The Kadison-Singer problem

Sourav Das

A Thesis Submitted to

Indian Institute of Technology Hyderabad

In Partial Fulfillment of the Requirements for

The Degree of Master of Science

Under the guidance of

Dr. Sukumar D

Department of Mathematics

June 2019

Declaration

This thesis entitled **"The Kadison-Singer Problem"** submitted by me to the Indian Institute of Technology, Hyderabad for the award of the degree of Master of Science in Mathematics contains a literature survey of the work done by some authors regarding this problem. The work presented in this thesis has been carried out under the supervision of **Dr. Sukumar D**, Department of Mathematics, Indian Institute of Technology, Hyderabad, Telangana.

I hereby declare that, to the best of my knowledge, the work included in this thesis has been taken from these books and articles ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [4]), mentioned in the References. No new results have been created in this thesis. I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

Sourav Das.

(Signature)

Sourav Das

MA17MSCST11011

(Roll No.)

Approval Sheet

This Thesis entitled **The Kadison-Singer problem** by **Sourav Das** is approved for the degree of Master of Science (Mathematics) from IIT Hyderabad.

(**Dr. Sukumar D**) Adviser
Dept. of Mathematics
IITH

Acknowledgements

I would like to express my deep sense of gratitude to my supervisor, **Dr. Sukumar D**, for his constant encouragement, co-operation and in-valuable guidance throughout this project. Due to his motivation and expert guidance, I was able to understand the concepts in a nice manner.

I thank the teachers of the department for imparting in me the knowledge and understanding of mathematics. Without their kind efforts I would not have reached this stage.

I would also like to extend my gratitude to my family and friends for helping me in every possible way and encouraging me during this Programme. Above all, I thank, The Almighty, for all his blessings.

Dedication

Dedicated to my parents

Abstract

In mathematics, the Kadison-Singer problem, posed in 1959, was a problem in C^*-algebra about whether certain extensions of certain linear functionals on certain C^*-algebras were unique. The uniqueness was proven in 2013.

The statement arose from work on the foundations of quantum mechanics done by Paul Dirac in the 1940s and was formalized in 1959 by Richard Kadison and Isadore Singer. The problem was subsequently shown to be equivalent to numerous open problems in pure mathematics, applied mathematics, engineering and computer science. Kadison, Singer, and most later authors believed the statement to be false, but, in 2013, it was proven true by Adam Marcus, Daniel Spielman and Nikhil Srivastava, who received the 2014 Polya Prize for the achievement.

We will discuss about the Kadison-Singer problem for a separable Hilbert space. First of all we will characterize all functions that can possibly have the Kadison-Singer property and then among these which class of functions fail to have the Kadison-Singer property and also finally which class will have the Kadison-Singer property.

Contents

Declaration ii

Approval Sheet iii

Acknowledgements iv

Abstract vi

0 List of symbols **1**

1 Preliminaries **3**

1.1 Linear Algebra 3

1.1.1 Positive operators and square Root 3

1.1.2 Characterization of positive operators 4

1.2 Functional Analysis 5

1.2.1 Weak*-topology 5

1.3 Hilbert Space 7

1.3.1 Operators on Hilbert spaces 8

1.3.2 Direct sums of Hilbert spaces 8

1.3.3 Projection lattice 10

1.4 C^*-algebras 11

1.4.1 Characters 14

1.5 von Neumann algebras 16

1.6 Topology 20

1.6.1 Compactness 20

1.6.2 Ultrafilters 22

1.6.3 Stone-Čech compactification of discrete spaces 27

2 KS Problem In Finite Dimension **33**

2.1 Basic definitions 33

2.2 Basic cases 34
2.2.1 KS problem in one dimension 34
2.2.2 KS problem in two dimension 34
2.2.3 KS problem in any finite dimension 37
2.3 The Kadison-Singer property 47

3 State space **51**
3.1 States on C^*-algebra 51
3.2 Pure states and characters 56
3.3 Extensions of pure states 61
3.4 Properties of extensions and restrictions 64

4 Maximal abelian C^*-subalgebras **66**
4.1 Maximal abelian C^*-subalgebras 66
4.2 Examples of maximal abelian C^*-subalgebras 70
4.2.1 The discrete subalgebra 71
4.2.2 The continuous subalgebra 76
4.2.3 The mixed subalgebra 83

5 Classification of MASA **89**
5.1 Unitary equivalence 89
5.2 Classification of MASA 93

6 The Kadison-Singer problem: An overview **97**
6.1 Normal states 97
6.2 Kadison-Singer conjecture 99

References **103**

Chapter 0

List of symbols

- V a normed vector space
- V^* the dual of the normed linear space V
- $\mathbb{C}$ the set of all complex numbers
- $co(S)$ the convex hull of S
- ∂K is the set of all boundary points of K
- H a Hilbert space
- $B(H_1, H_2)$ the set all bounded linear operators from H_1 to H_2
- $B(H)$ the set of all bounded linear operators on H
- $\mathcal{P}(H)$ the set of all projections in $B(H)$
- $M_n(\mathbb{C})$ the set of all $n \times n$ matrices over $\mathbb{C}$
- $D_n(\mathbb{C})$ the set of all $n \times n$ diagonal matrices over $\mathbb{C}$
- A is a unital C^*-algebra
- $\sigma(a)$ the spectrum of a
- $\Omega(A)$ the set of all characters on A
- S' the commutant of S

- $\mathcal{P}(X)$ the power set of X
- $\ell^2(\mathbb{N})$ the set of all square summable sequences in $\mathbb{C}$
- $\ell^\infty(\mathbb{N})$ the set of all bounded sequences in $\mathbb{C}$
- $\mathbb{D}(\ell^2(\mathbb{N}))$ the set of all bounded diagonal operators on $\ell^2(\mathbb{N})$
- $S(A)$ the set of all states on A
- $\mathcal{D}(M_n(\mathbb{C}))$ the set of all density operators in $M_n(\mathbb{C})$
- $\partial_e C$ the extreme boundary of C
- $\text{Ext}(f)$ the set of all extensions of f
- $L^2[0,1]$ the space of square integrable functions
- $L^\infty[0,1]$ the space of all essentially bounded functions
- $A_d(j)$ the discrete subalgebra of cardinality j
- A_c the continuous subalgebra
- $A_d(j) \oplus A_c$ the mixed subalgebra

Chapter 1

Preliminaries

To understand the Kadison-Singer problem, we need results from a wide range of areas in mathematics. In this chapter we briefly discuss the required results from linear algebra, topology, order theory, complex analysis, functional analysis, operator theory, C^*-algebra and operator algebra.

1.1 Linear Algebra

1.1.1 Positive operators and square Root

Definition 1.1.1 (Positive Operator)**.** *Let V be an inner product space. A linear operator $T \in L(V)$ is called* **positive** *if*

1. *T is self-adjoint ($T = T^*$) and*

2. *$\langle Tv, v\rangle \geqslant 0$ for all $v \in V$.*

Note: 1.1.1. *If V is a complex inner product space, then the requirement that T is self-adjoint can be dropped from the definition above. Because, in a complex inner product space V, $T \in L(V)$ is self adjoint if and only if $\langle Tv, v\rangle \in \mathbb{R}$ for every $v \in V$.*

Example 1.1.1. *If U is a subspace of V, then the orthogonal projection P_U on U is a positive operator.*

Definition 1.1.2 (Square root)**.** *An operator R is called a* **square root** *of an operator T if $R^2 = T$.*

Example 1.1.2. *If $T \in L(\mathbb{F}^3)$ is defined by $T(z_1, z_2, z_3) = (z_3, 0, 0)$, then the operator $R \in L(\mathbb{F}^3)$ defined by $R(z_1, z_2, z_3) = (z_2, z_3, 0)$ is a square root of T.*

1.1.2 Characterization of positive operators

The following theorem characterizes positive operators.

Theorem 1.1.1. *Let $T \in L(V)$, where V is a finite dimensional vector space over the field $\mathbb{F}$. Then the following are equivalent :*

1. *T is positive;*
2. *T is self-adjoint and all the eigen values are non-negative;*
3. *T has a positive square root;*
4. *T has a self-adjoint square root;*
5. *there exists an operator $R \in L(V)$ such that $T = R^*R$.*

Proof. We will prove that

$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (1).$$

First suppose (1) holds, so that T is positive. Obviously T is self-adjoint (by definition of a positive operator). To prove the other condition in (2), suppose λ is an eigen value of T. Let v be an eigenvector of T corresponding to λ. Then

$$0 \leq \langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle.$$

Thus λ is a nonnegative number. Hence (2) holds.

Now suppose (2) holds, so that T is self-adjoint and all the eigenvalues of T are nonnegative. By the Spectral theorem, there is an orthonormal basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of T. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T corresponding to $e_1, \ldots, e_n$; thus each λ_j is a nonnegative number. Let R be the linear map from V to V such that

$$Re_j = \sqrt{\lambda_j} e_j$$

for $j = 1, \ldots, n$.

Then R is a positive operator. Furthermore, $R^2 e_j = \lambda_j e_j = T e_j$ for each j, which implies that $R^2 = T$. Thus R is a positive square root of T. Hence (3) holds.

Now suppose (3) holds. Then T has a positive square root, say R. Since R is postive, it is self-adjoint by the definition of positiveness. Therefore T has a self-adjoint positive root. Hence (4) holds.

Now suppose (4) holds, meaning that there exists a self-adjoint operator R on V such that $T = R^2$. Then $T = R^* R$ (because $R^* = R$). Hence (5) holds.

Finally, suppose (5) holds. Let $R \in L(V)$ be such that $T = R^* R$. Then

$$T^* = (R^* R)^* = R^* (R^*)^* = T.$$

Hence T is self-adjoint. To complete the proof that (1) holds, note that

$$\langle Tv, v \rangle = \langle R^* Rv, v \rangle = \langle Rv, Rv \rangle \geq 0,$$

for every $v \in V$. Thus T is positive. □

1.2 Functional Analysis

1.2.1 Weak*-topology

For a normed vector space V, we can consider **bounded linear functionals** on V. These are linear maps $f : V \to \mathbb{C}$ such that

$$sup\{\ |f(v)| : \|v\| = 1\} < \infty.$$

We collect all such bounded linear functionals on V.This is itself a vector space, denoted by V^*, which we call the **dual space** of V,This dual space then has a natural norm itself, given by

$$\|f\| = sup\{|f(v)| : \|v\| = 1\},$$

for all $f \in V^*$. This gives the dual space a natural topology, but the dual space also has another topology. To describe this topology, we define for all $f \in V^*$, $v \in V$ and $\epsilon > 0$ the set

$$B(f, v, \epsilon) = \{g \in V^* : |f(v) - g(v)| < \epsilon\}$$

It is clear that these sets form a subbase for a topology on V^*. We call this topology the **weak*-topology**.

Theorem 1.2.1. *(Banach-Alaoglu) Suppose V is a normed vector space. Then the closed unit ball of the dual space V^*, i.e.*

$$\{f \in V^* : \|f\| \leq 1|\},$$

is **compact** *with respect to the weak*-topology.*

Proof. For the detailed proof of this please refer [4]. □

Theorem 1.2.2 (Hahn-Banach extension theorem)**.** *Let X be a normed linear space over $\mathbb{C}$ and Y be a linear subspace of X. If $g : Y \to \mathbb{C}$ is a bounded linear functional, then there $f \in X^*$ such that $f|_Y = g$ and $\|f\| = \|g\|$.*

Proof. For the detailed proof of this please refer [1]. □

The above Hahn-Banach theorem is the one we need in the main text.

Now there is one usual question that one can ask, when the abpve Hahn-Banach extention will be unique. Here we will state a theorem about unique Hahn-Banach extention. Before that we will a small note.

Note: 1.2.1. *If Y is a dense subspace of a normed linear space X and g is a continuous linear functional on Y, then the uniform continuty of g(since every continuous linear map between two normed linear spaces is uniformly continuous) enables us to conclude that g has a unique continuous linear extention f to X, and also $\|f\| = \|g\|$. Thus g has a unique Hahn-Banach extention to X.*

Theorem 1.2.3 (Taylor-Foguel, 1958)**.** *Let X be a normed linear space. For every nonzero subspace Y of X and every $g \in Y^*$, there is a unique Hahn-Banach extention of g to X if and only if X^* is strictly convex, that is, for $f_1 \neq f_2$ in X^* with $\|f_1\| = 1 = \|f_2\|$, we have $\|f_1 + f_2\| < 2$.*

Proof. For the detailed proof of this theorem please refer [1]. □

Definition 1.2.1. *Suppose X is a vector space over $\mathbb{C}$ and $S \subseteq X$. We define the convex hull of S to be:*

$$co(S) = \{\sum_{i=1}^{n} t_i s_i : n \in \mathbb{N}, t_i \geq 0, \sum_{i=1}^{n} t_i = 1, s_i \in S\}$$

i.e. the set of all finite convex combinbations of elements in K.

Using this definition, we have the following important result.

Theorem 1.2.4 (Krein-Milman)**.** *Suppose X is a normed linear space and $K \subseteq X$ is a convex compact subset. Then:*

$$K = \overline{co(\partial K)}.$$

Furthermore, if $M \subseteq V$ is such that $K = \overline{co(M)}$, then $\partial K \subseteq \overline{M}$.

Proof. For the proof please refer [3]. □

For more information regarding weak*-topology and topological vector spaces, please refer [1], [2], [3], [4].

1.3 Hilbert Space

One of the main concepts in the KS problem is that of a Hilbert space.

Definition 1.3.1. *A Hilbert space H is a complex vector space endowed with a complex inner product $\langle ., . \rangle$ which we take linear in the second coordinate, such that H is complete with respect to the norm $\|.\|$ induced by the inner product via $\|x\|^2 = \langle x, x \rangle$.*

Hilbert spaces can be seen as generalizations of Euclidean vector spaces. Therefore, we also want to consider bases for Hilbert spaces.

Definition 1.3.2. *Suppose H is a Hilbert space. Then a subset $E \subseteq H$ is called a* **basis** *for H if E is an orthonormal set whose linear span is dense in H.*

Note that if the cardinality of a basis of H is finite, then the Hilbert space is isomorphic to a complex Euclidean vector space. We have a special name for Hilbert spaces that have a countable basis.

Definition 1.3.3. *H is called* **separable** *if it has a countable basis.*

We also need the notion of orthogonal families.

Definition 1.3.4. *Let H be a Hilbert space. Two subsets $C, D \subseteq H$ are said to be* **orthogonal** *if for every $c \in C$ and $d \in D$, $\langle c, d \rangle = 0$. A family of subspaces $\{C_i\}_{i \in I}$ of H is said to be an* **orthogonal family** *if all pairs of members are orthogonal.*

1.3.1 Operators on Hilbert spaces

We now want to consider linear operators $T : H \to H'$ between two Hilbert spaces. In fact, we are only interested in bounded operators.

Definition 1.3.5. *Let H be a Hilbert space and $T : H \to H'$ be a linear operator. We say that T is* **bounded** *if there is a $k > 0$ such that $\|T(x)\| \leq k\|x\|$ for all $x \in H$. The set of all bounded operators from H to H' is denoted by $B(H, H')$.*

Now note that $B(H, H')$ is not just a set, but a normed vector space. Here scalar multiplication and addition are defined pointwise. The norm is naturally given by

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|,$$

Furthermore, for every $T \in B(H, H')$ there is a unique operator $T^* \in B(H, H')$ such that

$$\langle x, T(y) \rangle = \langle T^*(x), y \rangle$$

for every $x \in H'$ and $y \in H$. The operator T^* is called the **adjoint** of T.

When $H = H'$, we write $B(H) := B(H, H')$ and we observe that defining multiplication by composition, i.e. $(TS)(x) = T(S(x))$ for all $T, S \in B(H)$ and $x \in H$, gives $B(H)$ the structure of an algebra.

1.3.2 Direct sums of Hilbert spaces

Given two Hilbert spaces H_1 and H_2, we can form a Hilbert space $H = H_1 \oplus H_2$, which has an inner product $\langle , \rangle$ defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle_1 + \langle x_2, y_2 \rangle_2,$$

where $\langle\ ,\ \rangle_1$ and $\langle\ ,\ \rangle_2$ are the inner products on H_1 and H_2, respectively. H is called the **direct sum** of H_1 and H_2.

Conversely, given a Hilbert space H and a closed linear subspace $K \subseteq H$, one can realize H as a direct sum $H = K \oplus K^{\perp}$, where

$$K^{\perp} := \{x \in H : \langle x, y\rangle = 0 \ \forall \ y \in K\}$$

is called the **orthogonal compliment** of K.

Operators on direct sums

Note that for a given direct sum $H_1 \oplus H_2$, there are canonical inclusions and projection maps :

$$i_1 : H_1 \to H_1 \oplus H_2, \ \ i_1(x) = (x, 0)$$

$$i_2 : H_2 \to H_1 \oplus H_2, \ \ i_2(y) = (0, y)$$

$$\pi_1 : H_1 \oplus H_2 \to H_1, \ \ \pi_1(x, y) = x$$

$$\pi_2 : H_1 \oplus H_2 \to H_2, \ \ \pi_2(x, y) = y$$

Using this, for given $a_1 \in B(H_1)$ and $a \in B(H_2)$, one can define

$$(a_1, a_2) : H_1 \oplus H_2 \to H_1 \oplus H_2,$$

by $(a_1, a_2) = i_1 a_1 \pi_1 + i_2 a_2 \pi_2$, i.e. $(a_1, a_2)(x, y) = (a_1(x), a_2(y))$. Clearly, we then have $(a_1, a_2) \in B(H_1 \oplus H_2)$. Extending this idea, for subsets $A_1 \subseteq B(H_1)$ and $A_2 \subseteq B(H_2)$, $A_1 \oplus A_2 \subseteq B(H_1 \oplus H_2)$.

Conversely, one can ask the question whether for some $a \in B(H_1 \oplus H_2)$ there are $a_1 \in B(H_1)$ and $a_2 \in B(H_2)$ such that $a = (a_1, a_2)$. The following proposition answers this question.

Proposition 1.3.1. *Suppose H_1 and H_2 are Hilbert spaces and $a \in B(H_1 \oplus H_2)$. Then there are $a_1 \in B(H_1)$ and $a_2 \in B(H_2)$ such that $a = (a_1, a_2)$ if and only if $a(i_1(H_1)) \subseteq i_1(H_1)$ and $a(i_2(H_2)) \subseteq i_2(H_2)$.*

Proof. First, suppose that $a = (a_1, a_2)$ for some $a_1 \in B(H_1)$ and $a_2 \in B(H_2)$. Then let $x \in H_1$. Then

$$a(i_1(x)) = (a_1, a_2)(x, 0) = (a_1(x), 0),$$

so $a(i_1(H_1)) \subseteq i_1(H_1)$. Likewise, $a(i_2(H_2)) \subseteq i_2(H_2)$.

For the converse, suppose $a(i_1(H_1)) \subseteq i_1(H_1)$ and $a(i_2(H_2)) \subseteq i_2(H_2)$. Define $a_1 = \pi_1 a i_1$ and $a_2 = \pi_2 a i_2$. Then, for $(x, y) \in H_1 \oplus H_2$, $a(i_1(x)) = (x', 0)$ and $a(i_2(y)) = (0, y')$ for some $x' \in H_1$ and $y' \in H_2$. Then :

$$a(x, y) = a(i_1(x) + i_2(y)) = (x', 0) + (y', 0) = (x', y'),$$

and

$$\begin{aligned}(a_1, a_2)(x, y) &= (i_1\pi_1 a i_1 \pi_1 + i_2 \pi_2 a i_2 \pi_2)(x, y)\\ &= i_1\pi_1 a(i_1(x)) + i_2\pi_2 a(i_2(y))\\ &= i_1\pi_1(x', 0) + i_2\pi_2(0, y')\\ &= (x', y')\end{aligned}$$

Therefore, $a = (a_1, a_2)$. □

In the case that an operator $a \in B(H_1 \oplus H_2)$ can be written as $a = (a_1, a_2)$ for some $a_1 \in B(H_1)$ and $a_2 \in B(H_2)$, we say that a **decomposes over the direct sum** $H_1 \oplus H_2$. Likewise, if an algebra $A \subseteq B(H_1 \oplus H_2)$ satisfies $A = A_1 \oplus A_2$ for some $A_1 \subseteq B(H_1)$ and $A_2 \subseteq B(H_2)$, we say that A decomposes over the direct sum $H_1 \oplus H_2$.

1.3.3 Projection lattice

Definition 1.3.6. *Suppose H is a Hilbert space and $p \in B(H)$. Then p is a* **projection** *if $p^2 = p^* = p$.*

Note that a projection $p \in B(H)$ is always positive, since for any $x \in H$ we have

$$\langle x, px\rangle = \langle x, p^2 x\rangle = \langle x, p^* p x\rangle = \langle px, px\rangle = \|px\|^2 \geq 0.$$

Now, if we write $\mathcal{P}(H)$ for the set of all projections in $B(H)$ for a Hilbert space H, it is clear that for any $p \in \mathcal{P}(H)$, we have $1 - p \in \mathcal{P}(H)$.

We can introduce a partial order $\leq$ on $\mathcal{P}(H)$ by saying that $p \leq q$ if and only if $q - p \geq 0$. By the above it follows that (with respect to $\leq$) 0 is the minimal element of $\mathcal{P}(H)$ and 1 is the maximal element.

Furthermore, $p \leq q$ is equivalent to $p(H) \subseteq q(H)$.

Definition 1.3.7. *Let H be a Hilbert space and $p \in B(H)$ such that $p \neq 0$. Then p is called a* **minimal projection** *if $q \in \mathcal{P}(H)$ and $0 \leq q \leq p$ implies $q = 0$ or $q = p$.*

1.4 C^*-algebras

We know that for a given Hilbert space H the operator algebra $B(H)$ not only has the structure of an algebra, but also has an adjoint operation and a norm. Together, these properties give $B(H)$ a much more special algebraic structure, namely that of a C^*-algebra. Before that we will define algebra and $*$-algebra.

Definition 1.4.1 (Algebra)**.** *An* **algebra** *over a field $\mathbb{F}$ is a vector space $\mathcal{A}$ over $\mathbb{F}$ that also has a multiplication defined on it that makes $\mathcal{A}$ into a ring such that if $\alpha \in \mathbb{F}$ and $a, b \in \mathcal{A}$*

$$\alpha(ab) = (\alpha a)b = a(\alpha b).$$

Example 1.4.1. *$M_n(\mathbb{C})$ over $\mathbb{C}$ is an algebra.*

Definition 1.4.2 ($*$-algebra)**.** *An algebra A is called a *-algebra if it is a complex algebra with a conjugate linear involution $*$ which is an anti-isomorphism, i.e., for any $a, b \in A$ and $\alpha \in \mathbb{C}$,*

$$(a + b)^* = a^* + b^*,$$

$$(\alpha a)^* = \overline{\alpha} a^*,$$

$$a^{**} = a$$

and

$$(ab)^* = b^* a^*.$$

If $a \in A$, then a^* is called the adjoint of a. Let A be a *-algebra which is also a normed algebra. A norm on A that satisfies

$$\|a^* a\| = \|a\|^2$$

for all $a \in A$ is called a C^*-norm. If, with this norm, A is complete, then A is called a C^*-algebra. Then $M_n(\mathbb{C})$

Example 1.4.2. *$M_n(\mathbb{C})$ over $\mathbb{C}$ is an algebra. For $A = (a_{ij}) \in M_n(\mathbb{C})$, define $A^* = (\bar{a}_{ji})$. Then $M_n(\mathbb{C})$ is a C^*-algebra.*

Definition 1.4.3 (C^*-algebra)**.** *A C^*-algebra is a normed, associative algebra A endowed with an operation $* : A \to A, a \mapsto a^*$ (we call a^* the adjoint of a), with the following compatibility structure:*

1. *A is complete in the norm $\|.\|$, i.e. $(A, \|.\|)$ is a Banach space.*
2. *The norm is submultiplicative, i.e. $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.*
3. *The adjoint operation is an involution, i.e. $a^{**} = a$ for all $a \in A$.*
4. *The adjoint operation is conjugate-linear, i.e. $(\lambda a + b)^* = \bar{\lambda} a^* + b^*$ for all $\lambda \in \mathbb{C}$ and $a, b \in A$.*
5. *The adjoint operation is anti-multiplicative, i.e. $(ab)^* = b^* a^*$ for all $a, b \in A$.*
6. *The C^*-identity holds: $\|a^* a\| = \|a\|^2$ for all $a \in A$.*

A C^*-algebra A is called unital if it contains an algebraic unit 1 (i.e. a1=1a=a for all $a \in A$). Since the adjoint is an involution and is anti-multiplicative, automatically $1^* = 1$. By the C^*-identity it then also follows that $\|1\| = 1$.

Example 1.4.3. *$M_n(\mathbb{C})$ over $\mathbb{C}$ is a C^*-algebra.*

The C^*-identity together with submultiplicity also guarantees a more immediate compatibility between the adjoint operators and the norm.

Lemma 1.4.1. *Suppose A is a C^*-algebra. Then the adjoint preserves the norm, i.e. $\|a^*\| = \|a\|$ for all $a \in A$.*

Proof. For any $b \in A$, $\|b\|^2 = \|b^* b\| \leq \|b^*\|\|b\|$. So $\|b\| \leq \|b^*\|$. Using this for $b = a$ and $b = a^*$, we get $\|a\| \leq \|a^*\|$ and $\|a^*\| \leq \|a\|$, i.e. $\|a^*\| = \|a\|$ for any $a \in A$. □

We can also consider C^*-subalgebras.

Definition 1.4.4. *Let A be a C^*-algebra. A C^*-subalgebra S of A is a subalgebra $S \subseteq A$ that is topologically closed (with the topology coming from the norm $\|.\|$ of A) and closed under the adjoint operation, i.e. $a^* \in S$ for all $a \in S$.*

Note that by the conditions on a C^*-subalgebra, every C^*-subalgebra is a C^*-algebra in its own right, by restriction of the norm and adjoint operations to the subalgebra.

In KS problem we will study states and pure states. For the definition of states, we need the notion of positive elements of a C^*-algebra.

Definition 1.4.5 (Positivity)**.** *Suppose A is a C^*-algebra, and let $a \in A$. Then we say that a is* **positive** *if and only if there exists $b \in A$ such that $a = b^*b$. In this case, we write $a \geq 0$.*

In the case of unital C^*-algebra, we also have a characterization of positive elements in terms of the spectrum of an element. To define this, we write $Inv(A)$ for the set of all invertible elements in a C^*-algebra A.

Definition 1.4.6 (Spectrum)**.** *Suppose A is a unital C^*-algebra and let $a \in A$. Then we define*

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \notin Inv(A)\},$$

which we call the **spectrum** *of a.*

Then we have the following equivalence.

Proposition 1.4.1. *Suppose A is a unital C^*-algebra and let $a \in A$. Then a is positive if and only if $a = a^*$ and $\sigma(a) \subset [0, \infty)$.*

Proof. For the proof please refer [5]. □

In the case of an operator algebra $B(H)$ for some Hilbert space H, we have yet another description of positive elements, which resembles that of positive (semi-definite) matrices.

Proposition 1.4.2. *Let H be a Hilbert space and let $a \in B(H)$. Then a is positive if and only if for every $x \in H$ we have $\langle x, ax \rangle \geq 0$.*

Proof. For the proof please refer [5]. □

The set of positive elements in a C^*-algebra A is often denoted by A^+. This set has some special properties. First of all, we can decompose any element into positive elements.

Proposition 1.4.3. *Suppose A is a C^*-algebra. Then, for any $a \in A$, there are $a_k \geq 0$ such that $a = \sum_{k=0}^{3} i^k a_k$ and $\|a_k\| \leq \|a\|$.*

Proof. For the proof please refer [5]. □

We also have the following result.

Proposition 1.4.4. *Suppose A is a C^*-algebra and let $a \in A$ be positive. Then there is a $b \in A^+$ such that $a = b^2$.*

Proof. For the proof please refer [5]. □

Proposition 1.4.5. *Suppose A is a C^*-algebra and let $a \in A^+$ such that $\|a\| \leq 1$. Then $1 - a^2$ is positive and a commutes with b where $b^2 = 1 - a^2$.*

Proof. For the proof please refer [5] □

One crucial thing here is to noticed that the notion of positivity also induces a natural partial order $\leq$ on the self-adjoint elements of a C^*-algebra A, by defining $b \leq c$ if and only if $c - b \geq 0$. This partial order has the following properties.

Proposition 1.4.6. *If c, d are self-adjoint and $-d \leq c \leq d$, then $\|c\| \leq \|d\|$.*

Proof. Refer [5]. □

Proposition 1.4.7. *Suppose H is a Hilbert space and $d \in B(H)$ such that $d \geq 0$ and $\|d\| = 1$, then $d \leq 1$.*

Proof. Refer [5]. □

1.4.1 Characters

In KS problem we will consider abelian C^*-algebras, characters play a important role there.

Definition 1.4.7 (Characters)**.** *Let A be a C^*-algebra. A* **character** *is a non-zero algebra homomorphism $c : A \to \mathbb{C}$, i.e. c is multiplicative and linear. The set of all characters on A is denoted by $\Omega(A)$.*

First, we prove three lemmas that all give a certain property of characters.

Lemma 1.4.2. *Suppose that A is a unital C^*-algebra and $c \in \Omega(A.)$ Then $c(1) = 1$.*

Proof. First of all $c(1) = c(1^2) = c(1)^2$, so $c(1) \in \{0, 1\}$. If $c(1) = 0$, then for all $a \in A$, $c(a) = c(1.a) = c(1).c(a) = 0$, so $c = 0$. This is a contradiction with c being a character, so $c(1) = 1$. □

Lemma 1.4.3. *Suppose A is a C^*-algebra, $c \in \Omega(A)$ and $a = a^* \in A$. Then $c(a) \in \mathbb{R}$.*

Proof. We claim that $c(a) \in \sigma(a)$. To see this, suppose that $a - c(a)1$ is invertible. Then there is a $b \in A$ such that $(a - c(a)1)b = 1 = b(a - c(a)1)$. Then $1 = c(1) = c((a - c(a)1)b) = c(a - c(a)1)c(b) = (c(a) - c(a))c(b) = 0$. This is a contradiction, so $a - c(a)1$ is not invertible, i.e. $c(a) \in \sigma(a)$. A standard result in functional analysis is the fact that $\sigma(a) \subset \mathbb{R}$, since $a = a^*$. Therefore, $c(a) \in \mathbb{R}$. □

Lemma 1.4.4. *Suppose A is a C^*-algebra and $c \in \Omega(A)$. Then $c(a^*) = \overline{c(a)}$ for all $a \in A$.*

Proof. Suppose $a \in A$. Then $a = b + id$ for some $b = b^*, d = d^* \in A$. Then $c(a), c(d) \in \mathbb{R}$, by above lemma. Therefore,

$$c(a^*) = c(b - id) = c(b) - ic(d) = \overline{c(b) + ic(d)} = \overline{c(b + id)} = \overline{c(a)},$$

as desired. □

Because of the the following result, characters are important for abelian C^*-algebras.

Theorem 1.4.1 (Gelfand isomorphism)**.** *Suppose A is a non-zero abelian C^*-algebra. Then the map*

$$G : A \to C(\Omega(A)),$$

defined by

$$G(a)(f) = f(a),$$

is an isomorphism of C^*-algebras.

Proof. For the detailed proof please refer [5] □

The following lemma is an easy consequence of the Gelfand isomorphism.

Lemma 1.4.5. *Suppose A is an abelian C^*-algebra. Then $\Omega(A)$ separates points.*

Proof. Suppose $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ for all $f \in \Omega(A)$. Then

$$G(a_1)(f) = f(a_1) = f(a_2) = G(a_2)(f)$$

for all $f \in \Omega(A)$, so $G(a_1) = G(a_2)$, where $G : A \to C(\Omega(A))$ is the Gelfand isomorphism. However, since G is a isomorphism, $a_1 = a_2$. So, indeed, $\Omega(A)$ separates points. □

We can use this lemma to prove the following result about projections and characters.

Lemma 1.4.6. *Suppose A is a C^*-algebra. Then, for every $g \in \Omega(A)$ and projection $p \in A$, $g(p) = 1$. If $p \in A$ is a non-zero projection, there is a $f \in \Omega(A)$ such that $f(p) = 1$.*

Proof. Suppose $g \in \Omega(A)$ and $p \in A$ is a projection. Then $g(p)^2 = g(p^2) = g(p)$, whence $g(p) \in \{0, 1\}$. Now, $g(p) = 0$ for every $g \in \Omega(A)$ implies that $p = 0$, since $\Omega(A)$ separates points and $g(0) = 0$ for all $g \in \Omega(A)$. Therefore, if p is non-zero, then there is a $f \in \Omega(A)$ such that $f(p) = 1$. □

1.5 von Neumann algebras

In order to define von Neumann algebras, we first introduce the strong topology. We do this by means of a subbasis. For every $a \in B(H)$, $x \in H$ and $\epsilon > 0$, define:

$$S(a, x, \epsilon) := \{b \in B(H) : \|(a - b)x\| < \epsilon\}.$$

Collecting these sets together in

$$\mathcal{S} := \{S(a, x, \epsilon) : a \in B(H), x \in H, \epsilon > 0\},$$

we obtain a subbasis for a topology on $B(H)$, since $\bigcup \mathcal{S} = B(H)$. We call this topology the **strong topology** on $B(H)$. A basis for this topology is then given by:

$$\mathcal{B} := \{\bigcap_{i=0}^{n} S(a_i, x_i, \epsilon_i) : a_i \in B(H), x_i \in H, \epsilon_i > 0\}.$$

An important property of the strong topology is given in terms of convergent nets.

Proposition 1.5.1. *Let H be a Hilbert space and $\{a_i\}_{i \in I}$ be a net in $B(H)$. Furthermore, let $a \in B(H)$. Then the following are equivalent:*

1. *$\{a_i\}_{i \in I}$ converges to a with respect to the strong topology on $B(H)$.*

2. *For each $x \in H$, $\{a_i(x)\}$ converges to $a(x)$.*

Proof. First, suppose that $\{a_i\}_{i\in I}$ converges to a with respect to the strong topology. Let $x \in H$ and $\epsilon > 0$. Since $\{a_i\}_{i\in I}$ converges to a, there is a $i_0 \in I$ such that for all $i \geq i_0$, $a_i \in S(a, x, \epsilon)$, i.e. $\|a_i(x) - a(x)\| < \epsilon$. Therefore $\{a_i(x)\}_{i\in I}$ converges to $a(x)$.

For the converse, suppose that for all $x \in H$, $\{a_i(x)\}_{i\in I}$ converges to $a(x)$. Now, let U be a neighbourhood of a. Since $\mathcal{B}$ is a base for the strong topology, there is a $n \in \mathbb{N}$, $\{b_i\}_{i=1}^n \subseteq B(H)$, $\{x_i\}_{i=1}^n \subseteq H$ and $\{\epsilon_i\}_{i=1}^n \subseteq \mathbb{R}_{>0}$ such that

$$a \in \bigcap_{i=1}^{n} S(b_i, x_i, \epsilon_i) \subseteq U.$$

Since $a \in S(b_i, x_i, \epsilon_i)$ for all $i \in \underline{n}$, there are $\{\delta_i\}_{i=1}^n$ such that $S(a, x_i, \delta_i) \subseteq S(b_i, x_i, \epsilon_i)$ for all $i \in \underline{n}$. Then we have :

$$a \in \bigcap_{i=1}^{n} S(a, x_i, \delta_i) \subseteq S(b_i, x_i, \epsilon_i) \subseteq U.$$

By assumption, for every $i \in \underline{n}$ there is a $j_i \in I$ such that for all $j \geq j_i$ we have

$$\|a_j(x_i) - a(x_i)\| < \delta_i,$$

i.e. $a_j \in S(a, x_i, \delta_i)$. Now choose a $j_0 \in I$ such that $j_0 \geq j_i$ for all $i \in \underline{n}$, which exists because I is a directed set. Then, for every $j \geq j_0$,

$$a_j \in \bigcap_{i=1}^{n} S(a, x_i, \delta_i) \subseteq U.$$

So, the net $\{a_i\}_{i\in I}$ is eventually in U. Since U was an arbitrary neighbourhood of a, $\{a_i\}_{i\in I}$ converges to a.

□

Using the strong topology, we can directly define von Neumann algebras.

Definition 1.5.1. *Let H be a separable Hilbert space. Then a *-subalgebra $A \subseteq B(H)$ is called a* **von Neumann algebra** *if it is closed with respect to the strong topology.*

By now, we have two topologies on $B(H)$; the norm topology and the strong topology. C^*-subalgebras deal with the norm topology, whereas von Neumann algebras are defined using the strong topology. The following proposition gives a link between these two different viewpoints.

Proposition 1.5.2. *Let H be a Hilbert space and suppose that $A \subset B(H)$ is a Von Neumann algebra. Then A is a C^*-subalgebra of $B(H)$.*

Proof. Suppose $\{a_i\}_{i\in I}$ is a norm convergent net in A, say with limit $a \in B(H)$. Now let $x \in H$ such that $x \neq 0$ and $\epsilon > 0$. Then there is a $i_0 \in I$ such that for every $i \geq i_0$, $\|a - a_i\| < \frac{\epsilon}{\|x\|}$. Then for every $i \geq i_0$,

$$\begin{aligned}
\|(a-a_i)(x)\| &= \|x\|\|(a-a_i)(\frac{x}{\|\epsilon\|})\| \\
&\leq \|x\|\|a-a_i\| \\
&< \|x\|\frac{\epsilon}{\|x\|} \\
&= \epsilon.
\end{aligned}$$

Hence $a(x)$ is the limit of $\{a_i(x)\}_{i\in I}$ for every $x \in H$ such that $x \neq 0$. Since it clearly also holds for $x = 0$, it holds for every $x \in H$.

Therefore, a is the strong limit of $\{a_i\}_{i\in I}$. Since A is a Von Neumann algebra, $a \in A$.

Therefore, A is closed with respect to the norm topology and hence is a C^*-subalgebra of $B(H)$. □

There is an important result about von Neumann algebras that involves the commutant of an algebra.

So let us first define commutant of an algebra.

Definition 1.5.2 (Commutant)**.** *Suppose X is an algebra and $S \subset X$ is a subset. We define the commutant of S to be:*

$$S' := \{x \in X : sx = xs \ \ for \ \ all \ \ s \in S\},$$

i.e. the set of all $x \in X$ that commute with all of S.

We denote the double commutant of a subset S of an algebra X by $S'' := (S')'$ and likewise $S''' = (S'')'$.

Lemma 1.5.1. *Suppose X is an algebra and $S, T \subseteq X$ are subsets. Then:*

1. *$S \subseteq S'$ iff S is abelian*

2. *If $S \subseteq T$, then $T' \subseteq S'$.*

3. *$S \subseteq S''$.*

4. *$S' = S'''$.*

Proof. The proofs of the first three properties are trivial. For the last property, observe that $S' \subset (S')'' = S'''$ by the third property, and by combining property 2 and 3 one has $S''' = (S'')' \subset S'$. □

Proposition 1.5.3. *Let H be a Hilbert space and $A \subset B(H)$ a *-subalgebra. Then A' is a von Neumann algebra.*

Proof. We first prove that A' is a $*$-subalgebra of $B(H)$. To see this, let $u, v \in A'$, $\lambda \in \mathbb{C}$ and $a \in A$. Then

$$(uv)a = u(va) = u(av) = (ua)v = (au)v = a(uv),$$

$$(\lambda u)a = \lambda(ua) = \lambda(au) = a(\lambda u),$$

$$(u + v)a = ua + va = au + av = a(u + v)$$

and

$$u^*a = (a^*u)^* = (ua^*)^* = au^*,$$

where the latter follows from the fact that $a^* \in A$ too. Hence A' is indeed a $*$-subalgebra. Now suppose $\{v_i\}_{i \in I}$ is a net in A' that converges to $u \in B(H)$ in the strong topology. Now let $a \in A$ and $x \in H$ be arbitrary. Then:

$$(ua)(x) = u(a(x)) = \lim_i v_n(a(x)) = \lim_i a(v_n(x)) = a(lim_i(v_n(x))) = a(u(x)) = (au)(x),$$

whence $ua = au$ and therefore $u \in A'$. Hence A' is closed with respect to the strong topology. Therefore, A' is a von Neumann algebra. □

In our main conjecture we will make use of generated von Neumann algebras. So let us define generated von Neumann algebras.

Definition 1.5.3. *For any set $S \subset B(H)$ the von Neumann algebra generated by S is*

$$\langle S \rangle_{vN} := \bigcap \{A \subseteq B(H) : A \textit{ is a von Neumann algebra and } S \subseteq A\},$$

which is in fact a von Neumann algebra since an arbitrary intersection of von Neumann algebras is clearly again a von Neumann algebra.

1.6 Topology

Throughout our main text we need various concepts of topology. Here we discuss some of them.

1.6.1 Compactness

In a topological space, compactness is defined using open coverings. However, it can also be defined using closed sets. To show this, we first need the following.

Definition 1.6.1. *Let X be a topological space and $F \subseteq \mathcal{P}(X)$ a family of subsets. Then F has the finite intersection property if for every $\{A_i : i = 1, .., n\} \subseteq F$ we have that*

$$\bigcap_{i=0}^{n} A_i \neq \emptyset$$

Using this, we can give the equivalent definition of compactness.

Proposition 1.6.1. *Let X be a topological space. Then the following are equivalent:*

1. *X is compact.*
2. *Every family $F \subseteq \mathcal{P}(X)$ consisting of closed subsets with the finite intersection property satisfies $\bigcap F \neq \emptyset$.*

Proof. For the proof please refer [13]. □

In the next subsection we will use this equivalent definition of compactness to show that the space of ultrafilters is compact with respect to the ultra topology.

One of the most important theorems involving compactness is Tychonoff's theorem:

Theorem 1.6.1 (Tychonoff). *Suppose X_i is a non-empty topological space for every $i \in I$, where I is the index set. Then $\prod_{i \in I} X_i$ is compact if and only if every X_i is compact.*

Proof. Please refer [13]. □

The combination of compactness and the Hausdorff property often give strong results, for example in the following lemma.

Lemma 1.6.1. *Suppose X is a compact space and Y is a Hausdorff space. Furthermore, let $f : X \to Y$ be a continuous bijection. Then f is a homeomorphism.*

Proof. Refer [13]. □

Throughout the main text, we also need a few results from topology. The first concerns the separation axiom T_3.

Lemma 1.6.2. *If X is T_3, $U \subseteq X$ is open and $x \in U$, then there is a $V \subseteq X$ open such that $x \in V \subseteq \bar{V} \subseteq U$.*

Proof. Please refer [13]. □

Next, we have a well-known result about extensions of continuous functions.

Proposition 1.6.2. *Suppose X and Y are topological spaces, where Y is Hausdorff. Furthermore, suppose $A \subseteq X$ is dense and $f, g : X \to Y$ are continuous functions that coincide on A. Then $f = g$.*

Proof. Refer [13]. □

Most topological properties are preserved under finite products of topological spaces. However, with infinite products, this is not always the case. However, we do have the following two results, of which the second is the most famous one.

Theorem 1.6.2. *Countable products of metrizable topological spaces are metrizable.*

Proof. For the detailed proof of this theorem please refer [13]. □

1.6.2 Ultrafilters

In this subsection we develop the theory of ultrafilters and in the next subsection we will construct the Stone-Cech compactification of discrete spaces using ultrafilters.

Here the central objects of study are ultrafilters. First, we need the notion of a filter.

Definition 1.6.2 (Filter)**.** *Suppose X is a set. A family $F \subseteq \mathcal{P}(X)$ is called a* **filter** *if it satisfies the following axioms:*

1. $F \neq \emptyset$,
2. $\emptyset \notin F$,
3. *if* $A, B \in F$, *then* $A \cap B \in F$ *and*
4. *if* $A \in F$ *and* $A \subseteq B$, *then* $B \in F$.

An important non-trivial example of a filter is given by the set of neighbourhoods of a point of topological spaces. Note that filters are naturally partially ordered by inclusion. Hence we can consider maximal elements: these are the so-called ultrafilters.

Definition 1.6.3 (Ultrafilter)**.** *Suppose X is a set and $F \subseteq \mathcal{P}(X)$ is a filter. Then F is called an* **ultrafilter** *if the only filter $G \subseteq \mathcal{P}(X)$ that satisfies $F \subseteq G$ is F itself.*

The following lemma assures that for any set, ultrafilters are quite common.

Lemma 1.6.3. *Suppose X is a set and $F \subseteq \mathcal{P}(X)$ is a filter. Then there is an ultrafilter $G \subseteq \mathcal{P}(X)$ such that $F \subseteq G$.*

Proof. Consider $C := \{H \subseteq \mathcal{P}(X) : F \subseteq H, H \ is \ a \ filter\}$. Then C is partially ordered by set inclusion.

It is easy to observe that every chain in C has an upper bound.

By Zorn's lemma, C has a maximal element G.

Then $F \subseteq G$. We claim that G is an ultrafilter. To see this, suppose $G \subseteq K$, and K is a filter. Then $F \subseteq K$, so $K \in C$, so by maximality of G as an element of C, $K = G$.

Therefore, G is indeed an ultrafilter. □

One can describe ultrafilters with a few equivalent properties. To do this, we first define the notion of prime filters.

Definition 1.6.4 (Prime filter)**.** *Suppose X is a set and $F \subseteq \mathcal{P}(X)$ is a filter. F is called* **prime** *if for any $A, B \subseteq X$ such that $A \cap B \in F$, we have $A \in F$ or $B \in F$.*

The following lemma is easily proven with an inductive argument.

Lemma 1.6.4. *Suppose X is a set and $F \subseteq (X)$ is a prime filter and $\{A_i\}_{i=1}^{n} \subseteq \mathcal{P}(X)$ is a finite collection such that $\bigcup_{i=1}^{n} A_i \in F$. Then there is a $i \in \{1, ..., n\}$ such that $A_i \in F$.*

Using this, we can give three new descriptions of ultrafilters.

Lemma 1.6.5. *Suppose X is a set and $F \subseteq \mathcal{P}(X)$ is a filter. Then the following are equivalent:*

1. *F is an ultrafilter.*
2. *If $A \subseteq X$ and $A \subseteq B \neq \emptyset$ for all $B \in F$, then $A \in F$.*
3. *For every $A \subseteq X$ either $A \in F$ or $X \setminus A \in F$.*
4. *F is a prime filter.*

Proof. We first prove the equivalence between property 1 and 2.

For this, suppose that F ia an ultrafilter and $A \subseteq X$ is such that $A \cap B \neq \emptyset$ for all $B \in F$.

Then define

$$F' = F \cup \{C \subseteq X \mid \exists\ B \in F : A \cap B \subseteq C\}.$$

We claim that F' is a filter. First suppose that $Y_1, Y_2 \subseteq F'$. Then there are three cases.

Firstly, suppose $Y_1, Y_2 \in F$. Then $Y_1 \cap Y_2 \in F \subseteq F'$.

Next, suppose $Y_1 \in F, Y_2 \notin F$. Then there is a $B \in F$ such that $A \cap B \subseteq Y_2$. Then $A \cap B \cap Y_1 \subseteq Y_2 \cap Y_1$ and $B \cap Y_1 \subseteq F$, so $Y_1 \cap Y_2 \in F'$.

Lastly, suppose $Y_1 \notin F, Y_2 \notin F$. Then there are $B_1, B_2 \in F$ such that $A \cap B_1 \subseteq Y_1$ and $A \cap B_2 \subseteq Y_2$. Then $A \cap (B_1 \cap B_2) \subseteq Y_1 \cap Y_2$ and $B_1 \cap B_2 \in F$, so $Y_1 \cap Y_2 \in F'$.

Hence $Y_1 \cap Y_2 \in F'$ for all $Y_1, Y_2 \in F'$.

Next, suppose that $Y_1 \subseteq Y_2$ and $Y_1 \in F'$. Then there are two cases. Firstly, $Y_1 \in F$. Then $Y_2 \in F$, since F is a filter. Next, $Y_1 \notin F$. Then there is a $B \in F$ such that $A \cap B \subseteq Y_1 \subseteq Y_2$, whence $Y_2 \in F'$. Since $F \neq \emptyset$, $F' \neq \emptyset$. Furthermore $\emptyset \notin F$ and combining this with our assumption on $A, \emptyset \notin F'$. Hence F' is indeed a filter. By construction, $F \subseteq F'$ and F is an ultrafilter, so $F' = F$.

Now, take any $B \in F$. Then $A \cap B \subseteq A$, so $A \in F' = F$. Therefore, property 1 implies property 2.

For the converse, suppose that $A \subseteq X$ and $A \cap B \neq \emptyset$ for all $B \in F$ imply that $A \in F$.

Then suppose $G \subseteq \mathcal{P}(X)$ is a filter such that $F \subseteq G$. Then let $A \in G$. Then for any $B \in F, A, B \in G$, so $A \cap B \neq \emptyset$. Therefore, $A \in F$, by our assumption. Hence $G \subseteq F$, i.e. $G = F$. Therefore, F is an ultrafilter. Hence property 2 implies property 1.

Next suppose that F has property 2 and let $A \subseteq X$. Suppose $A \notin F$. Then, there is a $B \in F$ such that $A \cap B = \emptyset$, i.e. $B \subseteq X \setminus A$. Since $B \in F$, $X \setminus A \in F$. So, for all $A \subseteq X$, either $A \in F or X \setminus A \in F$. Hence the second property implies the third.

Now, suppose that F has property 3. Then suppose that $A \subseteq X$ is such that $A \cap b \neq \emptyset$ for all $B \in F$. Then $X \setminus A \notin F$, since $A \cap X \setminus A = \emptyset$. Therefore, $A \in F$, i.e. F has the second property.

Now, suppose that F is a prime filter. Let $A \subseteq X$. Then $A \cup (X \setminus A) = X \in F$, so $A \in F$ or $X \setminus A \in F$. Therefore, property 4 implies property 3.

Lastly, suppose F has property 3 and suppose that $A, B \subseteq X$ such that $A \cup B \in F$, but $A \notin F$ and $B \notin F$. Then, $X \setminus A, X \setminus B \in F$. Then also

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \in F,$$

so $\emptyset = (A \cup B) \cap X \setminus (A \cup B) \in F$. This is a contradiction with F being a filter, so whenever $A \cup B \in F$ we must have $A \in F$ or $B \in F$, i.e. F is prime. □

Now that we have four different descriptions of ultrafilters, it is time to introduce a very important class of examples of ultrafilters: those generated by a single element of a set.

Lemma 1.6.6. *Suppose X is a set and $x \in X$. Then $F_x := \{A \subset X : x \in A\}$ is an ultrafilter.*

Proof. First of all, for $A, B \in F_x$, $x \in A \cap B$, so $A \cap B \in F_x$. Next, if $A \in F_x$ and $A \subseteq B$, $x \in A \subseteq B$, so $B \in F_x$. Certainly $x \notin \emptyset$, so $\emptyset \notin F_x$ and $x \in X$, so $X \in F_x$, i.e. $F_x \neq \emptyset$. Hence F_x is a filter.

To see that F_x is in fact an ultrafilter, note that for any $A \subseteq X$ we either have $x \in A$ or $x \in X \setminus A$, i.e. $A \in F_x$ or $X \setminus A \in F_x$. Hence by earl.ier proposition, F_x is an ultrafilter. □

Definition 1.6.5 (Principal and free ultrafilter)**.** *A* **principal** *ultrafilter on a set X is an ultrafilter of the kind F_x for some $x \in X$. A* **free** *ultrafilter is an ultrafilter that is not principal.*

Filters and ultrafilters become especially interesting when they considered for topological spaces. For example, one can define the notion of convergence of a filter. For this, we use that notion $\mathcal{N}_x$ for the set of neighbourhoods of a point X in a topological space.

Definition 1.6.6. *Suppose X is a topological space, $x \in X$ and $F \in \mathcal{P}(X)$ is a filter. We say that F* **converges to** *X if $\mathcal{N}_x \subset F$.*

Like nets, filters in Hausdorff spaces behave nicely with respect to convergence.

Proposition 1.6.3. *Suppose X is a Hausdorff space and $F \subset \mathcal{P}(X)$ is a filter. Then F can converge to at most one point.*

Proof. Suppose that F converges to both $x, y \in X$. Then $\mathcal{N}_x \subseteq F$ and $\mathcal{N}_y \subseteq F$. If $x \neq y$, then by the Hausdorff property, there are open $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Then $U \in \mathcal{N}_x \subseteq F$ and $V \in \mathcal{N}_y \subseteq F$. whence $U, V \in F$ and $\emptyset = U \cap V \in F$.

This contradicts F being a filter, i.e. $x = y$. □

For compact spaces, ultrafilters also have a useful property.

Proposition 1.6.4. *Suppose X is a compact space and $F \subset \mathcal{P}(X)$ an ultrafilter. Then F converges to at least one point.*

Proof. Suppose F converges to no point. Then for all $y \in X$ there is a $N_y \in \mathcal{N}_y$ such that $N_y \notin F$. So, especially, for all $y \in X$, there is a open $U_y \subseteq X$ such that $y \in U_y$ and $U_y \notin F$.

Then clearly, $\bigcup_{y \in Y} U_y = X$, so by compactness of X, there is a finite set $\{y_i\}_{i=1}^n$ such that $\bigcup_{i=1}^n U_{y_i} = X$. However, $X \in F$ and F is prime, so there is a $i \in \{1, \ldots, n\}$ such that $U_{y_i} \in F$. This is a contradiction, since all $U_y \notin F$ by construction.

Hence F converges to at least one point. □

Combining these two propositions, we have an immediate corollary.

Corollary 1.6.1. *Suppose X is a compact Hausdorff space and $F \subset \mathcal{P}(X)$ is an ultrafilter. Then F converges to a unique point.*

The structure of a filter can also be transferred from one set to another by means of a function.

Definition 1.6.7. *Suppose X and Y are sets, $f : X \to Y$ is a function and U is a filter on X. Then the* **pushforward** *of U over f is defined as*

$$f_*(U) := \{Z \subset Y : f^{-1}(Z) \in U\}.$$

The following proposition assures that it is useful to consider pushforwards of filters.

Proposition 1.6.5. *Suppose X and Y are sets, $f : X \to Y$ is a function and U is a filter on X. Then the pushforward $f_*(U)$ of U over f is a filter on Y. In addition, if U is an ultrafilter, then $f_*(U)$ is an ultrafilter, too.*

Proof. Suppose $Z_1, Z_2 \in f_*(U)$. Then $f^{-1}(Z_1), f^{-1}(Z_2) \in U$, so

$$f^{-1}(Z_1 \cap Z_2) = f^{-1}(Z_1) \cap f^{-1}(Z_2) \in U,$$

since U is a filter. Hence $Z_1 \cap Z_2 \in f_*(U)$.

Now, suppose $Z_1 \in f_*(U)$ and $Z_1 \subseteq Z_2$. Then $f^{-1}(Z_2) \supseteq f^{-1}(Z_1) \in U$, so $f^{-1}(Z_2) \in U$, since U is a filter. Hence $Z_2 \in f_*(U)$.

Next, observe that $f^{-1}(Y) = X \in U$, so $Y \in f_*(U)$, i.e. $f_*(U) \neq \emptyset$.

Lastly, $f^{-1}(\emptyset) = \emptyset \notin U$, so $\emptyset \notin f_*(U)$. Hence $f_*(U)$ is a filter on Y.

Now, in addition, suppose that U is an ultrafilter. Then suppose $Z \subseteq Y$. Then $f^{1}(Z) \subseteq X$, so by earlier proposition either $f^{-1}(Z) \in U$ or $f^{-1}(Y \setminus Z) = X \setminus f^{-1}(Z) \in U$, i.e. either $Z \in f_*(U)$ or $Y \setminus Z \in f_*(U)$. Hence $f_*(U)$ is indeed an ultrafilter. □

1.6.3 Stone-Čech compactification of discrete spaces

In this section we construct the Stone-Čech compactification for discrete topological spaces using ultrafilters. Let us first recall the definition of the Stone-Čech compactification.

Definition 1.6.8 (Stone-Čech compactification)**.** *Suppose X is a topological space. The Stone-Čech compactification of X is a compact Hausdorff space βX together with a continuous map $S : X \to \beta X$ having the following universal property: for any compact Hausdorff space K and continuous function $f : X \to K$, there is a unique continuous $\beta f : \beta X \to K$ such that the following diagram commutes:*

We construct the Stone-Čech compactification for discrete spaces. To do this, first write $Ultra(X)$ for the collection of all ultrafilters on a set X. Our goal is to show that for a discrete space X, we can endow $Ultra(X)$ with a topology in such a way that it becomes the Stone-Čech compactification of X. Namely, for some set X and all subsets $A \subset X$, define

$$W(A) = \{U \in Ultra(X) : A \in U\}.$$

Note that $X \in U$ for any $U \in Ultra(X)$, so $W(X) = Ultra(X)$. Therefore, the collection $\{W(A) : A \in X\}$ forms a subbase of a topology on $Ultra(X)$. We call this topology the ultra topology. In order to understand this topology better, we investigate some properties of the subbase elements.

Lemma 1.6.7. *Suppose X is a set and let $A, B \subset X$. Then:*

1. *$W(A) = \emptyset$ if and only if $A = \emptyset$,*

2. *$W(A \cap B) = W(A) \cap W(B)$.*

3. *$W(X \setminus A) = Ultra(X) \setminus W(A)$.*

Proof. First, note that if $A = \emptyset$, then $A \notin U$ for any $U \in$Ultra(X), by definition of a filter. Therefore, $W(A) = \emptyset$. If $A \neq \emptyset$, then there is an $x \in A$, whence $A \in F_x$ and $F_x \in W(A)$, i.e. $W(A) \neq \emptyset$. So $W(A) = \emptyset$ if and only if $A = \emptyset$.

Next, suppose $U \in W(A \cap B)$. Then $A \cap B \in U$, so $A \in U$ and $B \in U$, since $A \cap B \subseteq A$ and $A \cap B \subseteq A$. Therefore, $U \in W(A)$ and $U \in W(B)$, i.e. $U \in$

$W(A) \cap W(B)$. Next, let $V \in W(A) \cap W(B)$. Then $A \in V$ and $B \in V$, so $A \cap B \in V$, so $V \in W(A \cap B)$.

Hence $W(A \cap B) = W(A) \cap W(B)$.

Lastly, suppose that $U \in W(X \setminus A)$. Then $X \setminus A \in U$, so by earlier proposition $A \notin U$, so $U \in Ultra(X) \setminus W(A)$. Conversely, if $U \in Ultra(X) \setminus W(A)$, then $A \notin U$, so $X \setminus A \in U$, by characterization of ultrafilter, so $U \in W(X \setminus A)$. So indeed, $W(X \setminus A) = Ultra(X) \setminus W(A)$. □

Using this lemma, we can describe the ultra topology through a base.

Corollary 1.6.2. *Suppose X is a set. Then $\{W(A) : A \subseteq X\}$ forms a base for the ultra topology on Ultra(X).*

Proof. By definition of the ultra topology, $\{W(A) : A \subseteq X\}$ forms a subbase for the ultra topology on Ultra topology. Hence

$$\{\bigcap_{i=1}^{n} W(A_i) : n \in \mathbb{N}, \{A_i\}_{1=1}^{n} \subseteq \mathcal{P}(X)\}$$

is a base for the Ultra topology. However, by using above lemma $n-1$ times, we see that

$$\bigcap_{i=1}^{n} W(A_i) = W(\bigcap_{i=1}^{n}),$$

which is a subbse element itself. Hence $\{W(A) : A \subseteq X\}$ is indeed a base for the ultra topology. □

Using lemma 1.14, we see that the base $\{W(A) : A \subseteq X\}$ of the ultra topology consists of elements that are both open and closed. From now on, for a set X, we will simply refer to the topological space Ultra(X), and imply that we are considering the ultra topology. We now come to an important property of the space of ultrafilters.

Proposition 1.6.6. *Suppose X is a set. Then Ultra(X) is Hausdorff.*

Proof. Suppose $U \neq V \in$Ultra(X). Then either $U \setminus V \neq \emptyset$ or $V \setminus U \neq \emptyset$. Without loss of generality, assume that $U \setminus V \neq \emptyset$ and let $A \subseteq X$ be such that $A \in U$ and $A \notin V$. Then, by characterization of ultrafilters, $X \setminus A \notin U$ and $X \setminus A \in V$, so $U \in W(A)$ and $V \in W(X \setminus A)$. Since $W(A)$ and $W(X \setminus A)$ are both open and

$$W(A) \cap W(X \setminus A) = W(A \cap X \setminus A) = W(\emptyset) = \emptyset.$$

Hence Ultra(X) is indeed Hausdorff. □

Lemma 1.6.8. *Suppose X is a set and suppose that $\{A_i\}_{i\in I} \subseteq \mathcal{P}(X)$ is a subset such that $\{W(A_i)\}_{i\in I}$ has the finite intersection property. Then*

$$\bigcap_{i\in I} W(A_i) \neq \emptyset.$$

Proof. Suppose that $\{i_k\}_{k=1}^n \subseteq I$. Then

$$W(\bigcap_{k=1}^{n} A_{i_k}) = \bigcap_{k=1}^{n} W(A_{i_k}) \neq \emptyset,$$

since $\{W(A_i)\}_{i\in I}$ has the finite intersection property. Then, $\bigcap_{k=1}^{n} A_{i_k} \neq \emptyset$. Now, define

$$F := \{B \subseteq X : \exists \{i_k\}_{k=1}^n \subseteq I \;\; s.t. \; \bigcap_{k=1}^{n} A_{i_k} \subseteq B\}.$$

Clearly, F is a filter. Hence, there is an ultrafilter $U \subseteq \mathcal{P}(X)$ such that $F \subseteq U$. Now, for all $i \in I$, $A_i \in F \subseteq U$, so $U \in W(A_i)$ for all $i \in I$. Therefore, $U \in \bigcap_{i\in I} W(A_i)$, so $\bigcap_{i\in I} W(A_i) \neq \emptyset$. □

Now we can use this to prove that Ultra(X) is compact for any set X.

Proposition 1.6.7. *Suppose X is a set. Then Ultra(X) is compact.*

Proof. Suppose that $\{C_i\}_{i\in I} \subseteq \mathcal{P}(Ultra(X))$ is a family of closed subsets that has the finite intersection property. We will prove that $\bigcap_{i\in I} C_i \neq \emptyset$ and thereby conclude that Ultra(X) is compact.

Since $\{W(A) : A \subseteq X\}$ is a base for Ultra(X) consisting of elements that are both open and closed, there is a set $\{A_j\}_{j\in J_i} \subseteq \mathcal{P}(X)$ for every $i \in I$ such that $C_i = \bigcap_{j\in J_i} W(A_j)$.

Now define $J = \bigcup_{i\in I} J_i$ and suppose that $\{j_k\}_{k=1}^n$ is a finite subset. Then for every

$k \in \{1, \ldots, n\}$, there is a $i_k \in I$ such that $j_k \in J_{i_k}$. Hence

$$\emptyset \neq \bigcap_{k=1}^{n} C_{i_k} = \bigcap_{k=1}^{n} \bigcap_{j \in J_{i_k}} W(A_j) \subseteq \bigcap_{k=1}^{n} W(A_{j_k}),$$

where we used the fact that $\{C_i\}_{i \in I}$ has the finite intersection property.

Therefore, $\{W(A_j)\}_{j \in J}$ has the finite intersection property, whence $\bigcap_{j \in J} W(A_j) \neq \emptyset$. Therefore,

$$\bigcap_{i \in I} C_i = \bigcap_{i \in I} \bigcap_{j \in J_i} W(A_j) = \bigcap_{j \in J} W(A_j) \neq \emptyset,$$

so Ultra(X) is compact. □

So we sec that for any set X, Ultra(X) is a compact Hausdorff space. Furthermore, there is a canonical map $S : X \to$ Ultra(X) defined by $S(x) = F_x$, where F_x is the principal ultrafilter generated by $x \in X$.

We will prove that Ultra(X) together with the map S gives the Stone-Čech compactification for discrete spaces, But we first need a few more things.

First of all, note that for any two sets X and Y and any function $f : X \to Y$, the pushforward operator gives a well-defined map $f_* :$ Ultra(X) $\to$ Ultra(Y).

Lemma 1.6.9. *Suppose X and Y are sets, and $f : X \to Y$ a function. Then the function f_* : Ultra(X) $\to$ Ultra(Y) is continuous.*

Proof. Suppose $A \subseteq Y$. Then :

$$\begin{aligned} f_*^{-1}(W(A)) &= \{U \in Ultra(X) : f_*(U) \in W(A)\} \\ &= \{U \in Ultra(X) : A \in f_*(U)\} \\ &= \{U \in Ultra(X) : f^{-1}(A) \in U\} \\ &= W(f^{-1}(A)). \end{aligned}$$

Therefore, the pre-image under f_* of any base element of the topology on Ultra(Y) is open in Ultra(X), so f_* is continuous. □

Next, we know that for a compact Hausdorff space K, there is a unique well-defined map ϕ_k; Ultra$(K) \to K$ such that for every $U \in$ Ultra(K), U converges to $\phi_k(U)$.

Lemma 1.6.10. *Suppose that K is a compact Hausdorff space and let ϕ_k : Ultra(K) $\to K$ be the map such that for every $U \in$ Ultra(K), U converges to $\phi_k(U)$. Then ϕ_k is continuous.*

Proof. Suppose $A \subseteq K$ is open and suppose that $U \in \phi_k^{-1}(A)$. Then U converges to $\phi_k(U) \in A$. Since A is open,there is an open $B \subseteq K$ such that

$$\phi_k(U) \in B \subseteq \overline{B} \subseteq A.$$

Then $B \in \mathcal{N}_{\phi_k(U)} \subseteq U$, so $U \in W(B)$. Now let $V \in W(B)$. Then $B \in V$ and $\mathcal{N}_{\phi_k(V)} \subseteq V$. We claim that $\phi_k(V)\overline{B}$.

If $\phi_k(V) \notin \overline{B}$, then $\phi_k(V) \in K \setminus \overline{B}$, and $K \setminus \overline{B}$ is open, so $K \setminus \overline{B} \in \mathcal{N}_{\phi_k(V)} \subseteq V$. Also, $K \setminus \overline{B} \subseteq K \setminus B$, so $K \setminus B \in V$. Then both $B \in V$ and $K \setminus B \in V$, which is a contradiction, since V is an ultrafilter.

Therefore, $\phi_k(V) \in \overline{B} \subseteq A$. Hence $V \in \phi_K^{-1}(A)$, so $W(B) \subseteq \phi_K^{-1}(A)$. Furthermore, $W(B)$ is open in Ultra(K), and $U \in \phi_K^{-1}(A)$ was arbitrary, so $\phi_K^{-1}(A)$ is open, i.e. ϕ_k is continuous. □

For the universal property of the Stone-Čech compactification we need a unique continuous extension of a continuous map. The following proposition is helpful for this.

Proposition 1.6.8. *Suppose X is a set. Then the image of the map $S : X \to$ Ultra(X) defined by $S(x) = F_x$, is dense in Ultra(X).*

Proof. Suppose $W(A)$ is any non-empty base element for the ultra topology Ultra(X). Then $A \neq \emptyset$, so there is a $x \in A$. Then $A \in F_x$, so $F_x \in W(A)$. Therefore, we have $W(A) \cap S(X) \neq \emptyset$, and $W(A)$ was an arbitrary non-empty base element of Ulta(X), so $S(X)$ is dense in Ultra(X). □

Now we come to the main point.

Theorem 1.6.3. *Suppose X is a discrete topological space and $S : X \to$ Ultra(X) is the function such that $S(x) = F_x$, the principal ultrafilter belonging to $x \in X$. Furthermore, let K be a compact Hausdorff space and suppose $f : X \to K$ is a function. Then the unique continuous function βf : Ultra(X) $\to K$ such that $\beta f \circ S = f$ is given by $\beta f = \phi_k \circ f_*$.*

Proof. Let $x \in X$, and suppose that $A \in \mathcal{N}_{f(x)}$. Then $f(x) \in A$, so $x \in f^{-1}(A)$, so $f^{-1}(A) \in F_x$, so $A \in f_*(F_x)$. Therefore, $N_{f(x)} \subseteq f_*(F_x)$. By uniqueness of the map ϕ_k, we conclude that $f(x) = \phi_K(f_*(F_x)) = (\phi_K \circ f_* \circ S)(x)$. Since $x \in X$ was arbitrary, $f = \phi_K \circ f_* \circ S = \beta f \circ S$. Furthermore, $\beta f = \phi_K \circ f_*$ is continuous, since both ϕ_K and f_* are continuous.

Now suppose that $g : \text{Ultra}(X) \to K$ is another continuous map that satisfies $f = g \circ S$. Then g coincides with βf on $S(X)$ and $S(X)$ is dense in Ultra(X), so $g = \beta f$, as desired. □

The above theorem gives the universal property of the Stone-Čech compactification, whence we have the following corollary.

Corollary 1.6.3. *Suppose X is a discrete topological space. Then the space Ultra(X) together with the map $S : X \to$ Ultra(X) defined by $S(x) = F_x$, is the Stone-Čech compactification of X.*

Proof. Since X is discrete, the map S is continuous. Since theorem 1.6.3 gives the universal properrty, Ultra(X) together with the map $S : X \to \text{Ultra}(X)$ is the Stone-Čech compactification of X. □

Chapter 2

KS Problem In Finite Dimension

2.1 Basic definitions

Let $\mathcal{B}(\ell^2(\mathbb{N}))$ be the set of all bounded linear operators on $\ell^2(\mathbb{N})$ and $\mathbb{D} \subseteq \mathcal{B}(\ell^2(\mathbb{N}))$ be the algebra of diagonal operators.

Definition 2.1.1 (Positive Operator)**.** *Let V be an inner product space. A linear operator $T \in L(V)$ is called* **positive** *if*

1. *T is self-adjoint ($T = T^*$) and*

2. *$\langle Tv, v\rangle \geqslant 0$ for all $v \in V$.*

A positive operator $T \in L(V)$ is denoted by $T \geq 0$.

Definition 2.1.2 (State)**.** *A linear function $s : B(\ell^2(\mathbb{N})) \to \mathbb{C}$ is a* **state** *if*

1. *s(I)=1, and*

2. *$A \geq 0 \implies s(A) \geq 0$.*

Later, we will see that states are automatically continuous.

Definition 2.1.3 (Pure state)**.** *A state $s : B(\ell^2(\mathbb{N})) \to \mathbb{C}$ is pure if it is not a convex combination of any other states.*

Problem 2.1.1 (The Kadison-Singer (KS))**.** *Does every pure state on the algebra of bounded diagonal operators on $\ell^2(\mathbb{N})$ have a unique extension to a state (pure state) on the algebra of all bounded operators on $\ell^2(\mathbb{N})$?*

2.2 Basic cases

2.2.1 KS problem in one dimension

Let $|N| = 1$. Here $\ell^2(N) \simeq \mathbb{C}$ (the set of all complex numbers). In this case KS problem has a positive answer trivially as $B(\ell^2(N)) = \mathbb{D}(\ell^2(N))$.

2.2.2 KS problem in two dimension

To understand the definition of state and pure state, let us begin by considering a two-dimensional analog of KS problem. That is when $|N| = 2$. In this case, the algebra of "bounded diagonal operators on $\ell^2(N)$" becomes the algebra of diagonal 2×2 matrices over $\mathbb{C}$. That is,

$$\mathbb{D}(\ell^2(N)) \simeq D_{2\times 2}(\mathbb{C}) := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathbb{C} \right\}.$$

Any linear functional on $\mathbb{D}(\ell^2(N))$ is a linear map $f : D_{2\times 2}(\mathbb{C}) \to \mathbb{C}$ in the form

$$f(M) = f\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = \alpha a + \beta d, \qquad \alpha, \beta \in \mathbb{C}.$$

A state is a linear functional f such that

1. $f(I) = 1$. So we must have $\beta = 1 - \alpha$.

2. $f(M)$ must be real and non-negative whenever M is positive semi definite(i.e., whenever a, d are both real and non-negative). So we must have α real and $\alpha \in [0, 1]$

With the above observations any state is of the form

$$f\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = \alpha a + (1 - \alpha)d, \qquad \alpha \in [0, 1].$$

Note that, in particular f is a state, when $\alpha = 0$ or $\alpha = 1$.

A pure state is a state f such that it cannot be written as a non-trivial convex combination of two different states. So, for f to be a pure state, we must have either

$\alpha = 0$ or $\alpha = 1$. Hence the only pure states on $D_{2\times 2}(\mathbb{C})$ are

$$f\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = a \text{ and } f\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = d.$$

Also, in this two dimensional case, the algebra of "bounded operators on $\ell^2(N)$" simply becomes the algebra of 2×2 matrices over $\mathbb{C}$. That is

$$B(\ell^2(N)) \simeq M_{2\times 2}(\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C} \right\}.$$

A linear functional on $M_{2\times 2}(\mathbb{C})$ is a function

$$g(M) = g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \alpha a + \beta b + \gamma c + \delta d, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C}.$$

Letting

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

we can observe $g(M) = tr(GM)$.

A state on $M_{2\times 2}(\mathbb{C})$ is a linear functional g satisfying :

1. $g(I) = 1$,
2. $g(M)$ must be real and non-negative whenever M is Hermitian and positive semi-definite.

The first condition is equivalent to

$$tr(G) = 1.$$

The second condition implies that

$$0 \leq g(vv^*) = tr(Gvv^*) = v^*Gv,$$

for all $v \in \mathbb{C}^2$. This implies G is Hermitian and positive semi-definite.

Conversely, for any complex matrix G that is positive semi-definite with $tr(G) = 1$, the function $g(M) := tr(GM)$ is a state on $M_{2\times 2}(\mathbb{C})$.

A state g on $M_{2\times 2}(\mathbb{C})$ is an extension of a state f on $D_{2\times 2}(\mathbb{C})$ if

$$g(M) = f(M) \text{ for all } M \in D_{2\times 2}(\mathbb{C}).$$

Since positive semi-definite diagonal matrices have non-negative diagonals, every state on $D_{2\times 2}(\mathbb{C})$ has a canonical extension to $M_{2\times 2}(\mathbb{C})$ as a state g obtained by

$$g(M) = g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right).$$

It remains to explore an answer to the question: *When is this extension unique?*

Consider the state $f(M) = \frac{a+d}{2}$ on $D_{2\times 2}(\mathbb{C})$. This is not a pure state. Consider the linear functional

$$h(M) = \frac{a+b+c+d}{2} = tr(GM).$$

Note that here $G = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. This is a state on $M_{2\times 2}(\mathbb{C})$, since G is a positive semi-definite matrix with $tr(G) = 1$. Clearly h is an extension of f. So the canonical extension g and h are distinct states on $M_{2\times 2}(\mathbb{C})$ that are extensions of f. This is not a counter example to the two dimensional KS problem, as f is not a pure state.

So, let us consider the pure state $f(M) = a$ on $D_{2\times 2}(\mathbb{C})$. Consider any linear functional $g(M) = tr(GM)$ that is a state on $M_{2\times 2}(\mathbb{C})$ and is an extension of f. Then we must have

$$G = \begin{pmatrix} 1 & \beta \\ \overline{\beta} & 0 \end{pmatrix}.$$

for some $\beta \in \mathbb{C}$ and G positive semi-definite. As the diagonal entries of G are non-negative, G is positive semi-definite iff $detG \geq 0$. As $detG = \beta\overline{\beta}$, that is non-negative only when $\beta = 0$. Thus $g(M) = a$ is the unique state on $M_{2\times 2}(\mathbb{C})$ that is an extension of $f(M) = a$.

Similarly, $f(M) = d$ has a unique extension to a state on $M_{2\times 2}(\mathbb{C})$.

So we may conclude that the two-dimensional analog of the KS problem is true.

Theorem 2.2.1. *Let $A \in B(H)$, where H is the n-dimensional Hilbert space $\mathbb{C}^n$. A is positive iff A is Hermitian and all its eigen values are positive.*

Proof. By theorem 1.1.1. □

Using this result, we replace positiveness of the matrix M by Hermitian and semi-definiteness in the two dimensional case of KS problem.

2.2.3 KS problem in any finite dimension

The Kadison-Singer conjecture is about infinite-dimentional Hilbert spaces H, but the underlying situation is already interesting in finite dimension.Hence we start with the Hilbert space

$$H = \mathbb{C}^n,$$

with standard inner product

$$\langle w, z \rangle = \sum_{i=1}^{n} \overline{w_i} z_i,$$

which we evidently take to be linear in the second entry. For the moment we identify operators with matrices.

Let $M_n(\mathbb{C})$ be the complex $n \times n$ matrices, regarded as an algebra (which we always assume to be complex and associative) with involution, namely the operation $a \mapsto a^*$ of Hermitian conjugation. Abstractly, an involution on an algebra A is an anti-linear anti-homomorphism $^* : A \to A$, so if we write $*(a) = a^*$, then for all $a, b \in A$ and $\lambda \in \mathbb{C}$ we have

$$(\lambda a + b)^* = \overline{\lambda} a^* + b^*;$$

$$(ab)^* = b^* a^*.$$

Here note that $M_n(\mathbb{C})$ has a unit, and the unit is I_n. An algebra with involution (and unit) is called a (unital) *-algebra. Beside $M_n(\mathbb{C})$, another unital *-algebra of interest to us is $D_n(\mathbb{C})$, i.e., the subalgebra of $M_n(\mathbb{C})$ consisting of all diagonal matrices, with the involution * inherited from $M_n(\mathbb{C})$.

In connection with the Kadison-Singer conjecture, the following concept is crucial.

A state on a unital *-algebra A (with unit I_A) is a linear map $\omega : A \to \mathbb{C}$ that satisfies

1. $\omega(I_A) = 1$;

2. $\omega(a^* a) \geq 0$, for all $a \in A$.

Note: 2.2.1. *This definition of state is equivalent to the earlier definition of state by the characterization of positive operators (theorem 1.1.1).*

Inspired by quantum mechanics, this concept was introduced by John von Neumann, albeit in the special case where A is the unital *-algebra $B(H)$ of all bounded operators on some Hilbert space H. The general notion of a state in the above sense is due to Gelfand and Naimark and Segal.

The states on A form a convex set $S(A)$, whose extremal points are called **pure states**. That is, ω is pure iff any decomposition

$$\omega = t\omega_1 + (1-t)\omega_2$$

for $\omega_1, \omega_2 \in S(A)$ and $t \in (0,1)$ is necessarily trivial, in that $\omega_1 = \omega_2 = \omega$. States that are not pure are mixed.

Before going further let us introduce some notations: We will denote $M_n(\mathbb{C})$ by M. We often denote an element $a \in M$ by

$$a = \sum_{i,j} a_{ij} |e_i\rangle\langle e_j|,$$

where $\{e_i\}$ is the standard basis of $\mathbb{C}^n$ and we use the shorthand notation $|x\rangle\langle y|$ for the operator which satisfies $|x\rangle\langle y|(z) = \langle y, z\rangle x$. This means that a_{ij} is the element in the i-th row and j-th column of the matrix a. Furthermore, we consider the diagonal matrices

$$D := \{a \in M : a_{ij} = 0, \quad if \quad i \neq j\},$$

which form a unital subalgebra of M. The algebra M also has a *-operation that is an involution, defined by:

$$a^* = \sum_{i,j} \overline{a_{ji}} |e_i\rangle\langle e_j|,$$

We call a^* the adjoint of a. Note that D is also closed under this operation.

von Neumann also defined a density matrix as an Hermitian matrix $\rho \in M_n(\mathbb{C})$ whose eigenvalues λ_i $(i = 1, \dots, n)$ are non-negative and sum to unity, or equivalently, as a positive (semi-definite) matrix (in that $\langle \psi, \rho\psi\rangle \geq 0$ for each $\psi \in \mathbb{C}^n$) with unit trace.

Definition 2.2.1 (Density Operator)**.** *A density operator $\rho \in M_n(\mathbb{C})$ is a positive op-*

erator that satisfies $Tr(\rho) = 1$. *We write* $\mathcal{D}(M_n(\mathbb{C}))$ *for the set of all density operators in* $M_n(\mathbb{C})$.

Theorem 2.2.2. *There is a bijective correspondence between states* f *on* $M_n(\mathbb{C})$ *and density operators* $\rho \in M_n(\mathbb{C})$. *Given by*

$$f(a) = Tr(\rho a),$$

for all $a \in M_n(\mathbb{C})$.

Proof. For sake of simplicity here we are denoting $M_n(\mathbb{C})$ by M.

We prove that $S(M) \cong \mathcal{D}(M)$ as sets.

We construct $\phi : S(M) \to \mathcal{D}(M)$ via

$$\phi(f) = \sum_{i,j} \rho_{ij} |e_i\rangle\langle e_j|,$$

where $\rho_{ij} = f(|e_i\rangle\langle e_j|)$.

To see that ϕ is well defined, note that

$$\begin{aligned} Tr(\phi(f)) &= \sum_i f(|e_i\rangle\langle e_i|) \\ &= f(\sum_{i,j} |e_i\rangle\langle e_i|) \\ &= f(I) \\ &= 1 \end{aligned}$$

and for $x \in \mathbb{C}^n$, say $x = \sum_i c_i e_i$,

$$\begin{aligned} \langle x, \phi(f)x\rangle &= \sum_{i,j} \bar{c}_i c_j \langle e_i, \phi(f) e_j\rangle \\ &= \sum_{i,j} \bar{c}_i c_j f(|e_j\rangle\langle e_i|) \\ &= f(|x\rangle\langle x|) \\ &\geq 0, \end{aligned}$$

which means $\phi(f)$ indeed a density operator.

Next, define $\psi : \mathcal{D}(M) \to S(M)$ by

$$\psi(\rho)(a) = Tr(\rho a)$$

for all $a \in M$.

To see that ψ is well defined, first note that $\psi(\rho)(I) = Tr(\rho) = 1$. Next, let $\rho \in \mathcal{D}(M)$ and $a \in M$ positive. Then ρ has a spectral decomposition

$$\rho = \sum_i p_i |v_i\rangle\langle v_i|,$$

for some orthonormal basis (v_i), where all $p_i \geq 0$. Since a is positive

$$a = \sum_{i,j} \lambda_{ij} |v_i\rangle\langle v_i|$$

with all $\lambda_{ii} \geq 0$. Then

$$\rho a = \sum_{i,j} p_i \lambda_{ij} |v_i\rangle\langle v_i|,$$

so

$$\psi(\rho)(a) = Tr(\rho a) = \sum_i p_i \lambda_{ij} \geq 0,$$

so $\psi(\rho)$ is positive, and hence a state. Now, note that

$$\begin{aligned}
\psi(\phi(f)(a) &= Tr(\phi(f)a) \\
&= Tr((\sum_{i,j} \rho_{ij} |e_j\rangle\langle e_j|)(\sum_{l,k} a_{lk} |e_l\rangle\langle e_k|) \\
&= \sum_{i,j} \rho_{ij} a_{ji} \\
&= \sum_{i,j} a_{ji} f(|e_J\rangle\langle e_i|) \\
&= f(\sum_{i,j} a_{ji} |e_J\rangle\langle e_i|) \\
&= f(a).
\end{aligned}$$

Therefore, $\psi \circ \phi = Id$.

$$\begin{aligned} \phi(\psi(\rho))_{ij} &= \psi(\rho)(|e_j\rangle\langle e_i|) \\ &= Tr(\rho|e_j\rangle\langle e_i|) \\ &= \langle e_i, \rho e_j\rangle \\ &= \rho_{ij} \end{aligned}$$

Therefore $\phi \circ \psi = Id$.

Hence $\mathcal{D}(M) \cong S(M)$ □

Note that $S(M_n(\mathbb{C}))$ and $\mathcal{D}(M_n(\mathbb{C}))$ have more structure than that of a set, since they are also convex.

Definition 2.2.2 (Affine). *A function $f : A \to B$ between two convex sets is called* **affine** *if it preserves the convex structure, i.e., if*

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y)$$

for all $t \in [0,1]$ and $x, y \in A$.

Here the bijection in the above theorem is an affine function.

For a convex set C, a point $a \in C$ is called extreme point if for any $a_1, a_2 \in C$ and $t \in (0,1)$ such that $a = ta_1 + (1-t)a_2$ we have $a_1 = a_2 = a$. The set of all extreme points of C is called the extreme boundary of C, often denoted by $\partial_e C$.

Lemma 2.2.1. *Suppose C and D are convex sets and that there is a affine isomorphism between them.Then $\partial_e C$ is isomorphic to $\partial_e D$.*

Proof. Suppose that the map $\phi : C \to D$ is an affine isomorphism. First of all, we claim that $\phi(\partial_e C) \subset \partial_e D$.

To see this, first note that ϕ^{-1} is an affine isomorphism as well. Now suppose $x \in \partial_e C$ and $t \in [0,1]$. $a, b \in D$ such that $\phi(x) = ta + (1-t)b$. Then

$$\begin{aligned} x &= \phi^{-1}(ta + (1-t)b) \\ &= t\phi^{-1}(a) + (1-t)\phi^{-1}(b) \end{aligned}$$

Then, since $x \in \partial_e C$, $x = \phi^{-1}(a) = \phi^{-1}(b)$, but then also $\phi(x) = a = b$, so $\phi(x) \in \partial_e D$.

Hence $\phi(\partial_e C) \subset \partial_e D$, so by the same token $\phi^{-1}(\partial_e D) \subset \partial_e C$, whence ϕ maps $\partial_e C$ bijectively to $\partial_e D$. Therefore $\partial_e C$ and $\partial_e D$ are isomorphic. □

We can now give an explicit description of the pure states on M

Corollary 2.2.1. *There is a bijective correspondence between pure states f on M and one-dimensional projections $|\psi\rangle\langle\psi|$, such that $f(a) = |\psi\rangle\langle a\psi|$, for all $a \in M$, where ψ is a unit vector.*

Proof. By theorem 2.2.2, we know that $S(M)$ corresponds bijectively to $\mathcal{D}(M)$ via the formula $f(a) = Tr(\rho a)$. Since this map is affine and the pure states on M are exactly $\partial S(M)$, we only need to determine $\partial\mathcal{D}(M)$, by lemma 2.2.1.

Suppose that $\rho \in \partial\mathcal{D}(M)$ and determine its spectral decomposition $\rho = \sum_i p_i|v_i\rangle\langle v_i|$, where $\{v_i\}$ are orthonormal. Then since ρ is positive and has unit trace, we have $p_i \geq 0$ and $\sum_i p_i = 1$. So, clearly all $p_i \in [0, 1]$.

Now suppose that there is a $j \in \{1, \ldots, n\}$ such that $p_j \in (0, 1)$. Then there must be a $k \neq j$ such that $p_k \in (0, 1)$ as well. Then there is a $\epsilon > 0$ such that $[p_j - \epsilon, p_k + \epsilon] \subset [0, 1]$ and $[p_k - \epsilon, p_k + \epsilon] \subset [0, 1]$. Now define

$$\begin{aligned} r_i &= p_i - \epsilon, \;\; if \;\; i = j \\ &= p_i + \epsilon, \;\; if \;\; i = k \\ &= p_i, \;\; if \;\; i \notin \{j, k\} \end{aligned}$$

and

$$\begin{aligned} q_i &= p_i + \epsilon, \;\; if \;\; i = j \\ &= p_i - \epsilon, \;\; if \;\; i = k \\ &= p_i, \;\; if \;\; i \notin \{j, k\} \end{aligned}$$

By construction,

$$\rho_1 := \sum_i r_i|v_i\rangle\langle v_i|$$

and

$$\rho_2 := \sum_i q_i|v_i\rangle\langle v_i|$$

are density operators too, and $\rho = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2$. However, $\rho_1 \neq \rho \neq \rho_2$, so ρ is not an extreme point of $\partial\mathcal{D}(M)$.

This is a contradiction, since $\rho \in \partial\mathcal{D}(M)$ by assumption. Therefore, all $p_i \in \{0, 1\}$. Combined with $\sum_i p_i = 1$, this gives a unique j such that $p_j = 1$ and $p_k = 0$ for all $k \neq j$. But then, $\rho = |v_j\rangle\langle v_j|$, so we see that every extreme point of $\mathcal{D}(M)$ is indeed a one-dimensional projection.

It is clear that every one-dimensional projection is positive and has unit trace, so every one-dimensional projections clearly a density operator. Now take a one-dimensional projection $\rho = |\psi\rangle\langle\psi|$, i.e. a unit vector ψ. Suppose that there are $\rho_1, \rho_2 \in \mathcal{D}(M)$ and a $t \in (0, 1)$ such that $\rho = t\rho_1 + (1 - t)\rho_2$.

Clearly, we have $\langle\psi, \rho\psi\rangle = 1$. If we write ρ_1 in its spectral decomposition $\rho_1 = \sum_i p_i|v_i\rangle\langle v_i|$, where $\{v_i\}$ are orthonormal, $p_i \geq 0$ and $\sum_i p_i = 1$, we see that

$$\begin{aligned}
\langle\psi, \rho_1\psi\rangle &= \sum_i p_i|\langle\psi, v_i\rangle|^2 \\
&\leq \sum_i p_i \\
&= 1,
\end{aligned}$$

by the Cauchy-Schwarz inequality.

By the same argument $\langle\psi, \rho_2\psi\rangle \leq 1$. Therefore,

$$\begin{aligned}
1 &= \langle\psi, \rho\psi\rangle \\
&= t\langle\psi, \rho_1\psi\rangle + (1 - t)\langle\psi, \rho_2\psi\rangle \\
&\leq t + (1 - t) \\
&= 1.
\end{aligned}$$

Therefore, we must have $\langle\psi, \rho_1\psi\rangle = 1$, so for all j such that $p_j \neq 0$, we have $|\langle\psi, v_j\rangle|^2 = 1$. Since ψ is a unit vector and $\{v_i\}$ is an orthonormal set, this means that there is a unique j such that $p_j \neq 0$ and $\psi = zv_j$ with $z \in \mathbb{C}$ with $|z| = 1$.

But then necessarily $p_j = 1$ and $\rho_1 = |v_j\rangle\langle v_j| = |\psi\rangle\langle\psi| = \rho$. Similarly, $\rho_2 = \rho$, so indeed, ρ is an extreme point.

So $\partial(M)$ consists exactly of the one-dimensional projections. Now, under the correspondence of states and density operators, we have

$$f(a) = Tr(|\psi\rangle\langle\psi|a) = \langle\psi, a\psi\rangle,$$

by evaluating the trace using an orthonormal basis with ψ as one of the basis vectors. □

In the same fashion we can also define (pure) states on D and derive their specific forms.Note that for $a \in D$ the notion of positivity when considering it as an element of M, i.e. $\langle x, ax\rangle \geq 0$ for all $x \in \mathbb{C}^n$, is equivalent to saying that all $a_i i \geq 0$.

Definition 2.2.3. *A state on D is a linear function $f : D \to \mathbb{C}$ that is positive and unital, meaning that $f(a) \geq 0$ for all $a \geq 0$ and $f(I) = 1$. The set of all states on D is denoted by $S(D)$ and is called the state space of D.*

In our discussion about the specific form of states on D, we need the notion of a probability distribution on finite sets.

Definition 2.2.4. *A* **probability distribution** *on a finite set X is a map $p : X \to [0, \infty)$ such that $\sum_x p(x = 1.)$ The set of all probability distributions on X is denoted by $Pr(X)$.*

Theorem 2.2.3. *There is a bijective correspondence between states f on D and probability distributions p on $\{1, \ldots, n\}$ such that $f(a) = \sum_i p(i)a_{ii}$ for all $a \in D$.*

Proof. We want to show that $S(D) \cong Pr(\{1, \ldots, n\})$ as sets. Define $\phi : S(D) \to Pr(\{1, \ldots, n\})$ by

$$\phi(f)(k) = f(|e_k\rangle\langle e_k|),$$

for all k. Then since f is a state, each $\phi(f)(k)$ is positive. Furthermore,

$$\begin{aligned}
\sum_i \phi(f)(i) &= \sum_i f(|e_i\rangle\langle e_i|) \\
&= f(\sum_i |e_i\rangle\langle e_i|) \\
&= f(I) \\
&= 1,
\end{aligned}$$

so $\phi(f)$ is indeed a probability distribution.

Next, define $\psi : Pr(\{1, \dots, n\}) \to S(D)$ by

$$\phi(p)(a) = \sum_i p(i) a_{ii}.$$

Since all $p(i)$ are positive, it is clear that $\psi(p)$ is positive too. Furthermore,

$$\psi(p)(I) = \sum_i p(i) = 1,$$

so $\psi(p)$ is indeed a state. Now note that

$$\begin{aligned}
\psi(\phi(f))(a) &= \sum_i \psi(f)(i) a_{ii} \\
&= \sum_i a_{ii} f(|e_i\rangle\langle e_i|) \\
&= f(\sum_i a_{ii} |e_i\rangle\langle e_i|) \\
&= f(a),
\end{aligned}$$

showing that $\psi \circ \phi = Id$.

Furthermore,

$$\begin{aligned}
\phi(\psi(p))(k) &= \psi(p)(|e_k\rangle\langle e_k|) \\
&= \sum_i p(i) |e_k\rangle\langle e_k|_{ii} \\
&= p(k),
\end{aligned}$$

whence $\phi \circ \psi = Id$.

So, indeed, $S(D) \cong Pr(1, \dots, n)$ as sets and writing $p = \phi(f)$, the given formula $f(a) = \sum_i p(i) a_{ii}$ holds for every $a \in D$.

Next, we note that just like in the case of M, the state space $S(D)$ is in fact a convex set, just like $Pr(\{1, \dots, n\})$. Hence we can again determine the boundary of $S(D)$ and call it the pure state space of D. Once again, these pure states have specific form.

□

Corollary 2.2.2. *For every pure state f on D there is an $i \in \{1, \dots, n\}$ such that*

$f(a) = a_{ii}$ *for all* $a \in D$.

Proof. By theorem 2.2.3, we know that the states on D corresponds to $Pr(\{1, \ldots, n\})$, and by lemma 2.2.1 we then know that we only need to determine the boundary of $Pr(\{1, \ldots, n\})$. If we show that these are exactly those probability distributions that have unique j such that $p(j) = 1$ and $p(k) = 0$ for all $k \neq j$, we are done.

So, suppose that $p \in \partial Pr(\{1, \ldots, n\})$. By definition of a probability distribution, we have $p(j) \in [0, 1]$ for all j. Suppose that $p(j) \in (0, 1)$ for some j. Then there must be a $k \neq j$ such that $p(k) \in (0, 1)$ as well. Then there is a $\epsilon \geq 0$ such that

$$[p(j) - \epsilon, p(j) + \epsilon] \subset [0, 1]$$

and

$$[p(k) - \epsilon, p(k) + \epsilon] \subset [0, 1].$$

By the same reasoning, p is not an extreme point.

This is a contradiction. Hence there is no j such that $p(j) \in (0, 1)$, so all $p(j) \in \{0, 1\}$. Therefore, there is a unique j such that $p(j) = 1$ and $p(k) = 0$ for all $k \neq j$.

Now suppose that p is a probability distribution such that there is a unique j such that $p(j) = 1$ and $p(k) = 0$ for all $k \neq j$. Then suppose there is a $t \in (0, 1)$ and $p_1, p_2 \in Pr(\{1, \ldots, n\})$ such that $p = tp_1 + (1 - t)p_2$. Suppose that $p_1(j) \neq 1$. Then $p_1(j) < 1$, since all $p_1(k) \geq 0$ and $\sum_k p_1(k) = 1$. Then $p_2(j) \geq 1$, which is a contradiction. Hence $p_1(j) = 1$. Likewise, $p_2(j) = 1$. Then, Since $p_1, p_2 \in Pr(\{1, \ldots, n\})$, $p_1(k) = 0 = p_2(k)$ for all $k \neq j$. Therefore $p = p_1 = p_2$ and p is an extreme point. □

From the definition of pure states we can say that the extreme points of $S(M_n(\mathbb{C}))$ are the pure states on $M_n(\mathbb{C})$.

The point, then is that states on $M_n(\mathbb{C})$ bijectively corresponds to density matrices through

$$\omega(a) = Tr(\rho a). \tag{2.1}$$

Upon the identification 2.2.2, pure states corresponds to one-dimensional projections, i.e., ω is pure iff

$$\omega(a) = \langle \psi, a\psi \rangle \tag{2.2}$$

for some unit vector $\psi \in \mathbb{C}^n$.

2.3 The Kadison-Singer property

Having introduced the basic definitions, let us now streamline the world of the Kadison-Singer conjecture by introducing the Kadison-Singer property.

Let H be a Hilbert space and denote the *-algebra of all bounded operators on H by $B(H)$, equipped with the adjoint as an involution, as above. In quantum mechanics one is particularly interested in abelian unital *-algebras $A \subseteq B(H)$.

Now both A and $B(H)$ have states, and states on $B(H)$ obviously restrict to states on A. In reverse direction, we can ask whether states on A extend to states on $B(H)$. It turns out that they always do due to Hahn-Banach extension theorem.

But what is at stake is the question whether this extension is unique. This question is particularly interesting for pure states, and hence we say that A has the Kadison-Singer property iff each pure state on A extends uniquely to a state on $B(H)$. Simple arguments in convexity theory shoe that if the extension is unique, then it is necessarily pure, so that one might as well say that:

A has the Kadison-Singer property iff each pure state on A extends uniquely to a pure state on $B(H)$.

Theorem 2.3.1. *For each $n \in \mathbb{N}$, the algebra $D_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ has the Kadison-Singer property.*

Proof. Consider the pure state ω_i on $D_n(\mathbb{C})$, where $i = 1, \ldots, n$ is arbitrary.Then $\omega_i(a) = a_{ii}$, for all $a \in D_n(\mathbb{C})$.

Let e_i be the i'th basis vector of $\mathbb{C}^n$.Then it is easy to observe that the functional $\mu : M_n(\mathbb{C}) \to \mathbb{C}$ defined by

$$\mu(a) = \langle e_i, ae_i \rangle = a_{ii},$$

is a pure state extension of ω.The only thing that is left to prove that μ is the unique pure state extension of ω_i.

Suppose that $\mu_1 : M_n(\mathbb{C}) \to \mathbb{C}$ is also a pure state extension of ω. Then $\mu_1(a) = \langle \psi, a\psi \rangle$ for some unit vector $\psi \in \mathbb{C}^n$.

We can write $\psi = \sum_{i=1}^{n} c_i e_i$.

Now since $|e_i\rangle\langle e_i| \in D_n(\mathbb{C})$ for all i,

$$\begin{aligned}|c_i|^2 &= \mu_1(|e_i\rangle\langle e_i|)\\ &= \omega_i(|e_i\rangle\langle e_i|)\\ &= \delta_{ki}.\end{aligned}$$

Therefore except c_i all are zero and $|c_i| = 1$.

Then $\psi = c_i e_i$.

Now

$$\begin{aligned}\mu_1(a) &= \langle \psi, a\psi\rangle\\ &= \langle c_i e_i, a c_i e_i\rangle\\ &= |c_i|^2\langle e_i, a e_i\rangle\\ &= \mu(a)\end{aligned}$$

for all $a \in M_n(\mathbb{C})$.

Therefore $\mu_1 = \mu$ and μ is the unique pure state extension of ω_i.

So, the algebra $D_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ has the Kadison-Singer property. □

Definition 2.3.1. *(Maximal unital abelian *-algebra)*

We say that a unital abelian *-algebras $A \subseteq B(H)$ is maximal if there is no abelian unital *-algebra $A_1 \subseteq B(H)$ that properly contains A.

If H is finite-dimensional, then the unital *-algebra generated by $a = a^*$.

Before going to next lemma let us introduced the concept of simultaneous diagonalization.

Definition 2.3.2. *(simultaneously diagonalizable)*

A set of matrices is said to be simultaneously diagonalizable if there exists a single invertible matrix P such that $P^{-1}AP$ is a diagonal matrix for every A in the set.

The following theorem characterizes simultaneously diagonalizable matrices:

Theorem 2.3.2. *(characterization of simultaneously diagonalizable matrices) A set of diagonalizable matrices commutes if and only if the set is simultaneously diagonalizable.*

By Spectral theorem and the above theorem, we can conclude that:

A set consists of commuting normal matrices if and only if it is simultaneously diagonalizable by a unitary matrix; that is, there exists a unitary matrix U such that U^*AU is diagonal for every A in the set.

Lemma 2.3.1. *Suppose H is finite dimensional Hilbert space and suppose that $A \subseteq B(H)$ is a maximal abelian unital *-algebra. Then A has the Kadison-Singer property.*

Proof. Since H is finite dimensional H is isomorphic to $\mathbb{C}^n$.And consequently, $B(H)$ is isomorphic to $M_n(\mathbb{C})$.

Therefore, there exists an isomorphism $\phi : B(H) \to M_n(\mathbb{C})$.

Now let $A \subseteq B(H)$ be a maximal abelian unital *-algebra. Then $\phi(A)$ is a maximal abelian unital *-algebra in $M_n(\mathbb{C})$.

We will prove that $D_n(\mathbb{C})$ is the unique maximal unital *-algebra in $M_n(\mathbb{C})$. Then $\phi(A) = D_n(\mathbb{C})$ and we know that $D_n(\mathbb{C})$ has the Kadison-Singer property and via isomorphism we can say that A also has the Kadison-Singer property.

So, the only thing remains to prove that $D_n(\mathbb{C})$ is the unique maximal unital *-algebra in $M_n(\mathbb{C})$.

First we will prove that $D_n(\mathbb{C})$ is a maximal abelian unital *-algebra.

Clearly, $D_n(\mathbb{C})$ is abelian unital $*$-algebra.

Let us assume that $D_n(\mathbb{C})$ is not maximal in $M_n(\mathbb{C})$. Then there exists an abelian unital *-algebra $\mathbb{D}$ such that $D_n(\mathbb{C}) \subseteq \mathbb{D}$.

Since $\mathbb{D}$ is abelian, every matrix in $\mathbb{D}$ is normal and so diagonalizable.

Let $A, B, C \in \mathbb{D}$ such that A is not in $D_n(\mathbb{C})$ (this is possible, since $D_n(\mathbb{C}) \subseteq \mathbb{D}$)

Now since $AB = BA$ and A, B are normal, A and B are simultaneously diagonalizable.

Therefore A and B have same set of eigen vectors.

By the same argument B and C also have the same set of eigen vectors.

Therefore A,B and C have same set of eigen vectors.

This implies every matrix of $\mathbb{D}$ has the same set of eigen vectors.

The set of all eigen vectors of every matrix of $D_n(\mathbb{C})$ is the basis vectors of $\mathbb{C}^n$.

But A is an non-diagonal matrix and so it can have the basis vectors of $\mathbb{C}^n$ as its eigen vectors.

This is a contradiction.

Therefore $D_n(\mathbb{C})$ is maximal abelian unital *-algebra in $M_n(\mathbb{C})$.

And $D_n(\mathbb{C})$ has the Kadison-Singer property and therefore A has the Kadison-Singer property. $\square$

So we are done with the finite dimensional case of the Kadison-Singer problem. In the next chapter we will discuss the problem in much broader sense. There we will define states and pure states on general unital C^*-algebra and will discuss its properties.

Chapter 3

State space

In last chapter we discussed the extension of pure states from the algebra of diagonal matrices $D_n(\mathbb{C})$ to the algebra of matrices $M_n(\mathbb{C})$. In this chapter, we formulate the question whether this is possible in a much broader setting. Instead of $M_n(\mathbb{C})$ we consider $B(H)$ for some Hilbert space H, and instead of $D_n(\mathbb{C})$ we consider abelian C^*-subalgebras A of $B(H)$. Having again defined (pure) states, we will likewise ask the question whether a unique extension property holds. This property is the so-called Kadison-Singer property.

3.1 States on C^*-algebra

Now we will define state in a general unital C^*-algebra.

Definition 3.1.1 (State)**.** *Let A be a unital C^*-algebra. A* **state** *on A is a linear map $f : A \to \mathbb{C}$ that is positive (i.e. $f(a) \geq 0$ for all $a \geq 0$) and unital (i.e. f(1)=1). The set of all states on A is denoted by $\mathbf{S}(\mathbf{A})$ and is called the* **state space** *of A.*

The condition of being positive has a very important consequence for states.

Proposition 3.1.1. *Suppose A is a unital C^*-algebra and $f \in S(A)$. Then*

$$sup\{|f(a)| : a \in A, \|a\| = 1\}$$

is finite, i.e. $S(A) \subseteq A^*$.

Proof. First suppose that $sup\{|f(a)| : \|a\| = 1, a \geq 0\}$ is infinite. Then there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $|f(a_n)| \geq 2^n, a_n \geq 0$ and $\|a_n\| = 1$ for all $n \in \mathbb{N}$. Then

$a = \sum_{n=1}^{\infty} 2^{-n} a_n$ exists and is positive too. Then, by linearity, $1 \leq f(2^{-n}a_n)$ for all $n \in \mathbb{N}$. Hence we have

$$N \leq \sum_{n=1}^{N} f(2^{-n}a_n) = f(\sum_{n=1}^{N} 2^{-n}a_n) \leq f(a).$$

i.e. $N \leq f(a)$ for all $N \in \mathbb{N}$. This is a contradiction, so

$$M := sup\{|f(a)| : \|a\| = 1, a \geq 0\}$$

is finite.

Now let $a \in A$ be an arbitary element such that $\|a\| = 1$. Then a can be written as $a = \sum_{k=0}^{3} i^k a_k$ where all $a_k \geq 0$ and $\|a_k\| \leq 1$. Therefore

$$\begin{aligned} |f(a)| &= |f(\sum_{k=0}^{3} i^k a_k)| \\ &= |\sum_{k=0}^{3} i^k f(a_k)| \\ &\leq \sum_{k=0}^{3} \|a_k\| f(\frac{a_k}{\|a_k\|}) \\ &\leq 4M, \end{aligned}$$

i.e. $sup\{|f(a)| : a \in A, \|a\| = 1\}$ is finite too. □

When considering states, the following result is often useful.

Lemma 3.1.1. *Suppose A is a C^*-algebra and $f \in S(A)$. Then the map*

$$\phi : A^2 \to \mathbb{C}, (a, b) \mapsto f(a^*b)$$

is a pre-inner product and hence for every $a, b \in A$ we have

$$|f(a^*b)| \leq f(a^*a)^{1/2} f(b^*b)^{1/2}.$$

Proof. Since f is positive, clearly it is a pre-inner product and second part follows from the Cauchy-Schwarz inequality for pre-inner products. □

Corollary 3.1.1. *Suppose A is a unital C^*-algebra and $f \in S(A)$. Furthermore, let $ainA$. Then $f(a^*) = \overline{f(a)}$.*

Proof. We use the above lemma to see that

$$f(a^*) = f(a^*1) = \overline{f(1^*a)} = \overline{f(a)}.$$

□

Since every state is bounded, we can consider its norm. Using this, we can give a different characterization of states.

Proposition 3.1.2. *Suppose that H is a Hilbert space and A is a unital C^*-algebra of $B(H)$. Furthermore, let $f : A \to \mathbb{C}$ be a bounded linear functional such that $f(1) = 1$. Then f is positive (and hence a state) iff $\|f\| = 1$.*

Proof. First suppose that f is positive. Since $\|1\| = 1$, $\|f\| \geq f(1) = 1$.

Now let $a \in A$ such that $\|a\| = 1$. Then

$$\begin{aligned} |f(a)|^2 &= |f(1a)|^2 \\ &\leq f(1^*1)f(a^*a) \\ &\leq f(1)\|f\|\|a^*a\| \\ &= \|f\|. \end{aligned}$$

Therefore,

$$\|f\|^2 = sup\{|f(a)|^2 : \|a\| = 1\} \leq \|f\|,$$

whence $\|f\| \leq 1$. So $\|f\| = 1$.

For the converse, suppose that $\|f\| = 1$. Let $a \in A$ be self-adjoint and $n \in \mathbb{Z}$. Since $f(a) \in \mathbb{C}$, we can write $f(a) = \alpha + i\beta$, with $\alpha, \beta \in \mathbb{R}$. Furthermore, denote $c := \|a^2\|$.

Then :

$$\begin{aligned}|f(a+in1)|^2 &\leq \|f\|^2\|a+in1\|^2 \\ &= \|(a+in1)^*(a+in1)\| \\ &= \|(a-in1)(a+in1)\| \\ &= \|a^2+n^21\| \\ &\leq \|a^2\|+n^2\|1\| \\ &= c+n^2\end{aligned}$$

Moreover,

$$\begin{aligned}|f(a+in1)|^2 &= |f(a)+inf(1)|^2 \\ &= |\alpha+i\beta+in|^2 \\ &= \alpha^2+(\beta+n)^2 \\ &= \alpha^2+\beta^2+2\beta n+n^2.\end{aligned}$$

Collecting this, we obtain the inequality :

$$\alpha^2+\beta^2+2\beta n+n^2 \leq c+n^2.$$

Rewriting this, we obtain :

$$2\beta n \leq c-\alpha^2-\beta^2.$$

If $\beta \neq 0$, then we obtain for every $n \in \mathbb{Z}$:

$$n \leq \frac{c-\alpha^2-\beta^2}{2\beta},$$

which is a contradiction since the right hand side is independent of n. Hence $\beta = 0$. So $f(a) = \alpha$, i.e. $f(a)$ is real.

Now, let $a \geq 0$, $a \neq 0$ and write $b = \dfrac{a}{\|a\|}$. Since a is self-adjoint and $\|b\| = 1$. We claim that $1-b$ is positive. To see this, let $x \in H$ and compute :

$$\begin{aligned}\langle x,(1-b)x\rangle &= \langle x,x\rangle - \langle x,bx\rangle \\ &\geq \|x\|^2 - \|x\|\|bx\| \\ &\geq \|x\|^2 - \|b\|\|x\|^2 \\ &\geq 0.\end{aligned}$$

So, indeed $1-b$ is positive and hence also self-adjoint. Since $0 \leq 1-b \leq 1$, we also have $\|1-b\| \leq 1$. Then :

$$\begin{aligned}1-f(b) = f(1)-f(b) &= f(1-b) \\ &\leq \|f(1-b)\| \\ &\leq \|f\|\|(1-b)\| \\ &\leq 1,\end{aligned}$$

whence $f(b) \geq 0$. Then also $f(a) = \|a\|f(b) \geq 0$. Since we obviously also have that $f(0) \geq 0$, f is positive. □

Since all states on a unital C^* -algebra A are bounded, $S(A)$ inherits the weak*-topology from A^*. With respect to this topology, $S(A)$ has an important property.

Proposition 3.1.3. *Let A be a unital C^*- algebra. Then $S(A) \subseteq A^*$ is a compact Hausdorff space.*

Proof. We first claim that $S(A) \subseteq A^*$ is closed with respect to the weak*-topology. To see this, suppose that $\{f_i\}$ is a net of states converging to a certain $f \in A^*$. By the definition of the **weak*-topology**, this means that $f(a) = lim f_i(a)$ for all $a \in A$.

So, certainly, when taking $a = 1$, it follows that $f(1) = lim f_i(1) = lim 1 = 1$, since every f_i is a state.

Furthermore, if $a \geq 0$, then $f_i(a) \geq 0$ for every i, so $f(a) = lim f_i(a) \geq 0$ as well. So, indeed, $f \in S(A)$, i.e. $S(A)$ is closed with respect to the weak*-topology on A^*.

Now, by the **Banach-Alaoglu theorem**, the closed unit ball A_1^* of A^* is compact with respect to the weak*-topology and we know that $S(A) \subseteq A_1^*$. Hence $S(A)$ is closed with respect to the relative topology on A_1^*, which is a compact space. Hence

$S(A)$ is **compact** with respect to the relative topology and therefore with respect to the weak*-topology.

Next to see that $S(A)$ is Hausdorff, suppose $f, g \in S(A)$ such that $f \neq g$. Then there is an $a \in A$ such that $f(a) \neq g(a)$.

Therefore, $\delta := |f(a) - g(a)| > 0$.

Now consider $U = B(f, a, \frac{\delta}{2}) \cap S(A)$ and $V = B(g, a, \frac{\delta}{2}) \cap S(A)$. Then both $U, V \in S(A)$ are open and $f \in U, g \in V$. Furthermore, $h \in U \cap V$ implies

$$\begin{aligned} |f(a) - g(a)| &\leq |f(a) - h(a)| + |h(a) - g(a)| \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta, \end{aligned}$$

which is a contradiction. Hence $U \cap V = \phi$. Therefore, $S(A)$ is Hausdorff. □

3.2 Pure states and characters

We note that $S(A)$ is **convex** for every unital C^*-algebra A. Therefore, we can consider its boundary $\partial_e S(A)$ and call this the pure state space of A. It turns out that in the case that A is abelian, the pure states are exactly the **characters**. To prove this we first need an equivalent definition of pure states in terms of positive functionals.

Lemma 3.2.1. *Suppose H is a Hilbert space and $A \subset B(H)$. Furthermore, suppose $f \in S(A)$. Then $f \in \partial_e S(A)$ if and only if for all $g : A \to \mathbb{C}$ such that $0 \leq g \leq f$ we have $g = tf$ for some $t \in [0, 1]$.*

Proof. Suppose $f \in \partial_e S(A)$ and $g : A \to \mathbb{C}$ such that $0 \leq g \leq f$. Since $1 \geq 0$, then $0 \leq g(1) \leq f(1) = 1$.

Now, there are a few cases. First of all, suppose $g(1) = 0$. Then let $a \in A$ be positive. Then $0 \leq \frac{a}{\|a\|} \leq 1$, whence $0 \leq a \leq \|a\|1$. Therefore,

$$\begin{aligned} 0 &\leq g(a) \\ &\leq g(\|a\|1) \\ &= \|a\|g(1) \\ &= 0. \end{aligned}$$

Since every $b \in A$ can be written as $b = \sum_{k=0}^{3} i^k b_k$ for some $b_k \geq 0$, $g(b) = 0$ for every $b \in A$, i.e. $g = 0$.

As a second case, suppose $g(1) = 1$. Then $f - g \geq 0$ and $(f - g)(1) = 0$, so by the same reasoning as in the firsty case, $f - g = 0$, i.e. g=f. Lastly, there is the case $0 < g(1) < 1$. In this case, define the functionals $g_1 = \frac{1}{1 - g(1)}(f - g)$ and $g_2 = \frac{1}{g(1)}g$. Then clearly, g_1 and g_2 are both positive and $g_1(1) = g_2(1) = 1$, so $g_1, g_2 \in S(A)$. Furthermore,

$$\begin{aligned} (1 - g(1))g_1 + g(1)g_2 &= f - g + g \\ &= f \end{aligned}$$

and $f \in \partial_e S(A)$, so $g_1 = g_2 = f$. Therefore, $g = g(1)g_2 = g(1)f$. In all cases, we see that $g = g(1)f$, and $g(1) \in [0, 1]$.

For the converse, suppose that for all $g : A \to \mathbb{C}$ such that $0 \leq g \leq f$ there is a $t \in [0, 1]$ such that $g = tf$. Then suppose that $h_1, h_2 \in S(A)$ and $s \in (0, 1)$ such that $f = sh_1 + (1 - s)h_2$. Then $f - sh_1 = (1 - s)h_2 \geq 0$, so $0 \leq sh_1 \leq f$. Hence there is a $t \in [0, 1]$ such that $sh_1 = tf$. However, $s = sh_1 = tf(1) = t$, so $h_1 = f$. Then also $h_2 = f$, so $f \in \partial_e S(A)$. □

Now we are all set to prove our main theorem in this section, that pure states are exactly the characters. We already proved that every pure state on $D_n(\mathbb{C})$ was of the form $f(a) = a_{ii}$, which is clearly multiplicative on the diagonal matrices, i.e. $\partial_e S(A) \subset \Omega(A)$. So, the following theorem can be seen as a generalization.

Theorem 3.2.1. *Suppose H is a Hilbert space and $A \subseteq B(H)$ be an abelian unital C^*-algebra. Then $\partial_e S(A) = \Omega(A)$.*

Proof. First we will prove that $\partial_e S(A) \subseteq \Omega(A)$.

Let $f \in \partial_e S(A)$. We will prove that f is multiplicative. Let $a, c \in A$ and first suppose that $0 \leq c \leq 1$. Now let $b \in A$ such that $b \geq 0$.

Then $c = d^*d, 1 - c = u^*u$ and $b = v^*v$ for some $c, u, v \in A$. Therefore,

$$\begin{aligned} bc &= v^*vd^*d \\ &= d^*v^*vd \\ &= (vd)^*vd \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} b - bc &= b(1-c) \\ &= v^*vu^*u \\ &= u^*v^*vu \\ &= (vu)^*vu \\ &\geq 0, \end{aligned}$$

so $0 \leq bc \leq b$.

Now define $g : A \to \mathbb{C}$ by $g(z) = f(zc)$ for all $z \in A$. Combining the fact that $f \geq 0$ and the above observation that $bc \geq 0$ for all $b \geq 0$, we see that $g \geq 0$.

Furthermore, for $b \geq 0$, $b \geq bc$ and hence

$$(f - g)(b) = f(b) - f(bc) = f(b - bc) \geq 0,$$

so $g \leq f$. Now using the earlier lemma, we know that $g = tf$ for some $t \in [0, 1]$. Now

$$f(ac) = g(a) = tf(a) = tf(1)f(a) = g(1)f(a) = f(c)f(a) = f(a)f(c).$$

If we now drop the requirement that $0 \leq c \leq 1$, we observe that we still have $c = \sum_{k=0}^{3} i^k c_k$ for some $c_k \geq 0$.

Then $c = \sum_{k=0}^{3} i^k \|c_k\| \frac{c_k}{\|c_k\|}$ and $0 \leq \frac{c_k}{\|c_k\|} \leq 1$, whence

$$
\begin{aligned}
f(ac) &= f(a\sum_{k=0}^{3} i^k \|c_k\| \frac{c_k}{\|c_k\|}) \\
&= \sum_{k=0}^{3} i^k \|c_k\| f(a\frac{c_k}{\|c_k\|}) \\
&= \sum_{k=0}^{3} i^k \|c_k\| f(a) f(\frac{c_k}{\|c_k\|}) \\
&= f(a) f(\sum_{k=0}^{3} i^k \|c_k\| \frac{c_k}{\|c_k\|}) \\
&= f(a) f(c).
\end{aligned}
$$

Since $f(1) = 1$ and hence $f \neq 0$, so $f \in \Omega(A)$. Therefore $\partial_e S(A) \subseteq \Omega(A)$

For the converse, suppose $c \in \Omega(A)$. Then $c(1) = 1$. Furthermore, for $a \in A$,

$$
c(a^*a) = c(a^*)c(a) = \overline{c(a)}c(a) = |c(a)|^2 \geq 0,
$$

so $c \geq 0$. Since c is also linear, $c \in S(A)$.

Now we claim that in fact $c \in \partial_e S(A)$. To see this, suppose that $t \in (0, 1)$ and $c_1, c_2 \in S(A)$ such that $c = tc_1 + (1 - t)c_2$. Furthermore, suppose that $a = a^* \in A$. Then $c_1(a) \in \mathbb{R}$, since $c_1 \geq 0$ and $c_1(a)^2 = |c_1(1^*a)|^2 \leq c_1(1^*1)c_!(a^*a) = c_1(a^2)$. Likewise, $c_2(a)^2 \leq c_2(a^2)$.

Since c is a character, we can compute :

$$
\begin{aligned}
0 &= c(a^2) - c(a)^2 \\
&= tc_1(a^2) + (1 - t)c_2(a^2) - (tc_1(a) + (1 - t)c_2(a))^2 \\
&= tc_1(a^2) + (1 - t)c_2(a^2) - t^2 c_1(a)^2 - (1 - t)^2 c_2(a)^2 - 2t(1 - t)c_1(a)c_2(a) \\
&\geq tc_1(a)^2 + (1 - t)c_2(a)^2 - t^2 c_1(a)^2 - (1 - t)^2 c_2(a)^2 - 2t(1 - t)c_1(a)c_2(a) \\
&= (t - t^2)c_1(a)^2 + ((1 - t) - (1 - t)^2)c_2(a)^2 - 2t(1 - t)c_1(a)c_2(a) \\
&= t(1 - t)(c_1(a)^2 + c_2(a)^2 - 2c_1(a)c_2(a)) \\
&= t(1 - t)(c_1(a) - c_2(a))^2 \geq 0,
\end{aligned}
$$

i.e. $c_1(a) = c_2(a)$ for all $a = a^* \in A$. Therefore, for any $b \in A$, $b = a_1 + ia_2$ with

$a_1 = a_1^*$, $a_2 = a_2^* \in A$, whence $c_1(b) = c_2(b)$ by linearity. Therefore $c_1 = c_2 = c$ and $c \in \partial_e S(A)$.

□

The above theorem is really beautiful, because the algebra $B(H)$ for a Hilbert space H of dimension at least 2 does not even admit any characters:

Proposition 3.2.1. *Let H be a Hilbert spacve of at least dimension 2. Then*

$$\Omega(B(H)) = \emptyset.$$

Proof. Suppose $\Omega(B(H)) \neq \emptyset$. Let $c \in \Omega(B(H))$ and let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. Let $i \in I$. By the hypothesis there is a $j \in I$ such that $j \neq i$.

Let $a = |e_i\rangle\langle e_j|$ and $B = |e_j\rangle\langle e_i|$. Then $a^2 = 0$, so $c(a)^2 = c(a^2) = c(0) = 0$, whence $c(a) = 0$. Likewise, $c(b) = 0$.

Now, $|e_i\rangle\langle e_i| = ab$, so

$$c(|e_i\rangle\langle e_i|) = c(ab) = c(a)c(b) = 0.$$

Since $i \in I$ was arbitary,

$$\begin{aligned} c(1) &= c(\sum_{i\in I} |e_i\rangle\langle e_j|) \\ &= \sum_{i\in I} c(|e_i\rangle\langle e_j|) \\ &= 0. \end{aligned}$$

Then for any $x \in B(H)$, $c(x) = c(x1) = c(x)c(1) = 0$, so $c = 0$, i.e. c is not a character. This a contradiction.

Therefore $\Omega(B(H)) = \emptyset$. □

Corollary 3.2.1. *Suppose A is an abelian unital C^*-algebra. Then $\partial_e S(A)$ is compact Hausdorff with respect to the weak*-topology.*

Proof. Since $\partial_e S(A) \subseteq S(A)$ and $S(A)$ is Hausdorff, we know that $\partial_e S(A)$ is Hausdorff too. In fact, we only need to show that $\Omega(A) = \partial_e S(A)$ is closed in $S(A)$, since $S(A)$

is compact. To prove this, we show that $U := S(A) \setminus \Omega(A)$ is open in S(A). For this, suppose $f \in U$. Then there are $a, b \in A$ such that $f(a)f(b) \neq f(ab)$. Since every element of A can be written as a sum of positive elements, we can then assume that a and b are positive.

Now, since A is abelian we then also know that ab is positive. Hence $f(a), f(b)$ and $f(ab)$ are positive numbers. Suppose that $f(a)f(b) > f(ab)$ and define $\delta = f(a)f(b) - f(ab) > 0$. Next, define $\epsilon_1 = \dfrac{\delta}{f(a) + f(b) + 1}$. Using this, define $\epsilon = min\{\epsilon_1, f(a), f(b)\} > 0$.

Then, take $g \in B(f, a, \epsilon) \cap B(f, b, \epsilon) \cap B(f, ab, \epsilon) \cap S(A)$. Then we have

$$\begin{aligned} g(a)g(b) - g(ab) \geq g(ab) &> (f(a) - \epsilon)(f(b) - \epsilon) - (f(ab) + \epsilon) \\ &= f(a)f(b) - f(ab) - \epsilon(f(a) + f(b) + 1) + \epsilon^2 \\ &> \delta - \epsilon(f(a) + f(b) + 1) \\ &\geq \delta - \delta \\ &= 0, \end{aligned}$$

i.e. $g(a)g(b) \neq g(ab)$. Hence $g \in U$. A similar argument works if $f(a)f(b) < f(ab)$. Hence U is open. Therefore $\partial_e S(A) = \Omega(A) \subseteq S(A)$ is closed and hence a compact Hausdorff space. □

3.3 Extensions of pure states

In our discussion we want to generalize the concept of pure states from the algebra of diagonal matrices $D_n(\mathbb{C})$ to the algebra of all matrices, $M_n(\mathbb{C})$. We have already generalized $D_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ to $A \subseteq B(H)$ for a Hilbert space H and an abelian unital C^*-subalgebra A. Now interesting thing is that we already characterize the pure states on A and they are in fact characters. These cannot be extended to characters on all of $B(H)$, since the latter do not exist. However, they might be extended to states (in fact pure states) on all of $B(H)$. The natural question is to ask that wheather this extension is unique or not ?

Definition 3.3.1. *Let H be a Hilbert space and A be an abelian unital C^*-subalgebra* *of $B(H)$. Furthermore, let $f \in S(A)$. We define the set of extensions of f to be :*

$$Ext(f) = \{g \in S(B(H)) : g|_A = f\}.$$

We already proved that for the case $H = \mathbb{C}^n$ and $A = D_n(\mathbb{C})$, for each $f \in \partial_e S(A)$ the set $\text{Ext}(f) \ \cap \ \partial_e S(M_n(\mathbb{C}))$ consists of exactly one element, i.e. every pure state on $D_n(\mathbb{C})$ extends to a unique pure state on $M_n(\mathbb{C})$. This motivates the following definition.

Definition 3.3.2. *Let H be a Hilbert space and A be an abelian unital C^*-subalgebra of $B(H)$. We say that A has the first Kadison-Singer property if for every $f \in \partial_e S(A)$, $Ext(f) \ \cap \ \partial_e S(B(H))$ consists of exactly one element.*

We may also drop the requirement that the unique extension must be pure. Then we obtain another property.

Definition 3.3.3. *Let H be a Hilbert space and A be an abelian unital C^*-subalgebra of $B(H)$. We say that A has the second Kadison-Singer property if for every $f \in \partial_e S(A)$, $Ext(f)$ consists of exactly one element.*

Now looking at these two definitions, it is unclear that whether the first Kadison-Singer property implies the second, since $\text{Ext}(f)$ might contain more elements than $\text{Ext}(f) \ \cap \ \partial_e S(B(H))$. Likewise, the one elment in $\text{Ext}(f)$ might not be in $\partial_e S(B(H))$, whence the second Kadison-Singer property might not imply the first.

However, it turns out that the first and second Kadison-Singer property are in fact equivalent. To prove this, we first need a lemma and note that for every $f \in S(A)$, $\text{Ext}(f)$ is a convex set, hence we can consider its boundary.

Lemma 3.3.1. *Let H be a separable Hilbert space and A an abelian unital C^*-subalgebra of $B(H)$. For every $f \in \partial_e S(A)$ we have the following identity:*

$$\partial_e Ext(f) = Ext(f) \ \cap \ \partial_e S(B(H)).$$

Proof. First we will show that $\partial_e Ext(f) \subseteq Ext(f) \cap \partial_e S(B(H))$. It is clear that $\partial_e Ext(f) \subseteq Ext(f)$. To see that $\partial_e Ext(f) \subseteq \partial_e S(B(H))$, suppose that $g \in \partial_e Ext(f)$ and $h_1, h_2 \in S(B(H))$, $t \in (0, 1)$ such that

$$g = th_1 + (1 - t)h_2.$$

Let k_1 and k_2 be the restrictions of h_1 and h_2 to A, respectively. Then, clearly, k_1 and k_2 are both states on A and we have $f = tk_1 + (1-t)k_2$. Since f is a pure state on A, this means that $k_1 = k_2 = f$.

Therefore, $h_1, h_2 \in Ext(f)$ and since $g \in \partial_e Ext(f)$, this means that $g = h_1 = h_2$. Therefore $g \in \partial_e S(B(H))$. Hence $\partial_e Ext(f) = Ext(f) \cap \partial_e S(B(H))$.

Now we will prove the other way set inclusion. Suppose that $g \in Ext(f) \cap \partial_e S(B(H))$ and $t \in (0,1)$ and $h_1, h_2 \in Ext(f)$ such that $g = th_1 + (1-t)h_2$. Then also $h_1, h_2 \in S(B(H))$ and since $g \in \partial_e S(B(H))$ we then have $h_1 = h_2 = g$. Therefore $g \in \partial_e Ext(f)$. □

Now we are all set to prove the main theorem of this section : the equivalence of the first and second Kadison-Singer property.

Theorem 3.3.1. *Let H be a Hilbert space and A an abelian unital C^*-subalgebra of $B(H)$. Then A has the first Kadison-Singer property if and only if it has the second Kadison-Singer property.*

Proof. Suppose A has the first Kadison-Singer property and let $f \in \partial_e S(A)$. Then, by assumption $\text{Ext}(f) \cap \partial_e S(A)$ consists of exactly one element, so by above lemma $\partial_e \text{Ext}(f)$ consists of exactly one element.

Now, clearly $\text{Ext}(f)$ is convex and is a closed subset of the compact set $S(B(H))$. Therefore, $\text{Ext}(f)$ is convex and compact and the Krein-Milman theorem can be applied to it, i.e. $\text{Ext}(f) = \overline{co(\partial_e Ext(f))}$. However, $\partial_e \text{Ext}(f)$ consists of exactly one element, whence $co(\partial_e Ext(f))$ consists of exactly one element. Therefore, $\text{Ext}(f)$ contains exactly one element, and A has the second Kadison-Singer property.

For the converse, suppose that A has the second Kadison-Singer property and let $f \in \partial_e S(A)$. Then $\text{Ext}(f)$ contains exactly one element, so $\partial_e Ext(f) = Ext(f)$ and hence $\partial_e Ext(f)$ consists of one element as well. By above lemma, then $\text{Ext}(f) \cap \partial_e S(B(H))$ consists of one element, i.e. A has the first Kadison-Singer property. □

By the above theorem, we can drop the adjectives 'first' and 'second' and just speak of one property.

Definition 3.3.4. *Let H be a Hilbert space and A an abelian unital C^*-subalgebra of $B(H)$. Then we say that A has the Kadison-Singer property if it has either (and hence both) the first and second Kadison-Singer property.*

From now on, the main goal of this text is to classify the examples of a Hilbert space H and an abelian unital C^*-subalgebra $A \subseteq B(H)$ that have the Kadison-Singer property.

3.4 Properties of extensions and restrictions

The Kadison-Singer property concerns two parts : existence and uniqueness. The following theorem shows that the first is never an issue.

Theorem 3.4.1. *Let H be a Hilbert space and A be an unital abelian C^*-subalgebra of $B(H)$. Furthermore, let $f \in S(A)$. Then Ext(f) $\neq \emptyset$.*

Proof. $f \in S(A)$, implies $\|f\| = 1$. Since $A \subseteq B(H)$ is a linear subspace, there is a functional $g : B(H) \to \mathbb{C}$ that is an extension of f and $\|g\| = \|f\| = 1$, by the Hahn-Banach theorem.

Since $1 \in A \subseteq B(H)$, $g(1) = f(1) = 1$. Using the characterization for states, it follows that $g \in S(B(H))$. Therefore, $g \in Ext(f)$, i.e. Ext(f)$\neq \emptyset$. □

Now that we know that an extension always exists, we only have to focus on uniqueness when we want to answer the question whether a given algebra has the Kadison-Singer property.

Lemma 3.4.1. *Suppose A is a C^*-algebra and $C \subseteq A$ is a C^*-subalgebra. Then the restriction map*

$$\phi : S(A) \to S(C), f \mapsto f|_C,$$

is continuous.

Proof. Note that the state spaces $S(A)$ and $S(C)$ are endowed with the weak*-topology. Therefore, let $f \in S(C)$, $c \in C$ and $\epsilon > 0$, i.e. let $B(f, c, \epsilon) \subseteq S(C)$ be an arbitrary subbase element. We prove that $\phi^{-1}(B(f, c, \epsilon)) \subseteq S(A)$ is open.

To do this, let $g \in \phi^{-1}(B(f, c, \epsilon))$. Then $|\phi(g)(c) - f(c)| < \epsilon$, so there is a $\delta > 0$

such that $|\phi(g)(c) - f(c)| < \epsilon - \delta$. Then let $h \in B(g, c, \delta)$. Then

$$\begin{aligned} |\phi(h)(c) - f(c)| &\leq |\phi(h)(c) - \phi(g)(c)| + |\phi(g)(c) - f(c)| \\ &< |h(c) - g(c)| + \epsilon - \delta \\ &< \delta + \epsilon - \delta \\ &= \epsilon, \end{aligned}$$

whence $h \in \phi^{-1}(B(f, c, \epsilon))$. Therefore,

$$B(g, c, \delta) \subseteq \phi^{-1}(B(f, c, \epsilon)),$$

i.e. $\phi^{-1}(B(f, c, \epsilon))$ is open.

Hence ϕ is continuous. □

So in this chapter we characterize some of the properties of state and pure state :

1. Every state is automatically continuous.
2. A bounded unital (i.e. f(1)=1) linear functional f is a state iff $\|f\| = 1$.
3. The set of all states on a unital C^*-algebra is a compact Hausdorff space with respect to weak*-topology.
4. If A is abelian unital C^*-algebra.Then the pure states on A are exactly the characters on A.
5. The set of all pure states is a compact Hausdorff space with respect to the weak*-topology.
6. Let H be a Hilbert space of at least dimension 2. Then characters does not exist on $B(H)$.
7. The first Kadison-Singer property and the second Kadison-Singer property are equivalent.

Chapter 4

Maximal abelian C^*-subalgebras

In last chapter we introduced the Kadison-Singer property and declared our main goal to be classifying Hilbert spaces H and abelian unital C^*-subalgebras $A \subseteq B(H)$ that have this property.

We already proved that $D_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ has the Kadison-Singer property in $M_n(\mathbb{C})$ and $D_n(\mathbb{C})$ is maximal abelian in $M_n(\mathbb{C})$. So from that we can guess that may be maximal abelian subalgebras are of special interest to characterize subalgebras that have the Kadison-Singer property.

In this chapter we are going to show that only maximal abelian subalgebras can possibly have the Kadison-Singer property and later we are going to characterize all maximal abelian subalgebras inside $B(H)$ for a separable Hilbert space H.

4.1 Maximal abelian C^*-subalgebras

For a fixed Hilbert space H, we can consider all unital abelian C^*-subalgebras of $B(H)$ and collect them in $C(B(H))$. For every element of $A \in C(B(H))$, we can ask ourselves whether A has the Kadison-Singer property with respect to $B(H)$. It turns out that only maximal elements of $C(B(H))$ can possibly have the Kadison-Singer property with respect to the canonical partial order $\leq$ on $C(B(H))$ given by inclusion, i.e. for $A_1, A_2 \in C(B(H))$ we have $A_1 \leq A_2$ iff $A_1 \subseteq A_2$. Since it would be tedious to use the symbol $\leq$, we just use the inclusion symbol $\subseteq$ to denote the partial order.

Since $C(B(H), \subseteq)$ is now a partially ordered set, we can consider its maximal elements.

Definition 4.1.1. *Suppose H is a Hilbert space and $A_1 \in C(B(H))$. Then A_1 is called maximal abelian if it is maximal with respect to the partial order $' \subseteq'$ on $C(B(H))$, i.e. if $A_1 \subseteq A_2$ for some $A_2 \in C(B(H))$, then necessarily $A_1 = A_2$.*

Maximal abelian elemwnts of $C(B(H)$ have a very nice description in terms of the commutant.

Definition 4.1.2 (Commutant)**.** *Suppose X is an algebra and $S \subseteq X$ is a subset. We define the commutant of S to be*

$$S' := \{x \in X| \quad sx = xs \quad \forall s \in S\},$$

i.e. the set of all $x \in X$ that commute with all of S.

We denote the double commutant of a subset S of an algebra X by $S'' := (S')'$ and likewise $S''' = (S'')'$.

Lemma 4.1.1. *Suppose X is an algebra and $S, T \subseteq X$ are subsets. Then :*

1. *$S \subseteq S'$ iff S is abelian.*
2. *If $S \subseteq T$, then $T' \subseteq S'$.*
3. *$S \subseteq S''$.*
4. *$S' = S'''$.*

Proof. The proofs of the first three properties follows directly from the definition of commutant. For the last property, observe that $S' \subseteq (S')' = S'''$ by the third property, and by combining property 2 and 3 we have $S''' = (S'')' \subseteq S'$. □

We can now give a nice description of maximal abelian subalgebra in terms of the commutant. This result is really useful to prove a subalgebra is maximal or not. We will use this result several times in the later part of this chapter to conclude various subalgerbras to be maximal.

Proposition 4.1.1. *Suppose A is a subalgebra of $B(H)$, for some Hilbert space H. Then the following are equivalent :*

1. *$A \in C(B(H))$ and A is maximal abelian;*

2. $A = A'$.

Proof. Suppose $A \in \mathcal{C}(B(H))$ is maximal abelian. Since A is abelian, $A \subseteq A'$.

Now let $b \in A'$ and let C be the smallest C^*-subalgebra of $B(H)$ that contains A and b. Then since b commutes with all of A, C is abelian and unital, since $1 \in A \subset C$. Therefore, $C \in \mathcal{C}(B(H))$ and $A \subseteq C$. However, A was assumed to be maximal, whence $C = A$.

Hence $b \in C = A$ and $A' \subseteq A$, so $A' = A$.

For the converse, suppose that $A = A'$. First note that $1 \in A' = A$ and $A \subseteq A'$, so $A \in \mathcal{C}(B(H))$. Now suppose that $C \in \mathcal{C}(B(H))$ such that $A \subseteq C$. Then C is abelian, so $C \subseteq C' \subseteq A' = A$, whence $A = C$ and A is maximal. □

The above result justifies dropping the adjevtive 'unital' when we defined maximal abelian subalgebras.

We now come to the main result in this chapter : only maximal abelian subalgebras can possibly have the Kadison-Singer property.

Theorem 4.1.1. *Suppose that H is a Hilbert space and that $A \in \mathcal{C}(B(H))$ has the Kadison-Singer property. Then A is maximal abelian.*

Proof. Suppose $C \in \mathcal{C}(B(H))$ such that $A \subseteq C$. To show A is maximal abelian it is sufficient to prove that $A = C$.

First we will show that the pure state spaces $\partial_e S(C)$ and $\partial_e S(A)$ are isomorphic. To do this, first construct the map :

$$\phi : \partial_e S(C) \to \partial_e S(A), \quad f \mapsto f|_A$$

Since the pure states are exactly characters on an abelian C^*-subalgebra and $f|_A$ is therefore a non-zero restriction of a character, $f|_A \in \Omega(A) = \partial_e S(A)$ for all $f \in \partial_e S(C)$. Therefore ϕ is well defined.

For any $g \in \partial_e S(A)$, we know that $\mathrm{Ext}(g)$ contains exactly one element. Denote this element by $\widetilde{g}$. Using this, we can construct the following map :

$$\psi : \partial_e S(A) \to \partial_e S(C), \quad g \mapsto \widetilde{g}|_C$$

To show that this map is well defined, let $g \in \partial_e S(A)$. Note that $\widetilde{g}$ is a state on $B(H)$, and $\widetilde{g}|_C$ is therefore a state on C, since positivity and unitality are clearly preserved under restriction. Now write $h = \widetilde{g}|_C$ and suppose $h = th_1 + (1-t)h_2$ for some $t \in (0,1)$ and $h_1, h_2 \in S(C)$. By Hahn-Banach extention theorem we can find $k_1 \in$Ext(h_1) and $k_2 \in$Ext(h_2). Then $k_1|_A = h_1|_A$ and $k_2|_A = h_2|_A$, so

$$\begin{aligned} g &= \widetilde{g}|_A \\ &= h|_A = th_1|_A + (1-t)h_2|_A \\ &= tk_1|_A + (1-t)k_2|_A \end{aligned}$$

However, $g \in \partial_e S(A)$, so $k_1|_A = k_2|_A = g$, i.e. $k_1, k_2 \in$Ext(g). So $k_1 = k_2 = \widetilde{g}$.

Then $h_1 = k_1|_C = \widetilde{g}|_C = h$ and likewise $h_2 = h$, i.e. $h \in \partial_e S(C)$, as desired.

The only thing left to show is that ϕ and psi are each other's inverse. First, let $g \in \partial_e S(A)$.

Then $(\phi \circ \psi)(g) = \widetilde{g}|_A = g$, since $\widetilde{g} \in$Ext(g). Hence $\phi \circ \psi = Id$.

Next, let $f \in \partial_e S(C)$. Choose $h \in$Ext(f), which exists by Hahn-Banach theorem. Then certainly $h \in$Ext$(f|_A)$. However, by assumption Ext$(f|_A)$ contains exactly one element, so $h = \widetilde{f|_A}$. Hence

$$(\psi \circ \phi)(f) = \widetilde{f|_A}|_C = h|_C = f,$$

since $h \in$Ext(f). Therefore, $\psi \circ \phi = Id$.

Hence $\phi : \partial_e S(C) \to \partial_e S(A)$ is a bijection. Also we proved earlier that it is also continuous. We know that $\partial_e S(A)$ and $\partial_e S(C)$ are both compact Hausdorff, so ϕ is in fact a homeomorphism. Therefore, ϕ induces an isomorphism

$$\phi^* : C(\partial_e S(A)) \to C(\partial_e S(C))$$

given by $\phi^*(F)(f) = F(\phi(f))$.

Using the Gelfand representation twice, i.e. using the isomorphisms

$$G_A : A \to C(\Omega(A)) = C(\partial_e S(A)), \quad (G_A(a))(f) = f(a)$$

and

$$G_C : C \to C(\Omega(C)) = C(\partial_e S(C)), \quad (G_C(c))(f) = f(c),$$

We can construct an isomorphism $F = G_C^{-1} \circ \phi^* \circ G_A$ such that the following diagram commutes :

$$\begin{array}{ccc} A & \xrightarrow{G_A} & C(\partial_e S(A)) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \phi^*} \\ C & \xrightarrow{G_C} & C(\partial_e S(C)) \end{array}$$

We now claim that F is in fact given by the inclusion map $i : A \to C$. To see this, let $a \in A$ and $f \in \partial_e S(C)$. Then :

$$\begin{aligned} ((\phi^* \circ G_A)(a))(f) &= \phi^*(G_A(a))(f) \\ &= G_A(a)(\phi(f)) \\ &= \phi(f)(a) \\ &= f|_A(a) \\ &= (f \circ i)(a) \\ &= f(i(a)) \\ &= G_C(i(a))(f) \\ &= ((G_C \circ i)(a))(f). \end{aligned}$$

Hence $\phi^* \circ G_A = G_C \circ i$, so indeed $i = G_C^{-1} \circ \phi^* \circ G_A = F$. So the inclusion map $i : A \to C$ is an isomorphism, i.e. $A = C$.

Therefore, A is maximal abelian. □

Thus, in our search for a classification of subalgebra with the Kadison-Singer property, we now merely have to focus on maximal abelian subalgebras.

4.2 Examples of maximal abelian C^*-subalgebras

It is time to give some key examples of maximal abelian C^*-subalgebras, since these are the only ones that can possess the Kadison-Singer property. Earlier we proved that $D_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ has the Kadison-Singer property and $D_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ is maximal abelian.

Proposition 4.2.1. *$D_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ is maximal abelian.*

Proof. Already done. □

4.2.1 The discrete subalgebra

One of the most inportant examples of a Hilbert space is the space $\ell^2(\mathbb{N})$, defined as

$$\ell^2(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{C} \quad | \quad \sum_{n\in\mathbb{N}} |f(n)|^2 < \infty\}.$$

This space has a natural inner product

$$\langle f, g\rangle = \sum_{n\in\mathbb{N}} \overline{f(n)} g(n),$$

which makes $\ell^2(\mathbb{N})$ a Hilbert space. $\ell^2(\mathbb{N})$ is separable because the functions $\{\delta_n\}_{n\in\mathbb{N}}$ defined by $\delta_n(m) = \delta_{nm}$ form a countable basis.

We can also consider the bounded functions on $\mathbb{N}$, given by

$$\ell^\infty(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{C} \quad | \quad sup_{n\in\mathbb{N}} |f(n)| < \infty\}.$$

It is clear that $\ell^\infty(\mathbb{N})$ is an abelian algebra under pointwise operations. Defining the adjoint operation pointwise as $f^*(n) = \overline{f(n)}$, $\ell^\infty(\mathbb{N})$ becomes a C^*-algebra in the norm

$$\|f\|_\infty = \sup_{n\in\mathbb{N}} |f(n)|.$$

Now we will state a very important theorem, by virtue of which we can identify $\ell^\infty(\mathbb{N})$ inside $B(\ell^2(\mathbb{N}))$ via multiplication operator.

Proposition 4.2.2. *The map* $M : \ell^\infty(\mathbb{N}) \to B(\ell^2(\mathbb{N})), \quad f \mapsto M_f$*, defined by*

$$(M_f(\phi))(n) = f(n)\phi(n),$$

is a well-defined norm-preserving injective **-homomorphism.*

Proof. First we check that the map is well defined, i.e. that $M_f \in B(\ell^2(\mathbb{N}))$ for each

$f \in \ell^\infty(\mathbb{N})$. Let $f \in \ell^\infty(\mathbb{N})$ and $\phi \in \ell^2(\mathbb{N})$. Now observe that

$$\begin{aligned}\|M_f(\phi)\|^2 &= \sum_{n\in\mathbb{N}} |(M_f(\phi))(n)|^2 \\ &= \sum_{\in\mathbb{N}} |f(n)|^2|\phi(n)|^2 \\ &\leq \|f\|_\infty^2 \sum_{n\in\mathbb{N}} |\phi(n)|^2 \\ &= \|f\|_\infty^2\|\phi\|^2,\end{aligned}$$

i.e.

$$\|M_f(\phi)\| \leq \|f\|_\infty\|\phi\|.$$

Hence $M_f \in B(\ell^2(\mathbb{N}))$ and $\|M_f\| \leq \|f\|_\infty$. Furthermore, for every $n \in \mathbb{N}$, $\|\delta_n\| = 1$, and $(M_f(\delta_n))(m) = f(m)\delta_{nm}$, so $\|M_f(\delta_n)\| = |f(n)|$. So for every $n \in \mathbb{N}$, $|f(n)| \leq \|M_f\|$.

Therefore, we also have $\|f\|_\infty \leq \|M_f\|$ and hence $\|f\|_\infty = \|M_f\|$. So, M is a well-defined norm-preserving map.

For injectivity, suppose that $f, g \in \ell^\infty(\mathbb{N})$ such that $M_f = M_g$. Then for any $n \in \mathbb{N}$,

$$\begin{aligned}f(n) &= M_f(\delta_n)(n) \\ &= M_g(\delta_n)(n) \\ &= g(n).\end{aligned}$$

Hence $f = g$, since $n \in \mathbb{N}$ was arbitrary.

By the following computations it follows that M is a homomorphism.

$$\begin{aligned}M_{\lambda f+g}(\phi)(n) &= (\lambda f + g)(n)\phi(n) \\ &= \lambda f(n)\phi(n) + g(n)\phi(n) \\ &= \lambda M_f(\phi)(n) + M_g(\phi)(n) \\ &= (\lambda M_f + M_g)(\phi)(n);\end{aligned}$$

$$\begin{aligned}M_{fg}(\phi)(n) &= (fg)(n)\phi(n)\\ &= f(n)g(n)\phi(n)\\ &= f(n)M_g(\phi)(n)\\ &= M_f(M_g(\phi))(n)\\ &= (M_f \circ M_g)(\phi)(n).\end{aligned}$$

To see that M preserves the *-operation, compute :

$$\begin{aligned}\langle \phi, M_{f^*}(\psi)\rangle &= \sum_{n\in\mathbb{N}} \overline{\phi(n)} M_{f^*}(\psi)(n)\\ &= \sum_{n\in\mathbb{N}} \overline{\phi(n) f(n)}\psi(n`)\\ &= \sum_{n\in\mathbb{N}} \overline{M_f(\phi)(n)}\psi(n)\\ &= \langle M_f(\phi), \psi\rangle.\end{aligned}$$

So, indeed, $M_{f^*} = (M_f)^*$. Hence M is a well-defined norm-preserving injective *-homomorphism.

□

By the above proposition we can identify $\ell^\infty(\mathbb{N})$ with the subalgebra $M(\ell^\infty(\mathbb{N}))$ of $B(\ell^2(\mathbb{N}))$. We will use this identification according to our need.

Proposition 4.2.3. *The subalgebra $\ell^\infty(\mathbb{N}) \subseteq B(\ell^2(\mathbb{N}))$ is maximal abelian.*

Proof. $\ell^\infty(\mathbb{N})$ is abelian, so $\ell^\infty(\mathbb{N}) \subseteq \ell^\infty(\mathbb{N})'$.

Now to prove the other way, let $T \in \ell^\infty(\mathbb{N})'$.

Define $f : \mathbb{N} \to \mathbb{C}$ by

$$f(n) := (T(\delta_n))(n).$$

For every $n \in \mathbb{N}$, $\|\delta_n\| = 1$, so

$$\begin{aligned}
|f(n)|^2 &= |(T(\delta_n))(n)|^2 \\
&\leq \sum_{m \in \mathbb{N}} |(T(\delta_n))(m)|^2 \\
&= \|T(\delta_n)\|^2 \\
&\leq \|T\|^2.
\end{aligned}$$

Therefore, $\underset{n \in \mathbb{N}}{sup} |f(n)| \leq \|T\|$, i.e., $f \in \ell^\infty(\mathbb{N})$.

Now take $\phi \in \ell^2(\mathbb{N})$. Then for any $n, m \in \mathbb{N}$ we have :

$$\begin{aligned}
(M_{\delta_n}(\phi))(m) &= \delta_{nm}\phi(m) \\
&= \phi(n)\delta_{nm} \\
&= \phi(n)\delta_n(m),
\end{aligned}$$

i.e. $M_{\delta_n}(\phi) = \phi(n)\delta_n$ for all $n \in \mathbb{N}$.

Therefore, for all $n \in \mathbb{N}$:

$$\begin{aligned}
(T(\phi))(n) &= ((M_{\delta_n}T)(\phi))(n) \\
&= ((TM_{\delta_n})(\phi))(n) \\
&= \phi(n)(T(\delta_n))(n) \\
&= \phi(n)f(n) \\
&= (M_f(\phi))(n),
\end{aligned}$$

where we used the fact that $T \in \ell^\infty(\mathbb{N})'$ and hence commutes with M_{δ_n}.

So, $T(\phi) = M_f(\phi)$, but $\phi \in \ell^2(\mathbb{N})$ was arbitary, so $T = M_f \in \ell^\infty(\mathbb{N})$. So $\ell^\infty(\mathbb{N})' \subseteq \ell^\infty(\mathbb{N})$.

Therefore $\ell^\infty(\mathbb{N}) = \ell^\infty(\mathbb{N})'$, so $\ell^\infty(\mathbb{N}) \subseteq B(\ell^2(\mathbb{N}))$ is maximal abelian. □

There is a considerable similarity between the case $D_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ and $\ell^\infty(\mathbb{N}) \subseteq B(\ell^2(\mathbb{N}))$; the latter can be viewed as the infinite-dimensional version of the first. We can make this observation more precise by rewriting the case $D_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ in a suitable fashion.

To do this, for every $n \in \mathbb{N}$ write $\underline{n} = \{1, \ldots, n\}$ and define

$$\ell(\underline{n}) = \{f \quad | \quad f : \underline{n} \to \mathbb{C}\}.$$

Note that in comparison with the infinite case, in this case it does not matter wheather we take all functions (like we did now), or the square-summable functions (which would give $\ell^2(\underline{n})$) or the bounded functions ($\ell^\infty(\underline{n})$), since these are all the same.

Furthermore, we can endow $\ell(\underline{n})$ with a canonical inner product

$$\langle f, g \rangle = \sum_{k \in \underline{n}} \overline{f(k)} g(k)$$

which makes $\ell(\underline{n})$ a Hilbert space. As a Hilbert space, $\ell(\underline{n})$ is clearly isomorphic to $\mathbb{C}^n$ under the canonical isomorphism

$$\ell(\underline{n}) \to \mathbb{C}^n, \quad f \mapsto (f(1), \ldots, f(n)).$$

This isomorphism induces an isomorphism between operators on $\ell(\underline{n})$ and operators on $\mathbb{C}^n$, explicitly given by

$$\phi : B(\ell(\underline{n})) \to M_n(\mathbb{C}), \quad \phi(T)_{ij} = (T(\delta_j))(i).$$

Just as in the infinite-dimensional case, we can define a multiplication operator

$$M : \ell(\underline{n}) \to B(\ell(\underline{n})), \quad f \mapsto M_f, \quad M_f(\phi)(m) = f(m)\phi(m).$$

Since we are now dealing with the finite case, there is no question whether this map is well defined, since all linear operators are automatically bounded. We can virtually copy the proof of earlier proposition and hence identify $\ell(\underline{n})$ with $M(\ell(\underline{n})) \subseteq B(\ell(\underline{n}))$.

We can now come to the main point : the diagonal matrices, as discussed urlier, exactly corresponds to the multiplication operators.

Proposition 4.2.4. *Suppose $n \in \mathbb{N}$. The restriction of the isomorphism $\phi : B(\ell(\underline{n})) \to M_n(\mathbb{C})$ to $\ell(\underline{n})$ gives an isomorphism between $\ell(\underline{n})$ and $D_n(\mathbb{C})$.*

Proof. Suppose $f \in \ell(\underline{n})$, then note that ϕ was given by

$$\phi(T)_{ij} = T(\delta_j)(i).$$

Hence

$$\begin{aligned}\phi(M_f)_{ij} &= (M_f(\delta_j))(i)\\ &= \delta_j(i)f(i)\\ &= \delta_{ji}f(i),\end{aligned}$$

so $\phi(M_f)_{ij} = 0$ if $i \neq j$, so $\phi(M_f) \in D_n(\mathbb{C})$. Next, let $N \in D_n(\mathbb{C})$ and note that there is an explicit inverse ψ of ϕ, given by

$$\psi(M)(f)(m) = \sum_k M_{mk} f(k).$$

So, since $N \in D_n(\mathbb{C})$, $\psi(N)(f)(m) = N_{nm}f(m) = M_g(f)(m)$, with $g \in \ell(\underline{n})$ given by $g(m) = N_{mm}$. Therefore $\psi(N) = M_g \in \ell(\underline{n})$. So, indeed, the restriction of ϕ gives an isomorphism between $\ell(\underline{n})$ and $D_n(\mathbb{C})$. □

Summarizing, we see that the finite-dimensional case and the infinite-dimensional case are not that different. Therefore, we introduce one general description.

Let $\aleph_0$ denote the cardinality of $\mathbb{N}$ and write $\underline{\aleph_0} = \mathbb{N}$. The expression $'1 \leq j \leq \aleph_0'$ means either $'j \in \mathbb{N}$ or $j = \aleph_0.'$ This can be made more precise by adding a maximal element $\aleph_0$ to the totally ordered set $\mathbb{N}$.

Definition 4.2.1. *Let $1 \leq j \leq \aleph_0$. Then $A_d(j)$ is the subalgebra $\ell^\infty(\underline{j}) \subseteq B(\ell^2(\underline{j}))$ that acts on the Hilbert space $\ell^2(\underline{j})$ via multiplication operator. We call $A_d(j)$ the discrete subalgebra of cardinality j.*

Note that we have used the identification $\ell(\underline{j} = \ell^2(\underline{j}) = \ell^\infty(\underline{j})$ for $j \in \mathbb{N}$. Discrete subalgebras provide key examples of maximal abelian subalgebras and will play a major role in our further discussion.

4.2.2 The continuous subalgebra

Another important example of a maximal abelian subalgebra is non-discrete. As an introduction to this example, we consider all measurable functions from $[0,1]$ to $\mathbb{C}$:

$$\mathcal{F}[0,1] := \{f : [0,1] \to \mathbb{C} \quad | \quad f \quad is \quad measurable\},$$

where we use the standard Lebesgue measure μ on $[0,1]$. We define a relation $\sim$ on $\mathcal{F}[0,1]$ by

$$f \sim g \iff \mu(\{x \in [0,1] : f(x) \neq g(x)\}) = 0$$

We sometimes denote the latter condition $\mu(f \neq g) = 0$. It is clear that $\sim$ is an equivalence relation on $\mathcal{F}[0,1]$, so we can define :

$$F[0,1] := \mathcal{F}[0,1]/\sim .$$

We denote equivalence classes in $F[0,1]$ by $[f]$, where $f \in \mathcal{F}[0,1]$ is a representative.

$F[0,1]$ is an algebra under the canonical operations

$$\lambda f + [g] = [\lambda f + g]$$

and

$$[f][g] = [fg].$$

Lemma 4.2.1. *The function*

$$I_2 : F[0,1] \to [0,\infty], \quad [f] \mapsto \int_{[0,1]} |f(x)|^2 dx$$

is well defined.

Proof. All we need to do is show that if $[f] = [g]$, then $I_2([f]) = I_2([g])$, i.e. the definition of I_2 is independent of the choice of representative. However, if $[f] = [g]$, then $\mu(f \neq g) = 0$, so there is an $A \subset [0,1]$ such that $f(x) = g(x)$ for all $x \in X \setminus A$ and $\mu(A) = 0$, so :

$$\begin{aligned}\int_{[0,1]} |f(x)|^2 dx &= \int_{X\setminus A} |f(x)|^2 dx \\ &= \int_{X\setminus A} |g(x)|^2 dx \\ &= \int_{[0,1]} |g(x)|^2 dx.\end{aligned}$$

So, indeed, $I_2([f]) = I_2([g])$, i.e. I_2 is well defined.

□

Using this lemmma, we can define a new space, which we call the space of square-integrable functions :

$$L^2[0,1] := \{\psi \in F[0,1] \quad | \quad I_2(\psi) < \infty\}.$$

One of the most important results of basic functional analysis is that $L^2[0,1]$ is a Hilbert space with respect to the inner product $\langle , \rangle$, given by :

$$\langle [f],[g] \rangle = \int_{[0,1]} \overline{f(x)} g(x) dx.$$

The equivalence relation $\sim$ is necessary in the construction of $L^2[0,1]$ in order for the inner product on $L^2[0,1]$ to be positive definite. Note that the norm induced by this inner product satisfies $\|\psi\|^2 = I_2(\psi)$.

There is a certain kind of analogy between $L^2[0,1]$ and $\ell^2(\mathbb{N})$, by replacing sums by integrals. Just as in the case of $\ell^2(\mathbb{N})$ one could again want to define the space of bounded functions. Because we are dealing with equivalence classes of functions, we need to define this property : we put

$$L^\infty[0,1] := \{\psi \in F[0,1] \quad | \quad \exists f \in \psi : \underset{x \in [0,1]}{sup} |f(x)| < \infty\}.$$

This is called the space of essentially bounded functions, coming with a natural norm:

$$\|\psi\|_\infty^{(ess)} = \underset{f \in \psi}{inf} \{k \in [0, \infty) : |f(x)| \leq k \forall x \in [0,1]\}.$$

If we include the operation $[f]^* = [\overline{f}]$, then $L^\infty[0,1]$ becomes a C^*-algebra.

Now we have made our set-up : similar to the previous example, we want to regard $L^\infty[0,1]$ as a subalgebra of $B(L^2[0,1])$. Again, we do this by means of a multiplication operator :

$$M : L^\infty[0,1] \to B(L^2[0,1]), \quad \psi \mapsto M_\psi,$$

where $M_{[f]}([g]) = [fg]$.

Proposition 4.2.5. *M is a well-defined injective, norm-preserving, *-homomorphism.*

Proof. First of all, we check that the definition is independent of choice of representatives. So suppose $[f_1] = [f_2] \in L^\infty[0,1]$ and $[g_1] = [g_2] \in L^2[0,1]$. Then

$$\begin{aligned} M_{[f_1]}([g_1]) &= [f_1 g_1] \\ &= [f_1][g_1] \\ &= [f_2][g_2] \\ &= [f_2 g_2] \\ &= M_{[f_2]}([g_2]), \end{aligned}$$

so indeed, the definition is independent of choice of representatives.

Next, we need to check that $M_\psi \in B(L^2[0,1])$ for all $\psi \in L^\infty[0,1]$. So let $\psi \in L^{infty}[0,1]$ and let $f \in \psi$ be such that $\underset{x\in[0,1]}{sup} |f(x)| < \infty$, say $\underset{x\in[0,1]}{sup} |f(x)| = k$. Then for any $[g] \in L^2[0,1]$, we have :

$$\begin{aligned} I_2([fg]) &= \int_{[0,1]} |f(x)g(x)|^2 dx \\ &= \int_{[0,1]} |f(x)|^2 |g(x)|^2 \\ &\leq k^2 \int_{[0,1]} |g(x)|^2 \\ &= k^2 I_2([g]). \end{aligned}$$

Since $[g] \in L^2[0,1]$, we therefore have $I_2([fg]) < \infty$, i.e. $[fg] \in L^2[0,1]$, so indeed $M_\psi : L^2[0,1] \to L^2[0,1]$.

Furthermore, for the same f and g.

$$\begin{aligned} \|[fg]\|^2 &= I_2([fg]) \\ &\leq k^2 I_2([g]) \\ &= k^2 \|[g]\|^2, \end{aligned}$$

whence $\|M_\psi([g])\| = \|[fg]\| \leq k\|[g]\|$, so in fact $M_\psi \in B(L^2[0,1])$. Also, by the above inequality, $\|M_\psi\| \leq \|\psi\|_\infty^{(ess)}$ for all $\psi \in L^\infty[0,1]$. Now let $\psi \in L^\infty[0,1]$ and let $\epsilon > 0$. Furthermore, let $f \in \psi$ and define :

$$A_f = \{x \in [0,1] : |f(x)| \geq \|\psi\|_\infty^{(ess)} - \epsilon\}.$$

We claim that $\mu(A_f) \neq 0$. We argue by contraposition, so suppose $\mu(A_f) = 0$. Then define $h = f1_{A_f}$. Since $\mu(A_f) = 0$, $[h] = [f] = \psi$. However, for all $x \in [0,1]$, we then have $|h(x)| < \|\psi\|_\infty^{(ess)} - \epsilon$, so

$$\underset{x\in[0,1]}{sup} |h(x)| \leq \|\psi\|_\infty^{(ess)} - \epsilon.$$

Since $[h] = \psi$, then : $\|\psi\|_\infty^{(ess)} \leq \|\psi\|_\infty^{(ess)} - \epsilon$. This is a contradiction, so indeed $\mu(A_f) \neq 0$.

Therefore, $1_{A_f} \neq [0]$. Furthermore, $[1_{A_f}] \in L^2[0,1]$, so we can compute :

$$\begin{aligned} \|M_{[f]}([1_{A_f}])\|^2 &= \|[f1_{A_f}]\|^2 \\ &= \int_{A_f} |f(x)|^2 dx \\ &\geq (\|\psi\|_\infty^{(ess)} - \epsilon)^2 \mu(A_f) \\ &= (\|\psi\|_\infty^{(ess)} - \epsilon)^2 \|1_{A_f}\|^2 \end{aligned}$$

Since $[1_{A_f}] \neq 0$, then $\|M_\psi\| = \|M_{[f]}\| \geq \|\psi\|_\infty^{(ess)} - \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\|M_\psi\| \geq \|\psi\|_\infty^{(ess)}$.

Therefore $\|M_\psi\| = \|\psi\|_\infty^{(ess)}$ for all $\psi \in L^\infty[0,1]$, so M is indeed norm-preserving.

M is clearly a homomorphism, by definition of the algebraic operations on $F[0,1]$ (i.e. $\lambda[f] + [g] = [\lambda f + g]$ and $[f][g] = [fg]$). To see that M also preserves the adjoint operation, compute :

$$\begin{aligned} \langle M_{[f]^*}([g]), [h]\rangle &= \langle M_{[\overline{f}]}([g]), [h]\rangle \qquad\qquad = \langle[\overline{f}g], [h]\rangle \\ &= \int_{[0,1]} f(x)\overline{g(x)}h(x)dx \\ &= \int_{[0,1]} \overline{g(x)}f(x)h(x)dx \\ &= \langle[g], [fh]\rangle \\ &= \langle[g], M_{[f]}([h])\rangle. \end{aligned}$$

So, indeed, $M_{\psi^*} = (M_\psi)^*$ for all $\psi \in L^\infty[0,1]$. Therefore M is indeed a *-homomorphism.

Lastly, for injectivity, suppose that $\phi, \psi \in L^\infty[0,1]$ such that $M_\phi = M_\psi$. Then

$M_{\phi-\psi} = 0$, so $\|\phi - \psi\|_\infty^{(ess)} = \|M_{\phi-\psi}\| = 0$. Hence $\phi - \psi = 0$, i.e. $\phi = \psi$ and M is injective.

□

So, we can regard $L^\infty[0,1]$ as a C^*-subalgebra of $B(L^2[0,1])$, where we tacitly identify $L^\infty[0,1]$ with its image under M. Of course, $L^\infty[0,1]$ is an abelian subalgebra. We introduced this example since it is maximal abelian.

Theorem 4.2.1. $L^\infty[0,1] \subseteq B(L^2[0,1])$ *is maximal abelian.*

Proof. $L^\infty[0,1]$ is abelian, so $L^\infty[0,1] \subseteq L^\infty[0,1]'$.

For the other inclusion, suppose that $T \in L^\infty[0,1]'$. Note that $I_2([1]) = 1$, so $[1] \in L^2[0,1]$. Therefore, we can define $\psi = T([1]) \in L^2[0,1]$. We claim that $\psi \in L^\infty[0,1]$.

To see this, we argue by contraposition, so we suppose that $\psi \in L^\infty[0,1]$. Now let $f \in \psi$ and for every $N \in \mathbb{N}$, define :

$$A_N := \{x \in [0,1] : |f(x)| \geq N\}.$$

Since $\psi \notin L^\infty[0,1]$, for every $N \in \mathbb{N}$, $\mu(A_N) \neq 0$. Since $1_{A_N} \in L^\infty[0,1]$, we can compute :

$$T([1_{A_N}]) = T(M_{[1_{A_N}]}([1])) = M_{[1_{A_N}]}(T([1])) = M_{[1_{A_N}]}([f]) = [f1_{A_N}].$$

Therefore, we also have :

$$\begin{aligned} N^2\mu(A_N) &\leq \int_{A_N} |f(x)|^2 dx \\ &= \|[f1_{A_N}]\|^2 \\ &= \|T([1_{A_N}])\|^2 \\ &\leq \|T\|^2\|[1_{A_N}]\|^2 \\ &= \|T\|^2\mu(A_N). \end{aligned}$$

Since $\mu(A_N) \neq 0$, $N \leq \|T\|$ for all $N \in \mathbb{N}$. However, $T \in B(L^2[0,1])$, so this is a contradiction. Hence $\psi \in L^\infty[0,1]$.

We now claim that $T + M_\psi$. To see this, let $\phi \in L^2[0,1]$ and let $g \in \phi$. For each $n \in \mathbb{N}$ define

$$U_n := \{x \in [0,1] : |g(x)| \le n\},$$

and $g_n := g1_{U_n}$. Note that the sequence of functions $f_i : [0,1] \to [0,\infty)$ defined by $f_i(x) = |g_i(x)|^2$ is pointwise non-decreasing and has $f : [0,1] \to [0,\infty)$, $f(x) = |g(x)|^2$, as its pointwise limit. Hence, by Lebesgue's monotone convergence theorem,

$$\begin{aligned} \underset{n\to\infty}{lim} \|[g_n]\|^2 &= \underset{n\to\infty}{lim} \int_{[0,1]} |g_n(x)|^2 dx \\ &= \int_{[0,1]} |g(x)|^2 dx \\ &= \|[g]\|^2. \end{aligned}$$

Furthermore,

$$\begin{aligned} \|[g] - [g_n]\|^2 &= \int_{[0,1]\setminus U_n} |g(x)|^2 dx \\ &= \int_{[0,1]} |g(x)|^2 dx - \int_{U_n} |g(x)|^2 dx \\ &= \|[g]\|^2 - \|[g_n]\|^2, \end{aligned}$$

whence $\underset{n\to\infty}{lim} \|[g] - [g_n]\| = 0$, i,e, $\underset{n\to\infty}{lim} [g_n] = [g]$.

Choose $h \in \psi$. Since $[g_n] \in L^\infty[0,1]$, we can compute :

$$\begin{aligned} T([g_n]) &= T(M_{[g_n]}([1])) \\ &= M_{[g_n]}(T([1])) \\ &= M_{[g_n]}([h]) \\ &= [g_n h] \\ &= M_{[h]}([g_n]) \\ &= M_\psi([g_n]). \end{aligned}$$

Then also, by continuity of both T and M_ψ,

$$\begin{aligned} T([g]) &= T(\lim_{n\to\infty}[g_n]) \\ &= \lim_{n\to\infty} T([g_n]) \\ &= \lim_{n\to\infty} M_\psi([g_n]) \\ &= M_\psi(\lim_{n\to\infty}[g_n]) \\ &= M_\psi([g]). \end{aligned}$$

Therefore, $T(\phi) = M_\psi(\phi)$. Since $\phi \in L^2[0,1]$ was arbitrary, $T = M_\psi$. So, $T \in L^\infty[0,1]$.

Hence $L^\infty[0,1]' \subseteq L^\infty[0,1]$.

Therefore, $L^\infty[0,1]' = L^\infty[0,1]$, i.e. $L^\infty[0,1]$ is maximal abelian.

□

Along the lines of the definition of the discrete subalgebra of cardinality j (i.e. $A_d(j)$), we introduce a special short notation for the subalgebra $L^\infty[0,1] \subseteq B(L^2[0,1]$.

Definition 4.2.2. *We denote the maximal abelian subalgebra $L^\infty[0,1]$ of $B(L^2[0,1])$ by A_c, realized via multiplication operator. We call A_c the continuous subalgebra.*

4.2.3 The mixed subalgebra

Combining two different examples of maximal abelian subalgebras, one can construct another example of a maximal abelian subalgebra.

Proposition 4.2.6. *Suppose $A_1 \subseteq B(H_1)$ and $A_2 \subseteq B(H_2)$ are both maximal abelian C^*-subalgebras. Then $A_1 \oplus A_2 \subseteq B(H_1 \oplus H_2)$ is maximal abelian.*

Proof. Since $A_1 \oplus A_2(j)$ is a pointwise defined subalgebra of $B(H_1 \oplus H_2)$ and both A_1 and A_2 are abelian, $A_1 \oplus A_2$ is abelian.

Therefore $A_1 \oplus A_2 \subseteq (A_1 \oplus A_2)'$.

For the other way, suppose that $T \in (A_1 \oplus A_2)'$. Define $T_1 = \pi_1 \circ T \circ i_1$ and $T_2 = \pi_2 \circ T \circ i_2$. Since T is bounded, $T_1 \in B(H_1)$ and $T_2 \in B(H_2)$.

Now note that for any $x \in H_1$ and $y \in H_2$,

$$\begin{aligned} T(x,y) &= T(i_1(x) + i_2(y)) \\ &= T(i_1(x)) + T(i_2(y)) \\ &= (T \circ (1,0) \circ i_1)(x) + (T \circ (0,1) \circ i_2)(y) \\ &= ((1,0) \circ T \circ i_1)(x) + ((0,1) \circ T \circ i_2)(y) \\ &= ((\pi_1 \circ T \circ i_1)(x), 0) + (0, (\pi_2 \circ T \circ i_2)(y)) \\ &= (T_1(x), 0) + (0, T_2(y)) \\ &= (T_1(x), T_2(y)), \end{aligned}$$

where we used the fact that T commutes with (1,0) and (0,1), since $T \in (A_1 \oplus A_2)'$.

Therefore, $T = (T_1, T_2)$. Now, for all $a \in A_1$,

$$(T_1 \circ a, 0) = T \circ (a, 0) = (a, 0) \circ T = (a \circ T_1, 0)$$

Therefore, $T_1 \in A_1' = A_1$. Likewise, $T_2 \in A_2$. Hence $T = (T_1, T_2) \in A_1 \oplus A_2$, i.e. $(A_1 \oplus A_2)' \subseteq A_1 \oplus A_2$. Therefore

$$(A_1 \oplus A_2)' = A_1 \oplus A_2,$$

i.e. $A_1 \oplus A_2 \subseteq B(H_1 \oplus H_2)$ is maximal abelian.

□

Since we are interested in the question whether a maximal abelian subalgebra possesses the Kadison-Singer property, we would like to make a connection between the Kadison-Singer property for a direct sum $A_1 \oplus A_2$ and the Kadison-Singer property of A_1 and A_2 separately. It turns out that we can do this. First of all, we need to describe the characters (and hence the pure states) of a direct sum. For this, note that for a state $f \in S(A_i)$, the pullback over the projection $\pi_i : A_1 \oplus A_2 \to A_i$, i.e. $\pi_i^*(f) = f \circ \pi_i$, gives a map $\pi_i : A_1 \oplus A_2 \to \mathbb{C}$.

Proposition 4.2.7. *Suppose A_1 and A_2 are both C^*-algebras. Then*

$$\Omega(A_1 \oplus A_2) = \pi_1^*(\Omega(A_1)) \cup \pi_2^*(\Omega(A_2)).$$

Proof. Suppose $f \in \Omega(A_1 \oplus A_2)$. Then

$$f((0,1))^2 = f((0,1)^2) = f((0,1)),$$

so $f((0,1)) \in \{0,1\}$. Likewise $f((1,0)) \in \{0,1\}$. However, we also have

$$f((0,1)) + f((1,0)) = f((1,1)) = f(1) = 1,$$

so there are two cases. Either $f((1,0)) = 1$ and $f((0,1)) = 0$, or $f((1,0)) = 0$ and $f((0,1)) = 1$.

Suppose the first case is true. Then define $g_1 : A_1 \to \mathbb{C}$ by $g(a) = f(a,0)$. Then $g(1) = 1$, so g is non-zero and for any $a_1, A_2 \in A_1$ we have

$$g(a_1 a_2) = f((a_1 a_2, 0)) = f((a_1, 0))f((a_2, 0) = g(a_1)g(a_2)),$$

so $g \in \Omega(A_1)$. Furthermore, for any $(a_1, a_2) \in A_1 \oplus A_2$ we have

$$\begin{aligned} f((a_1, a_2)) &= f((a_1, 0)) + f((0, a_2)) \\ &= f((a_1, 0))f((1, 0)) + f((0, a_2)) \\ &= f(a_1, 0) \\ &= g(a_1) \\ &= (g \circ \pi_1)((a_1, a_2)), \end{aligned}$$

i.e. $f = \pi_1^*(g)$, so $f \in \pi_1^*(\Omega(A_1))$.

If the second case is true, it follows likewise that $f \in \pi_2^*(S(A_2))$. Hence

$$\Omega(A_1 \oplus A_2) \subseteq \pi_1^*(\Omega(A_1)) \cup \pi_2^*(\Omega(A_2)).$$

Now suppose that $h \in \pi_1^*(\Omega(A_1))$. Then $h = k \circ \pi_1$ for some $k \in \Omega(A_1)$, so

$$h(1) = h((1,1)) = k(1) = 1,$$

i.e. h is non-zero. Furthermore, h is clearly linear and for any $(a_1, a_2), (b_1, b_2) \in A_1 \oplus A_2$, we have

$$h((a_1, a_2)(b_1, b_2)) = h((a_1 b_1, a_2 b_2)) = k(a_1 b_1) = k(a_1)k(b_1) = h((a_1, a_2))h((b_1, b_2)),$$

i.e. $h \in \Omega(A_1 \oplus A_2)$. Therefore, $\pi_1^*(A_1) \subseteq \Omega(A_1 \oplus A_2)$.

Likewise, $\pi_2^*(A_2) \subseteq \Omega(A_1 \oplus A_2)$.

So indeed,

$$\Omega(A_1 \oplus A_2) = \pi_1^*(\Omega(A_1)) \cup \pi_2^*(\Omega(A_2)).$$

□

The above proposition gives us information about the pure states on a direct sum of abelian subalgebras, since the pure states are exactly the characters. Next, we need to make a connection between the concepts of positivity and direct sums of operator algebras.

Lemma 4.2.2. *Suppose H_1 and H_2 are Hilbert spaces and $b \in B(H_1 \oplus H_2)$ is positive. Then for $j \in \{1, 2\}$, $\pi_j b i_j \in B(H_j)$ is positive.*

Proof. Let $(x, y) \in H_1 \oplus H_2$. Then compute :

$$\begin{aligned}\langle(\pi_1 b i_1)(x), x\rangle &= \langle(\pi_1 b)(x, 0), x\rangle \\ &= \langle(\pi_1 b)(x, 0), x\rangle + \langle(\pi_2 b)(x, 0), 0\rangle \\ &= \langle b(x, 0), (x, 0)\rangle \geq 0,\end{aligned}$$

since b is positive. Therefore, $\pi_1 b i_1$ is positive. Likewise, $\pi_2 b i_2$ is positive. □

We use these results to prove the following theorem about the connection between direct sums and the Kadison-Singer property.

Theorem 4.2.2. *Suppose H_1 and H_2 are Hilbert spaces. Furthermore, let $A_1 \subseteq B(H_1)$ and $A_2 \subseteq B(H_2)$ be abelian C^*-subalgebras such that $A_1 \oplus A_2 \subseteq B(H_1 \oplus H_2)$ has the Kadison-Singer property. Then $A_1 \subseteq B(H_1)$ and $A_2 \subseteq B(H_2)$ have the Kadison-Singer property.*

Proof. Suppose $f \in \partial_e S(A_1)$ and $g_1, g_2 \in$Ext$(f)\subseteq B(H_1)$. Then $f \in \Omega_1$, so by lemma $\pi_1^*(f) \in \Omega(A_1 \oplus A_2) = \partial_e S(A_1 \oplus A_2)$.

Now define the linear functional $k_1, k_2 : B(H_1 \oplus H_2) \to \mathbb{C}$ by $k_j(b) = g_j(\pi_1 b i_1)$ for all $b \in B(H_1 \oplus H_2)$ and $j \in \{1, 2\}$, $k_j(1) = g_j(\pi_1 i_1) = g_j(1) = 1$, since g_j is a state. Furthermore for a positive $b \in B(H_1 \oplus H_2)$, $\pi_1 b i_1 \in B(H_1$ is positive by previous lemma.

Therefore, $k_j(b) = g_j(\pi_1 b i_1) \geq 0$, since g_j is positive.

Hence $k_1, k_2 \in S(B(H_1 \oplus H_2))$.

Now, for an element $(a_1, a_2) \in A_1 \oplus A_2$, $\pi_1(a_1, a_2)i_1 = a_1$, so

$$\begin{aligned} k_j((a_1, a_2)) &= g_j(\pi_1(a_1, a_2)i_1) \\ &= g_j(a_1) \\ &= f(a_1) \\ &= (f \circ \pi_1)(a_1, a_2) \\ &= \pi_1^*(f)((a_1, a_2)), \end{aligned}$$

i.e. $k_1, k_2 \in$Ext$(\pi_1^*(f))$. However, by assumption, $A_1 \oplus A_2 \subseteq B(H_1 \oplus H_2)$ has the Kadison-Singer property, so Ext$(\pi_1^*(f))$ has at most one element, i.e. $k_1 = k_2$.

For any $b \in B(H_1)$, $b = \pi_1(b, 0)i_1$, so we have

$$g_1(b) = g_1(\pi_1(b, 0)i_1) = k_1((b, 0)) = k_2((b, 0)) = g_2((\pi_1(b, 0)i_1)) = g_2(b),$$

i.e. $g_1 = g_2$. Therefore, Ext(f) has at most one element. Hence, Ext(f) has exactly one element.

Therefore, $A_1 \subseteq B(H_1)$ has the Kadison-Singer property.

Likewise, $A_2 \subseteq B(H_2)$ has the Kadison-Singer property.

□

As a special example of a direct sum, we can combine the discrete subalgebra $A_d(j)$ for some $1 \leq j \leq \aleph_0$ with the continuous example A_c. To do this, define

$$H_j := L^2[0, 1] \oplus \ell^2(\underline{j}).$$

We will call the maximal abelian subalgebra $A_c \oplus A_d(j) \subseteq B(H_j)$ the mixed subalgebra.

As it will turn out later, this is in some way the only direct sum that we need to consider.

By now, we have constructed three different examples : the discrete, continuous and mixed subalgebra. These are all examples with a separable Hilbert space. In our search for examples of maximal abelian subalgebras that satisfy the Kadison-Singer property, we will restrict ourselves to this kind of Hilbert spaces, since it turns out that we can make a complete classification of abelian subalgebras with the Kadison-Singer property when we only consider separable Hilbert spaces.

So far we prove some important results in this chapter. These are :

1. A subalgebra is maximal abelian iff it is self commutant.

2. Only maximal abelian unital C^*-subalgebras possibly can have the Kadison-Singer property.

   So it reduces our job. Now we only need to classify all possible maximal abelian unital C^*-subalgebras and check among these which has the Kadison-Singer property.

3. The subalgebra $\ell^\infty(\mathbb{N}) \subseteq B(\ell^2(\mathbb{N}))$ is maximal abelian.

4. The subalgebra $L^\infty[0,1] \subseteq B(L^2[0,1])$ is maximal abelian.

5. Suppose $A_1 \subseteq B(H_1)$ and $A_2 \subseteq B(H_2)$ are both maximal abelian C^*-subalgebras. Then $A_1 \oplus A_2 \subseteq B(H_1 \oplus H_2)$ is maximal abelian.

6. Suppose H_1 and H_2 are Hilbert spaces. Furthermore, let $A_1 \subseteq B(H_1)$ and $A_2 \subseteq B(H_2)$ be abelian C^*-subalgebras such that $A_1 \oplus A_2 \subseteq B(H_1 \oplus H_2)$ has the Kadison-Singer property. Then $A_1 \subseteq B(H_1)$ and $A_2 \subseteq B(H_2)$ have the Kadison-Singer property.

In the next chapter we are going classify the maximal abelian unital C^*-subalgebras upto unitary equivalence.

Chapter 5

Classification of MASA

Recall that we are considering maximal abelian C^*-subalgebras of $B(H)$, for some Hilbert space H. Note that a maximal abelian C^*-subalgebra $A \subseteq B(H)$ satisfies $A' = A$ and A' is a von Neumann algebra. Therefore, every maximal abelian C^*-subalgebra is a von Neumann algebra. Furthermore, every von Neumann algebra is a C^*-algebra, so certainly every maximal abelian von Neumann algebra (i.e. a von Neumann algebra A that satisfies $A' = A$) is a maximal abelian C^*-algebra. Hence we see that the maximal abelian von Neumann algebras are exactly the maximal abelian C^*-algebras.

We will first show that it is only necessary to classify all maximal abelian subalgebras up to unitary equivalence, in order to determine whether they satisfy the Kadison-Singer property. Next, we restrict ourselves to separable Hilbert spaces and by considering maximal abelian subalgebras to be von Neumann algebras, we can classify these subalgebras up to unitary equivalence, by using the existence and properties of minimal projections. Together, this greatly simplifies the classification of subalgebras with the Kadison-Singer property in the case of separable Hilbert spaces.

5.1 Unitary equivalence

The classification of maximal abelian von Neumann algebras is up to so-called unitary equivalence. For this, we need unitary elements.

Definition 5.1.1. *Suppose H and H' are Hilbert spaces. Then $u \in B(H, H'$ is called unitary if for all $x, y \in H$, $\langle ux, uy \rangle = \langle x, y \rangle$ and $u(H) = H'$.*

The above conditions for being unitary are not always the easiest to check. However, there is an equivalent definition.

Proposition 5.1.1. *Suppose H, H' are Hilbert spaces and $u \in B(H, H')$. Then u is unitary if and only if $u^*u = 1$ and $uu^* = 1$.*

Proof. Suppose u is unitary. Then $\langle u^*ux, x\rangle = \langle ux, ux\rangle = \langle x, x\rangle$ for every $x \in H$, so we have $u^*u = 1$.

Next, let $x' \in H'$. Then $x' = u(y)$ for some $y \in H$, so

$$\begin{aligned}\langle uu^*x', x'\rangle &= \langle uu^*uy, uy \\ &= \langle uy, uy\rangle \\ &= \langle x', x'\rangle.\end{aligned}$$

Since $x' \in H'$ was arbitrary, $uu^* = 1$.

For the converse, suppose that $uu^* = 1$ and $u^*u = 1$. Then for any $x, y \in H$,

$$\begin{aligned}\langle ux, uy\rangle &= \langle u^*ux, y\rangle \\ &= \langle x, y\rangle.\end{aligned}$$

Furthermore, for $x' \in H'$, $x' = u(u^*x')$, so $x' \in u(H)$, i.e. $H' = u(H)$. So u is indeed unitary.

□

Using unitary elements, we can define the notion of unitary equivalence of subalgebras of $B(H)$.

Definition 5.1.2. *Suppose H_1 and H_2 are Hilbert spaces and $A_1 \subseteq B(H_1)$, $A_2 \subseteq B(H_2)$ are subalgebras. Then A_1 is called unitarily equivalent to A_2 if there is a unitary $u \in B(H_1, H_2)$ such that $uA_1u^* = A_2$. We denote this by $A_1 \cong A_2$.*

The following lemma is easily proven, but it is essential for our classification.

Lemma 5.1.1. *Unitary equivalence is an equivalence relation.*

Proof. Suppose $A_1 \subseteq B(H_1)$, $A_2 \subseteq B(H_2)$ and $A_3 \subseteq B(H_3)$ such that $A_1 \cong A_2$ and $A_2 \cong A_3$.

Then $1 \in B(H_1)$ is unitary and $1A_11^* = A_1$.

Therefore unitary equivalence is reflexive.

Since $A_1 \cong A_2$, there is a unitary $u \in B(H_1, H_2)$ such that $uA_1u^* = A_2$. Now $u^* \in B(H_1, H_2)$ is unitary too, and

$$u^*A_2u = u^*uA_1u^*u = A_1.$$

Hence unitary equivalence is symmetric.

Since $A_2 \cong A_3$, there is a unitary $v \in B(H_2, H_3)$ such that $vA_2v^* = A_3$. Then $vu \in B(H_1, H_3)$ is a unitary too, and

$$vuA_1(vu)^* = vuA_1u^*v^* = vA_2v^* = A_3.$$

So, unitary equivalence is transitive.

Hence $A_1 \cong A_1$, $A_2 \cong A_1$ and $A_1 \cong A_3$.

Therefore, unitary equivalence is an equivalence relation.

□

One of the crucial steps in this chapter is the following theorem: it shows that we only have to consider subalgebras up to unitary equivalence when determining whether the subalgebra satisfies the Kadison-Singer property.

Theorem 5.1.1. *Suppose that H_1 and H_2 are Hilbert spaces and $A_1 \subseteq B(H_1)$ and $A_2 \subseteq B(H_2)$ are unital abelian subalgebras that are unitarily equivalent. Then A_1 has the Kadison-Singer property if and only if A_2 has the Kadison-Singer property.*

Proof. Suppose that A_1 has the Kadison-Singer property. By assumption, there is a unitary $u \in B(H_1, H_2)$ such that $uA_1u^* = A_2$.

Now let $f \in \partial_e S(A_2)$. Then define $g : A_1 \to \mathbb{C}$ by $g(a) = f(uau^*)$. We first claim that $g \in S(A_1)$. To see this, first let $a \in A_1$ and observe that

$$g(a^*a) = f(ua^*au^*) = f((au^*)^*(au)) \geq 0,$$

since f is positive. Hence g is positive. Furthermore, $g(1) = f(uu^*) = f(1) = 1$, so g is unital too. Hence, indeed $g \in S(A_1)$.

Next, we prove that in fact $g \in \partial_e S(A_1)$. To see this, suppose that $h_1, h_2 \in S(A_1)$ and $t \in (0, 1)$ such that $g = th_1 + (1 - t)h_2$.

Now define $k_1 : A_2 \to \mathbb{C}$ by $k_1(a) = h_1(u^*au)$ for all $a \in A_2$ and likewise define $k_2 : A_2 \to \mathbb{C}$ by $k_2(a) = h_2(u^*au)$ for all $a \in A_2$. Then by the same reasoning as above, $k_1, k_2 \in S(A_2)$. Furthermore, for $a \in A_2$,

$$\begin{aligned} f(a) &= f(uu^*auu^*) \\ &= g(u^*au) \\ &= th_1(u^*au) + (1-t)h_2(u^*au) \\ &= tk_1(a) + (1-t)k_2(a), \end{aligned}$$

i.e. $f = tk_1 + (1-t)k_2$. However, $f \in \partial_e S(A_2)$ by assumption, so $f = k_1 = k_2$. Then for $a \in A_1$:

$$\begin{aligned} h_1(a) &= h_1(u^*uau^*u) \\ &= k_1(uau^*) \\ &= f(uau^*) \\ &= g(a), \end{aligned}$$

i.e. $h_1 = g$. Likewise, $h_2 = g$, so indeed $g \in \partial_e S(A_1)$.

We want to prove that Ext(f) contains exactly one element. By Hahn-Banach theorem, we know that Ext$(f) \neq \emptyset$.

Therefore, suppose that $c, d \in$Ext$(f) \subseteq S(B(H_2))$. Then define $\widetilde{c} : B(H_1) \to \mathbb{C}$ by $\widetilde{c}(b) = c(ubu^*)$ and likewise $\widetilde{d} : B(H_1) \to \mathbb{C}$ by $\widetilde{d}(b) = d(ubu^*)$. Then by the same reasoning as above, $\widetilde{c}, \widetilde{d} \in S(B(H_1))$.

Now for $a \in A_1$, $uau^* \in A_2$, so $\widetilde{c}(a) = c(uau^*) = f(uau^*) = g(a)$, since $c \in$ Ext(f). Hence $\widetilde{c} \in$ Ext(g). Likewise, $\widetilde{d} \in$ Ext(g). However, A_1 has the Kadison-Singer property, so Ext(g) has exactly one element, i.e. $\widetilde{c} = \widetilde{d}$,

Let $b \in B(H_2)$. Then

$$\begin{aligned} c(b) &= c(uu^*buu^*) \\ &= \widetilde{c}(u^*bu) \\ &= \widetilde{d}(u^*bu) \\ &= d(uu^*buu^*) \\ &= d(b), \end{aligned}$$

i.e. $c = d$. Hence $\text{Ext}(f)$ contains exactly one element, so A_2 has the Kadison-Singer property.

Likewise, if A_2 has the Kadison-Singer property, then A_1 has the Kadison-Singer property. □

One of the crucial steps in this chapter is the following theorem: it shows that we only have to consider subalgebras up to unitary equivalence when determining whether the subalgebra satisfies the Kadison-Singer property.

5.2 Classification of MASA

In this section we are going to give the classification of maximal abelian unital C^*-subalgebra upto unitary equivalence. Here we are going to use two key concepts here :

1. the number of minimal projections of the algebra.
2. the question whether the whole algebra is generated by these minimal projections.

Beforeing proving the main theorem, we are going give some results, that will useful in our discussion :

To be more precise, write $P(A) = \mathcal{P}(H) \cap A$ for the set of projections in some maximal abelian von Neumann algebras $A \subseteq B(H)$. Now write, $P_m(A)$ for the set of minimal projections in $P(A)$. The important results are given below :

1. Let $1 \leq j \leq \aleph_0$. Then the minimal projections of the discrete subalgebra is given by
$$P_m(A_d(j)) = \{\delta_n : \underline{j} \to \mathbb{C} \mid n \in \underline{j}\},$$
where $\delta_n(m) = \delta_{nm}$.
2. A_c has no minimal projections.
3. Let $1 \leq j \leq \aleph_0$. Then combining the above results the minimal projections in the mixed subalgebra is given by
$$P_m(A_c \oplus A_d(j)) = \{(0, \delta_n) : n \in \underline{j}\}.$$

4. Suppose H is a separable Hilbert space and $A \subseteq B(H)$ a maximal abelian von Neumann algebra. Then $P_m(A)$ is countable.

5. Let $1 \leq j \leq \aleph_0$. Then the discrete subalgebra is generated by its minimal projections, i.e.
$$\langle P_m(A_d(j))\rangle_{vN} = A_d(j),$$
where $\langle X\rangle_{vN}$ denotes the von Neumann algebra generated by the set X.

6. Suppose H is a separable Hilbert space and $A \subseteq B(HJ)$ is a maximal abelian von Neumann algebra that has no minimal projection. Then A is unitarily equivalent to A_c.

7. Suppose H is a separable Hilbert space and $A \subseteq B(H)$ is a maximal abelian von Neumann algebra that is generated by its minimal projections. Furthermore, let j be the cardinality of $P_m(A)$. Then A is unitarily equivalent to $A_d(j)$.

8. Let H be a separable Hilbert space and $A \subseteq B(H)$ a maximal abelian von Neumann algebra. Furthermore, let $1 \leq j \leq \aleph_0$ and suppose that the cardinality of $P_m(A) = j$ and $\langle P_m(A)\rangle_{vN} \neq A$. Then A is unitarily equivalent to $A_d(j) \oplus A_c$.

For detailed proof of these results please refer [6].

So, we can distinguish our three examples (the discrete, continuous and mixed subalgebras) by considering minimal projections and the question whether they generate the whole algebra. Note that these two properties together divide up the collection of maximal abelian subalgebras in three classes:

- There is no minimal projection (like A_c).
- There are minimal projections that do not generate the whole algebra (like $A_c \oplus A_d(j)$)
- There are minimal projections that do generate the whole algebra (like $A_d(j)$).

Theorem 5.2.1. *Suppose H is a separable Hilbert space and $A \subseteq B(H)$ is a maximal abelian C^*-subalgebra. Then A is unitarily equivalent to exactly one of the following :*

1. $A_c \subseteq B(L^2[0,1])$

2. $A_d(j) \subseteq B(\ell^2(j))$ *for some* $1 \leq j \leq \aleph_0$

3. $A_d(j) \oplus A_c \subseteq B(\ell^2(j) \oplus L^2[0,1])$ *for some* $1 \leq j \leq \aleph_0$.

Proof. We are not going to proved this theorem in minor detains. We will give the key ideas of the proof and will use certain results of to prove this theorem.

The proof of this theorem relies on the notion of minimal projections.

A projection p on a Hilbert space H is a linear operator satisfying $p^2 = p^* = p$

It is well known that such operators bijectively corresponds to the closed linear subspaces $p(H)$ of H that form their images.

More generally, a projection in a C^*-algebra A ia an element $p \in A$ that satisfies the same equalities (i.e. $p^2 = p^* = p$). On the set $P(A)$ consisting of the projections in A, we can define a natural order, which coincides with the notion of positivity for A.

For example, in the algebra $\ell^\infty(\mathbb{N})$, the projections are exactly the indicator functions 1_W of subsets $W \subseteq \mathbb{N}$ and $1_W \leq 1_Y$ if and only if $W \subseteq Y$.

The zero element of A is the minimal element of $P(A)$ with respect to this order, but we say aprojection is a minimal projection if it is a minimal element of the ordered set $P(A) \setminus \{0\}$.

One can easily check that in the case of $\ell^\infty(\mathbb{N})$, the minimal projections are then exactly the indicator functions of single points. Furthermore, the whole algebra is generated by these indicator functions of single points. For the finite dimensional case, i.e. $D_n(\mathbb{C})$ where $n \in \mathbb{N}$, that is exactly the same.

However, for the continuous subalgebra $L^\infty[0,1]$ the situation is different. Again, the projections are indicator functions, but since for any (measurable) set $A \subseteq [0,1]$ such that $\mu(A) > 0$, there is a $B \subseteq A$ such that $0 < \mu(B) < \mu(A)$, this algebra has no minimal projections and is therefore certainly not generated by them.

The mixed subalgebra keeps the middle ground between the discrete and the continuous case : it does have minimal projections (coming from the discrete part), but is not generated by them.

Hence we see that the discrete, continuous and mixed cases can be distinguished by considering the number of minimal projections and the question wheather the whole algebra is generated by these minimal projections. As it turns out these two pieces of information classify all maximal abelin unital C^*-subalgebras on separable Hilbert spaces.

After giving an overview of the theorem, it's time to prove it.

Consider $P_m(A)$. Define $j :=$ the cardinality of $P_m(A)$. By result (4), $0 \leq j \leq \aleph_0$.

If $j = 0$, then by (6), A is unitarily equivalent to A_c.

If $1 \leq j \leq \aleph_0$, there is a distinction between the cases $\langle P_m(A)\rangle_{vN} = A$ and $\langle P_m(A)\rangle_{vN} \neq A$. In the first case, by (7), A is unitarily equivalent to $A_d(j)$. In the second case, A is unitarily equivalent to $A_d(j) \oplus A_c$ by (8).

So A is indeed unitary equivalent to one of the three mentioned cases. Since the three cases have different properties concerning its minimal projections, they are mutually unitarily inequivalent, so A is unitary equivalent to exactly one of them.

□

This classification has the following very important corollary for our main goal of classifying all subalgebras with the Kadison-Singer property.

Corollary 5.2.1. *Suppose H is a separable Hilbert space and $A \subseteq B(H)$ a unital abelian subalgebra that has the Kadison-Singer property. Then A is unitarily equivalent to either $A_d(j)$ for some $1 \leq j \leq \aleph_0$, A_c or $A_d(j) \oplus A_c$ for some $1 \leq j \leq \aleph_0$.*

Proof. We know that A is a maximal abelian C^*-algebra, since A has the Kadison-Singer property. Hence it is also a maximal abelian von Neumann algebra. Therefore, by above theorem A is unitarily equivalent to either $A_d(j)$ for some $1 \leq j \leq \aleph_0$, A_c or $A_d(j) \oplus A_c$ for some $1 \leq j \leq \aleph_0$. □

In rest of this text, we will determine whether the discrete, continuous and mixed subalgebra have the Kadison-Singer property. So far, we only proved that $A_d(j)$ has the Kadison-Singer property if $j \in \mathbb{N}$.

Chapter 6

The Kadison-Singer problem: An overview

In the last chapter we classify the possible subalgebras that can have the Kadison-Singer property. In this chapter we will discuss whether the discrete, continuous and the mixed subalgebrs has the Kadison-Singer property. Here we are not going prove the results in detail. We will give a whole overview of the things.

Before going to other things we will define **normal state** and **singular state**. Actually in the finite dimensional case we can easily characterize the states and pure states. All the states in finite dimension are normal states. But certainly this is not the case for infinite dimension. There are states on $B(H)$, which are not of the form as in finite dimension.

6.1 Normal states

In chapter 2, we described all states on the matrix algebra $M_n(\mathbb{C})$ using density operators. In fact, using the spectral decomposition of density operators, we saw that every state on $M_n(\mathbb{C})$ was given by

$$\omega(a) = \sum_{i=1}^{n} p_i \langle v_i, av_i \rangle,$$

where $\{v_i\}_{i=1}^n$ is some orthonormal basis of $\mathbb{C}^n$ and $\{p_i\}_{i=1}^n \subseteq [0,1]$ is such that $\sum_{i=1}^n p_i = 1$. We can generalize these states to the infinite dimensional case.

Definition 6.1.1. *For any orthonormal base* $\{v_i\}_{i=1}^{\infty}$ *of* $\ell^2(\mathbb{N})$ *and sequence* $\{p_i\}_{i=1}^{\infty} \subseteq [0,1]$ *such that* $\sum_{i=1}^{\infty} p_i = 1$, *the functional* $f : B(\ell^2(\mathbb{N})) \to \mathbb{C}$ *defined by*

$$f(a) = \sum_{i=1}^{\infty} p_i \langle v_i, a v_i \rangle,$$

is a state on $B(\ell^2(\mathbb{N}))$. *Such states are called* **normal states**.

In contrary to the finite dimensional case, the set of normal states do not exhaust the set of all states on $B(\ell^2(\mathbb{N}))$.

It is clear that for any orthogonal set of projections $\{e_i\}_{i \in I}$, we have

$$f(\sum_{i \in I} e_i) = \sum_{i \in I} f(e_i)$$

for any normal state f.

Theorem 6.1.1 (von Neumann). *A sate* ω *on* $B(H)$ *is normal if and only if*

$$\omega(\sum_{i \in I} e_i) = \sum_{i \in I} \omega(e_i)$$

for any countable family $e_{i i \in I}$ *of mutually orthogonal projections (this is similar to the countable additivity condition in the definition of a measure)*

In contrast to normal states, there are singular states.

Definition 6.1.2. Singular states *are states that annihilate (i.e. vanishes) all one-dimensional projections and thereby all compact operators.*

So we can check that, singular states are not normal. In fact, any state is either normal, or singular, or it can be written as a convex combination of a normal and a singular state.

Corollary 6.1.1. *Every pure state is either normal or singular*

Proof. This corollary follows from the above argument. □

It is however a non-trivial matter to write down states on $B(H)$ that are not normal.

Proposition 6.1.1. *Let $A \subseteq B(H)$ be a MASA (and hence a von Neumann algebra). Then any normal pure state on A has a unique extension to $B(H)$.*

Proof. Using density operators, this can be proved as in the finite-dimensional case. □

6.2 Kadison-Singer conjecture

From last section, it follows that in looking for possible pure states on A without unique extensions to $B(H)$, one necessarily enters the realm of singular states. As we noted, these are hard to grasp, and having already encountered the Hahn-Banach theorem in this context, it may not be surprising that the world of ultrafilters and the like plays a role in the analysis of the Kadison-Singer property. Furthermore, we are not able to treat the singular states on two different MASA's in the same way; each MASA needs a diffirent approach.

Let us start with the continuous case, Kadison-Singer proved in their original article from 1959 that the continuous subalgebra does not have the Kadison-Singer property. Twenty years latter, in 1979, Joel Anderson gave a more straightforward proof of the same fact, and also improved upon it. He proved that there is no pure state on the continuous subalgebra at all that extends in a unique way to a pure state on $B(L^[0, 1])$, which is definitely stronger than the negation of having the Kadison-Singer property. Anderson used the Stone-Cech compactification of $\mathbb{N}$ (realized via ultrafilters) in order to able to describe all pure states on A_c. A careful and tricky argument then gave the desired result.

Theorem 6.2.1. *A_c does not have the Kadison-Singer property.*

Proof. For the detailed proof of this theorem please refer chapter 6 [6] □

So, in light of the above theorem, we have now eliminated the continuous subalgebra from the list of algebras that could possibly have the Kadison-Singer property, However, we can also eliminate the mixed subalgebra by this following corollary.

Corollary 6.2.1. *Suppose $1 \leq j \leq \aleph_0$. Then $A_d(j) \oplus A_c \subseteq B(H_j)$ does not have the Kadison-Singer property.*

Proof. We know that if a direct sum of algebras has the Kadison-Singer property, then all summands must have the Kadison-Singer property too. Hence the fact that the continuous subalgebra does not have the Kadison-Singer property implies that the mixed subalgebra does not have the Kadison-Singer property. □

Now that we have eliminated the continuous and mixed subalgebra of our list, we can make a new step towards our classification of abelian C^*-subalgebras with the Kadison-Singer property: only the discrete subalgebra can possibly have this property. The proof of the following corollary mainly serves as a summary of our results so far.

Corollary 6.2.2. *Suppose H is a separable Hilbert space and $A \subseteq B(H)$ is a unital abelian C^*-subalgebra that has the Kadison-Singer property. Then A is unitarily equivalent to $A_d(j) \subseteq B(\ell^2(\underline{j}))$ for some $1 \leq j \leq \aleph_0$.*

Proof. The proof this immediately follows by combining the earlier results. □

The natural question that now arises is whether we can reduce our list of abelian C^*-algebras that possibly have the Kadison-Singer property even further. Note that we have already proven in lemma 2.3.1 that $A_d(j)$ has the Kadison-Singer property for $j \in \mathbb{N}$. Hence the only open question is whether $A_d(\aleph_0) = \ell^\infty(\mathbb{N}) \subseteq B(\ell^2(\mathbb{N}))$ has the Kadison-Singer property. Richard Kadison and Isadore Singer ([8]) formulated this question in 1959 and believed that the answer was negative.

This open question became known as the Kadison-Singer conjecture and was answered in 2013, by Adam Marcus, Daniel Spielman and Nikhil Srivastava. Despite the belief of Kadison and Singer, it was proven that $\ell^\infty(\mathbb{N}) \subseteq B(L^2[0,1])$ in fact does have the Kadison-Singer property.

Theorem 6.2.2. *Any pure state on the abelian von Neumann algebra $\ell^\infty(\mathbb{N})$, realized as multiplication operators on the Hilbert space $\ell^2(\mathbb{N})$, has a unique extension to a (necessarily pure) state on $B(\ell^2(\mathbb{N}))$.*

In other words, $\ell^\infty(\mathbb{N}) \subseteq B(\ell^2(\mathbb{N}))$ has the Kadison-Singer property.

Proof. For the detailed proof please refer [6], and [7] □

We will end this chapter by giving the history of the Kadison-Singer problem. In the years that followed, many people worked on this problem. Before the turn of the century, the most notable progress was made by the aforementioned Anderson. He straightened out some of the details in the article by Kadison and Singer and reformulated what later became known as the **paving conjecture**. This is a statement that is equivalent to the Kadison–Singer conjecture and says the following:

For every $\epsilon > 0$ there is an $l_\epsilon \in \mathbb{N}$ such that for all $a \in B(\ell^3(\mathbb{N}))$ that satisfy $diag(a) = 0$, there exists a set of projections $\{p_i\}_{i=1}^{l_\epsilon} \subseteq \ell^\infty(\mathbb{N})$ such that

$$\sum_{i=1}^{l_\epsilon} p_i = 1$$

and

$$\|p_i a p_i\| \leq \epsilon \|a\|$$

for every $i \in \{1, \ldots, l_\epsilon\}$.

Here, we used the function

$$diag(a) : \mathbb{N} \to \mathbb{C}$$

which is defined by

$$diag(a)(n) = \langle e_n, ae_n \rangle.$$

The strength and difficulty in proving this conjecture is contained in the uniformity of l_ϵ: there is one fixed l_ϵ that should work for all a.

In turn, using Tychonoff's theorem, it can be shown that this paving theorem for operators on $\ell^2(\mathbb{N})$ is equivalent to a paving theorem for matrices. To be more precise, the Kadison–Singer conjecture is equivalent to:

For every $\epsilon > 0$ there is an $l_\epsilon \in \mathbb{N}$ such that for all $a \in M_n(\mathbb{C})$ such that $diag(a) = 0$, there is a set of diagonal projections

$$\{p_i\}_{i=1}^{l_\epsilon} \subseteq D_n(\mathbb{C})$$

such that

$$\sum_{i=1}^{l_\epsilon} p_i = 1$$

and

$$\|p_i a p_i\| \leq \epsilon \|a\|.$$

This equivalence is quite remarkable, since we can now use tools of linear algebra to draw conclusions about the infinite dimensional discrete algebra.

In 2004, Nik Weaver formulated a new conjecture, which he showed was equivalent to the paving conjecture. Weaver's conjecture was reformulated by Terence Tao and finally, this problem was proven by Adam Marcus, Daniel Spielman and Nikhil Srivastava in 2013, by using the theory of random matrices, real stable polynomials and some new tools from linear algebra.

References

[1] B. Limaye, *Functional analysis*. J. Wiley, 1981. [Online]. Available: https://books.google.co.in/books?id=Pj3vAAAAMAAJ

[2] J. Conway, *A Course in Functional Analysis*, ser. Graduate Texts in Mathematics. Springer New York, 2013. [Online]. Available: https://books.google.co.in/books?id=ccEGCAAAQBAJ

[3] W. Rudin, *Functional Analysis*, ser. International series in pure and applied mathematics. McGraw-Hill, 2006. [Online]. Available: https://books.google.co.in/books?id=l7XFfDmjp5IC

[4] R. Bhatia, *Notes on Functional Analysis*, ser. Texts and Readings in Mathematics. Hindustan Book Agency, 2009. [Online]. Available: https://books.google.co.in/books?id=x_JdDwAAQBAJ

[5] E. Kaniuth, *A Course in Commutative Banach Algebras*, ser. Graduate Texts in Mathematics. Springer New York, 2008. [Online]. Available: https://books.google.co.in/books?id=Sp5uwaZsQLAC

[6] M. Stevens, "The kadison-singer property," Master Thesis, Radboud University.

[7] S. D. Marcus, A. and N. Srivastava, "Interlacing families ii: Mixed characteristic polynomials and the kadison-singer problem." *ArXiv: 1306.3969.*, 2013.

[8] R. V. Kadison and I. Singer, "Extensions of pure states," *American Journal of Mathematics*, vol. 81(2):383–400., 1959.

[9] K. Landsman and M. Stevens, "The Kadison-Singer conjecture," *Nieuw Arch. Wiskd. (5)*, vol. 17, no. 1, pp. 41–46, 2016.

[10] M. Stevens, *The Kadison-Singer property*, ser. SpringerBriefs in Mathematical Physics. Springer, Cham, 2016, vol. 14, with a foreword by Klaas Landsman. [Online]. Available: https://doi.org/10.1007/978-3-319-47702-2

[11] B. Tanbay, "A letter on the Kadison-Singer problem," *Rev. Roumaine Math. Pures Appl.*, vol. 59, no. 2, pp. 293–302, 2014.

[12] D. Timotin, "The solution of the Kadison-Singer problem," in *Recent advances in operator theory and operator algebras.* CRC Press, Boca Raton, FL, 2018, pp. 117–148.

[13] J. Munkres, *Topology*, ser. Featured Titles for Topology Series. Prentice Hall, Incorporated, 2000. [Online]. Available: https://books.google.co.in/books?id=XjoZAQAAIAAJ