Introduction to Fredholm, Toeplitz and Hankel Operator

Subhajit Roy

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Under the guidance of
Dr. Sukumar D
Department of Mathematics

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Declaration

This thesis entitled **"Introduction to Fredholm, Toeplitz and Hankel operator"** submitted by me to the Indian Institute of Technology, Hyderabad for the award of the degree of Master of Science in Mathematics contains a literature survey of the work done by some authors in this area. The work presented in this thesis has been carried out under the supervision of **Dr. Sukumar D**, Department of Mathematics, Indian Institute of Technology, Hyderabad, Telangana.

I hereby declare that, to the best of my knowledge, the work included in this thesis has been taken from the books mentioned in the References. No new results have been created in this thesis. I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

Subhajit Roy

(Signature)

Subhajit Roy

MA17MSCST11012

(Roll No.)

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Approval Sheet

This Thesis entitled **Introduction to Fredholm, Toeplitz and Hankel Operator** by **Subhajit Roy** is approved for the degree of Master of Science from IIT Hyderabad.

(Dr. Sukumar D) Adviser
Dept. of Mathematics
IITH

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Dedication

Dedicated to my parents

Abstract

The whole discussion of the thesis is about **Fredholm**, **Toeplitz**, **and Hankel** operator. **Atkinson's theorem** gives an equivalent definition of Fredholm operator, there we get a relation between Compact and Fredholm operator.

We will learn the **Calkin algebra** on a Hilbert space and what is the relation between the Fredholm operator and this algebra, the index of a Fredholm operator. The sum of Compact and Fredholm operator is Fredholm and the index is the same as the index of the Fredholm operator. The characteristic of Fredholm operator of index 0 and we will show that the index map is a locally constant map.

We will learn about the **Hardy space** and **Toeplitz operator on Hardy space**. All the Toeplitz operator on the Hardy space is not Fredholm, there is a class of Toeplitz operator which are Fredholm and the Toeplitz operator which is compact is only the zero operators. Also, we will learn about some properties about the set of all Toeplitz operator.

Finally, we will learn **Hankel operator on Hardy space** and some algebraic properties of the Hankel operator. In general, all Hankel does not commute with a Toeplitz operator, if a Hankel commute with a symmetric Toeplitz operator, then the Toeplitz operator is a constant multiple of the identity operator.

Contents

Chapter 0

List of symbols

- $B(X, Y)$ the set of all bounded operators from X to Y
- $\mathcal{K}(X, Y)$ the set of all compact operators from X to Y
- $\mathcal{F}(H)$ the set of all finite rank operator on H
- $cl\{A\}$ closure of A
- $R(T)$ range of T
- $kerT$ kernel of T
- I the identity operator
- X^* dual of X
- $\xrightarrow{w}$ weakly converges
- S_1 unit circle on complex plane
- $C(S_1)$ the set of all continuous function on S_1
- $\mathbb{N}$ the set of all natural number
- $\mathbb{Z}$ the set of all integer
- $\mathbb{C}$ the set of all complex number
- $\mathbb{R}$ the set of all real number
- $\mathbb{K} = \mathbb{C}$ *or* $\mathbb{R}$

Chapter 1

Compact operators

1.1 Introduction:

Definition 1.1.1 (Compact Operator)**.** *Let X and Y be normed linear spaces. A linear operator $K : X \to Y$ is said to be a compact operator if the set $cl\{Kx : ||x|| \leq 1\}$ is compact in Y. In other words, closure of image of closed unit ball $U = \{x \in X : ||x|| \leq 1\}$ under K is compact*

1.2 Examples:

Example 1.2.1. *Every bounded linear operator $K : X \to Y$ of finite rank is a compact operator.*

Proof. Since K is a finite rank operator, $R(K)$ is a finite-dimensional normed linear space, so it is a closed supspace of Y. Since K is continuous and $\{x \in X : ||x|| \leq 1\}$ is a bounded set, $cl\{Kx : ||x|| \leq 1\}$ is a bounded subset of Y. Thus, $cl\{Kx : ||x|| \leq 1\}$ is closed and bounded subspace of the finite dimensional space $R(K)$. By "**Heine Borel theorem**", $cl\{Kx : ||x|| \leq 1\}$ is compact. Hence K is a compact operator. □

Example 1.2.2. *The identity operator on a normed linear space is a compact operator iff the space is of finite dimension.*

Proof. Let $I : X \to X$ be an identity operator. We know that closed unit ball $U = \{x \in X : ||x|| \leq 1\}$ is compact iff X is finite dimensional space. Now $cl\{I(U)\} = U$ is compact iff X is finite dimensional. Therefore I is compact iff X is finite dimensional. □

Example 1.2.3. *If X_0 is a subspace of a normed linear space X, then the inclusion operator $I_0 : X_0 \to X$, that is, $I_0x = x$ for all $x \in X_0$, is a compact operator iff X_0 is finite dimensional.*

Proof. Let I_0 is a compact operator. Since the closure of image of the closed unit ball under I_0 that is, $cl(I_0(\{x \in X_0 : ||x|| \leq 1\})) = \{x \in X_0 : ||x|| \leq 1\}$ is compact in X. Hence it is also compact in X_0. Therefore X_0 is finite dimensional.

Conversely, let X_0 is finite dimensional. Since $X_0 = R(I_0)$, I_0 is a finite rank operator. Hence, by Example 1.2.1, the inclusion operator $I_0 : X_0 \to X$ is a compact operator. □

Example 1.2.4. *If $P : X \to X$ is a projection operator, then it is a compact operator iff $R(P)$ is finite dimensional.*

Proof. Since P is projection operator, the restriction of P on $R(P)$ is a identity operator on $R(P)$. Hence by Example 1.2.2 $P : X \to X$ is a compact operator iff $R(P)$ is finite dimensional. □

Example 1.2.5. *Let $N \in \mathbb{N}$. Consider the linear operator $T_N : \ell^2 \to \ell^2$ defined by*

$$T_Nx = (x_1, x_2, x_3, \dots, x_N, 0, 0, \dots), \qquad x \in \ell^2.$$

Then T_N is compact operator.

Proof. Image of T_N is finite dimensional and $\dim R(T_N) = N$. So T_N is finite rank bounded linear operator, hence compact. □

1.3 Properties:

Theorem 1.3.1. *Let X and Y be normed linear spaces and $A : X \to Y$ be a linear operator. Then the following are equivalent:*

1. *A is a compact operator.*
2. *$cl\{Ax : ||x|| < 1\}$ is compact in Y.*
3. *For every bounded subset E of X , $clA(E)$ is compact in Y.*
4. *For every bounded sequence (x_n) in X , the sequence (Ax_n) has a convergent sub-sequence in Y.*

Proof. Clearly 3 implies 1 and 2. Now assume that 1 holds. That is, $cl\{Ax : ||x|| \le 1\}$ is compact. Let E be a bounded subset of X. Then, we know that there exists $r > 0$ such that $E \subseteq \{x \in X : ||x|| < r\}$. First we want to show that if $A : X \to Y$ is a linear operator between normed linear spaces X and Y, then for any $r > 0$,

$$cl\{Ax : ||x|| \le r\} \text{ is compact} \Leftrightarrow cl\{Ax : ||x|| \le 1\} \text{ is compact.}$$

Let $S = cl\{Ax : ||x|| \le r\}$ is compact and (y_n) be a sequence in $D = cl\{Ax : ||x|| \le 1\}$ then $\exists$ a sequence (x_n) in $U = \{x \in X : ||x|| \le 1\}$ such that $y_n = Ax_n$. Now, $(A(rx_n))$ is a sequence in S, since $||rx_n|| \le r$ and S is compact, so it has a convergent sub-sequence, say (Az_n) where (z_n) is a subsequence of (rx_n). Consequently the sequence $(A(z_n/r))$ is a convergent subsequence of the sequence (y_n) in D. Hence $D = cl\{Ax : ||x|| \le 1\}$ is a compact set. By similar, way we can show that if $D = cl\{Ax : ||x|| \le 1\}$ is a compact set then $S = cl\{Ax : ||x|| \le r\}$ is compact set.

Now, from the relations $cl\{Ax : x \in X, ||x|| < r\} \subseteq cl\{Ax : x \in X, ||x|| \le r\}$, and the fact that a closed and bounded subset of a compact set is compact, it follows that 1 implies 2 and 3, and 2 implies 3 by the above result.

Now to complete the proof we prove the equivalence of 3 and 4. Assume that 3 holds, and let (x_n) be a bounded sequence in X. So there exists $c > 0$ such that $||x_n|| \le c$ for every $n \in \mathbb{N}$, and let

$$E = \{x \in X : ||x|| \le c\}.$$

Then (Ax_n) is a sequence in the compact set $cl(A(E))$, so that it has a convergent sub-sequence by "**Bolzano Weierstrass theorem**". Thus, 4 holds.

Conversely, assume that 4 holds, and let E be a bounded subset of X. To show that $cl(A(E))$ is compact, it is enough to show that every sequence in it has a convergent sub-sequence. To show this, suppose that (y_n) is a sequence in $cl(A(E))$. Then there exists (x_n) in E such that $||y_n - Ax_n|| \le 1/n$ for all $n \in \mathbb{N}$. Now by the hypothesis in 4, (Ax_n) has a convergent sub-sequence, say (Ax_{n_j}). Then it follows that (y_{n_j}) is a convergent sub-sequence of (y_n). □

Theorem 1.3.2. *Let X, Y and Z be normed linear spaces. Let $B(X,Y)$ and $\mathcal{K}(X,Y)$ are the set of all bounded operators from X to Y and the set of all compact operators from X to Y respectively. Then the following results hold.*

1. *$\mathcal{K}(X,Y)$ is a subspace of $B(X,Y)$*

2. *If $P \in B(X,Y)$ and $Q \in B(Y,Z)$ and if one of them is compact, then $QP \in \mathcal{K}(X,Z)$*

Proof. (1) Clearly $\mathcal{K}(X,Y)$ is a subset of $B(X,Y)$ and not an empty subset because zero operator lies in both.

Now let $P, Q \in \mathcal{K}(X,Y)$ and $c \in \mathbb{K}$. We prove that the operator $P + cQ$ is a compact operator.

Let (x_n) be a abounded sequence in X. Since P is a compact operator, there is a sub-sequence (y_n) of (x_n) such that (Py_n) converges in Y.

Also since (y_n) is a bounded sequence and Q is compact operator, there is a sub-sequence (z_n) of (y_n) such that (Qz_n) converges.

Hence the sub-sequence $(Pz_n + cQz_n)$ of the sequence $(Px_n + cQx_n)$ converges.

(2) Let (x_n) be a bounded sequence in X. If $P \in \mathcal{K}(X,Y)$ then there is a sub-sequence (y_n) of (x_n) such that (Py_n) converges in Y. Then by the continuity of Q, (QPy_n) converges in Z, which is sub-sequence of (QPx_n). Hence QP is a compact operator.

Let $Q \in \mathcal{K}(Y,Z)$. Since $(P \in B(X,Y))$, the sequence (Px_n) is bounded in Y, so using compactness of Q, (QPx_n) has a convergent sub-sequence in Z. So QP is a compact operator. □

Remark 1.3.1. *Converse of second part of the previous theorem is not true. That is, composition of two bounded operators may be compact operator.*

Example 1.3.1. *Let H be a Hilbert space and (e_n) be orthonormal basis of H. Let A and B be projection operator onto the span $\{e_n : n$ is odd $\}$ and onto span $\{e_n$: n is even $\}$ respectively. Then A,B both are not compact operators. But their composition is zero operator,which is a compact operator. That can be seen from the following example.*

Theorem 1.3.3. *Let X be a normed linear space and Y be a Banach space and $A \in B(X,Y)$. If (A_n) is a sequence in $\mathcal{K}(X,Y)$ such that $\|A_n - A\| \to 0$ as $n \to \infty$, that is sequence (A_n) converge to A. Then $A \in \mathcal{K}(X,Y)$.*

Proof. Let (x_n) be a bounded sequence in X. We will prove that (Ax_n) has a convergent sub-sequence in Y.

Since A_1 is compact, (x_n) has a sub-sequence (x_n^1) such that $A_1(x_n^1)$ converges. Also A_2 is compact and (x_n^1) is bounded sequence so (x_n^1) has a sub-sequence (x_n^2) such that $(A_2x_n^2)$ converges.

Continuing this process we get sub-sequence (x_n^k) of (x_n^{k-1}) such that (Ax_n^k) converges, which hold for all $k \in \mathbb{N}$. Therefore by taking $y_n = x_n^n, n \in \mathbb{N}$, is a bounded sequence, so there exists $c > 0$ such that $\|y_n\| \leq c$, for all $n \in \mathbb{N}$. Also the sequence $(A_k y_n)$ converges for any $k \in \mathbb{N}$.

Now, let us choose $\epsilon > 0$. Since $\|A_n - A\| \to 0$, there exists $k \in \mathbb{N}$ such that $\|A - A_k\| < \epsilon$. Also since for any k, $(A_k y_n)$ is Cauchy sequence so there exists $N \in \mathbb{N}$ such that

$$\|A_k y_n - A_k y_m\| < \epsilon, \forall n, m \geq N$$

.

Therefore, for all $m, n \in \mathbb{N}$,

$$\|Ay_n - Ay_m\| \leq \|(A - A_k)y_n\| + \|A_k y_n - A_k y_m\| + \|(A_k - A)y_m\|$$

$$\leq c\epsilon + \epsilon + c\epsilon = (2c + 1)\epsilon$$

Thus, (Ay_n) is a Cauchy sub-sequence of (Ax_n). Since Y is Banach space so this sub-sequence converges in Y. Hence A is compact operator. □

Remark 1.3.2. *$\mathcal{K}(X, Y)$ is closed subspace of $B(X, Y)$ if Y is Banach space.*

Remark 1.3.3. *If X is Banach space, and if (A_n) is a sequence of finite rank operators in $B(X)$ such that $\|A - A_n\| \to 0$, as $n \to \infty$ for some $A \in B(X)$, then A is a compact operator. The converse is true if $X = Y = H$, where H is a separable Hilbert space.*

Theorem 1.3.4. *Let K be a compact operator on a separable Hilbert space H and suppose that $(T_n) \subseteq B(H)$ and $T \in B(H)$ are such that for each $x \in H$, the sequence $(T_n x)$ converges to Tx. Then $(T_n K)$ converges to TK in the norm of $B(H)$.*

Proof. Suppose that the sequence $(T_n K)$ does not converges to TK. Then there exists a $\delta > 0$ and a sub-sequence (T_{n_j}) of the sequence (T_n) such that

$$\|T_{n_j} K - TK\| > \delta.$$

Choose unit vectors (y_n) of H such that

$$\|(T_{n_j} K - TK)(y_n)\| > \delta$$

Since K is compact, we get a sub-sequence (y_{n_k}) of (y_n) such that (Ky_{n_k}) is convergent. Assume that (Ky_{n_k}) converges to y. Then

$$\delta < \|(T_{n_j}K - TK)y_{n_k}\| \leq \|(T_{n_j} - T)(Ky_{n_k} - y)\| + \|(T_{n_j} - T)y\| \tag{1.1}$$

Since $(T_n) \subseteq B(H)$ is bounded, then there exists $C > 0$, such that $\|T_n\| \leq C$ and

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq C.$$

Hence $\|T - T_{n_j}\| \leq 2C$.

Since (Ky_{n_k}) converges to y, so there exists $N \in \mathbb{N}$ such that for $n_j > N$,

$$\|Ky_{n_k} - y\| < \frac{\delta}{8C}$$

Also as $(T_{n_j}y)$ converges to Ty, for each $y \in H$, so there exists $m \in \mathbb{N}$ such that for all $n_j > m$

$$\|(T - T_{n_j})y\| < \frac{\delta}{4}$$

Now from Equation 1.1

$$\delta < \|(T_{n_j} - TK)x_{n_j}\| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

a contradiction.

□

Theorem 1.3.5. *Let H be separable Hilbert space. Then $cl\{\mathcal{F}(\mathcal{H})\} = \mathcal{K}(\mathcal{H})$*

Proof. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for H and $H_n := span\{e_1, e_2, e_3, \ldots, e_n\}$.

Then the orthogonal projections, $P_n : H \to H$ defined by

$$P_n x = \sum_{j=1}^{n} \langle x, e_j \rangle e_j.$$

Then $P_n x$ converges to x for every $x \in H$, since $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for H. Now if K be a compact operator on H then by theorem 1.3.4 $(P_n K)$, converges to K in the operator norm of $B(H)$. We want to show $P_n K$ is finite rank operator, for every $n \in \mathbb{N}$. Here $R(P_n K) \subseteq R(P_n) = H_n$ is finite dimensional. Hence the proof. □

Example 1.3.2. *Let $T : \ell^2 \to \ell^2$ be an operator defined by*

$T_n(x_n)_{n\in N} = (\lambda_n x_n)_{n\in N}$, where $(\lambda_n)_{n\in N}$ is a sequence converges to 0. Then T is a compact operator.

Proof. For $k \in N$, consider the operators, $T_k : \ell^2 \to \ell^2$ defined by

$$T_k(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_k x_k, 0, 0, 0, 0, , \dots)$$

Then for each $k \in N$, T_k is a finite rank linear operator, so by Example 1.2.1 for each $k \in N$, T_k is a compact operator.

Now for any $x \in \ell^2$,

$$||(T - T_k)x|| = sup_{n>k}(|\lambda_n x_n|) \leq ||x||sup_{n>k}|\lambda_n|$$

$\implies ||T - T_n|| \leq sup_{n>k}(|\lambda_n|)$

Since λ_n tends to 0 as, $n \to \infty$. So $||T - T_n||$ tends to 0. as $n \to \infty$.

So by Theorem 1.3.3, T is compact operator.

□

Definition 1.3.1. *Let X be a normed linear space, and let $x_n, x \in X$. We say that x_n, "**converges strongly**", or converges in norm to x, and write $x_n \to x$, if $\|x_n - x\|$ tends to 0, as $n \to \infty$.*

Definition 1.3.2. *Let X be a normed linear space, and let $(x_n) \in X$ be a sequence in X, we say that x_n "**converges weakly**" to $x \in X$, and write $x_n \xrightarrow{w} x$, if $f(x_n)$ converge to $f(x)$ as $n \to \infty$, for all $f \in X^*$, where X^* is dual of X.*

Theorem 1.3.6. *Show that strong convergence implies weak convergence.*

Proof. Let (x_n) be a strongly convergent sequence in a normed linear space X converges to $x \in X$.

For $f \in X^*$, $||f(x_n) - f(x)|| \leq ||f||||x_n - x||$ tends to 0, as $n \to \infty$.

So $x_n \xrightarrow{w} x$. □

Remark 1.3.4. *Converges of above theorem is not true.*

Proof. Let X be an infinite dimensional Hilbert space and (e_n) be an orthonormal set of X. Then $||e_n|| = 1$, for every $n \in \mathbb{N}$ and $||e_n - e_m|| = \sqrt{2}$, for all $n, m \in \mathbb{N}, n \neq m$. So it is not Cauchy sequence and consequently not convergent. By "**Riesz representation theorem**", for every $f \in X^*$, there is a unique $y \in X$ such that

$$f(x) = \langle x, y \rangle, \ \ x \in X.$$

Again by "**Bessel inequality**",

$$\sum_{i=1}^{\infty} |\langle y, e_n \rangle|^2 \leq ||y||^2,$$

Thus the series is convergent, so

$$|f(e_n)| = |\langle e_n, y \rangle| = |\langle y, e_n \rangle| \to 0, \ \ as \ \ n \to \infty.$$

$$\implies f(e_n) \to 0, \ \ as \ \ n \to \infty.$$

Hence the sequence (e_n) is weakly converges to 0, but not converges to 0. □

Theorem 1.3.7. *weakly convergence sequence has unique limit.*

Proof. Let (x_n) be a sequence in X, weakly converges to different limits x and y. Since $x \neq y$, so by a consequence of "**Hahn Banach theorem**" there exists $f \in X^*$ such that $f(x-y) = ||x-y|| \neq 0$. That is $f(x) \neq f(y)$. But $f(x) = f(y)$, since $f(x_n)$ converges to $f(x)$ and $f(y)$, a contradiction. Hence limit is unique. □

Theorem 1.3.8. *(Resonance theorem) Let E be a subset of Banach space X. Then E is bounded in X iff $f(E)$ is bounded, for all $f \in X^*$.*

Proof. Since E is bounded in X, so there exists $c > 0$ such that $||x|| \leq c$, for all $x \in E$.

Now for $f \in X^*$, $|f(x)| \leq ||f||||x|| \leq c||f||$, for all $x \in E$. So $f(E)$ is bounded for all $f \in X^*$.

Conversely, Let $f(E)$ is bounded for all $f \in X^*$.

For $x \in E$, let $\varphi_x : X^* \to F$ defined by

$$\varphi_x(f) = f(x), \ \ f \in X^\star.$$

Clearly φ_x is a linear functional, since f is linear. Now

$$||\varphi_x|| = sup_{||f||=1}||\varphi_x(f)|| = sup_{||f||=1}||f(x)|| \leq ||x||.$$

Again by "**Hahn Banach theorem**" for $x \in X$ there exists $f \in X^*$, such that $f(x) = ||x||$ and $||f|| = 1$, so $||\varphi_x|| = ||x||$. Hence φ_x is a bounded linear functional.

Now by our assumption, $\{\varphi_x(f) : x \in E\}$ is a bounded set, for all $f \in X^*$. Since X^* is Banach space, so by "**Uniform bounded principle**", $\{\varphi_x : x \in E\}$ is uniformly bounded.

That is, $\{||\varphi_x|| : x \in X\}$ is bounded. Hence E bounded. □

Theorem 1.3.9. *Let $x_n \xrightarrow{w} x$. Then (x_n) is a bounded sequence.*

Proof. Let $E = \{x_n : n \in N\}$. Since for all $f \in X^\star$, $f(x_n)$ converges to $f(x)$ and every convergent sequence in $\mathbb{K}$ is bounded, so $f(E)$ is bounded for all $f \in X^*$.

Hence by Theorem 1.3.8 E is bounded set. That is the sequence (x_n) is a bounded sequence. □

Theorem 1.3.10. *Let $A \in \mathcal{K}(X, Y)$ is compact operator. Then $x_n \xrightarrow{w} x$ implies (Ax_n) converges to Ax. In otherwords compact operator map weakly converges sequence to strongly converges sequence.*

Proof. Let us assume, $x_n \xrightarrow{w} x$ and (Ax_n) does not converges to Ax. Since $x_n \xrightarrow{w} x$, so by Theorem 1.3.9 (x_n) is a bounded sequence.

Now since (Ax_n) does not converges to Ax, so there exists $\epsilon > 0$ and a sub-sequence (y_n) of (x_n) such that $||Ay_n - Ax|| \geq \epsilon$, for all $n \in \mathbb{N}$.

Since A is compact operator and (y_n) is bounded sequence, so the sequence (Ay_n) has a convergent sub-sequence. Let (Az_n) be the such sub-sequence converging to $z \in Y$, where (z_n) is a sub-sequence of (y_n).

Since $||Az_n - Ax|| \geq \epsilon$, so

$$Ax \neq y \tag{1.2}$$

Now let $f \in Y^*$, then $fA \in X^*$. Since $(z_n) \xrightarrow{w} y \in Y$, so

$$\begin{aligned}(fA)x &= \lim_{x\to\infty} fA(z_n)\\ &= f(\lim_{x\to\infty} Az_n)\\ &= f(z),\end{aligned}$$

since f is uniformly continuous.

Since $f(Ax) = f(z)$ for all $f \in Y^*$, so by "Hahn Banach theorem" $Ax = y$, which is contradict to (1.2).

So our assumption is wrong, hence (Ax_n) converges to Ax. □

Remark 1.3.5. *The converge of the above theorem is not true.*

Example 1.3.3. *Since in ℓ^1 every weakly convergent sequence is convergent by Theorem 1.3.11, so for the identity operator I, $I(x_n)$ converges to $I(x)$, for all $x_n \xrightarrow{w} x$.*

But by Example 1.2.1 identity operator on ℓ^1 is not a compact operator.

Theorem 1.3.11. *In ℓ^1 every weakly convergent sequence is converges.*

Proof. Let us assume that sequence $(x_n) \in \ell^1$ weakly converges to $x \in \ell^1$ that is, $x_n \xrightarrow{w} x$ but (x_n) does not converges to x.

Without lose of generality $x = 0$. Since (x_n) does not converges to 0, so $\exists\ \epsilon > 0$ such that,

$$||x_n||_1 = \sum_{j=1}^{\infty} |\ x_n(j)\ | \geq \epsilon, \quad \forall n \in \mathbb{N}$$

For $j \in \mathbb{N}$, let f_j denote the jth co-ordinate functional. Then $f_j(x) = x(j)$, $x \in \ell^1$. Since $f(x_n)$ converges to 0, for all $f \in {\ell^1}^*$, so for each $j \in \mathbb{N}$, $x_n(j) = f_j(x_n)$ converges to 0.

Now since $x_n(1)$ converges to 0, so $\exists\ n_1 \in \mathbb{N}$, such that $|x_n(1)| < \epsilon\ \forall n \geq n_1$.

Let $m_1 \in \mathbb{N}$, since for each fixed $j \in \mathbb{N}$, $x_n(j)$ converges to 0, so $\exists\ k_j \in \mathbb{N}$ such that for each $j \in \mathbb{N}$,

$$|\ x_n(j)\ | \leq \frac{\epsilon}{2m_1}, \quad \forall n \geq k_j$$

$$\implies |\ x_n(j)\ | \leq \frac{\epsilon}{2m_1}, \quad \forall n \geq max\{k_1, k_2, \ldots, k_{m_1}\}$$

$$\implies \sum_{j=1}^{m_1} |x_n(j)| < \epsilon, \quad \forall n \geq max\{k_1, k_2, \ldots, k_{m_1}\} \tag{1.3}$$

Now if $max\{k_1, k_2, ..., k_{m_1}\} = k > n_1$, then put $k = n_2$. Otherwise if $k \leq n_1$, then choose $n_2 \in \mathbb{N}$, such that $n_2 > n_1$. So in both cases inequality (1.3) is holds.

Therefore $\exists\ n_2 > n_1$ such that

$$\sum_{j=1}^{m_1} |x_n(j)| < \epsilon, \forall n \geq n_2$$

$$\implies \sum_{j=1}^{m_1} |x_{n_2}(j)| < \epsilon$$

Since $x_{n_2} \in \ell^1$, so $\exists\ \ m_2 > m_1, m_2 \in \mathbb{N}$, such that

$$\sum_{j=m_2+1}^{\infty} |x_{n_2}(j)| < \epsilon$$

Continuing this process we get strictly increasing sequences of natural numbers (n_k) and (m_k) such that

$$\sum_{j=1}^{m_{k-1}} |x_{n_k}(j)| < \frac{\epsilon}{5}$$

and

$$\sum_{j=m_k+1}^{\infty} |x_{n_k}(j)| < \frac{\epsilon}{5}$$

Put $m_0 = n_0 = 1$ define

$$y(j) = \begin{cases} 1 & \text{if } j = 1 \\ sgnx_{n_k} & \text{if } m_{k-1} + 1 \leq j \leq m_k, k \in \mathbb{N} \end{cases}$$

Then $y \in \ell_\infty$ and $||y|| \leq 1$.

Define $f_y : \ell^1 \to \mathbb{K}$ by

$$f_y(x) = \sum_{j=1}^{\infty} x(j)y(j), x \in l^1$$

Then $f_y \in \ell^{1^*}$. Now for each fixed $k \in \mathbb{N}$,

$x_{n_k}(j)y(j) = |x_{n_k}(j)|$, for $m_{k-1} + 1 \leq j \leq m_k$.

Since

$$sgnx = \begin{cases} 0 & \text{if } x = 0 \\ \frac{|x|}{x} & \text{if } m \neq 0 \end{cases}$$

So ,

$$\begin{aligned} |f_y(x_{n_k}) - \sum_{j=1}^{\infty} |x_{n_k}(j)|| &\leq 2 \sum_{j=1}^{m_{k-1}} |x_{n_k}| + 2 \sum_{j=_{m_k}+1}^{\infty} |x_{n_k}(j)| \\ &< \frac{4\epsilon}{5} \end{aligned}$$

$$\implies |f_y(x_{n_k})| \geq ||x_{n_k}||_1 - \frac{4\epsilon}{5} \geq \epsilon - \frac{4\epsilon}{5} = \frac{\epsilon}{5} \tag{1.4}$$

But since $x_n \xrightarrow{w} 0$, so $x_{n_k} \xrightarrow{w} 0$. Since $f_y \in \ell^{1^*}$, so $f_y(x_{n_k})$ converges to 0, which is contradictory to (1.4). Therefore our assumption is wrong. So every weakly convergent sequence in ℓ^1 is convergent. $\square$

Definition 1.3.3. *(Adjoint of an Operator): Let X and Y are Hilbert spaces. An adjoint operator $T^* \in B(X,Y)$ of $T \in B(X,Y)$ is an operator such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, $\forall x \in X, y \in Y$.*

Theorem 1.3.12. *Let X and Y be Hilbert spaces, and $T \in B(X,Y)$. Then $T \in \mathcal{K}(X,Y) \iff T^* \in \mathcal{K}(Y,X)$*

Proof. Let (y_n) be a bounded sequence in Y. Then there exists $c > 0$, such that $\|y_n\| \leq c$, $\forall n \in \mathbb{N}$

We prove that (T^*y_n) has a convergent sub-sequence.

Since $T \in \mathcal{K}(X,Y)$ and $T^* \in B(Y,X)$, so $TT^* \in \mathcal{K}(Y)$, so (TT^*y_n) has a convergent sub-sequence, say $(TT^*y_{n_k})$.

Then we have,

$$\begin{aligned}
(||T^*y_{n_k} - T^*y_{n_m}||)^2 &= \langle T^*y_{n_k} - A^*y_{n_m}, T^*y_{n_k} - T^*y_{n_m}\rangle \\
&= \langle TT^*y_{n_k} - TT^*y_{n_m}, y_{n_k} - y_{n_m}\rangle \\
&\leq ||TT^*y_{n_k} - TT^*y_{n_m}||||y_{n_k} - y_{n_m}|| \\
&\leq 2c||TT^*y_{n_k} - TT^*y_{n_m}||
\end{aligned}$$

So $(T^*y_{n_k})$ is a Cauchy sequence in X so it is converges to X. Hence $T^\star$ is compact operator.

Conversely , let $T^* \in (Y,X)$ is compact operator. Since $T = (T^*)^*$, so replacing T by T^* in the proof we get $T \in B(X,Y)$ is compact operator. □

Theorem 1.3.13. *Let X and Y be normed linear spaces and $T : X \to Y$ be an injective compact linear operator. Then $T^{-1} : R(T) \to X$ is continuous iff $rankT < \infty$.*

Proof. Let $T^{-1} : R(T) \to X$ is continuous. Since T is compact operator, so the inclusion operator on $R(T)$, namely $TT^{-1} : R(T) \to Y$, is a compact operator. Hence $R(T)$ is finite dimensional.

Conversely, let $rankT < \infty$, then $T^{-1} : R(T) \to X$ is linear operator defined on finte dimensional space, so continuous. □

Theorem 1.3.14. *Let X and Y be normed linear spaces, and $T : X \to Y$ be a linear operator. Then T is bounded below, i.e., there exists $c > 0$ such that*

$$||Tx|| \geq c||x||, \forall x \in X$$

if and only if T is injective and $T^{-1} : R(T) \to X$ is continuous, and in this case

$$||T^{-1}y|| \leq \frac{1}{c}||x||, \forall y \in R(T).$$

Proof. Let T is bounded below, that is there exists $c > 0$, such that $||Tx|| \geq c\|x\|, \forall x \in X$. Then $kerT = \{0\}$, so T is injective. To prove $T^{-1} : R(T) \to X$ is continuous, let $y \in R(T)$, then there exists $x \in X$ such that $y = Tx$. Then we have

$$||y|| = ||Tx|| \geq c||x|| = c||T^{-1}y||.$$

So,

$$||T^{-1}y|| \leq \frac{1}{c}||y||, \forall y \in R(T)$$

Thus, $||T^{-1}|| < \infty$, so $T^{-1} : R(T) \to X$ is continuous and $||T^{-1}y|| \leq \frac{1}{c}||y||, \forall y \in R(T)$.

Conversely, let T is injective and $T^{-1} : R(T) \to X$ is continuous. Then there exists $d > 0$, such that

$$||T^{-1}y|| \leq d||y||, \forall y \in R(T).$$

For $x \in X$, let $y = Tx$. Then by above relation, we have $||x|| \leq c||Tx||$. So $||x|| \leq c||Tx\|, \forall x \in X$ □

Remark 1.3.6. *Let X and Y be normed linear spaces and $T : X \to Y$ is an infinite rank operator which is bounded below. Then T is not a compact operator.*

Proof. Since T is bounded below, so by above theorem T is injective and T^{-1} : $R(T) \to X$ is continuous, so if T is compact operator then $R(T)$ is finite dimensional, which is contradicts that, T is infinite rank operator. □

Theorem 1.3.15. *If $T \in \mathcal{K}(X, Y)$ and if T is bounded below on a subspace M of X, then M is finite-dimensional.*

In particular, if N is a closed subspace of Y such that N is contained in the range of T, then N is finite-dimensional.

Proof. Since T is bounded below on M, so T is also bounded below (by the same constant) on $cl\{M\}$; we may therefore assume, without loss of generality, that M is closed.

Let us assume that M is infinite dimensional. Then M contains an infinite orthonormal set, say $\{e_n : n \in \mathbb{N}\}$.

Now if T is bounded below by ϵ on M, then note that

$$\|Te_n - Te_m\| \geq \epsilon\sqrt{2}, \ \ \forall n \neq m.$$

Hence e_n is a bounded sequence in H such that (Te_n) has no Cauchy subsequence, contradicting our assumption that the compactness of T; hence M must be finite-dimensional.

As for the second assertion, let $M = T^{-1}(N) \cap (kerT)^{\perp}$. Then T maps M 1-1 onto N so by the "Open mapping theorem", T must be bounded below on M; hence by the first assertion, M is finite-dimensional, and so also is N . □

Definition 1.3.4 (Quotient operator)**.** *We know if $T : X \to Y$ is a bounded linear operator and X_0 is closed subspace X, then the restriction of T to X_0 is a bounded linear operator. It is also clear that if T is compact operator, then for every subspace X_0 of X restriction of T on X_0 is compact operator. Also if X_0 is closed subspace of X such that $X_0 \subseteq N(T)$, then the quotient operator $\widetilde{T} : X/X_0 \to Y$ defined by*

$$\widetilde{T}(x + X_0) = Tx, x \in X$$

is a bounded operator. Do such results hold if boundedness condition is removed by compactness ?

Theorem 1.3.16. *Let X and Y be normed linear spaces and $X_0 \subseteq N(T)$ be a closed subspace of X. If $T : X \to Y$ is compact operator, then $\widetilde{T} : X/X_0 \to Y$ is also a compact operator.*

Proof. Suppose $T : X \to Y$ is compact operator. Let (x_n+X_0) be a bounded sequence in X/X_0, that is there exists $c > 0$, such that

$$||x_n + X_0|| := inf\{||x_n + u|| : u \in X_0\} \leq c, \forall n \in \mathbb{N}.$$

Then there exists a sequence (u_n) in X_0 such that

$$||x_n + u_n|| \leq 2c, \forall n \in \mathbb{N}.$$

Since T is compact, so $(T(x_n + u_n))$ has a convergent sub-sequence. Also since $T(x_n + u_n) = \widetilde{T}(x_n + X_0)$, $\forall n \in \mathbb{N}$, so $(\widetilde{T}(x_n + X_0))$ has convergent sub-sequence. Hence $\widetilde{T}$ is a compact operator. □

Theorem 1.3.17. *Let X and Y are Banach spaces, and $T : X \to Y$ is a compact operator with $R(T)$ closed in Y. Then $rankT$ is finite.*

Proof. Since $N(T)$ is closed subspace of X, so $\widetilde{T} : X/N(T) \to Y$ defined by $\widetilde{T}(x + N(T)) = Tx$, is a compact operator. Since X is Banach space and $N(T)$ is closed subspace of X, so the quotient space $X/N(T)$ is Banach space. Now $ker\widetilde{T} = \{x + N(T) : \widetilde{T}(x + N(T)) = Tx = 0\} = \{N(T)\}$, so $\widetilde{T}$ is injective and $R(T) = R(\widetilde{T})$ is Banach space, since $R(T)$ is closed subspace of Banach space Y.

Thus $\widetilde{T} : X/N(T) :\to R(\widetilde{T})$ is bijective bounded linear operator between two Banach spaces, so by "**Bounded inverse theorem**" inverse of $\widetilde{T}$ is continuous. Since $\widetilde{T}$ is compact operator, so by theorem 1.3.13 $R(\widetilde{T}) = R(T))$ is finite dimensional. ☐

Definition 1.3.5. *(Transpose of an operator): Let $T \in B(X, Y)$ be a bounded linear operator and X^* and Y^* are Dual spaces of X and Y are respectively. For $f \in Y^*$ and $x \in X$, we define*

$$(T'f)(x) = f(Tx).$$

Now

$$|(T^*f)(x)| = |f(Tx)| \leq ||f||||Tx|| \leq ||f||||T||||x||$$

for all $x \in X$ and $f \in Y^*$, so $T'f \in X^*$ for all $f \in Y^*$.

Clearly, $T' : Y^* :\to X^*$ is linear operator. Also from above $||T'f|| \leq ||f||||T|| \implies ||T'|| \leq ||T||$, so $T' \in B(Y^\star, X^\star)$. Since by "Hahn Banach extension theorem" there exists $f \in Y^*$ such that, $||f|| = 1$ thus $||T'|| = ||T||$.

Definition 1.3.6. *(Equicontinuous) Let (Ω, d) be a metric space with metric d. Then a subset $\mathcal{S} \subseteq \mathcal{F}(\Omega, \mathbb{K}) = \{f : \Omega \to \mathbb{K} : f \text{ is continuous}\}$ is said to be equi-continuous if for every $\epsilon > 0$, there exists $\delta > 0$, such that*

$$s, t \in \Omega, d(s, t) \ < \delta \implies | x(s) - x(t) | < \epsilon, \forall x \in \mathcal{S}.$$

Theorem 1.3.18. *For a norm linear space X,*

$$||x|| = sup\{|f(x)| : f \in X^*, ||f|| \leq 1\}.$$

Proof. Let $f \in X^*$ such that $||f|| = 1$, then $|f(x)| \leq ||x||$. Therefore,

$$sup\{|f(x)| : f \in X^*, ||f|| \leq 1\} \leq ||x||.$$

Again by "**Hahn Banach theorem**", for every $x \in X$, there exists $f_x \in X^*$ such that $||f_x|| = 1$ and $f_x(x) = ||x||$. Therefore

$$||x|| = |f_x(x)| \leq sup\{|f(x)| : f \in X^*, ||f|| \leq 1\}.$$

Hence for every $x \in X$,

$$||x|| = sup\{|f(x)| : f \in X^*, ||f|| \leq 1\}.$$

□

Definition 1.3.7. *(Canonical linear isometry)*

Let X be a normed linear space. For each $x \in X$, consider the evaluation functional

$$\varphi_x(f) = f(x), f \in X^*.$$

Since $f \in X^*$ is linear, so φ_x is a linear functional on X^*. Also

$$|\varphi_x(f)| = |f(x)| \leq ||x||||f||, \forall f \in X^*,$$

so $\varphi_x \in (X^*)^*$ and $||\varphi_x|| \leq ||x||$.

We denote the space $(X^*)^*$ is dual of dual space X^*, by X^{**}, and called second dual of X. Similarly we can define third dual, the forth dual and so on.

Now let $J : X \to X^{**}$ be defined by

$$J(x) = \varphi_x, \ \ x \in X$$

Then, since φ is linear so J is linear operator and $||J(x)|| \leq ||x||$ for all $x \in X$. Thus J is bounded linear operator. Also J is linear isometry, the isometry $J : X \to X^{**}$ is called the "**canonical linear isometry**" from X into X^{**}.

Theorem 1.3.19. *Let X be a normed linear space and $J : X \to X^{**}$ be defined by $J(x) = \varphi_x$, where*

$$\varphi_x(f) = f(x), \ \ \forall f \in X^*, \ \ \forall x \in X.$$

Then J is a linear isometry.

Proof. From above $J : X \to X^{**}$ be defined by $J(x) = \varphi_x$, where

$$\varphi_x(f) = f(x) \ \ \forall f \in X^*, \ \ \forall x \in X.$$

is bounded linear operator and $||J(x)|| \leq ||x||$ for every $x \in X$. Now we will show that $||x|| \leq ||J(x)||$ for every $x \in X$. By "**Hahn Banach theorem**" for every $x \in X$, there exists $f_x \in X^*$ such that $||f_x|| = 1$ and $f_x(x) = ||x||$. Hence ,

$$||x|| = |f_x(x)| = |(Jx)(f_x) \leq ||Jx||, \ \ \forall x \in X.$$

Hence complete the proof. □

Theorem 1.3.20. *Let X and Y be normed linear spaces and $T \in B(X,Y)$. If $T$is compact operator then the transpose T' of T is compact operator and converse, is hold if Y is Banach space.*

Proof. Let $T \in B(X,Y)$ be a compact operator and (f_n) is a bounded sequence in Y^*, then there exists $c > 0$ such that $||f_n|| \leq c, \forall n \in \mathbb{N}$. For $y, z \in Y$

$$|f_n(y) - f_n(z)| \leq ||f_n||||y - z||$$

Let $\Omega = cl\{Tx : ||x|| \leq 1\}$, then Ω is compact, since T is compact operator. Let $g_n = f_n|_\Omega, n \in \mathbb{N}$, then $\{g_n : n \in \mathbb{N}\}$ is a set of uniformly bounded equicontinuous function on the compact metric space Ω. Hence by "**Arzera – Ascoli's theorem**" $g_n = f_n|_\Omega, n \in \mathbb{N}$, has convergence subsequence, say $(h_n)_{n\in\mathbb{N}}$

Now, for all $n, m \in \mathbb{N}$

$$\begin{aligned} ||T'h_n - T'h_m|| &= sup\{|T'(h_n - h_m)x| : ||x|| \leq 1\} \\ &= sup\{|(h_n - h_m)(Tx) : ||x|| \leq 1\} \\ &\leq sup\{|(h_n - h_m)y| : y \in \Omega\} \end{aligned}$$

Since the sequence (h_n) is uniformly Cauchy on Ω, so $(T'h_n)$ is cauchy sequence in X^* and it is convergent as X^* is Banach space. Thus $(T'f_n)$ has convergent subsequence $(T'h_n)$ and so $T' \in B(Y^*, X^*)$ is a compact operator.

Conversely, let Y is Banach space and $T' \in B(Y^*, X^*)$ is a compact operator. Then $T'' : X^{**} :\to Y^{**}$ is compact operator. Let (x_n) be a bounded sequence in X. Since $Y$is Banach space, we will show that (Tx_n) has a cauchy subsequence. Let $\varphi_{x_n} = Jx_n$, $n \in \mathbb{N}$, where $J : X \to X^{**}$ is the linear isometry. Then $||\varphi_{x_n}|| = ||x_n||$, so (φ_{x_n}) is a bounded sequence in X^{**}. Therefore by compactness of T'', the sequence

$(T''\varphi_{x_n})$ has a convergent subsequence, say $(T''\varphi_{x_{n_k}})$. Now

$$\begin{aligned}
||Tx_{n_k} - Tx_{n_m}|| &= sup\{|f(Tx_{n_k} - Tx_{n_m})| : f \in Y^*, ||f|| \leq 1\} \\
&= sup\{|(T'f)(x_{n_k} - x_{n_m})| : ||f|| \leq 1\} \\
&= sup\{|(\varphi_{x_{n_k}} - \varphi_{x_{n_m}})(T'f)| : ||f|| \leq 1\} \\
&= sup\{||T''(\varphi_{x_{n_k}} - \varphi_{x_{n_m}})(f)|| : ||f|| \leq 1\} \\
&= ||T''\varphi_{x_{n_k}} - T''\varphi_{x_{n_m}}||
\end{aligned}$$

for all $k, m \in \mathbb{N}$. Thus, (Tx_{n_k}) is a Cauchy subsequence of (Tx_n), which is convergent.

$\square$

Chapter 2

Fredholm Operator

2.1 Introduction:

Definition 2.1.1. *A Fredholm operator is a bounded linear operator $S : X \to Y$ between two Banach spaces with finite dimensional kernel and co-kernel, where co-kernel is the dimension of $cokerT = Y/R(T)$.*

Definition 2.1.2. *Let $T \in B(X,Y)$ be a Fredholm operator. Define the "**Index of T**" by*

$$indexT := dim(kerT) - dim(Y/R(T)).$$

2.2 Examples:

Example 2.2.1. *If $A : X \to Y$ be a linear operator, where X and Y are both finite dimensional Hilbert spaces, then A is Fredholm and ind A= dim X - dim Y.*

Proof. $A \in B(X,Y)$ is Fredholm, since domain and range spaces both are finite dimensional. From linear algebra we know that,

$$dimX = dimR(A) + dim(kerA) = dimY - dim(Y/R(A)) + dim(kerA).$$

Thus $indA = dim(kerA) - dimR(A) = dimX - dimY$. □

Example 2.2.2. *Let $S_{-1} : \ell^2 \to \ell^2$ be the unilateral shift operator given by*

$$S_{-1}(a_0, a_1, a_2, \dots) = (0, a_0, a_1, \dots).$$

Then it is a Fredholm operator.

Proof. S_{-1} is injective, since $kerS_{-1} = \{0\}$, so dim $kerS_{-1} = 0$. Also, $cokerS_{-1} = \ell^2/R(S_{-1}) = Span\{e_1 + R(S_{-1})\}$, and so dim $cokerS_{-1} = 1 < \infty$.

Hence, S_{-1} is a Fredholm operator, and
$ind(S_{-1}) = dim(kerS_{-1}) - dim(cokerS_{-1}) = 0 - 1 = -1$. □

Example 2.2.3. *Let $S_1 : \ell^2 \to \ell^2$ be a left shift operator given by $S_1(a_0, a_1, a_2, \dots) = (a_1, a_2, \dots)$. Then S_1 is Fredholm operator*

Proof. We know S_1 is adjoint of S_{-1}. In this case, $kerS_1 = Span(e_1)$ and $cokerS_1 = \{R(S_1)\}$. Hence, S_2 is also Fredholm operator and

$$ind(S_2) = dim(kerS_1) - dim(cokerS_1) = 1 - 0 = 1$$

. □

Remark 2.2.1. *It is clear that $S_{-n} := (S_{-1})^n$ and $S_n := (S_1)^n$. are both Fredholm with index $-n$ and n, respectively. Hence, for any $k \in \mathbb{Z}$, there are Fredholm operators on ℓ^2 with index k.*

2.3 Properties:

Lemma 2.3.1. *Let H_1 and H_2 are Hilbert spaces and $T \in B(H_1, H_2)$ such that Kernel and Co-kernel of T both are finite dimensional, then $R(T)$ is closed.*

Proof. Let $\widetilde{T}$ be restriction of T to $(KerT)^\perp$. Then $\widetilde{T}$ is bounded linear operator. Since T is Fredholm operator, so dimension of co-kernel of T is finite say, $codimT = n$. Now if R is algebraic complement of $R(T)$, then $H_2 = R(T) \oplus R$ and R is isomorphic to $H_2/R(T)$. So dimension of R is n. Let $S : C^n \to R$ be a linear bijective map, such a map exists since dim $C^n = dimR$.

Define $T_1 : (kerT)^\perp \oplus C^n \to H_2$ by $T_1(x, y) = \widetilde{T} + Sy$. Then T_1 is one-one and onto, since $H_2 = R(T) \oplus R$. So By "**Bounded inverse theorem**", T^{-1} is bounded and continuous. Hence $R(T) = T_1((kerT)^\perp \oplus \{0\})$ is closed, since $(kerT)^\perp \oplus \{0\})$ is closed. □

Theorem 2.3.1. *Let X is a Banach space and T is a normed linear space and $T : X \to Y$ is a bounded linear operator. If T is bounded below i.e., there exists $c > 0$, $||Tx|| \geq c||x||$, $\forall x \in X$. Then $R(T)$ is closed subspace of Y.*

Proof. Let (y_n) be a sequence in $R(T)$ converges to $y \in Y$. Since $y_n \in R(T)$, so there exists $x_n \in X$ such that $Tx_n = y_n$.

Now Since (y_n) is Cauchy sequence in Y and T is bounded below, so (x_n) is a Cauchy sequence in X. Since X is Banach space, (x_n) converges to $x \in X$, for some $x \in X$.

So by continuity of T, (Tx_n) converges to Tx. Therefore $y = Tx \in R(T)$. Hence $R(T)$ is closed subspace of Y. □

Theorem 2.3.2. *Let X and Y are Hilbert spaces and $T \in B(X,Y)$. Then R(T) is closed iff $R(T^*)$ is closed.*

Proof. Let Y$_0$:=R(T) is closed and $T_0 : X \to Y_0$ be defined by $T_0x = Tx$, $\forall x \in X$. Then $T_0 \in B(X,Y)$. We want to proof $R(T_0^*)$ is closed and $R(T_0^*) = R(T^*)$.

Since $T_0 : X \to Y_0$ is surjective bounded linear operator between two Banach spaces , By consequence of "**Open mapping theorem**", there exists $c > 0$, such that for all $y \in Y_0$, there exists $x \in X$ satisfy $T_0x = y$ and $||x|| \leq c||y||$.

Now for every $u \in Y_0$,

$$\begin{aligned}
|\langle y, u\rangle| &= |\langle Tx, u\rangle| \\
&= |\langle x, T_0^*u\rangle| \\
&\leq ||x||||T_0^*u|| \\
&\leq c||y||||T_0^*u||
\end{aligned}$$

which hold for all, $y \in Y_0$.

Since for an element $z \in X^*$, in an inner product space X^*,

$$\|z\| = sup\{|\langle z, u\rangle| : u \in X_1, ||u|| \leq 1\}$$

So $||u|| \leq c||T_0^*u||$, for all $u \in Y_0$.

Therefore T_0^* is bounded below and so by theorem 2.3.1 $R(A_0^*)$ is closed.

It remains to prove that $R(T^*) = R(A_0^*)$.

Now

$$\langle T_0x, y\rangle = \langle Tx, y\rangle = \langle x, T^*y\rangle, \forall x \in X, y \in Y$$

$$\implies T_0^*y = T^*y, \forall y \in Y_0$$

Since $Y_0 = R(T)$ is closed in Y. So by "Projection Theorem", we have $Y = R(T) \oplus R(T)^{\perp}$ and $R(T)^{\perp} = N(T^*)$.

Therefore $R(T_0^*) = \{T_0^* y : Y \in R(T)\}$=$\{T^* y : Y \in R(T)\} = \{T^* y : y \in Y\} = R(T^*)$.

Hence $R(T^\star)$ is closed.

Conversely, let $R(T^*)$ is closed. Since, $T^{**} = T$. So by above result $R(T) = R(T^{**})$ is closed. □

Proposition 2.3.1. *Let X and Y be Hilbert spaces, and $T \in B(X, Y)$. Then following are holds.*

i) $N(T) = R(T^\star)^{\perp}$

ii) $N(T^*) = R(T)^{\perp}$

iii) $N(T)^{\perp} = cl\{R(T^*)\}$

iv) $N(T^*)^{\perp} = cl\{R(T)\}$

Proof. Note that for $x \in X$,

$x \in R(T^*)^{\perp} \iff \langle x, T^* y\rangle = 0, \forall y \in Y \iff \langle Tx, y\rangle = 0, \forall y \in Y \iff Tx = 0 \iff x \in N(T)$.

Hence i) is proved.

Since $(T^*)^* = T$, replacing T by T^* in proof i) we get prove ii).

Now we know that for any subset S of X $(S^{\perp})^{\perp} = cl\{spanS\}$.

Taking both side of i) perpendicular we get result iii) and taking both side of ii) perpendicular we get result iv). □

Theorem 2.3.3. *Adjoint of a finite rank bounded linear operator is finite rank operator and dimension is same.*

Proof. Let X and Y are Hilbert spaces and $T : X \to Y$ be a finite rank bounded linear operator. Consider the restrict of T to $ker(T)^{\perp}$,

$$T|_{(kerT)^{\perp}} : (kerT)^{\perp} \to ran(T)$$

is an bijective bounded linear operator between Hilbert spaces, since $(kerT)^{\perp}$ is closed subspace of Hilbert space. So by "**Bounded inverse theorem**", the inverse of the restriction is continuous, hence the restriction of T on $(kerT)^{\perp}$ is an isomorphism. Therefore $dim(kerT)^{\perp} = dimR(T)$. Also we know $(kerT)^{\perp} = cl\{R(T^*)\}$. So $dim\ cl\{R(T^*)\} < \infty$ and $R(T^*)$ is closed subspace of X. Since $cl\{R(T^*)\}$ is smallest closed set containing $R(T^*)$, so $cl\{R(T^*)\} = R(T^*)$, thus $dim\ R(T^*) = dim\ R(T)$. □

Theorem 2.3.4. *Let H_1, H_2 are Hilbert spaces and $T \in B(H_1, H_2)$ is Fredholm operator. Then the adjoint T^* of T is Fredholm and $indexT^* = -indexT$.*

Proof. We know $H_2 = cl\{R(T)\} \oplus kerT^*$. Since $R(T)$ is closed, so $H_2 = R(T) \oplus kerT^*$. As quotient space $H_2/kerT$ is isomorphic to $kerT^*$ and dimension of $H_2/R(T)$ is finite, so dim $kerT^*$ is finite.

Similarly $H_1 = R(T^*) \oplus kerT$, since $R(T^*)$ is closed. As $dim(kerT)$ is finite, so dimension of $H_1/R(T^*)$, that is dimension of $cokerT^*$ is finite.

Hence adjoint of T is Fredholm operator. Now

$$indexT^* = dim(kerT^*) - codimT^* = codimT - dim(kerT) = -indexT.$$

□

Remark 2.3.1. *$T \in B(H_1, H_2)$ is Fredholm iff $T^* \in B(H_2, H_1)$ is Fredholm.*

Theorem 2.3.5. *Let $T \in B(H_1, H_2)$ be bijective, and let $K \in B(H_1, H_2)$ be compact. Then $T + K$ is a Fredholm operator.*

Proof. $ker(T + K)$ is a subspace of Hilbert space H_1, and in particular it is a linear space, so for $x \in ker(T + K)$ we have $Tx = -Kx$. Let $(x_n)_{n\in\mathbb{N}} \subseteq ker(T + K)$ be a bounded sequence. Since $K \in B(H_1, H_2)$ is a compact operator, so the sequence $(Kx_n)_{n\in\mathbb{N}}$ has a convergent sub-sequence, say $(Kx_{n_k})_{k\in\mathbb{N}}$. But $x_{n_k} \in ker(T + K)$ for each $k \in \mathbb{N}$, so $(Kx_{n_k})_{k\in N} = (-Tx_{n_k})_{k\in\mathbb{N}}$, which tells us that $(x_{n_k})_{k\in\mathbb{N}}$ is convergent, since by Bounded inverse theorem T^{-1} is bounded linear operator.

Hence any bounded sequence in $ker(T+K)$ has a convergent sub sequence, which means that $dim(ker(T + K)) < \infty$, since an infinite dimensional Hilbert space has an infinite orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ with no convergent sub sequences as it is not Cauchy sequence.

We know that $H_2 = cl\{R(T+K)\} \oplus ker(T^* + K^*)$, since $H_2 = (ker(T+K)^*)^{\perp} \oplus ker(T+K)^*$ and $cl\{R(T+K)\} = (ker(T+K)^*)^{\perp}$ and since T^* is invertible and K^* is compact, we get by the above that $dim(ker(T^* + K^*)) < \infty$.

This means that we only have to check that $R(T + K)$ is closed in order to see that $codim(T + K) < \infty$.

To see this we split H_1 into the direct sum $H_1 = \widetilde{H_1} \oplus ker(T+K)$, and we consider the restriction of $T + K$ to $\widetilde{H_1}$. We want to show that for all $x \in \widetilde{H_1}$, the inequality

$$||x|| \leq c||k(T+K)x||$$

holds for some $c > 0$. In order to show this inequality, we assume that for all $c > 0$ there exists $x \in \widetilde{H_1}$ such that $||x|| \geq c||(T+K)x||$. Then there exist sequences $(c_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$, $(x_n)_{n\in\mathbb{N}} \subseteq \widetilde{H_1}$ such that $||x_n|| = 1$ for all $n \in \mathbb{N}$, $c_n \to \infty$ as $n \to \infty$, and $1 = \|x_n\| \geq c_n\|(T+k)x_n\|$, for all $n \in \mathbb{N}$. Hence $\|(T+K)x_n\| \leq 1\ \frac{1}{c_n} \to 0$ for $n \to \infty$. Since K is compact operator, and x_n has norm 1 for each $n \in \mathbb{N}$, so there exists a sub sequence $(Kx_{n_k})_{k\in\mathbb{N}}$ of $(Kx_n)_{n\in\mathbb{N}}$ which is convergent.

Let us assume that (Kx_{n_k}) converges to $v \in H_2$. This means (Tx_{n_k}) converges to $-v \in H_2$. Thus

$$x_{n_k} = T^{-1}Tx_{n_k} \to -T^{-1}v = w$$

for $k \to \infty$, where $w \in \widetilde{H_1}$. Since

$$\lim_{k\to\infty} x_{n_k} = w \implies ||lim_{k\to\infty}x_{n_k}|| = ||w||$$

$$\implies \lim_{k\to\infty} ||x_{n_k}|| = ||w|| \implies ||w|| = 1$$

since, $||x_{n_k}|| = 1$ for each $k \in \mathbb{N}$.

Now since (x_{n_k}) converges to w and $(Tx_{n_k} + Kx_{n_k})$ converges to 0, so

$$(T+K)w = \lim_{k\to\infty}(Tx_{n_k} + Kx_{n_k}) = 0$$

Hence $w \in ker(T+K)$, which is contradicting that $\widetilde{H_1} \perp ker(T+K)$. Therefore for all $x \in \widetilde{H_1}$,

$$||x|| \leq c||(T+K)x||$$

holds for some $c > 0$. So by theorem 2.3.1 range of the restriction of $T+K$ on $\widetilde{H_1}$ is closed. Since range of the restriction and range of $T+K$ is equal, so $R(T+K)$ is closed set and $H_2 = im(T+K) \oplus ker(T^*+K^*)$. So $coker(T+K)$ is finite dimensional, since $coker(T+K)$ is isomorphic to $ker(T^*+K^*)$. Hence $T+K$ is Fredholm operator. □

Definition 2.3.1. *(Algebra) Let $\mathcal{A}$ be a non-empty set. Then $(\mathcal{A},+,.,\circ)$ is called an algebra*

if

1. *($\mathcal{A}$, +, .) is a vector space over a field F.*

2. *($\mathcal{A}$, +, ∘) is a ring and*

3. $(\alpha a) \circ b = \alpha(a \circ b) = a \circ (\alpha b)$ *for every* $\alpha \in F$, *for every* $a, b \in \mathcal{A}$.

An algebra $\mathcal{A}$ *is called real or complex acoording as* $F = \mathbb{R}$ *or* $F = \mathbb{C}$ *and commutative if* $(\mathcal{A}, +, \circ)$ *is a commutative ring.*

An algebra $\mathcal{A}$ *is said to be unital if* $(\mathcal{A}, +, \circ)$ *has a unit, usually denoted by* 1. *Let* $\mathcal{A}$ *be unital and* $a \in \mathcal{A}$. *If there exists an element* $b \in \mathcal{A}$ *such that* $ab = ba = 1$, *then* b *is called an inverse of* a.

Definition 2.3.2. *(Banach algebra) Let* $\mathcal{A}$ *be an algebra over the field* F *and* $||.||$ *is a norm on* $\mathcal{A}$, *then* $(\mathcal{A}, ||.||)$ *is called "***Banach algebra***" if*

1. $||ab|| \leq ||a||||b||$, *for all* $a, b \in \mathcal{A}$

2. $\mathcal{A}$ *is complete with respect to the norm* $||.||$

$(\mathcal{A}, ||.||)$ *is commutative Banach algebra if* $\mathcal{A}$ *is commutative.*

Example 2.3.1.

1. *The set of real (or complex) numbers is a commutative unital Banach algebra with norm given by the absolute value.*

2. *Take the Banach space* $\mathbb{R}^n$(*or* $\mathbb{C}^n$) *with norm* $||x|| = max\{| x_i |: 1 \leq i \leq n\}$ *and define multiplication point wise:* $(x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_1y_1, \ldots, x_ny_n)$, *is a commutative Banach algebra.*

3. *Let* $H^\infty(D) := \{f : D \to \mathbb{C} : f \text{ is bounded and analytic}\}$, *where* $D = \{z \in \mathbb{C} : ||z|| < 1\}$ *is a commutative unital Banach algebra with respect to the point wise addition, multiplication of functions and usual scalar multiplication of functions and the supremum norm.*

4. *Let* X *be a complex Banach space and* $B(X)$ *is the Banach space of bounded linear operators on* X *with respect to the operator norm. With composition of operators as multiplication,* $B(X)$ *is a non commutative, unital Banach algebra.*

5. *Let* (X, μ) *be a finite measure space and* $L^\infty(\mu) := \{f : X \to \mathbb{C} : ess\ \ sup|f| < \infty\}$. *With the point wise multiplication of functions and* $||f||_\infty = ess\ \ sup|f|$, $L^\infty(\mu)$ *is a commutative, unital Banach algebra.*

6. $\mathcal{K}(X) = \{T \in B(X) : T \text{ is compact}\}$ *is a subalgebra of B(X). Hence it is a Banach algebra. It can be verified that* $\mathcal{K}(X)$ *is unital if and only if* $dim(X) < \infty$.

Definition 2.3.3. *(Clakin algebra) Let H be a Hilbert space.We know $K(H)$ is a ideal of $B(H)$.*

*Now the quotient $B(H)/\mathcal{K}(H)$ is a Banach algebra called "***Calkin algebra***", denoted by $\mathcal{C}(H)$.*

Let $\pi : B(H) \to \mathcal{C}(H)$ be the natural projection map defined by

$$\pi(T) := T + \mathcal{K}(H).$$

$I + \mathcal{K}(H)$ is the identity element in The Calkin algebra. An element $T + \mathcal{K}(H)$ is invertible in $C(H)$ if there exists an element $S + \mathcal{K}(H)$ in $\mathcal{C}(H)$ such that

$$(T + \mathcal{K}(H))(S + \mathcal{K}(H)) = (S + \mathcal{K}(H))(T + \mathcal{K}(H)) = I + \mathcal{K}(H)$$

equivalently, $ST = I + K_1$ and $TS = I + K_2$, for some $K_1, K_2 \in \mathcal{K}(H)$.

Theorem 2.3.6. *An operator $T \in B(H)$ is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra. In particular, an operator T is Fredholm if and only if there exists an $S \in B(H)$ such that*

$ST = I + K_1$ and $TS = I + K_2$, for some $K_1, K_2 \in \mathcal{K}(H)$.

Proof. Let $T \in B(H)$ is Fredholm operator.That is $kerT$ and $cokerT$ are finite dimensional. Let $P \in B(H)$ be the or-thogonal projection onto $kerT$, and let $Q \in B(H)$ be the orthogonal projection onto $R(T)$.

The restriction of T at $(kerT)^\perp$ onto $R(T)$ is invertible:

$$T : (kerT)^\perp \to R(T)$$

Let the inverse of the above restriction is

$$S : R(T) \to (kerT)^\perp$$

Now we can extend the domain of S to all of H by defining $S((imT)^\perp) = 0$.

With this extension, it follows that $ST+P$ and $TS+Q$ are the identity operators on H. Since $kerT$ and $(R(T))^\perp$ are finite dimensional, P and Q are compact operators. Hence, $ST = I - P$ and $TS = I - Q$, and so S is the inverse of T modulo compact operators. Thus, $\pi(T)$ is invertible in $\mathcal{C}(H)$.

Conversely, let there exists an $S \in B(H)$ such that $ST = I + K_1$ and $TS = I + K_2$, for some $K_1, K_2 \in \mathcal{K}(H)$. Then $I - ST$ is compact.

Now we have $kerT \subseteq kerST = ker(I + K_1)$. Since $I + K_1$ is Fredholm, so $dim(ker(I + K_1))$ is finite. Therefore dimension of $kerT$ is finite.

Again $R(T) \subseteq R(TS) = R(I + K_2)$, and $I + K_2$ is Fredholm so $codimT \leq codim(I + K_2) < \infty$. Hence T is Fredholm a operator. □

Proposition 2.3.2. *Let H be a Hilbert space, then the set of all Fedholm operators $\mathcal{F}(H)$ is an open set of $B(H)$.*

Proof. We know $\mathcal{F}(H) = \pi^{-1}(\mathcal{C}^{\times}(H))$, where $\mathcal{C}^{\times}(H)$ is the invertible element in the Calkin algebra which is an open set, so $\mathcal{F}(H)$ is a open set of $B(H)$. □

Theorem 2.3.7. *Let $T \in B(X, Y)$ is Fredholm iff there exists $R, S \in B(X, Y)$ and operators K_1 and K_2 which are compact operators on X and Y respectively such that $RT = I + K_1$ and $TS = I + K_2$*

Proof. Let $T \in B(X, Y)$ is a Fredholm operator. Define $\widetilde{T} : \widetilde{X} :\to \widetilde{Y}$, where $\widetilde{X} = (kerT)^{\perp}$ and $\widetilde{Y} = R(T) = (kerT^{*})^{\perp}$ a restriction of T on $(kerT)^{\perp}$ is a bijective operator.

Define $S_2 \in B(Y, X)$, by

$$S_2 = I_{\widetilde{X}} \circ (\widetilde{T})^{\perp} \circ P_{R(T)},$$

where $I_{\widetilde{X}} : \widetilde{X} \to X$ is an inclusion operator. and $P_{R(T)} : Y \to Y$ is projection operator on $R(T)$.

Then $T \circ S_2 = T \circ (\widetilde{T})^{-1} \circ P_{R(T)} = P_{R(T)} = I - P_{kerT^{*}}$, since if $P : X \to X$ is projection operator on $Y \subset X$, where $X = Y \oplus Y^{\perp}$, then $I - P$ is projection operator on $Y^{\perp}$, I is identity operator on X.

Put $K_2 = -P_{kerT^{*}}$, since $dimKerT^{*}$ is finite so K_2 is finite rank bounded linear operator. Therefore K_2 is compact operator.

Also, T^{*} is Fredholm operator, so using same way as above we get that there exists operators $S_3 \in B(Y, X)$ and $K_3 \in K(X)$ such that $T^{*} \circ S_3 = I + K_3$.

$\implies (T^{*} \circ S_3)^{*} = (I + K_3)^{*}$

$\implies S_3^{*} \circ T = I + K_3^{*}$

$\implies S_1 \circ T = I + K_1$, where $S_1 = S_3^{*} \in B(Y, X)$ and $K_1 = K_3^{*} \in B(X)$ is compact operator, since adjoint of a compact operator is compact operator.

Conversely, let the condition is true. Now we have $kerT \subseteq kerS_1 \circ T = ker(I + K_1)$. Since $I + K_2$ is Fredholm, so $dimker(I + K_2)$ is finite. Therefore dimension of $kerT$ is finite.

Again $R(T) \subseteq R(T \circ S_2) = R(I + K_2)$,so $codimT \leq codim(I + K_2) < \infty$, since $(I + K_2)$ is Fredholm.

Hence T is Fredholm operator. □

Remark 2.3.2. *Let $T \in B(X, Y)$ is Fredholm iff there exists $R, S \in B(X, Y)$ and operators K_1 and K_2 which are finite rank operators on X and Y respectively such that $RT = I + K_1$ and $TS = I + K_2$*

Proof. Let $T \in B(X, Y)$ is a Fredholm operator. Define $\widetilde{T} : \widetilde{X} :\rightarrow \widetilde{Y}$, where $\widetilde{X} = (kerT)^{\perp}$ and $\widetilde{Y} = R(T) = (kerT^*)^{\perp}$ a restriction of T on $(kerT)^{\perp}$ is a bijective operator.

Define $S_2 \in B(Y, X)$, by $S_2 = I_{\widetilde{X}} \circ (\widetilde{T})^{\perp} \circ P_{R(T)}$,

where $I_{\widetilde{X}} : \widetilde{X} \rightarrow X$ is an inclusion operator. and $P_{R(T)} : Y \rightarrow Y$ is projection operator on $R(T)$.

Then $T \circ S_2 = T \circ (\widetilde{T})^{-1} \circ P_{R(T)} = P_{R(T)} = I - P_{kerT^*}$, since if $P : X \rightarrow X$ is projection operator on $Y \subset X$ where $X = Y \oplus Y^{\perp}$, then $I - P$ is projection operator on $Y^{\perp}$, I is identity operator on X.

Put $K_2 = -P_{kerT^*}$, since $dimKerT^*$ is finite so K_2 is finite rank bounded linear operator.

Also, T^* is Fredholm operator, so using same way as above we get that there exists operators $S_3 \in B(Y, X)$ and $K_3 \in B(X)$ a finite rank operator such that $T^* \circ S_3 = I + K_3$.

$\implies (T^* \circ S_3)^* = (I + K_3)^*$

$\implies S_3^* \circ T = I + K_3^*$

$\implies S_1 \circ T = I + K_1$, where $S_1 = S_3^* \in B(Y, X)$ and $K_1 = K_3^* \in B(X)$ is finite rank operator, since adjoint of a finite rank operator is finite rank operator.

Conversely, let the condition is true. Now we have $kerT \subseteq kerS_1 \circ T = ker(I + K_1)$. Since $I + K_2$ is Fredholm, so $dimker(I + K_2)$ is finite. Therefore dimension of $kerT$ is finite.

Again $R(T) \subseteq R(T \circ S_2) = R(I + K_2)$, so $codimT \leq codim(I + K_2) < \infty$, since $(I + K_2)$ is Fredholm.

Hence T is Fredholm operator. □

Theorem 2.3.8 (Atkinson's Theorem)**.** *Let H_1 and H_2 are Hilbert spaces. If $T \in B(H_1, H_2)$, then the following conditions are equivalent:*

1. *$T \in B(H_1, H_2)$ is Fredholm.*

2. *There exists operators* $S_1, S_2 \in B(H_2, H_1)$ *and compact operators* $K_i \in K(H_i), i = 1, 2,$ *such that*
$$S_1 T = I_{H_1} + K_1 \text{ and } TS_2 = I_{H_2} + K_2.$$

3. *There exists* $S \in B(H_2, H_1)$, *and finite dimensional sub-spaces* $N_i \subset H_i$, $i = 1, 2,$ *such that*
$$ST = I_{H_1} - P_{N_1} \text{ and } TS = I_{H_2} - P_{N_2}$$
where P_{N_i}, *i=1,2, is the projection on* N_i.

Proof. The proof of all equivalents condition are clear from the above theorems. □

Definition 2.3.4. *Let* $V_0, V_1, V_2, \ldots, V_n$ *be vector spaces, and let* $T_j : V_j \to V_{j+1}$. $0 \le j \le n-1$, *be linear mappings. Then the sequence*
$V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \ldots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n$

*is called "***exact***" if* $R(T_j) = kerT_{j+1}$,$j = \{0, 1, 2, \ldots, n-2\}$.

Lemma 2.3.2 (). *Let* $V_0 = 0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \ldots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n = 0$
be an exact sequence with $dimV_j < \infty$, *for all* $j = \{0, 1, 2, \ldots, n\}$.
Then
$$\sum_{j=1}^{n-1} (-1)^j dimV_j = 0$$
.

Proof. Let decompose of V_j is $V_j = N_j \oplus Y_j$, where Y_j is a algebraic complement of $N_j = kerT_j$ for each j. Then $T_j : Y_j \to n_{j+1}$ is a isomorphism for each j. Hence $dimY_j = dimN_{j+1}$

Again by "**Rank − nullity theorem**", for $j \in 0, 1, 2, \ldots, n-1$

$dimV_j = dimN_j + dimY_j = dimN_j + dimN_{j+1}$. Now $dimN_0 = 0$, and $dimV_{n-1} = dimN_{n-1}$. So by calculation,
$$\sum_{j=1}^{n-1} (-1)^j dimV_j = 0$$
□

Proposition 2.3.3. *If* $V \xrightarrow{f} E \xrightarrow{g} W$ *is an exact sequence and* V *and* W *are finite dimensional, then* E *is finite dimensional.*

Proof. Since V is finite dimensional, so by "Rank nullity theorem", $R(f)$ is finite dimensional. So by exactness of sequences $kerg$ is finite dimensional. Also $R(g)$ is subspace of W, so finite dimensional. We want to prove E is finite dimensional.

Let us assume that E is infinite dimensional. Since W is finite, so there is finitely many elements in E whose images are linearly independent in $R(g)$, otherwise $R(g)$ is infinite dimensional, so $kerg$ contains infinitely many element of E. Therefore $kerg$ is infinite dimensional, which is a contradiction. □

Theorem 2.3.9. *(Multiplicative property of the index). Let $T_1 \in B(H_1, H_2)$ and $T_2 \in B(H_2, H_3)$ are given two Fredholm operators, then $T_2 \circ T_1 \in B(H_1, H_3)$ is also a Fredholm operator, and $indexT_2T_1 = indexT_1 + indexT_2$.*

Proof. Consider the following sequence:

$$0 \to kerT_1 \xrightarrow{i} kerT_2T_1 \xrightarrow{T_1} kerT_2 \xrightarrow{q} H_2/imT_1 \xrightarrow{T_2} H_3/imT_2T_1 \xrightarrow{E} H_3/imT_2 \to 0.$$

Where $i : kerT_1 \to kerT_2T_1$ denotes the inclusion, $q : H_2 \supseteq kerT_2 \to H_2/imT_1$ is the map defined by, $q(x) = x + imT_1$, and E maps equivalence classes modulo imT_2T_1 into equivalence classes modulo imT_2.

Then the above sequence is exact sequence. Since $kerT_1$, $kerT_2$, $cokerT_1$, and $cokerT_2$ are all finite dimensional, by above Proposition , $kerT_2T_1$ and $cokerT_2T_1$ are finite dimensional. Thus, $T_2T_1 \in B(H_1, H_3)$ is Fredholm operator. By Lemma 2.4.2, we have:

$0 = -dimkerT_1 + dimkerT_2T_1 - dimkerT_2 + dimcokerT_1 - dimcokerT_2T_1 + dimcokerT_2 = ind(T_2T_1) - ind(T_1) - ind(T_2)$.

Thus, $ind(T_2T_1) = ind(T_2) + ind(T_1)$. □

Theorem 2.3.10. *Let $T_1 \in B(H_1, H_2)$ and $T_2 \in B(H_2, H_3)$ such that $T_2T_1 \in B(H_1, H_3)$ is Fredholm operator, then T_1 Fredholm iff T_2 Fredholm.*

Proof. We know the sequence:

$$0 \to kerT_1 \xrightarrow{i} kerT_2T_1 \xrightarrow{T_1} kerT_2 \xrightarrow{q} H_2/R(T_1) \xrightarrow{T_2} H_3/R(T_2T_1) \xrightarrow{E} H_3/R(T_2) \to 0$$

is exact, where where $i : kerT_1 \to kerT_2T_1$ denotes the inclusion, $q : H_2 \supseteq kerT_2 \to H_2/imT_1$ is the map defined by, $q(x) = x + imT_1$, and E maps equivalence classes modulo imT_2T_1 into equivalence classes modulo imT_2.

Since T_2T_1 is Fredholm so, $dimH_3/R(T_2T_1) < \infty$. Also $R(T_2T_1) \subseteq R(T_2)$ so, $H_3/R(T_2) \subseteq H_3/R(T_2T_1)$ and $dimH_3/R(T)$ is finite.

Now if T_1 is Fredholm then $kerT_1$ and $cokerT_1$ are finite dimensional, since $KerT_2T_1$ and $CokerT_1$ are finite so by proposition (2.4.2), $kerT_2$ is finite dimensional.

Hence T_2 is fredholm operator.

Conversely, let T_2 is Fredholm. Since $kerT_2$ and $CokerT_2T_1$ are finite dimensional, so by proposition (2.4.2) $dimcokerT_1$ is finite. Also $dimkerT_2$ is finite, since $KerT_2 \subseteq KerT_2T_1$.

Hence T_1 is Fredholm operator. □

Proposition 2.3.4. *Let $F \in B(H)$ is a finite rank operator on Hilbert space H, then $index(I + F) = 0$*

Proof. Define $L := R(F) + (kerF)^{\perp}$, then $dimL < \infty$, since restriction of F on $(kerF)^{\perp}$ onto $R(F)$ is bijective operator. Then $H = L + L^{\perp}$, and $(I+F)L = L$, since we see that $(I + F)L \subseteq L + FL \subseteq L$, also for $u \in L$, $u = Fx + y$ for some $x \in H$ and for some $y \in (kerF)^{\perp}$ implies $u \in (I + F)L$ by definition of L.

Since $L \supseteq (kerF)^{\perp}$, so $L^{\perp} \subseteq kerT$ thus $(I+F)|_L = I|_L$. So L and $L^{\perp}$ are invarient under $I + F$, and we have

$$index(I + F) = index((I + F)|_L) + index((I + F)|_{L^{\perp}})$$

Here clearly $index((I + F)|_L) = 0$ and since $dimL < \infty$, so by "Rank Nullity theorem" $dimker((I + F)|_L) = 0$, $codim((I + F)|_L) = 0$, so $index((I + F)|_L) = 0$.

Hence $index(I + F) = 0$. □

Theorem 2.3.11. *(Invariance of Fredholm property and index under small pertubations). Let $T \in B(H_1, H_2)$ be a Fredholm operator. Then there exists a constant $c > 0$ such that for all operators $S \in B(H_1, H_2)$ with norm $< c$, $T + S$ is a Fredholm operator, which satisfies $index(T + S) = indexT$.*

Proof. Let $T \in B(H_1, H_2)$ is Fredholm operator. The restriction of T on $(kerT)^{\perp}$ $\widetilde{T} : (kerT)^{\perp} \to R(T)$ is a bijectve operator.

Define $R \in B(H_2, H_1)$ by $R = pr_{R(T)^*} \circ (\widetilde{T})^{-1} \circ i_{R(T)}$, where $pr_{R(T)^*}$ is projection operator on $R(T)^*$ and $i_{R(T)}$ is inclution operator on $R(T)$. Then $R \in B(H_2, H_1)$ and

$$R \circ T = pr_{R(T)^*} \circ (\widetilde{T})^{-1} \circ i_{R(T)} \circ T = pr_{R(T)^*} = I - pr_{kerT}$$

Now $R(T + S) = RT + RS + I - pr_{kerT} + RS$. Let $S \in B(H_1, H_2)$ such that $||S|| < (||R||)^{-1}$, so $||RS|| < 1$. Since for any $T \in B(X, Y)$ such that $||T|| < 1$, where

X is non zero norm linear space and Y is Banach space $I+T$ is invertible, so $I+RS$ is invertible.

Since, $RT = I - pr_{kerT}$, so

$$
\begin{aligned}
indR + indT &= ind(I - pr_{kerT}) \\
&\iff indT = -indR + \underbrace{ind(I - pr_{kerT})}_{=0} \\
&= -ind((I+RS)^{-1}R) + \underbrace{ind(I - (I+RS)^{-1}pr_{kerT})}_{=0} \\
&= -ind((I+RS)^{-1}R) + \underbrace{ind(I - (I+RS)^{-1}pr_{kerT})}_{=0} + ind(I+RS) \\
&= -indR + ind(I + RS_p r_{kerT}) \\
&= ind(T+S)
\end{aligned}
$$

Where we use that $-pr_{kerT}$ and $-(I-RS)^{-1}pr_{kerT}$ are finite rank operators, $I+RS$ is invertible operator, so its index is zero, $I - pr_{kerT} + RS$ is Fredholm operator.

Hence $T+S$ is Fredholm operator and $indexT = index(T+S)$. □

Theorem 2.3.12. *(Invariance of Fredholm property and index under compact pertubations). Let $T \in B(H_1, H_2)$ be a Fredholm operator. Then for any compact operator $K \in B(H_1, H_2)$, $T+K$ is a Fredholm operator, and $index(T+K) = indexT$ holds.*

Proof. Let $T \in B(H_1, H_2)$ be a Fredholm operator, and let $K \in B(H_1, H_2)$ be a compact operator. Then there exist $S_1, S_2 \in B(H_2, H_1)$ and operators K_1 and K_2 which are finite rank operators on H_1 and H_2 respectively such that $S_1T = I + K_1$ and $TS_2 = I + K_2$. We see that

$$S_1(T+K) = S_1T + S_1K = I + K_1 + S_1K = I + \widetilde{K_1}$$

$$(T+K)S_2 = TS_2 + KS_2 = I + K_2 + KS_2 = I + \widetilde{K_2}$$

where $\widetilde{K_1}$ and $\widetilde{K_2}$ are finite rank operators.

Hence by by previous theorem $T+K$ is a Fredholm operator and by theorem (2.3.16) index same with index of T. □

Theorem 2.3.13. *Let X be Banach space and $T \in \mathcal{B}(X)$, if $||T|| < 1$, then $I - T$ is invertible and*

$$(I-T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Proof. We have $||T^n|| = ||\underbrace{T \circ T \cdots \circ T}_{\text{n times}}|| \leq ||T||^n$, for all $T \in B(X)$. So

$$\sum_{n=0}^{\infty} ||T^n|| \leq \sum_{n=0}^{\infty} ||T||^n$$

Since $||T|| < 1$, the series

$$\sum_{n=0}^{\infty} ||T||^n$$

is geometric series with common ratio less that 1 , so it is convergent series. Thus

$$\sum_{n=0}^{\infty} T^n$$

is absolutely convergence series in the Banach space $B(X)$, so the series

$$\sum_{n=0}^{\infty} T^n$$

is convergent. So

$$\sum_{n=0}^{\infty} T^n \in B(X) \ \ and \ \ (I - T)\sum_{n=0}^{\infty} T^n = \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n = I.$$

Hence

$$(I - T) \ \ is \ \ invertible \ \ and \ \ (I - T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

□

Theorem 2.3.14. *Let X be a Banach space and $S, T \in B(X)$. If T is invertible and $||T - S|| < \frac{1}{||T^{-1}||}$, then S is invertible and $||S^{-1} - T^{-1}|| \leq ||T^{-1}||^2 ||S - T||$.*

Proof. We have $||T^{-1}(T - S)|| \leq ||T - S|| ||T^{-1}|| < 1$, so $I - T^{-1}(T - S) = T^{-1}S$ is invertible and hence $T(T^{-1}S) = S$ is invertible, since composition of two invertible maps is invertible. □

Theorem 2.3.15. *Let X be a Banach space. Then the set of all invertible element say $G(X)$ in $B(X)$ is open set.*

Proof. Let $T \in G(X)$ and $D = \{S \in B(X) : ||T - S|| < \frac{1}{||T^{-1}||}\}$ Then by above theorem every element of D is invertible and so $D \subseteq G(X)$. Therefore T is an interior point of $G(A)$. Hence $G(X)$ is an open set. □

Theorem 2.3.16. *The map, ind* $: \mathcal{F}(H) \to \mathbb{Z}$ *is locally constant i.e., if* $T_0 \in \mathcal{F}(H)$ *and for* $T \in \mathcal{F}(H)$*, there exists* $\delta > 0$*, such that,* $\|T - T_0\| < \delta$*, then* $indT = indT_0$ *Hence the map* $ind : B(H) \to Z$ *is continous.*

Proof. Let $T \in B(H)$ is Fredholm operator. Let $J : (kerT)^{\perp} \to H$ be the inclusion of $(kerT)^{\perp}$ into H and let $Q : H \to R(T)$ be the orthogonal projection of H onto $R(T)$. Since $kerJ = 0$ and $cokerJ = H/(kerT)^{\perp} = kerT$, J is Fredholm operator with index

$$ind(J) = dimkerJ - dimcokerJ = -dimkerT$$

Similarly, since $kerQ = (R(T))^{\perp} = cokerT$ and $cokerQ = R(T)/R(T) = 0$, so Q is Fredholm operator. with index

$$ind(Q) = dimkerQ - dimcokerQ = dimcokerT$$

Hence,

$$ind(T) + ind(J) + ind(Q) = 0 \tag{2.1}$$

Now $QTJ : (kerT)^{\perp} \to imT$ is invertible, since it is bijective. Fix $\epsilon = \frac{1}{(\|QTJ\|)^{-1}} > 0$,

Let $T_1 \in B(H)$ is a Fredholm operator, such that $||T - T_1|| < \frac{\epsilon}{||Q||||J||}$. Then

$$||QTJ - QT_1J|| = ||Q(T - T_1)J|| \leq ||Q||||T - T_1||||J|| < \epsilon$$

Since $B(H)$ is Banach algebra, so by above theorem QT_1J is invertible element in $B(H)$.

Hence $ind(QT_1J) = 0$, so

$$indQ + indT_1 + indJ = 0 \tag{2.2}$$

Since all Q, T_1 and J are Fredholm operators so, from (2.1) and (2.2) we get $indT = indT_1$.

Hence index map is locally constant, so index map is continuous. $\square$

Proposition 2.3.5. *Let* H *be a Hilbert space and* $T \in B(H)$ *is a Fredholm operator then,* $\forall K \in \mathcal{K}(H)$ *and* $\forall c \in \mathbb{C}$, $T + cK$ *is a Fredholm operator and*

$$indT = ind(T + cK), \;\; \forall K \in \mathcal{K}(H), \forall c \in \mathbb{C}.$$

Proof. Clearly $T + cK$ is a Fredholm operator, since $\forall K \in \mathcal{K}(H), \forall c \in \mathbb{C}$, $T + cK$ is invertiable element in the calkin algebra. Let $T_t = T + tK$, $t \in \mathbb{C}$. Define a map

$\varphi : \mathbb{C} \to \mathbb{Z}$ by

$$\varphi(t) = ind(T_t)$$

We need to prove that φ is locally constant map.

Let us choose $t_0 \in \mathbb{C}$, such that $|t - t_0| < \frac{\delta}{||K||}$, then

$$\begin{aligned} ||T_t - T_{t_0}|| &\leq |t - t_0|||K|| \\ &< \delta \end{aligned}$$

Since the map $ind : B(H) \to Z$ is locally constant, so $indT_t = indT_{t_0}$, whenever $|t - t_0| < \frac{\delta}{||K||}$.

Therefore, φ is locally constant map on connected set, so constant map.

Hence $indT = ind(T + cK), \ \ \forall K \in \mathcal{K}(H), \forall c \in \mathbb{C}$. □

Now we will discus about "**characteristic of Fredholm operator of index 0**".

Proposition 2.3.6. *Let $T \in B(H)$ is a Fredholm operator on a Hilbert space H, then*

$$indT = 0 \iff T = K + S,$$

for some, $K \in \mathcal{K}(H)$ and invertible operator $S \in B(H)$.

Proof. Let us assume that $T \in B(H)$ is a Fredholm operator with $indT = 0$. We khow that $kerT$ is isomorphic to $R(T)^{\perp}$, so there exists a map say, $S : kerT \to R(T)^{\perp}$ which is one one and onto.

Consider an operator, $K : H \to H$, defined by $K(x) = S(x_1)$, where $x = x_1 + x_2$ $x_1 \in kerT, x_2 \in (kerT)^{\perp}$. Since S is finite rank operator, so K is compact operator.

Now the map $(T - K) : H \to H$ is one one, since

$$(T - K)x = 0 \implies T(x_2) = S(x_1) \implies x_1 = x_2 = 0.$$

Also from proposition 2.4.5, we have ,

$$indT = ind(T - K) = 0 \implies dim(ker(T - K)) - coker(T - K) = 0 \implies dim(R(T)^{\perp}) = 0$$

Therefore $T - K$ is onto. Hence $T - K = S_1$, for some invertible operator $S_1 \in B(H)$.

Conversely, if $T = K + S$, for some $K \in \mathcal{K}(H)$ and invertible operator $S \in B(H)$, then T is Fredholm and $indT = indS = 0$ □

Chapter 3

Toeplitz Operator

3.1 Introduction:

Definition 3.1.1. *Let H be a separable Hilbert space and $(e_n)_{n=0}^{\infty}$ be orthonormal basis for H. A linear operator T on H is said to be a Toeplitz operator if*

$$\langle Te_n, e_m\rangle = a_{m-n}$$

for some complex sequence $(a_n)_{n=-\infty}^{\infty}$. This means that the matrix, with respect to the orthonormal basis $(e_n)_{n=0}^{\infty}$, is constant along each diagonal parallel to the main one.

That is, the matrix of the Toeplitz operator is

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

*is called "***Toeplitz matrix***".*

Proposition 3.1.1. *$T \in B(H)$ is a Toeplitz operator on a separable Hilbert space H if and only if it satisfies the operator equation $S^*TS = T$, where S is the right shift operator.*

Proof. Let $(a_{n-m})_{n,m=0}^{\infty}$ be the matrix of T with respect to its orthonormal basis, then clearly matrix of S^*TS and T are same. Hence $S^*TS = T$.

Conversely let $S^*TS = T$. Then

$$\langle S^*TSe_n, e_m\rangle = \langle Te_n, e_m\rangle$$

$$\implies \langle Te_{n+1}, Se_m \rangle = \langle Te_n, e_m \rangle$$

$$\implies \langle Te_{n+1}, e_{m+1} \rangle = \langle Te_n, e_m \rangle$$

So the matrix of T with respect to the orthonormal basis $(e_n)_{n=0}^{\infty}$, is constant along each diagonal parallel to the main one. Hence T is a Toeplitz operator. □

3.2 Examples:

1. Let (a_n) be any sequence of complex number, then the Diagonal operator $D : \ell^2 \to \ell^2$ defined by,

$$D(x_1, x_2, x_3 \dots) = (a_1x_1, a_2x_2, a_3x_3, \dots)$$

   is a Toeplitz operator.

2. The right shift operator $R : \ell^2 \to \ell^2$ defined by

$$R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$$

   is a Toeplitz operator.

3. The Left shift operator $R : \ell^2 \to \ell^2$ defined by

$$R(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$$

   is a Toeplitz operator.

3.3 Hardy sapce:

Let S_1 be the unit circle in $\mathbb{C}$ and S_1 be the circle group and endow S_1 with the normalised arc length measure(= Haar measure), denoted by $d\lambda$. We write $L^p(S_1)$ for $L^p(S_1; d\lambda)$. Since $d\lambda(S_1) \leq 1$, $L^q(S_1) \subset L^p(S_1)$ if $1 \leq p < q$. If $f \in L^1(S_1)$, then

$$\int f(\lambda)d\lambda = \int_0^{2\pi} f(e^{i\theta})d\theta$$

For each $n \in \mathbb{Z}$, we define the continuous function e_n to be

$$e_n : T \to T, \lambda \to \lambda^n$$

We denote by Γ and Γ_+ linear spans of the sets $\{e_n : n \in \mathbb{Z}\}$ and $\{e_n : n \in \mathbb{N}\}$ respectively. We call the elements in Γ and Γ_+ trigonometric polynomials and analytic trigonometric polynomials, respectively.

Lemma 3.3.1. *(1) Γ is a $*$-subalgebra of $C(S_1)$.*

(2) For $1 \leq p \leq +\infty$, is L^p-norm dense in $L^p(S_1)$.

(3) $(e)_{n\in\mathbb{Z}}$ is an orthonormal basis of the Hilbert space $L^2(S_1)$.

Proof. Since $e_n^* = e_{-n}$, clearly Γ is a $*$-subalgebra of $C(S_1)$.

By the Stone-Weierstrass theorem, Γ is norm-dense in $C(S_1)$, and since $C(S_1)$ is L^p-norm dense in $L^p(S_1)$, one gets the statement (2). (3) follows immediately from (2). □

Definition 3.3.1. *If $f \in L^1(S_1)$ and $n \in \mathbb{Z}$, the n-th Fourier coefficient of f is defined to be*

$$\hat{f}(n) = \int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta$$

and the function

$$\hat{f} : \mathbb{Z} \to \mathbb{C}, n \to \hat{f}(n)$$

is called the Fourier transform of f.

Definition 3.3.2. *Suppose $1 \leq p \leq +\infty$. We define the Hardy space H^p by setting,*

$$H^p = \{f \in L^p(S_1) : \hat{f}(n) = 0(n < 0)\}.$$

H^p is a $L^p -$ norm closed vector subspace of $L^p(S_1)$.

Definition 3.3.3. *We define Hardy space on unit disk $\mathbb{D}$ by*

$$H^2(\mathbb{D}) = \{f(z) = \sum_{n=0}^{\infty} a^n z^n : ||f||^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

It is clear that the vector $f = (a_o, a_1, a_2, \dots) \in \ell^2$ is identified with the analytic fundion $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\mathbb{D})$ and vice versa.

Also the Hardy space on unit disk $\mathbb{D}$ by

$$H^2(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \ \ is \ \ holomorphic \ \ : \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\}.$$

Definition 3.3.4. *(Hardy space on $H^2(S_1)$)*

Recall that $L^2(S_1)$ is the complex Hilbert space of square-integrable functions on the unit circle. There is a natural identification of $L^2(S_1)$ with $L^2([0, 2\pi])$. For $f, g \in L^2(S_1)$ the inner product is

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{S_1} f(z)\overline{g(z)}dz = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta.$$

For $n \in \mathbb{Z}$, let

$$e_n : S_1 \to \mathbb{C}, e_n(e^{i\theta}) := e^{in\theta},$$

then $\{z^n = e^{in\theta}\}$ form a orthonormal basis for it.

The Hardy space is defined as

$$H^2(S_1) = \{f \in L^2(S_1) : \langle f, z^n\rangle = 0, for \ \ n < 0\}.$$

Any function $f \in H^2(S_1)$ is in the form,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \ \ with \ \ \sum_{n=0}^{\infty} |a_n|^2 < \infty$$

.

Definition 3.3.5. *(Multiplication operator) Let $\phi \in L_\infty(S_1)$, then the multiplication operator $M_\phi : L^2(S_1) \to L^2(S_1)$ defined by*

$$M_\phi f(z) = \phi(z)f(z)$$

is a bounded operator on $L^2(S_1)$.

Proposition 3.3.1.

1. *$M_\phi^* = M_{\overline{\phi}}$.*

2. *$M_{\phi+\psi} = M_\phi + M_\psi$.*

3. *$M_\phi M_\psi = M_{\phi\psi}$.*

4. *M_ϕ is invertible iff ϕ is invertible.*

3.4 Toeplitz operator on Hardy space:

Definition 3.4.1. *(Toeplitz operator on Hardy space) we let $P : L^2(S_1) \to H^2(S_1)$ be the orthogonal projection onto $H^2(S_1)$. Then*

$$P(\sum_{n\in\mathbb{Z}} c_n e^{in\theta}) = \sum_{n=0}^{\infty} c_n e^{in\theta}.$$

For each, $\phi \in L_\infty(S_1)$, define the Toeplitz operator $T_\phi : H^2(S_1) \to H^2(S)$ by the formula

$$T_\phi = PM_\phi.$$

Let $(a_{i,j})$ be the matrix of T_ϕ, then

$$\begin{aligned} a_{m,n} &= \langle T_\phi e^{in\theta}, e^{im\theta}\rangle \\ &= \langle PM_\phi e^{in\theta}, e^{im\theta}\rangle \\ &= \langle M_\phi e^{in\theta}, Pe^{im\theta}\rangle \\ &= \langle M_\phi e^{in\theta}, e^{im\theta}\rangle \\ &= \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{i\theta})e^{in\theta}e^{-im\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{i\theta})e^{-i(m-n)\theta}d\theta \\ &= \hat{\phi}(m-n) \end{aligned}$$

where $\hat{\phi}(n)$ is the n'th fouries coefficient of ϕ. Thus the matrix of T_ϕ is

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $a_n = \hat{\phi}(n)$ is the n'th fouries coefficient of ϕ.

The function ϕ is called the symbol of T_ϕ.

Proposition 3.4.1. *1. $T_\phi^* = T_{\overline{\phi}}$*

2. $T_{\phi+\psi} = T_\phi + T_\psi$

3. $T_\phi T_\psi \neq T_{\phi\psi}$

4. T_ϕ *invertible implies that* ϕ *invertible.*

Remark 3.4.1. *Converse of proposition (4) is not true.*

Proof. Let $T_z = T_{e^{i\theta}}$. Then $T_z(z^n) = z^{n+1}$ for $n \geq 0$. So, $1 = z^0 \notin range(T_z)$, so T_z is not invertible. The operator T_z is the unilateral shift operator. □

Lemma 3.4.1. *Let* $\phi \in C(S_1)$*, then* $PM_\phi - M_\phi P$ *is a compact operator on* $L^2(S_1)$*.*

Proof. We consider the case when $\phi(z) = z$, and apply the operator to each element of the standard basis for $L^2(S_1)$.

If $n \geq 0$, we have

$$(PM_z - M_zP)z^n = PM_zz^n - M_zPz^n = Pz^{n+1} - z^{n+1} = 0.$$

If $n < -1$,

$$(PM_z - M_zP)z^n = Pz^{n+1} - M_z0 = 0 - 0 = 0.$$

If $n = -1$,

$$(PM_z - M_zP)z^n = PM_zz^{-1} - M_zPz^{-1} = Pz^0 = z^0.$$

Hence, the image of the set of polynomials under the operator $PM_z - M_zP$ has dimension 1. By the Stone-Weierstrauss theorem, the image of $L^2(S_1)$ has dimension 1. Thus, $PM_z - M_zP$ is finite rank, hence compact.

Now consider the collection

$$E = \{\phi \in C(S_1) : PM_\phi - M_\phi P \ \textit{is compact}\}$$

is a C^* subalgebra of $C(S_1)$. To see this, consider

$$(PM_\phi - M_\phi P)^* = M_\phi^* P^* - P^* M_\phi^*$$

$$= M_{\overline{\phi}}P - PM_{\overline{\phi}} = -(PM_{\overline{\phi}} - M_{\overline{\phi}}P).$$

So compactness of $PM_\phi - M_\phi P$ implies compactness of $(PM_{\overline{\phi}} - M_{\overline{\phi}}P)$. Hence, E is closed under conjugation. That E is closed under addition and scalar multiplication is clear. Since $PM_\phi - M_\phi P$ depends linearly on ϕ, the map $\phi \to PM_\phi - M_\phi P$ is continuous and so E is norm-closed. It remains to verify tha E is closed under multiplication. Let $\psi \in E$. Then

$$PM_{\phi\psi} - M_{\phi\psi}P = PM_{\phi\psi} - M_\phi PM_\psi + M_\phi PM_\psi - M_{\phi\psi}P$$

$$= (PM_\phi - M_\phi P)M_\psi + M_\phi(PM_\psi - M_\psi P)$$

Since $PM_\phi - M_\phi P$ and $PM_\psi - M_\psi P$ are compact and $K(L^2(S_1))$ is an ideal of $B(L^2(S_1))$, E is closed under multiplication. Hence, E is a C^* -subalgebra of $C(S_1)$ containing z. By the Stone-Weierstrauss theorem, z generates $C(S_1)$, so $E = C(S_1)$. Therefore, $PM_\phi - M_\phi P$ is compact for all $\psi \in C(S_1)$. □

Proposition 3.4.2. *Let $\phi \in C(S_1)$ is nowhere-zero, then T_ϕ is Fredholm.*

Proof. Suppose that $\phi \in C(S_1)$ is nowhere-zero. We show that T_ϕ is invertible up to a compact operator. Since ϕ is nonzero on S_1, the function $1/\phi$ is well-defined. Then

$$T_\phi T_{1/\phi} = PM_\phi PM_{1/\phi}$$

$$= PM_\phi M_{1/\phi} + PM_\phi PM_{1/\phi} - PM_\phi M_{1/\phi}$$

$$= P + P(M_\phi P - PM_\phi)M_{1/\phi}.$$

Since $P = I$ on $H^2(S_1)$, and since $M_\phi P - PM_\phi$ is compact by above Lemma, $T_\phi T_{1/\phi} = I + K_1$ where K_1 is compact. A similar calculation gives $T_{1/\phi}T_\phi = I + K_2$, for some compact operator K_2. Hence, T_ϕ is invertible modulo compact operators, and is therefore Fredholm. □

Remark 3.4.2. *On $L^2(S_1)$, the map $\phi :\to M_\phi$ is multiplicative in that $M_\phi M_\psi = M_{\phi\psi}$. This is not the case for Toeplitz operators. However, generalizing the calculation in the proof above gives us the following fact.*

Proposition 3.4.3. *Suppose that $\phi, \psi \in C(S_1)$. Then $T_{\phi\psi} = T_\phi T_\psi + K$, for some compact operator K.*

Proof. Suppose that $\phi, \psi \in C(S_1)$. Then

$$T_\phi T_\psi = PM_\phi PM_\psi$$

$$= PM_\phi M_\psi + PM_\phi PM_\psi - PM_\phi M_\psi$$

$$= PM_{\phi\psi} + P(M_\phi P - PM_\phi)M_\psi$$

$$= T_{\phi\psi} + P(M_\phi P - PM_\phi)M_\psi$$

Since $M_\phi P - PM_\phi$ is compact by Lemma, $T_{\phi\psi} = T_\phi T_\psi + K$, for some compact operator K.

□

Theorem 3.4.1. *Let $\phi, \psi \in L^\infty(S_1)$. If $\psi \in H^\infty$, then $T_{\phi\psi} = T_\phi T_\psi$ and $T_{\overline{\psi}\phi} = T_{\overline{\psi}} T_\phi$. Conversely, if $T_\phi T_\psi$ is a Toeplitz operator, then $\overline{\phi} \in H^\infty$ or $\psi \in H^\infty$, and $T_\phi T_\psi = T_{\phi\psi}$*

Proof. If $\psi \in H^\infty$, then clearly $\psi H^2(S_1) \subseteq H^2(S_1)$. Let $f \in H^2(S_1)$, then

$$T_\phi T_\psi(f) = P(\phi P(\psi f)) = P(\phi\psi f) = T_{\phi\psi}(f),$$

so $T_\phi T_\psi = T_{\phi\psi}$.

Therefore, $T_{\overline{\phi}} T_\psi = T_{\overline{\phi}\psi}$. so by taking adjoints,

$$T_{\overline{\psi}} T_\phi = T_\psi^* T_{\overline{\phi}}^* = T_{\overline{\phi}\psi}^* = (T_{\overline{\phi}} T_\psi)^* = T_{\overline{\psi}\phi}$$

We now suppose conversely that $T_\phi T_\psi$ is a Toeplitz operator and $(a_n), (b_n), (c_n)$ are matrix of $T_\phi, T_\psi, T_\phi T_\psi$ respectively. Then its matrix is a Toeplitz matrix, so by the product matrix formula, $a_{i+1} b_{-j-1} = 0$, for all i, j. It follows that $a_{i+1} = 0$ for each $i \geq 0$ or $b_{-j-1} = 0$ for each $j \geq 0$, which is equivalent to the desired conclusion □

The following example shows the picture of above Theorem. Let a, b, c, d *and* $h \in \mathbb{C}$, and set $\psi = ae^0 + be^{i\theta}$ and $\phi = ce^{-i\theta} + de^0 + he^{i\theta}$. Then, $\psi \in \subseteq H^\infty$ and $\psi \in L^\infty(S_1)$. So, the matrix Ψ of T_ψ looks like this:

$$\begin{bmatrix} a & 0 & 0 & 0 & \cdots \\ b & a & 0 & 0 & \cdots \\ 0 & b & a & 0 & \cdots \\ 0 & 0 & b & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Similarly, the matrix of T_ϕ looks like this

$$\begin{bmatrix} d & c & 0 & 0 & \cdots \\ h & d & c & 0 & \cdots \\ 0 & h & d & c & \cdots \\ 0 & 0 & h & d & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Clearly, these are Toeplitz matrices. Then, the product

$$\begin{bmatrix} ad+bc & ac & 0 & 0 & \cdots \\ ah+bd & ad+bc & ac & 0 & \cdots \\ bh & ah+bd & ad+bc & ac & \cdots \\ 0 & bh & ah+bd & ad+bc & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is a Toeplitz matrix. Since $\phi\psi = ace^{-i\theta} + (ad+bc)e^0 + (ae+bd)E^{i\theta} + bhe^{2i\theta}$, this is the matrix of the Toeplitz operator $T_{\phi\psi}$.

Proposition 3.4.4. *(Brown-Halmos) There are no zero divisiors in the set of all Toeplitz operators. Specifically, if $\phi, \psi \in L^\infty(S_1)$, then $T_\phi T_\psi = 0 \iff T_\phi = 0$ or $T_\psi = 0$*

Proof. The implication $\Longleftarrow$ is trivial, so we prove the converse. Since 0 operator is a Toeplitz operator, it follows that $\overline{\phi}$ or $\psi \in H^\infty(\subset H^2(S_1))$ and $\phi\psi = 0$ a.e. by the F. and M. Riesz theorem. Now if $\overline{\phi} = 0$ a.e. and if $\psi \in H^2$, then $\psi = 0$ a.e., and if $\psi \in H^2$, then $\phi = 0$ a.e. Thus, $T_\phi = 0$ or $T_\psi = 0$.

□

Remark 3.4.3. *Suppose, $\phi_i \in L^\infty(S_1)$ $(i = 1, 2, \ldots n)$ and*

$$T_{\phi_1} T_{\phi_2} T_{\phi_3} \ldots T_{\phi_n} = 0$$

is it necessary that there exists an index i such that $T_{\phi_i} = 0$? This problem, which is a natural generalization of above proposition, has been solved completely by Alexandru Aleman and Dragan $Vukoti'c$ [4] in 2009, by showing that the question above has an affirmative answer for all n.

3.5 Elementary Spectral Theory of Toeplitz Operators:

Now, we study the elementary spectral theory of Toeplitz operators. First, we apply the F. and M. Riesz theorem to Toeplitz operators.

Proposition 3.5.1. *If $\phi \in H^\infty$ and ϕ is not a scalar a.e., then T_ϕ has no eigenvalues.*

Proof. Suppose that $f \in H^2(S_1)$ and $\lambda \in \mathbb{C}$ and

$$(T_\phi - \lambda)(f) = 0.$$

Then, $(\phi - \lambda)f = 0$ a.e. Since $(\phi - \lambda) \in H^2(S_1)$ and is not the zero element, the set $\{\gamma \in \mathbb{C} : (\phi - \lambda)(\gamma) = 0\}$ is of measure 0 by the F. and M. Riesz theorem. Thus $f = 0$ a.e. $\square$

Theorem 3.5.1. *(Hartman-Wintner) Let $\phi \in L^\infty(S_1)$ and let $\sigma(\phi)$ denote the spectrum of ϕ in $L^\infty(S_1)$. Then $\sigma(\phi) \subseteq \sigma(T_\phi)$ and $r(T_\phi) = ||T_\phi|| = ||\phi||_\infty$.*

Proof. To show that $\sigma(\phi) \subseteq \sigma(T_\phi)$, it suffices to show that if T_ϕ is invertible in $B(H^2)$, then ϕ is invertible in $^\infty(S_1)$. Indeed, this reduction follows from the equality $T_\phi - \lambda = T_{\phi-\lambda}$ if $\lambda \in \mathbb{C}$. So, we now suppose T_ϕ is invertible and set $m := ||T_\phi^{-1}||$. For all $f \in H^2(S_1)$,

$$||f|| = ||T_\phi^{-1}T_\phi(f) \leq m||T_\phi||$$

One then infers that for any $n \in \mathbb{Z}$

$$||M_\phi(e^n f)|| = ||\phi e^n f|| = ||\phi|| \geq ||T_\phi(f)|| \geq \frac{||f||}{m} = \frac{||e^n f||}{m}$$

However, the functions $e^n f$ are L^2 -norm dense in $L^2(S_1)$, since $\{e^n\}_{n\in\mathbb{Z}}$ is L^2 -norm dense in $L^2(S_1)$. Hence, for all $g \in L^2(S_1)$ we have

$$||M_\phi(g)|| \geq \frac{||g||}{m}$$

, and so $M_\phi^* M_\phi \geq m^{-2} > 0$. It follows that $M_\phi^* M_\phi$ is invertible, and by the normality of M_ϕ, M_ϕ is invertible. Since the map $M^* : L^\infty(S_1) \to B(L^2(S_1))$ is an isometric $*$ -homomorphism (and then injective), ϕ is invertible in $L^\infty S_1)$ Now, suppose that ϕ is an arbitrary element of $L^\infty(S_1)$. Then, since $\sigma(\phi) \subseteq \sigma(T_\phi)$, we get

$$||T_\phi|| \leq ||\phi|| = r(\phi) \leq r(T_\phi) \leq ||T_\phi||,$$

so we have $||T_\phi|| = r(T_\phi) = ||\phi||_\infty$. $\square$

Proposition 3.5.2. *If $\phi \in C(S_1)$, then T_ϕ is compact if and only if $\phi = 0$.*

Proof. The part $\Longleftarrow$ is clearly hold.

For converse let $\phi \in C(S_1)$ and let $\sigma(\phi)$ denote the spectrum of ϕ in $L^\infty(S_1)$. Then we know that $\sigma(\phi) \subseteq \sigma(T_\phi)$ and $r(T_\phi) = ||T_\phi|| = ||\phi||_\infty$. Since spectrum of a compact operator is a discrete set and $\phi \in C(S_1)$, so image of ϕ is connected set, hence ϕ is a constant function. Now if ϕ is non zero constant function then T_ϕ is constant multiple of identity operator which is non constant in $H^2(S_1)$. Hence $\phi = 0$. $\square$

Chapter 4

Hankel Operator

4.1 Introduction:

Definition 4.1.1. *A Hankel operator* $\mathbf{H} : H \to H$ *is a linear operator on Hilbert space* H *such that,*

$$\langle \mathbf{H} e_m, e_n \rangle = a_{n+m}$$

for some complex sequence $\{a_n\}_{n=0}^{\infty}$. *This means that the matrix, with respect to the orthonormal basis* $\{e_n\}_{n=0}^{\infty}$, *is constant along each diagonal perpendicular to the main one.*

The matrix of the Hankel operator is

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Proposition 4.1.1. *An operator* $\mathbf{H} : H \to H$ *is Hankel operator on* H *if and only if* H *satisfies the operator equation* $S^*\mathbf{H} = \mathbf{H}S$.

Proof. Clearly matrix of $S^*\mathbf{H}$ and $\mathbf{H}S$ are same, so $S^*\mathbf{H} = \mathbf{H}S$.

Conversely, let $S^*\mathbf{H} = T\mathbf{H}$ and $\{e_n\}$ is a orthonormal basis for H. Then

$$\langle S^*\mathbf{H} e_n, e_m \rangle = \langle \mathbf{H} S e_n, e_m \rangle$$

$$\implies \langle \mathbf{H} e_n, S e_m \rangle = \langle \mathbf{H} e_{n+1}, e_m \rangle$$

$$\implies \langle \mathbf{H} e_n, e_{m+1} \rangle = \langle \mathbf{H} e_{n+1}, e_m \rangle$$

So entries of matrix of T is same perpendicular to main one. Hence $\mathbf{H}$ is Hankel operator. $\square$

4.2 Hankel operator on Hardy space:

The basic theorem in the theory of Hankel operators is the classical theorem of Nehari. In the following, P the orthogonal projection fiom $L^2(S_1)$ to $H^2(S_1)$. Now consider an another operator $J : L^2(S_1) \to L^2(S_1)$ is the (self-adjoint) flip operator,

$$Jf = \tilde{f}.$$

where $\tilde{f}(z) = f(\bar{z})$

Theorem 4.2.1. *(Nehari's Theorem) A Hankel operator* $\mathbf{H}$ *on* $H^2(S_1)$ *is bounded if and only if there exists a function* $\phi \in L^\infty$ *such that*

$$\mathbf{H} = PJM_\phi$$

In this case, it Is possible to clioose p in such a way that $||\mathbf{H}|| = ||\phi||_\infty$.

In other words, what Nehari's theorem says is that the Hankel operator H *with matrix* $(a_{m+n})_{n,m=0}^\infty$ *is bounded if and only if there exists a function* $\phi \in L^\infty$ *such that* a_n *is the (-n)-th Fourier coefficient of* ϕ*, for* $n \geq 0$*. That is, the matrix*

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_{-2} & a_{-3} & a_{-4} & a_{-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is a matrix of a bounded Hankel operator if and only if there exists a function $\phi \in L^\infty$ *such that* a_n *is the (n)-th Fourier coefficient of* ϕ*, for* $n \geq 0$.

We will use the following fact (a straightforward caldation) fiequently If $\phi \in L^\infty$ and $\mathbf{H} = PJM_\phi$, then, for $g, h \in H^2(S_1)$

$$\langle Hg, h \rangle = \langle PJM_\phi, h \rangle = \langle \phi g, h \rangle = \frac{1}{2\pi} \int_0^\infty \phi(e^{i\theta}) g(e^{i\theta}) h^*(e^{i\theta}) d\theta.$$

We refer to the function $\phi \in L^\infty$, given by Nehari's theorem as a symbol of $\mathbf{H}$ and we write

$$\mathbf{H} = H_\phi.$$

Proposition 4.2.1. *Linear with respect to its symbol, i.e., for any $\phi, \psi \in L^\infty$ and complex number $a, b \in \mathbb{C}$,*

$$H_{a\phi+b\psi} = aH_\phi + bH_\psi$$

Proof. It is clear from the definition of Bounded Hankel operator on the Hardy space $H^2(S_1)$. □

Proposition 4.2.2. *The symbol of the Hankel operator is not unique, i.e.,*

$$H_\phi = H_\psi \ iff \ \phi - \psi \in zH^2.$$

Proof. Let $\psi \in zH^2$, then $\psi(z) = \sum_{n=1}^{\infty} c_n z^n$, for some sequence complex sequence $(c_n) \subseteq \mathbb{C}$, so $H_\psi = PJM_\psi = 0$ on $H^2(S_1)$. hence by the linearity of Hankel with respect to its symbol we have,

$$H_\phi = H_\psi \ iff \ \phi - \psi \in zH^2.$$

□

Definition 4.2.1. *Given a set A of Hankel operators $\mathcal{A}$ we define the set of symbols of $\mathcal{A}$ as*

$$Sp\mathcal{A} = \{\phi \in L^\infty : H_\phi \in \mathcal{A}\}.$$

It is clear that if $\mathcal{A}$ is a vector space of Hankel operators, the symbol of $\mathcal{A}$ is a subspace of L^∞. In this case, it will always contain zH^∞.

The is also a complete characterization of compact Hankel operators. Here $C(S_1)$ is the subalgebra of continuous functiom on S_1 and $H^\infty + C$, is the sum of function in H^∞ and in $C(S_1)$.

Theorem 4.2.2 (Hartman's Theorem)**.** *A Hankel operator H on H^2 is compact if and only if there exists a function $\psi \in H^\infty + C$ such that,*

$$H = H_\psi.$$

A characterization of those Hankel operators of finite-rank is given by Kronecker's theorem.

Theorem 4.2.3. *Let the Hankel operator H have matrix, $(a_{n+m})_{n,m=0}^{\infty}$. Then H is a finte-rank matrix if and only if the function, $\frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_3}{z^3} + \ldots$ rational. Furthermore, H is bounded if the poles of $\frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_3}{z^3} + \ldots$ are contained in the open unit disk $\mathbb{D}$.*

4.3 Algebraic properties of Hankel operators:

In this section we investïgate the question of when a HankeI operator and a Toeplitz operator cornmute- That is, we will completely answer the question of when the commutant of a Hankel operator contains a Toeplitz operator.

Lemma 4.3.1. *Let H and T_g be a non-zero Hankel operator and a non-zero Toeplitz operator respectively. Then $T_g^* H = HT_{g^*}$, if and only if g is analytic.*

Notice that when $g(z) = z$, the condition $T_g^* H = HT_{g^*}$ is equivalent to the alternative definition of a Hankel operator: namely, H is Hankel if and only if $S^* H = HS$.

Proof. Let g have Fourier series coefficients $a_k = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})e^{-ik\theta}d\theta$ for $k \in \mathbb{Z}$. This means that T_g has matrix

$$(a_{n-m})_{n,m=0}^{\infty}.$$

Let $H = H_f$, where f has Fourier coefficiets, $b_{-k} = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})e^{-ik\theta}d\theta$, for $k \in \mathbb{Z}$, that is, H has matrix

$$(b_{n+m})_{n,m=0}^{\infty}.$$

A straightforward calculation (just remember that, for an operator C, Ce_m is the m-th column of the matrix of C with respect to the basis $(e_n)_{n=0}^{\infty}$) shows that

$$\langle T_g^* e_n, e_m\rangle = \langle H_f e_n, T_g e_m\rangle = \sum_{k=0}^{\infty} b_{k+n}\overline{a}_{k-m} \tag{4.1}$$

and that

$$\langle H_f T_{g^*} e_n, e_m\rangle = \langle T_{g^*} e_n, H_f^* e_m\rangle = \sum_{k=0}^{\infty} \overline{a}_{k-n} b_{k+m} \tag{4.2}$$

for $m \geq 0$ and $m \geq 0$.

Let us suppose that g is analytic. Then, since $a_k = 0$ if $k < 0$, equation (3.1) becomes

$$\langle T_g^* H_f e_n, e_m\rangle = \sum_{k=m}^{\infty} \overline{a}_{k-m} b_{k+n} = \sum_{s=0}^{\infty} \overline{a}_s b_{m+n+s}$$

and equation (3.2) becomes

$$\langle H_f T_{g^*} e_n, e_m\rangle = \sum_{k=n}^{\infty} \overline{a}_{k-n} b_{k+m} = \sum_{s=0}^{\infty} \overline{a}_s b_{m+n+s}.$$

Since, the right hand side of the both equation are equal, $T_g^* H = HT_{g^*}$.

Conversely, assume that $T_g^* H = HT_{g^*}$ and rewrite the equation (3.1) as

$$\langle T_g^* e_n, e_m \rangle = \langle H_f e_n, T_g e_m \rangle = \sum_{k=0}^{m-1} b_{k+n}\overline{a}_{k-m} = \sum_{k=m}^{\infty} b_{k+n}\overline{a}_{k-m} \tag{4.3}$$

and equation (3.2) as

$$\langle H_f T_{g^*} e_n, e_m \rangle = \langle T_{g^*} e_n, H_f^* e_m \rangle = \sum_{k=0}^{n-1} \overline{a}_{k-n} b_{k+m} = \sum_{k=n}^{\infty} \overline{a}_{k-n} b_{k+m} \tag{4.4}$$

where the first term of the right hand side of the equations is just through of as zero if m=0 or n=0.

A change of variables as before shows that both second summands in the right hand side of equation (3.3) and (2.4) are equal. and thus

$$\sum_{k=0}^{m-1} \overline{a}_{n-m} b_{k+n} = \sum_{k=0}^{n-1} \overline{a}_{k-n} b_{k+m}, \;\; when \;\; both \;\; m, n > 0. \tag{4.5}$$

If n=0 and $m > 0$ we obtain,

$$\sum_{k=0}^{m-1} \overline{a}_{k-m} b_k = 0. \tag{4.6}$$

We will assume for the rest of this proof that $m > n$. The left hand side of the equation (3.5) can then be written as

$$\begin{aligned}
\sum_{k=0}^{m-1} \overline{a}_{k-m} b_{k+n} &= \sum_{k=0}^{m-n-1} \overline{a}_{k-m} b_{k+n} \\
&= \sum_{k=m-n}^{m-1} \overline{a}_{k-m} b_{k+n} \\
&= \sum_{k=0}^{m-n-1} \overline{a}_{k-m} b_{k+n} \\
&= \sum_{s=0}^{n-1} \overline{a}_{s-n} b_{s+m}
\end{aligned}$$

but the last summand is equal to teh right hand side of the equation (3.5). It follows that

$$\sum_{k=0}^{m-n-1} \overline{a}_{k-m} b_{k+n} = 0, \;\; for \;\; m > n > 0.$$

This last equation is also valid for n=0, since it then reduce to equation (3.6). Thus we have

$$\sum_{s=0}^{m-n-1} \overline{a}_{k-m} b_{k+n} = 0, \;\; for \;\; m > n \geq 0. \tag{4.7}$$

Now, clearly there must exists a non negative integer n_0 such that $b_{n_0} \neq 0$(otherwise H would be a zero operator). We use equation (3.7) with $n = n_0$, i.e.,

$$\sum_{k=0}^{m-n_0-1} \overline{a}_{k-m} b_{k+n_0} = 0, \;\; for \;\; m > n_0. \tag{4.8}$$

We use strong induction to prove that $a_{-(n_0+s)} = 0$, for all $s > 0$. If $m = n_0 + 1$ in the equation (3.8), we obtain $\overline{a}_{-(n_0+1)} b_{n_0} = 0$, which in turn implies $a_{-(n_0+1)} = 0$ (recall again that $b_{n_0} \neq 0$).

Now assume that $a_{-(n_0+1)} = a_{-(n_0+2)} = a_{-(n_0+3)} \dots a_{-(n_0+s)} = 0$. Then equation (3.8), with $m = n_0 + s + 1$ becomes $\overline{a}_{-(n_0+s+1)} b_{n_0} = 0$, which implies $a_{-(n_0+s+1)} = 0$. Thus $a_{-(n_0+s)} = 0$, for $s > 0$.

Now go back to equation (3.7). If we set $m = n_0 + 1$, we get

$$\sum_{k=0}^{n_0-n} \overline{a}_{k-n_0-1} b_{k+n} = \sum_{k=1}^{n_0-n} \overline{a}_{k-(n_0-1)} b_{k+n}, \;\; for \;\; n_0 + 1 > n > 0 \tag{4.9}$$

since $a_{(n_0+1)} = 0$(as proven in the previous paragraph). If we set $n = n_o - 1$ in equation (3.9), we get $\overline{a}_{-n_0} b_{n_0} = 0$ and thus $a_{-n_0} = 0$. Proceeding in this fashion (set $n = n_0 - 2, n = n_0 - 3, \dots n = n = n_0 - (n_0 - 1), n = n_0 - n_0$ in equation (3.9), we get $a_{-n_0 =} a_{-(n_0-1)} = a_{-(n_0-2)} \cdots = a_{-2} = a_{-1} = 0$

Therefore $a_{-s} = 0$, for all $s > 0$ i.e., g is analytic. □

Now we can prove a result concerning commutativity. What this theorem says is that if a non-zero Hankel operator commutes with a "symmetric" (equal to its transpose) Toepiitz operator, then the Toeplitz operator is just a multiple of the identity operator.

Theorem 4.3.1. *Let $\psi \in L^\infty$. Suppose that $\tilde{\psi} = \psi$ and that $HT_\psi = T_\psi H$ for a non-zero Hankel operator H. Then ψ is constant function (ie., T_ψ is a constant multiple of the identity).*

Proof. Since $\psi = \tilde{\psi}$, it follows that $(\overline{\psi})^* = \tilde{\psi} = \psi$. Also, we known that, $T_\psi = T^*_{\overline{\psi}}$. Putting these two facts together, we see that $T_\psi H = HT_\psi$, is equivalent to $T^*_{\overline{\psi}} H = HT_{(\tilde{\psi})^*}$. By Lemma 4.3.1 that $\overline{\psi} \in H^\infty$, and, since $\psi = \tilde{\psi}$, it follows that ψ is constant. □

We now prove the following theorem. What it says is that if a Hankel operator commutes with a Toeplitz operator, it must also commute with the"transpose" of the Toeplitz operator, since Hankel operators are 'symmetric' (equal to their transpose).

Theorem 4.3.2. *If $\psi \in L^\infty$, then $T_\psi H = HT_\psi$ if and only if $T_{\tilde{\psi}} H = HT_{\tilde{\psi}}$*

Proof. Define the anti-unitary involution V on $H^2(S_1)$ by

$$Vf = f^*, \ \ where \ \ f^*(z) = \overline{f(\overline{z})}.$$

Then we have, $VT_fV = T_{f^*}$, for $f \in L^\infty$, and $VHV = H^*$, for any Hankel operator H. ClearIly, $H^2 = I$.

Thus $T_f H = HT_f$ implies that $VT_fVVHV = VHVVT_fV$, which in turn implies that $T_{f^*}H^* = H^*T_{f^*}$. Taking adjoints we get, $HT_{\tilde{f}} = T_{\tilde{f}}H$, where $\widetilde{f(z)} = f(\overline{z})$. Applying the previous caldation to $\tilde{f}$, it follows that $T_{\tilde{f}}H = HT_{\tilde{f}}$, implies $T_fH = HT_f$, □

The last two theorems allow us to get the following corollary. This is the first necessary condition for a Hankel and a Toeplitz operator to commute.

Proposition 4.3.1. *If $\psi \in L^\infty$, then $T_\psi H = HT_\psi$, for some $\psi \in L^\infty$ and a non zero Hankel operator H, then $\psi + \tilde{\psi}$ is constant function.*

Proof. If $T_\psi H = HT_\psi$, then by above theorem $T_{\tilde{\psi}} H = HT_{\tilde{\psi}}$. Using both equation we get, $T_{\psi+\tilde{\psi}} H = HT_{\psi+\tilde{\psi}}$ (we know that a Toeplitz operator is linear with respect to its symbol). Since $\widetilde{\psi + \tilde{\psi}} = \psi + \tilde{\psi}$, so $\psi + \tilde{\psi}$ is constant. □

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