PRESERVATION OF THE EXCHANGE PRINCIPLE UNDER LATTICE OPERATIONS ON FUZZY IMPLICATIONS

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Summary

In this work, we solve an open problem related to the exchange principle of fuzzy implications [Problem 3.1, Fuzzy Sets and Systems 261(2015) 112-123]. We show that two important generalizations of the exchange principle, namely, the generalized exchange principle(GEP) and the mutual exchangeability(ME) are sufficient conditions for the solution of the problem. We also show that, under some conditions, these are necessary too. Finally, we investigate the pairs (I, J) from different families of fuzzy implications such that the exchange principle is preserved under the join and meet operations.

Keywords: Fuzzy implication, the exchange principle, the generalized exchange principle, the mutual exchangeability, lattice operations.

1 INTRODUCTION

Fuzzy implications are one of the important logical connectives in fuzzy logic. These operators generalize the classical implication from $\{0, 1\}$-setting to the $[0, 1]$- setting. They are defined as follows:

Definition 1.1 ([1], Definition 1.1.1). *A function $I\colon [0, 1]^2 \longrightarrow [0, 1]$ is called a* fuzzy implication *if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, the following conditions:*

$$\text{if } x_1 \le x_2, \text{ then } I(x_1, y) \ge I(x_2, y), \tag{I1}$$

$$\text{if } y_1 \le y_2, \text{ then } I(x, y_1) \le I(x, y_2), \tag{I2}$$

$$I(0, 0) = 1,\ I(1, 1) = 1,\ I(1, 0) = 0. \tag{I3}$$

Let $\mathbb{I}$ denote the set of all fuzzy implications defined on $[0, 1]$. Fuzzy implications have many applications in fuzzy logic, approximate reasoning, decision making, fuzzy image processing, fuzzy control etc. Due to their applicational value it is always essential to generate fuzzy implications that satisfy various properties and functional equations.

However, it is always not straight-forward to generate fuzzy implications that preserve the desirable basic properties. For example, the lattice operations proposed by Bandler and Kohout as follows

$$(I \vee J)(x, y) := \max(I(x, y), J(x, y)),$$
$$(I \wedge J)(x, y) := \min(I(x, y), J(x, y)),$$

do not always preserve the exchange principle, which is defined as follows:

Definition 1.2 ([1], Definition 1.3.1). *A fuzzy implication I is said to satisfy the exchange principle* (EP)*, if for all $x, y, z \in [0, 1]$*

$$I(x, I(y, z)) = I(y, I(x, z)). \tag{EP}$$

For more about the lattice operations of fuzzy implications and the preservation of the basic properties, please see Chapter 6 of [1].

Thus this fact has become the main motivation to propose the following open problem.

Problem 1.3 ([3], Problem 3.1). *Characterize the subfamily of all fuzzy implications ((S,N)-implications, R-implications, etc.) which preserve* (EP) *for lattice operations.*

In this paper, we investigate the solutions of Problem 1.3 in a more general context, i.e., we attempt to characterize all fuzzy implications which preserve (EP) under the lattice operations.

Towards this end, in Section 2, we present some examples of $I, J \in \mathbb{I}$ such that $I \vee J$ and $I \wedge J$ preserve (EP) and investigate some basic conditions for a pair (I, J) to satisfy the same. We also recall some important generalizations of the exchange principle, viz., the generalized exchange principle(GEP) and the mutual exchangeability(ME). Following this, in Section 3, we show that either of (GEP) or (ME) is a sufficient condition for $I \vee J$ and $I \wedge J$ to preserve (EP). Later on, in Section 4, we show that the properties (GEP) and (ME) are also necessary under some conditions, namely, the Lattice Exchangeability Inequlaities (LEI). Finally, we present some results pertaining to the solutions of fuzzy implications I, J that satisfy (ME) and (GEP) separately in Sections 5 and 6, respectively.

2 PRELIMINARIES

In this section, we first show that there exist solutions of Problem 1.3. Following this, we investigate the basic characterizations of pairs (I, J) of fuzzy implications that become the solutions of Problem 1.3. Finally, we recall two important generalizations of (EP), namely, (GEP) and (ME), that will be helpful in obtaining the solutions of Problem 1.3.

Example 2.1. *(i) Let $I, J \in \mathbb{I}$ satisfy* (EP) *and $I \leq J$ under the usual point-wise ordering of functions. Clearly, $I \vee J = J$ and $I \wedge J = I$, which satisfy* (EP)*. Thus when I, J are comparable, $I \vee J$ and $I \wedge J$ always preserve* (EP)*.*

(ii) Let $I, J \in \mathbb{I}$ be defined as follows:

$$I(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ \sin(\frac{\pi y}{2}), & \text{if } x > 0, \end{cases}$$

$$\text{and } J(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ y^2, & \text{if } x > 0. \end{cases}$$

Then it is easy to see that the implications $I, J, I \vee J$, and $I \wedge J$ satisfy (EP)*. Note also that I, J are not comparable.*

In fact, one can generalize Example 2.1(ii) to obtain further solutions of Problem 1.3 as in the following.

Remark 2.2. *Let $I, J \in \mathbb{I}$ be defined as follows:*

$$I(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(y), & \text{if } x > 0, \end{cases}$$

$$\text{and } J(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ \psi(y), & \text{if } x > 0, \end{cases}$$

where $\varphi, \psi : [0, 1] \longrightarrow [0, 1]$ are increasing bijections such that $\varphi(0) = 0 = \psi(0)$ and $\varphi(1) = 1 = \psi(1)$. Then it is easy to see that the implications $I, J, I \vee J, I \wedge J$ satisfy the exchange principle. In the case, if φ, ψ are incomparable then I, J are also incomparable.

From Example 2.1, it follows that lattice operations of comparable fuzzy implications always preserve (EP) and there exist some incomparable fuzzy implications whose lattice operations also preserve the same.

Now, in the following we present some important results that will be useful in the investigations of pairs (I, J) of fuzzy implications such that $I \vee J$ and $I \wedge J$ preserve (EP).

Proposition 2.3 ([1], Propositions 7.2.15 and 7.2.26)**.** *For a function $I\colon [0, 1]^2 \to [0, 1]$ the following statements are equivalent:*

(i) I is increasing in the second variable, i.e., I satisfies (I2)*.*

(ii) I satisfies $I(x, \min(y, z)) = \min(I(x, y), I(x, z))$ for all $x, y, z \in [0, 1]$.

(iii) I satisfies $I(x, \max(y, z)) = \max(I(x, y), I(x, z))$ for all $x, y, z \in [0, 1]$.

From the above result the following Lemma follows directly.

Lemma 2.4. *Let $I, J \in \mathbb{I}$ satisfy* (EP)*. Then the following statements are equivalent:*

(i) $A_i(I, J)$ satisfies (EP)*, where $A_1(I, J) = I \wedge J$ and $A_2(I, J) = I \vee J$.*

(ii) $A_i(I(x, I(y, z)), I(x, J(y, z)), J(x, I(y, z)), J(x, J(y, z))) = A_i(I(y, I(x, z)), I(y, J(x, z)), J(y, I(x, z)), J(y, J(x, z)))$, for $i = 1, 2$, where $A_1 = \min$ and $A_2 = \max$.

In the following, we recall two important generalizations of (EP) proposed in different contexts, which play an important role in the sequel.

Definition 2.5 (cf. [4], Proposition 5.5)**.** *A pair (I, J) of fuzzy implications is said to satisfy the* generalized exchange principle (GEP)*, if for all $x, y, z \in [0, 1]$,*

$$\left.\begin{aligned} I(x, J(y, z)) = I(y, J(x, z)), \\ J(x, I(y, z)) = J(y, I(x, z)). \end{aligned}\right\} \qquad \text{(GEP)}$$

Remark 2.6. *Note that, in the original definition of* (GEP) *in [4], the pair (I, J) satisfies* (GEP) *if only the first of the above two conditions, viz., $I(x, J(y, z)) = I(y, J(x, z))$, is true. In that sense, given $I, J \in \mathbb{I}$, our definition requires both the pairs (I, J) and (J, I) to satisfy* (GEP)*. However, to avoid cumbersome repetitions, we continue to consider the definition given in Definition 2.5 in this work.*

Example 2.7. *Let $I, J \in \mathbb{I}$ be defined as follows:*

$$I(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ y^3, & \text{if } x > 0, \end{cases}$$

$$\text{and } J(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ y^4, & \text{if } x > 0. \end{cases}$$

Then it is easy to see that the pair (I, J) *satisfies* (GEP).

Definition 2.8 ([8], Definition 3.9)**.** *A pair (I, J) of fuzzy implications is said to be* mutually exchangeable*, if for all* $x, y, z \in [0, 1]$*,*

$$I(x, J(y, z)) = J(y, I(x, z)). \quad \text{(ME)}$$

From Remark 3.10 in [7], it follows that (GEP) is different from (ME).

3 SUFFICIENT CONDITIONS ON I, J SUCH that $I \vee J, I \wedge J$ PRESERVE (EP)

In this section, we show that either of (GEP) and (ME) is a sufficient condition for a pair (I, J) to be a solution of Problem 1.3.

Theorem 3.1. *Let* $I, J \in \mathbb{I}$ *satisfy* (EP)*. If the pair* (I, J) *satisfies either* (GEP) *or* (ME)*, then both* $I \vee J$ *and* $I \wedge J$ *satisfy* (EP)*.*

Proof. Let $I, J \in \mathbb{I}$ satisfy (EP).

(i) Let the pair (I, J) satisfy (GEP). Let $K_1 = I \vee J$ amd $x, y, z \in [0, 1]$. Now, from (EP), (GEP) of I, J and Lemma 2.4, it follows that

$$\begin{aligned} K_1(x, K_1(y, z)) &= \max(I(x, I(y, z)), I(x, J(y, z)), \\ &\quad J(x, I(y, z)), J(x, J(y, z))) \\ &= \max(I(y, I(x, z)), I(y, J(x, z)), \\ &\quad J(y, I(x, z)), J(y, J(x, z))) \\ &= K_1(y, K_1(x, z)), \end{aligned}$$

or equivalently, $K_1 = I \vee J$ satisfies (EP). Similarly, one can show that $I \wedge J$ also satisfies (EP).

(ii) Let the pair (I, J) satisfy (ME). Now, from (ME) it follows that

$$I(y, J(x, z)) = J(x, I(y, z)), \qquad x, y, z \in [0, 1].$$

Now let $x, y, z \in [0, 1]$. Then, once again, by using (EP) and (ME) of I, J and Lemma 2.4, we get

$$\begin{aligned} K_1(x, K_1(y, z)) &= \max(I(x, I(y, z)), I(x, J(y, z)), \\ &\quad J(x, I(y, z)), J(x, J(y, z))) \\ &= \max(I(y, I(x, z)), J(y, I(x, z)), \\ &\quad I(y, J(x, z)), J(y, J(x, z))) \\ &= K_1(y, K_1(x, z)). \end{aligned}$$

Thus $K_1 = I \vee J$ satisfies (EP). Similarly, one can show that $I \wedge J$ also satisfies (EP).

□

4 NECESSARY CONDITIONS ON I, J SUCH THAT $I \vee J, I \wedge J$ PRESERVE (EP)

In Theorem 3.1, we have shown that either (GEP) and (ME) of I, J is a sufficient condition for $I \vee J$ and $I \wedge J$ to preserve (EP). In this section, we show that these properties also become necessary under some conditions.

Towards this end, we define the following:

Definition 4.1. *Let* $I, J \in \mathbb{I}$ *satisfy* (EP)*. Then we say that the pair* (I, J) *satisfies* ***Lattice Exchangeable Inequalities*** *(LEI) if it satisfies the following inequalities: For all* $x, y, z \in [0, 1]$*,*

$$\max(I(x, I(y, z)), J(x, J(y, z))) \leq \max(I(x, J(y, z)), J(x, I(y, z))), \quad \text{(LEI-1)}$$

$$\min(I(x, I(y, z)), J(x, J(y, z))) \geq \min(I(x, J(y, z)), J(x, I(y, z))). \quad \text{(LEI-2)}$$

Example 4.2. *It can be easily verified that the pair of fuzzy implications* (I_1, J_1) *does satisfy the (LEI) inequalities, while the pair* (I_2, J_2) *does not:*

$$I_1(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ 0, & \text{if } x = 1 \text{ and } y = 0, \\ 0.4, & \text{otherwise}, \end{cases}$$

$$J_1(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1, \\ 0, & \text{if } x = 1 \text{ and } y = 0, \\ 0.6, & \text{otherwise}, \end{cases}$$

$$I_2(x, y) = \begin{cases} 1, & \text{if } x \leq 0.4, \\ y^2, & \text{if } x > 0.4, \end{cases}$$

$$J_2(x, y) = \begin{cases} 1, & \text{if } x \leq 0.6, \\ y^4, & \text{if } x > 0.6. \end{cases}$$

Lemma 4.3. *Let* $I, J \in \mathbb{I}$ *satisfy* (EP)*.* (LEI-1) *is equivalent to* (LEI-1')*:*

$$\max(I(x, I(y, z)), J(x, J(y, z))) \leq \max(I(y, J(x, z)), J(y, I(x, z))), \quad \text{(LEI-1')}$$

and (LEI-2) *is equivalent to* (LEI-2')*:*

$$\min(I(x, I(y, z)), J(x, J(y, z))) \geq \min(I(y, J(x, z)), J(y, I(x, z))). \quad \text{(LEI-2')}$$

Proof. Let $I, J \in \mathbb{I}$ satisfy (EP). In the following we show that (LEI-1) is equivalent to (LEI-1'), since the proof for the other can be similarly obtained.

(LEI-1) $\Longrightarrow$ (LEI-1'): Let the pair (I, J) satisfy (LEI-1). From (LEI-1), one can always write

$$\max(I(y, I(x, z)), J(y, J(x, z))) \leq \max(I(y, J(x, z)), J(y, I(x, z))). \quad (1)$$

Since I, J satisfy (EP), the inequality (1) becomes

$$\max(I(x, I(y, z)), J(x, J(y, z))) \leq \max(I(y, J(x, z)), J(y, I(x, z))),$$

which is equal to (LEI-1').

(LEI-1') $\Longrightarrow$ (LEI-1): Follows similarly. □

Theorem 4.4. *Let $I, J, I \vee J, I \wedge J \in \mathbb{I}$ satisfy* (EP). *If the pair (I, J) satisfies (LEI) then it also satisfies the following equations:*

$$\max(I(x, J(y, z)), J(x, I(y, z))) = \max(I(y, J(x, z)), J(y, I(x, z))), \quad (2)$$

$$\min(I(x, J(y, z)), J(x, I(y, z))) = \min(I(y, J(x, z)), J(y, I(x, z))). \quad (3)$$

Proof. Let $I, J, I \vee J, I \wedge J \in \mathbb{I}$ satisfy (EP). Let the pair (I, J) also satisfy (LEI). In the following, we prove only the equation (2), since the proof for (3) can be obtained similarly. Let $x, y, z \in [0, 1]$. Since $I \vee J$ satisfies (EP), from Lemma 2.4, the pair (I, J) satisfies the equation in Lemma 2.4(ii), with $i = 2$. Thus we have, for all $x, y, z \in [0, 1]$,

$$\max\Big\{I(x, I(y, z)), I(x, J(y, z)), J(x, I(y, z)), J(x, J(y, z))\Big\} = \max\Big\{I(y, I(x, z)), I(y, J(x, z)), J(y, I(x, z)), J(y, J(x, z))\Big\}. \quad (*)$$

Since the pair (I, J) also satisfies (LEI), from (LEI-1), we get

$$\textbf{L.H.S.} \text{ of } (*) = \max(I(x, J(y, z)), J(x, I(y, z))) = \textbf{L.H.S.} \text{ of } (2).$$

Since I, J satisfy (EP) from Lemma 4.3, it follows that (LEI-1) is equivalent to (LEI-1'), from whence we obtain that

$$\textbf{R.H.S.} \text{ of } (*) = \max(I(y, J(x, z)), J(y, I(x, z))) = \textbf{R.H.S.} \text{ of } (2).$$

□

Let $I \vee J, I \wedge J$ preserve (EP) and I, J satisfy (LEI). Then from Theorem 4.4, it follows that the pair (I, J) satisfies (2) and (3). In other words, this fact implies that the solutions of the equations (2), (3) also become the solutions of Problem 1.3. In the following, we investigate the solutions of (2) and (3). Before doing so, we recall the following important result which is useful in the sequel.

Lemma 4.5 ([2], page. 366)**.** *Let L be any distributive lattice. Let $a, b, c \in L$ satisfy*

$$\max(a, b) = \max(a, c), \quad (4)$$
$$\min(a, b) = \min(a, c). \quad (5)$$

Then $b = c$.

Remark 4.6. *Since $([0, 1], \leq, \vee, \wedge)$ is also a distributive lattice, Lemma 4.5 is also true for all $a, b, c \in [0, 1]$.*

Remark 4.7. *Let $a, b, c, d \in [0, 1]$ satisfy*

$$\max(a, b) = \max(c, d), \quad (6)$$
$$\min(a, b) = \min(c, d). \quad (7)$$

Then either $a = c$ or $a = d$. Further,

(i) if $a = c$ then $b = d$.

(ii) if $a = d$ then $b = c$.

Lemma 4.8. *Let the pair $(I, J) \in \mathbb{I}$ satisfy the equations* (2) *and* (3). *Then it satisfies either* (GEP) *or* (ME).

Proof. Follows from Remark 4.7. □

Theorem 4.9. *Let $I, J, I \vee J, I \wedge J \in \mathbb{I}$ satisfy* (EP) *and let the pair (I, J) satisfy (LEI). Then the pair (I, J) satisfies either* (GEP) *or* (ME).

Proof. Follows from Theorem 4.4 and Lemma 4.8. □

Theorem 4.10. *Let $I, J \in \mathbb{I}$ satisfy* (EP) *and (LEI). Then the following statements are equivalent:*

(i) $I \vee J, I \wedge J$ satisfy (EP).

(ii) The pair (I, J) satisfies either (GEP) *or* (ME).

Proof. Follows from Theorems 4.9 and 3.1. □

Remark 4.11. *Let $I, J \in \mathbb{I}$ satisfy* (EP). *If the pair (I, J) satisfies (LEI) then from Theorem 4.10, it follows that $I \vee J, I \wedge J$ satisfy* (EP). *However, the converse need not be true. For example, take $I = I_2$ and $J = J_2$ of Example 4.2.*

Since (ME) and (GEP) play an important role in the characterizations of solutions of Problem 1.3, it is of interest to know the pairs (I, J) of fuzzy implications that do satisfy (ME) or (GEP). We take up this investigation in the following sections.

5 PAIRS OF FUZZY IMPLICATIONS SATISFYING (ME)

Due to the variety of fuzzy implications and the complexity of the functional equation, it is not an easy task to investigate the pairs of fuzzy implications that do satisfy (ME). However, Vemuri [5] has investigated the solutions of (ME), but only for the families of fuzzy implications whose characterizations are well established. In the following, we recall some of the most important results that give the solutions of (ME) and thus the solutions of Problem 1.3. For details about definitions, properties, characterizations and representations of different families of fuzzy implications, please see [1].

5.1 (S, N)-implications satisfying (ME)

Proposition 5.1 ([5], Proposition 4.1)**.** *Let I be an (S, N)-implication whose negation N has trivial range, i.e., $N(x) \in \{0, 1\}$ for all $x \in [0, 1]$. Then I satisfies* (ME) *with every $J \in \mathbb{I}$.*

From Proposition 5.1, it follows that if at least one of I, J is an (S, N)-implication with trivial range negation N then the pair (I, J) satisfies (ME) and hence becomes the solution of Problem 1.3.

In the case if I, J are two (S, N)-implications with continuous negations and satisfy (ME) then as the following result suggests the two t-conorms involved in the definition must be the same.

Theorem 5.2 ([7], Theorem 6.7)**.** *Let $I(x, y) = S_1(N_1(x), y), J(x, y) = S_2(N_2(x), y)$ be two (S, N)-implications such that N_1, N_2 are continuous negations. Then the following statements are equivalent:*

(i) The pair (I, J) satisfies (ME).

(ii) $S_1 = S_2$.

5.2 R-implications satisfying (ME)

Theorem 5.3 ([5], Theorem 5.1)**.** *Let $I = I_{T_1}$ and $J = I_{T_2}$ be two R-implications generated from left-continuous t-norms T_1, T_2 respectively. Then the following statements are equivalent:*

(i) The pair (I, J) satisfies (ME).

(ii) $I = J$.

Before presenting the solutions of f and g-implications that satisfy (ME), we recall two important definitions that will be useful in the sequel.

Definition 5.4 ([6, 7])**.** *For any $I, J \in \mathbb{I}$, we define $I \circledast J \colon [0, 1]^2 \to [0, 1]$ as*

$$(I \circledast J)(x, y) = I(x, J(x, y)), \qquad x, y \in [0, 1].$$

Definition 5.5 ([8], Definition 5.1)**.** *Let $I \in \mathbb{I}$. For any $n \in \mathbb{N}$, we define the n-th power of I w.r.t. the binary operation $\circledast$ as follows: For $n = 1$,*

$$I_{\circledast}^{[n]} = I,$$

and for $n \geq 2$,

$$I_{\circledast}^{[n]}(x, y) = I\left(x, I_{\circledast}^{[n-1]}(x, y)\right) = I_{\circledast}^{[n-1]}(x, I(x, y)),$$

for all $x, y \in [0, 1]$.

5.3 f-implications satisfying (ME)

Theorem 5.6 ([5], Theorem 6.6)**.** *Let I, J be two f-implications. Then the following statements are equivalent:*

(i) The pair (I, J) satisfies (ME).

(ii) $J = I_{\circledast}^{[n]}$ for some $n \in \mathbb{N}$.

5.4 g-implications satisfying (ME)

Theorem 5.7 ([5], Theorem 7.4)**.** *Let I, J be two g-implications. Then the following statements are equivalent:*

(i) The pair (I, J) satisfies (ME).

(ii) $J = I_{\circledast}^{[n]}$ for some $n \in \mathbb{N}$.

6 PAIRS OF FUZZY IMPLICATIONS SATISFYING (GEP)

In this section, we attempt to find the pairs (I, J) of fuzzy implications that do satisfy (GEP). Once again keeping the complexity of the functional equation (GEP) in mind, we restrict ourselves to do so for the families (S, N)-, R-, f- and g- of fuzzy implications.

Note that all of these families of fuzzy implications satisfy the following left neutrality property (NP):

Definition 6.1 (cf. [1], Definition 1.3.1)**.** *An $I \in \mathbb{I}$ is said to satisfy the left neutrality property* (NP) *if*

$$I(1, y) = y, \qquad y \in [0, 1]. \tag{NP}$$

Lemma 6.2. *Let $I, J \in \mathbb{I}$ satisfy* (NP)*. If the pair (I, J) satisfies* (GEP) *then $I = J$.*

Proof. The substitution of $x = 1$ in (GEP) and (NP) of $I, J \in \mathbb{I}$ will yield $I = J$. □

From the above results, it is clear that if $I, J \in \mathbb{I}$ belong to one of the following families of fuzzy implications, viz.,(S, N)-, R-, f-, g- implications, and satisfy (GEP), then $I = J$ and hence it trivially follows that both $I \vee J$ and $I \wedge J$ preserve (EP).

7 CONCLUSIONS

In this paper, we have investigated the solutions of an open problem [Problem 3.1, Fuzzy Sets and Systems 261(2015) 112-123] related to the preservation of the exchange principle (EP) of fuzzy implications under lattice operations. Our study has shown the importance of two of the generalizations of (EP), viz., (GEP) and (ME) in obtaining the solutions of the problem.

While (GEP), (ME) are independently sufficient for the lattice operations of fuzzy implications to preserve (EP), these conditions are not necessary. However, the newly proposed pair of inequalities, namely the Lattice Exchangeable Inequalities (LEI-1) and (LEI-2) make (GEP) and (ME) also a necessity for a pair of fuzzy implications to be a solution of Problem 1.3.

Since the pairs (I, J) of fuzzy implications satisfying either (GEP) or (ME) are the most general solutions of the problem, we have investigated them but for some well known families of fuzzy implications. However, this problem has to be investigated in the most general setting. Further, the solutions of (LEI) are worthy of study. We intend to explore these in detail in the near future.

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