COMMUTATIVE ALGEBRA

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under the supervision of
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DECLARATION

This thesis entitled **COMMUTATIVE ALGEBRA** submitted by me to the Indian Institute of Technology, Hyderabad for the award of the degree in Master of Science in Mathematics contains a literature survey of the work done by some authors in this area. The work presented in this thesis has been carried out under the supervision of **Dr. Pradipto Banarjee**, Department of Mathematics, Indian Institute of Technology, Hyderabad, Telangana.

I hereby declare that, to the best of my knowledge, the work included in this thesis has been taken from the books, "An introduction to commutative algebra" by Atiyah Mac Donald, and "Jmaes Milney Notes". No new results have been created in this thesis. The definitions, notations and results in Commutative algebra are learnt from the above mentioned sources and are presented here. I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

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This Thesis entitled **Commutative Algebra** by **Mohd Shahvez Alam** is approved for the degree in Master of Science from IIT Hyderabad.

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ABSTRACT

The main aim of this project is to learn a branch of Mathematics that studies commutative rings with unity.
The central notion in commutative algebra is that of prime ideal. This provides common generalization of primes of airthmetics and points of geometry. The geometric notion of concentrating attention near a point has as its algebraic analogue the important process localizing a ring at prime ideal, therefore result about lacalization can be thought in term of geometry.

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Chapter 1

Rings And Algebra

1.1 Introduction

In this course,we shall consider ring to be commutative and with unity.

Definition 1.1.1. ***Algebra***
Let A be any ring.An A-algebra is a ring B together with a homomorphism $\phi : A \rightarrow B$

Example 1.1.2. *Let A be any non zero ring then $f : \mathbb{Z} \rightarrow A$ defined by $f(n)=n.1_A$ is a ring homomorphism so A become a $\mathbb{Z}$-algebra*

An A-subalgebra C of B is a subring C of B together with homomorphism ϕ:$A \rightarrow B$
Let B be an A-algebra with a homomorphism ϕ: $A \rightarrow B$ and let S be subset of B.The intersection of all A subalgebras of$B$.It is denoted by $A[S]$ and called the A subalgebra generated by S.If B=$A[S]$ then S is called a set of algebra generators of B,and $A[S]$ is the smallest subring of B containing S.If B=$A[S]$ for a finite set S then B is called finitely generated A-algebra

Let $b \in B$.The subalgebra $A[b]$ generated by the singleton $\{b\}$ consists precisely of all polynomial expression in b with coefficients in A,i.e.elements of the form $\sum_{i=0}^{\infty} a_i x^i$ with n a non-negative integer and $a_i \in A$ for every i.

Definition 1.1.3. *:* ***Prime Ideal***
An ideal P in A is prime if $P \neq A$ and $ab \in P \Rightarrow a \in P$ or $b \in P$

Definition 1.1.4. ***Maximal Ideal***
An ideal M in A is maximal if $M{\neq}A$ and if there an ideal J of such that $M{\subseteq}J{\subseteq}A \Rightarrow M{=}J$ or $J{=}A$

Definition 1.1.5. ***Multiplcative Subset***
A set S is said be multiplicative subset if $1{\in}S$, $a{,}b{\in}S{\Rightarrow}ab{\in}S$

for example,the following are multiplicative subsets.
The multiplicative set $\langle f\rangle$ generated by an element f of A,the complimrnt of a prime ideal is also an example of multiplicative set

Theorem 1.1.6. *Every proper ideal in a ring A is contained in some maximal ideal.*

Proof. Proof is by using Zorne's Lemma,
Let $F = \{\ J \mid J\ is\ an\ ideal\ in\ A\ with\ I \subseteq J \neq A\}$.
Clearly $I{\in}F{\Rightarrow}\ F{\neq}\phi$
Let $J_1{,}J_2{\Rightarrow}F$ and define $J_1{\leq}J_2{\Leftrightarrow}J_1{\subseteq}J_2$ then $(F, \leq)$ is a poset.Let C be a chain in F and define
$T_0{=}\bigcup_{T\in C} T$ and $T{\leq}T_0\ \forall\ T{\in}C$.So T_0 is an upper bound for C and T_0 is an ideal in A containing I and also we note that T_0 can not be equal to A therefore $T_0{\in}F$ and T_0 is a upper bound for C.Now using Zorn's lemma there exist a maximal element say M in F and now it is easy to show M is maximal ideal in A by using maximality of M in F

□

Proposition 1.1.7. *Let S be a subset of a ring A and I be an ideal of A disjoint from S.Then the set of ideals in A containing I and disjoint from S contain a maximal element and if S is multiplicative then every such maximal element is prime*

Proof. : The set F of ideals of A containing I disjoint from S is non empty because it contains I.Now by previous theorem we define
$T_0 = \bigcup_{T\in C} T$ where C is a chain in F and $T_0 \in F$ otherwise some element of S lies in T_0 and hence in hence in T for some T which is a contradiction to the defination of F then by Zorn's lemma F has maximal element

Now assume S is multplicative subset of A and let M be maximal element in F Let bb$\prime \in M$ and if b $\notin M$ then $M \subset M + (b)$
$\Rightarrow M + (b) \notin F$, therefore S contains an element of $M + (b)$ say, f=c+ab where c$\in M$,a$\in A$ similarly if b$\prime \notin M$ then S contain an element f$\prime$=c$\prime$+a$\prime$b, where c$\prime \in M$,a$\prime \in A$
Now we have ff$\prime$=cc$\prime$+abc$\prime$+a$\prime$b$\prime$c+aba$\prime$b$\prime \in M$ which contradicts to ff$\prime \in S$.Hence M is prime ideal in A

$\square$

1.2 Radical

Let A be a ring and I be an ideal of A then radical of I is
$\{f \in A : f^r \in I, some\ r \in \mathbb{N}\}$

Remark 1.2.1. *Prime ideals are radical*

Proposition 1.2.2. *Let I be an ideal in a ring A then,*
(a) The radical of I is an ideal
(b) $rad(rad(I))$=$rad(I)$

Proof. First part is easy i shall prove second one.Let $a \in$rad(I) then $a^r \in I$
$\Rightarrow (a^r)^s \in I$ for some r,s $\in \mathbb{N}$
$\Rightarrow a^r \in$ rad(I)
$\Rightarrow$ a$\in$rad$(rad(I))$
Conversely,let a$\in$rad$(rad(I))$ then $a^r \in$rad(I)
$\Rightarrow (a^r)^s \in I$ and so $a^t \in I$ for some t$\in \mathbb{N}$ which means a$\in$rad(I) $\square$

Remark 1.2.3. *If I and J be two radical then $I \cap J$ is also a radical but $I + J$ need not be an radical,for example let $I = (X^2 - Y)$ and $J = (X^2 + Y)$ both are prime ideals in $\mathbb{K}[X, Y]$ then $I + J = (X^2, Y)$ which is not radical because it contains X^2 but not X*

Proposition 1.2.4. *The radical of an ideal I is equal to the intersection of prime ideals containing it.In perticular,the nilradical of a ring A is equal the intersection of the prime ideals of A*

Proof. Claim: rad(I)=$\bigcap P$,where $I \subseteq P$.If I=A then there is no prime ideal and set of all prime ideal is $\emptyset$ and then intersection over empty set is full ring then we are done.Let $I \subseteq A$ then rad$(I) \bigcap P$ where $I \subseteq P$ because prime ideals are radical and rad(I) is the smallest ideal containg I

Conversely,let f$\notin$rad(I) and let S=$\{1$,f,f^2......$\}$ be multiplicative set and we know rad(I) is an ideal that contain I and rad(I)$\cap S=\phi$ and then by prop1 $\exists$ a prime ideal say P disjoint from S, therefore $f\notin P$ and hence f does not belong to the intersection of prime ideals.Hence we are done □

Definition 1.2.5. *The Jacobson radical J of a ring is the intersection of the maximal ideals of the ring*

$J(A)$=$\bigcap\{m \mid m$ *is maximal ideal in* A$\}$
A ring is local if it has exactly one maximal ideal,for such a ring,the Jacobson radical is m

Proposition 1.2.6. *An element c of A is in the jacobson radical of A if and only if* 1-ac *is a unit for all* $a\in A$

Proof. : We prove the contrapositive,$\exists$ a maximal ideal M such that $c\notin M$ iff $\exists$ $a\in A$ such that $1-ac$ is not a unit.Let $1-ac$ is not a unit then $(1-ac)\subset M$ and $1-ac$ $\in(1-ac)$ then $c\notin M$ otherwise,
$1 = 1-ac+ac\in M$
$\Rightarrow 1\in M$ that is not possible
Conversely,let $c\notin M$ then $M\subset M+(c)$
$\Rightarrow M+(c)=A$,since M is maximal ideal,therefore $1=m+ac$,$m\in M$,$a\in A$
$\Rightarrow$ $1-ac\in M$
$\Rightarrow$ $1-ac$ is not a unit □

Theorem 1.2.7. ***Prime Avoidance***
Let P_1,P_2,....,P_r,$r\geq 1$ *be ideals in A such that* P_i *are prime ideals for* $i\geq 3$.*If an ideal I is not contained any of* P_i *then I is not contained in the union of* P_i

Proof. :I shall prove it by induction on r.The idea is to find an element in I but not in any of P_i's
For $r=1$ nothing to prove.Next suppose $r\geq 2$ and for each i choose

$$z_i\in I\setminus \bigcup P_j \text{ where } i\neq j$$

Where the set on right is nonempty by inductive hypothesis.We can assume $z_i\in P_i$ for all i,otherwise ,if some z_i does not lie in P_i, then $z_i\in I\setminus\bigcup P_i$ for all $i=1,2,..,r$.

Now put

$$z = z_1....z_{r-1}+z_r$$

Then z in I but not in any of P_i's.If z is in any of P_i for some $i\leq r-1$ then $z_r\in P_i$ which contradict to $z_r\in P_r$.Now suppose z is inP_r.Then $z_1...z_{r-1}$ is in P_r

If r is 2,we are done.If $r\geq 3$,then,since P_r is a prime ideal,some z_i, $i\leq r-1$ is in P_r,a contradiction so our assumption $z_i\in P_i$ for all i is wrong.So we are done. □

1.3 Contraction and extension ideals

Let ϕ :A $\rightarrow B$ be a ring homomorphism.For an ideal b of B, $\phi^{-1}(b)$ is an ideal in A called the contraction of b to A and denoted by b^c,and for an ideal a of A the ideal in B generated by $\phi(a)$ is called the extension of a to B and denoted by a^e,when ϕ is surjective then $\phi(a)$ is an ideal in B and when A is subring of B then $b^c = b \cap A$

Properties of contraction and extension of ideals

Let a,$a\prime$ be ideals of A and b,$b\prime$ be ideals of B then

$(a + a\prime)^e = a^e + a\prime^e$, $(aa\prime)^e = a^e a\prime^e$, $(b \cap b\prime)^c = b^c \cap b\prime^c$, $rad(b^c) = rad(b)^c$

Theorem 1.3.1. *Correspondence Theorem*

Let $f : A \rightarrow B$ be a ring homomorphism then

(1)*for any ideal I of$A$ we have $I \subseteq I^{ec}$ and $I^{ece} = I^e$. For any ideal J of B we have $J^{ce} \subseteq J$ and $J^c = J^{cec}$*

(2) *There is a bijection between contracted ideals in A and extended ideals in B*

Proof. Let $r \in I$ then $f(r) \in I^e$ so $r \in I^{ec}$.The ideal J^{ce} generated by $f(J^c)$.If $r \in J^c$ then $f(r) \in J$,so the ideal generated by $f(J^c)$ is contained in the ideal J.Since $I \subseteq I^{ec}$ we get $I^e \subseteq I^{ece}$ and since $I^{ece} \subseteq I^e$, we conclude that $I^{ece} = I^e$, similarly we can prove $J^{cec} = J^c$

(2) Let C denote the set of contracted ideal in A and E denote the set of extented ideals in B and every in C is of the form J^c for some ideal J of B and every ideal of E is of the form I^e for some ideal I of A.Since $I^{ece} = I^e$ and $J^{cec} = J^c$.Now the map $\varphi : C \rightarrow E$ given by $\varphi(I) = I^e$ is clearly bijective with the given condition above □

Remark 1.3.2. *If J is a prime ideal of B then J^c is a prime ideal A but if I is prime idel of A then I^e need not be prime ideal for example take identity map $\mathbb{Z} \to \mathbb{Q}$ then for any p prime $p\mathbb{Z}$ is ideal of $\mathbb{Z}$ but $(p\mathbb{Z})^e = \mathbb{Q}$ which is not prime in $\mathbb{Q}$*

Theorem 1.3.3. *Chinese Remainder Theorem Let A be a ring and $I_1, I_2, ..., I_k$ be ideals of A such that I_i and I_j are coprime for $i \neq j$ then,*

$$A/I_1 \cap I_2 \cap ... \cap I_k \cong A/I_1 \times ..... \times A/I_k$$

Proof. I shall prove it for $k = 2$,one can show for finitely many such ideals Define a map $\phi; A/IJ \to A/I \times A/J$ by

$\phi(x + IJ) = (x + I, x + J)$,ϕ is well define since IJ is an ideal of A and let

$$x + IJ = y + IJ$$
$$x - y \in IJ = I \cap J \text{ , since } I, J \text{ are co prime}$$
$$\Rightarrow x + I = y + J,\ x + J = y + J \text{ therefore, } \phi \text{ is well define}$$
$$\phi \text{ is one one clearly,for onto let } (x + I, y + J) \in A/I \times A/J$$
$$\text{Now we want to find } \alpha \in A \textit{ such that } \phi(\alpha + IJ) = (x + I, y + J)$$
$$\text{Since } I, J \text{ co prime therefore } 1 = a + b,\ a \in I, b \in J, \text{ now define}$$
$$\alpha = ay + bx \text{ then we have}$$

$$\begin{aligned}\phi(\alpha + IJ) &= (\alpha + I, \alpha + J)\\ &= (bx + I, ay + J)\\ &= (x + I, y + J)\end{aligned}$$

therefore ϕ is onto, and also clearly homomorphim so we are done

∎

Chapter 2

Noetherian Rings

Proposition 2.0.1. *T.F.A.E on a ring A*
(a) *Every ideal in A is finitely generated*
(b) *Every ascending chain of ideals $I_1 \subset I_2 \subset .....$ eventually become constant*
(c) *Every non empty set of ideals in A has a maximal element*

Proof. $(a) \Rightarrow (b)$ Let $I_1 \subsetneq I_2 \subsetneq ....$ be ascending chain of ideals of A.Now set $I = \bigcup_{i=1}^{\infty} I_i$,then I is an ideal of A therefore I is finitely generated say $I = (x_1, x_2, ..., x_n)$ then $\exists m$ such that $x_i \in I_m$ for all i so $x_i \; in \; I \Rightarrow x_i \in \; I_m$ then we are done
$(b) \Rightarrow (c)$
Let $F = \{I_i, i \in \wedge\}$ be a non empty family of non empty family of ideals of A .Pick any index i_1 and look at I_{i_1} if this is maximal in F then we are done.If not then choose $i_2 \in \wedge$ such that $I_{i_1} \subsetneq I_{i_2}$ if this one is maximal then we are done if not repeat this process after finite stage it stop surely

$(c) \Rightarrow (a)$ Let I be an ideal of A.Consider the family of F of all finitely generted ideals of I then $F \neq \phi$ since $(0) \in F$,then F has maximal element say $I_0 = (x_1, ..., x_n)$ .If $I \neq I_0$ then pick $x \in$ I but not in I_0 then $I_1 = I_0 + (x)$ $\Rightarrow I_1 \in F$ which is a contraction since I_0 is max so we are done

□

Definition 2.0.2. *A ring A is said to be noetherian if it is satisfies the above equivalent condition*

Proposition 2.0.3. *Let A be a ring.The following conditions on an A-module M are equivalent*
(a) *Every submodule of M is finitely generated*

(b) Every ascending chain of submodules $M_1 \subsetneq M_2 \subsetneq .....$ eventually become constant
(c) every non empty set of submodules of M has A maximal element

Proof. Essentially same as the prop 2.0.1 □

Theorem 2.0.4. ***Hilbert Basis Theorem*** *If A is Noetherian , then $A[x]$ is Noetherian*

Proof. Let$I$ be an ideal of $A[x]$.We shall show I is finitely generated.Choose a sequence $f_1, f_2, ... \subsetneq I$ as follows,let f_1 be non zero element of least degree in I.For $i \geq 1$, if $(f_1, ..., f_i) \neq I$ then choose f_{i+1} to be an element of least degree amomg those in I but not in $(f_1, ..., f_i)$ otherwise if $I = (f_1, ..., f_i)$ then we are done.
Let a_j be the leading coefficient of f_j,since A is noetherian then ideal $J = (a_1, a_2, ...)$ is finitely generated,so $J = (x_1, ..., x_m)$ and again J can be written $J = (a_1, ..., a_m)$.Now we claim $I = (f_1, ..., f_m)$
Otherwise, consider $f_{m+1} \cdot a_{m+1} \in J$, so we can write $a_{m+1} = \sum_{j=1}^{m} u_j a_j$ for some $u_j \in A$.Define

$$g = \sum_{j=1}^{m} u_j f_j x^{def_{m+1} - deg f_j} \in (f_1, ..., f_m)$$

and notice that this is of the same degree as f_{m+1} ,with the same initial term.The difference $f_{m+1} - g$ is in I but not $(f_1, ..., f_m)$, and has degree less than that of f_{m+1}.But f_{m+1} was something of minimal degree with this property,so we have contradiction □

Remark 2.0.5. *Converse of the above theorem is also true and result also true for finitely many variables*

Example of Noetherian rings
Any field , PID,finite ring,$\mathbb{Z}$ and the ring $\mathbb{Z}[x_1, x_2, ...]$ is not noetherian because we have non terminating ascending chain

Subring of noetherian need not be noetherian
Take above infinite variable ring which is a subring of its field of fraction and field of fraction of a ring is noetherian

Lemma 2.0.6. ***Nakayama's Lemma*** *Let A be a ring and I be an ideal in A.Let M be an A- module and assume that I is contained in all maximal ideal of A and M is also finitely generated then*

(a) If $M = IM$ then $M = 0$ (b) If N is a submodule of M such that $M = N + IM$ then $M = N$

Proof. Suppse M is non zero,choose a minimal gener ating set $x_1, x_2, ..., x_n$ for M .Now $x_{1\in} \in M$ so $x_1 \in IM$ therefore , $x_1 = a_1m_1 + ... + a_nm_n$, $a_i \in I, m_i \in M$
now each m_i can be written in form of x_i
$\Rightarrow (1 - a_1)x_1 = a_2x_2 + ... + a_nx_n$, but $(1 - a_1)$ is unit in A therefore x_1 is a linear combination of remaining x_i which contradict minimality of generating set for M.Hence M is zero module

(b) Since N is submodule of M then M/N makes sense then we note

$$I(M/N) = \{\sum_{i=1}^{n} a_i(m_i + N) \mid a_i \in I, m_i \in M\}$$

$$= (IM + N)/N$$

$$= M/N$$

then by part one

$$M/N = 0$$

$$\Rightarrow M = N$$

□

Note 2.0.7. *Let A be a local ring with maximal ideal m.Let $K = A/m$ be the residue field of A. Let M be finitely generated A- module then $m \subseteq Ann(M/mM)$*

Now we note that M/mM is a vector space over K where scalar multiplication is define as follows

$A/m \times M/mM \to M/mM$
$(x+m, y+mM) \mapsto xy+mM$ and this is well define can be proved by using $m \subseteq Ann(M/mM)$

Proposition 2.0.8. *Let A be local ring with maximal ideal m and residue field $K = A/m$. And let M be finitely generated module over A the action of A on M/mM factor through K and elements $a_1, a_2, ..., a_n$ of M generate it as an A module iff the elements $a_1 + mM, ..., a_n + mM$ spanM/mM as a vector space over K*

Proof. If $a_1, .., a_n$ generates M then their images generate the vector space M/mM
Conversely,suppose that $a_1 + mM, ..., a_n + mM$ span M/mM and let N be a submodule of M then the composite map $N \to M \to M/mM$ is onto and so $M = N + mM$ then by lemma $M = N$ □

Proposition 2.0.9. *Let A be noetherian local ring with maximal ideal m. Elements $a_1, ..., a_n$ of m generate m as an ideal if and only if $a_1+m^2, ..., a_n+m^2$ generate m/m^2 as a vector space over A/m. In particular, the minimum number of generators for the maximal ideal is equal to the dimension of the vector space m/m^2.*

Proof. Because A is noetherian so m is finitely generated then apply previous proposition for $M = m$ we are done □

Definition 2.0.10. *Let A be a noetherian ring.*
(a) The height $ht(p)$ of a prime ideal p in A is the greatest length d of a chain of distinct prime ideals $p = p_d \supseteq ..... \supseteq p_0$
(b) The (krull) dimension of A is $\sup\{ht(p) \mid p \subset A, p \text{ is prime ideal}\}$

Example 2.0.11. *The hight of a non-zero prime ideal in PID is one because $(0) = p_0 \subsetneq (x) = p_1$, so such a ring has krull dim one unless it is not field*

Note 2.0.12. *It is sometimes convenient to define the Krull dimension of the zero ring to be -1*
Let A be an integral domain then $dim(A) = 0$ iff (0) is maximal ideal of A iff A is field

Proposition 2.0.13. *Every set of generators for a finitely generated ideal contains a finite generating set.*

Proof. Let $S = \{S_1, S_2, ...\}$ be a set of generators for an ideal I and suppose that I is generated by a finite set $\{a_1, ..., a_n\}$. Each a_i lies in the ideal generated by a finite subset S_i of S, and so I is generated by a finite subset $\cup S_i$ of S .Since the set $\{a_1, ..., a_n\} \subseteq \cup S_i$ □

Theorem 2.0.14. ***Krull Intersection Theorem*** *Let I be an ideal in a noetherian ring A. If I is contained in all maximal ideals of A, then* $\bigcap_{n\geq} I^n = (0)$

Proof. We shall show that, for every I in a noetherian ring A
$\bigcap_{n\geq 1} I^n = I \bigcap_{n\geq 1} I^n$
Since A is noetherian ,let $a_1, a_2, ..., a_r$ generate I and
$I^n = \{g(a_1, ..., a_r) \mid g \in A[x_1, ...x_r], g\ is\ homogeneous\ of\ degree\ n\}$
Let S_m denote the set of homogeneous polynomials f of such that $f(a_1, ..., a_r) \in \bigcap_{n\geq 1} I^n$ and let J be an ideal in $A[x_1, .., x_r]$ generated by the set $\bigcup_{m\geq 1} S_m$.Since $A[x_1, .., x_r]$ is noetherian so J is finitely generated and generated by the srt $\{f_1, ..., f_s\}$ of elements of $\bigcup_{m\geq 1} S_m$.Let $d_i = degf_i$ and $d = max\ d_i$

Let $b \in \bigcap_{n\geq 1} I^n$ then $b \in I^{d+1}$, and so $b = f(a_1, ..., a_r)$ for some homogeneous polynomial f of degree $d + 1$ therefore by definition $f \in S_{d+1} \subseteq J$ so $f = g_1 f_1 + ... + g_s f_s$ for some $g_i \in A[x_1, ..., x_n]$
As f and the f_i are homogeneous, we can omit from each g_i all terms not of degree $degf - degf_i$, since these terms cancel out. In other words, we can choose the g_i to be homogeneous of degree $degf - degf_i = d + 1 - d_i > 0$, in perticular the constant term of g_i is zero and so $g_i(a_1, ..., a_r) \in I$.Now $b = f(a_1, ..., a_r) = \sum_i g_i(a_1, .., a_r) f_i(a_a, ..., a_r) \in I \bigcap I^n$ and this complete the our requirement $\square$

Chapter 3

Rings of fraction

Let S be a multiplicative subset of a ring A. Define a relation $\equiv$ on $A \times S$ as follows , for $a, b \in A, s, t \in S$

$$(a, s) \equiv (b, t)$$

iff $\exists u \in S$ such that $(at - bs)u = 0$
This is an equivalence relation
Write a/s for the equivalence class containing (a, s) and define addition and multiplication of equivalence classes according to the rules

$$a/s + b/t = (at + bs)/st$$
$$(a/s) \cdot (b/t) = (ab/st)$$

The operations addition and multiplication defined above are well define.Now first we shall prove multiplication is well define
Let $(a_1, s_1) \equiv (a_2, s_2)$ and $(b_1, t_1) \equiv (b_2, t_2)$ then for some $u, v \in S$ we have $(a_1 s_2 - a_2 s_1)u = 0$ and $(b_1 t_2 - b_2 t_1)v = 0$
Want to show $(a_1 b_1, s_1 t_1) \equiv (a_2 b_2, s_2 t_2)$

$$\begin{aligned}[(a_1 b_1)(s_2 t_2) - (a_2 b_2)(s_1 t_1)]uv &= (a_1 s_2 - a_2 s_1)u b_1 t_2 v + (b_1 t_2 - b_2 t_1)v a_2 s u \\ &= 0 + 0 = 0\end{aligned}$$

Similarly we can show addition is also well define
Now we define a set $S^{-1}A = \{a/s : a \in A, s \in S\}$ and this is ring with the

operation defined above with identity $1 = s/s, \forall s \in S$ .We call $S^{-1}A$ the ring of fractions of A with respect to S
If A is an integral domain and $S = A \setminus \{0\}$ then $S^{-1}A$ is the familiar field of fractions of A

Let $f : A \to S^{-1}A$, where $f(x) = x/1$ then clearly f is a ring homomorphism
Observation if A is an integral domain and S any multiplicatively closed subset not containing 0 then$f$ is injective.
proof Suppose A is an integral domain, $0 \notin S \subseteq A$, and S multiplicatively closed. Let $x_1, x_2 \in A$ such that $x_1/1 = x_2/1$,then $(x_1, 1) \equiv (x_2, 1)$,so

$$(x_1 - x_2)u = 0$$

for some $u \in S$

$$\Rightarrow x_1 - x_2 = 0$$

, since A is an integral domain and $u \neq 0$ thus f is injective

$S^{-1}A$ has following universal property

Theorem 3.0.1. *Let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for each $s \in S$.*
Then there is a unique homomorphism h such that this diagrame

$$\begin{tikzcd} & S^{-1}A \arrow[dr, "h"] & \\ A \arrow[ur, "f"] \arrow[rr, "g"] & & B \end{tikzcd}$$

commutes

Proof. Define $h : S^{-1}A \to B$ by

$$h(a/s) = g(a)g(s)^{-1}$$

where $a \in A$,$s \in S$. Now i will show h is well define
Suppose $a/s = a'/s'$,then

$$(as' - a's)t = 0$$

for some $t \in S$ thus
$0 = g(0) = g((as\prime - a\prime s)t)$

$$0 = [g(a)g(s\prime) - g(a\prime)g(s)]g(t)$$

and $g(t)$ is unit in B , then

$$g(a)g(s\prime) - g(a\prime)g(s) = 0$$

and since $g(s), g(s\prime)$ are unit in B and this prove that h is well define map also we note as g is ring homomorphism so is h
Further if $a \in A$ then
$(h \circ f)(a) = h(a/1) = g(a)g(1)^{-1} = g(a)$ so that the diagrame

$$\begin{tikzcd} & S^{-1}A \arrow[dr, "h"] & \\ A \arrow[ur, "f"] \arrow[rr, "g"] & & B \end{tikzcd}$$

commutes
Suppose also that $h\prime : S^{-1}A \to B$ is a ring homomorphism such that this diagrame

$$\begin{tikzcd} & S^{-1}A \arrow[dr, "h\prime"] & \\ A \arrow[ur, "f"] \arrow[rr, "g"] & & B \end{tikzcd}$$

commutes and for all $s \in S$, $g(s)$ is unit in B,then
$h\prime(a/s) = h\prime(a/1 \cdot 1/s) = h\prime(a/1)h\prime(1/s)$ .But $1/s$ is unit in $S^{-1}A$ with inverse $s/1$, so that $h\prime(1/s)$ is a unit in B and
$h\prime(1/s) = [h\prime(s/1)]^{-1}$
Hence
$h\prime(a/s) = h\prime(a/1)[h\prime(s/1)]^{-1} = g(a)g(s)^{-1} = h(a/s)$ and this proves h is unique with this property $\square$

3.1 Localization

Let P be a prime ideal of A , and put $S = A \backslash P$ which is multiplicatively closed , form $A_P = S^{-1}A$ and put $M = \{a/s \in A_P : a \in P\}$
Claim A_P is a local ring with unique maximal ideal M. The process of passing from A to A_P is called localization at P . e.g. If $A = \mathbb{Z}$ and $P = p\mathbb{Z}$

where p is a prime integer, then localization at P produces $A_P = \{a/b : a, b \in \mathbb{Z}, p \nmid b\}$

Proof of claim We first prove $\forall b \in A, \forall t \in S$, $b/t \in M \Rightarrow b \in P$
Suppose
$b/t = a/s$ where $b \in A, a \in P$ and $s, t \in S$. Then $(at - bs)u = 0$ for some $u \in S$

So , $(at - bs) \in S$ since P is prime, $0 \in P$ and $u \notin P$. Hence $bs = at - (at - bs) \in P$. But $s \notin P$, so $b \in P$, and above sub claim is proved. By subclaim, certainly $1 = 1/1 \notin M$, (since $1 \notin P$) so $M \neq A_P$ and M is ideal of A_P
Now if $b \in A, t \in S$ and $b/t \notin M$, then, by definition of M , $b \notin P$, so $b \in S$, yielding $t/b \in A_P$, where b/t is a unit of A_P, therefore M is the set of all unit of A_P so M is maximal ideal so A_P is local ring

Example 3.1.1. *$S^{-1}A$ is the zero ring iff $0 \in S$*
Solution :$\Leftarrow$ If $0 \in S$ then, for all $a, b \in A, s, t \in S$

$$a/s = b/t$$

since $(at - bs)0 = 0$,so that all elements of $S^{-1}A$ are equal
$\Rightarrow$ If $S^{-1}A$ contains only one element then $(0, 1) \equiv (1, 1)$ so that
$0 = (0 \cdot 1 - 1 \cdot 1)t = -t$ for some $t \in S$ so that $0 = t \in S$

Proposition 3.1.2. *For an ideal I of A , $S^{-1}I$ is a proper ideal of $S^{-1}A \Leftrightarrow I \cap S = \phi$.Further if P is a prime ideal of A with $P \cap S = \phi$ then*
(1) For $a \in A, s \in S$ we have $a/s \in S^{-1}P \Leftrightarrow a \in P$
(2) $S^{-1}P$ is a prime ideal of $S^{-1}A$

Proof. If $s \in I \cap S$ then $1 = s/s \in S^{-1}I$, so $S^{-1}I$ is not proper ideal , so

$$I \cap S = \phi$$

Conversely,suppose that $S^{-1}I$ is not proper ideal of $S^{-1}A$ then $1 \in S^{-1}I$ so $1/1 = a/s$ with $a \in I, s \in S$

$$\Rightarrow at = st$$

for some $t \in S$ .Also $at \in I$ since I is an ideal ,therefore $st \in I \cap S \neq \phi$
(1) If P is prime ideal disjoint from S and if $a/s \in S^{-1}P$ then $a/s = p/u$ for some $p \in P, u \in S$ therefore, $a{\cdot}u{\cdot}t = s{\cdot}p{\cdot}t \in P$ for some $u, s, t \in S$ but $ut \notin P$

$$\Rightarrow a \in P$$

since P is a prime ideal

(2) Let $(a/s)(b/t) \in S^{-1}P$

$$\Rightarrow ab/st \in S^{-1}P$$

then by previous part $ab \in P$ so either $a \in P$ or $b \in P$
$\Rightarrow$ either $a/s \in S^{-1}P$ or $b/t \in S^{-1}P$ .Hence $S^{-1}P$ is prime ideal □

Example 3.1.3. *Every ideal of $S^{-1}A$ is of the form $S^{-1}I$ for some ideal I of A*
Let J be an ideal of $S^{-1}A$. Put $I = \{x \in A : x/1 \in J\}$.
Claim $J = S^{-1}I$
If $a/s \in J$ then $a/1 = s/1 \cdot a/s \in J$ so $a \in I$ implies $a/s \in S^{-1}I$.
Conversely, $a/s \in S^{-1}I$ then $a \in I$ so $a/1 \in J$ therefore $(1/s)(a/1) = a/s \in J$

Theorem 3.1.4. *The map $P \mapsto S^{-1}P$ is bijective from the set of prime ideals of A and disjoint from S onto the set of all prime ideals of $S^{-1}A$*

Proof. If P is prime then $S^{-1}P$ is prime. Let P, Q be prime ideals of A disjoint from S . If $P \subseteq Q$ then $S^{-1}P \subseteq S^{-1}Q$
Conversely, suppose $S^{-1}P \subseteq S^{-1}Q$ then for $p \in P$ we have $p/1 \in S^{-1}P$

$$\Rightarrow p/1 \in S^{-1}Q$$

$$\Rightarrow p \in Q$$

Hence $P \subseteq Q$ .This proves that $P \subseteq Q \Leftrightarrow S^{-1}P \subseteq S^{-1}Q$.Consequently $P = Q \Leftrightarrow S^{-1}P = S^{-1}Q$, therefore the given map is injective.
Let P be prime ideal of $S^{-1}A$ then $P = S^{-1}P_1$ for some ideal P_1 of A and P_1 is prime since $S^{-1}P_1$ is prime then we are done □

Proposition 3.1.5. *Let S be a multiplicative subset of the ring A, and consider extension $I \mapsto I^e = S^{-1}I$ and contraction $I \mapsto I^c$ of ideals with respect to the homomorphism $\phi : A \to S^{-1}A$. Then*
$I^{ce} = I$ for all ideals of $S^{-1}A$ and $P^{ec} = P$, if P is a prime ideal of A and disjoint from S

Proof. Let I be an ideal in $S^{-1}A$ then $I^{ce} \subseteq I$. Now $b \in I$ then $b = a/s, a \in A, s \in S$

So $a/1 = s(a/s) \in I$ this implies $a \in I^c$. Hence $b \in I^{ce}$
Now let P be prime ideal of A then $P \subseteq P^{ec}$. Let $a \in P^{ec}$ so that $a/1 = a\prime/s$ for some $a\prime \in P, s \in S$. Then $(as - a\prime)t = 0$ for some $t \in S$ and therefore $ast \in P$ implies $a \in P$ since $st \notin P$ and P is prime and this complete the proof $\square$

Chapter 4

Modules of fractions

Let A be a ring, S a multiplicatively closed subset of A , and M be an A-module.
Define a relation $\equiv$ on $M \times S = \{(m,s)|m \in M, s \in S\}$ by, for $m, m\prime \in M, s, s\prime \in S$

$(m,s) \equiv (m\prime, s\prime)$
iff $\exists t \in S, t(sm\prime - s\prime m) = 0$
If $m \in M$ and $s \in S$ then write $m/s =$ *equivalence class of* (m,s) and put

$$S^{-1}M = \{m/s : m \in M, s \in S\}$$

Define addition and scalar multiplication on $S^{-1}M$ by, for $m, m\prime \in M, s, s\prime \in S, a \in A, t \in S$
$(m/s) + (m\prime/s\prime) = (sm\prime + ms\prime)/ss\prime$
$(a/t)(m/s) = am/ts$

And $S^{-1}M$ is an $S^{-1}A$-module, referred to as the module of fractions with respect to S

Since the mapping $a \mapsto a/1$ is a ring homomorphism from $A \to S^{-1}A$, by restriction of scalars we have
$S^{-1}M$ is an A-module with scalar multiplication $(\forall a \in A, m \in M, s \in S)$
$a \cdot (m/s) = (a/1)(m/s) = am/s$

Notation

Let M be an A-module. (1) Write $M_P = S^{-1}M$ if $S = A \setminus P$ where P is a prime ideal of A

Think S^{-1} as an "operator" which manufactures $S^{-1}A$-modules from A-modules.
Also S^{-1} "operates" on module homomorphisms. Let $u : M \to N$ be an A-module homomorphism.
Define , $S^{-1}u : S^{-1}M \to S^{-1}N$ by

$$m/s \to u(m)/s \quad , m \in M, s \in S$$

$S^{-1}u$ is well define as u is an A module homomorphism .Now we observe S^{-1} preserve addition and multiplication

$$(S^{-1}u)(m_1/s_1 + m_2/s_2) = (S^{-1}u)((m_1s_2 + m_2s_1)/s_1s_2)$$

$$= u(m_1s_2 + m_2s_1)/s_1s_2$$

$$= [s_2u(m_1) + s_1u(m_2)]/s_1s_2$$

$$= u(m_1)/s_1 + u(m_2)/s_2$$

Similarly we can show S^{-1} preserve scalar multiplicatation
Hence $S^{-1}u$ is $S^{-1}A$ module homomorphism
(and also, by restriction of scalars, an A-module homomorphism).

Further if $M_1 \xrightarrow{u} M_2 \xrightarrow{v} M_3$ are A-module homomorphisms, then,
for all $x \in M_1, s \in S$
$[S^{-1}(v \circ u)](x/s) = (v \circ u)(x)/s = v(u(x))/s$

$$= (S^{-1}v)(S^{-1}u)(x/s)$$

$$= [(S^{-1}v) \circ (S^{-1}u)(x/s), which\ shows\ S^{-1}(v \circ u) = (S^{-1}v) \circ (s^{-1}u)$$

Theorem 4.0.1. *Suppose $M_1 \xrightarrow{f} M \xrightarrow{g} M_2$ be exact sequence of A- modules at M . Then*

$$S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M_2$$

is exact sequence of $S^{-1}A$ modules at $S^{-1}M$

Proof. Since the given sequence is exact so we have $g \circ f = 0$ the zero homomorphism,therefore
$(S^{-1}g \circ S^{-1}f) = S^{-1}(g \circ f) = S^{-1}(0) = 0$, which proves
$Im(S^{-1}f) \subseteq ker(S^{-1}g)$. Suppose $m/s \in ker(S^{-1}g)$, so $g(m)/s$ is the zero of $S^{-1}M_2$. Hence $(g(m), s) \equiv (0, 1)$, so $0 = tg(m) = g(tm), for\ some\ t\ \in S$, yielding $tm \in kerg = Imf$.

Hence ,$tm = f(m\prime)$ for some $m\prime \in M_1$,and $(S^{-1}f)(m\prime/st)$
$= f(m\prime)/st = tm/ts = m/s$, proving $m/s \in Im(S^{-1}f)$.
Thus $ker(S^{-1}g) \supseteq Im(S^{-1}f)$, completing the proof exactness at $S^{-1}M$ □

Example 4.0.2. *Let M be an A-module. For $h \in A$, let $M_h = S_h^{-1}M$ where $S_h = \{1, h, h^2, ...\}$. Then every element of M_h can be written in the form $m/h^r, m \in M, r \in \mathbb{N}$ and $m/h^r = m\prime/h^{r\prime}$ if and only if $h^N(mh^{r\prime} - m\prime h^r) = 0$ for some $N \in \mathbb{N}$*

Proposition 4.0.3. *Let M be a finitely generated A-module. If $S^{-1}M = 0$, then there exists an $h \in S$ such that $M_h = 0$.*

Proof. $S^{-1}M = 0$ means that, for each $x \in M$, there exists an $s_x \in S$ such that $s_x x = 0$. Let $x_1, ..., x_n$ generate M. Then define $h = s_{x_1}.....s_{x_n}$ in S and observe $hM = 0$ by using M is finitely generated. Now let $a/s \in M_h$ then $a/s = ha/hs = 0$, therefore $M_h = 0$ □

Proposition 4.0.4. *Let M be an A module then the canonical map*

$$M \to \prod\{M_m : m \text{ is maximal ideal in } A\}$$

is injective

Proof. Let $x \in M$ map to zero in all M_m then we shall show x is zero.
Here $M_m = S_m^{-1}M, S_m = A \setminus m$
Let $I = Ann(x) = \{a \in A : ax = 0\}$ is an ideal of A.
Because x maps to zero in all M_m so $\exists s \in S_m$ such that $sx = 0, s \notin m, s \in A$
$\Rightarrow s \in I$ but $s \notin m$ and therefore I is not contained in m and this is true for all m so I is equal to A itself
$\Rightarrow 1 \in Ann(x)$, therefore $x = 1 \cdot x = 0$ so given map is injective □

Proposition 4.0.5. *Let A - module $M = 0$ if $M_m = 0$ for all maximal ideal m*

Proof. Let $x \in M$ and $I = Ann(x) = \{a \in A : ax = 0\}$, then I is an ideal of A , since $M_m = 0$ for all m so $\exists s \in A \setminus m$ such that $sx = 0$, doing same as previous proposition we get $x = 0$ □

Chapter 5

Integral Extentions

Let A be a subring of B.An element b of B is said be integral over A if it is a root of a non zero monic polynomial with coefficients in A it means it satisfies the equaton

$b^n + a_1b^{n-1} + ... + a_n = 0, a_i \in A$. Such an equation is called an integral equation of b over A

Proposition 5.0.1. *For an element b of B , T.F.A.E*

(1) *b is integral over A*
(2) *$A[b]$ is finitely generated as an A module*
(3) *There exist a subring C of B containing $A[b]$ such that C is finitely generated as an A module*
(4) *There exist a finitely generated A submodule M of B such that $bM \subseteq M$ and $ann_B(M) = 0$*

Proof. (1) $\Rightarrow$ (2) Let $b^n + a_1b^{n-1} + ... + a_n = 0, a_i \in A$ be an integral equation of b over A . Let M be an A - submodule of $A[b]$ generated by $1, b, b^2, ..., b^{n-1}$. We claim that $b^r \in M$ for every $r \geq 0$ .This is clear for $r \leq n-1$. If $r \geq n$ then multiplying the integral equation by b^{r-n} we get
$b^r = -(a_1b^{r-1} + a_2b^{r-2} + .. + a_nb^{r-n}) \in M$
Therefore $b^r \in M$ for all non-negative and thus $M = A[b]$. Thus $A[b]$ is finitely generated as an A module

(2) $\Rightarrow$ (3) Take $C = A[b]$

(3) $\Rightarrow$ (4) Take $M = C$,and M has the property $bM \subseteq M$ since $y \in bM$ implies $y = bm \in M$ for some $m \in M$, and note that $1 \in C$ implies that $ann_B(C) = 0$

(4) $\Rightarrow$ (1). Let M be an A module in B with a finite set of generators $\{e_1, ..., e_r\}$ such that $bM \subseteq M$ and $ann_B(M) = 0$ then for all $1 \leq i \leq r$ $be_i = \sum_{j=1}^{r} a_{ij}e_j$ for some $a_{ij} \in A$, and we can rewrite these equation as $\sum_{j=1}^{r}(b\delta_{ij} - a_{ij})e_j = 0$ where δ_{ij} is Kronecker delta and put $d = det(b\delta_{ij} - a_{ij})$ then using cramer rule we get integral equation for b over A

□

Corollary 5.0.2. *Let $b_1, ..., b_r \in B$ be integral over A . Then $A[b_1, ..., b_r]$ is finitely generated as an A-module*

Proof. For $r = 1$, we are done by previous proposition .Inductively assume that $B^{'} = A[b_1, ..., b_{r-1}]$ is finitely generated as an A-module .Since b_r is integral over A , it also integral over $B^{'}$. Now $B^{'}[b_r]$ is finitely generated as a $B^{'}$ module by the case $r = 1$. Now if $x_1, ..., x_m$ are A-module generators of $B^{'}$ and $y_1, .., y_n$ are $B^{'}$-module generators of $B^{'}[b_r]$ then the set $\{x_iy_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ generators of $B^{'}[b_r]$ as an A-module

□

Corollary 5.0.3. *The set $A^{'}$ of elements of B which are integral over A is a subring of B containing A*

Proof. Clearly $A \subseteq A^{'}$. If $b_1, b_2 \in A^{'}$ then by previous corollary $A[b_1, b_2]$ is finitely generated as an A-module . Since $b_1 + b_2$ and $b_1 \cdot b_2 \in A[b_1, b_2]$ then by first proposition both are integral over A □

Note 5.0.4. *The subring $A^{'}$ defined above is called the integral closure of A in B . We say B is integral over A if $A^{'} = B$, and that A is integrally closed in B if $A^{'} = A$*

Proposition 5.0.5. *Let $A \subseteq B \subseteq C$ be integral extentions . If C is integral over B and B is integral over A then C is integral over A*

Proof. Let $c \in C$ and let $c^n + b_1c_{n-1} + ... + b_n = 0$ be an integral equation of c over B . Let $B^{'} = A[b_1, ..., b_n]$. Then c is integral over $B^{'}$ then $B^{'}[c]$ is finitely generated as an $B^{'}$ module by one of the above result .Therefore

$B^{'}[c]$ is finitely generated as an A-module by using B is integral over A , and so c is integral A □

Proposition 5.0.6. *Let A be an integral domain with field of fractions F , and let E be a field containing F . If $x \in E$ is algebraic over F then there exist a non zero $d \in A$ such that $d \cdot x$ is integral over A*

Proof. Since x is algebraic over F we have

$$x^n + a_1 x^{n-1} + ... + a_n = 0$$

where $a_i \in F$. Now using common dinominator, $a_i = b_i/d, \forall i, 1 \leq i \leq n$ .So $b_i = da_i \in A, \forall i$. Now

$$d^n x^n + a_1 d^n x^{n-1} + ... + a_n d^n = 0$$

this implies

$$(dx)^n + a_1 d_1 (dx)^{n-1} + ... + a_n d^n = 0$$

where $a_1 d_1, .., a_n d^n \in A$. So $d.x$ is integral over A

□

Definition 5.0.7. *An integral domain A is said be integrally closed or normal if it is equal to its integral closure in its field of fraction F it mean if $x \in F$, x is integral over A implies $x \in A$*

Proposition 5.0.8. *Every unique factorization domain is integrally closed.*

Proof. Let A be UFD . An element of the field of fractions of A not in A can be written a/b with $a, b \in A$ and b divisible by some prime element p not dividing A , then

$$(a/b)^n + a_1 (a/b)^{n-1} + ... + a_m = 0$$

where $a_i \in A$

$$\Rightarrow a_1 b a^{n-1} + ... + a_m b^n = -a^n$$

then p divides every term in LHS and hence a^n but p does not divide a so we got a contradiction □

Proposition 5.0.9. *Let $A \subseteq B$ be rings, and let $A^{'}$ be the integral closure of A in B. For every multiplicative subset S of A, $S^{-1}A^{'}$ is the integral closure of $S^{-1}A$ in $S^{-1}B$*

Proof. Let $b/s \in S^{-1}A'$ with $b \in A'$ and $s \in S$, then
$b^n + a_1b^{n-1} + ... + a_n = 0$ then , b/s is integral over $S^{-1}A$ this implies that $S^{-1}A'$ is contained in closure of $S^{-1}A$
Conversely let $b/s, b \in B, s \in S$ be integral over $S^{-1}A$ then
$(b/s)^n + a_1/s_1(b/s)^{n-1} + ... + a_n/s_n = 0$. Now multiplying $s^n s_1^n .....s_n^n$ and observe that $s_1s_2....s_nb \in A'$ and therefore
$b/s = (s_1s_2....s_nb)/(s_1s_2....s_ns) \in S^{-1}A'$ □

Corollary 5.0.10. *$A \subseteq B$ be rings and S a multiplicative subset of A. If A is integrally closed in B, then $S^{-1}A$ is integrally closed in $S^{-1}B$.*

Proof. A is integrally closed in B implies $A' = A$ then by proposition $S^{-1}A' = S^{-1}A$ □

5.1 Prime ideal in an integral extention

Proposition 5.1.1. *Let B be an integral domain and the extension $A \subseteq B$ is integral . Then*

(1) *If I is non zero ideal of B then $A \cap I \neq \phi$*

(2) *An element $a \in A$ is a unit of $A \Leftrightarrow$ it is a unit in B*

(3) *A is field $\Leftrightarrow$ B is field*

Proof. (1) Let $0 \neq b \in I$ and let $b^n + a_1b^{n-1} + ... + a_n = 0$ be an itegral equation of b over A .Then choose n to be the least such that $a_n \neq 0$ and we see $a_n \in I$ this implies $a_n \in A \cap I$ since $a_n \in A$

(2) Suppose a is a unit in B . Let $b = a^{-1} \in B$ and
$b^n + a_1b^{n-1} + ... + a_n = 0$ where $a_i \in A$, $(a^{-1})^n + a_1(a^{-1})^{n-1} + ... + a_n = 0$. Now multiplying this equation by a^{n-1} and see $a^{-1} \in A$ and this implies a is a init of A
Converse is trivally hold
(3) If B is a field then from (2) A is field
Conversely , suppose A is a field . Let be a non zero element of B and let $b^n + a_1b^{n-1} + ... + a_n = 0$ be integral equation of b where $a_i \in A$. Now

assume $a_n \neq 0$ and we have
$b(b^{n-1} + a_1 b^{n-2} + ... + a_{n-1}) = -1a_n$
$a_n^{-1} b(b^{n-1} + a_1 b^{n-2} + ... + a_{n-1}) = -1$ since A is field and b is unit so B is field □

Proposition 5.1.2. *Let $A \subseteq B$ be an integral extention and let P, Q be prime ideals of B then*
(1) *P is maximal ideal of $B \Leftrightarrow A \cap P$ is maximal ideal of A*
(2) *If $P \subseteq Q$ and $A \cap P = A \cap Q$ then $P = Q$*

Proof. Put $p = A \cap P$ and define a map $\phi : A/p \to B/P$ by

$$\phi(a + p) = a + P$$

then ϕ is well define and one-one

$$\begin{aligned} Ker\phi &= \{a + p : a + P = P\} \\ &= \{a + p : a \in P\} \\ &= \{a + A \cap P : a \in P\} \\ &= A \cap P = p \end{aligned}$$

And B/P is integral over A/p
Now P is maximal $\Leftrightarrow B/P$ is field $\Leftrightarrow A/p$ is field
$\Leftrightarrow p$ is maximal ideal of A
(2) Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A_p = S^{-1}A & \xrightarrow{h} & S^{-1}B \end{array}$$

where $S = A \setminus p$ and $S \cap P = \phi$. Suppose $p = A \cap P = A \cap Q$ then $S^{-1}p = S^{-1}A \cap S^{-1}P = S^{-1}A \cap S^{-1}Q$ and A_p is local ring so $S^{-1}p = pA_p$ is unique maximal ideal of $S^{-1}A$ and since $S^{-1}B$ is integral over $S^{-1}A$ by first part $S^{-1}P$ is maximal ideal of $S^{-1}B$ and $S^{-1}P \subseteq S^{-1}Q \Rightarrow S^{-1}P = S^{-1}Q$ Now take an element in Q and not hard to see this element belong to P and thus $P = Q$

□

Theorem 5.1.3. *Let $A \subseteq B$ be rings and B is integral over A and if $p \in spec(A)$ then $\exists q \in spec(B)$ such that $q \cap A = p$*

Proof. Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\beta"'] & B \arrow[d, "\alpha"] \\ A_p = S^{-1}A \arrow[r, "f_p"] & S^{-1}B \end{tikzcd}$$

Here A_p is local ring . Let M be a maximal ideal in $S^{-1}B$, then $M \cap A_p$ is maximal ideal in A_p and A_p has unique maximal ideal so $M \cap A_p = pA_p$ Now define $\alpha^{-1}(M) = q$ and

$$f_p^{-1}(M) = \{a/s \in A_p : a/s \in M\}$$
$$= M \cap A_p = pA_p$$

Now calculate $\beta^{-1}(pA_p) = \{x \in A : \beta(x) \in pA_p\}$

$$= \{x \in A : x/1 \in pA_p\}$$
$$= \{x \in A : x \in p\}$$
$$= A \cap p = p$$

And

$$f^{-1}(q) = \{x \in A : f(x) \in q\}$$
$$= \{x \in A : x \in q\}$$
$$= A \cap q$$

Now by the commutative diagram we have

$$f^{-1}(\alpha^{-1}(M)) = \beta^{-1}(f_p^{-1}(M))$$
$$\Rightarrow f^{-1}(q) = \beta^{-1}(pA_p)$$
$$\Rightarrow A \cap q = p$$

and we are done since q is prime and satisfied the required condition

$\square$

Remark 5.1.4. *Thus result is true for integral extention but neet not be true for general rings*

Example 5.1.5. *Let* $f : \mathbb{Z} \to \mathbb{Q}$ *defined by* $f(x) = x$ *and let* $I = 2\mathbb{Z}$ *then* $I^e = \mathbb{Q}$ *and* $(I^e)^c = \mathbb{Z} \neq I$

5.2 Going Up Going Down Thereom

Theorem 5.2.1. *Going up theorem*
Let $A \subseteq B$ *be an integral extension . Let* $p_1 \subseteq p_2 \subseteq ... \subseteq p_n$ *be a chain of prime ideals of* A *and* $q_1 \subseteq q_2 \subseteq ... \subseteq q_m$ *be chain of prime ideals in* B $m < n$ *such that* $q_i \cap A = p_i$ *then there exists* $q_{m+1}, ..., q_n \in spec(B)$ *such that* $q_i \cap A = p_i$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \psi\downarrow & & \downarrow\phi \\ A_p = S^{-1}A & \xrightarrow{g} & S^{-1}B \end{array}$$

Let $n = 2, m = 1, q_1 \in spec(B)$ such that $q_1 \cap A = p_1$ already we know $A/p_1 \subseteq B/q_1$ is an integral extension and $p_2/p_1 \in spec(A/p_1)$ therefore $\exists$ a prime ideal $q_2/q_1 \in spec(B/q_1)$ such that $g^{-1}(q_2/q_1) = p_2/p_1$, by previous theorem

And by commutativity of diagram we have

$$f^{-1}(\phi^{-1}(q_2/q_1)) = \psi^{-1}(g^{-1}(q_2/q_1))$$
$$\Rightarrow f^{-1}(q_2) = \psi^{-1}(p_2/p_1)$$
$$\Rightarrow q_2 \cap A = p_2$$

Here p_2/p_1 is prime ideal one can check by taking element or one famous charactrisation for checking prime ideal .And now inductively we are done

□

Lemma 5.2.2. *Let* M *be a finitely generated* A *module and* I *be an ideal of* A *and* $\phi : M \to M$ *be an* A *module homomorphism such that* $\phi(M) \subseteq IM$ *then* $\exists a_1, ...a_n \in I$ *such that* $\phi^n + a_1\phi^{n-1} + ... + a_{n-1}\phi + a_n = 0$

Proof. Let $\{x_1, ..., x_n\}$ be a generating set for M.
Let $\phi(x_i) = \sum_{j=1}^{n} a_{ij}x_j$ where $a_{ij} \in I$. Now we write this another form

$$\sum_{j=1}^{n}(\phi\delta_{ij} - a_{ij})x_j = 0$$

where δ_{ij} is kronecker delta. Now consider $\phi\delta_{ij} - a_{ij} \in A^{'}[\phi]$ where $A^{'}[\phi]$ is the subring of $End_A(M)$ containing $A^{'} = \{image\ of\ A\ in\ End_A(M)\}$ and ϕ where

$$A^{'}[\phi] = \{\sum_{i=0}^{n} a_i\phi^i : n \in \mathbb{N}, a_i \in A\}$$

where ,$(a_i : M \to M, a_i(x) = a_ix)$ and note that $A^{'}[\phi]$ is a commutative subring of $End_A(M)$. Consider the matrix $B = (\phi\delta_{ij} - a_{ij}) \in M_n(A^{'}[\phi])$ Let b_{ik} denote the cofactor of B. Now

$$\sum_{j=1}^{n}(\phi\delta_{ij} - a_{ij})x_j = 0$$

Take cofactor , $\sum_i b_{ij}(\sum_{j=1}^{n}(\phi\delta_{ij} - a_{ij}))(x_j) = 0$

$$\Rightarrow det(B)(x_j) = 0, \forall j$$

$\Rightarrow Det(B)$ is zero map as an element of $A^{'}[\phi]$
$\Rightarrow det(B) = \phi^n + a_1\phi^{n-1} + ... + a_{n-1}\phi + a_n = 0$, where $a_i \in I$

□

Proposition 5.2.3. *Let $A \subseteq B$ be rings and I be an ideal of A and C be the ingral closure be the integral closure of A in B . Then the set of all elements in B which are integral over I is the radical of $IC = I^e$*

Proof. Let $x \in C$ be integral over I then we have $x^n + a_1x^{n-1} + ... + a_n = 0$ where $a_i \in I$
$x^n \in I^e = IC$ so $x \in rad(I^e)$
Conversely , let $x \in rad(I^e)$ implies that $x^n \in I^e$ for some $n \in \mathbb{N}$
So $x^n = \sum_{i=1}^{m} b_ix_i$,where $b_i \in C, x_i \in I$

Consider the ring $M = A[b_1, ..., b_n]$, and this is finitely generated A module and $x^nM \subseteq IM$ and consider the map $\phi_{x^n}; M \to M$ by,

$\phi_{x^n}(m) = x^n m$ and $\phi_{x^n}(M) \subseteq IM$, therefore by lemma $\exists a_1, ..., a_r \in I$ such that $(\phi_{x^n})^r + a_1(\phi_{x^n})^{r-1} + ... + a_r = 0$

$$x^{nr} + a_1 x^{n(r-1)} + ... + a_r = 0$$

and this implies x is integral over I

□

Proposition 5.2.4. *Let $A \subseteq B$ be integral domain and A is integrally closed. Let $b \in B$ be integral over an ideal $I \subseteq A$. Then b is algebraic over field of fraction of A say K and its minimal poynomial has coefficients in $rad(I)$ except foe leading coefficient* 1

Proof. Clearly b is algebraic over K. Let $f(X)$ be the minimal polynomial of b over K. Let $x_1, ...x_n$ be roots of $f(X)$ in some field F containing K
Then $f(X) = \prod_{i=1}^{n}(X - x_i)$, moreover x_i are all integral over I implies all polynomial in $x_1, ..x_n$ are integral over I it means the coefficients $a_i^{'}s$ are all integral over I, therefore they are all in K and integral over A,
Hence $a_i \in A$ implies $a_i \in rad(I)$ □

Theorem 5.2.5. ***Going Down*** *Let A be an integrally closed domain and $A \subseteq B$ be an integral extension . Let $p_1 \subseteq p_2$ be two prime ideals of A and q_2 be prime ideal of B such that $q_2 \cap A = p_2$ then there exist a prime ideal of q_1 contained in q_2 such that $q_1 \cap A = p_1$*

Proof. We need to show that $p_1 B_{q_2} \cap A = p_1$. Let $x/s \in p_1 B_{q_2}$
Then $x \in p_1 B$ so $x \sum_{i=1}^{n} b_i x_i$ for some $b_i \in B, x_i \in p_1$
Let $A^{'} = A[b_1, ..., b_n]$. Consider the multiplication map $\phi_x : A^{'} \to A^{'}$ sending $(a \mapsto ax)$ where $a \in A^{'}$ and $\phi_x(A^{'}) = xA^{'} \subseteq p_1 A^{'}$, therefore ny lemma $\exists a_1, ..., a_n \in p_1$ such that $x^n + a_1 x^{n-1} + ... + a^n = 0$ and this implies x is integral over p_1

Now suppose $x/s \in p_1 B_{q_2} \cap A, s \in B \backslash q_2$, and let $x^n + a_1 x^{n-1} + ... + a^n = 0$ be the minimal integral equation of x over A
Let $x/s = y \Rightarrow s = xy^{-1} \in frac(A) = K$. Also $s \in B \Rightarrow s$ is integral over A and now multiplying above equation y^{-n} that gives the equation $s^n + (a_1/y)s^{n-1} + ... + a_n/y^n = 0$, and since above equatin is minimal then this also minimal equation for s
Now as $x \in B$ is integral over p_1 then we have $x^n + a_1 x^{n-1} + ... + a^n = 0$ where $a_i \in rad(p_1) = p_1$ since p_1 is prime ideal
Now let $a_i/y^i = u_i$ then $y^i u_i = a_i \in p_1$ and since $s \in B$ is integral over A

implies that $u_i \in A$ and $y^i u_i \in p_1$
Now if $y \notin p_1 \Rightarrow u_i \in p_1, \forall i$ and the equation in s becomes

$$s^n + u_1 s^{n-1} + ... + u_n = 0$$

So $s^n \in p_1 B \subseteq p_2 B \subseteq q_2$ this implies $s \in q_2$ a contradiction therefore $y \in p_1$ and hence $p_1 B_{q_2} \cap A = p_1$ implies p_1 is contracted ideal

□

5.3 Noether Normalization Thereom

Lemma 5.3.1. *Let $f(x_1, .., x_n) \in K[x_1, .., x_n]$ be a non zero polynomial over an infinte field K . Then there are $\lambda, a_1, ..., a_{n-1} \in K$ such that the polynomial $\lambda f(y_1 + a_1 y_n, ..., y_{n-1} + a_{n-1} y_n, y_n) \in K[y_1, ...y_n]$ is monic in y_n*

Proof. Let f_d be the homogeneous partof f of highest degree where d is the degree of f . Since K is infinite we can always find $a_1, ..., a_{n-1}, 1$ such that $f_d(a_1, ..., a_{n-1}, 1) \neq 0$
Now let $x_i = y_i + a_i y_n, i = 1, 2, .., n-1$ and $y_n = x_n$ and let $\lambda = [f_d(a_1, ..., a_{n-1}, 1)]^{-1}$
Now $f(x_1, .., x_n) = f_d(x_1, .., x_n) + ... + f_0(x_1, .., x_n)$ and look at

$$f_d(x_1, .., x_n) = \sum_{k_1 + .. + k_n = d} C_{k_1 \cdots k_n} x_1^{k_1} ... x_n^{k_n}$$

$$f_d(y_1 + a_1 y_n, ..., y_{n-1} + a_{n-1} y_n, y_n) = \sum_{k_1 + .. + k_n = d} C_{k_1 \cdots k_n} (y_1 + a_1 y_n)^{k_1} ... (y_{n-1} + a_{n-1} y_n)^{k_{n-1}} y_n^{k_n}$$

$$= \sum_{k_1 + .. + k_n = d} C_{k_1 \cdots k_n} a_1^{k_1} ... a_{n-1}^{k_{n-1}} 1^{k_n} y_n^d + O(y_n^{d-1})$$

$$= f_d(a_1, ..., a_{n-1}, 1) y_n^d + O(y_n^{d-1})$$

And multiply λ we get what we want

□

Theorem 5.3.2. *Let R be finitely generated algebra over an infinite field K with generators $x_1, ..., x_n \in R$. Then there is an injective K algebra homomorphism $\phi : K[t_1, ..., t_r] \to R$ from a polynomial ring to R , such that R is integral over $K[t_1, ..., t_r]$*

Proof. Since R is finitely generated implies $R = K[x_1, ..., x_n]$. WE shall prove this result by induction on n
If $n = 1$ then $R = K[x_1]$ and let $x_1 = t_1$ then $K[t_1] = R$ and every ring is integral over itself so we are done. Assume $n > 1$, if the generators $x_1, ..., x_n$ are algebraically independent , we choose $t_i = x_i$ and $r = n$ and we are done

Suppose there an algebraic dependence between the generators it means a non zero polynomial f over K such that $f(x_1, .., x_n) = 0$. Let f_d be the homogeneous part of the highest degree of f . Then by previous lemma we can find $a_1, ..., a_{n-1}$ such that
$\lambda, a_1, ..., a_{n-1} \in K$ such that the polynomial $\lambda f(y_1+a_1y_n, ..., y_{n-1}+a_{n-1}y_n, y_n) \in K[y_1, ...y_n]$ is monic in y_n . The new coordinates are given by $y_i = x_i - a_ix_n, y_n = x_n$
$\lambda\lambda f(y_1 + a_1y_n, ..., y_{n-1} + a_{n-1}y_n, y_n) = \lambda f(x_1, ..., x_n) = 0$

$$\Rightarrow y_n^d + O(y_n^{d-1}) = 0$$

This implies y_n is integral over $K[y_1, ...y_{n-1}]$, and $K[y_1, ...y_n] = K[x_1, ...x_n]$ by using the relation $x_i = y_i+a_iy_n$. Therefore by induction hypothesis there is an injective K algebra homomorphism $\phi : K[t_1, ..., t_r] \to K[y_1, .., y_{n-1}]$ such that $K[y_1, .., y_{n-1}]$ is integral over $K[t_1, ..., t_r]$. But y_n is integral over $K[y_1, .., y_{n-1}]$
Now $K[t_1, ..., t_r] \subseteq K[y_1, .., y_{n-1}] \subseteq K[y_1, .., y_n]$, and by tower law of integrality $K[y_1, .., y_{n-1}]$ is integral over $K[t_1, ..., t_r]$ and thus $K[x_1, ...x_n]$ is integral over $K[t_1, ..., t_r]$ $\square$

Chapter 6

Tensor Products

6.1 Axiomatic definition of tensor products

In linear algebra we have many types of products. For example,
(1)The scalar product: $V \times \mathbb{F} \to V$
(2) The dot product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
(3) The cross product $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$
(4) The matrix product $M_{m\times k} \times M_{k\times n} \to M_{m\times n}$

Note 6.1.1. *Note that the three vector spaces involved aren't necessarily the same. What these examples have in common is that in each case, the product is a bilinear map. The tensor product is just another example of a product like this. If V_1 and V_2 are any two vector spaces over a field $\mathbb{F}$, the tensor product is a bilinear map:$V_1 \times V_2 \to V_1 \otimes V_2$ where $V_1 \otimes V_2$ is a vector space over $\mathbb{F}$.*
The tricky part is that in order to define this map,we first need to construct this vector space $V_1 \otimes V_2$ We give two definitions. The first is an axiomatic definition, in which we specify the properties that $V_1 \otimes V_2$ and the bilinear map must have. In some sense, this is all we need to work with tensor products in a practical way. Later we shall show that such a space actually exists, by constructing it.

Definition 6.1.2. *Let V_1, V_2 be vector spaces over a field $\mathbb{F}$. A pair (Y, μ) , where Y is a vector space over $\mathbb{F}$ and $\mu : V_2 \times V_2 \to Y$ is a bilinear map, is called the tensor product of V_1 and V_2 if the following condition holds $(*)$*

whenever β_1 *is a basis for* V_1 *and* β_2 *is basis for* V_2*, then* $\mu(\beta_1 \times \beta_2) = \{\mu(x_1, x_2) : x_1 \in \beta_1, x_2 \in \beta_2\}$ *is a basis for* Y

Notation

We write $V_1 \otimes V_2$ for the vector space Y , and $x_1 \otimes x_2$ for $\mu(x_1, x_2)$. The condition $(*)$ does not actually need to be checked for every possible pair of bases β_1, β_2 it is enough to check it for any single pair of basis

Working with tensor products

Let V and W be two vector space over $\mathbb{F}$. There are two ways to work with the tensor product. One way is to think of the space $V \otimes W$ abstractly, and to use the axioms to manipulate the objects. In this context, the elements of $V \otimes W$ just look like expressions of the form $\sum_i a_i(v_i \otimes w_i)$ where $a_i \in \mathbb{F}, v_i \in V, w_i \in W$

The other way is to actually identify the space $V_1 \otimes V_2$ and the map $V_1 \times V2 \to V_1 \otimes V_2$ with some familiar object. There are many examples in which it is possible to make such an identification naturally. Note, when doing this, it is crucial that we not only specify the vector space we are identifying as $V \otimes V_2$, but also the product (*bilinear map*) that we are using to make the identification

Example 6.1.3. *Let* $V = \mathbb{R}^2_{row}$ *and* $W = \mathbb{R}^2_{col}$ *then* $V \otimes W = M_{2\times 2}(\mathbb{R})$ *. Define a map* $\mu : \mathbb{R}^2_{row} \times \mathbb{R}^2_{col} \to \mathbb{R}^2_{row} \otimes \mathbb{R}^2_{col}$ *by*

$$\mu(v, w) = v \otimes w = w \cdot v$$

Then μ *is bilinear map and* $(*)$ *condition holds clearly. Similary we can do for* $V = \mathbb{R}^n_{row}$ *and* $W = \mathbb{R}^n_{col}$ *then* $V \otimes W = M_{n\times n}(\mathbb{R})$

Example 6.1.4. *Let* $V = \mathbb{F}[X]$ *and* $W = \mathbb{F}[Y]$ *then* $V \otimes W = \mathbb{F}[X, Y]$*.*
Define a map $\mu : \mathbb{F}[X] \times \mathbb{F}[Y] \to \mathbb{F}[X] \otimes \mathbb{F}[Y]$ *by*
$\mu(f(X), f(Y) = f(X) \otimes g(Y) = f(X)g(Y)$ *, then* μ *is bilinear map easy to see and* $(*)$ *condition holds easily. Note that this is NOT a commutative product because in general* $f(X) \otimes g(Y) = f(X)g(Y) \neq g(X)f(Y) = g(X) \otimes f(Y)$

Example 6.1.5. *If* V *is any vector space over* $\mathbb{F}$*, then* $V \otimes \mathbb{F} = V$*. In this case,* $\otimes$ *is just scalar multiplication.Both the condition obiviously hold good*

Example 6.1.6. *Let* $V = \mathbb{Q}^n(\mathbb{Q})$ *and* $W = \mathbb{R}(\mathbb{Q})$ *then* $V \otimes W = \mathbb{Q}^n \otimes \mathbb{R} = \mathbb{R}^n$ *as vector space over* $\mathbb{Q}$*. Then define a map* $\mu : \mathbb{Q}^n \times \mathbb{R} \to \mathbb{Q}^n \otimes \mathbb{R}$ *by*

$$\mu(x, y) = x \otimes y = xy$$

where $x \in \mathbb{Q}^n, y \in \mathbb{R}$ *and* μ *is a bilinear map. Now i shall prove condition* $(*)$*. Let* $\beta = \{e_1, ..., e_n\}$ *be standard basis of* $\mathbb{Q}^n(\mathbb{Q})$ *and* γ *be a basis of* $\mathbb{R}(\mathbb{Q})$*.*

First we show that $\mu(\beta \times \gamma)$ *spans* $\mathbb{R}^n$*. Let* $(a_1, ..., a_n) \in \mathbb{R}^n$ *where* $a_i \in \mathbb{R}$

$$a_i = \sum_j b_{ij} x_j, b_{ij} \in \mathbb{Q}, x_j \in \gamma$$

where j *runs over finite set of* γ
Now $(a_1, ..., a_n) = (\sum_j b_{1j} x_j, \sum_j b_{2j} x_j, ..., \sum_j b_{nj} x_j)$

$$= \sum_j b_{1j} e_1 x_j + ... + \sum_j b_{nj} e_n x_j$$

$$= \sum_{i,j} b_{ij} e_i x_j$$

Next we show $\beta \otimes \gamma$ *is linearly independent.Suppose* $\sum_{i,j} b_{ij} e_i \otimes x_j = (0, .., 0)$

$$(\sum_j b_{1j} x_j, ..., \sum_j b_{nj} x_j) = (0, ..., 0)$$

$\Rightarrow b_{i,j} = 0$ *. Since* x_j *linearly independent*

6.2 Constructive definition of tensor product

To give a construction of the tensor product, we need the notion of a free vector space.

Definition 6.2.1. *Let* A *be a set, and* $\mathbb{F}$ *be field. The free vector space over* $\mathbb{F}$ *generated by* A *is the vector space* $Free(A)$ *consisting of all formal finite linear combinations of elements of* A*. Thus,* A *is always a basis of* $Free(A)$

Note 6.2.2. *When the elements of the set* A *are numbers or vectors, the notation get tricky, because there is a danger of confusing the operations of addition and scalar multiplication and the zero-element in the vector space* $Free(A)$*, and the operations of addition and multiplication and the zero* *element in A and (which are irrelevant in the definition of $Free(A)$). To help keep these straight in situations where there is a danger of confusion, we'll write $\boxplus$, and $\boxdot$ when we mean the operation in $Free(A)$. We shall denote zero vector of $Free(A)$ by $0_{Free(A)}$*

Example 6.2.3. *Let $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{F} = \mathbb{R}$. Then $Free(\mathbb{N})$ is an infinite dimentional vector space whose elements are of the form $(a_0 \boxdot 0) \boxplus (a_1 \boxdot 1).....(a_m \boxdot m)$ for some $m \in \mathbb{N}, a_i \in \mathbb{R}$*
Note that the element 0 here is not the zero vector in $Free(\mathbb{N})$. It's called 0 because it happens to be the zero element in $\mathbb{N}$, but this is completely irrelevant in the construction of the free vector space. If we wanted we could write this a little differently by putting x^i in place of $i \in \mathbb{N}$. In this new notation, the elements $Free(\mathbb{N})$ would look like $a_0x^0 + ... + a_nx^n$
For some some $m \in \mathbb{N}$, in other words elements of the vector space of polynomials in a single variable

Definition 6.2.4. *Let V and W be two vector space over $\mathbb{F}$*
Let $P := Free(V \times W)$, the free vector space over $\mathbb{F}$ generated by the set $V \times W$. Let $R \subseteq P$ be the subspace spanned by all vectors of the form $(u+kv, w+lx) \boxplus (-1 \boxdot (u, w)) \boxplus (-k \boxdot (v, w)) \boxplus (-l \boxdot (u, x)) \boxplus (-kl \boxdot (v, x))$, with $k, l \in \mathbb{F}, u, v \in V, x, w \in W$

Let $\pi : P \to P/R$ be the quotient and let $\mu : V \times W \to P/R$ be the map defined by

$$\mu(v, w) = \pi((v, w))$$

The pair $(P/R, \mu)$ is the tensor product of V and W and we write $V \otimes W$ for P/R and $v \otimes w$ for $\mu(v, w)$

Note 6.2.5. *We need to show that two definitions agree, i.e.. that tensor product as defined in the definition above satisfies the conditions of definition above In particular, we need to show thatμ is bilinear,and that the pair $(P/R, \mu)$ satisfies condition $(*)$*
We can show the bilinearity immediately. Essentially bilinearity is built into the definition.

If P is the space of all linear combinations of symbols (v, w), then R is the space of all those linear combinations that can be simplified to the zero vector using bilinearity. Thus P/R is the set of all expressions, where two expressions are equal iff one can be simplified to the other using bilinearity

Proposition 6.2.6. *The map μ is bilinear $\mu : V \times W \to P/R$ defined by $\mu(v,w) = \pi((v,w))$*

Proof. Aim: $\mu(u+kv, w+lx) = \mu(u,w) + k\mu(v,w) + l\mu(u,x) + kl\mu(v,x)$. We know that $\pi(z) = 0_R, \forall z \in R$ and this implies

$$\pi((u+kv, w+lx) \boxplus (-1 \boxdot (u,w)) \boxplus (-k \boxdot (v,w)) \boxplus (-l \boxdot (u,x)) \boxplus (-kl \boxdot (v,x))) = 0$$

And so $\mu((u+kv, w+lx) - \mu(u,w) - k\mu(v,w) - l\mu(u,x) - kl\mu(v,x) = 0$ $\square$

Now to prove the condition $(*)$ holds we use the following important lemma from the theory of quotient spaces

Lemma 6.2.7. *Suppose V and W are vector spaces over a field $\mathbb{F}$ and $T : V \to W$ is a linear transformation. Let S be a subspace of V . Then there exists a linear transformation $\bar{T} : V/S \to W$ such that $\bar{T}(x+S) = T(x)$ for all $x \in V$ if and only if $T(s) = 0$ for all $s \in S$. Moreover, if $\bar{T}$ exists it is unique*

Proof. Suppose $\bar{T}$ exist then $\bar{T}(x+S) = T(x), \forall X \in V$ then $\forall s \in S$ we have $T(s) = \bar{T}(s+S) = \bar{T}(0) = 0$
Conversely , suppose that $T(s) = 0, \forall s \in S$. Now define a map $\bar{T} : V/S \to W$ such that $\bar{T}(x+S) = T(x)$ for all $x \in V$, then $\bar{T}$ is well define and linear and clearly unique $\square$

6.3 Universal mapping property of tensor product

Theorem 6.3.1. *Let V, W, M be vector spaces over a field $\mathbb{F}$.Let $V \otimes W = P/R$ be the tensor product, as defined in above definition then For any bilinear map $\phi : V \times W \to M$,there is a unique linear transformation $\bar{\phi} : V \otimes W \to M$, such that $\bar{\phi}(v \otimes w) = \phi(v,w)$ for all $v \in V, w \in W$.*

Proof. Since $V \times W$ be a basis for P . We can extend any map $\phi : V \times W \to M$ to a linear map $\psi : P \to M$ defined by $\psi(v,w) = \phi(v,w), \forall v \in V, w \in W$
Claim : ψ is bilinear $\Leftrightarrow$ $\phi(s) = 0, \forall s \in R$. Let $\phi(s) = 0, \forall, s \in R$,then $\psi(s) = \phi(s) = 0, \forall, s \in R$, then write s in the form of spanning vectors of R and using bilinearity of ϕ we see that ψ is bilinear.
Coversely suppose ψ is bilinear , and we have $\psi(z) = \phi(z), \forall z \in R$. Now calculate $\psi(z) = \psi((u+kv, w+lx) \boxplus (-1 \boxdot (u,w)) \boxplus (-k \boxdot (v,w)) \boxplus (-l \boxdot (u,x)) \boxplus (-kl \boxdot (v,x))$ and since ψ is bilinear implies $\psi(z) = 0, \forall z \in R$ and

so $\phi(z) = 0, \forall z \in R$, then by previous lemma there exist unique LINEAR map $\bar{\phi} : V \otimes W \to M$ such that $\bar{\phi}((v \otimes w)) = \phi(v, w)$ and uniqueness is clear □

Theorem 6.3.2. *Condition $(*)$ holds for the tensor product as defined in the above definition*

Proof. Let β be a basis for V and γ be a basis for W then we must show that $\beta \otimes \gamma$ is a basis of $V \otimes W$. First we show that it spans .Let $z \in P/R$ then $z = x + R, x \in P$ where $x = a_1(u_1, x_1) + ... + a_m(u_m, x_m)$ and π is a quotient map such that $\pi(y) = y + R$,and $\mu((v, w)) = \pi((v, w))$, then therefore we have

$$z = a_1\pi(u_1, x_1) + ... + a_m\pi(u_m, x_m)$$

$$= a_1\mu(u_1, x_1) + ... + a_m\mu(u_m, x_m)$$

where $a_i \in \mathbb{F}, u_i \in V, x_i \in W$. But now $u_i = \sum_j b_{ij}v_j, v_j \in \beta$ and $x_i = \sum_k c_{ik}w_k, w_k \in \gamma$ and putting these values in z we are done

Next we show linear independence ,suppse $\sum_{ij} d_{ij}\mu(v_i, w_j) = 0$
where $v_i \in \beta, w_j \in \gamma$ .Let $f_k \in V^*$ be the linear functional defined by $f_k(k) = 1$ and $f_k(v) = 0$ for $v \in \beta \setminus \{v_k\}$
Define a map $F_k : V \times W \to W$ by $F_k(v, w) = f_k(v)w$, then F_k is bilinear map then by universal mapping property there exist a map $\bar{F_k} : V \otimes W \to W$ such that $\bar{F_k}(\mu(u, x)) = f_k(u)x$.
Now apply $\bar{F_k}$ to the equation $\sum_{ij} d_{ij}\mu(v_i, w_j) = 0$, therefore

$$0 = \bar{F_k}(\sum_{ij} d_{ij}\mu(v_i, w_j))$$

$$= \sum_{ij} d_{ij}(\bar{F_k}(\mu(v_i, w_j)))$$

$$= \sum_{ij} d_{ij}f_k(v_i)w_j$$

$$= \sum_{j} d_{kj}w_j$$

and thus $d_{kj} = 0$ since w_j are liniearly independent □

6.4 Tensor product on modules

Introduction Let R be a commutative ring and M and N be R-modules. We always work with rings having a multiplicative identity and modules are assumed to be unital , $1 \cdot m = m, \forall m \in M$

Theorem 6.4.1. *Let M and N be two R- module then tensor product of M and N exists*

Proof. Consider $M \times N$ as a set simply and form a free R- module on this set

$$F_R(M \times N) := \oplus_{(m,n)\in M\times N} R\delta_{(m,n)}$$

The direct sum runs over all pairs of $M \times N$ not just pairs coming from a basis

Let D be the submodule of $F_R(M \times N)$ spanned by all elements

$$\delta(m+m^{'},n) - \delta(m,n) - \delta(m^{'},n)$$
$$\delta(m,n+n^{'}) - \delta(m,n) - \delta(m,n^{'})$$
$$\delta(rm,n) - r\delta(m,n)$$
$$\delta(m,rn) - r\delta(m,n)$$
$$\delta(rm,n) - \delta(m,rn)$$

Now define $M \otimes N := F_R(M \times N)/D$

We write the coset $\delta_{(m,n)} + D$ in $M \otimes N$ as $m \otimes n$ and from the definition of D

$$\delta_{(m+m',n)} \equiv \delta_{(m,n)} + \delta_{(m',n)}, \ modD$$

Which is same as

$$(m+m^{'}) \otimes n = m \otimes n + m^{'} \otimes n$$

and also we have

$$m \otimes (n+n^{'}) = m \otimes n + m \otimes n^{'}$$
$$rm \otimes n = r(m \otimes n) = m \otimes rn$$

Suppose P is an any R- module and $B : M \times N \to P$ be a bilinear map and then extend it linearly $l : F_R(M \times N) \to P$ by $l(\delta_{(m,n)}) = B(m,n)$ so the diagram

$$\begin{tikzcd} & F_R(M \times N) \arrow[dr, "l"] & \\ M \times N \arrow[ur, "f"] \arrow[rr, "B"] & & P \end{tikzcd}$$

Where $f(m,n) = \delta_{(m,n)}$ Now we want to show l makes a sense as a function on $M \otimes N$ which means showing $Kerl$ contains D and using bilinearity B and linearity of l we are done . So l induces a linear map
$L : F_R(M \times N)/D \to P$ such that $L(\delta_{(m,n)} + D) = l(\delta_{(m,n)}) = B(m,n)$, which means the diagram

$$\begin{tikzcd} & F_R(M \times N)/D \arrow[dr, "L"] & \\ M \times N \arrow[ur, "\bar{f}"] \arrow[rr, "B"] & & P \end{tikzcd}$$

commutes it means $L \circ \bar{f} = B$. Since $F_R(M \times N)/D = M \otimes N$ and $\delta_{(m,n)} + D = m \otimes n$, the above diagram become

$$\begin{tikzcd} & M \otimes N \arrow[dr, "L"] & \\ M \times N \arrow[ur, "\otimes"] \arrow[rr, "B"] & & P \end{tikzcd}$$

and $L(m \otimes n) = B(m,n)$
And this shows every bilinear B out of $M \times N$ comes from a linear map L out of $M \otimes N$ such that $L(m \otimes n) = B(m,n), \forall m \in M, n \in N$

□

6.5 Properties of Tensor products

Example 6.5.1. *If A is a finite abelian group,then $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$. Since every elementary tensor is 0 as*
Let $a \in A$ such that $na = 0, n \in \mathbb{Z}^+$ and $r \otimes a = n(r/n) \otimes a$

$$= (r/n) \otimes (na)$$

$$= (r/n) \otimes 0 = 0$$

NOTE to show that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$ we dont need A to be finite but rather than each element of A has finite order and thus $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$

Example 6.5.2. *Let $(m, n) = 1$ then $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} = 0$*

Theorem 6.5.3. *Let $a, b \in \mathbb{Z}^+$ with $d = gcd(a, b)$ then $\mathbb{Z}/a\mathbb{Z} \otimes \mathbb{Z}/b\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ as abelian group*

Proof. Since 1 spans $\mathbb{Z}/a\mathbb{Z}$ and $\mathbb{Z}/b\mathbb{Z}$ then $1 \otimes 1$ spans $\mathbb{Z}/a\mathbb{Z} \otimes \mathbb{Z}/b\mathbb{Z}$. Now $a(1 \otimes 1) = 0$ and $b(1 \otimes 1) = 0$, the additive order of $1 \otimes 1$ divides a and b and therefore also d so $|\mathbb{Z}/a\mathbb{Z} \otimes \mathbb{Z}/b\mathbb{Z}| \leq d$, To show $\mathbb{Z}/a\mathbb{Z} \otimes \mathbb{Z}/b\mathbb{Z}$ has size atleast d , we create a $\mathbb{Z}$ bilinear map from $\mathbb{Z}/a\mathbb{Z} \otimes \mathbb{Z}/b\mathbb{Z}$ onto $\mathbb{Z}/d\mathbb{Z}$

Consider a map $\phi : \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ by

$$\phi(x, y) = xy$$

then this is bilinear map and then by using by UMP (*universal mapping property*) there exist unique $\mathbb{Z}$ linear map $f : \mathbb{Z}/a\mathbb{Z} \otimes \mathbb{Z}/b\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ such that $f(x \otimes y) = xy$ in perticular $f(x \otimes 1) = x$, so f is onto map then we are done

$\square$

Theorem 6.5.4. *For an ideal I in R and M is an R module then there is unique R- module isomorphism $(R/I) \otimes M \cong M/IM$. In perticular , taking $I = 0$ then $R \otimes M \cong M$*

Proof. We shall start with a bilinear map $\phi : (R/I) \times M \to M/IM$ by

$$\phi(\bar{r}, m) = \overline{rm}$$

Then ϕ is well define clearly , then by universal mapping property we get a linear map $f : (R/I) \otimes M \to M/IM$ such that the diagram commutes

$$\begin{tikzcd}
& (R/I) \otimes M \arrow[dr, "f"] & \\
(R/I) \times M \arrow[ur, "\mu"] \arrow[rr, "\phi"] & & M/IM
\end{tikzcd}$$

it means $f \circ \mu = \phi$ or $f(\bar{r} \otimes m) = \overline{rm}$
To create an inverse map start with a function $\psi : M \to (R/I) \otimes M$ given by

$$\psi(m) = \bar{1} \otimes m$$

Then ψ is linear in m and observe $\psi(im) = \bar{1} \otimes im = 0$ it means kills IM therefore there exist a linear map $g : M/IM \to (R/IM) \otimes M$ given by $g(\bar{m}) = \bar{1} \otimes m$
To check $f(g(\bar{m})) = \bar{m}$ and $g(f(t)) = t, for\ all\ \bar{m} \in M/IM, t \in (R/I) \otimes M$ first one clear $f(g(\bar{m})) = f(\bar{1} \otimes m) = \bar{m}$ To show $g(f(t)) = t$ we shall show all tensor in $R/I \otimes M$ are elementary tensor .
An elementary tensor look like $\bar{r} \otimes m = \bar{1} \otimes rm$, and the sum of tensors $\bar{1} \otimes m_i$ is $\bar{1} \otimes \sum_i^n m_i$, thus all tensors look like $\bar{1} \otimes m$ so we have $g(f(\bar{1} \otimes m)) = g(\bar{m}) = \bar{1} \otimes m$

□

Theorem 6.5.5. *For ideals I and J in R , there is a unique R module isomorphism*

$$R/I \otimes R/J \cong R/(I + J)$$

Proof. We shall start with a bilinear map $\phi : R/I \times R/J \to R/(I + J)$ by

$$\phi(\bar{x}, \bar{y}) = \overline{xy}$$

Then ϕ is well define clearly , then by universal mapping property we get a linear map $f : R/I \otimes R/J \to R/(I + J)$ such that the diagram commutes

$$\begin{tikzcd} & R/I \otimes R/J \arrow[dr, "f"] & \\ R/I \times R/J \arrow[ur, "\mu"] \arrow[rr, "\phi"] & & R/(I+J) \end{tikzcd}$$

it means $f(\bar{x} \otimes \bar{y}) = \overline{xy}$ Now our aim is to create inverse map ,let $h : R \to R/I \otimes R/J$ by

$$h(r) = r(\bar{1} \otimes \bar{1})$$

and h h is l well define and linear and when $r \in I$ then $r(\bar{1} \otimes \bar{1}) = 0$
Similarly , when $r \in J$ then $r(\bar{1} \otimes \bar{1}) = 0$
And note that $I + J \subseteq Ker(h)$, then we get a linear map

$$g : R/(I + J) \to r(\bar{1} \otimes \bar{1})$$

Defined by $g(\bar{r}) = r(\bar{1} \otimes \bar{1})$, And now we can check by like in previous theorem arguement that f and g are inverses to each other □

Remark 6.5.6. *When f and g are additive functions you can check $f(g(t)) = t$ for all tensors t by only checking it on elementary tensors, but it would be wrong to think you have proved injectivity of a linear map $f : M \otimes N \to P$ by only looking at elementary tensors. That is, if $f(m \otimes n) = 0 \Rightarrow m \otimes n = 0$, there is no reason to believe $f(t) = 0 \Rightarrow t = 0, \forall t \in M \otimes N$, since injectivity of a linear map is not an additive identity.*

Example 6.5.7. *Let $f : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ be the R - linear map defined by*

$$f(z \otimes w) = zw$$

on elemetary tensor . If $f(z \otimes w) = 0$ then $zw = 0 \Rightarrow z = 0, or, w = 0$
So $z \otimes w = 0$,but the map is not injective because ,$1 \otimes i - i \otimes 1 \mapsto 0$ but $1 \otimes i - i \otimes 1 \neq 0$, since $1 \otimes i$ and $i \otimes 1$ belong to basis of $\mathbb{C} \otimes \mathbb{C}$

Theorem 6.5.8. *Let R be a domain with fraction field K and V be vector space over K then there is an R module isomorphism $K \otimes V \cong V$*

Proof. Define a map $\phi : K \times V \to V$, defined by

$$\phi(r, x) = rx$$

then ϕ is R bilinear map , so by universal mapping property there exist a linear map

$$f : K \otimes V \to V$$

Such that $f(x \otimes v) = xv$, on elementary tensor and that says diagram commute

$$\begin{tikzcd} & K \otimes V \arrow[dr, "f"] & \\ K \times V \arrow[ur, "\mu"] \arrow[rr, "\phi"] & & V \end{tikzcd}$$

And since $f(1 \otimes v) = v$ implies f is onto
To show f is one one , first we show every tensor in $K \otimes V$ is elementary with 1 in first component

For an elementary tensor

$$x \otimes v = a/b \otimes v = 1/b \otimes av = 1/b \otimes (ab/b)v = 1 \otimes xv$$

Notice how we moved $x \in K$ across even though x need not be in R, we used K-scaling in V to create b and $1/b$ on the right side of $\otimes$ and bring b

across from right to left, which cancels $1/b$ on the left side of $\otimes$. This has the efeect of moving $1/b$ from left to right. Thus all elementary tensors in $K \otimes V$ have the form $1 \otimes v$ for some $v \in V$, so by adding, every tensor is $1 \otimes v$ for some v. Now we can show f has trivial kernel if $f(t) = 0$ then, writing $t = 1 \otimes v$, we get $v = 0$, so $t = 1 \otimes 0 = 0$. $\square$

6.6 Questions

Questions
(1) What is $m \otimes n$?

(2) What does it mean to say $m \otimes n = 0$?

(3) What does it mean to say $M \otimes N = 0$?

(4) What does it mean to say $m_1 \otimes n_1 + ... + m_k \otimes n_k = m_1^{'} \otimes n_1^{'} + ... + m_k^{'} \otimes n_k^{'}$?

(5) Where do tensor products arise outside of mathematics?

(6) Is there a way to picture the tensor product ?

Answers
(1)$m \otimes n$ is the image of $(m, n) \in M \times N$ under the canonical bilinear map $\otimes : M \times N \to M \otimes N$ in the definition of tensor product

(2) We have $m \otimes n = 0 \Leftrightarrow$ every bilinear map out of $M \times N$ vanishes at (m, n) , indeed if $m \otimes n = 0$, then for every bilinear map $B : M \times N \to N$ we have commutative diagram

$$\begin{tikzcd} & M \otimes N \arrow[dr, "L"] & \\ M \times N \arrow[ur, "\otimes"] \arrow[rr, "B"] & & P \end{tikzcd}$$

for some linear map L, so $B(m, n) = L(m \otimes n) = L(0) = 0$. Conversely, if every bilinear map out of $M \times N$ sends (m, n) to 0 then the canonical bilinear map $M \otimes N \to M \times N$ which is a particular example, sends (m, n)

to 0. Since this bilinear map actually sends (m, n) to $m \otimes n$, we obtain $m \otimes n = 0$.

(3) The tensor product $M \otimes N$ is 0 if and only if every bilinear map out of $M \times N$ (*to all modules*) is identically 0. First suppose $M \otimes N = 0$. Then all elementary tensors $m \otimes n$ are 0, so $B(m, n) = 0$ for any bilinear map out of $M \times N$ by the answer to the second question.Thus B is identically 0. Next suppose every bilinear map out of $M \times N$ is identically 0. Then the canonical bilinear map $M \times N \to M \otimes N$ which is a particular example, is identically 0. Since this function sends (m, n) to $m \otimes n$ we have $m \otimes n = 0$ for all m and n. Since $M \otimes N$ is additively spanned by all $m \otimes n$, the vanishing of all elementary tensors implies $M \otimes N = 0$.

(4) It is based on above two answers

(5) Tensors are used in physics and engineering (stress, elasticity, electromagnetism, *metrics*, diffusion MRI), where they transform in a multilinear way under a change in coordinates.

(6) There isn't a simple picture of a tensor (even an elementary tensor) analogous to how a vector is an arrow.

Theorem 6.6.1. *Let M and N be R-modules with respective spanning sets $\{x_i\}_{i \in I}$ and $\{y_j\} j \in J$. The tensor product $M \otimes N$ is spanned linearly by the elementary tensors $x_i \otimes x_j$*

Proof. An elementary tensor in $M \otimes N$ has the form $m \otimes n$. Write $m = \sum_i a_i x_i$ and $n = \sum_i b_j y_j$, where the a_i *'s* and b_j *'s* are 0 for all but fnitely many i and j. From the bilinearity of $\otimes$

$$m \otimes n = \sum_i a_i x_i \otimes \sum_i b_j y_j = \sum_{ij} a_i b_j (x_i \otimes y_j)$$

is a linear combination of the tensors $x_i \otimes y_j$.

So every elementary tensor is a linear combination of the particular elementary tensors $x_i \otimes y_j$. Since every tensor is a sum of elementary tensors, the $x_i \otimes y_j$ *'s* span $M \otimes N$ as an R-module. $\square$

6.7 Primary Decompositions

Definition 6.7.1. *An ideal in a ring A is primary if* $Q \neq A$ *and if* $xy \in Q \Rightarrow$ *either* $x \in Q$ *or* $y^n \in Q$ *for some* $n > 0$

Observation: Q is primary iff A/Q is not trivial and every zero-divisor in A/Q is nilpotent

Example 6.7.2. *Every prime ideal is primary, contraction of a primary ideal is primary*

Proposition 6.7.3. *The radical of a primary ideal Q is the smallest prime ideal containing it.*

Proof. Let Q be a primary ideal of A.We know that the radical of Q is the intersection of all the prime ideals containing Q.Now it suffices to show that $r(Q)$ is prime,an this is obivious since Q is primary □

Remark 6.7.4. *Let A be a UFD and let* $x \in A$ *be prime. Then all powers of* xA *are primary.*

We give an example to show that primary ideals need not be powers of prime ideals.

Example 6.7.5. *Let* $A = \mathbb{F}[X, Y], Q = (X, Y^2)$*,define a map*

$$\phi : A \to \mathbb{F}[Y]/(Y^2)$$

by

$$\phi(p(X, Y)) = p(0, Y) + (Y^2)$$

Then ϕ *is an onto ring homomorphism and* $Ker\phi = Q = (X, Y^2)$*,then FTH we have* $A/Q \cong \mathbb{F}[Y]/(Y^2)$*, then by remark* (Y^2) *is primary ideal of* $\mathbb{F}[Y]$ *then this shows that Q is primary and further* $r(Q) = (X, Y)$
Also we have $r(Q)^2 \subsetneq Q \subsetneq r(Q)$*, thus Q is not a power of its radical. Now our next claim is Q is not a power of prime ideal ,first suppose* $Q = P^n$ *for somr prime ideal P and also note that* $r(Q) = P$ *and* $P^2 \subsetneq P^n \subsetneq P$ *which is impossible ,thus Q is not a power of prime ideal*

We now give an example to show that powers of prime ideals need not be primary.

Example 6.7.6. *Let $A = \mathbb{F}[X, Y, Z]$, where $\mathbb{F}$ is a field, and put $I = (XY - Z^2)A, B = A/I, P = (X + I, Z + I)$.*
Claim:P is prime ideal of B but P^2 is not primary ideal .Idea is B/P is integral domain implies P is prime. Now we shall show that P^2 is not primary
Observe that $(x + I)(y + I) = xy + I = xy - (xy - z^2) + I = z^2 + I = (z + I)^2 \in P^2$. Also $P^2 = (x^2 + I, xz + I, z^2 + I)$.

If P^2 is primary then $x + I \in P^2$ or $y^k + I = (y + I)^k \in P^2$ for some k so that x or $y^k \in (x^2, xz, z^2, xy - z^2)$ which is impossible, by inspecting monomials in $\alpha x^2 + \beta xz + \gamma z^2 + \delta(xy - z^2)$ for $\alpha, \beta, \gamma, \delta \in A$.

Proposition 6.7.7. *If $Q \lhd A$ and $r(Q)$ is maximal, then Q is primary. In particular, all powers of a maximal ideal M are $M-$primary.*

Proof. We have an epimorphism $\phi : A/Q \to A/r(Q)$ and A/M is field.
Claim: Every zero divisors of A/Q is nilpotent.Let if possible $x = a + Q \in A/Q$ is a zero divisor but not nilpotent.Then $x \mapsto \bar{x} \neq 0 \in A/M$,which is not a zero divisor implies x is not a zero divisor contradiction so x is nilpotent so Q is primary .If M is any maximal ideal of A then $r(M^n) = M$ implies M^n is primary. □

Definition 6.7.8. *Let $Q \lhd A$ and $x \in A$ then $(Q : x) = \{y \in A : xy \in Q\}$.*

Lemma 6.7.9. *Let P be prime, Q be P-primary and $x \in A$. Then*

1. *$x \in Q \Rightarrow (Q : x) = A$*
2. *$x \notin Q \Rightarrow (Q : x)$ is P-primary*
3. *$x \notin P \Rightarrow (Q : x) = Q$.*

Proof. (1) and (3) are easy.We shall porve (2).We have $Q \subseteq (Q : x)$.And observe $(Q : x) \subseteq P$ and conclude $r(Q : x) = P$.Now suppose $yz \in (Q : x)$ with $y \notin P$ then $xyz \in Q \Rightarrow y(xz) \in Q \Rightarrow xz \in Q \Rightarrow z \in (Q : x)$.So $(Q : x)$ is primary. □

Lemma 6.7.10. *Let P be a prime ideal and $Q_1, ..., Q_n$ be P-primary ideals. Then $\bigcap_{i=1}^n Q_i$ is also P-primary.*

Proof. By induction we can see easily. □

Definition 6.7.11. *A primary decomposition of $I \lhd A$ is an expression as a finite intersection of primary ideals: $I = \bigcap_{i=1}^n Q_i \quad (*)$*

Primary decomposition above may not exist always

Definition 6.7.12. *A decomposition* $(*)$ *is minimal if*

1. $r(Q_1), ..., r(Q_n)$ *are distinct*
2. $Q_i \not\supseteq \bigcap_{i \neq j} Q_j, \forall, i = 1,,,n$

Theorem 6.7.13. ***First Uniqueness Theorem*** *Let I be a decomposable ideal and let* $(*)$ *be a minimal primary decomposition. Put* $P_i = r(Q_i), \forall, i = 1,..,n$*,then*
$\{P_1, ..., P_n\} = \{Prime\ ideals\ P : P = r(I : x)\ for\ some\ x\}$.

We say that the prime ideals $P_1, ..., P_n$ belong to I or are associated to I . In particular, I is primary iff I has exactly one associated prime ideal. The minimal elements of $P_1, ..., P_n$ with respect to $\subseteq$ are called minimal or isolated prime ideals belonging to I ; the nonminimal ones are called embedded prime ideals

The set $\{P_1, ..., P_n\}$ in the conclusion of the Theorem is independent of the particular minimal decomposition chosen for I

Proof. Consider $(I : x) = (\bigcap_{i=1}^n Q_i : x) = \bigcap_{i=1}^n (Q_i : x)$

$$\Rightarrow r(I : x) = \bigcap_{i=1}^{n} r(Q_i : x)$$

But $r(Q_i : x) = A,\ if\ x \in Q_i, and\ P_i\ if\ x \notin Q_i$ by lemma.So $r(I : x) = \bigcap_{i=1}^n P_i, when\ x \notin Q_i$
If $r(I : x)$ is prime say P then $P = \bigcap_{i=1}^n P_i$ when $x \notin Q_i$,then $P = P_i$ for some i and this implies $r(I : x) = P_i$.On the other hand ,$\forall i$ choose $x_i \in Q_j, \forall j \neq i$,and so $x_i \in \bigcap_{j \neq i} Q_j$,therefore $r(I : x_i) = P_i$ □

Note 6.7.14. *Primary components need not be unique.*

Example 6.7.15. *Let* $A = \mathbb{F}[X, Y], I = (X^2, XY)$ *, then we observe*

$$I = (X) \cap (X, Y)^2$$

and

$$I = (X) \cap (X^2, Y)$$

Lemma 6.7.16. *Let S be a multiplicatively closed subset of A , P a prime ideal and Q a P-primary ideal. Then*

1. $S \cap P \neq \phi \Rightarrow S^{-1}Q = S^{-1}A$

2. $S \cap P = \phi \Rightarrow, S^{-1}Q$ *is* $S^{-1}P-$*primary ideal and* $(S^{-1}Q)^c = Q$

Theorem 6.7.17. *Primary ideals of A which avoid S are in a one-one correspondence with primary ideals in $S^{-1}A$ under the map $Q \mapsto S^{-1}Q$*

Proof. Put $P_1 = \{Primary\ ideals\ Q\ of\ A : Q \cap S = \phi\}$ and $P_2 = \{Primary\ ideals\ of\ S^{-1}A\}$.Now we define

$$\phi : P_1 \to P_2, Q \mapsto S^{-1}Q$$

$$\psi : P_2 \to P_1, I \mapsto I^c$$

And easy to see both are inverse of each others. □

Notation Let $J \lhd A$,write $S(J) = J^{ec} = \{a \in A : a/1 \in S^{-1}J\}$

Theorem 6.7.18. *If S is a multiplicatively closed subset of A and $I \lhd A$ has a minimal primary decomposition $I = \bigcap_i Q_i$ and we put $P_i = r(Q_i), \forall i$. We suppose further that the ideals have been arranged so that, for some m where $1 \leq m \leq n$, $S \cap P_i = \phi, \forall i = 1, .., m$,and $S \cap P_j \neq \phi, \forall j = m+1, ..., n$,then we have the following minimal primary decompositions.*

$$S^{-1}I = \bigcap_{i=1}^{m} S^{-1}Q_i$$

and

$$S(I) = \bigcap_{i=1}^{m} Q_i$$

Notation

Consider a decomposable ideal I and put $L = \{prime\ ideals\ belonging\ to\ I\}$. Call a subset N of L isolated if $\forall P \in N, \forall P' \in L, P' \subseteq P \Rightarrow P' \in N$

Theorem 6.7.19. *Let I be a decomposable ideal and $P_1, .., P_n$ be the prime ideal associated to I.Suppose $m \leq n$ and $N = \{P_1, ...P_m\}$ is isolated,then for any two minimal primary decompositions $I = \bigcap_{i=1}^{n} Q_i = \bigcap_{i=1}^{n} Q_i'$ where $r(Q_i) = r(Q_i'), \forall i$ then we have $\bigcap_{i=1}^{m} Q_i = \bigcap_{i=1}^{m} Q_i'$.*

Definition 6.7.20. *An ideal I is irreducible if $I = J_1 \cap J_2$, then $I = J_1$ or $I = J_2$*

Lemma 6.7.21. *In a Noetherian ring A ,every ideal is a finite intersection of irreducible ideals.*

Proof. Let S be the set of ideals which are not finite intersections of irreducible ideals. If $S = \phi$,then we are done ; If $S \neq \phi$,then S has a maximal element, I ($since\ R\ is\ Noetherian$). Then I is not irreducible, therefore $I = J_1 \cap J_2$ with $I \subsetneq J_1, J_2$.So $J_1, J_2 \notin S$, hence they are finite intersection of irreducible ideals. Since the intersection of two finite intersection of irreducible ideals, I is the intersection of irreducible ideals, i.e., $I \notin S$. This is a contradiction. Hence $S = \phi$. □

Lemma 6.7.22. *In a Noetherian ring R, all irreducible ideals are primary.*

Proof. Let I be an irreducible ideal. Let $x, y \in R$,with $xy \in I$.Define $I_n = (I : y^n)$ for $m = 1, 2, 3, ..$,then $I \subseteq I_1 \subseteq I_2 \subseteq ...$ and sice R is Noetherian $I_n = I_{n+1}$ for some n.
Claim: $I = (I + (x)) \cap (I + (y^n))$.Let $z \in (I + (x)) \cap (I + (y^n))$,and observe that $yz \in I$ and $z \in I$.So $I = (I + (x)) \cap (I + (y^n))$ and since I is irreducible then we are done. □

Theorem 6.7.23. *In a Noetherian ring R, every ideal I has a primary decomposition.*

Proof. This follows directly from the previous two lemma. □

6.8 Discrete Valuation rings

Definition 6.8.1. *Suppose* $\mathbb{F}$ *is a field .A discrete valuation on* $\mathbb{F}$ *is a function* $v : \mathbb{F}^* \to \mathbb{Z}$ *such that*

1. *v is onto*
2. $v(ab) = v(a) + v(b)$
3. $v(a + b) \geq min(v(a), v(b))$ *if* $a + b \neq 0$

Proposition 6.8.2. *The set* $R = \{0\} \cup \{r \in F : v(r) \geq 0\}$, *is a ring ,which we call the valuation ring of v.*

Proof. Observe that $v(1) = 0, \Rightarrow 1 \in R$ also $ab \in R$. □

Example 6.8.3. *The field* $\mathbb{C}((t)) = \{\sum_{n=N}^{\infty} a_n t^n : N \in \mathbb{Z}, a_n \in \mathbb{C}\}$*,of Laurent series without an essential singularity at* $t = 0$
Define $v : \mathbb{C}((t)) \to \mathbb{Z}$ *by*

$$v(f(t)) = N$$

Where $f(t) = \sum_{n=N}^{\infty} a_n t^n$ *and we can write* $f(t) = a_N t^N g(t)$ *with* $a_N \neq 0$ *,*$g(t) \in A[[t]]$

Definition 6.8.4. *An integral domain A is called a valuation ring if for every element* $a \in (FracA)^*$*,we have* $a \in a$ *or* $a^{-1} \in A$

Lemma 6.8.5. *For any discrete valuation v on a field* $\mathbb{F}$ *with valuation ring A ,we have* $A^* = v^{-1}(0)$.

Proof. We have $v(x^{-1}) = v(x), \forall x \in \mathbb{F}^*$ and this implies either $x \in A$ or $x^{-1} \in A$.Now $x \in A$ is invertible in A implies $v(x) = 0$.Conversely if $v(x) = 0$ then x is invertible in A. □

Lemma 6.8.6. *A valuation ring a with* $Frac(A) = K$ *is a discrete valuation ring iff the quotient group* $K^*/A^* \cong \mathbb{Z}$

Proof. Let A be a valuation ring and $K^*/A^* \cong \mathbb{Z}$ and $(A - \{0\})/A^* \subseteq K^*/A^*$ is a submonoid □

Lemma 6.8.7. *Every valuation ring is normal and local.*

Proof. Let A be the valuation ring and $K = Frac(A)$.Let $f(X) = X^{d+1} + \sum_{i=0}^{d} a_i X^i$ be monic in $A[X]$.Let $b \in K$ st $f(b) = 0$.If $b \in A$ then we are done .If $b^{-1} \in A$, then we have $f(b) = 0$ and thus $b^{d+1} = -\sum a_i b^i$.So $b = -\sum a_i b^i / b^d \in A$
A is local: we shal show that the set $A - A^*$ of non units is an ideal .If $a \in A - A^*$ and $b \in A$ then clearly $ab \in A - A^*$ since otherwise $a^{-1} = b(ab)^{-1} \in A$. Let $a, b \in A - A^*$.Suppose WLOG $a/b \in A$. If $a + b \in A^*$, then $(a/b + 1)(1/a + b) = (a + b/b)1/a + b = 1/b \in A$ contradiction. □

Theorem 6.8.8. *Let A be a subring of a field* $\mathbb{F}$ *then ,T.F.A.E*

1. *A is valuation ring.*
2. *The set of principal ideals of A is totally order by inclusion.*
3. *The set of ideal of A is totally order by inclusion.*
4. *A is local ring and every finitely generated ideal of A is principal.*

6.9 Topologies and completions

Definition 6.9.1. *Let G be an abelian group then G is said to be topological abelian group if both the maps $G \times G \to G$ and $G \to G$ defined by $(x, y) \mapsto x + y$ and $x \mapsto -x$ respectively are continuous.*

Lemma 6.9.2. *Let H be the intersection of all neighbourhood's of 0 in G . Then*

1. *$H \leq G$.*
2. *H is the closure of zero.*
3. *G/H is Hausdorff.*
4. *G is Hausdorff $\Leftrightarrow H = 0$.*

Definition 6.9.3. *An inverse system of groups is a sequence of $\{A, \theta\}$ and $\theta_{n+1} : A_{n+1} \to A_n$ where the transition maps $\forall n$ are homomorphisms of groups.*

Definition 6.9.4. *Let $\{A, \theta\}$ be an inverse system of groups and inverse limit is a subset of $\prod_{i\geq 0} A_i$ and define by*

$$\varprojlim\{A_n\} = \{(a_1, a_2, ..) : \theta_{n+1}a_{n+1} = a_n, \forall n \geq 1\} \subseteq \prod_{i\geq 0} A_i$$

Proposition 6.9.5. *The map $\bar{G} \to G/G_n$ define by*

$$\{x_i\} \mapsto (\varinjlim\{x_i + G_1\}, \varinjlim\{x_i + G_2\}, ...)$$

is an isomorphism.

Proposition 6.9.6. *If $\{0\} \to \{A_n\} \to \{B_n\} \to \{C_n\} \to \{0\}$ is an exact sequence of inverse system and $\{A_n\}$ is a surjective system then*

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$$

is exact.

Let $A = \prod_{i=1}^{\infty}$ and define $d^A : A \to A$,by $d^A(a_n) = a_n - \theta_{n+1}a_{n+1}$ and $ker(d^A) = \varprojlim A_n$.Define B ,C and d^B, d^C similarly.The exact sequence of inverse system defines commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "d^A"] & B \arrow[r, "g"] \arrow[d, "d^B"] & C \arrow[r] \arrow[d, "d^C"] & 0 \\
0 \arrow[r] & A \arrow[r, "f'"] & B \arrow[r, "g'"] & C \arrow[r] & 0
\end{tikzcd}$$

then by snake lemma and d^A is onto we are done.

Corollary 6.9.7. *Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of groups . Let G have the topology defined by the sequence $\{G_n\}$ of subgruops and G', G'' have induces topology i.e. by the sequences $\{G'_n \cap G_n\}, \{f(G_n)\}$, then*

$$0 \to \bar{G'} \to \bar{G} \to \bar{G''} \to 0$$

is exact.

Proof. Exactness of given sequence implies that the diagram below is commutative with exact rows

$$\begin{tikzcd}
0 \arrow[r] & G'/(G' \cap G_{n+1}) \arrow[r, "f"] \arrow[d, "\theta'_{n+1}"] & G/G_{n+1} \arrow[r, "g"] \arrow[d, "\theta_{n+1}"] & G''/g(G_{n+1}) \arrow[r] \arrow[d, "\theta''_{n+1}"] & 0 \\
0 \arrow[r] & G'/(G' \cap G_n)' \arrow[r, "f'"] & G/G_n \arrow[r, "g'"] & G''/g(G_n) \arrow[r] & 0
\end{tikzcd}$$

and clearly θ_n is surjective $\forall n$ now use here snake lemma and $Ker(\theta_{n+1}) = G_n$. We have an exact sequence $0 \to G'_n \to G_n \to G''_n \to 0$.Now applying previous proposition we are done. □

Corollary 6.9.8. *$\bar{G_n}$ is a subgroup of $\bar{G}$ and $\bar{G}/\bar{G_n} \cong G/G_n$.*

Proof. Let $G' = G_n$ and $G'' = G/G_n$,now applying these in previous corollary we get an exact sequence

$$0 \to \bar{G_n} \to \bar{G} \to G\bar{/}G_n \to 0$$

So $\bar{G_n}$ is a subgroup of $\bar{G}$ and G'' has discrete topology so $G'' \cong \bar{G''}$ and $\bar{G}/\bar{G_n} \cong G\bar{/}G_n$, this complete the proof.

□

Definition 6.9.9. *Let $I \lhd A$ be an ideal. Then the completion of A with respect to the I-adic filtration $A \supseteq I \supseteq I^2 \supseteq ...$ is called the I-adic completion of A. It is denoted by $\bar{A}$*

Definition 6.9.10. *Let $I \lhd A$ be an ideal, M an A-module with filtration $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ The filtration is called I-filtration if $IM_n \subseteq M_{n+1}$.*

Definition 6.9.11. *An I-filtration M on an A-module M is called stable if $\ni N$ such tha $\forall n \geq N$, $IM_n = M_{n+1}$*

Definition 6.9.12. *A graded ring is a ring A together with abelian subgroups $A_n \subseteq A$ such that $A = \oplus_{n\geq 0} A_n$, and $A_n A_m \subseteq A_{n+m}$. The elements of A_n in a graded ring A are called homogeneous elements of degree n.*

Lemma 6.9.13. *If A is a Noetherian ring, $I \lhd A$, then the graded ring $A = \oplus_{n\geq 0} I^n$ is also Noetherian.*

Proof. A being Noetherian implies I is a finitely generated A-module, say by $x_1, ..x_n$. Then the A-algebra map $A[X_1, .., X_n] \to \oplus_{n\geq 0} I^n$ defined by $X_i \mapsto x_i$ is surjective. It is surjective because $x_1, ..., x_n$ generates I. Since A is Noetherian, Hilbert's Basis Theorem implies $A[X_1, ..., X_n]$ Noetherian and hence any quotient of $A[X_1, ..., X_n]$ is Noetherian. Hence we have $\oplus_{n\geq 0} I^n$ is Noetherian.

□

Lemma 6.9.14. *Let A be a Noetherian ring, I be an ideal of A, M a finitely generated A-module together with an I-filtration $M = M_0 \subseteq M_1 \subseteq \ldots$ Then the filtration M is stable if and only if $\oplus_{n\geq 0} M_n$ is a finitely generated $A = \oplus_{n\geq 0} I^n$-module.*

Proof. Assume M is a stable I-filtration . Then $\ni n, \forall k \geq 0$ such that $I^k M_n = Mn_{n+k}$. This implies $\oplus M_n = M_0 \oplus M1 \oplus \ldots \oplus M_n \oplus IM_n \oplus I^2 M_n \oplus ..$ is finitely generated by $M_0 \oplus \ldots \oplus M_n$ as A-module. Since A is Noetherian and M is finitely generated implies $M_i \subseteq M$ are all finitely generated. Hence $M_0 \oplus \ldots \oplus M_n$ generated by finitely many elements and so $\oplus M_n$ is generated by these finitely many elements as A-modules.
Conversely Assume $\oplus M_n$ is a finitely generated $A = \oplus I^n$ module. Let $P_K = M_0 \oplus ..M_K \oplus IM_K \oplus I^2 M_K \oplus ..$. Now P_K is a graded A-submodule of $\oplus M_n$, we have $P_0 \subseteq P_1 \subseteq P_2 \subseteq \ldots \subseteq \oplus M_n$ an ascending chain of A-submodules. Now R is Noetherian implies A is Noetherian by lemma . By assumption $\oplus M_n$ is a finitely generated A-module, hence a Noetherian A-module, so the chain P_K has to stop, i.e.,$\ni N$ such that $P_N = P_{N+1} = \ldots.$
But $\cup P_K = \oplus M_n$ implies $\oplus M = P_N$ implies $M_n = I^{n-N} M_N, \forall n \geq N$ i.e., the filtration is stable. □

Lemma 6.9.15. *Let A be a Noetherian ring, I be an ideal of A and M a finitely generated A-module with stable I-fultration M. Let N be a submodule of M. Then the filtration $\{N \cap M_n\}$ on N is a stable I-filtration of N*

Proof. A Noetherian, I be an ideal of A an ideal, then $A' = \oplus_{n\geq 0} I^n$ is Noetherian. So M_n is a stable I-filtration on M implies $\oplus_{n\geq 0} M_n$ is a finitely generated A'-module. Now $\oplus M_n \cap N \subseteq \oplus M_n$ is a A'-submodule. Since A' is Noetherian and $\oplus M_n$ is a finitely generated A'-module, the submodule $\oplus M_n \cap N$ is also a finitely generated A'-module. Hence $M_n \cap N$ is a stable I-filtration □

Theorem 6.9.16. *Let A be a Noetherian ring,$\lhd$ A. Let $0 \to M \to N \to P \to 0$ be an exact sequence of finitely generated A-module. Then the sequence of I-adic completions $0 \to \bar{M} \to \bar{N} \to \bar{P} \to 0$ is exact*

$\bar{M}, \bar{N}, \bar{P}$ are the completion of M, N, P with respect to the filtrations $I^n M, I^n N, I^n P$. So we have the exact sequence $\forall n.0 \to M/(M \cap I^n N) \to N/I^n N \to P/I^n P \to 0$ Now $M \cap I^n N$ is a stable I-filtration (*Artin-Rees lemma*). Hence by Lemma the completion of M with respect to $M \cap I^n M$ is the completion $\bar{M}$ of M with respect to $I^n M$. Now $M/(M \cap I^n N)$ is a surjective inverse system, so by above equatio we are done.

Lemma 6.9.17. *Let A be a Noetherian ring,$I \lhd A$, M a finitely generated A-module . Then $\bar{A} \otimes_A M \to \bar{M}$ defined by $\{a_i\} \otimes x \mapsto \{a_i x\}$ is an isomorphism*

Definition 6.9.18. *If $I \lhd A$, then we set $gr(A) = \oplus_{n\geq 0} I^n/I^{n+1}$. This is a graded ring with multiplication $I^n/I^{n+1} \times I^m/I^{m+1} \to I^{n+m}/I^{n+m+1}$ defined by $(a + I^{n+1}, b + I^{m+1}) \mapsto ab + I^{n+m+1}$ The ring grA is called the associated graded ring of $A \supseteq I \supseteq I^2 \supseteq ....$*

Lemma 6.9.19. *Let $A = A_0 \supseteq A_1 \supseteq ...$ and $B = B_0 \supseteq B_1 \supseteq ...$ be filtered modules and $f : A \to B$ a map of filtered modules (that is $f(A_i) \subseteq B_i$).Then*

1. *If $gr(f) : gr(A) \to gr(B)$ is surjective (injective) then $\bar{f} : \bar{A} \to \bar{B}$ is surjective (injective), where $gr(A) = \oplus_{i\geq 0} A_i/A_{i+1}$*

Proof. Since $f : A \to b$ is a homomorphism of filtered modules,then $\phi(M_n) \subseteq N_n$ and consider commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_n/M_{n+1} & \xrightarrow{f} & M/M_{n+1} & \xrightarrow{g} & M/M_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta'_{n+1}} & & \downarrow{\scriptstyle \theta_{n+1}} & & \downarrow{\scriptstyle \theta''_{n+1}} & & \\ 0 & \longrightarrow & N_n/N_{n+1} & \xrightarrow{f'} & N/N_{n+1} & \xrightarrow{g'} & N/N_n & \longrightarrow & 0 \end{array}$$

By snake lemma and assuming $gr(f)$ is injective ($surjective$) we are done.

□

Lemma 6.9.20. *Let $I \lhd A$ which is I-adically complete. Let M be an A-module with an I-filtration $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq ...$ such that $\bigcap M_i = 0$. Then if $gr(M) = \oplus_{i\geq 0} M_i/M_{i+1}$ is a finitely generated $gr(A) = \oplus_{i\geq 0} I^i/I^{i+1}$-module, then M itself is a finitely generated A-module.*

Proof. Choose a finite generating set of $gr(M)$ over $gr(R)$ consisting of homogeneous elements $y_1, ..., y_t$ where $deg(y_i) = n_i, i = 1, .., t$.Choosing $x_i \in M_{n_i}$ with $y_i = x_i + M_{n_{i+1}}$. Let $F = R \oplus R \oplus ... \oplus R$,($t\ times$).And $F_n = \{(a_i) : a_i \in I^{n-n_i}, i = 1, 2, .., t\}$,where $I^k = R$ if $k \leqq 0$ and this define a filtration on F and the map $\phi : F \to M$ given by $\phi[(a_i)] = \sum a_i x_i$ is a homomorphism of filtered R modules,thus associated gradded homomorphism $gf(\phi) : gr(F) \to gr(M)$ is surjective as y_i generates $gr(M)$ implies $\bar{\phi} : \bar{F} \to \bar{M}$ is surjective Consider the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\phi} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \bar{F} & \xrightarrow{\bar{\phi}} & \bar{M} \end{array}$$

,since R is complete and F is free module of finite rank , f is an isomorphism since intersection of M_i is zero, g is injective this implies ϕ is onto as $\bar{\phi}$ is onto so M is finitely generated R module . □

Proposition 6.9.21. *Let A be nowetherian ring $I \lhd A$,$\bar{A}$ the I-adic completion.Then*

$$I^n/I^{n+1} \cong \bar{I^n}/\bar{I^{n+1}}$$

Theorem 6.9.22. *Let A be a Noetherian ring and $I \lhd A$. Then its I-adic completion $\bar{A}$ is Noetherian.*

Proof. Let M be an $\bar{A}$ ideal.Equip M with the filtration$\{M \cap \bar{I^n}\}$,then $gr(M) = \oplus_{i\geq 0}(M \cap \bar{I^n})/M \cap \bar{I^{n+1}}$ is submodule of $gr(\bar{A}) = \oplus_{n\geq 0} \bar{I^n}/\bar{I^{n+1}}$.Then by proposition we have $gr(\bar{A}) \cong gr(A)$ and A being noetherian $\Rightarrow gr(A)$ is notherian hence the submodule $gr(M)$ is also finitely generated as $gr(\bar{A})$ module and $\bigcap_{n\geq 0} M \cap \bar{I^n} \subseteq \bigcap_{n\geq 0} \bar{I^n} = 0$ then by previous lemma we are done. $\square$

Corollary 6.9.23. *If A is noetherian then $A[[X_1, ..., X_n]]$ is noetherian.*

Proof. Since A is noetherian then $A[X_1, ..., X_n]$ is noetherian and let $I = (X_1, X_2, ..., X_n)$ I adic filtration then the polynomial ring has $A[[X_1, ..., X_n]]$ completion with this filtration then by theorem we are done. $\square$