Sobolev Space On Riemannian Manifolds

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Declaration

This thesis entitled **"Sobolev Space On Riemannian Manifolds"** submitted by me to the Indian Institute of Technology, Hyderabad for the award of the degree of Master of Science in Mathematics contains a literature survey of the work done by some authors in this area. The work presented in this thesis has been carried out under the supervision of **Dr. Bhakti Bhusan Manna**, Department of Mathematics, Indian Institute of Technology, Hyderabad, Telangana.

I hereby declare that, to the best of my knowledge, the work included in this thesis has been taken from the books ([1] [2] [3] [4] [5] [6] [7]) mentioned in the References. No new results have been created in this thesis. I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

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Approval Sheet

This Thesis entitled **Sobolev Space On Riemannian Manifolds** by **Arnab Kayal** is approved for the degree of Master of Science (Mathematics) from IIT Hyderabad.

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I would also like to extend my gratitude to my family and friends for helping me in every possible way and encouraging me during this Programme. Above all, I thank, The Almighty, for all his blessings.

Dedication

Dedicated to my parents

Abstract

The main aim of this thesis is to study the theory of Sobolev spaces on Riemannian manifolds. This thesis is divided into three parts, 1st we will learn Riemannian Geometry then Sobolev space on $\mathbb{R}^n$ at last we will define Sobolev space on Riemannian Manifolds and we will learn some properties and embeddings of Sobolev space on Riemannian Manifolds.

The Sobolev space over $\mathbb{R}^n$ is a vector space of functions that have weak derivatives. Motivation for studying these spaces is that solutions of partial differential equations, when they exist, belong naturally to Sobolev spaces. The functions of Sobolev space is not easy to handle, we shall approximate this functions by smooth functions. We have calculated some inequalities on Sobolev space. With the help of this inequalities we will embedded the Sobolev space in some L^p space and Hölder continuous space. Similarly on the manifold using covarient derivative we define Sobolev Space Over Riemannian Manifold. Riemannian manifolds are natural extensions of Euclidean space, the naive idea that what is valid for Euclidean space must be valid for manifolds is false. But Sobolev embedding theorem for $\mathbb{R}^n$ does hold for compact manifolds.

Contents

Chapter 0

List of symbols

- A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is called a multi-index of order

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

- $D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}$
- $C(U) = \{u : u \to \mathbb{R} \mid u \; continuous\}$
- $C(\overline{U}) = \{u \in C(U) \mid u \; uniformly \; continuous\}$
- $C^k(U) = \{u : U \to \mathbb{R} \;\mid u$ is k times continuously differentiable$\}$
- $C^k(\overline{U}) = \{u \in C^k(U) \mid D^\alpha u$ is uniformly continuous for all $|\alpha| \le k\}$
- $C^\infty(U) = \{u : U \to \mathbb{R} \mid u$ is indefinitely differentiable.$\}$
- $C^\infty(\overline{U}) = \cap_{k=0}^\infty C^k(\overline{U})$
- $C_c(U),\; C_c^k(U)$, etc denote the functions in $C(U),\; C^k(U)$ with compact support.
- $V \subset\subset U$ means $V \subset K \subset U$, where K is compact (compactly contained)
- $L^p_{loc}(U) = \{u : U \to \mathbb{R} \mid u \in L^p(V) \; for \; each \; V \subset\subset U\}$
- $W^{k,p}(U)$, $H^k(U)$, etc denotes Sobolev spaces: see Section 6.2

PART 1

Riemannian Geometry

Chapter 1

Introduction

1.1 Differentiable manifold

Definition 1.1.1 (Differentiable manifold)**.** *A Differentiable manifold of dimension n is a set M and a family of injective mappings $x_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ of open sets U_α of $\mathbb{R}^n$ into M such that:*

1. $\bigcup_\alpha x_\alpha(U_\alpha) = M$.

2. *for any pair α, β, with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W(\neq \phi)$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings*
   $x_\alpha^{-1} \circ x_\beta : X_\beta^{-1}(W) \to X_\alpha^{-1}(W)$ *are differentiable* ($Figure$ 1.1).

3. *The family $\mathcal{A} = \{(U_\alpha, x_\alpha)\}$ is maximal relative to the conditions (1) and (2), meaning that if $x_0 : U_0 \subset \mathbb{R}^n \to M$ is a map such that $x_0^{-1} \circ x$ and $x \circ x_0^{-1}$ are differentiable for all x in $\mathcal{A}$, then (U_0, x_0) is in $\mathcal{A}$.*

The pair (U_α, x_α) (or the mapping x_α) with $p \in x_\alpha(U_\alpha)$ is called a parametrazation, (or *system of coordinates*) of M at p; $x_\alpha(U_\alpha)$ is then called a coordinate neighbourhood at p. A family $\{(U_\alpha, x_\alpha)\}$ satisfying (1) and (2) is called a differentiable structure(or Atlas) on M.

Remark. A differentiable structure on a set M induces a topology on M. Define $A \subset M$ to be *open set* in M iff ${x_\alpha}^{-1}(A \cap x_\alpha(U_\alpha))$ is an open set in $\mathbb{R}^n$ for all α. The empty set and M are open sets, the union of open sets is again an open set and that the finite intersection of open sets remain a open set. The topology defined in such a way that the sets $x_\alpha(U_\alpha)$ are open and that mapping x_α is continuous.

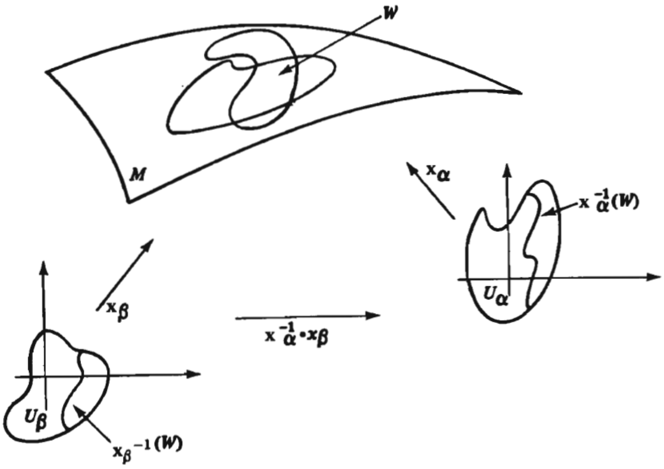

Figure 1.1:

Example 1.1.1. *The Euclidean space* $\mathbb{R}^n$*, The differentiable structure given by* $\{\mathbb{R}^n, id\}$.

1.2 Differentiable Maps

Definition 1.2.1. *Let* M_1^n *and* M_2^m *be differentiable manifolds. A mapping* $\phi : M_1 \to M_2$ *is differentiable at* $p \in M_1$ *if given a parametrization* $y : V \subset \mathbb{R}^m \to M_2$ *at* $\phi(p)$ *there exists a parametrazation* $x : U \subset \mathbb{R}^n \to M_1$ *at* p *such that* $\phi(x(U)) \subset y(V)$ *and the mapping*

$$y^{-1} \circ \phi \circ x : U \subset \mathbb{R}^n \to \mathbb{R}^m \tag{1.1}$$

is differentiable at $x^{-1}(p)$ $(Figure 1.2)$*. The map* ϕ *is differentiable on an open set of* M_1 *if it is differentiable at all of the points of this open set.*

As coordinate changes are smooth, this definition is independent of the parametrizations chosen at $\phi(p)$ and p. The mapping (1.1) is called the *expression* of ϕ in the parametrazation x and y.

1.3 Tangent Space

We would like to extend the idea of tangent vector to the differentiable manifolds. For regular surface in $\mathbb{R}^3$, a tangent vector at a point p of the surface is defined as the "*velocity*" in $\mathbb{R}^3$ of a curve in the surface passing through p. Since we do not

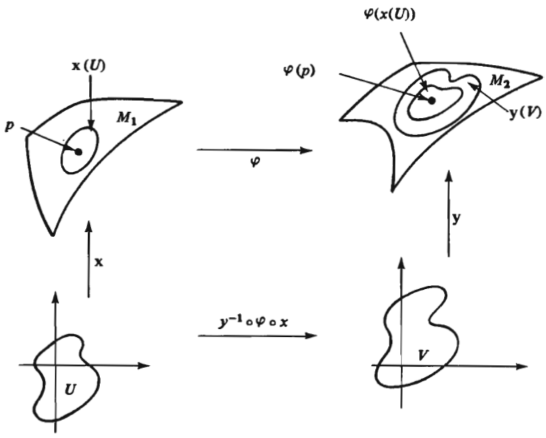

Figure 1.2:

have the support of the ambient space, we have to find a characteristic property of the tangent vector which will substitute for the idea of velocity.

Let $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^n$ be a differentiable curve in $\mathbb{R}^n$, with $\alpha(0) = p$. Write

$$\alpha(t) = (x_1(t), \ldots, x_n(t)),\ t \in (-\epsilon, \epsilon),\ \ (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Then $\alpha'(0) = (x_1'(0), \ldots, x_n'(0)) = v \in \mathbb{R}^n$. Now let f be a differentiable function defined in a neighborhood of p. We can restrict f to the curve α and express the directional derivative with respect to the vector $v \in \mathbb{R}^n$ as

$$\frac{d(f \circ \alpha)}{dt}\Big|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_{t=0} \frac{dx_i}{dt}\Big|_{t=0} = \Big(\sum_i x_i'(0)\frac{\partial}{\partial x_i}\Big) f$$

Therefore, the directional derivative with respect to v is an operator on differentiable functions that depends uniquely on v. This is the characteristics property that we are going to use to define tangent vectors on manifold.

Definition 1.3.1. *Let $\alpha : (-\epsilon, \epsilon) \to M$ be a differentiable curve on a smooth manifold M. Consider the set $\mathcal{D}$ of all functions $f : M \to \mathbb{R}$ that are differentiable at $\alpha(0) = p$ (i.e., C^∞ on a neighborhood of p). The tangent vector to the curve α at $t = 0$ is the operator $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ given by*

$$\alpha'(0)(f) = \frac{d(f \circ \alpha)}{dt}\Big|_{t=0},\ \ f \in \mathcal{D}$$

A tangent vector to M at p is a tangent vector at $t = 0$ to some differentiable curve $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$. The tangent space at p is the space T_pM of all tangent vectors at p.

Choosing a parametrization $x : U \to M^n$ at $p = x(0)$, we can express the function f and the curve α in this parametrazation by $f \circ x(q) = f(x_1, \dots, x_n)$, $q = (x_1, \dots, x_n) \in U$, and the curve $\alpha(t), \alpha : (-\epsilon, \epsilon) \to M$ $\alpha(0) = p$ by $\hat{\alpha}(t) = (x^{-1} \circ \alpha)(t) = (x_1(t), \dots, x_n(t))$, respectively. Therefore,

$$\alpha'(0)(f) = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = \frac{d}{dt}(f \circ x \circ x^{-1} \circ \alpha)\Big|_{t=0}$$

$$= \frac{d}{dt}(\hat{f}(x_1(t), ..., x_n(t)))\Big|_{t=0}, \hat{f} = f \circ x$$

$$= \sum_{i=1}^{n} \left(\frac{\partial \hat{f}}{\partial x_i}\right)_0 x_i'(0) = \left(\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right) x_i'(0)\right)(\hat{f})$$

Hence we can write the tangent vector $\alpha'(0)$ in the parametrazation x by

$$\alpha'(0) = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)_0 x_i'(0) \tag{1.2}$$

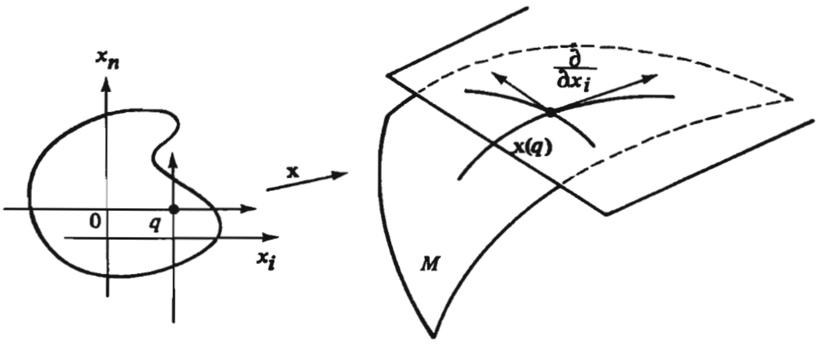

Figure 1.3:

Note: 1.3.1. *The tangent vector to the curve α at p only depends on the derivative of α in the local coordinate system.*

Note: 1.3.2. *Let* $\beta : (-\epsilon, \epsilon) \to U(\subset \mathbb{R}^n)$, *be a curve on* U, $\beta(0) = 0$, $\beta(x_i) = (0,..0,x_i,0,..0)$, $\alpha(x_i) = x \circ \beta(x_i)$. *then* $\alpha(x_i)$ *is the coordinate curve on* M $\alpha(0) = p$, $\alpha^{'}(0) = \big(\frac{\partial}{\partial x_i}\big)$ *by* (1.2). *Hence* $\left(\frac{\partial}{\partial x_i}\right)_0$ *is a tangent vector at* p *of the coordinate curves* $x_i \to x(0,..,0,x_i,0,..0)$ $(Figure\ 1.3)$.

Note: 1.3.3. $T_pM = \{\alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to M$, *differentiable curve with* $\alpha(0) = p\}$ *Define (i)* $\big(\alpha^{'}(0) + \beta'(0)\big)f = \alpha'(0)(f) + \beta^{'}(0)(f)$, $\alpha^{'}(0), \beta'(0) \in T_pM, f \in \mathcal{D}$. *(ii)* $\big(\lambda\alpha^{'}(0)\big)(f) = \lambda\big(\alpha^{'}(0)(f)\big)$, $\lambda \in \mathbb{R}$. *with this addition and multiplication* T_pM *is a vector space over* $\mathbb{R}$.

Note: 1.3.4. *Every element of* T_pM *can be written as linear combination of* $\big(\frac{\partial}{\partial x_i}\big)_0$, $i = 1, 2, ..., n$ *from* (1.2). *and* $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ *is linearly independent. So,* $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ *is a basis of* T_pM. *Hence* T_pM *is vector space of dimension* n.

Note: 1.3.5. *It is clear from the definition of* T_pM *the linear structure in* T_pM *defined above does not depend on the parametrazation* x. *The vector space* T_pM *is called the tangent space of* M *at* p.

Definition 1.3.2. *Let* $\phi : M_1 \to M_2$ *be a differentiable map between two smooth manifolds of dimension n and m respectively. For* $p \in M$, *the differential of* ϕ *at* p *is the map*

$$d\phi_p : T_pM_1 \to T_{\phi/(p)}M_2$$

given by $d\phi_p(v) = (\phi \circ \alpha)^{'}(0) = \beta^{'}(0)$ *where* $\alpha : (-\epsilon, \epsilon) \to M_1$ *is a curve satisfying* $\alpha(0) = p$ *and* $\alpha'(0) = v$, $\beta = \phi \circ \alpha$

Proposition 1.3.1. *The map* $d\phi_p : T_pM_1 \to T_{\phi(p)}M_2$ *defined above is a linear mapping does not depend on the choice of the curve* α $(Figure 1.4)$.

proof. Let $x : U \to M_1$ and $y : V \to M_2$ be parametrazation at p and $\phi(p)$ respectively ,such that $\phi(x(U)) \subset y(V)$. Now the expression of ϕ in these parametrization, We can write

$$y^{-1} \circ \phi \circ x(q) = \big(y_1(x_1, \ldots, x_n), \ldots, y_m(x_1, \ldots, x_n)\big)$$

$$q = (x_1, \ldots, x_n) \in U, (y_1, \ldots, y_m) \in V$$

Now the expression of α in the parametrization x We obtain

$$x^{-1} \circ \alpha(t) = (x_1(t), \ldots, x_n(t))$$

. Hence,

$$y^{-1} \circ \beta(t) = y^{-1} \circ \phi \circ \alpha = y^{-1} \circ \phi \circ x \circ x^{-1} \circ \alpha$$

$$= \big(y_1(x_1(t), \ldots, x_n(t)), \ldots, y_m(x_1(t), \ldots, x_n(t))\big)$$

$\beta^{'}(0)$ is the tangent vector at $\phi(p)$. The expression for $\beta^{'}(0)$ with respect to basis $\{(\frac{\partial}{\partial y_i})_0\}$ of $T_{\phi(p)}M_2$, associated to the parametrazation y, is given by

$$\beta^{'}(0) = \sum_{i=1}^{m} \frac{d}{dt}\big(y_i(x_1(t), \ldots, x_n(t))\big)_{t=0}\left(\frac{\partial}{\partial y_i}\right)_0$$

$$\beta^{'}(0) = \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n} x_j^{'}(0)\left(\frac{\partial y_i}{\partial x_j}\right)_p \right\}\left(\frac{\partial}{\partial y_i}\right)_0$$

where $x_j^{'}(0), j = 1, \ldots, n$ are the component of $v\,(= \alpha^{'}(0))$ in the local coordinate system. Hence $\beta^{'}(0)$ does not depends on The choice of γ as long as $\alpha^{'}(0) = v$

$$\beta^{'}(0) = \left(\sum_{j=1}^{n} \frac{\partial y_1}{\partial x_j} x_j^{'}(0), \ldots, \sum_{j=1}^{n} \frac{\partial y_m}{\partial x_j} x_j^{'}(0) \right)$$

$$\beta^{'}(0) = d\phi_p(v) = \left(\frac{\partial y_i}{\partial x_j}\right)(x_j^{'}(0))$$

$$i = 1, \ldots, m; j = 1, \ldots, n,$$

where $\left(\frac{\partial y_i}{\partial x_j}\right)$ denotes an $m \times n$ matrix and $x_j^{'}(0)$ denotes a column matrix with n elements. Therefore, $d\phi_p$ is a linear mapping of T_pM_1 into T_pM_2 whose matrix in the associated bases obtained from the parametrazation x and y is the matrix $\left(\frac{\partial y_i}{\partial x_j}\right)$.

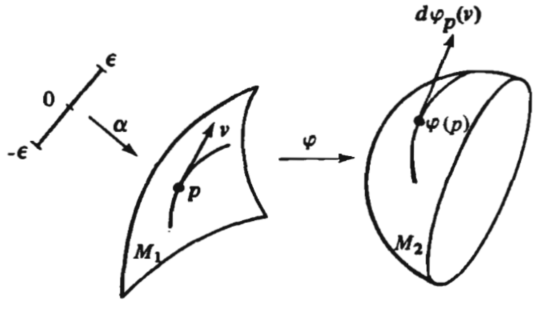

Figure 1.4:

Definition 1.3.3 (Diffeomorphism)**.** *Let M and N be differentiable manifolds. A mapping $\phi : M \to N$ is a diffeomorphism if it is differentiable, bijective, and it's inverse ϕ^{-1} is differentiable. ϕ is said to be local diffeomorphism at $p \in M$ if there exists neighbourhood U of p and V of $\phi(p)$ such that $\phi : U \to V$ is a diffeomorphism.*

Definition 1.3.4 (Alternate Definition of Differential of a map)**.** *Let $\phi : M_1 \to M_2$ be a differentiable map between two smooth manifolds of dimension n and m respectively. For $p \in M$, the differential of ϕ at p is the map*

$$d\phi_p : T_pM_1 \to T_{\phi(p)}M_2$$

given by

$$d\phi_p(\alpha^{'}(0))(f) = \alpha^{'}(0)(f \circ \phi)$$

where $\alpha : (-\epsilon, \epsilon) \to M_1$ is a curve satisfying $\alpha(0) = p$ and $f \in C^{\infty}(\phi(p))$.

Now,

$$d\phi_p(\alpha^{'}(0))(f) = \alpha^{'}(0)(f \circ \phi) = \frac{d}{dt}(f \circ \phi \circ \alpha(t)) \,|_{t=0}$$

$$= \frac{d}{dt}(f \circ \beta(t)) \,|_{t=0} = \beta^{'}(0)(f)$$

$$i.e., \; d\phi_p(\alpha^{'}(0)) = \beta^{'}(0) = (\phi \circ \alpha)^{'}(0)$$

So, the Definition 1.3.2. and Definition 1.3.4 are equivalent.

Proposition 1.3.2. *Let $\phi : M_1 \to M_2$ and $\psi : M_2 \to M_3$ be a differentiable map between differentiable manifolds then:*

1. $d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p$

2. $i_M : M \to M$, *identity map then* $d(i_M)_p = i_{T_pM}$

Proof 1. Let $\alpha^{'}(0) \in T_pM$, $f \in C^{\infty}(\psi \circ \phi(p))$

$$\begin{aligned}
&(d\psi_{\phi(p)} \circ d\phi_p)(\alpha^{'}(0))(f) \\
&= \big(d\psi_{\phi(p)}(d\phi_p(\alpha^{'}(0))\big)(f) \\
&= \big(d\phi_p(\alpha^{'}(0))\big)(f \circ \psi) \\
&= \alpha^{'}(0)(f \circ \psi \circ \phi) \\
&= d(\psi \circ \phi)_p(\alpha^{'}(0))(f)
\end{aligned}$$

$$i.e., \; d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p$$

Proof 2.

$$\begin{aligned}
&d(i_M)_p(\alpha^{'}(0))(f) \\
&= \alpha^{'}(0)(f \circ i_M) \\
&= \alpha^{'}(0)(f)
\end{aligned}$$

Therefore $d(i_M)_p$ is the identity map between the tangent spaces.

Proposition 1.3.3. *Let $\phi : M_1 \to M_2$ is a diffeomorphism then $d\phi_p : T_pM_1 \to T_{\phi(p)}M_2$ is an isomorphism $forall\, p \in M_1$.*

Proof. Let $\psi = \phi^{-1} : M_2 \to M_1$, $p \in M_1$.
We shall prove that $(d\phi_p)^{-1} = d\phi^{-1}_{\phi(p)}$
Now,

$$d\psi_{\phi(p)} \circ d\phi_p = d(\psi \circ \phi)_p = d(i_{M_1})_p = i_{T_pM_1}$$

$$d\phi_p \circ d\psi_{\phi(p)} = d(\phi \circ \psi)_{\phi(p)} = d(i_{M_2})_{\phi(p)} = i_{T_{\phi(p)}M_2}$$

Hence, $d\phi_p$ is an isomorphism for all $p \in M_1$.

Theorem 1.3.1. *Let $\phi : M_1^n \to M_2^n$ be a differentiable mapping and let $p \in M_1$ be such that $d\phi_p : T_pM_1 \to T_{\phi(p)}M_2$ is isomorphism. Then ϕ is a local diffeomorphism at p.*

Proof. Let $x : U_1 \to M_1$ and $y : V_2 \to M_2$ be parametrazation at p and $\phi(p)$ respectively ,such that $\phi(x(U_1)) \subset y(V_2)$. Let $x^{-1}(p) = q$. Now the expression of ϕ in these parametrization, We can write

$$\hat{\phi} = y^{-1} \circ \phi \circ x : U_1 \to V_2$$

$$\hat{\phi} = y^{-1} \circ \phi \circ x(q_1) = \big(y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n)\big)$$

$$q_1 = (x_1, \ldots, x_n) \in U_1, (y_1, \ldots, y_n) \in V_2$$

Since, $d\phi_p$ is is an isomorphism then (by Proposition 1.3.1.)

$$\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}(q) \neq 0$$

Then, by the inverse function theorem in $\mathbb{R}^n$, $\exists$ two open sets U and V of $\mathbb{R}^n$ such that $q \in U$, $\hat{\phi}(q) \in V$, $\hat{\phi}$ is one-one, $\hat{\phi}(U) = V$, $\hat{\phi}^{-1}$ exists and $\hat{\phi}^{-1}(V) = U$ and $\hat{\phi}^{-1} = x^{-1} \circ \phi^{-1} \circ y$ is differentiable in V.

Hence, ϕ is bijection between $x(U)$ and $y(V)$, and ϕ^{-1} is differeniable on $y(V)$. So, ϕ is a local diffeomorphism at p.

1.4 Immersions and Embeddings

Definition 1.4.1. *Let M^m and N^n be differentiable manifolds. A differentiable mapping $\phi : M \to N$ is said to be an* **immersion** *if $d\phi_p : T_pM \to T_{\phi(p)}N$ is injective for all $p \in M$. If in addition , ϕ is homeomorphism into $\phi(M) \subset N$, where $\phi(M)$ has the subspace topology induce from N, we say that ϕ is an embedding. If $M \subset N$ and the inclusion map $i : M \to N$ is an embedding, We say that M is a sub-manifold of N.*

It can be seen that if $\phi : M^m \to N^n$ is an immersion , then $m \leq n$; the difference $n - m$ is called the *codimension* of the immersion ϕ.

Definition 1.4.2 (Regular surface in $\mathbb{R}^n$)**.** *A subset $M^k \subset \mathbb{R}^n$ is a regular surface of dimension k, $k \leq n$ if for every $p \in M^k$ there exists a neighborhood V of p of $\mathbb{R}^n$ and a mapping $x : U \subset \mathbb{R}^k \to M \cap V$ of an open set $U \subset \mathbb{R}^k$ onto $M \cap V$ such that.*

1. *x is differentiable homeomorphism.*

2. *$(dx_q) : \mathbb{R}^k \to \mathbb{R}^n$ is injective for all $q \in U$*

Example 1.4.1. *The curve* $\alpha : \mathbb{R} \to \mathbb{R}^2$ *given by* $\alpha(t) = (t^3, t^2)$ *is a differentiable mapping but is not an immersion because* $\alpha^{'}(0) = 0$.

Example 1.4.2. *The curve* $\alpha : \mathbb{R} \to \mathbb{R}^2$ *given by* $\alpha(t) = (t^3 - 4t, t^2 - 4)$ *is a differentiable mapping and an immersion but not an embedding because* $\alpha(2) = \alpha(-2) = 0$, *not injective.*

Example 1.4.3. *The map* $f : \mathbb{R} \to \mathbb{R}^2$ *given by* $f(t) = (e^t cos(t), e^t sin(t))$ *is an embedding of* $\mathbb{R}$ *to* $\mathbb{R}^2$.

Example 1.4.4. *Let* $M^k \subset \mathbb{R}^n$ *be a k dimension regular surface. The inclusion* $i : M^k \to \mathbb{R}^n$ *is an embedding, that is,* M^k *is a submanifold of* $\mathbb{R}^n$.

Proof: For all $p \in M^k$ there exists a parametrization $x : U \subset \mathbb{R}^k \to M^k$ of M^k at p. Let V be a neighborhood of p in $\mathbb{R}^n$ and a parametrization $j : V \subset \mathbb{R}^n \to V$ of $\mathbb{R}^n$ at $i(p)$ (j is the identity mapping). $j^{-1} \circ i \circ x = x$ is differentiable, so i is differentiable for all $p \in M^k$. From the condition (2) of the definition of regular surface $(di)_p$ is injective so i is an immersion and From the condition (1) of the definition of regular surface i is homeomorphism onto its image. Hence M^k is submanifold of $\mathbb{R}^n$.

Proposition 1.4.1. *Let* $\phi : M_1^n \to M_2^m$, $n \leq m$ *be an immersion of the differentiable manifold* M_1 *into the differentiable manifold* M_2. *For every* $p \in M_1$ $\exists$ *a neighbourhood* $V \subset M_1$ *of* p *such that the restriction* $\phi : V \to M_2$ *is an embedding.*

Proof. Let $x_1 : U_1 \subset \mathbb{R}^n \to M_1$ and $x_2 : U_2 \subset \mathbb{R}^m \to M_2$ be a system of coordinate at p and $\phi(p)$ respectively. Let us denote by $(x_1, ..., x_n)$ the coordinate of $\mathbb{R}^n$ and by $(y_1, ..., y_m)$ the coordinate of $\mathbb{R}^m$. In this coordinate , The expression for ϕ is the mapping $\tilde{\phi} = x_2^{-1} \circ \phi \circ x_1$ can be written as,

$$\tilde{\phi} = \big(y_1(x_1, ..., x_n), ..., y_m(x_1, ..., x_n)\big).$$

Let, $q = x_1^{-1}(p)$, since ϕ is an immersion then $d\phi_p$ is injective for all $p \in M_1$, that is $\big(\frac{\partial y_i}{\partial x_j}\big)$, $1 \leq i \leq m$, $1 \leq i \leq n$ is injective , so has rank n. Hence,

$$\frac{\partial(y_1, ..., y_n)}{\partial(x_1, ..., x_n)}(q) \neq 0$$

To apply inverse function theorem, we introduce the mapping, $\Phi : U_1 \times \mathbb{R}^{m-n=k} \to \mathbb{R}^m$ given by ,

$$\Phi(x_1, ..., x_n, t_1, ..., t_k) =$$

$$(y_1(x_1,...,x_n),...,y_n(x_1,...,x_n),y_{n+1}(x_1,...,x_n)+t_1,...,y_{n+k}(x_1,...,x_n)+t_k)$$

Where ,$(t_1,...,t_k) \in \mathbb{R}^{m-n=k}$. Here if we restricts Φ to U_1 then Φ coincide with $\tilde{\phi}$. $d\Phi_q$ is the $m \times m$ matrix given by

$$\begin{pmatrix} (\frac{\partial y_i}{\partial x_j})_{n\times n} & 0 \\ 0 & I_{k\times k} \end{pmatrix}$$

$$det(d\Phi_q) = \frac{\partial(y_1,...,y_n)}{\partial(x_1,...,x_n)}(q) \neq 0$$

It follows from the inverse function theorem ,that there exists a neighbourhood $W_1 \subset U_1 \times \mathbb{R}^k$ of q and $W_2 \subset \mathbb{R}^m$ of $\Phi(q)$ such that , the restriction $\Phi|_{w_1}$ is a diffeomorphism onto W_2. let $\tilde{V} = W_1 \cap U_1$, $\Phi|_{\tilde{V}} = \tilde{\phi}|_{\tilde{V}}$ and X_1 ,x_2 are diffeomorphism. We conclude that, the restriction to $V = x_1(\tilde{V})$ of the mapping $\Phi = x_2 \circ \tilde{\phi} \circ x_1^{-1} : V \to \phi(V) \subset M_2$ is a diffeomorphism , Hence an embedding. □

1.5 Examples of manifolds.

Example 1.5.1 (The tangent bundle). *Let M^n be a differentiable manifold and let $TM = \{(p,v); p \in M, v \in T_pM\}$. We are going to provide the set TM with a differentiable structure (of dimension 2n); with such a structure TM will be called the $Tangent\,bundle$ of M.*

Proof: Let $\{(U_\alpha, x_\alpha)\}$ be the maximal differentiable structure on M. Denoted by $(x_1^\alpha,...,x_n^\alpha)$ the coordinates of U_α and by $\{\frac{\partial}{\partial x_1^\alpha},...,\frac{\partial}{\partial x_n^\alpha}\}$ the associated bases to the tangent spaces of $x_\alpha(U_\alpha)$. For every α, Define

$$y_\alpha : U_\alpha \times \mathbb{R}^n \to TM,$$

by

$$y_\alpha(x_1^\alpha,...,x_n^\alpha,u_1,...,u_n) =$$

$$(x_\alpha(x_1^\alpha,...,x_n^\alpha), \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i^\alpha}), \quad (u_1,...,u_n) \in \mathbb{R}^n$$

Geometrically, this means that we are taking as coordinates of a point $(u.v) \in TM$ the coordinates of $x_1^\alpha,...,x_n^\alpha$ of p together with the coordinates of v in the basis

$\{\frac{\partial}{\partial x_1^\alpha}, \ldots, \frac{\partial}{\partial x_n^\alpha}\}$.

We are going to show that $\{(U_\alpha \times \mathbb{R}^n, y_\alpha)\}$ is a differentiable structure on TM. Since $\bigcup_\alpha x_\alpha(U_\alpha) = M$ and $(dx_\alpha)_q(\mathbb{R}^n) = T_{x_\alpha(q)}M$, $q \in U_\alpha$, $q \in U_\alpha$, we have that

$$\bigcup_\alpha y_\alpha(U_\alpha \times \mathbb{R}^n) = TM,$$

which verifies the condition (1) of Definition 1.1.1. Now let

$$(p, v) \in y_\alpha(U_\alpha \times \mathbb{R}^n) \cap y_\beta(U_\beta \times \mathbb{R}^n).$$

then

$$(p, v) = (x_\alpha(q_\alpha), dx_\alpha(v_\alpha)) = (x_\beta(q_\beta), dx_\beta(v_\beta)),$$

where $q_\alpha \in U_\alpha$, $q_\beta \in U_\beta$, $v_\alpha, v_\beta \in \mathbb{R}^n$. Therefore,

$$y_\beta^{-1} \circ y_\alpha(q_\alpha, v_\alpha) = y_\beta^{-1}(x_\alpha(q_\alpha), dx_\alpha(v_\alpha))$$

$$= ((x_\beta^{-1} \circ x_\alpha)(q_\alpha), d(x_\beta^{-1} \circ x_\alpha)(v_\alpha)).$$

Since $x_\beta^{-1} \circ x_\alpha$ is differentiable, $d(x_\beta^{-1} \circ x_\alpha)$ is as well. It follows that $y_\beta^{-1} \circ y_\alpha$ is differentiable, which verifies condition (2) of the definition 1.1.1. and completes the example.

Example 1.5.2. *The real projective space $P^n(\mathbb{R})$. Let us denote by $P^n(\mathbb{R})$ the set of straight lines of $\mathbb{R}^{n+1}$ which pass through the origin $0 = (0, \ldots, 0) \in \mathbb{R}^{n+1}$; that is, $P^n(\mathbb{R})$ is the set of "direction" of $\mathbb{R}^{n+1}$.*

Proof. We shall introduce a differentiable structure on $P^n(\mathbb{R})$. For this, on the set $\mathbb{R}^{n+1} - \{0\}$ we define the equivalence relation $\sim$ by

$p \sim q$ if and only if there exists a $\lambda \in \mathbb{R} - \{0\}$ such that $p = \lambda q$ where $p, q \in \mathbb{R}^{n+1}$

Then $P^n(\mathbb{R})$ be the quotient space $\mathbb{R}^{n+1} - \{0\}|_\sim$. The points of $P^n(\mathbb{R})$ will be denoted by $[x_1, \ldots, x_{n+1}]$. Observe that if $x_i \neq 0$,

$$[x_1, \ldots, x_{n+1}] = \left[\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots \frac{x_{n+1}}{x_i}\right]$$

Define subsets $V_1, \ldots, V_{n+1}$, of $P^n(\mathbb{R})$, by:

$$V_i = \{[x_1, \ldots, x_{n+1}] : x_i \neq 0\}, \quad i = 1, \ldots, n+1$$

Geometrically, V_i is the set of straight lines $\mathbb{R}^{n+1}$ which pass through the origin and do not belong to the hyperplane $x_i = 0$. We are now going to show that we can take the V_i's as coordinate neighborhoods, where the coordinates on V_i are

$$y_1 = \frac{x_1}{x_i}, \ldots, y_{i-1} = \frac{x_{i-1}}{x_i}, , y_i = \frac{x_{i+1}}{x_i}, \ldots, y_n = \frac{x_{n+1}}{x_i}.$$

For this, we will define mappings $x_i : \mathbb{R}^n \to V_i$ by

$$x_i(y_1, \ldots, y_n) = [y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n], \quad (y_1, \ldots, y_n) \in \mathbb{R}$$

and we will show that the family $\{(\mathbb{R}^n, x_i)\}$ is a differentiable structure on $P^n(\mathbb{R})$.

Indeed, any mapping x_i is clearly bijective while $\cup x_i(\mathbb{R}^n) = P^n(\mathbb{R})$. It remains to show that that $x_i^{-1}(V_i \cap V_j)$ is an open set in $\mathbb{R}^n$ and that $x_j^{-1} \circ x_i,\ j = 1, \ldots, n+1$, is differentiable there. Now, if $i > J$, the points in $x_i^{-1}(V_i \cap V_j)$ are of the form:

$$\{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_j \neq 0\}.$$

Therefore $x_i^{-1}(V_i \cap V_j)$ is open set in $\mathbb{R}^n$, and supposing that $i > j$(the case for $i < j$ is similar),

$$\begin{aligned} x_j^{-1} \circ x_i(y_1, \ldots, y_n) &= x_j^{-1}[y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n] \\ &= x_j^{-1}\left[\frac{y_1}{y_j}, \ldots, \frac{y_{j-1}}{y_j}, 1, \frac{y_{j+1}}{y_j}, \ldots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_i}{y_j}, \ldots, \frac{y_n}{y_j}\right] \\ &= \left(\frac{y_1}{y_j}, \ldots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \ldots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_i}{y_j}, \ldots, \frac{y_n}{y_j}\right) \end{aligned}$$

Which is clearly differentiable.

In summary, the space of directions of $\mathbb{R}^{n+1}$ (real projective space $P^n(\mathbb{R})$) can be covered by $n+1$ coordinate neighborhood V_i, where V_i are made up of those directions of $\mathbb{R}^{n+1}$ that are not in the hyperplane $x_i = 0$; in addition, in each V_i we have coordinates

$$\left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_{n+1}}{x_i}\right)$$

where $(x_1, \ldots, x_{n+1})$ are the coordinates of $\mathbb{R}^{n+1}$. □

Example 1.5.3. *Let M be a differentiable manifold of dimension m and N be a differentiable manifold of dimension n. Then the cartesian product $M \times N$ is a differentiable manifold of dimension $m + n$.*

Example 1.5.4. *Regular surface of dimension k is a differentiable manifold of dimension k.*

Proof: Let a subset $M^k \subset \mathbb{R}^n$ is a regular surface of dimension k, $p \in M^k$. Let $x : U(\subset \mathbb{R}^k) \to M^k$ and $y : V(\subset \mathbb{R}^k) \to M^k$ are two parametrazation at p with $x(U) \cap y(V) = W \neq \phi$, then consider the mapping $h = x^{-1} \circ y : y^{-1}(W) \to x^{-1}(W)$, we have to show that h is a diffeomorphism.

Let $r \in y^{-1}(W)$ and put $q = h(r)$. Let $(u_1, ..., u_k) \in U$ and $(v_1, ..., v_n) \in \mathbb{R}^n$, and write x in these coordinate as

$$x(u_1, ..., u_k) = (v_1(u_1, .., u_k), ..., v_n(u_1, ..., u_k)).$$

From condition (2) of the Definition of Regular surface we have,

$$\frac{\partial(v_1, ..., v_k)}{\partial(u_1, ..., u_k)}(q) \neq 0$$

Extend x to the mapping $F : U \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ given by

$$F(u_1, .., u_k, t_{k+1}, ..., t_n)$$

$$= (v_1(u_1, ..., u_k), ..., v_k(u_1, ..., u_k), v_{k+1}(u_1, ..., u_k) + t_{k+1}, ..., v_n(u_1, ..., u_k) + t_n),$$

Where $(t_{k+1}, ..., t_n) \in R^{n-k}$. It is clear that F is differentiable and restriction of F to $U \times \{(0, ..., o)\}$ coinside with x. And we have,

$$det(dF_q) = \frac{\partial(v_1, ..., v_k)}{\partial(u_1, ..., u_k)}(q) \neq 0$$

. Then by inverse function theorem, which guarantees the existence of a neighborhood Q of $x(q)$ where F^{-1} exists and is differentiable. By the continuity of y, there exists a neighborhood $R \subset V$ of r such that $y(R) \subset Q$. The restriction of h to R, $h|_R = F^{-1} \circ y|_R$ is a composition of differentiable mappings. Thus h is differentiable at r, hence in $y^{-1}(W)$. Similarly we can show that h^{-1} is differentiable. Hence regular surface is a differentiable manifold.

Definition 1.5.1 (Regular value)**.** *Let $F : U(\subset \mathbb{R}^n) \to \mathbb{R}^m$ be a differentiable mapping of an open set U of $\mathbb{R}^n$. A point $p \in U$ is defined to be a critical point of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective. The image $F(p)$ of a critical point* *is called a critical value of F. A point $a \in \mathbb{R}^m$ that is not a critical value is said to be is said to be a regular value of F. Any point $a \notin F(U)$ is trivially a regular value of F and if there exists a regular avlue of F in $\mathbb{R}^m$, then $n \geq m$.*

Example 1.5.5. *Let $F : U(\subset \mathbb{R}^n) \to \mathbb{R}^m$ be a differentiable mapping of an open set U of $\mathbb{R}^n$, $a \in F(U)$ be a regular value of F. Then $F^{-1}(a) \subset \mathbb{R}^n$ is a regular surface of dimension $m - n = k$. Hence $F^{-1}(a) \subset \mathbb{R}^n$ is differentiable manifold of dimension k.*

Proof. To prove this We use the inverse function theorem. Let $p \in F^{-1}(a)$. Denote by $q = (y_1, \ldots, y_m, x_1, \ldots, x_k)$ an arbitary point of $\mathbb{R}^{n=m+k}$ and by $F(q) = (f_1(q), \ldots, f_m((q))$ its image by the mapping F. Since a is a regular value of F, dF_p is surjactive. therefore, we have

$$\frac{\partial(f_1, \ldots, f_m)}{\partial(y_1, \ldots, y_m)}(p) \neq 0$$

Define a mapping $\phi : U \subset \mathbb{R}^n \to \mathbb{R}^{n+k}$ by

$$\phi(y_1, \ldots, y_m, x_1, \ldots, x_k) = (f_1(q), \ldots, f_m(q), x_1, \ldots, x_k).$$

Then,

$$det(d\phi)_p = \frac{\partial(f_1, \ldots, f_m)}{\partial(y_1, \ldots, y_m)}(p) \neq 0.$$

By the inverse function theorem, ϕ is a diffeomorphism of a neighborhood Q of p onto a neighborhood W of $\phi(p)$. Let $K^{m+k} \subset W \subset \mathbb{R}^{n+k}$ be a cube of center $\phi(p)$ and put $V = \phi^{-1}(K^{m+k} \cap Q)$. Then ϕ maps the neighborhood V diffeomorphically onto $K^{m+k} = K^m \times K^k$. Define a mapping $x : K^k \to V$ by

$$x(x_1, \ldots, x_k) = \phi^{-1}(a_1, \ldots, a_m, x_1, \ldots, x_k),$$

Where $(a_1, \ldots, a_m) = a$. Here ϕ satisfies conditions (1) and (2) of the definition of regular surface. Since p is arbitary, $F^{-1}(a)$ is a regular surface in $\mathbb{R}^n$. $\square$

1.6 Orientation

Definition 1.6.1. *A smooth manifold M is orientable if M admits a differentiable structure $\{(U_\alpha, x_\alpha)\}$ such that:*

1. *for every pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \phi$, the differential of the change of coordinates $x_\beta \circ x_\alpha$ has positive determinant.*

In the opposite case M is nonorientable. If M is orientable, then an choice of differentiable structure satisfying (1) is called an orientation of M. Furthermore, M (equipped with such differentiable structure) is said to be oriented. We say that two differentiable structure satisfying (1) determine the same orientation if their union satisfies (1) too.

Note: 1.6.1.

1. *An orientable and connected smooth manifold has exactly two distinct orientations.*
2. *If M and N are smooth manifolds and $f : M \to N$ is a diffeomorphism, then M is orientable if and only if N is orientable.*
3. *Let M and N be connected oriented smooth manifolds and $f : M \to N$ a diffeomorphism. Then f induces an orientation on N. Which may or may not coincide with the initial orientation of N. In the 1st case we say that f preserves the orientation in the second case , f reverse the orientation.*

Example 1.6.1. *If M can be covered by two coordinate neighborhoods V_1 and V_2 in such a way that the intersection $V_1 \cap V_2$ is connected, then M is orientable.*

Proof: Suppose that there exists an atlas $\{(V_1, x_\alpha),\ (V_2, y_\beta)\}$ of M such that $W = V_1 \cap V_2$ is connected. The mapping $y_\beta^{-1} \circ x_\alpha : x_\alpha^{-1}(W) \to y_\beta^{-1}(W)$ is diffeomorphic. So $det(y_\beta^{-1} \circ x_\alpha)'(x) \neq 0\ \forall x \in x_\alpha^{-1}(W)$. Since $x \to det(y_\beta^{-1} \circ x_\alpha)'(x)$ is continuous and $x_\alpha^{-1}(W)$ is connected, the determinant can not change its sign. If the sign is positive, we are done. If the sign is negative, replace the chart (V_2, y_β),$y_\beta = (y_1, ..., y_n)$, by the chart $(V_2, \tilde{y_\beta})$, $\tilde{y_\beta} = (-y_1, y_2, ..., y_n)$. Then the atlas $\{(V_1, x_\alpha),(V_2, \tilde{y_\beta})\}$ satisfies (1).

Example 1.6.2. *The sphere S^n is orientable.*

$$S^n = \left\{ (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}; \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$$

Proof. Let $N = (0, .., 0, 1)$ be the nort pole and $S = (0, ..., 0, -1)$ the south pole of S^n. Define a mapping $\pi_1 : S^n - \{N\} \to \mathbb{R}^n$ (stereographic projection from the north pole) that takes $p = (x_1, ..., x_{n+1})$ in $S^n - \{N\}$ into the intersection of the hyperplane $x_n + 1 = 0$ with the lin ethat passes through p and N.

$$\pi_1(x_1, ..., x_{n+1}) = \big(\frac{x_1}{1 - x_{n+1}}, ..., \frac{x_n}{1 - x_{n+1}}\big).$$

The mappng π_1 is differentiable, injective and the maps $S^n - \{N\}$ into the hyperplane $x_{n+1} = 0$. The stereographic projection $\pi_2 : S^n - S \to \mathbb{R}^n$ from the south pole onto the hyperplane $x_{n+1} = 0$ has the same properties.

Therefore, the parametrizations $(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})$ cover S^n. In addition, the change of coordinates:

$$y_j = \frac{x_j}{1 - x_{n+1}} \leftrightarrow y_j^{'} = \frac{x_j}{1 + x_{n+1}},$$

$$(y_1, ..., y_n) \in \mathbb{R}^n, \quad j = 1, ..., n$$

is given by

$$y_j^{'} = \frac{y_j}{\sum_{i=1}^{n} y_i^2}$$

(Using the fact $\sum_{k=1}^{n+1} x_k^2 = 1$). Therefore, the family $\{(\mathbb{R}^n, \pi_1^{-1}), (\mathbb{R}^n, \pi_2^{-1})\}$ is a differentiable structure on S^n. The intersection $\pi_!^{-1}(\mathbb{R}^n) \cap \pi_2^{-1}(\mathbb{R}^N) = S^n - \{N \cup S\}$ is connected, thus S^n is orierntable and family given determines an orientation of S^n. □

1.7 Vector fields; brackets.

Definition 1.7.1. *A Vector field X on a differentiable manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. In terms of mappings, X is a mapping from M into the tangent bundle TM The Field is differentiable if the mapping $X : M \to TM$ is differentiable.*

Consider a parametrazation $x : U(\subset \mathbb{R}^n) \to M$ we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \left(\frac{\partial}{\partial x_i}\right)_p,$$

where each $a_i : U \to R$ is a function on U and $\{\frac{\partial}{\partial x_i}\}$ is the basis of the tangent space associated to x, $i = 1, ..., n$.

Proposition 1.7.1. *A vector field X is differentiable if and only if the functions a_i are differentiable*

Proof. Let, $X : M \to TM$ be a vector field. Let $x : U \to M$ and $y : U \times \mathbb{R}^n \to TM$ be the parametrizations at p and $X(p)$,respectively. Expressing X in these parametrization, we can write

$$\hat{X}(x_1, ..., x_n) = (x_1, ..., x_n, a_1(x_1, ..., x_n), ..., a_n(x_1, ..., x_n))$$

Therefore X is differentiable if and only if the functions $a_i : U \to R$ are differentiable. □

We can also think of a vector field as a mapping $X : \mathcal{D} \to \mathcal{F}$, where $\mathcal{D}$ is the set of differentiable function on M and $\mathcal{F}$ is the set of function on M. Let $f \in \mathcal{D}$ and $X : \mathcal{D} \to \mathcal{F}$ is defined by

$$f \to Xf$$

Where, $Xf : M \to R$ is defined by,

$$p \to (Xf)(p) = \sum_{i=1}^{n} a_i(p)\left(\frac{\partial f}{\partial x_i}\right)(p),$$

So, the vector field is differentiable if and only if $X : \mathcal{D} \to \mathcal{D}$, that is $Xf \in \mathcal{D}$ for all $f \in \mathcal{D}$. Let $\mathfrak{X}(M)$ be the set of all vector fields of class C^∞ on M. The interpretation of X as an operator on $\mathcal{D}$ permits us to consider the iterates of X. For example, if X and Y are differentiable vector fields on M and $f \in \mathcal{D}$, we can consider the functions $X(Yf)$ and $Y(Xf)$. In general, the operators XY , YX will involve derivatives of order two, and will not lead to vector fields. However, $XY - YX$ does define a vector field.

Lemma 1.7.1. *Given two differentiable vector fields $X, Y \in \mathfrak{X}(M)$ on a smooth manifold M, there exists a unique differentiable vector field $Z \in \mathfrak{X}(M)$ such that $Zf = (XY - YX)f$, for every differentiable function $f \in \mathcal{D}$*

Proof. First, we prove that if Z exists, then it is unique. Assume the existence of such Z. Let $p \in M$ and let $x : U \to M$ be a parametrization at p, and let,

$$X = \sum_i a_i \frac{\partial}{\partial x_i} \quad Y = \sum_j b_j \frac{\partial}{\partial x_j}$$

be the expression of X and Y in these parametrization. Then for all $f \in \mathcal{D}$,

$$X(Yf) = X\left(\sum_j b_j \frac{\partial f}{\partial x_j}\right) = \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j},$$

$$Y(Xf) = Y\left(\sum_i a_i \frac{\partial f}{\partial x_i}\right) = \sum_{i,j} b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} + \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

$$Zf = XYf - YXf = \sum_{i,j} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right) \frac{\partial f}{\partial x_j}$$

Which proves the uniqueness of Z.

To show existence, define Z_α in each coordinate neighbourhood $X_\alpha(U_\alpha)$ of a differentiable structure $\{(U_\alpha), x_\alpha\}$ on M by the previous expression. By uniqueness, $Z_\alpha = Z_\beta$ on $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) \neq \phi$, which allows us to define Z over the entire manifold M. □

The vector field Z given by Lemma 1.5.1 is called the bracket $[X, Y] = XY - YX$ of X and Y.

Proposition 1.7.2. *If $X, Y, Z \in \mathfrak{X}(M)$, $a, b \in \mathbb{R}$ and $f, g \in \mathcal{D}$ then:*

1. $[X, Y] = -[Y, X]$ *(anticommutativity),*

2. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ *(linearity),*

3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *(jacobi identity),*

4. $[fX, gY] = fgf[X, Y] + fX(g)Y - gY(f)X$.

Proof. 1. $[X, Y] = XY - YX = -(YX - XY) = -[Y, X]$

2. $[aX + bY, Z] = (aX + bY)Z - Z(aX + bY) = aXZ + bYZ - aZX - bZY = a(XZ - ZX) + b(YZ - ZY) = a[X, Z] + b[Y, Z]$

3. $[[X,Y],Z] = [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX$ So,

$$[X,[Y,Z]] + [Y,[Z,X]] =$$

$$XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY$$

Hence,

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

4. $[fX, gY] = fX(gY) - gY(fX) = fgXY + fX(g)Y - gfYX - gY(f)X = fg[X,Y] + fX(g)X - gY(f)X$.

□

The bracket $[X,Y]$ can be also interpreted as a derivation of Y along the "*trajectories*" of X.

Since a differentiable manifold is locally diffeomorphic to $\mathbb{R}^n$, the fundamental theorem on existence, uniqueness, and dependence on initial conditions of ordinary differential equation (which is a local theorem) extends naturally to differentiable manifolds. Which is stated below.

Theorem 1.7.1. *Let X be a differentiable vector field on a differentiable manifold M, and let $p \in M$. Then there exists a neighborhood $U \subset M$ of p, an interval $(-\delta, \delta)$, $\delta > 0$, and a differentiable mapping $\phi : (-\delta,\delta) \times U \to M$ such that the curve $t \to \phi(t,q)$, $t \in (-\delta,\delta)$, $q \in U$, is the unique curve which satisfies $\frac{\partial \phi}{\partial t} = X(\phi(t,q))$ and $\phi(0,q) = q$.*

A curve $\alpha : (-\delta,\delta) \to M$ which satisfies the conditions $\alpha'(t) = X(\alpha(t))$ and $\alpha(0) = q$ is called a *trajectory* of the field X that passes through q for $t = 0$. The above theorem guarantees that for each point of a certain neighborhood there passes a unique trajectory of X and that the mapping so obtained depends differentiably on t and on the "*initial condition*" q. The mapping $\phi_t : U \to M$ is called the local flow of X where $\phi_t(q) = \phi(t,q)$.

Lemma 1.7.2. *Let $h : (-\delta,\delta) \times U \to \mathbb{R}$ be differentiable mapping with $h(0,q) = 0$ for all $q \in U$. Then there exists a differentiable mapping $g : (\delta,\delta) \times U \to \mathbb{R}$ with $h(t,q) = tg(t,q)$; in particular,*

$$g(0,q) = \frac{\partial h(t,q)}{\partial t}\Big|_{t=0}$$

Proof. Define for fixed t,

$$g(t,q) = \int_0^1 \frac{\partial h(ts,q)}{\partial(ts)} ds$$

$$tg(t,q) = \int_0^1 t\frac{\partial h(ts,q)}{\partial(ts)} d(s)$$

and, after change of variables, observe that

$$tg(t,q) = \int_0^t \frac{\partial h(ts,q)}{\partial(ts)} d(ts) = h(t,q).$$

□

We can also express the bracket in following form.

Proposition 1.7.3. *Let X, Y be differentiable vector fields on a differentiable manifold M, let $p \in M$, and let ϕ_t be the local flow of X in a neighborhood U of p. Then*

$$[X,Y](p) = \lim_{t\to 0} \frac{1}{t}[Y - d\phi_t Y](\phi_t(p))$$

Proof. Let f be a differentiable function in a neighborhood of p. Putting

$$h(t,q) = f(\phi_t(q)) - f(q),$$

now applying the previous lemma we obtain a differentiable function $g(t,q)$ such that

$$f \circ \phi_t(q) = f(q) + tg(t,q) \ \ \textit{and} \ \ g(0,q) = Xf(q).$$

Now,

$$((d\phi_t Y)f))(\phi_t(p)) = (Y(f \circ \phi_t))(p) = Yf(p) + t(Yg(t,p)).$$

Therefore,

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{t}[Y - d\phi_t Y]f(\phi_t p) &= \lim_{t\to 0} \frac{(Yf)(\phi_t p) - Yf(p)}{t} - (Yg(0,p)) \\
&= (X(Yf))(p) - (Y(X)f)(p) \\
&= ([X,Y]f)(p).
\end{aligned}$$

This completes the proof. □

Chapter 2

Riemannian Metrics

We have natural way of measuring the length of vectors tangent to a surface $S \subset \mathbb{R}^n$, using the inner product $\langle v, w \rangle$ of two vectors tangent to S at a point p of S is simply the inner product of these vectors in $\mathbb{R}^n$. For abstract differentiable manifolds there is no ambient space so, we have to define inner product in the tangent space at each point. The definition of $\langle\ ,\ \rangle$ permits us to measure not only the length of the curve but also volume, angle between two curves and all the other "metric" ideas used in geometry.

Definition 2.0.1 (Riemannian Metric)**.** *A Riemannian Metrics (or Riemannian structure) on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $<,>_p$ (that is , a symmetric, bilinear, positive-definite form) on the tangent space T_pM*

$$< \ . \ >_p : T_P M \times T_p M \to \mathbb{R}$$

Which varies differentiably with $p \in M$.

The last condition means if $x : U \subset \mathbb{R}^n \to M$ is a system of coordinate around $p \in M$ with $x(x_1, ..., x_n) = q \in x(U)$ and $\frac{\partial}{\partial x_i}(q) = dx_q(0, .., 1, ..., 0)$. Then each function $g_{ij} : U \to \mathbb{R}$ defined by

$$g_{ij}(x_1, ..., x_n) = < \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) >_{(q)}$$

is differentiable.

The function g_{ij} is called the local representation of the Riemannian metric (or the g_{ij} of the metric) in the coordinate system $x : U \subset \mathbb{R}^n \to M$. A differentiable manifold with the Riemannian metric will be called a $Riemannian\, manifold$.

Definition 2.0.2. *Let M and N be Riemannian manifolds. A diffeomorphism $f : M \to N$ is called an isometry if*

$$< u, v >_{(p)} = < df_p(u), df_p(v) >_{f(p)}, \; for\, all\, p \in M, u, v \in T_pM$$

Definition 2.0.3. *Let M and N be Riemannian manifolds. A diffeomorphism $f : M \to N$ is a local isometry at $p \in M$ if there is a neighborhood $U \subset M$ of p such that $f : U \to f(U)$ is a diffeomorphism satisfying*

$$< u, v >_{(p)} = < df_p(u), df_p(v) >_{f(p)}, \; for\, all\, p \in M, u, v \in T_pM$$

Example 2.0.1. *The almost trivial example. $M = \mathbb{R}^n$ with $\frac{\partial}{\partial x_i}$ identified with $e_i = (0, \ldots, 1, \ldots, 0)$. The metric is given by $\langle e_i, e_j \rangle = \delta_{ij}$.*

Example 2.0.2. *$M = \mathbb{R}^2$ the local expression of the previous metric in polar coordinate.*

$$\frac{\partial}{\partial r}(r, \theta) = (cos\theta, sin\theta) \;\; and \;\; \frac{\partial}{\partial \theta}(r, \theta) = (-rsin\theta, rcos\theta)$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Example 2.0.3. *Let $f : M^n \to N^{n+k}$ be an immersion (that is f is differentiable and $df_p : T_pM \to T_{f(p)}M$ is injective for all $p \in M$. If N has a Riemannian structure, f induces Riemannian structure on M. Defining,*

$$< u, v >_{(p)} = < df_p(u), df_p(v) >_{f(p)}, \; for\, all\, p \in M, u, v \in T_pM.$$

Since, $< u, v >_p$ is symmetric, $< u, v >_p \geq 0$ and $< u, u >_p = 0 \implies < df_p(v), df_p(v) >_{f(p)} = 0 \implies df_p(v) = 0$ (since, df_p is injective). The metric on M is then called the metric induced by f, and f is isometric immersion.

In particular, when we have a differentiable function $h : M^{n+k} \to N^k$ and $q \in N$ is a regular value of h (that is , $dh_p : T_PM \to T_{h(p)}N$ is surjective for all $p \in h^{-1}(q)$). It is known that $h^{-1}(q) \subset M$ is a submanifold of M of dimension n; hence, We can put a Riemannian metric on it induced by the inclusion.

Example 2.0.4. *(The product metric) Let* (M_1, g_1) *and* (M_2, g_2) *are Riemannian manifolds, the product* $M_1 \times M_2$ *has a natural Riemannian metric, the product metric, defined by*

$$g(X_1 + X_2, Y_1 + Y_2) := g_1(X_1, Y_1) + g_2(X_2, Y_2),$$

where $X_i, Y_i \in T_p(M_i)$ *and* $T_{(p,q)}(M_1 \times M_2) = T_pM_1 \oplus T_qM_2$ *for all* $(p, q) \in M_1 \times M_2$.

If $(x_1, \ldots, x_n)$ *is a chart on* M_1 *and* $(x_{n+1}, \ldots, x_{n+m})$ *is a chart on* M_2*, then* $(x_1, \ldots, x_{n+m})$ *is a chart on* $M_1 \times M_2$*. In these coordinates the local representation of the product metric,* g_{ij}*, can be written as,*

$$\begin{pmatrix} (g_1)_{11} & \cdots & (g_1)_{1n} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ (g_1)_{n1} & \cdots & (g_1)_{nn} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (g_2)_{11} & \cdots & (g_2)_{1m} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & (g_2)_{m1} & \cdots & (g_2)_{nm} \end{pmatrix}$$

A Riemannian metric allows us to compute the length of curves.

Definition 2.0.4. *A differentiable mapping* $c : I \to M$ *of an open interval* $I \subset \mathbb{R}$ *into a differentiable manifold* M *is called a (parametrized) curve.*

Definition 2.0.5. *A vector field* V *along a curve* $c : I(\subset \mathbb{R}) \to M$ *is mapping* $V : I \to TM$ *that associates to every* $t \in I$ *a tangent vector* $V(t) \in T_{c(t)}M$*.* V *will be differentiable if the mapping* $V : I \to TM$ *is differentiable (that is for any differentiable function* f *on* M*, the function* $t \to V(t)f$ *is a differentiable function on* I*).*

The vector field $dc(\frac{d}{dt})$, denoted by $\frac{dc}{dt}$, is called the *velocity field* (or tangent vector field). A vector field along c can not necessarily be extended to a vector field on an open set of M.

Definition 2.0.6. *let* M *be a Riemannian manifold and* $c : I(\subset \mathbb{R}) \to M$ *be a curve. The restriction of a curve* c *to closed interval* $[a, b] \subset I$ *is called a* **segment**. *We define the length of a segment by*

$$l_a^b(c) = \int_a^b \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle^{1/2} dt$$

Definition 2.0.7. *Let M be a differentiable manifold. A family of open sets $V_\alpha \subset M$ with $\bigcup_\alpha V_\alpha = M$ is said to be locally finite if every point $p \in M$ has a neighborhood W such that $W \cap V_\alpha \neq \phi$ for only a finite number of indices.*

Definition 2.0.8 (Differential partition of unity)**.** *We say that a family f_α of differentiable functions $f_\alpha : M \to \mathbb{R}$ is a Differential partition of unity if:*

1. *For all α, $f_\alpha \geq 0$ and the support of f_α is contained in a coordinate neighborhood $V_\alpha = x_\alpha(U_\alpha)$ of a differentiable structure $\{(U_\beta, x_\beta)\}$ of M.*
2. *The family $\{V_\alpha\}$ is locally finite.*
3. *$\sum_\alpha f_\alpha(p) = 1$, for all $p \in M$ (This condition make sense because for each p, $f_\alpha(p) \neq 0$ only for finite number of indices).*

We say that the partition of unity $\{f_\alpha\}$ is subordinate to the covering $\{V_\alpha\}$.

Theorem 2.0.1. *A differentiable manifold M has a Differential partition of unity if and only if every connected component of M is Hausdorff and has a countable basis.*

Proposition 2.0.1. *A differentiable manifold M(Hausdorff with countable basis) has a Riemannian metric.*

Proof. Let $\{f_\alpha\}$ be a differentiable partition of unity on M subordinate to a covering $\{V_\alpha\}$ of M by coordinate neighborhood. We can define a Riemannian metric $<,>^\alpha = g^\alpha$ on each $V\alpha$ induce by the system of local coordinate, whose local representation is $(g^\alpha_{ij}) = \delta_{i,j}$. Let us define

$$< u, v >_p = \sum_\alpha f_\alpha(p) < u, v >^\alpha_p \; \forall p \in M, u, v \in T_P M.$$

since, the family of supports of f_α is locally finite the above sum is finite. Hence $<,>_p$ is well defined and smooth. It is bilinear and symmetric at each point. Since, $f_\alpha \geq 0$ and $\sum_\alpha f_\alpha = 1$ it follows that $<,>_p$ is positive definite. So, this defines a Riemannian metric on M. □

Riemannian metric permits us to define a notion of volume element on a given oriented manifold M^n.

Let $p \in M$ and let $x : U \subset \mathbb{R}^n \to M$ be a parametrization about p which belongs to a family of parametrization consists with the orientation of M(We say that such parametrization are positive). Consider a positive orthonormal basis $\{e_1, ..., e_n\}$ of T_pM and write $X_i(p) = \frac{\partial}{\partial x_i}(p)$ in the basis e_i, $X_i(p) = \sum_{i,j} a_{i,j}e_J$. Then

$$g_{ik}(p) =< X_i, X_k > (p) = \sum_{jl} a_{ij}a_{kl} < e_j, e_l >= \sum_{j} a_{ij}a_{kj}$$

Since the volume $vol(X_1(p), ..., X_n(p))$ of the parallelepiped formed by the vectors $(X_1(p), ..., X_n(p)$ in T_pM is equal to $vol(e_1, ..., e_n) = 1$ multiplied by the determinant of the matrix $(a_{i,j})$, we obtain

$$vol(X_1(p), ..., X_n(p)) = det(a_{i,j}) = \sqrt{det(g_{i,j})(p)}.$$

If $y : V \subset \mathbb{R}^n \to M$ is another positive parametrization about p, with $Y_i(p) = \frac{\partial}{\partial y_i}(p)$ and $h_{i,j} =< Y_i, Y_j > (p)$, we obtain

$$\sqrt{det(g_{i,j})(p)} = vol(X_1(p), ..., X_n(p))$$

$$= \mathbf{J}vol(Y_1(p), ..., Y_n(p)) = \mathbf{J}\sqrt{det(h_{i,j}(p))}$$

where $\mathbf{J} = det(\frac{\partial y_i}{\partial x_j}) = det(dy^{-1} \circ dx)(p) > 0$ is the determinant of the derivative of the change of coordinates.

Now let $R \subset M$ be a region (an open and connected), whose closer is compact. WE suppose that R is contained in a coordinate neighborhood $x(U)$ with a positive parametrization $x : U \to M$, and that the boundary of $x^{-1}(R) \subset U$ has measure zero in $\mathbb{R}^n$.

Let us define the volume $vol(R)$ of R by the integral in $\mathbb{R}^n$

$$vol(R) = \int_{x^{-1}(R)} \sqrt{det(g_{i,j})}dx_1...dx_n.$$

The expression is well-defined. Because if R is contained in another coordinate neighborhood $y(V)$ with a positive parametrization $y : V \subset \mathbb{R}^n \to M$, we obtain from the change of variable theorem for multiple integral, we have

$$\int_{x^{-1}(R)} \sqrt{det(g_{i,j})}dx_1...dx_n = \int_{y^{-1}(R)} \sqrt{det(h_{i,j})}dy_1...dy_n = vol(R)$$

which proves that the definition of volume does not depend on the choice of the coordinate system. The hypothesis of the orientability of M guarantee that $vol(R)$ does not change sign.

The integrand in the formula for the volume expression that is,

$$\sqrt{det(g_{i,j})}dx_1...dx_n$$

is a positive differential form of degree n, which is called a **volume element** ν on M.

To define the the volume of a compact region R, which is not contained in a coordinate neighborhood it is necessary to consider a partition of unity ϕ_i subordinate to a (finite) covering of R consisting of coordinate neighbourhoods $x(U_i)$ and to take

$$vol(R) = \sum_i \int_{x^{-1}(R)} \phi_i \nu.$$

The above expression does not depend on the choice of the partition of unity. The existence of a globally defined positive differential form of degree n (volume element) leads to a notion of volume on a differentiable manifold. A Riemannian metric notion is the only on of the ways through which a volume element can be obtained.

Chapter 3

Connections

If X and Y are vector fields in Euclidean space, we can define the directional derivative $\nabla_X Y$ of Y along X. This definition, however, no longer holds in a general manifold, because let $S \subset \mathbb{R}^3$ be a surface and let $c : I \to S$ be a parametrized curve in S. The vector $\frac{dV}{dt}(t)$, $t \in I$, does not in general belongs to the tangent space of S. The concept of differentiating a vector field is not a "*intrinsic*" geometric notion on S. To overcome this problem we consider, instead of the usual derivative $\frac{dV}{dt}(t)$, the orthogonal projection of $\frac{dV}{dt}(t)$ on $T_{c(t)}S$. This orthogonally projected vector is called the covarient derivative of V and denoted by $\frac{DV}{dt}(t)$. The covarient derivative of V is the derivative of V as seen from the "viewpoint of S".

3.1 Affine Connection

Let us indicate by $\mathfrak{X}(M)$ the set of all vector field of class C^∞ on M and by $\mathcal{D}(M)$ the ring of real-valued functions of class C^∞ defined on M.

Definition 3.1.1 (Affine Connection)**.** *An Affine connection ∇ on a differentiable manifold M is a mapping*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

which is denoted by $(X, Y) \to \nabla_X Y$ and which satisfies the following properties :

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.

3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

in which $X, Y, Z \in \mathfrak{X}(M)$ *and* $f, g \in \mathcal{D}(M)$

Proposition 3.1.1. *Let* ∇ *be an affine connection on* M*,* $X, Y \in \mathfrak{X}(M), p \in M$*, then* $(\nabla_X Y)_p \in T_pM$ *depends only on* X_p *and the value of* Y *along a curve tangent to* X *at* p*.*

Proof. Let $x : U(\subset \mathbb{R}^n) \to M$ be a system of coordinate at p, and $X = \sum_i x_i X_i$, $Y = \sum_j y_j Y_j$, where $X_i = \frac{\partial}{\partial x_i}$. Now,

$\nabla_X Y = \nabla_X(\sum_j y_j X_j) = \sum_j y_j \nabla_X X_j + \sum_j X(y_j)X_j$

$= \sum_j y_j \nabla_{\sum_i x_i X_i} X_j + \sum_j (\sum_i x_i X_j(y_j))X_j$

$= \sum_{i,j} x_i, y_j \nabla_{X_i} X_j + \sum_{i,j} x_i X_i(y_j)X_j$

Let $\nabla_{X_i} X_j = \sum_k \Gamma^k_{i,j} X_k$, $\Gamma^k_{i,j}$ are differentiable function defined on U in a coordinate system (U, x). So we have

$$\nabla_X Y = \sum_k (\sum_{i,j} x_i y_j \Gamma^k_{i,j} + X(y_k))X_k \tag{3.1}$$

So, $\nabla_X Y(p)$ depends only on $x_i(p), y_j(p)$ and $X(y_k)(p)$. However $x_i(p), y_j(p)$ depends on X_p, Y_p. and $X(y_k)(p) = \frac{d}{dt} y_k(c(t))|_{t=0}$, depend on the the value of Y_k along the curve c whose tangent vector at $p = c(0)$ is X_p. □

Proposition 3.1.2. *Let* M *be a differentiable manifold with an affine connection* ∇*. There exists a unique correspondence which associates to a vector field* V *along the differentiable curve* $c : I \to M$ *another vector field* $\frac{DV}{dt}$ *along* c*, called the covariant derivative of* V *along* c*, such that:*

1. $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$
2. $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$*, where* W *is a vector field along* c *and* f *is differentiable function on* I*.*
3. *If* V *is induced by a another vector field* $Y \in \mathfrak{X}(M)$*, i.e.,* $V(t) = Y(c(t))$*, then* $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$*.*

Proof. Let us suppose initially that there exists a correspondence satisfying (1), (2) and (3). Let $X : U \subset \mathbb{R}^n \to M$ be a system of coordinate with $c(I) \cap x(U) \neq \phi$ and let $(x_1(t), ..., x_n(t))$ be the local expression of $c(t), t \in I$. Let $X_i = \frac{\partial}{\partial x_i}$. Then we can express the field V locally as $V = \sum_j v^j X_j, j = 1, ..., n$, where $v^j = v^j(t)$ and $X_j = X_j(c(t))$. By (1) *and* (2) we have

$$\frac{DV}{dt} = \sum_j \frac{dv^j}{dt} X_j + \sum_j v^j \frac{DX_j}{dt}.$$

By (3) of Definition 3.1.1

$$\frac{DX_j}{dt} = \nabla_{dc/dt} X_j = \nabla_{(\sum \frac{dx_i}{dt} X_i)} X_j = \sum_i \frac{dx_i}{dt} \nabla_{X_i} X_j, \quad i, j = 1, ..., n$$

Therefore,

$$\frac{DV}{dt} = \sum_j \frac{dv^J}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} v^j \nabla_{x_i} X_j. \tag{3.2}$$

The expression shows that if there is a correspondence satisfying the condition of proposition 2.2, then such a correspondence is unique.

To show the existence , define $\frac{DV}{dt}$ in $X(U)$ by 3.2. $\frac{DV}{dt}$ possesses the desire properties. If $y(W)$ is another coordinate neighborhood, with $y(W) \cap x(U) \neq \phi$ and we define $\frac{DV}{dt}$ in $y(W)$ by 3.2 the definition agree in $y(W) \cap x(U)$, by the uniqueness of $\frac{DV}{dt}$in $x(U)$. It follows that the definition can be extended over all of M, and this conclude the proof. □

Definition 3.1.2. *Let M be a differentiable manifold with an affine connection ∇. A vector field V along a curve $c : I \to M$ is called parallel when $\frac{DV}{dt} = 0$, for all $t \in I$.*

Proposition 3.1.3. *Let M be a differentiable manifold with an affine connection ∇. LEt $c : I \to M$ be a differentiable curve in M and let V_0 be a vector tangent to M at $c(t_0), t_0 \in I$(i.e., $V_0 \in T_{c(t_0)}M$). Then there exists a unique parallel vector field V along c, such that $V(t_0) = V_0$,($V(t)$ is called the parallel transport of $V(t_0)$ along c).*

Proof. Suppose that the theorem was proved for the case in which $c(I)$ is contained in a local neighborhood. By compactness, for any $t_1 \in I$, the segment $c([t_0, t_1]) \subset M$ can be covered be a finite number of coordinate neighborhoods, in each of which V

can be defined, by hypothesis. From uniqueness, the definition coincide when the intersections are not empty, thus allowing the definition of V along all of $[t_0, t_1]$.

We have only, therefore, to prove the theorem when $c(I)$ is contained in a coordinate neighborhood $x(U)$ of a system of coordinates $x : U \subset \mathbb{R}^n \to M$. Let $x^{-1}(c(t)) = (x_1(t), \ldots, x_n(t))$ be the local expression for $c(t)$ and let $V_0 = \sum_j v_0^j X_j$, where $X_j = \frac{\partial}{\partial x_j}(c(t_0))$.

Suppose that there exists a vector field V in $X(U)$ which is parallel along c with $V(t_0) = V_0$. Then $V = \sum v^j X_j$ satisfies

$$0 = \frac{DV}{dt} = \sum_j \frac{dv^j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} v^j \nabla_{X_i} X_j.$$

Putting $\nabla_{X_i}^{X_j} = \sum_k \Gamma_{ij}^k X_k$, and replacing j with k in the first sum, We obtain,

$$\frac{DV}{dt} = \sum_k \left\{ \frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx_i}{dt} \Gamma_{ij}^k \right\} = 0$$

The system of n differential equation in $v^k(t)$,

$$\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx_i}{dt} \Gamma_{ij}^k = 0, \quad k = 1, \ldots, n, \tag{3.3}$$

possesses a unique solution satisfying the initial conditions $v^k(t_0) = v_0^k$. It then follows that, if V exists, it is unique. Moreover, since the system is linear, any solution is defined for all $t \in I$, which then proves the existence(and uniqueness) of V with the desired properties.

□

3.2 Riemannian Connection

Definition 3.2.1. *Let M be a differentiable manifold with an affine connection ∇ and a Riemannian metric $\langle\ ,\ \rangle$. A connection is said to be compatible with the metric $\langle\ ,\ \rangle$, when for any smooth curve c and any pair of parallel vector fields P and P' along c, we have $\langle P, P' \rangle$ =constant.*

Proposition 3.2.1. *Let M be a differentiable manifold with an affine connection ∇*

is compatible with the metric if and only if for any vector fields V and W along the differentiable curve $c : I \to M$ we have

$$\frac{d}{dt}\langle V, W\rangle = \langle\frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle, \quad t \in I \tag{3.4}$$

Proof. Let V and W be two parallel vector field along c then $\frac{DV}{dt} = \frac{DW}{dt} = 0$. Then $\frac{d}{dt}\langle V, W\rangle = 0$, so $\langle V, W\rangle$ =constant. Hence ∇ is compatible with the $\langle\ ,\ \rangle$.

To prove converse let us choose an orthonormal basis $\{P_1(t_0), \dots, P_n(t_0)\}$ of $T_{x(t_0)}M, t_0 \in I$. Using Proposition 3.1.3, we acn extend the vectors $P_i(t_0), i = 1, \dots, n$, along c by parallel transport. Because ∇ is compatible with the metric,$\{P_1(t), \dots, P_n(t)\}$is an orthonormal basis of $Tc(t)M$, for any $t \in I$. Therefore, We can write

$$V = \sum_i v^i P_i, \quad W = \sum_i w^i P_i, \quad i = 1, \dots, n.$$

where $v^i$and w^i are differentiable functions on I. It follows that

$$\frac{DV}{dt} = \sum_i \frac{dv^i}{dt} P_i, \quad \frac{DW}{dt} = \sum_i \frac{dw^i}{dt} P_i.$$

Therefore,

$$\begin{aligned}\langle\frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle &= \sum_i \left\{\frac{dv^i}{dt} w^i + \frac{dW^i}{dt} v^i\right\}\\ &= \frac{d}{dt}\left\{\sum_i v^i w^i\right\} = \frac{d}{dt}\langle V, W\rangle.\end{aligned}$$

□

Proposition 3.2.2. *A connection ∇ on a Riemannian manifold M is compatible with the metric if and only if*

$$X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \quad X, Y, Z \in \mathfrak{X}(M) \tag{3.5}$$

Proof. Let, ∇ is compatible with the metric. Let $p \in M$ and let $c : I \to M$ be a differentiable curve with $c(t_0) = p,\ t_0 \in I$, and with $\frac{dc}{dt}|_{t=0} = X(p)$. Then

$$\begin{aligned} X(p)\langle Y, Z\rangle &= \frac{d}{dt}\langle Y, Z\rangle|_{t=t_o} \\ &= \langle \nabla_{X(p)} Y, Z\rangle_p + \langle Y, \nabla_{X(p)} Z\rangle_p \end{aligned}$$

So,

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \quad X, Y, Z \in \mathfrak{X}(M).$$

Converse automatically follows from the defination.

□

Definition 3.2.2. *An affine connection ∇ on smooth manifold M is said to be symmetric when $\nabla_x Y - \nabla_Y X = [X, Y]$ for all $X, Y \in \mathfrak{X}(M)$.*

Remark. In a coordinate system (U, x) the fact that ∇ is symmetric implies that for all $i, j = 1, ..., n$,

$$\nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j] = 0, \;\; X_i = \frac{\partial}{\partial x_i},$$

So, $\Gamma^k_{i,j} = \Gamma^k_{j,i}$

Definition 3.2.3. *(Riemannian Connection)*
Given a Riemannian manifold M, with the metric g. An affine connection ∇ on M is called a Riemannian (or Levi-Civita) connection on M if ∇ satisfies the following conditions:

1. *∇ is symmetric.*

2. *∇ is compatible with the Riemannian metric.*

Theorem 3.2.1 (Levi-Civita)**.** *Given a Riemannian manifold M, there exists a unique affine connection ∇ on M satisfying the following conditions:*

1. *∇ is symmetric.*

2. *∇ is compatible with the Riemannian metric.*

Proof. Suppose initially the existence of such a ∇. Then

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X, Z\rangle$$

$$Y\langle Z,X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle$$

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_z Y\rangle$$

adding first two equation and subtracting the last equation, we have, using the symmetry of ∇, we get,

$$\begin{aligned}
&X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle\\
&= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X, Z\rangle + \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle\\
&\quad - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_z Y\rangle\\
&= \langle \nabla_X Y, Z\rangle - \langle \nabla_Z X, Y\rangle + \langle \nabla_Y Z, X\rangle - \langle \nabla_Z Y, X\rangle + \langle \nabla_X Y, Z\rangle - \langle \nabla_Y X, Z\rangle\\
&\quad + \langle Z, \nabla_Y X\rangle + \langle Z, \nabla_Y X\rangle
\end{aligned}$$

Hence,

$$\begin{aligned}
&\langle Z, \nabla_Y X\rangle\\
&= \frac{1}{2}\Big\{ X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle [X,Z],Y\rangle\\
&\quad - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle \Big\}
\end{aligned}$$

Suppose ∇^1 and ∇^2 are Riemannian connections. Since the right-hand side of the previous equation is independent of the connection, we have

$$\langle Z, \nabla^1_Y X - \nabla^2_Y X\rangle = 0$$

Hence, $\nabla^1 = \nabla^2$

To prove existence, define ∇ by

$$\begin{aligned}
\langle Z, \nabla_Y X\rangle = \frac{1}{2}\Big\{ &X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle [X,Z],Y\rangle\\
&- \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle \Big\}
\end{aligned} \tag{3.6}$$

It suffices to show that such ∇ exists in each coordinate chart since, the uniqueness guarantees that connections agree if the charts overlap. Let $(U,x), x = (x_1,\ldots,x_n)$, be a chart. Using 3.6 and $[X_i, X_j] = 0$, where $X_i = \frac{\partial}{\partial x_i}$. We have

$$\langle \nabla_{X_i} X_j, X_k \rangle = \frac{1}{2}\big(X_i\langle X_j, X_k\rangle + X_j\langle X_k, X_i\rangle - X_k\langle X_i, X_j\rangle\big) \tag{3.7}$$

This is the same as

$$\Gamma_{ij}^l g_{lk} = \frac{1}{2}(X_i g_{jk} + X_j g_{ki} - X_k g_{ij}).$$

Let (g^{ij}) be the inverse matrix of (g_{ij}), i.e. $g_{lk}g^{km} = \delta_{lm}$. Multiplying both sides of the above equality by g^{km} and summing over $k = 1, 2, \ldots, n$, we get

$$\Gamma_{ij}^m = \frac{1}{2}\sum_k (X_i g_{jk} + X_j g_{ki} - X_k g_{ij})g^{km}. \tag{3.8}$$

This formula defines ∇ in U. Furthermore, from (3.8) we get $\Gamma_{ij}^m = \Gamma_{ji}^m$, i.e. ∇ is symmetric. And it is compatible with the metric, since from 3.6 we get

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \quad X, Y, Z \in \mathfrak{X}(M).$$

This completes the proof. □

Remark: For the Euclidean space $\mathbb{R}^n$, we have $\Gamma_{ij}^k = 0$. And covarient derivative have this expression

$$\frac{DV}{dt} = \sum_k \left\{ \frac{dv^k}{dt} + \sum_{i,j}^k \Gamma_{ij}^k v^j \frac{dx_i}{dt} \right\} X_k.$$

Hence, in Euclidean space covarient derivative coincides with the usual derivative.

Chapter 4

Geodesics

In this chapter we will discuss about the curve geodesic as a curve with zero acceleration. And a geodesic minimizes arc length for points “sufficiently close”.

4.1 The geodesic flow

Definition 4.1.1. *Let M be a Riemannian manifold, together with its Riemannian connection. A parametrized curve $\gamma : I \to M$ is a geodesic at $t_0 \in I$ if $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$ at the point t_0; if γ is a geodesic at t, for all $t \in I$, we say that γ is a geodesic. If $[a, b] \subset I$ and $\gamma : I \to M$ is a geodesic, the restriction of γ to $[a, b]$ is called a geodesics segment joining $\gamma(a)$ to $\gamma(b)$.*

If $\gamma : I \to M$ is a geodesic, then

$$\frac{d}{dt}\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle = \langle\frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle + \langle\frac{d\gamma}{dt}, \frac{D}{dt}\frac{d\gamma}{dt}\rangle = 0$$

that is, the length of the tangent vector $\frac{d\gamma}{dt}$ is constant. We assume, from now on , that $|\frac{d\gamma}{dt}| = c \neq 0$, that is, we exclude the geoidesic which reduce to a points. The arc length s of γ, starting from a fixed origin, say $t = t_0$, is given by

$$s(t) = \int_{t_0}^{t} |\frac{d\gamma}{dt}| dt = c(t - t_0).$$

Therefore, the parameter of the geodesic is proportional to arc length. When the parameter is actually arc length, that is, $c = 1$, we say that the geodesic γ is normalized.

Now we are going to determine the local equation satisfied by a geodesic γ in a system of coordinates (U, x) about $\gamma(t_0)$. In U, a curve γ

$$\gamma(t) = (x_1(t), \ldots, x_n(t)).$$

will be geodesic if and only if

$$0 = \frac{D}{dt}(\frac{d\gamma}{dt}) = \sum_k \left(\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} \right) \frac{\partial}{\partial x^k}.$$

Hence the second order system

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} = 0 \qquad k = 1, \ldots, n, \tag{4.1}$$

yields the desire equation.

To study the system (4.1), it is convenient to consider the tangent bundle TM, which will also be useful in future situation.

TM is the set of pairs $(q, v), q \in M, v \in T_qM$. If (U, x) is system of coordinates on M, then any vector in T_qM, $q \in x(U)$, can be written as $\sum_{i=1}^n y_i \frac{\partial}{\partial x_i}$. Taking $(x_1, \ldots, x_n, y_1. \ldots, y_n)$ as a coordinates of (q, v) in TU

Observe that $TU = U \times \mathbb{R}^n$, that is, the tangent bundle is locally a product. In addition, the canonical projection $\pi : TM \to M$ given by $\pi(q, v) = q$ is differentiable.

Any differentiable curve $t \to \gamma(t)$ in M determines a curve $t \to (\gamma(t), \frac{d\gamma}{dt}(t))$ in TM. If γ is a geodesic then, on TU, the curve

$$t \to (x_1(t), \ldots, x_n(t), \frac{dx_1(t)}{dt}, \ldots, \frac{dx_n(t)}{dt})$$

satisfies the system

$$\begin{cases} \frac{dx_k}{dt} = y_k \\ \frac{dy_k}{dt} = -\sum_{i,j} \Gamma_{ij}^k y_i y_j \qquad k = 1, \ldots, n \end{cases} \tag{4.2}$$

in terms of coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on TU. Therefore the second order system (4.11) on U is equivalent to the first order system (4.2) on TU.

Theorem 4.1.1. *Let X be a differentiable vector field on the open set V in the* *manifold M, and let $p \in M$. Then there exists an open set $V_0 \subset V$, $p \in V_0$, a number $\delta > 0$, and a differentiable mapping $\phi : (-\delta, \delta) \times V_0 \to V$ such that the curve $t \to \phi(t, q)$, $t \in (-\delta, \delta)$, $q \in V_0$, is the unique trajectory of X which at the instant $t = 0$ passes through the point q, for every $q \in V_0$*

The mapping $\phi_t : V_0 \to V$ is given by $\phi_t(q) = \phi(t, q)$ is called the flow of X on V.

Lemma 4.1.1. *There exists a unique vector field G on TM whose trajectories are of the form $t \to (\gamma(t), \gamma^{'}(t))$, where γ is a geodesic on M.*

Proof. We shall first prove the uniqueness of G, supposing its existence. consider a system of coordinates (U, x) on M. From the hypothesis, the trajectories of G on TU are given by $t \to (\gamma(t), \gamma^{'}(t))$ where γ is a geodesic. It follows that $t \to (\gamma(t), \gamma^{'}(t))$ where γ is a solution of the system of differentiable equation (4.2). From the uniqueness of the trajectories of such system, we conclude that if G exists, then it is unique.

To prove the existence of G, define it locally by the system (4.2). Using the uniqueness, we conclude that G is well-defined on TM. □

Definition 4.1.2. *The vector field G defined above is called the geodesic field on TM and its flow is called the geodesic flow on TM.*

Applying Theorem 4.1.1 to the geodesic field G at teh point $(p, 0) \in T_pM$, we obtain the following fact:

For each $p \in M$ there exists an open set $\mathcal{U}$ in TU, where (U, x) is a system of coordinates at p and $(p, 0) \in \mathcal{U}$, a number $\delta > 0$ and a C^∞ mapping, $\phi : (-\delta, \delta) \times \mathcal{U} \to TU$, such that $t \to \phi(t, q, v)$ is a unique trajectory of G which satisfies the initial condition $\phi(0, q, v) = (q, v)$, for each $(q, v) \in \mathcal{U}$.

It is possible to choose $\mathcal{U}$ in the form

$$\mathcal{U} = \{(q, v) \in TU : q \in V \text{ and } v \in T_qM \text{ with } |v| < \epsilon_1\},$$

where $V \subset U$ is a neighborhood of $p \in M$. Putting $\gamma = \pi \circ \phi$, where $\pi : TM \to M$ is the canonical projection, we can describe the previous result in the following way.

Proposition 4.1.1. *Given $p \in M$, there exists an open set $V \subset M$, $p \in V$, numbers $\delta > 0$and $\epsilon_1 > 0$ and a C^∞ mapping*

$$\gamma : (-\delta, \delta) \times \mathcal{U} \to M, \ \ \mathcal{U} = \{(q, v) \in TU : q \in V \text{ and } v \in T_qM \text{ with } |v| < \epsilon_1\}$$

such that the curve $t \to \gamma(t, q, v)$, $t \in (-\delta, \delta)$ *is the unique geodesic of* M *which, at the instant* $t = 0$*, passes through* q *with velocity* v*, for each* $q \in V$ *and for each* $v \in T_qM$ *with* $|v| < \epsilon_1$.

Proposition 4.1.1. asserts that if $|v| < \epsilon_1$, the geodesic $\gamma(t, q, v)$ exists in an interval (δ, δ) and is unique. Actually, it is possible to increase the velocity of a geodesic by decreasing its interval of definition, or vice-versa. This follows from the following lemma.

Lemma 4.1.2. *(Homogeneity of a geodesic.)*
If the geodesic $\gamma(t, q, v)$ *is defined on the interval* (δ, δ)*, then the geodesic* $\gamma(t, q, av)$, $a \in \mathbb{R}, a > 0$*, is defined on the interval* $(-\frac{\delta}{a}, \frac{\delta}{a})$ *and*

$$\gamma(t, q, av) = \gamma(at, q, v).$$

Proof. Let, $h : (-\frac{\delta}{a}, \frac{\delta}{a}) \to M$ be a curve given by $h(t) = \gamma(at, q, v)$. Then $h(0) = q$ and $\frac{dh}{dt}(0) = av$. In addition, since $h^{'}(t) = a\gamma^{'}(at, q, v)$ In addition, since $h^{'}(t) = a\gamma^{'}(at, q, v)$,

$$\frac{D}{dt}(\frac{dh}{dt}) = \nabla_{h^{'}(t)}h^{'}(t) = a^2\nabla_{\gamma^{'}(at,q,v)}\gamma^{'}(at, q, v) = 0$$

where, for the 1st equality, we extend $h^{'}(t)$ to a neighborhood of $h(t)$ in M. Therefore, h is a geodesic passing through q with velocity av at instant $t = 0$. By the uniqueness,

$$h(t) = \gamma(at, q, v) = \gamma(t, q, av).$$

□

Proposition 4.1.1, together with this lemma of homogeneity, permits us to make the interval of definition of geodesic uniformly large in a neighborhood of p. More precisely, we have the following fact.

Proposition 4.1.2. *Given* $p \in M$*, there exists an open set* $V \subset M$*,* $p \in V$*, a numbers* $\epsilon > 0$ *and a* C^∞ *mapping*

$$\gamma : (-2, 2) \times \mathcal{U} \to M, \;\; \mathcal{U} = \{(q, w) \in TM : q \in V \textit{ and } w \in T_qM \textit{ with } |w| < \epsilon\}$$

such that the curve $t \to \gamma(t, q, v)$, $t \in (-2, 2)$ *is the unique geodesic of* M *which, at the instant* $t = 0$*, passes through* q *with velocity* w*, for each* $q \in V$ *and for each*

$w \in T_qM$ with $|w| < \epsilon_1$.

Proof. The geodesic $\gamma(t, q, v)$ of Proposition 4.1.1 is defined for $|t| < \delta$ and for $|v| < \epsilon_1$. From the lemma of homogenicity, $\gamma(t, q, \frac{\delta v}{2})$ is defined for $|t| < 2$. Taking $\epsilon < \frac{\delta \epsilon_1}{2}$, we obtain that the geodesic $\gamma(t, q, w)$ is defined for $|t| < 2$ and $|w| < \epsilon$. □

By analogous argument, we can make the velocity of geodesic uniformly large in a neighborhood of p. Proposition 4.1.2. permits us to introduce the concept of exponential map in the following manner.

Definition 4.1.3. *Let $p \in M$ and let $\mathcal{U} \subset TM$ be an open set. Then the map $exp : \mathcal{U} \to M$ is given by the above proposition*

$$exp(q, v) = \gamma(1, q, v), \quad (q, v) \in \mathcal{U}$$

is called the exponential map on $\mathcal{U}$, where γ is a geodesic. We define, $exp_q : B_\epsilon(0) \subset T_qM \to M$ by $exp_q(v) = exp(q, v)$.

Where $B_\epsilon(0)$ an open ball with center at origin 0 of T_qM and of radius ϵ.

Geometrically, $exp_q(v)$ is the point of M obtain by going out the length equal to $|v|$, starting from q, along a geodesic which passes through q with velocity equal to $\frac{v}{|v|}$.

Proposition 4.1.3. *Given $q \in M$, there exists an $\epsilon > 0$ such that $exp_q : B_\epsilon(0) \subset T_qM \to M$ is diffeomorphism of $B_\epsilon(0)$ onto an open subset of M.*

Proof. Let us calculate $d(exp_q)_0$:

$$\begin{aligned} d(exp_q)_0(v) &= \frac{d}{dt}(exp_q(tv))|_{t=0} \\ &= \frac{d}{dt}(\gamma(1, q, tv))|_{t=0} \\ &= \frac{d}{dt}(\gamma(1, q, v))|_{t=0} = v \end{aligned}$$

Hence, $d(exp_q)_0$ is the identity of T_qM, and it follows form the inverse function theorem that exp_q is a local diffeomorphism on a neighborhood of 0. □

Example 4.1.1. *Let $M = \mathbb{R}^n$, since the covarient derivative coincide with the usual derivative, the geodesics are straight lines parametrized proportionally to arc length. The exponential map is clearly the identity map.*

4.2 Minimizing Properties of Geodesics

Definition 4.2.1. *A segment of the geodesic $\gamma : [a, b] \to M$ is called minimizing if $l(\gamma) \leq l(c)$, where $l(\ )$ denotes the length of the curve and c is an arbitrary piecewise differentiable curve joining $\gamma(a)$ and $\gamma(b)$*

In the proof of Gauss lemma, we shall use the following terminology.

Definition 4.2.2. *Let A be a connected set in $\mathbb{R}^2$, $U \subset A \subset \overline{U}$, U open, such that the boundary ∂A of A is piecewise differentiable curve with vertex angles different from π. A parametrized surface in M is a differentiable mapping $s : A \subset \mathbb{R}^2 \to M$. (Observe that to say that s is differentiable on A means that there exists an open set $U \supset A$ to which s can be extended differentiably. The condition on the vertex angles of A is necessary to ensure that the differential of s does not depend on the given extension.)*

A vector field V along s is a mapping which associates to each $q \in A$ a vector $V(q) \in T_{s(q)}M$, and which is differentiable in the following sense: if f is a differentiable function on M ,then the mapping $q \to V(q)f$ is differentiable.

Let (u, v) be cartesian coordinates on R^2 For v_0 fixed, the mapping $u \to s(u, v_0)$, where u belongs to a connected component of $A \cap v = v_0$, is a curve in M, and $ds(\frac{\partial}{\partial u})$, which we indicate by $\frac{\partial s}{\partial u}$ is a vector field along this curve. This defines $\frac{\partial s}{\partial u}$ for all $(u, v) \in A$ and $\frac{\partial s}{\partial u}$ is a vector field along s. The vector field $\frac{\partial s}{\partial u}$, is defined analogously.

If V is a vector field along $s : A \to M$, let us define the covariant derivative $\frac{DV}{\partial u}$ and $\frac{DV}{\partial v}$ in the following way. $\frac{DV}{\partial u}(u, v_0)$ is the covariant derivative along the curve $u \to s(u, v_0)$ of the restriction of V to this curve. This defines $\frac{DV}{\partial u}(u, v)$ for all $(u, v) \in A$. $\frac{DV}{\partial u}$ is defined analogously.

Lemma 4.2.1. *(symmetry)*
If M is a differentiable manifold with a symmetric connection and $s : A \to M$ is a parametrized surface then:

$$\frac{D}{\partial u}\frac{\partial s}{\partial v} = \frac{D}{\partial v}\frac{\partial s}{\partial u}$$

Lemma 4.2.2. *(Gauss)*
Let $p \in M$ and let $v \in T_pM$ such that exp_pv is defined. Let $w \in T_pM \approx T_v(T_pM)$. Then

$$\langle (dexp_p)_v(v), (dexp_p)_v(w) \rangle = \langle v, w \rangle$$

Proof. First we shall prove that $(dexp_q)_v = v$. Let us consider a curve $\alpha : I \to T_pM$ st. $\alpha(0) = v$, $\alpha'(0) = v \in T_v(T_pM) \cong T_pM$, so $\alpha(t) = v + tv$. Now,

$$\begin{aligned} (dexp_p)_v(v) &= \frac{d}{dt}(exp_p \circ \alpha(t))|_{t=0} \\ &= \frac{d}{dt}(exp_p(v + tv))|_{t=0} \\ &= \frac{d}{dt}(\gamma(1, p, v + tv))|_{t=0} \\ &= \frac{d}{dt}(\gamma(1 + t, p, v))|_{t=0} \\ &= \frac{d}{dt}(\gamma(s, p, v))|_{s=1} = v \end{aligned}$$

since, velocity of γ is constant. Now, let $w = w_T + w_N$, where w_T is the parallel to v and w_N is normal to v, so, $w_T = av, a \in \mathbb{R}$.

$$\begin{aligned} \langle (dexp_p)_v(v), (dexp_p)_v(w) \rangle &= \langle (dexp_p)_v(v), (dexp_p)_v(w_T) \rangle \\ &\quad + \langle (dexp_p)_v(v), (dexp_p)_v(w_N) \rangle \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \langle (dexp_p)_v(v), (dexp_p)_v(w_T) \rangle &= a\langle (dexp_p)_v(v), (dexp_p)_v(w_T) \rangle \\ &= a\langle v, v \rangle = \langle v, w_T \rangle \end{aligned}$$

$$\langle (dexp_p)_v(v), (dexp_p)_v(w_T) \rangle = \langle v, w_T \rangle \tag{4.3}$$

Let, $v(s)$ is a curve in T_pM with $v(0) = v$ $V'(0) = w_N$ and $|v(s) = constant$. Since, exp_pv is defined, there exists $\epsilon > 0$ such that $exp_p(u)$ is defined for $u = tv(s)$, $0 \le t \le 1$, $-\epsilon < s < \epsilon$. Consider a parametrized surface $f : A \to M$, $A = \{(t, s) : 0 \le t \le 1, -\epsilon < s < \epsilon\}$ given by $f(t, s) = exp_ptv(s)$, the curve

$t \to f(t, s_0)$ are geodesics

Now, $\frac{\partial f}{\partial s}(1,0)$ is the tangent vector to the curve $s \to f(1,s)$ at $s = 0$ (By defn.). Hence,

$$\frac{\partial f}{\partial s}(1,0) = \frac{d}{ds} f(1,s)|_{s=0} = \frac{d}{ds}(exp_p(v(s)))|_{s=0}$$

and

$$(dexp_p)_v(w_N) = (dexp_p)_{v(0)}(w_N) = \frac{d}{ds}(exp_p(v(s)))|_{s=0}$$

Hence,

$$\frac{\partial f}{\partial s}(1,0) = (dexp_p)_v(w_N).$$

Similarly we can get,

$$\frac{\partial f}{\partial t}(1,0) = (dexp_p)_v(v).$$

Now,

$$\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle (1,0) = \langle (dexp_p)_v(w_N), (dexp_p)_v(v) \rangle$$

In addition for all (t, s) we have

$$\frac{\partial}{\partial t} \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle = \langle \frac{D}{dt} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle + \langle \frac{\partial f}{\partial s}, \frac{D}{dt} \frac{\partial f}{\partial t} \rangle$$

The last term of the equation is zero, since $\frac{\partial f}{\partial t}$ is the tangent vector to a geodesic. From the symmetry lemma the 1st term of the sum transformed into

$$\langle \frac{D}{dt} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle = \langle \frac{D}{ds} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle = \frac{1}{2} \frac{\partial}{\partial s} \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle = 0.$$

So, $\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \rangle$ is independent of t. Now,

$$\lim_{t \to 0} \frac{\partial f}{\partial s}(t, 0) = \lim_{t \to 0} (dexp_p)_{tv}(tw_N) = 0 \tag{4.4}$$

Since, $\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle$ is independent of t. So, from 4.4

$$\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle (1,0) = \langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \rangle (0.0) = 0$$

So, form 4.2

$$\langle (dexp_p)_v(v), (dexp_p)_v(w) \rangle = \langle v, w_T \rangle + 0 = \langle v, w_T \rangle + \langle v, w_N \rangle = \langle v, w \rangle.$$

This completes the proof. □

If exp_p is a diffeomorphism of a neighborhood V of the origin in in T_pM, $exp_pV = U$ is called a normal neighborhood of p. If $B_\epsilon(0)$ is such that $\overline{B_\epsilon(0)} \subset V$, we call $exp_pB_\epsilon(0) = B_\epsilon(p)$ the normal ball (or geodesic ball) with center p and radius ϵ. From the Gauss lemma, the boundary of a normal ball is a hypersurface (submanifold of codimension 1) in M orthogonal to the geodesics that start from 4, it is denoted by $S_\epsilon(p)$ and called the normal sphere (or geodesic sphere) at 4. The geodesics in $B_\epsilon(p)$ that begin at p are referred to as radial geodesics.

We now show that geodesics locally minimize the arc length. More precisely, we have the following fact.

Proposition 4.2.1. *Let $p \in M$, U a normal neighborhood of p, and $B \subset U$ a normal ball of center p. Let $\gamma : [0, 1] \to B$ a geodesic segment with $\gamma(0) = p$. If $c : [0, 1] \to M$ is any piecewise differentiable curve joining $\gamma(0)$ to $\gamma(1)$ then $l(\gamma) \le l(c)$ and if equality holds then $\gamma([0, 1]) = c([0, 1])$.*

It should be noted that the proposition above is not global. If we consider a sufficiently large arc of a geodesic it can cease minimizing the arc length after awhile. For example the geodesics on the sphere which start at a point p are no longer minimizing after they pass through the antipode of p.

Chapter 5

Curvature

In this chapter we introduce the Riemann curvature, sectional and the Ricci and scalar curvature of a Riemannian manifold. Riemann curvature intuitively measures how much a Riemannian manifold deviates from being Euclidean.

5.1 Curvature

Definition 5.1.1 (Curvature). *The curvature R of a Riemannian manifold M is a correspondence that associates to every pair $X, Y \in \mathfrak{X}(M)$ a mapping $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by*

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \quad Z \in \mathfrak{X}(M),$$

where ∇ is the Riemannian connection of M.

Observe that if $M = R^n$, then $R(X,Y)Z = 0$ for all $X, Y, Z \in \mathfrak{X}(\mathbb{R}^n)$. In fact, if the vector field Z is given by $Z = (z_1, \dots, z_n)$, with the components of Z coming from the natural coordinates of $\mathbb{R}^n$, since for $\mathbb{R}^n$ all $\Gamma_{ij}^k = 0$, we obtain

$$\nabla_X Z = (Xz_1, \dots, Xz_n),$$

hence

$$\nabla_Y \nabla_X Z = (YXz_1, \dots, YXz_n),$$

which implies that

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = ([X,Y]z_1, \dots, [X,Y]z_n) = \nabla_{[x,y]} Z$$

So,

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[x,y]} Z = 0.$$

We are able to think of R as a way of measuring how much M deviates from being Euclidean.

If we consider a system of coordinates around $p \in M$, and we have $[X_i, X_j] = 0$, and we obtain

$$R(X_i, X_j)X_k = (\nabla_{X_j} \nabla_{X_i} - \nabla_{X_i} \nabla_{X_i})X_k$$

from this we can say that, the curvature measures the non-commutativity of the covarient derivative.

Proposition 5.1.1. *The curvature R of a Riemannian manifold has the following properties.*
(1) R is bilinear in $\mathfrak{X}(M) \times \mathfrak{X}(M)$, that is,

$$R(fX_1 + gX_2, Y_1) = fR(X_1, Y_1) + gR(X_2, Y_1)$$

$$R(X_1, fY_1 + gY_2) = fR(X_1, Y_1) + gR(X_2, Y_2)$$

where $f, g \in \mathcal{D}(M)$, and $X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(M)$
(2) For any $X, Y \in \mathfrak{X}(M)$, the curvature operator $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is linear, that is,

$$R(X,Y)(Z+W) = R(X,Y)Z + R(X,Y)W.$$

$$R(X,Y)fZ = fR(X,Y)Z$$

Proposition 5.1.2. *(Bianchi Identity)*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

Proof. Its follows directly form the definition of Riemannian curvature and the symmetry of the Riemannian connection. □

From now on we shall write $\langle R(X,Y)Z, T\rangle = (X,Y,Z,T)$.

Proposition 5.1.3. *(1)* $(X, Y, Z, T) + (Y, Z, X, T) + (Z, X, Y, T) = 0$
(2) $(X, Y, Z, T) = -(Y, X, Z, T)$
(3) $(X, Y, Z, T) = -(X, Y, T, Z)$
(4) $(X, Y, Z, T) = (Z, T, X, Y)$.

Now, let us consider a coordinate system (U, x) at the point $p \in M$ and $\frac{\partial}{\partial x_i} = X_i$. Then,

$$R(X_i, X_j)X_k = \sum_l R^l_{ijk} X_l.$$

Thus R^l_{ijk} are the components of the curvature R in (U, x). If

$$X = \sum_i u^i X_i, \quad Y = \sum_j v^j X_j, \quad Z = \sum_k w^k X_k,$$

we obtain from the linearity of R,

$$R(X, Y)Z = \sum_{i,j,k,l} R^l_{ijk} u^i v^j w^k X_l.$$

To express R^l_{ijk} in term of the coefficients γ^k_{ij} of the Riemannian connection, we have,

$$\begin{aligned} R(X_i, X_j)X_k &= \nabla_{X_j} \nabla_{X_i} X_k - \nabla_{X_i} \nabla_{X_j} X_k \\ &= \nabla_{X_j} (\sum_l \Gamma^l_{ik} X_l) - \nabla_{X_i} (\sum_l \Gamma^l_{jk} X_l), \end{aligned}$$

Then By direct calculation we get,

$$R^s_{ijk} = \sum_l \Gamma^l_{ik} \gamma^s_{jl} - \sum_l \Gamma^l_{jk} \gamma^s_{il} + \frac{\partial}{\partial x_j} \Gamma_{ik} s - \frac{\partial}{\partial x_i} \Gamma^s_{jk}.$$

Now,

$$\langle R(X_i X_j)X_k, X_s \rangle = \sum_l R^l_{ijk} g_{ls} := R_{ijks},$$

We can write the identities of the Proposition 5.1.3. as

$$R_{ijks} + R_{jkis} + R_{kijs} = 0$$

$$R_{ijks} = -R_{jiks}$$

$$R_{ijks} = -Rijsk$$

$$R_{ijks} = R_{ksij}.$$

5.2 Sectional curvature

let V be a real vector space (of dimension at least 2) equipped with an inner-product $\langle \ , \ \rangle$, for each $x, y \in V$ denote the area of the parallelogram determined by the pair of vectors x and y by

$$|x \wedge y| := \sqrt{|x|^2|y|^2 - |\langle x, y\rangle|^2}$$

Proposition 5.2.1. *Let $\sigma \subset T_pM$ be a two-dimensional subspace of the tangent space T_pM and let $x, y \in \sigma$ be two linearly independent vectors. Then*

$$K(x, y) = \frac{(x, y, x, y)}{|x \wedge y|^2}$$

does not depend on the choice of the vectors $x, y \in \sigma$.

Proof. First, observe that it is possible to transform the basis $\{x, y\}$ for σ into any other basis for σ using compositions of the operations:

1. $\{x, y\} \to \{y, x\}$
2. $\{x, y\} \to \{\lambda x, y\}$
3. $\{x, y\} \to \{x + \lambda y, y\}$

Hence, it suffices to prove that K is invariant under these operations

(1) Clearly, $|x \wedge y| = |y \wedge x|$, and so it suffices to show that $\langle R(y, x)y, x\rangle = \langle R(x, y)x, y\rangle$, which follows by applying Proposition 5.1.3.

$$\langle R(y, x)y, x\rangle = -\langle R(x, y)y, x\rangle = \langle R(x, y)x, y\rangle.$$

(2) Suppose $\lambda \in \mathbb{R} \setminus \{0\}$. Since $|\lambda x \wedge y| = |\lambda||x \wedge y|$, it suffices to note that $\langle R(\lambda x, y)(\lambda x), y\rangle = \lambda^2\langle R(x, y)x, y\rangle$ by the bilinearity of R on $\mathfrak{X}(M) \times \mathfrak{X}(M)$ and linearity of $R(.,.)$ on $\mathfrak{X}(M)$.

(3) Suppose $\lambda \in \mathbb{R}$, Then we have

$$|(x + \lambda y) \wedge y|^2 = |x + \lambda y|^2|y|^2 - |\langle x + \lambda y, y\rangle|^2$$

$$= (|x|^2 + 2\lambda\langle x, y\rangle + \lambda^2|y|^2)|y|^2 - (\langle x, y\rangle^2 + \lambda^2|y|^4 + 2\lambda\langle x, y\rangle|y|^2)$$

$$= |x|^2|y|^2 - |\langle x, y\rangle|^2$$

$$|x \wedge y|^2$$

and so it remains to show that $\langle R(x + \lambda y, y)(x + \lambda y), y\rangle = \langle R(x, y)x, y\rangle$ For this, observe that the bilinearity of R on $\mathfrak{X}(M) \times \mathfrak{X}(M)$ and linearity of $R(.,.)$ on $\mathfrak{X}(M)$ yield

$$\langle R(x + \lambda y, y)(x + \lambda y), y\rangle = \langle R(x, y)(x + \lambda y), y\rangle + \lambda\langle R(y, y)(x + \lambda y), y\rangle$$

$$= \langle R(x, y)x, y\rangle + \lambda\langle R(x, y)y, y\rangle + \lambda\langle R(y, y)x, y\rangle + \lambda^2\langle R(y, y)y, y\rangle$$

and, hence, the result follows by applying parts (2) and (3) of Proposition (2.5) to obtain $\langle R(y, y)y, y\rangle = 0$ and $\langle R(y, y)x, y\rangle = \langle R(x, y)y, y\rangle = 0$.

□

Definition 5.2.1 (Sectional Curvature)**.** *Let $(M, \langle\ ,\ \rangle)$ be any Riemannian manifold equipped with the Levi-Civita connection. For a point $p \in T_pM$ and a two-dimensional subspace $\sigma \subset T_pM$, the real number $K(x, y) = K(\sigma)$, where $\{x, y\}$ is any basis of σ, is called the sectional curvature of σ at p, is given by*

$$K(\sigma) = K(x, y) = \frac{\langle R(x, y)x, y\rangle}{|x \wedge y|^2}$$

Where, $|x \wedge y|^2 = \langle x, x\rangle\langle y, y\rangle - \langle x, y\rangle^2$.

Sectional curvature is important because of its relationship to the curvature operator R. In particular for any $p \in M$, knowing the values $K(\sigma)$ for all two-dimensional subspaces of σ of T_pM completely determines R. We make this precise with the following lemma:

Lemma 5.2.1. *Let V be a vector space of dimension ≥ 2, provided with an inner product $\langle\ ,\ \rangle$. Let $R : V \times V \times V \to V$ and $R' : V \times V \times V \to V$ be tri-linear mappings such that conditions:*

1. $\langle R(x, y)z, t\rangle + \langle R(y, z)x, t\rangle + \langle R(z, x)y, t\rangle = 0$

2. $\langle R(x, y)z, t\rangle = -\langle R(y, x)z, t\rangle$

3. $\langle R(x,y)z,t\rangle = -\langle R(x,y)t,z\rangle$

4. $\langle R(x,y)z,t\rangle = \langle R(z,t)x,y\rangle$

are satisfied by

$$(x,y,z,t) = \langle R(x,y)z,t\rangle,\ (x,y,z,t)^{'}\langle R^{'}(x,y)z,t\rangle$$

. If x,y are two linearly independent vectors, We may write,

$$K(\sigma) = \frac{\langle R(x,y)x,y\rangle}{|x\wedge y|^2},\quad K^{'}(\sigma) = \frac{\langle R^{'}(x,y)x,y\rangle}{|x\wedge y|^2},$$

where σ is the bi-dimensional subspace generated by two linearly independent vectors $\{x,y\}$. If for all $\sigma \subset V$, $K(\sigma) = K^{'}(\sigma)$, then $R = R^{'}$.

Proof. It suffices to prove that $(x,y,z,t) = (x,y,z,t)^{'}$ for any $x,y,z,t \in V$. Observe first that, by hypothesis, we have $(x,y,x,y) = (x,y,x,y)^{'}$, for all $x,y \in V$. Then

$$(x+z,y,x+z,y) = (x+z,y,x+z,y)^{'}$$

hence

$$(x,y,x,y) + 2(x,y,z,y) + (z,y,z,y) = (x,y,x,y)^{'} + 2(x,y,z,y)^{'} + (z,y,z,y)^{'}$$

and, therefore

$$(x,y,z,y) = (x,y,z,y)^{'}$$

Using what we have just proved, we obtain

$$(x,y+t,z,y+t) = (x,y+t,z,y+t)^{'}$$

hence

$$(x,y,z,t) + (x,t,z,y) = (x,y,z,t)^{'} + (x,t,z,y)^{'}$$

which can be written further as

$$(x,y,z,t) - (x,y,z,t)^{'} = (y,z,x,t) - (y,z,x,t)^{'}$$

It follows that, the expression $(x,y,z,t) - (x,y,z,t)^{'}$ is invariant by the cyclic permutation of the first three elements. Therefore, by (1) of Proposition 1.5, we have

$$3[(x, y, z, t) - (x, y, z, t)^{'}] = 0,$$

hence

$$(x, y, z, t) = (x, y, z, t)^{'} \qquad \text{for all } x, y, z, t \in V$$

□

Lemma 5.2.2. *Let M be a Riemannian manifold and p a point of M. Define a tri-linear mapping $R^{'} : T_pM \times T_pM \times T_pM \to T_pM$ by*

$$\langle R^{'}(X, Y, W), Z\rangle = \langle X, W\rangle\langle Y, Z\rangle - \langle Y, W\rangle\langle X, Z\rangle,$$

for all $X, Y, W, Z \in T_pM$. Then M has constant sectional curvature equal to K_0 if and only if $R = K_0R^{'}$, where R is the curvature of M.

Proof. Assume that the sectional curvature at p is constant, $K(p, \sigma) = K_0$, for all $\sigma \subset T_pM$ and set $\langle R^{'}(X, Y, W,), Z\rangle = (x, y, w, z)^{'}$. $R^{'}$ satisfies the following properties:

1. $(X, Y, W, Z)^{'} + (Y, W, X, Z)^{'} + (W, X, Y, Z)^{'} = 0$
2. $(X, Y, W, Z)^{'} = -(Y, X, W, Z)^{'}$
3. $(X, Y, W, Z)^{'} = -(X, Y, Z, W)^{'}$
4. $(X, Y, W, Z)^{'} = (W, Z, X, Y)^{'}$

Then by definition we have $\langle R(X, Y)X, Y\rangle = K_o|x \wedge y|^2$ for all $X, Y \in T_pM$ since,

$$(X, Y, X, Y)^{'} = \langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2,$$

we have for all pair of vectors $X, Y \in T_pM$,

$$R(X, Y, X, Y) = K_0(|X|^2|Y|^2 - \langle X, Y\rangle^2) = K_0R^{'}(X, Y, X, Y).$$

By Lemma 5.2.1, it implies that for all X, Y, W, Z,

$$R(X, Y, W, Z) = R^{'}(X, Y, W, Z)$$

hence $R = K_0 R'$.

Conversely, assume that $K_0 R'(X,Y,W) = R(X,Y,W)$ for all $X, Y, W \in T_pM$ and $K_0 \in \mathbb{R}$.
as $\langle R'(X,Y,X), Y\rangle = |X \wedge Y|^2$ for any two dimensional subspace $\sigma \subset T_pM$ and any pair $\{X, Y\}$ of linearly independent vectors in T_pM. We have,

$$K(X,Y) = \frac{\langle R(X,Y,X), Y\rangle}{|X \wedge Y|^2} = \frac{K_0 |X \wedge Y|^2}{|X \wedge Y|^2} = K_0$$

□

corollary 5.2.1. *Let M be a Riemannian manifold, p a point of M and $\{e_1, \dots, e_n\}$, $n = dim\ M$, an orthonormal basis of T_pM. Define $R_{ijkl} = \langle R(e_i, e_j)e_k, e_l\rangle$, $i, j, k, l = 1, \dots, n$. Then $K(p, \sigma) = K_0$ for all $\sigma \subset T_pM$, iff*

$$R_{ijkkl} = K_0(\delta_{ik}\delta_{jk} - \delta_{il}\delta_{jk}),$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

In other words, $K(p, \sigma) = K_0$ for all $\sigma \subset T_pM$ if and only if $R_{ijij} = -R_{ijji} = K_0$ for all $i \neq j$, and $R_{ijkl} = 0$ in others cases.

5.3 Ricci and scalar curvature

We conclude this chapter by defining the Ricci and scalar curvatures of a Riemannian manifold. These are obtained by taking certain combination of sectional curvature and these play an important role in Riemannian geometry

Definition 5.3.1. *(Ricci and scalar curvature)*
Let $p \in M$ and $x = z_n$ be a unit vector in T_pM, we take an orthonormal basis $\{z_1, z_2, \dots, z_{n-1}\}$ of the hyperplane in T_pM orthogonal to x. The Ricci curvature at p in the direction x is defined by

$$Ric_p(x) = \frac{1}{1-n} \sum_i \langle R(x, z_i)x, z_i\rangle, \quad i = 1, \dots, n-1.$$

The scalar curvature at p is defined by

$$K(p) = \frac{1}{n}\sum_{j} Ric_p(z_j) = \frac{1}{n(n-1)}\sum_{ij}\langle R(z_i, z_j)z_i, z_j\rangle \quad j = 1, \dots, n.$$

These expressions are called the Ricci curvature in the direction x and the scalar curvature at p, respectively.

PART 2

Sobolev Space on $\mathbb{R}^n$

Chapter 6

Weak derivatives and Sobolev Spaces

In this part we will study the theory of Sobolev space on $\mathbb{R}^n$, which turn out to be the proper setting in which to apply ideas of functional analysis to glean information concerning partial differential equation.

6.1 Weak derivatives

Definition 6.1.1. *Weak derivatives Suppose $u, v \in L^1_{loc}(U)$, and α is a multi index. We say that v is the α^{th}- weak partial derivative of u, written $D^\alpha u = v$, provided*

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v dx \tag{6.1}$$

for all test functions $\phi \in C_c^\infty(U)$.

Remark: Classical derivatives are defined pointwise as limit of difference quotients. Weak derivatives, on the other hand, are defined in an integral sense. By changing a function on a set of measure zero we do not affect its weak derivatives.

Lemma 6.1.1. *A weak α^{th}th-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

Proof. Assume that $v, \tilde{v} \in L^1_{loc}(U)$ satisfies

$$\int_U u(x) D^\alpha \phi(x) dx = (-1)^\alpha \int_U u(x)\phi(x) dx = (-1)^\alpha \int_U \tilde{v}(x)\phi(x) dx$$

for all $\phi \in C_c^\infty(U)$. This implies

$$\int_U (v(x) - \tilde{v}(x))\phi(x)dx = 0 \quad \forall \phi \in C_c^\infty(U).$$

Hence, $v - \tilde{v} = 0$ almost everywhere.

□

Example 6.1.1. *Let $n = 1$, $U = (0, 2)$ and*

$$u(x) = \begin{cases} x & \text{if } \ 0 < x \leq 1 \\ 1 & \text{if } \ 1 < x < 2 \end{cases} \qquad v(x) = \begin{cases} 1 & \text{if } \ 0 < x \leq 1 \\ 0 & \text{if } \ 1 < x < 2 \end{cases}$$

Now for all $\phi \in C_c^\infty(U)$

$$\begin{aligned}
\int_0^2 u(x)\phi'(x)dx &= \int_0^1 u(x)\phi'(x)dx + \int_1^2 u(x)\phi'(x)dx \\
&= \int_0^1 x\phi'(x)dx + \int_1^2 1\phi'(x)dx \\
&= x\phi(x)|_0^1 - \int_0^1 \phi(x)dx + \int_1^2 \phi'(x)dx \\
&= \phi(1) - \int_0^1 \phi(x)dx + \phi(2) - \phi(1) \\
&= -\int_0^1 \phi(x)dx = \int_0^2 v(x)\phi(x)dx.
\end{aligned}$$

Hence, $\int_0^2 u(x)\phi'(x)dx = -\int_0^2 v(x)\phi(x)dx$, for all $\phi \in C_c^\infty(U)$. $v(x)$ is the weak derivative of $u(x)$.

Example 6.1.2. *$n = 1$, $U = (0, 2)$ and*

$$u(x) = \begin{cases} x & \textit{if } 0 < x \leq 1 \\ 2 & \textit{if } 1 < x < 2 \end{cases}$$

In order to check, that u does not have a weak derivative we have to show that there does not exist any function $u \in L^1_{loc}(U)$ satisfying

$$\int_0^2 u(x)\phi'(x)dx = -\int_0^2 v(x)\phi(x)dx$$

for all $\phi \in C_c^\infty(U)$*. Assume there exists a* $v \in L^1_{loc}(U)$ *satisfying the previous equation. Then,*

$$-\int_0^2 v(x)\phi(x)dx = \int_0^2 u(x)\phi^{'}(x)dx = \int_0^1 x\phi^{'}(x)dx + \int_1^2 2\phi^{'}(x)dx$$

$$= x\phi(x)|_0^1 - \int_0^1 \phi(x)dx + 2(\phi(2) - \phi(1)) = -\phi(1) - \int_0^1 \phi(x)dx$$

is valid for all $\phi \in C_c^\infty(U)$*. We choose a sequence* $(\phi_m)_{m=1}^\infty$ *of smooth functions satisfying*

$$0 \leq \phi_m \leq 1, \ \ \phi_m(1) = 1 \ and \ \ \phi_m(x) \to 0 \ as \ m \to \infty, \ \ \forall x \neq 1$$

Now, replacing ϕ *by* ϕ_m *we get*

$$1 = \phi_m(1) = \int_0^2 v(x)\phi_m(x)dx - \int_0^1 \phi_m(x)dx.$$

We take the limit for $m \to \infty$

$$1 = lim_{m\to\infty}\phi_m(1) = lim_{m\to\infty}\left[\int_0^2 v(x)\phi_m(x)dx - \int_0^1 \phi_m(x)dx\right] = 0.$$

a contradiction.

6.2 The Sobolev spaces $W^{k,p}(U)$

Let $U \subseteq \mathbb{R}^n$ open. Let $1 \leq p \leq \infty$ and k be a non-negative integer.

Definition 6.2.1. *The Sobolev space The Sobolev space* $W^{k,p}(U)$ *is the space of all locally integrable functions* $u : U \to \mathbb{R}$ *such that for every multiindex* α *with* $|\alpha| \leq k$ *the weak derivative* $D^\alpha u$ *exists and* $D^\alpha \in L^p(U)$*.*

Definition 6.2.2. *We define the norm of* $u \in W^{k,p}(U)$ *to be*

$$\|u\|_{W^{k,p}(U)} = \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha u(x)|^p dx\right)^{\frac{1}{p}}, \ \ if \ 1 \leq p < \infty,$$

$$\|u\|_{W^{k,\infty}(U)} = \sum_{|\alpha|\le k} ess\ sup_{x\in U}|D^\alpha u(x)|.$$

Theorem 6.2.1. *For each $k \in \mathbb{N}_0$ and $1 \le p \le \infty$ the Sobolev space $W_{k,p}(U)$ is a Banach space.*

Remark. (1) If $p = 2$, we usually write

$$H^k(U) = W^{k,2}(U), \quad (k = 0, 1, \dots)$$

H^k is a Hilbert space with respect to the inner product

$$\langle u, v\rangle = \sum_{|\alpha|\le k} \int_U D^\alpha u(x) D^\alpha v(x) dx.$$

Definition 6.2.3. *(1) Let $\{u_m\}_{m=1}^\infty$, $u \in W^{k,p}(U)$. We say that u_m converges to u in $W^{k,p}(U)$, written as*

$$u_m \to u \quad in\ W^{k,p}(U)$$

provided $\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0$.
(2) We write

$$u_m \to u \quad in\ W^{k,p}_{loc}(U)$$

to mean

$$u_m \to u \quad in\ W^{k,p}(V)$$

for each $V \subset\subset U$

Definition 6.2.4. *We denote by $W_0^{k,p}(U)$, the closer of $C_c^\infty(U)$ in $W^{k,p}(U)$.*

Theorem 6.2.2. *(Properties of Weak derivatives) Assume $u, v \in W^{k,p}(U)$, $|\alpha| \le k$, $|\alpha| \le k$. Then,*

1. *$D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ for all multiindices α, β with $|\alpha| + |\beta| \le k$.*

2. *For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu u \in W^{k,p}(U)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \le k$*

3. *If V is open subset of U, then $u \in W^{k,p}(V)$.*

4. *If* $\zeta \in C_c^\infty(U)$, *than* $\zeta u \in W^{k,p}(U)$ *and*

$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \left(\begin{array}{c} \alpha \\ \beta \end{array} \right) D^\beta \zeta D^{\alpha-\beta} u \quad (Leibniz's formula)$$

Chapter 7

Approximation in Sobolev Spaces

In order to study the deeper properties of Sobolev spaces, without returning continually to the definition of weak derivatives, we need procedures for approximating a function in a Sobolev space by smooth functions. These approximation procedures allow us to consider smooth functions and then extend the statements to functions in the Sobolev space by density arguments. We have to prove that smooth functions are in fact dense in $W^{k,p}(U)$. The method of mollifiers provides the tool.

7.1 Smoothing by convolution

Definition 7.1.1. *(1) Let $\eta \in C^\infty(\mathbb{R}^n)$ be given by*

$$\eta(x) = \begin{cases} Ce^{1/(|x|^2-1)} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

with constant $C > 0$ chosen such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$.
(2) For each $\epsilon > 0$ we define

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon}).$$

We call η the standard mollifier. The functions η_ϵ are C^∞ and satisfy

$$\int_{\mathbb{R}^n} \eta_\epsilon dx = 1, \ \ spt(\eta_\epsilon) \subset B(0, \epsilon).$$

Definition 7.1.2. *Let $U \subseteq \mathbb{R}^n$ be open and $\epsilon > 0$. Let*

$$U_\epsilon = \{x \in U : d(x, \delta U) > \epsilon\} = \{x \in U : \overline{B(x,\epsilon)} \subseteq U\},$$

where $B(x,\epsilon) = \{y \in \mathbb{R}^n : |x-y| < \epsilon\}$.

Definition 7.1.3. *If $f : U \to \mathbb{R}$ is locally integrable, define its mollification*

$$f^\epsilon := \eta_\epsilon * f \quad in\ U_\epsilon.$$

That is, for $x \in U_\epsilon$

$$f^\epsilon(x) = \int_U \eta_\epsilon(x-y) f(y) dy = \int_{B(0,\epsilon)} \eta_\epsilon(y) f(x-y) dy$$

Theorem 7.1.1. *(Properties of mollifiers)*
(1) $f^\epsilon \in C^\infty(U_\epsilon)$.
(2) $f^\epsilon \to f$ a.e. as $\epsilon \to 0$
(3) If $f \in C(U)$, then $f^\epsilon \to f$ uniformly on compact subsets of U.
(4) If $1 \le p < \infty$ and $f \in L^p_{loc}(U)$, then $f^\epsilon \to f$ in $L^p_{loc}(U)$.

7.2 Local approximation by smooth functions

Lemma 7.2.1. *Let $U_\epsilon \subset U$. Assumed that $f \in L^1_{loc}(U)$ admits a weak derivative $D^\alpha f$ for some multiindex α Then*

$$D^\alpha(f * \eta_\epsilon)(x) = \eta_\epsilon * D^\alpha f(x), \quad for\ all\ x \in U_\epsilon$$

Note that the derivative of the mollification $D^\alpha(f * \eta_\epsilon)$ exists in the classical sense.

Proof.

$$\begin{aligned} D^\alpha(f * \eta_\epsilon)(x) &= \int_U D_x^\alpha \eta_\epsilon(x-y) f(y) dy \\ &= (-1)^{|\alpha|} \int_u D_y^\alpha \eta_\epsilon(x-y) f(y) dy \\ &= (-1)^{|\alpha|+|\alpha|} \int_U \eta_\epsilon(x-y) D^\alpha f(y) dy \\ &= \int_U \eta_\epsilon(x-y) D^\alpha f(y) dy \\ &= \eta_\epsilon * D^\alpha f(x). \end{aligned}$$

□

Theorem 7.2.1. *(Local Approximation by Smooth function)*
Let $u \in W^{k,p}(U)$, $1 \le p < \infty$. Let $\epsilon > 0$ and set

$$u^\epsilon(x) = (\eta_\epsilon * u)(x), \quad x \in U_\epsilon,$$

where η_ϵ is the mollifier, then

$$(1)\ \ u^\epsilon \in C^\infty(U_\epsilon)\ for\ each\ \epsilon > 0,$$

$$(2)\ \ u^\epsilon \to u\ in\ W_{loc}^{k,p}(U),\ as\ \epsilon \to 0.$$

Proof. $u \in W^{k,p}(U)$, therefore $u \in L^1_{loc}(U)$. Hence, by previous lemma for all $|\alpha| \le k$

$$D^\alpha(f * \eta_\epsilon)(x) = \eta_\epsilon * D^\alpha f(x), \quad for\ all\ x \in U_\epsilon$$

Now, for all $V \subset\subset U$, by previous lemma and Properties of mollifiers we have,

$$\|u^\epsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha| \le k} \|D^\alpha u^\epsilon - D^\alpha u\|^p_{L^p(V)} \to 0$$

as $\epsilon \to 0$. This completes the proof. □

7.3 Global approximation by smooth functions

Now we will approximate the function of $W^{k,p}(U)$ globally and we do not assume anything about the smoothness of boundary.

Theorem 7.3.1. *Let $U \subset \mathbb{R}^n$ be open and bounded. Let $u \in W^{k,p}(U)$, $1 \leq p < \infty$. Then there exists a sequence $(U_m)_{m\in\mathbb{N}}$ in $C^\infty(U) \cap W^{k,p}(U)$ such that*

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0$$

Proof. Let,

$$U_i = \{x \in U : d(x, \partial U) > \frac{1}{i}\}, \quad i \in \mathbb{N}$$

Then $U_i \subseteq U_{i+1}$ and

$$U = \bigcup_{i=1}^{\infty} \{x \in U : d(x, \partial U) > \frac{1}{i}\}.$$

Let $V_i = U_{i+3} - \overline{U}_i$. Then $\#\{j \in \mathbb{N} : V_i \cap V_j \neq 0\} \leq 3$. Therefore each $x \in U$ is an element of at least one and at most three sets of the family $(V_i)_{i\in\mathbb{N}}$. We choose $V_0 \subset\subset U$ such that

$$U = \bigcup_{i\in\mathbb{N}_0} V_i$$

Let $(\xi_i)_{i=0}^{\infty}$ be a smooth partition of unity subordinate to the family of open sets $(V_i)_{i=0}^{\infty}$, i.e.

$$0 \leq \xi_i \leq 1, \; \xi_i \in C_c^\infty(V_i), \; for\ all\ i \in \mathbb{N}_0, \quad \sum_{i=0}^{\infty} \xi_i = 1 \; on\ U$$

Let $u \in W^{k,p}(U)$. Then we have that $\xi_i u \in W^{k,p}(U)$ and support $\xi_i u \subset\subset V_i$. Let $\delta > 0$ be fixed. By Theorem 2.2.1 we can choose $\epsilon_i > 0$ such that $u^i = \eta_{\epsilon_i} * (\xi_i u)$ satisfies

$$\|u^i - \xi_i u\|_{W^{k,p}(U)} \leq \frac{\delta}{2^{i+1}}$$

$$supp\ u^i \subset W_i := U_{i+4}\overline{U}_i \supset V_i.$$

We define

$$v(x) := \sum_{i=0}^{\infty} u^i(x), \quad x \in U$$

$v \in C^\infty(U)$, since for every $x \in U$ we have that $\#\{i \in \mathbb{N}_0 : u^i(x) \neq 0\} \leq 3$. We have

$$u = u.1 = \sum_{i=0}^{\infty} \xi_i u$$

Therefore,

$$\begin{aligned}\|u - v\|_{W^{k,p}(U)} &= \|\sum_{i=0}^{\infty} \xi u - \sum_{i=0}^{\infty} u^i\|_{W^{k,p}(U)} \\ &\leq \sum_{i=0}^{\infty} \|\xi_i u - u^i\|_{W^{k,p}(U)} \leq \sum_{i=0}^{\infty} \delta 2^{-i-1} = \delta.\end{aligned}$$

Note that $\|v\|_{W^{k.p}(U)} \leq \|v - u\|_{W^{k,p}(U)} + \|u\|_{W^{k,p}(U)} < \infty$. Summarizing we have that

$$\forall \delta > 0,\ \exists v \in W^{k,p}(U) \cap C^\infty(U) \ : \ \|u - v\|_{W^{k,p}(U)} \leq \delta$$

□

Chapter 8

Extensions

In general, many properties of $W^{k,p}(U)$ can be inherited from $W^{k,p}(\mathbb{R}^n)$ provided U is "nice". The goal of this section is to extend functions in the Sobolev space $W^{k,p}(U)$ to become functions in the Sobolev space $W^{k,p}(\mathbb{R}^n)$. Indeed, we need a strong theorem. Observe for instance that extending $u \in W^{k,p}(U)$ by setting it zero in $\mathbb{R}^n - U$ will not in general work, as we thereby create such a discontinuity along ∂U that the extended function no longer has a weak partial derivative. We must invent a way to extend u that preserves the weak derivatives across ∂U.

Theorem 8.0.1. *(Extension Theorem) Assume $U \subset \mathbb{R}^n$ is open and bounded and ∂U is C^1. Let $V \subset \mathbb{R}^n$ be open and bounded such that $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

such that for all $u \in W^{1,p}(U)$.

1. *$Eu = u$ a.e. in U*
2. *Eu has support within V,*
3. *$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$.*

The constant C depending only on $p, U,$ and V.

Chapter 9

Sobolev inequalities and Embeddings

In this chapter we prove a class of inequalities of the form

$$\|u\|_X \leq C\|u\|_{W^{k,p}(U)} \tag{9.1}$$

where X is a Banach space, i.e. we consider the question: "If $u \in W^{k,p}(U)$, does u belong automatically to a certain other Banach space X? Inequalities of the form (4.1) are called Sobolev type inequalities. This kind of estimates give us information on the embeddings of Sobolev spaces into other spaces.

We say that a Banach space E is continuously embedded into another Banach space F, written $E \hookrightarrow F$ if there exists a constant C such that for all $x \in E$.

$$\|x\|_F \leq \|x\|_E$$

This means that the natural inclusion map $i : E \to F$, $x \to x$ is continuous.

We start the investigations with the Sobolev spaces $W^{1,p}(U)$ and will observe that these Sobolev spaces indeed embed into certain other spaces, but which other spaces depends upon whether

(1) $1 \leq p < n$

(2) $p = n$

(3) $n < p \leq \infty$

9.1 Gagliardo-Nirenberg-Sobolev inequality

For this section lat us assume

$$1 \leq p < n$$

Motivation. We first demonstrate that if any inequality of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \tag{9.2}$$

for certain constants $C > 0$, $1 \leq q < \infty$ and functions $u \in C_c^\infty(\mathbb{R}^n)$ holds, then the number q cannot be arbitary. Let $u \in C_c^\infty(\mathbb{R}^n)$, $u \neq 0$ and define for $\lambda > 0$

$$u_\lambda(x) := u(\lambda x) \quad (x \in \mathbb{R}^n)$$

We assume that (4.2) holds and apply it to u_λ, i.e. there exists a constant C such that for all $\lambda > 0$

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C\|Du_\lambda\|_{L^p(\mathbb{R}^n)} \tag{9.3}$$

Now,

$$\int_{\mathbb{R}^n} |u_\lambda(x)|^q dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u(y)|^q dy$$

and

$$\int_{\mathbb{R}^n} |Du_\lambda(x)|^p dx = \lambda^p \int_{\mathbb{R}^n} |Du(\lambda x)|^p dx = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |Du(y)|^p dy$$

Hence, by (4.3) we get

$$\Big(\frac{1}{\lambda^n}\Big)^{\frac{1}{q}} \|u\|_{L^q(\mathbb{R}^n)} \leq C\Big(\frac{\lambda^p}{\lambda^n}\Big)^{\frac{1}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

and therefore

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-\frac{n}{p}+\frac{n}{q}} \|Du\|_{L^p(\mathbb{R}^n)}.$$

If $1 - \frac{n}{p} + \frac{n}{q} \neq 0$ we acn obtain a contradition by sending λ to 0 or ∞, depending on whether $1 - \frac{n}{p} + \frac{n}{q} > 0$ *or* $1 - \frac{n}{p} + \frac{n}{q} < 0$. Thus, if in fact the desired inequality (4.2) holds, we must necessarily have $1 - \frac{n}{p} + \frac{n}{q} = 0$. This implies that $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and therefore $q = \frac{np}{n-p}$.

Definition 9.1.1. *If $1 \leq p < n$, the Sobolev conjugate of p is $p^* = \frac{np}{n-p}$, $(p^* > p)$*

Theorem 9.1.1. *(Gagliardo-Nirenberg-Sobolev inequality)*

Let $1 \leq p < n$. There exists a constant C, depending only on n and p such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Proof. Assume $p = 1$. Note that u has compact support. Therefore, we have for each $i = 1, \dots, n$ and $x \in \mathbb{R}^n$

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)dy_i$$

and

$$|u(x)| \leq \int_{-\infty}^{\infty} Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|dy_i$$

Then

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|dy_i \right)^{\frac{1}{n-1}}.$$

We integrate the above inequality with respect to x_1 and obtain:

$$\begin{aligned} &\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \\ &\leq \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)|dy_i \right)^{\frac{1}{n-1}} dx_1 \\ &= \left(\int_{-\infty}^{\infty} |Du|dy_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du|dy_i \right)^{\frac{1}{n-1}} dx_1 \end{aligned}$$

Applying the general Hölder inequality with $p_i = \frac{1}{n-1},\ i = 1, \dots, n-1$ we obtain

$$\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \leq \left(\int_{-\infty}^{\infty} |Du|dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du|dx_1 dy_i \right)^{\frac{1}{n-1}}$$

Now we integrate with respect to x_2 and obtain.

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 \\
&\leq \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|Du|dy_1\right)^{\frac{1}{n-1}} dx_2 \int_{-\infty}^{\infty}\left(\prod_{i=2}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_i\right)^{\frac{1}{n-1}} dx_2 \\
&= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_2\right)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\prod_{i=1,i\neq 2}^{n} I_i^{\frac{1}{n-1}} dx_2
\end{aligned}$$

where

$$I_1 = \int_{-\infty}^{\infty}|Dy|dy_1|;\ and\ \ I_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_i\ for\ i=3,\ldots,n.$$

Applying the general Hölder inequality once more we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 dx_2 \\
&\leq \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dy_2\right)^{\frac{1}{n-1}}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dy_1dx_2\right)^{\frac{1}{n-1}} \\
&\prod_{i=3}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|dx_1dx_2dy_i\right)^{\frac{1}{n-1}}
\end{aligned}$$

We continue by integrating with respect to $x_3,\ldots,x_n$ and and using Hölder's general inequality to obtain finally

$$\begin{aligned}
\int_{\mathbb{R}^n}|u|^{\frac{n}{n-1}}dx &\leq \prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|Du|dx_1\ldots dy_i\ldots dx_n\right)^{\frac{n}{n-1}} \\
&= \left(\int_{\mathbb{R}^n}|Du|dx\right)^{\frac{n}{n-1}}
\end{aligned}$$

This is the Gagliardo-Nirenberg-Sobolev inequality for $p=1$.

We consider now the case $1<p<n$. Let $v := |u|^\gamma$ for some $\gamma>1$. We apply

Gagliardo-Nirenberg-Sobolev inequality for $p=1$ to v. Then, by Hölder inequality

$$\begin{aligned}\Big(\int_{\mathbb{R}^n}|u|^{\frac{\gamma n}{n-1}}dx\Big)^{\frac{n-1}{n}} &\le \int_{\mathbb{R}^n}|D|u|^\gamma|dx=\gamma\int_{\mathbb{R}^n}|u|^{\gamma-1}|Du|dx\\ &\le \gamma\Big(\int_{\mathbb{R}^n}|u|^{(\gamma-1)\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^n}|Du|^p dx\Big)^{\frac{1}{p}}.\end{aligned}$$

We choose γ so that $\frac{\gamma n}{n-1}=(\gamma-1)\frac{p}{p-1}$. That is, we set

$$\gamma=\frac{p(n-1)}{n-p}>1$$

in which case $\frac{\gamma n}{n-1}=(\gamma-1)\frac{p}{p-1}=\frac{np}{n-p}=p^*$. Therefore, we get

$$\Big(\int_{\mathbb{R}^n}|u|^{p^*}dx\Big)^{\frac{n-1}{n}}\le\gamma\Big(\int_{\mathbb{R}^n}|u|^{p^*}dx\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^n}|Du|^p dx\Big)^{\frac{1}{p}}.$$

what is equal to

$$\Big(\int_{\mathbb{R}^n}|u|^{p^*}dx\Big)^{\frac{n-1}{n}-\frac{p-1}{p}}\le\gamma\Big(\int_{\mathbb{R}^n}|Du|^p dx\Big)^{\frac{1}{p}}$$

Hence, we get

$$\Big(\int_{\mathbb{R}^n}|u|^{p^*}dx\Big)^{\frac{1}{p^*}}\le C\Big(\int_{\mathbb{R}^n}|Du|^p dx\Big)^{\frac{1}{p}}$$

This completes the proof. □

Note. $\overline{C_c^\infty(\mathbb{R}^n)}:=W_0^{k,p}(\mathbb{R}^n)=W^{k,p}(\mathbb{R}^n)$ and from the Gagliardo-Nirenberg-Sobolev inequality we get,

$$\|u\|_{L^{p^*}(\mathbb{R}^n)}\le C\|Du\|_{L^p(\mathbb{R}^n)}\le K\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Hence, $W^{1,p}(\mathbb{R}^n)$ continuously embedded in $L^{p^*}(\mathbb{R}^n)$.

Theorem 9.1.2. *(Estimates for $W^{1,p}$, $1\le p<n$)*
Let $U\subseteq\mathbb{R}^n$ open and bounded and suppose ∂U is C^1. Assume $1\le p<n$, and $u\in W^{1,p}(U)$. Then $u\in L^{p^*}(U)$, with the estimate

$$\|u\|_{L^{p^*}(U)}\le C\|u\|_{W^{1,p}(U)}$$

the constant C depending only on p, n and U.

Proof. The Extension Theorem yields that there exists an extension $\overline{u} = Eu \in W^{1,p}(\mathbb{R}^n)$, such that

$$\overline{u} = u \textit{ in } U, \ \overline{u} \text{ has compact support}$$

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}.$$

Because $\overline{u}$ has compact support we know from Theorem 7.2.1 that there exists a sequence $(U_m)_{m=1}^{\infty}$ of functions in $C_c^{\infty}(\mathbb{R}^n)$ such that

$$u_m \to \overline{u} \ \textit{ in } W^{1,p}(\mathbb{R}^n)$$

Now according to Theorem 9.1.1 we have that for all $l, m \geq 1$

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq \|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}.$$

Thus, by last last two equation we get

$$u_m \to \overline{u} \quad \textit{in } L^{p^*}.$$

By the Gagliardo-Nirenberg-Sobolev inequality we have

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m\|_{L^p(\mathbb{R}^n)}$$

and hence,

$$\|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\overline{u}\|_{L^p(\mathbb{R}^n)} \leq C\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)}.$$

Therefore, by the properties of the extension $\overline{u}$ we have

$$\|u\|_{L^{p^*}(U)} = \|\overline{u}\|_{L^{p^*}(U)} \leq \|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} \ \leq C\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}.$$

This completes the proof.

□

Theorem 9.1.3. *(Estimates for $W_0^{1,p}$, $1 \leq p < n$)*
Let $U \subseteq \mathbb{R}^n$ open and bounded. Assume $1 \leq p < n$, and $u \in \overline{C_c^{\infty}(\mathbb{R}^n)} := W_0^{1,p}(U)$.

Then we have the estimate

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$$

for each $q \in [1, p^*]$, the constant C depending only on p, q, n and U.

Proof. Let $u \in W_0^{1,p}(U)$. Then there exists a sequence $(u_m)_{m=1}^{\infty}$ in $C_c^{\infty}(U)$ such that $u_m \to u$ in $W^{1,p}(U)$. Now we extend each function u_m to be 0 on $\mathbb{R}^n \setminus \overline{U}$. Analogously to the above proof we get from the Gagliardo-Nirenberg-Sobolev inequality (Theorem 9.1.1) the following estimate

$$\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)}$$

Since U is bounded, then for every $1 \leq q \leq p^*$ the following estimate holds

$$\|u\|_{L^q(U)} \leq \|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)}$$

□

Remark. Let $u \in W_0^{1,p}(U)$, U is bounded, then we have $\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$ for each $q \in [1, p^*]$, and $p^* > p$, Hence $\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$. Now,

$$\|Du\|_{L^p(U)} \leq \|u\|_{W^{1,p}(U)} \leq (1 + C)\|Du\|_{L^p(U)}.$$

So, on $W_0^{1,p}(U)$ the norm $\|Du\|_{L^p(U)}$ and $\|u\|_{W^{1,p}(U)}$ are equivalents.

9.2 Morrey's Inequality

Morrey's inequality gives the continuous embedding of the Sobolev spaces $W^{1,p}(U)$, $p > n$ into spaces of Hölder continuous functions, the so called Hölder spaces.
Throughout this chapter let $U \subseteq \mathbb{R}^n$ be open and $0 < \gamma \leq 1$

Definition 9.2.1. *(Hölder continuous)*
A function $u : U(\subseteq \mathbb{R}^n) \to \mathbb{R}$ is said to be Hölder continuous with exponent γ $(0 < \gamma \leq 1)$, if there exists a constant $C > 0$ such that for all $x, y \in U$. $\quad |u(x) - u(y)| \leq C|x - y|^{\gamma}$

Example 9.2.1. $f(x) = \sqrt{x}$, $x \in [0, 1]$ *is a Hölder continuous function with exponent* $\gamma = \frac{1}{2}$.

Definition 9.2.2. *(1) If* $u : U \to \mathbb{R}$ *is bounded and continuous, we write*

$$\|u\|_\infty = sup_{x\in U}|u(x)|$$

(2) The γ^{th}*- Hölder seminorm is defined by*

$$[u]_{0,\gamma} := sup_{x\neq y\in U}\frac{|u(x) - u(y)|}{|x - y|^\gamma}$$

The γ^{th}*- Hölder norm is defined by*

$$\|u\|_{0,\gamma} = \|u\|_\infty + [u]_{0,\gamma}.$$

Definition 9.2.3. *Let* $k \in \mathbb{N}_0$ *and* $0 < \gamma \leq 1$*. The Hölder space* $C^{k,\gamma}(\bar{U})$ *consists of all functions* $C^k(\bar{U})$ *for which the norm*

$$\|u\|_{k,\gamma} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_\infty + \sum_{|\alpha|=k} [D^\alpha u]_{0,\gamma}$$

is finite. where $[u]_{0,\gamma} = \sup_{x\neq y\in U} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$

The Hölder space consists of all the functions that are C^k and whose k-th partial derivatives are bounded and Hölder continuous.

Theorem 9.2.1. $(C^{k,\gamma}\overline{U}, \|.\|_{k,\gamma})$ *is a Banach Space.*

Theorem 9.2.2. *(Morrey's inequality)*
Let $n < p \leq \infty$ *Then there exists a constant* C*, depending only on* n *and* p *such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$*, where* $\gamma = 1 - n/p$.

Proof. We will show that there exists a constant $C(n)$ such that for any $B(x, r) \subseteq \mathbb{R}^n$

$$\frac{1}{|B(x, r)|}\int_{B(x,r)} |u(y) - u(x)|dy \leq C\int_{B(x,r)} \frac{|Du(y)|}{|x - y|^{n-1}}dy. \tag{9.4}$$

Let $x \in \mathbb{R}^n, r > 0$ be fixed. Let $w \in \partial B(0,1)$ and $s < r$. Then

$$
\begin{aligned}
|u(x+sw)-u(x)| &\leq \int_0^s |\frac{d}{dt}u(x+tw)|dt \\
&= \int_0^s |Du(x+tw).w|dt = \int_0^s |Du(x+tw)|dt
\end{aligned}
$$

Hence,

$$
\int_{\partial B(0,1)} |u(x+sw)-u(x)|dS(w) \leq \int_{\partial B(0,1)} \int_0^s |Du(x+tw)|dt\, dS(w). \tag{9.5}
$$

We apply Fubini to the right hand side and apply integration in polar coordinates to obtain

$$
\begin{aligned}
\int_{\partial B(0,1)} \int_0^s |Du(x+tw)|dt\, dS(w) &= \int_{\partial B(0,1)} \int_0^s |Du(x+tw)|dt\, dS(t) \\
&= \int_{B(x,s)} \frac{|Du(y)}{|y-x|^{n-1}}dy
\end{aligned}
$$

Now, multiplying equation (9.5) by s^{n-1} and integrating from 0 to r with respect to s, yields the inequality:

$$
\int_0^r \int_{\partial B((0,1)} |u(x+sw)-u(x)|dS(w)s^{n-1}\, ds \leq \int_0^r s^{n-1} \int_{\partial B(0,1)} \frac{|Du(y)|}{|y-x|^{n-1}}dy\, ds. \tag{9.6}
$$

On the left-hand side of (9.6) we apply integration in polar coordinates to obtain

$$
\begin{aligned}
\int_{B(x,r)} |u(v)-u(x)|dv &\leq \int_0^r s^{n-1}ds \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}dy \\
&= \frac{r^n}{n} \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}dy.
\end{aligned}
$$

Note that $|B(x,r)| = r^n|B(0,1)| = r^nC(n)$. Hence we have

$$
\int_{B(x,r)} |u(v)-u(x)|dv \leq C(n)|B(x,r)| \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}dy.
$$

So, equation (9.4) is proved.

Now, fix, $x \in \mathbb{R}^n$. We apply equation (9.4) as follows

$$\begin{aligned}
|u(x)| &\leq \frac{1}{|B(x,1)|}\int_{B(x,1)} |u(x)-u(y)|dy + \frac{1}{|B(x,1)|}\int_{B(x,1)} |u(y)|dy \\
&\leq \int_{B(x,1)} \frac{|Du(y)|}{|y-x|^{n-1}}dy + \frac{1}{|B(x,1)|}\int_{B(x,1)} |u(y)|dy \\
&= \int_{B(x,1)} \frac{|Du(y)|}{|y-x|^{n-1}}dy + \int_{B(x,1)} |u(y)|\frac{dy}{|B(x,1)|} \\
&\leq \int_{B(x,1)} \frac{|Du(y)|}{|y-x|^{n-1}}dy + \left(\int_{B(x,1)} |u(y)|^p\frac{dy}{|B(x,1)|}\right)^{\frac{1}{p}}.
\end{aligned}$$

The last inequality holds, since $(B(x,1), \frac{dy}{|B(x,1)|})$ is a probability space. We apply Hölder's inequality to the first term on the right-hand side and obtain

$$|u(x)| \leq \left(\int_{B(x,1)} |Du(y)|^p dy\right)^{\frac{1}{p}} \left(\int_{B(x,1)} \frac{1}{|y-x|^{\frac{(n-1)p}{p-1}}}dy\right)^{\frac{p-1}{p}} + C\|u\|_{L^p(B(x,1))}$$

Hence, by integration in polar coordinates we have

$$\left(\int_{B(x,1)} \frac{1}{|y-x|^{\frac{(n-1)p}{p-1}}}dy\right)^{\frac{p-1}{p}} = C(n)\int_0^1 \frac{r^{n-1}}{r^{\frac{(n-1)p}{p-1}}}dr = \int_0^1 r^{-\frac{n-1}{p-1}}dr$$

Since, $P > n$, we have $\frac{n-1}{p-1} < 1$. Therefore,

$$\int_0^1 r^{-\frac{n-1}{p-1}}dr = C(n,p)r^{\frac{p-n}{p-1}}|_0^1 = C(n,p)$$

Summarizing we have

$$|u(x)| \leq C(n,p)\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Since x was arbitrary, we can conclude

$$sup_{x\in\mathbb{R}^n}|u(x)| \leq C\|u\|_{w^{1,p}(\mathbb{R}^n)} \tag{9.7}$$

Choose any two points $x, y \in \mathbb{R}^n$ andf write $r := |x-y|$. Let $W = B(x,r)\cap B(y,r)$.

Then

$$\begin{aligned}
&|u(x) - u(y)| \\
&\leq \frac{1}{|W} \int_W |u(x) - u(z)| dz + \frac{1}{|W} \int_W |u(x) - u(z)| dz \\
&\leq \frac{C}{|B(x,r)} \int_{B(x,r)} |u(x) - u(z)| dz + \frac{C}{|B(y,r)|} \int_{B(y,r)} |u(y) - u(z)| dz \\
&=: A + B
\end{aligned}$$

By the inequality (9.4) we obtain

$$\begin{aligned}
A &\leq C \int_{B(x,r)} \frac{|Du(z)|}{|x-z|^{n-1}} dz \\
&\leq \left(\int_{\mathbb{R}^n} |Du(z)|^p dz \right)^{\frac{1}{p}} \left(\int_{B(x,r)} \frac{1}{|x-z|^{\frac{(n-1)p}{p-1}}} dz \right)^{\frac{p-1}{p}} \\
&\leq C(n,p) \|Du\|_{L^p(\mathbb{R}^n)} r^{1-\frac{n}{p}} \\
&= C(n,p) \|Du\|_{L^p(\mathbb{R}^n)} |x-y|^{1-\frac{n}{p}}
\end{aligned}$$

The same estimate holds for B. Therefore, we have the following estimate

$$|u(x) - u(y)| \leq C \|Du\|_{L^p(\mathbb{R}^n)} |x-y|^{1-\frac{n}{p}}$$

which implies

$$\frac{|u(x) - u(y)|}{|x-y|^{1-\frac{n}{p}}} \leq C \|Du\|_{L^p(\mathbb{R}^n)}$$

for all $x, y \in \mathbb{R}^n$. Thus,

$$[u]_{0,\gamma} = \sup_{x \neq y \in \mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^\gamma} \leq C \|Du\|_{L^p} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}. \tag{9.8}$$

The inequalities (9.7) and (9.8) yields the statement. □

Theorem 9.2.3. *(Estimates for $W^{1,p}$, $n < p \leq \infty$)*
Let $U \subseteq \mathbb{R}^n$ open and bounded and suppose ∂U is C^1. Assume $n < p \leq \infty$, and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\bar{U})$ for $\gamma = 1 - \frac{n}{p}$ with estimate

$$\|u^*\|_{C^{0,\gamma}(\bar{U})} \leq C \|u\|_{W^{1,p}(U)}$$

The constant C depends only on n, p and U.

Proof. According to Extension theorem there exists a compactly supported function $\overline{u} = Eu \in W^{1,p}(\mathbb{R}^n)$ such that $u = \overline{u}$ on U and

$$\|u\|_{W^{1,p}(\mathbb{R}^n)} \leq \|u\|_{W^{1,p}(U)}$$

Since $\overline{u}$ has compact support, we obtain from Theorem 7.2.1 the existence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|u_m - \overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \to 0$$

Now according to Morrey's inequality we have for all $m, l \in \mathbb{N}$

$$\|u_m - u_l\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u_m - u_l\|_{W^{1,p}(\mathbb{R}^n)}$$

$(u_m)_{m=1}^\infty$ converges to $\overline{u}$ in $W^{1,p}(\mathbb{R}^n)$, therefore it is Cauchy sequence in $C^{0,\gamma}(\mathbb{R}^n)$. Since this is a complete Banach space, there exists a function $u^* \in C^{0,\gamma}(\mathbb{R}^n)$ such that

$$\|u_m - u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \to 0.$$

From previous two equation we see that $\overline{u} = u^*$ a.e. on $\mathbb{R}^n$, i.e. u^* is a version of $\overline{u}$. Note that $\overline{u} = u$ a.e. on U hence, u^* is a version of u on U.

Applying Morrey's inequality to the functions $u_m \in C_c^\infty(\mathbb{R}^n)$ i.e.

$$\|u_m\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u_m\|_{W^{1,p}(\mathbb{R}^n)}$$

ade therefore we have,

$$\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}.$$

By the definition of the norm $\|.\|C^{0,\gamma}$ we have

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)}.$$

This completes the proof.

□

9.3 Compact Embedding

The Gagliardo-Nirenberg-Sobolev inequality shows that $W^{1,p}(U)$ is continuously embedded into $L^{p^*}(U)$, if $1 \leq p < n$.Now we show that $W^{1,p}(U)$ is in fact compactly embedded into some $L^q(U)$ space.

Definition 9.3.1. *(compactly embedded)*
Let X and Y be Banach spaces, $X \subset Y$. We say X is compactly embedded in Y ($X \subset\subset Y$) if and only if the operator

$$Id : X \to Y \ , \ \ x \to x$$

is continuous and compact, i.e.

1. *$\exists C \ \forall x \ \in X, \|x\|_Y \leq C\|x\|_X$*
2. *for all sequences $(x_n)_{n=1}^{\infty}$ in X with $sup_n \|x_n\|_X \leq \infty$ there exists a subsequence $(x_{n_i})_{i=1}^{\infty}$ and $y \in Y$ such that $\|I(x_{n_i}) - y\|_Y \to \infty$ as $i \to \infty$.*

Theorem 9.3.1. *(Rellich-Kondrachov Compactness Theorem)*
Let $U \subseteq \mathbb{R}^n$ open and bounded and suppose ∂U is C^1. Assume $1 \leq p < n$. Then

$$W^{1,p}(U) \subset\subset L^q(U),$$

for all $1 \leq q < p^*$, $p^* = \frac{np}{n-p}$

Sketch the proof :
(1) Take $\{u_m\}$ bounded sequence in $W^{1,p}(U)$. We need to find a subsequence which is Cauchy in $L^q(U)$.
(2) Use the extension theorem to extend $\{u_m\}$ to a larger set V and such that $\{u_m\}$ vanishes outside V.
(3) Now let $u_m^\epsilon = \eta_\epsilon * u_m$. It turns out that

$$u_m^\epsilon \to u_m \ in \ L^q(V) \ as \ \epsilon \to 0, \ uniformly \ in \ m.$$

(4) for each $\epsilon > 0$, $\{u_m^\epsilon\}$ is uniformly bounded and equicontinuous. Thus by the Arzela-Ascoli theorem, for each fixed $\epsilon > 0$, there is a subsequence of $\{u_m^\epsilon\}$ converges

uniformly, and thus converges in $L_q(V)$.
(5) from (3) and (4) we will get

$$lim\, sup_{j,k\to\infty}\|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta,$$

(6) Now taking $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and repeatly subtract subsequences, we obtain a Cauchy sequence via the standard diagonal argument.

Proof. We fix $q \in [1, p^*)$. Let $u \in W^{1,p}(U)$. From Theorem 9.1.2 we get

$$\|u\|_{L^p} \leq C\|u\|_{W^{1,p}(U)}.$$

Hence, the operator $Id : W^{1,p} \to L^q$ is continuous.
We have to show compactness. Let $(\hat{u}_m)_{m=1}^{\infty} \in W^{1,p}(U)$, $sup_m\|\hat{u}_m\|_{W^{1,p}(U)} \leq A$. We show that there exists a subsequence $(\hat{u}_{m_k})_{k=1}^{\infty}$ of the bounded sequence $(\hat{u}_m)_{m=1}^{\infty}$ and a $u \in L^q(U)$ so that $\|\hat{u}_{m_k} - u\|_{L^q(U)} \to 0$ as $k \to \infty$. By the extension theorem we may assume that

1. $(u_m)_{m=1}^{\infty}$ is in $W^{1,p}(\mathbb{R}^n)$ with $u_m|_U = \hat{u_m}$

2. for all $m \in \mathbb{N}$ there exists V with $U \subset\subset V$ such that supp$u_m \subset V$,

3. $sup_m\|\hat{u_m}\|_{W^{1,p}(\mathbb{R}^n)} \leq \infty$

We first consider the smooth functions

$$u_m^{\epsilon} = \eta_{\epsilon} * u_m \ \in C_c^{\infty}(\mathbb{R}^n). \quad (\epsilon > 0, \ \ m \in \mathbb{N}).$$

We may assume that for all $m \in \mathbb{N}$ the support of u_m^{ϵ} is in V.

Claim 1.

$$u_m^{\epsilon} \to u_m \ in\ L^q(V)\ as\ \epsilon \to 0,\ uniformly\ in\ m.$$

Verification: If u_m is smooth then

$$\begin{aligned} u_m^\epsilon(x) - u_m(x) &= \int_{B(0,1)} \eta(y)(u_m(x-\epsilon y) - u_m(x))\, dy \\ &= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt} u_m(x - \epsilon t y)\, dt\, dy \\ &= -\epsilon \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\epsilon t y).y \, dt\, dy \end{aligned}$$

Thus,

$$\begin{aligned} \int_V |u_m^\epsilon(x) - u_m(x)| dx &\leq \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x - \epsilon t y)|\, dx\, dt\, dy \\ &\leq \epsilon \int_V |Du_m(z)| dz. \end{aligned}$$

Summarizing we have for $u_m \in C_c^\infty(\mathbb{R}^n)$ with supp $u_m^\epsilon \in V$ the estimate

$$\|u_m^\epsilon - u_m\|_{L^1(V)} \leq \epsilon \|Du_m\|_{L^1(V)} \tag{9.9}$$

By approximation Theorem this estimate holds for $u_m \in W^{1,p}(V)$. Since V is open and bounded, we obtain

$$\|u_m^\epsilon - u_m\|_{L^1(V)} \leq \epsilon \|Du_m\|_{L^1(V)} \leq \epsilon C \|Du_m\|_{L^p(V)}$$

By assumption we have $sup_m \|u\|_{W^{1,p}(V)} < \infty$. Therefore,

$$\lim_{\epsilon \to 0} sup_m \|u_m^\epsilon - u_m\|_{L^1(V)} = 0 \tag{9.10}$$

Note that $1 \leq q < p^*$. Let $0 \leq \theta \leq 1$ such that

$$\frac{1}{q} = \frac{1-\theta}{1} + \frac{\theta}{p^*}$$

We apply the interpolation inequality for L^p-norms to obtain

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - u_m\|_{L^1(V)}^{1-\theta} \|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{\theta}.$$

Theorem (9.1.2) gives

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - u_m\|_{L^1(V)}^{1-\theta}\|u_m^\epsilon - u_m\|_{W^{1,p}(V)}^{\theta}.$$

By equation (9.10)

$$\lim_{\epsilon\to 0} sup_{m\in\mathbb{N}}\|u_m^\epsilon - u_m\|_{L^q(V)} = 0 \tag{9.11}$$

Claim 2: for each $\epsilon > 0$, $\{u_m^\epsilon\}$ is uniformly bounded and equicontinuous. **Verification**: Let, $x \in \mathbb{R}^n$.

$$\begin{aligned} |u_m^\epsilon(x)| &\leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u_m(y)|dy \\ &\leq sup_{x\in\mathbb{R}^n}|\eta_\epsilon(x)| \int_V |u_m(y)|dy \\ &\leq \frac{1}{\epsilon^n}\|u_m\|_{L^1(V)} \\ &\leq \frac{C}{\epsilon^n}\|u_m\|_{L^p(V)} \leq \frac{C}{\epsilon^n} < \infty \end{aligned}$$

Hence,

$$sup_{m\in\mathbb{N}}\|u_m^\epsilon\| \leq \frac{C}{\epsilon^n}. \tag{9.12}$$

Similarly, for $m = 1, 2, \ldots$

$$|Du_m^\epsilon(x)| \leq \int_{B(x,\epsilon)} |D\eta_\epsilon(x-y)||u_m(y)|dy \leq \frac{C}{\epsilon^{n+1}} \tag{9.13}$$

Hence,

$$sup_{m\in\mathbb{N}}\|Du_m^\epsilon\|_\infty \leq \frac{C}{\epsilon^{n+1}} \tag{9.14}$$

Equation (9.12) and (9.14) proves the claim.

Now, fix $\delta > 0$. we will show that there exists a subsequence $(u_{m_j})_{j=1}^\infty \subset (u_m)_{m=1}^\infty$ such that

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta,$$

From the first claim, to select $\epsilon 0$ so small that

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \frac{\delta}{2}$$

for $m = 1, 2, \ldots$

We now observe that since the functions $(u_m)_{m=1}^{\infty}$, and thus the functions $(u_m^{\epsilon})_{m=1}^{\infty}$, have support in some fixed bounded set $V \subset \mathbb{R}^n$, Now using claim 2 and Arzela-Ascoli compactness criterion to obtain a subsequence $(u_{m_j})_{j=1}^{\infty} \subset (u_m^{\epsilon})_{m=1}^{\infty}$ which converges uniformly on V. In particular therefore

$$\limsup_{j,k\to\infty} \|u_{m_j}^{\epsilon} - u_{m_k}^{\epsilon}\|_{L^q(V)} = 0 \tag{9.15}$$

Now, from last two equations imply

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta, \tag{9.16}$$

Now taking $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and by standard diagonal argument, we extract a subsequence $(u_{m_l})_{l=1}^{\infty} \subset (u_m)_{m=1}^{\infty}$ satisfying

$$\limsup_{l,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

This completes the proof. □

9.4 Poincaré‘s inequality

For all $u \in W_0^{1,p}(U)$. Then we have the estimate $\|u\|_{L^{p^*}(U)} \leq C\|Du\|_{L^p(U)}$ $(1 \leq p < n)$. But for all $u \in W^{1,p}(U)$ this does not hold, where $U \subseteq R^n$ open and bounded. However when the boundary ∂U is C^1 for all $u \in W_0^{1,p}(U)$ we can get this kind of inequality with some extra term.
Notation. $(u)_U = ⨍ u dy :=$ average of u over U.

Proposition 9.4.1. *Let $U \subseteq \mathbb{R}^n$ open and bounded and connected. Let $u \in W^{1,p}(U)$ and $Du = 0$ a.e. in U. Then u is constant a.e. on U.*

Theorem 9.4.1. *(Poincaré‘s inequality)*
Let $U \subseteq \mathbb{R}^n$ open, bounded and connected. suppose ∂U is C^1. Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n, p and U, such that

$$\|u - (u)_U\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

for all $u \in W^{1,p}(U)$.

Proof. By contradiction. We assume that the statement is not true, i.e.

$$\forall k \in \mathbb{N}\ \exists u_k \in W^{1,p}(U)\ :\ \|u_k - (u_k)_U\|_{L^p(U)} > k\|Du_k\|_{L^p(U)}. \tag{9.17}$$

We define

$$v_k := \frac{u_k - (u_k)_U}{\|u_k - (u_k)_U\|_{L^p(U)}},$$

Then $\|v_k\|_{L^p(U)} = 1$ and $(v_k)_U = 0$. The gradient of v_k

$$Dv_k = \frac{Dv_k}{\|u_k - (u_k)_U\|_{L^p(U)}},$$

satisfies by the assumption (9.17)

$$\|Dv_k\|_{L^p(U)} = \frac{\|Du_k\|_{L^p(U)}}{\|u_k - (u_k)_U\|_{L^p(U)}} < \frac{1}{k}.$$

Hence,

$$\|v_k\|_{W^{1,p}(U)} \leq C(n,p)(\|Dv_k\|_{L^p(U)} + \|v_k\|_{L^p(U)}) \leq C(n,p)\big(1 + \frac{1}{k}\big)$$

and

$$sup_{k\in\mathbb{N}}\|v_k\|_{W^{1,p}(U)} \leq 2C(n,p).$$

By Rellich-Kondrachov Compactness Theorem there exists a subsequence $(v_{k_j})_{j=1}^{\infty}$ and a $v \in L^p(U)$ with $\|v\|_{L^p(U)} = 1$ and $(v)_U = 0$ such that

$$\lim_{j\to\infty} \|v_{k_j} - v\|_{L^p(U)} = 0$$

Let, $\phi \in C_c^\infty(U)$. Then, using Lebesgue's Theorem and the definition of the weak derivative, we have

$$\int v\phi_{x_i}dx = \lim_{j\to\infty}\int v_{k_j}\phi_{x_j}dx = -\lim_{j\to\infty}\int (v_{k_j})_{x_i}\phi dx = 0$$

where the last equality follows from $\lim_{j\to\infty}\|Dv_{k_j}\|_{L^p(U)} = 0$. Hence, $Dv = 0$. Since U is connected, from the previous Proposition it implies that v is constant a.e on U. As $(v)_U = 0$ we have $v = 0$ a.e. on U, which is a contradiction to $\|u\|_{L^p(U)} = 1$. □

Theorem 9.4.2. *(Poincaré's inequality for a ball)*
Let $1 \leq p \leq \infty$, then there exists a constant C, that depends only on n and p, such that

$$\|u - (u)_{B(x,r)}\|_{L^p(B(x,r))} \leq Cr\|Du\|_{L^p(B(x,r))}$$

for each ball $B(x,r) \subseteq \mathbb{R}^n$ and each function $u \in W^{1,p}(B(x,r))$.

Remark. Let $u \in W^{1,n}(\mathbb{R}^n)$ and $B(x,r) \subseteq \mathbb{R}^n$. Then by the Theorem 4.4.2. we get,

$$\begin{aligned}\Big(\int_{B(x,r)} |u(y) - (u)_{B(x,r)}|^n \frac{dy}{|B(x,r)|}\Big)^{\frac{1}{n}} &\leq Cr\Big(\int_{B(x,r)} |Du(y)|^n dy\Big)|^{\frac{1}{n}} \\ &\leq \frac{Cr}{|B(x,r)|^{\frac{1}{n}}}\|Du\|_{L^n(\mathbb{R}^n)} \\ &= \frac{C}{|B(0,1)|^{\frac{1}{n}}}\|Du\|_{L^n(\mathbb{R}^n)}.\end{aligned}$$

By Hölder's inequality we obtain for the left-hand side

$$\int_{B(x,r)} |u(y) - (u)_{B(x,r)}| \frac{dy}{|B(x,r)|} \leq \Big(\int_{B(x,r)} |u(y) - (u)_{B(x,r)}|^n \frac{dy}{|B(x,r)|}\Big)^{\frac{1}{n}}.$$

Hence,

$$\int_{B(x,r)} |u(y) - (u)_{B(x,r)}| \frac{dy}{|B(x,r)|} \leq C\|Du\|_{L^n(\mathbb{R}^n)},$$

where C only depends on n.

Definition 9.4.1. *(Space of bounded mean oscillation)*
A function $f \in L^1_{loc}(\mathbb{R}^n)$ is called of bounded mean oscillation if

$$sup_{B(x,r)\subseteq\mathbb{R}^n} \int_{B(x,r)} |f(y) - (f)_{B(x,r)}| \frac{dy}{|B(x,r)|} < \infty$$

The space of all such functions is called the space of functions of bounded mean oscillation ($BMO(\mathbb{R}^n)$) and the left-hand side of equation defines a norm $\|u\|_{BMO(\mathbb{R}^n)}$ on this space.

$$\|u\|_{BMO(\mathbb{R}^n)} \leq C\|Du\|_{L^n(\mathbb{R}^n)} \leq C\|u\|_{W^{1,n}(\mathbb{R}^n)}$$

Therefore, we have $W^{1,n}(\mathbb{R}^n)$ is continuously embedded into $BMO(\mathbb{R}^n)$.

PART 3

Sobolev Space On Riemannian Manifolds

Chapter 10

Sobolev Spaces on Riemannian Manifolds

In this chapter we shall define Sobolev Spaces on Riemannian Manifolds, then we shall some density properties of the Sobolev Spaces on Riemannian Manifolds and some embedding.

10.1 Definitions

Let (M, g) be a smooth Riemannian manifold and $\{(\Omega_k, \phi_k\}$ is a differentiable structure(or Atlas) on M, where $\phi_k : \Omega_k(\subset M) \to \mathbb{R}^n$. For k integer, and $u : M \to \mathbb{R}$ smooth, we denote by ∇^k the covarient derivative of u. The component of ∇u in local coordinates are given by $(\nabla u)_i = \partial_i u$ and the component of $\nabla^2 u$ in local coordinates are given by $(\nabla^2 u)_{ij} = \partial_{ij} u - \Gamma_{ij}^k \partial_k u$. Now $|\nabla^k u|$, the norm of $\nabla^k u$ defined in the local chart by

$$|\nabla^k (u)|^2 = g^{i_1,j_1} \dots g^{i_k,j_k} (\nabla^k u)_{i_1,\dots,i_k} (\nabla^k u)_{j_1,\dots,j_k}$$

Definition 10.1.1. *For an integer k and $p \geq 1$ real, we denoted by $\mathcal{C}_k^p(M)$ the space of smooth functions $u \in C^\infty(M)$ such that $|\nabla^k u| \in L^p(M)$ for any $j = 0, 1, \dots, k$. Hence,*

$$\mathcal{C}_k^p(M) = \big\{u \in C^\infty(M) \ s.t. \ \forall j = 0, 1, \dots, k, \quad \int_M |\nabla^k u|^p d\nu(g) \ < \ \infty \big\}$$

Where, in local coordinates, $d\nu(g) = \sqrt{det(g_{i,j})}dx$, and where dx stands for the

Lebesgue's volume element of $\mathbb{R}^n$

Note. So if M is compact, then $\mathcal{C}_k^p(M) = \mathcal{C}^\infty(M)$ for all k and $p \geq 1$.

Definition 10.1.2. *(Sobolev Spaces on Riemannian Manifold)*
The Sobolev space $H_k^p(M)$ *is the completion of* $\mathcal{C}_k^p(M)$ *with respect to the norm*

$$\|u\|_{H_k^p} = \sum_{j=0}^{k} \left(\int_M |\nabla^k u|^p d\nu(g) \right)^{1/p} \qquad 1 \leq p < \infty$$

Note. (1) any Cauchy sequence in $(C_k^p(M), \|.\|_{H_p^k})$ is a Cauchy sequence in the Lebesgue space $(L^p(M), \|.\|_p)$.

(2) any Cauchy sequence in $(C_k^p(M), \|.\|_{H_p^k})$ that converges to 0 in the Lebesgue space $(L^p(M), \|.\|_p)$ also converges to 0 in $(C_k^p(M), \|.\|_{H_p^k})$.

(3) As a consequence of (1) and (2) One can look at $H_k^p(M)$ as a subspace of $L^p(M)$ made of functions $u \in L^p(M)$ which are limits in $(L^p(M), \|.\|_p)$ of a Cauchy sequence (u_m) in $(C_k^p(M), \|.\|_{H_p^k})$. and define $\|u\|_{H_k^p}$ as before, where $|\nabla^j u|$, $0 \leq j \leq k$, is now the limit in $(L^p(M), \|.\|_p)$ of the Cauchy sequence $(\nabla^j u_m)$.

(4) $H_k^p(M)$ is a Banach space.

Proposition 10.1.1. *If* $1 < p < \infty$, $H_k^p(M)$ *is reflexive Banach space.*

Proof. $H_k^p(M)$ is closed subspace of a finite Cartesian product space of spaces $L^p(M)$. And $L^p(M)$ is a reflexive Banach space for $(1 < p < \infty)$, finite Cartesian product space of reflexive space is reflexive space. Since a closed subspace of a reflexive Banach space is also reflexive, thus $H_k^p(M)$ is reflexive if $1 < p < \infty$. □

Proposition 10.1.2. *If* $p = 2$, $H_k^2(M)$ *is a Hilbert space when equipped with the equivalent norm*

$$\|u\| = \sqrt{\sum_{j=0}^{k} \int_M |\nabla^k u|^2 d\nu(g)}$$

The scalar product $\langle ., . \rangle$ *associated to* $\|.\|$ *is defined by*

$$\langle u, v \rangle = \sum_{m=0}^{k} \int_M \left(g^{i_1 j_1} \ldots g^{i_m, j_m} (\nabla^m u)_{i_1 \ldots i_m} (\nabla^m v)_{j_1 \ldots j_m} \right) d\nu(g)$$

Proof. Here in $H_k^2(M)$, $\|.\|$ and $\|.\|_{H_k^p}$ are equivalents. And $\langle \, . \, \rangle$ satisfies the following four properties.

1. $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

2. $\langle au, w\rangle = a\langle u, w\rangle$

3. $\langle u, w\rangle = \langle w, u\rangle$

4. $\langle u, u\rangle = 0$ if $u = 0$

for $u \in H_k^2(M)$. Now, if $\langle u, u\rangle = o$ then,

$$\langle u, u\rangle = \sum_{m=0}^{k} \int_M \left(g^{i_1 j_1} \dots g^{i_m, j_m} (\nabla^m u)_{i_1 \dots i_m} (\nabla^m u)_{j_1 \dots j_m} \right) d\nu(g) = 0$$

for, $m = 0$ we get,

$$\int_M u^2 d\nu(g) = 0$$

Hence, if $\langle u, u\rangle = o$ then $u = 0$ a.e. on M.
So, $\langle\ ,\ \rangle$ is a inner product. And $H_k^2(M)$ is complete so is a Hilbert space. □

Proposition 10.1.3. *If M is compact, $H_k^p(M)$ does not depend on the Riemannian metric.*

Proof. M be a compact manifold endowed with two Riemannian metrics g and $\tilde{g}$. Now since M is compact, M can be covered by a finite number of charts $(\Omega_m, \phi_m)_{m=1,2,\dots,N}$ such that for any m the components g_{ij}^m of g in (Ω_m, ϕ_m) satisfy

$$\frac{1}{C}\tilde{g}_{i,j} \le g_{ij}^k \le C\tilde{g}_{ij},$$

as bilinear forms, where $C > 1$. Let η_m be a smooth partition of unity subordinate to the covering (Ω_m). Now let $u \in H_k^p(M_{\tilde{g}})$ with respect to the metric $\tilde{g}$ then

$$\begin{aligned}
&\|\eta_k u\|_{H_k^p(M_g)} \\
&= \sum_{j=0}^{k} \left(\int_M |\nabla^j \eta_k u|^p d\nu(g) \right)^{\frac{1}{p}} \\
&= \sum_{j=0}^{m} \left(\int_{\phi_k(\Omega_k)} |\nabla^j \eta_k u \circ \phi_k^{-1}|^p \sqrt{det(g_{ij}^k)}\; dx \right)^{\frac{1}{p}} \\
&\leq K \sum_{j=0}^{m} \left(\int_{\phi_k(\Omega_k)} |\nabla^j \eta_k u \circ \phi_k^{-1}|^p \sqrt{det(\tilde{g}_{ij}^k)}\; dx \right)^{\frac{1}{p}} \\
&= \sum_{j=0}^{k} \left(\int_M |\nabla^j \eta_k u|^p d\nu(\tilde{g}) \right)^{\frac{1}{p}} \\
&= \|\eta_k u\|_{H_k^p(M_{\tilde{g}})}
\end{aligned}$$

Now,

$$u = \sum_{k=1}^{N} \eta_k u.$$

Hence, $u \in H_k^p(M_g)$ with respect to the metric g. This completes the proof.

□

Theorem 10.1.1. *If Ω is bounded, open subset of $\mathbb{R}^n$, and if $u : \Omega \to \mathbb{R}$ is Lipschitz, then $u \in H_1^p(\Omega)$ for all $p \geq 1$*

Lemma 10.1.1. *Let (M, g) be a smooth Riemannian manifold, and $u : M \to \mathbb{R}$ a Lipschitz function on M with compact support. Then $u \in H_1^p(M)$ for any $p \geq 1$. In particular, if M is compact, any Lipschitz function on M belongs to the Sobolev spaces $H_1^p(M)$, $p \geq 1$.*

Proof. Let $u : M \to \mathbb{R}$ a Lipschitz function on M with compact support. Let $(\Omega_k, \phi_k)_{k=1,2,\ldots,N}$ be a family of charts such that $K \subset \cup_{k=1}^N \Omega_k$ and such that for any $k = 1, \ldots, N$,

$$\phi_k(\Omega_k) = B_0(1)$$

and

$$\frac{1}{C}\delta_{i,j} \leq g_{i,j}^k \leq C\delta_{i,j},$$

Where $C > 0$ and $B_0(1)$ is the Euclidean ball of $\mathbb{R}^n$ of center 0 and radius 1. g_{ij}^k are the component of g in $(\Omega_k)_{k=1,2,\ldots,N}$. For any $k = 1, \ldots, N$

$$u_k = (\eta_k u) \circ \phi_k^{-1} : \mathbb{R}^n \to \mathbb{R}$$

is a Lipschitz on $B_0(1)$ for the Euclidean metric. By previous theorem we get $u_k \in H_1^p(B_0(1))$ for any $p \geq 1$. Now,

$$\begin{aligned}
&\|\eta_k u\|_{H_1^p(M)} \\
&= \sum_{j=0}^{1} \left(\int_M |\nabla^j \eta_k u|^p d\nu(g) \right)^{\frac{1}{p}} \\
&= \sum_{j=0}^{1} \left(\int_{\Omega_k} |\nabla^j u_k \circ \phi_k|^p d\nu(g^k) \right)^{\frac{1}{p}} \\
&= \left(\int_{\Omega_k} |u_k \circ \phi_k|^p d\nu(g^k) \right)^{\frac{1}{p}} + \left(\int_{\Omega_k} |\nabla u_k \circ \phi_k|^p d\nu(g^k) \right)^{\frac{1}{p}} \\
&= \left(\int_{\phi_k(\Omega_k)} |u_k|^p \sqrt{det(g_{ij}^k)}\, dx \right)^{\frac{1}{p}} + \left(\int_{\phi_k(\Omega_k)} |\nabla u_k|^p \sqrt{det(g_{ij}^k)}\, dx \right)^{\frac{1}{p}} \\
&\leq K \left(\left(\int_{\phi_k(\Omega_k)} |u_k|^p dx \right)^{\frac{1}{p}} + \left(\int_{\phi_k(\Omega_k)} |\nabla u_k|^p dx \right)^{\frac{1}{p}} \right) \quad < \infty
\end{aligned}$$

So, $\eta_k u \in H_1^p(M)$. Now,

$$u = \sum_{k=1}^{N} \eta_k u \in H_1^p(M).$$

This completes the proof.

□

10.2 Density Properties

Definition 10.2.1. *The Sobolev space $\mathring{H}_k^p(M)$ is the closer of the set $\mathcal{D}(M)$ of smooth functions with compact support in M in $H_k^p(M)$.*

We know that $\mathring{H}_k^p(\mathbb{R}^n) = H_k^p(\mathbb{R}^n)$. Now, does it holds for the manifolds? For complete manifolds it does hold. We shall prove this for $k = 1$, however the situation is more complicated when $k \geq 2$ and we need some assumption on the manifold.

Theorem 10.2.1. *If (M, g) is complete, then, for any $p \geq 1$, $\mathring{H}_1^p(M) = H_1^p(M)$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
$f(t) = 1$ when $t \leq 0$ $f(t) = 1 - t$ when $0 \leq t \leq 1$, $f(t) = 0$ when $t \geq 1$
and let $u \in C_1^p(M)$ where $p \geq 1$ is some given real number. Let x be some point of M and set

$$u_j(y) = u(y) f(d_g(x, y) - j)$$

where d_g is the distance associated to g, j is an integer, and $y \in M$. By the previous proposition $u_j \in H_1^p(M)$ for any j, and since $u_j = 0$ outside a compact subset of M, one easily gets that for any j, u_j is the limit in $H_1^p(M)$ of some sequence of functions in $\mathcal{D}(M)$. One just has to note here that if $(u_m) \in C_1^p(M)$ converges to u_j in $H_1^p(M)$, and if $\alpha \in \mathcal{D}(M)$, then (αu_m) converges to αu_j in $H_1^p(M)$. Then, one can choose $\alpha \in \mathcal{D}(M)$ such that $\alpha = 1$ where $u_j \neq 0$. Independently, one clearly has that for any j,

$$\left(\int_M |u_j - u|^p d\nu(g) \right)^{\frac{1}{p}} \leq \left(\int_{M \setminus B_x(j)} |u|^p d\nu(g) \right)^{\frac{1}{p}}$$

and

$$\left(\int_M |\nabla(u_j - u)|^p d\nu(g) \right)^{\frac{1}{p}} \leq \left(\int_{M \setminus B_x(j)} |\nabla u|^p d\nu(g) \right)^{\frac{1}{p}} + \left(\int_{M \setminus B_x(j)} |u|^p d\nu(g) \right)^{\frac{1}{p}}$$

where $B_x(j)$ is the geodesic ball of center x and radius j. Hence, (u_j) converges to u in $H_1^p(M)$ as j goes to $+\infty$. This ends the proof of the theorem. □

10.3 Sobolev Embeddings

In Euclidean space $\mathbb{R}^n$ we have seen the Sobolev embeddings, in this section we shall discuss on what condition that kind of embeddings hold for the manifolds.

Lemma 10.3.1. *Let (M, g) be a complete Riemannian n-manifold. Suppose that the embedding $H_1^1(M) \subset L^{n/(n-1)}(M)$ is valid. Then for any real numbers $q \in [1, n)$ satisfying $1/p = 1/q - 1/n$, then $H_1^q(M) \subset L^P(M)$.*

Proof. Let $A \in \mathbb{R}$, from the given condition for any $u \in H_1^1(M)$

$$\Big(\int_M |u|^{n/(n-1)} d\nu(g)\Big)^{(n-1)/n} \leq A \int_M \big(|\nabla u| + |u|\big) d\nu(g).$$

Let $q \in (1, n)$, $p = nq/(n-q)$ and $u \in \mathcal{D}(M)$. Set $\phi = |u|^{p(n-1)/n}$ Applying Hölder‘s inequality, we get that

$$\begin{aligned}
&\Big(\int_M |u|^p d\nu(g)\Big)^{(n-1)/n} \\
&= \Big(\int_M |\phi|^{n/(n-1)} d\nu(g)\Big)^{(n-1)/n} \\
&\leq A \int_M \Big(|\nabla \phi| + |\phi|\Big) d\nu(g) \\
&= \frac{Ap(n-1)}{n} \int_M |u|^{p'} |\nabla u| d\nu(g) + A \int_M |u|^{p(n-1)/n} d\nu(g)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{Ap(n-1)}{n} \Big(\int_M |u|^{p'q'} d\nu(g)\Big)^{1/q'} \Big(\int_M |\nabla u|^q d\nu(g)\Big)^{1/q} \\
&\quad + A\Big(\int_M |u|^{p'q'} d\nu(g)\Big)^{1/q'} \Big(\int_M |u|^q d\nu(g)\Big)^{1/q}
\end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $q' = \frac{p(n-1)}{n} - 1$. And $p'q' = p$ since $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$.
For any $u \in \mathcal{D}(M)$.

$$\begin{aligned}
\Big(\int_M |u|^p d\nu(g)\Big)^{(n-1)/n} &\leq \frac{Ap(n-1)}{n} \Big(\int_M |u|^p d\nu(g)\Big)^{1/q'} \times \\
&\Big(\Big(\int_M |\nabla u|^q d\nu(g)\Big)^{1/q} + \Big(\int_M |u|^q d\nu(g)\Big)^{1/q}\Big)
\end{aligned}$$

Now,

$$\begin{aligned}
&\Big(\int_M |u|^p d\nu(g)\Big)^{\frac{n-1}{n} - \frac{1}{q'}} \\
&\leq \frac{Ap(n-1)}{n} \Big(\Big(\int_M |\nabla u|^q d\nu(g)\Big)^{1/q} + \Big(\int_M |u|^q d\nu(g)\Big)^{1/q}\Big).
\end{aligned}$$

Hence,

$$\|u\|_{L^p(M)} \leq C\big(\|u\|_{L^q(M)} + \|\nabla u\|_{L^q(M)}\big)$$

Since, (M, g) is complete, then, for any $p \geq 1$, $\overset{\circ}{H}{}_1^p(M) = H_1^p(M)$. this completes the proof. $\square$

Theorem 10.3.1. *(Sobolev Embedding for Compact manifold)*
Let (M, g) be a compact Riemannian n-manifold. Then for any real numbers $q \in [1, n)$ satisfying $1/p = 1/q - 1/n$, then $H_1^q(M) \subset L^P(M)$.

Proof. By previous lemma, we just have to prove that the embedding

$$H_1^1(M) \subset L^{n/(n-1)}(M)$$

is valid. Now since M is compact, M can be covered by a finite number of charts $(\Omega_m, \phi_m)_{m=1,2,\ldots,N}$ such that for any m the components g_{ij}^m of g in (Ω_m, ϕ_m) satisfy $\frac{1}{2}\delta_{i,j} \leq g_{i,j}^m \leq 2\delta_{i,j}$ as bilinear forms. Let η_m be a smooth partition of unity subordinate to the covering (Ω_m). For any $u \in C^\infty(M)$ and any m, we have

$$\int_M |\eta_m u|^{n/(n-1)} d\nu(g) \leq 2^{n/2} \int_{\mathbb{R}^n} |(\eta_m u) \circ \phi_m^{-1}(x)|^{n/n-1} dx$$

and

$$\int_M |\nabla(\eta_m u)| d\nu(g) \geq 2^{-(n+1)/2} \int_{\mathbb{R}^n} |\nabla((\eta_m u) \circ \phi_m^{-1})(x)| dx$$

By G.N.S inequality ($\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq C\|Du\|_{L^1(\mathbb{R}^n)}$) we have,

$$\left(\int_{\mathbb{R}^n} |(\eta_m u) \circ \phi_m^{-1}(x)|^{n/(n-1)} dx\right)^{(n-1)/n} \leq \frac{1}{2}\left(\int_{\mathbb{R}^n} |\nabla((\eta_m u) \circ \phi_m^{-1})(x)| dx\right).$$

Then for any m and $u \in C^\infty(M)$

$$\begin{aligned}
&\left(\int_M |u|^{n/(n-1)} d\nu(g)\right)^{(n-1)/n} \\
&\leq \sum_{m=1}^{N} \left(\int_M |\eta_m u|^{n/(n-1)} d\nu(g)\right)^{(n-1)/n} \\
&\leq 2^{n-1} \sum_{m=1}^{N} \int_M |\nabla(\eta_m u)| d\nu(g) \\
&\leq 2^{n-1} \int_M |\nabla u| d\nu(g) + 2^{n-1}\left(max_M \sum_{m=1}^{N} |\nabla \eta_m|\right) \int_M |u| d\nu(g) \\
&\leq A\left(\int_M |\nabla u| d\nu(g) + \int_M |u| d\nu(g)\right)
\end{aligned}$$

Hence,

$$\|u\|_{L^{n/(n-1)}(M)} \leq A\big(\|u\|_{L^1(M)} + \|\nabla u\|_{L^1(M)}\big)$$

Since, M is compact, then $\mathcal{C}_k^p(M) = \mathcal{C}^\infty(M)$ for all k and $p \geq 1$. So, this completes the proof. □

10.4 Example of PDE on Riemmannian Manifold

Yamabe Problem

(M, g) is smooth compact Riemannian manifold, to find a metric $\tilde{g}$ conformal to g such that the scalar curvature of $(M, \tilde{g})$ is a function $K =$ *constant*. If $\tilde{g} = u^{4/(n-2)} g$ $(n \geq 3)$, $u > 0$, one has to solve

$$-\Delta_g u + R_g u = K u^{\frac{n+2}{n-2}}, \quad u \in H^1(M), \;\; u > 0$$

where Δ_g denotes the Laplace-Beltrami operator and R_g is the scalar curvature of (M, g).

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