent uninorm obtained from N. Then the following statements are equivalent:*

(i) *RU-implication I_U satisfies* (EP).

(ii) *N is strong.*

Proof. $(i) \implies (ii)$ Let N be a continuous negation with a fixed point $e \in]0,1[$ and U be an idempotent uninorm obtained from N. If the RU-implication obtained from U satisfies (EP), then N satisfies the condition (8). We show that $N(N(x)) \geq x$ for all $x \in [0,1]$ by discussing the following two cases.

If $x \leq e$, then by (8) we see that $N(N(x)) \geq x$.

Let us suppose that there exists $x > e$ such that $N(N(x)) < x$. By Theorem 6.6 and (8), we have $N(x) = e$. Since N is continuous and decreasing, there exists $y < e$ such that $N(y) = x > e = N(e) = N(x)$. Once again, by the continuity of N, if z' is such that $N(y) = x > z' > e = N(e)$, then there exists z such that $y < z < e$ and $N(z) = z'$. Now, from these two inequalities and formula for U_I in Theorem 3.5 we get

$$U(x,z) = \min(x,z) = z,$$

since $z < N(x) = e$. By the commutativity of U we have $U(z,x) = z$. But $x \neq N(z) = z'$, which implies $x < N(z) = z'$, a contradiction.

Hence there does not exist any $x \in [0,1]$ such that $N(N(x)) < x$, i.e., N is super-involutive. From Lemma 6.4, we have that N is strong.

$(ii) \implies (i)$ This follows from Corollary 6.7. □

The above investigations lead us to the fact that we should consider only two cases: N is non-continuous or N is strong. Let us consider the case when N is strong. In fact, De Baets and Fodor ([9, Proposition 12]) were the first to obtain a sufficient condition in this case, which was later strengthened by Ruiz and Torrens ([24, Proposition 8]). The following result is a further generalization made possible by Proposition 6.8 above.

Theorem 6.9. *Let U_I be an idempotent uninorm obtained from a continuous function g, N a fuzzy negation and U a uninorm. Then the following statements are equivalent:*

(i) *The RU-implication I_{U_I} is also a (U,N)-implication $I_{U,N}$.*

(ii) *$g = N$ is a strong negation and U is given by the formula*

$$U(x,y) = \begin{cases} U_I(x,y), & \text{if } y \neq g(x), \\ \max(x,y), & \text{if } y = g(x), \end{cases} \quad \text{for all } x,y \in [0,1]. \tag{9}$$

Proof. $(ii) \implies (i)$ If g is a strong negation, then in particular $g(0) = 1$, so I_{U_I} is an RU-implication given by the formula (6). When g is a strong negation, then we get (cf. Theorem 3.5)

$$U_I(x,y) = \begin{cases} \min(x,y), & \text{if } y < g(x), \\ \max(x,y), & \text{if } y > g(x), \\ \max(x,y) & \\ \text{ or} & \text{if } y = g(x), \\ \min(x,y), & \end{cases} \quad \text{for all } x,y \in [0,1].$$

Therefore the function U, given by the formula (9), is a well defined right-continuous idempotent uninorm. Now for any $x,y \in [0,1]$ we get

$$I_{U,g}(x,y) = U(g(x),y) = \begin{cases} U_I(g(x),y), & \text{if } y \neq x \\ \max(g(x),y), & \text{if } y = x \end{cases} = \begin{cases} \min(g(x),y), & \text{if } y < x \\ \max(g(x),y), & \text{if } y > x \\ \max(g(x),y), & \text{if } x = y \end{cases}$$

$$= \begin{cases} \min(g(x),y), & \text{if } y < x \\ \max(g(x),y), & \text{if } y \geq x \end{cases} = I_{U_I}(x,y).$$

$(i) \Longrightarrow (ii)$ Since I_U is a fuzzy implication, we know that $g(0) = 1$ by Proposition 5.7. Lemma 6.5 implies that $g = N$ is a continuous fuzzy negation. From Proposition 6.8 we deduce that g is a strong negation. From the just proved implication $(ii) \Longrightarrow (i)$ and the uniqueness of the representation of (U,N)-implications generated from strong negations we get the claim. □

Corollary 6.10 (cf. [9, Proposition 12])**.** *Let N be a strong negation and let U be the disjunctive right-continuous idempotent uninorm obtained from N. Then the corresponding (U,N)- and RU-implications are identical, i.e., $I_{U,N} = I_U$.*

Remark 6.11. It immediately follows that the RU-implication given in Example 5.9(ii) is a (U,N)-implication, while the RU-implication presented in Example 5.9(iii) is not.

Let us denote by

- $\mathbb{I}_{\mathbb{U}_{\mathbf{I}^*}}$ – the family of all RU-implications generated from uninorms in $\mathcal{U}_{\mathbf{Idem}}$ whose generator is a strong negation.
- $\mathbb{I}_{\mathbb{U}_{\mathbf{I^*d}_N},N}$ – the family of all (U,N)-implications obtained from right-continuous disjunctive idempotent uninorms, whose generator g is a strong negation N and this N.

Using the above notations, the presented results can be summarized as follows:

$$\mathbb{I}_{\mathbb{U}_\mathbf{I}} \cap \mathbb{I}_{\mathbb{U},\mathbb{N}_\mathbb{C}} = \mathbb{I}_{\mathbb{U}_{\mathbf{I}^*}} = \mathbb{I}_{\mathbb{U}_{\mathbf{I^*d}_N},N}.$$

6.4. Intersection between $\mathbb{I}_{\mathbb{U}_\mathbf{M}}$ and $\mathbb{I}_{\mathbb{U}_\mathbf{R}}$

From Proposition 6.1, we know that $\mathbb{I}_{\mathbb{U}_\mathbf{R}} \subset \mathbb{I}_{\mathbb{U},\mathbb{N}}$, while from Sect. 6.1 we know that $\mathbb{I}_{\mathbb{U},\mathbb{N}} \cap \mathbb{I}_{\mathbb{U}_\mathbf{M}} = \emptyset$. Hence

$$\mathbb{I}_{\mathbb{U}_\mathbf{M}} \cap \mathbb{I}_{\mathbb{U}_\mathbf{R}} = \emptyset.$$

6.5. Intersection between $\mathbb{I}_{\mathbb{U}_\mathbf{M}}$ and $\mathbb{I}_{\mathbb{U}_\mathbf{I}}$

From Examples 5.5(ii) and 5.9(i) with $e = 0.5$, we see that $\mathbb{I}_{\mathbb{U}_\mathbf{M}} \cap \mathbb{I}_{\mathbb{U}_\mathbf{I}} \neq \emptyset$. In fact, since $\mathcal{U}_{\mathbf{Min}} \bigcap \mathcal{U}_{\mathbf{Idem}} = \mathcal{U}_{I,G_c}$ (see Remark 3.10(iii)), it can be easily seen that

$$\mathbb{I}_{\mathbb{U}_\mathbf{M}} \cap \mathbb{I}_{\mathbb{U}_\mathbf{I}} = \mathbb{I}_{\mathbb{U}_{\mathbf{I}_*}},$$

where $\mathbb{I}_{\mathbb{U}_{\mathbf{I}_*}}$ denotes the family of RU-implications generated from uninorms in $\mathcal{U}_{I,G_c}$.

6.6. Intersection between $\mathbb{I}_{\mathbb{U}_\mathbf{R}}$ and $\mathbb{I}_{\mathbb{U}_\mathbf{I}}$

From Proposition 5.12(i), we see that if $I \in \mathbb{I}_{\mathbb{U}_\mathbf{R}}$, then $I(x,x) = e$, for all $x \in]0,1[$. If $I \in \mathbb{I}_{\mathbb{U}_\mathbf{I}}$ then, from Theorem 5.8 and since g is decreasing and $e \in]0,1[$, there exists $x > e = g(e) \geq g(x)$, i.e., $I(x,x) \neq e$. Hence

$$\mathbb{I}_{\mathbb{U}_\mathbf{R}} \cap \mathbb{I}_{\mathbb{U}_\mathbf{I}} = \emptyset.$$

Remark 6.12. From the listed examples the following observations can be made.

(i) The (U,N)-implication $I_{U_{\mathbf{LK}},N_\mathbf{C}}$ in Example 4.5(i) shows that $\mathbb{I}_{\mathbb{U},\mathbb{N}_\mathbb{C}} \supsetneq \mathbb{I}_{\mathbb{U}_{\mathbf{I}^*}} \cup \mathbb{I}_{\mathbb{U}_\mathbf{R}}$.
(ii) The RU-implication $I_{U_{N_\mathbf{K}}}$ from Example 5.9(iii) shows that $\mathbb{I}_{\mathbb{U}_\mathbf{I}} \supsetneq \mathbb{I}_{\mathbb{U}_{\mathbf{I}^*}} \cup \mathbb{I}_{\mathbb{U}_{\mathbf{I}_*}}$.
(iii) The RU-implication $I_{U_{\mathbf{LK}}}$ from Example 5.5(i) shows that $\mathbb{I}_{\mathbb{U}_\mathbf{M}} \supsetneq \mathbb{I}_{\mathbb{U}_{\mathbf{I}_*}}$.
(iv) Let us consider the idempotent uninorm $U_\mathbf{M}$ from the class $\mathcal{U}_{\mathbf{Max}}$ and the discontinuous Gödel negation

$$N_\mathbf{D}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0, \end{cases} \qquad \text{for all } x \in [0,1].$$

Then the (U,N)-implication obtained from them is given by

$$I_{U_{\mathbf{M}},N_{\mathbf{D}}}(x,y) = \begin{cases} 0, & \text{if } y \leq 0.5, \\ 1, & \text{if } x = 0, \\ y, & \text{otherwise,} \end{cases} \qquad \text{for all } x, y \in [0,1].$$

Clearly, $I_{U_{\mathbf{M}},N_{\mathbf{D}}} \in \mathbb{I}_{\mathbb{U},\mathbb{N}} \setminus \mathbb{I}_{\mathbb{U},\mathbb{N}_{\mathbb{C}}}$. Moreover, it is not an RU-implication generated from any idempotent uninorm since $I_{U_{\mathbf{M}},N_{\mathbf{D}}}(0.5, 0.5) = 0$, while $I_U(0,5,0,5) \geq 0.5$ for any idempotent uninorm U.

The main results presented in this section are also diagrammatically represented in Figure 1. We also see that this discussion leaves us with the following open problem.

Problem 6.13. Is the intersection $(\mathbb{I}_{\mathbb{U},\mathbb{N}} \setminus \mathbb{I}_{\mathbb{U},\mathbb{N}_{\mathbb{C}}}) \cap \mathbb{I}_{\mathbb{U}_{\mathbf{I}}}$ non-empty? If yes, then characterize this intersection.

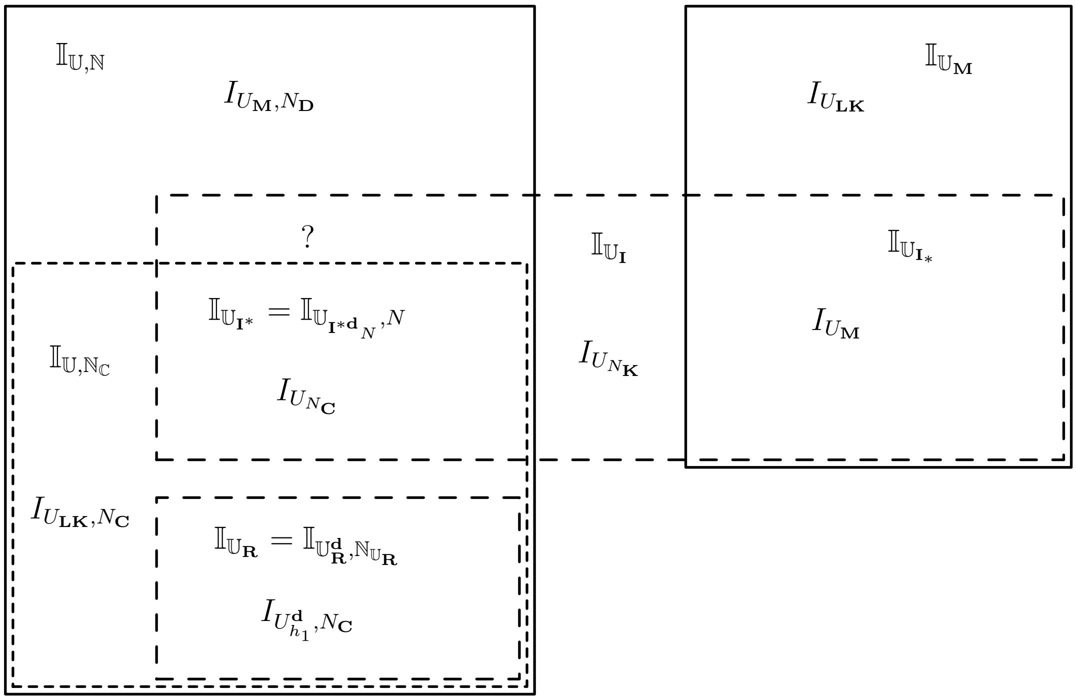

Figure 1: Intersections between some families of (U,N)- and RU-implications

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