

REPRESENTATION OF FINITE GROUPS

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under the supervision of
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DECLARATION

This thesis entitled **REPRESENTATION OF FINITE GROUPS** submitted by me to the Indian Institute of Technology, Hyderabad for the award of the degree of Master of Science in Mathematics contains a literature survey of the work done by some authors in this area. The work presented in this thesis has been carried out under the supervision of **Dr. Venku Naidu Dogga**, Department of Mathematics, Indian Institute of Technology, Hyderabad, Telangana.

I hereby declare that, to the best of my knowledge, the work included in this thesis has been taken from the books, "Representation of finite and compact groups" by Barry Simon, "Representation of finite groups" by Benjamin Steinberg, and "Linear Representations of finite groups" by Jean Pierre Serre. No new results have been created in this thesis. The definitions, notations and results in representation theory are learnt from the above mentioned sources and are presented here. I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

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This Thesis entitled **Representation of finite groups** by **Mansi Bajaj** is approved for the degree of Master of Science from IIT Hyderabad.



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ABSTRACT

The main aim of this project is to learn a branch of Mathematics that studies abstract algebraic structures by representating their elements as linear transformations of vector spaces. This branch is known as Representation Theory. A representation makes an abstract algebraic object more concrete by describing its elements by matrices and the algebraic operations in terms of matrix addition and matix multiplication. The most prominent is the representation theory of groups which is discussed in this thesis.

In representation of groups, elements of group are represented by invertible elements in such a way that the group operation is matrix multiplication. If say, R is a representation of a group G on vector space V , then V is called as the representation space for G . There are two ways to describe a representation. First idea is to use action. That is, representation of a group G on a vector space V is a map $\phi : G \times V \rightarrow V$ with property that $\phi_g : V \rightarrow V$ defined as $v \mapsto \phi(g, v)$ is linear over the field \mathbb{F} . Further, if $\phi(g, v) = g.v$ then the properties of a group action hold for any elements $g_1, g_2 \in G$ and $v \in V$. The second way to define a representation focuses on map ϕ defined as $\phi(g) : V \rightarrow V$ to be a homomorphism. This approach is more concise and hence is being used in this project.

Moreover, the irreducible representations play an important role in Representation theory since every finite dimensional representation is a direct sum of irreducible representations which is an important result known as Maschke's theorem in this theory. This theory has some extremely useful fundamental results. One of them is Schur's lemma which says that if we have two irreducible representations ϕ and ψ of a finite group G on vector spaces V and W respectively and T , a linear transformation from V to W that commutes with action of group, then T is either invertible or is zero. This lemma is named after Issai Schur who used it to prove Schur's Orthogonality Relations and develop the basics of Representation theory of finite groups. Schur's lemma admits generalizations to Lie groups and Lie Algebras but this part is not mentioned in this thesis.

Further, the Dimension theorem is proved here which states that degree of any irreducible representation of a group divides the order of that group. Then, using this theorem, again the most fundamental result of group theory known as the Burnside's lemma is proved using Representation theory. It states that no finite abelian group of order $p^a q^b$ where p, q are primes and a, b are non negative, is simple. Apart from these results, the permutation representations, regular representations and fourier analysis of finite groups have also been discussed here in this project.

Preface

Motivation

In the mathematical field of representation theory, group representations describe abstract groups in terms of linear transformations of vector spaces. In particular, they can be used to represent group elements as matrices so that the group operation can be represented by matrix multiplication.

Representations of groups are important because they allow many theoretical problems to be reduced to problems in linear algebra which is well understood. They are also important in physics because, for example, they describe how the symmetry group of physical system affects the solutions of equations describing that system. The term 'representation of a group' is also used in a more general sense to mean any description of a group as a group of transformations of some mathematical object. Or we can say that, a representation means a homomorphism from the group to automorphism group of an object. If this object is a vector space, we have a linear representation.

This thesis is divided into five chapters. Chapter one consists of the definitions, notations, some fundamental results of Representation theory, the irreducible representations, Schur's lemma and ends up with a short introduction to Tensor products. Chapter two consists of the Orthogonality Relations, Characters, class functions, the Regular representation and Representation of finite abelian groups. Chapter three contains some results of Fourier analysis on finite groups. Chapter four describes the most fundamental result in Representation Theory, The Burnside's theorem. It uses some results from field theory to prove this theorem. Chapter five talks about the group actions and the permutation representations ending up with the introduction and some representation related results of Centralizer Algebra and the Gelfand Pair.

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Chapter 1

Introduction to Representation

1.1 Definition and Unitarity

Let V be a finite dimensional vector space. Let $\text{Hom}(V)$ be the set of linear maps 'A' from 'V' to 'V'. For a pair of vectors, V and W , let $\text{Hom}(V, W)$ be the space of linear maps from V to W . $GL(V) \subset \text{Hom}(V)$ are those maps A which are invertible. GL is known as the general group under composition with group identity I , the identity matrix. If V is a real or complex vector space, $GL(V)$ is precisely the set of continuous automorphisms (since any bounded linear operator is continuous) of V as a group under addition.

Definition 1.1.1. *A group representation is a homomorphism of G to $GL(V)$ for some 'V'. The dimension of 'V' is called the degree of representation.*

Example 1.1.2. *Following are some well known group representations.*

1. *Trivial Representation: $\phi: G \rightarrow \mathbb{C}^*$ given by $\phi(g) = 1 \forall g \in G$.*
2. *$\phi: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}^*$ given by*

$$\phi([m]) = \exp(2\pi im/n)$$

is also a representation .

Definition 1.1.3. *Let $\phi: G \rightarrow GL(V)$ and $\psi: G \rightarrow GL(W)$ be two representations of a finite group G on vector spaces V and W respectively. Then they are said to be equivalent if there exists an isomorphism $T: V \rightarrow W$ such that $\psi_g = T\phi_g T^{-1} \forall g \in G$, that is, $\psi_g T = T\phi_g, \forall g \in G$.*

Remark 1.1.4. The map T is called the intertwining map for ϕ_g and ψ_g .

Example 1.1.5. Define $\phi : \mathbb{Z}/n\mathbb{Z} \rightarrow GL_2(\mathbb{C})$ by,

$$\phi([m]) = \begin{pmatrix} \cos\left(\frac{2\pi im}{n}\right) & -\sin\left(\frac{2\pi im}{n}\right) \\ \sin\left(\frac{2\pi im}{n}\right) & \cos\left(\frac{2\pi im}{n}\right) \end{pmatrix}.$$

This matrix is known as the matrix of rotation by angle $\frac{2\pi im}{n}$ and define $\psi : \mathbb{Z}/n\mathbb{Z} \rightarrow GL_2(\mathbb{C})$ by,

$$\psi([m]) = \begin{pmatrix} \exp\left(\frac{2\pi im}{n}\right) & 0 \\ 0 & \exp\left(\frac{-2\pi im}{n}\right) \end{pmatrix}.$$

Now, let

$$A = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

then,

$$A^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$

so, $A^{-1}\phi([m])A =$

$$\begin{aligned} & \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \begin{pmatrix} \cos\left(\frac{2\pi im}{n}\right) & -\sin\left(\frac{2\pi im}{n}\right) \\ \sin\left(\frac{2\pi im}{n}\right) & \cos\left(\frac{2\pi im}{n}\right) \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2i} \begin{pmatrix} \exp\left(\frac{2\pi im}{n}\right) & i\exp\left(\frac{2\pi im}{n}\right) \\ -\exp\left(\frac{-2\pi im}{n}\right) & i\exp\left(\frac{2\pi im}{n}\right) \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2i} \begin{pmatrix} 2i\exp\left(\frac{2\pi im}{n}\right) & 0 \\ 0 & 2i\exp\left(\frac{-2\pi im}{n}\right) \end{pmatrix} \\ &= \psi([m]). \end{aligned}$$

Note 1.1.6. We consider complex vector spaces $V \cong \mathbb{C}^n$. An inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ such that,

1. $\langle v, \cdot \rangle$ is linear, \forall fixed v .
2. $\langle w, v \rangle = \overline{\langle v, w \rangle}$.
3. $\langle v, v \rangle \geq 0$, with equality if and only if $v=0$.

Definition 1.1.7. A unitary operator is a surjective bounded operator on a Hilbert space preserving inner product. Or we can say that, A Unitary Operator is a bounded linear operator $U : \mathbb{H} \rightarrow \mathbb{H}$ on a Hilbert space \mathbb{H} that satisfies $UU^* = U^*U = I$ where $I : \mathbb{H} \rightarrow \mathbb{H}$ is the identity operator.

Theorem 1.1.8. Let $U : G \rightarrow GL(V)$ be a representation of a finite group G . Then, V has an inner product $\langle \cdot, \cdot \rangle$ in which each $U(g)$ is unitary i.e., $\langle U(g)v, U(g)w \rangle = \langle v, w \rangle \forall v, w \in V, g \in G$.

Proof. Let $\langle \cdot, \cdot \rangle_0$ be an inner product on V .

Define $\langle v, w \rangle = \frac{1}{O(G)} \sum_{g \in G} \langle U(g)v, U(g)w \rangle_0$.

We need to show that $\langle v, w \rangle$ defined above is an inner product.

1. $\langle v, v \rangle \geq 0 \forall v \in V$.
2. By definition $\langle v, v \rangle = 0$ if and only if we have the following.

$$\langle v, v \rangle = 0 \text{ if and only if } \frac{1}{O(G)} \sum_{g \in G} \langle U(g)v, U(g)v \rangle_0 = 0$$

$$\text{if and only if } \langle U(g)v, U(g)v \rangle_0 = 0$$

$$\text{if and only if } U(g)v = 0 \text{ if and only if } v=0.$$

3. We have,

$$\begin{aligned} \langle w, v \rangle &= \frac{1}{O(G)} \sum_{g \in G} \langle U(g)v, U(g)w \rangle_0 \\ &= \frac{1}{O(G)} \sum_{g \in G} \overline{\langle U(g)v, U(g)w \rangle_0} \\ &= \overline{\langle v, w \rangle}. \end{aligned}$$

4. We have,

$$\begin{aligned}
\langle u + v, w \rangle &= \frac{1}{O(G)} \sum_{g \in G} \langle U(g)u + U(g)v, U(g)w \rangle_0 \\
&= \frac{1}{O(G)} \sum_{g \in G} [\langle U(g)u, U(g)w \rangle_0 + \langle U(g)v, U(g)w \rangle_0] \\
&= \frac{1}{O(G)} \sum_{g \in G} \langle U(g)u, U(g)w \rangle_0 + \frac{1}{O(G)} \sum_{g \in G} \langle U(g)v, U(g)w \rangle_0 \\
&= \langle u, w \rangle + \langle v, w \rangle.
\end{aligned}$$

Hence, $\langle \cdot, \cdot \rangle$ is an inner product on V .

Moreover,

$$\begin{aligned}
\langle U(h)v, U(h)w \rangle &= \frac{1}{O(G)} \sum_{g \in G} \langle U(g)U(h)v, U(g)U(h)w \rangle_0 \\
&= \frac{1}{O(G)} \sum_{g \in G} \langle U(gh)v, U(gh)w \rangle_0 \quad (\text{since, } U \text{ is homomorphism}) \\
&= \frac{1}{O(G)} \sum_{g \in G} \langle U(g)v, U(g)w \rangle_0 \\
&= \langle v, w \rangle.
\end{aligned}$$

since, for each fixed h , $g \mapsto gh$ is a bijection, so as g runs through G , so does gh . \square

Example 1.1.9. Let $G = \mathbb{Z}$. Let $a \in \mathbb{C}^*$ be fixed and $U : G \rightarrow GL(\mathbb{C})$ be defined by

$$U(n) = a^n.$$

If $|a| \neq 1$, U is not unitary in the only inner product that \mathbb{C} supports as, $\langle U(n)x, U(n)y \rangle = \langle a^n x, a^n y \rangle = |a^n|^2 \langle x, y \rangle \neq \langle x, y \rangle$ if $|a| \neq 1$.

Therefore, it implies that each $U(g)$ has eigenvalues ω with

$$|\omega| = 1.$$

(since, g^n for some $n \Rightarrow U(g)^n = \mathbb{I} \Rightarrow \omega^n = 1$ that is, eigen values are roots of unity).

Note 1.1.10. 1. *Hilbert Space is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.*

2. *Unitary Representation of a group G is a linear representation π of G on a complex Hilbert space V such that $\pi(g)$ is a unitary operator $\forall g \in G$.*

Definition 1.1.11. *Let V be a finite dimensional complex space with inner product (hilbert space). $\mathbb{U}(V)$, the unitary operators on V is a group (under composition).*

A unitary representation of a finite group G is a homomorphism of G to $\mathbb{U}(V)$ (when we say representation, we mean unitary representation).

Note 1.1.12. *One way of rephrasing Thm 2.2 is:*

If $U : G \rightarrow GL(V)$; V is a hilbert space and G is finite, then there is a $T \in GL(V)$ such that $TU(g)T^{-1}$ is unitary.

Definition 1.1.13. *Let $U : G \rightarrow \mathbb{U}(X)$, $V : G \rightarrow \mathbb{U}(Y)$ be two representations of same group G , where, $\mathbb{U}(X)$ and $\mathbb{U}(Y)$ are unitary operators on X and Y respectively.*

We say that U and V are unitarily equivalent \Leftrightarrow there exist a unitary $W : X \rightarrow Y$ such that $V(g) = WU(g)W^{-1} \forall g \in G$.

Note 1.1.14. *We classify the representations upto equivalence.*

Definition 1.1.15. *A symmetric $n \times n$ real matrix M is said to be **positive definite** if $z^T M z$ is strictly positive \forall non zero column vector z of n real numbers.*

Or

an $n \times n$ hermitian matrix M is said to be positive definite if the scalar $z^ M z$ is strictly positive \forall non zero column vector z of n complex numbers.*

Note 1.1.16. 1) $z^* M z$ is automatically real since, M is hermitian.
2) Entries on main diagonal of any hermitian matrix are real.

Definition 1.1.17. *A matrix M is called positive semi definite if $z^T M z$ or $z^* M z$ are positive or zero.*

Definition 1.1.18 (Polar Decomposition of a matrix:). *Right polar decomposition of a matrix A is, $A=UP$, where, $A \in C^{m \times n}; m \geq n$, $U \in C^{m \times n}$ with orthonormal columns and $P \in C^{n \times n}$ is positive definite.*

Left polar decomposition of a matrix A is, $A=PU$, where, $A \in C^{n \times m}; m \geq n$, $U \in C^{n \times m}$ with orthonormal columns and $P \in C^{n \times n}$ is positive definite.

Remark 1.1.19. *If $T=W|T|$ is a polar decomposition of T then, W is the partial isometry and $|T|$ is positive semi definite.*

A partial isometry is a linear map between Hilbert spaces such that it is an isometry on the orthogonal complement of kernel.

If the kernel is $\{0\}$, then, in that case, partial isometry is unitary.

Theorem 1.1.20. *Let U, V be unitary representations of G on X and Y respectively. Suppose, there exists an invertible map $T : X \rightarrow Y$ such that $U(g) = T^{-1}V(g)T, \forall g \in G$. Then, \exists a unitary map $W : X \rightarrow Y$ such that $U(g) = WV(g)W^{-1}, \forall g \in G$.*

Proof. Since $U(g)$ is a unitary operator,

$$U(g)U(g)^* = U(g)^*U(g) = \mathbb{I}.$$

This implies,

$$U(g^*) = U(g)^{-1} = U(g^{-1})$$

(since, U is a homomorphism).

Now, given that, \exists an invertible map $T : X \rightarrow Y$ such that

$$U(g) = T^{-1}V(g)T.$$

Taking adjoint and using $U(g)^* = U(g^{-1})$, and replacing g by g^{-1} , we get,

$$U(g)^* = T^*V(g)^*(T^{-1})^*$$

$$i.e., U(g)^{-1} = T^*V(g)^{-1}(T^*)^{-1}$$

$$i.e., U(g^{-1}) = T^*V(g)^{-1}(T^*)^{-1}$$

$$i.e., U(g) = T^*V(g)(T^*)^{-1}$$

(since, true for all g , in general)

$$i.e., V(g) = (T^*)^{-1}U(g)T^*$$

$$i.e., V(g) = TU(g)T^{-1}.$$

Now,

$$\begin{aligned} T^*TU(g)(T^*T)^{-1} &= T^*TU(g)(T^{-1}(T^*)^{-1}) \\ &= T^*(TU(g)T^{-1})(T^*)^{-1} \\ &= T^*U(g)(T^*)^{-1} \\ &= U(g). \end{aligned}$$

This implies, $U(g)^{-1}(T^*T)U(g) = T^*T$. Now, by uniqueness of square root,

$$U(g)^{-1}|T|U(g) = |T|.$$

Let $T = W|T|$ be the polar decomposition of T , where W is unitary from X to Y . Since, $|T|$ is invertible,

$$\begin{aligned} WU(g)W^{-1} &= T|T|^{-1}U(g)|T|T^{-1} \\ &= TU(g)T^{-1} \\ &= V(g). \end{aligned}$$

Hence proved. □

Note 1.1.21. In above theorem, T is an invertible linear map (automorphism) $\Rightarrow KerT = \{0\}$, hence, W is unitary.

Example 1.1.22. Standard Representation of S_n Define $\phi : S_n \rightarrow GL_n(\mathbb{C})$ on standard basis by

$$\phi_{\sigma(e_i)} = e_{\sigma(i)}.$$

For instance,

$$\text{when } n=3, \phi(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \phi(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\phi_\sigma(e_1 + e_2 + \dots + e_n) = e_{\sigma(1)} + e_{\sigma(2)} + \dots + e_{\sigma(n)} = e_1 + e_2 + \dots + e_n.$$

Thus, $\mathbb{C}(e_1 + e_2 + \dots + e_n)$ is invariant under ϕ_σ with $\sigma \in S_n$.

1.2 Irreducibility and Complete Reduction

Definition 1.2.1. Let X and Y be vector spaces. Their Direct sum $X \oplus Y$ is defined as the cartesian product $X \times Y$ with coordinatewise operation.

For example, $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

Note: If X and Y are Hilbert spaces, so is $X \oplus Y$, with inner product as follows:

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y.$$

Definition 1.2.2. If $A \in \text{Hom}(X), B \in \text{Hom}(Y)$ then, $A \oplus B \in \text{Hom}(X \oplus Y)$ by:

$$(A \oplus B)(x, y) = (Ax, By).$$

Also, $(A \oplus B)(C \oplus D) = (AC \oplus BD)$. And if, $A \in \mathbb{U}(X), B \in \mathbb{U}(Y)$, then,

$$A \oplus B \in \mathbb{U}(X \oplus Y).$$

Verification I:

We need to show that $A \oplus B \in \text{Hom}(X \oplus Y)$ if $A \in \text{Hom}(X)$ and $B \in \text{Hom}(Y)$. That means to show $A \oplus B : X \oplus Y \rightarrow X \oplus Y$ is linear. That is, we show, $A \oplus B(\alpha x + y) = \alpha(A \oplus B)(x) + (A \oplus B)(y)$.

Consider, $(A \oplus B)(\alpha x + y)$; $x, y \in X \oplus Y$.

Let $x = (x_1, x_2), y = (y_1, y_2)$.

$$\begin{aligned} (A \oplus B)((\alpha x_1, \alpha x_2) + (y_1, y_2)) &= (A \oplus B)(\alpha x_1 + y_1, \alpha x_2 + y_2) \\ &= (A(\alpha x_1 + y_1), B(\alpha x_2 + y_2)) \\ &= (\alpha(Ax_1, Bx_2) + (Ay_1, By_2)) \\ &= \alpha(A \oplus B)(x) + (A \oplus B)(y). \end{aligned}$$

Hence $A \oplus B$ is linear.

Verification II:

We need to show $A \in \mathbb{U}(X), B \in \mathbb{U}(Y) \Rightarrow A \oplus B \in \mathbb{U}(X \oplus Y)$. That means to show $A \oplus B$ is a unitary operator. That is, we show $\langle (A \oplus B)(x_1, y_1), (A \oplus B)(x_2, y_2) \rangle = \langle (x_1, y_1), (x_2, y_2) \rangle$.

$$\begin{aligned} \text{Consider } \langle (A \oplus B)(x_1, y_1), (A \oplus B)(x_2, y_2) \rangle \\ &= \langle (Ax_1, By_1), (Ax_2, By_2) \rangle \\ &= \langle Ax_1, Ax_2 \rangle + \langle By_1, By_2 \rangle \end{aligned}$$

(since, $X \oplus Y$ is a hilbert space , inner product can be defined like this)

$$\begin{aligned} &= \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle \\ &= \langle (x_1, y_1), (x_2, y_2) \rangle. \end{aligned}$$

Hence, $A \oplus B$ is a unitary operator on $X \oplus Y$.

Definition 1.2.3. *If U and V are representations of G on X and Y , respectively, then, $U \oplus V$ defined as,*

$$(U \oplus V)(g) = U(g) \oplus V(g)$$

is a representation of G , also known as Diresct sum representation.

Verification I:

We need to show that $(U \oplus V)(g) = U(g) \oplus v(g)$ is also representation of G ; where, U and V are representations of G on X and Y respectively. That is, to show $U \oplus V$ is a homomorphism from G to $\mathbb{U}(X \oplus Y)$ or to show $U \oplus V(g_1g_2) = U \oplus V(g_1)U \oplus V(g_2)$.

$$\begin{aligned} U \oplus V(g_1g_2)(x, y) &= (U(g_1g_2) \oplus V(g_1g_2))(x, y) \\ &= (U(g_1g_2)(x), V(g_1g_2)(y)) \\ &= (U(g_1)U(g_2)(x), V(g_1)V(g_2)(y)). \end{aligned}$$

and

$$\begin{aligned} U \oplus V(g_1)U \oplus V(g_2)(x, y) &= U \oplus V(g_1)((U(g_2) \oplus V(g_2))(x, y)) \\ &= U \oplus V(g_1)(U(g_2)(x), V(g_2)(y)) \\ &= U(g_1) \oplus V(g_1)(U(g_2)(x), V(g_2)(y)) \\ &= (U(g_1)U(g_2)x, V(g_1)V(g_2)y) \\ &= U \oplus V(g_1g_2)(x, y); \quad \forall (x, y) \in X \oplus Y. \end{aligned}$$

Hence , $U \oplus V(g_1g_2) = U \oplus V(g_1)U \oplus V(g_2)$; $\forall g_1, g_2 \in G$. Therefore $U \oplus V$ is a homomorphism. This shows that it is a representation of G on $X \oplus Y$.

Definition 1.2.4. Let U be a representation of G on X . A subspace $Y \subset X$ is called invariant if $\forall g \in G$ and $y \in Y$, $U(g)y \in Y$.

Theorem 1.2.5. If Y is an invariant subspace , so is

$$Y^\perp = \{x \in X | \langle x, y \rangle = 0, \forall y \in Y\}.$$

$U \upharpoonright Y$ and $U \upharpoonright Y^\perp$ define representations U_1 and U_2 and U is equivalent to $U_1 \oplus U_2$.

Conversely, $(y, 0)$ is an invariant subspace for any direct sum representation $U_1 \oplus U_2$.

Proof. First , note that , $U : G \rightarrow \mathbb{U}(X)$ is a representation of G on Hilbert space X .

Let $x \in Y^\perp, y \in Y, g \in G$. Then,

$$\begin{aligned} \langle U(g)x, y \rangle &= \langle x, U^*(g)(y) \rangle \\ &= \langle x, U(g^{-1})y \rangle \\ &= 0 \end{aligned}$$

since, $U(g^{-1})y \in Y$. Thus, $U(g)x \in Y^\perp$ so, Y^\perp is invariant. This implies that in $Y \oplus Y^\perp$ block decomposition of X ,

$$U(g) = \begin{pmatrix} U_1(g) & 0 \\ 0 & U_2(g) \end{pmatrix} \text{ That means, } U \cong U_1 \oplus U_2.$$

Conversely, let $U = U_1 \oplus U_2$. Then $\forall g \in G$,

$$\begin{aligned} U(g)((y, 0)) &= (U_1 \oplus U_2)(y, 0) \\ &= (U_1(g) \oplus U_2(g))((y, 0)) \\ &= (U_1(g)(y), U_2(g)(0)) \\ &= (y_1, 0) \in (y, 0). \end{aligned}$$

Hence $(y, 0)$ is an invariant subspace. □

Note 1.2.6. In general, $Y \subset X$ (invariant), may not have a complementary subspace that is invariant, if U is not unitary and G is infinite (if G is finite, U is equivalent to unitary).

Example 1.2.7. The non unitary representation of \mathbb{R} ,

$$U(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

has an invariant subspace $\left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right\}$ but no complementary subspace since, say if $U : \mathbb{R} \rightarrow GL(\mathbb{R}^2)$ then given $U(x)$ is not unitary representation as,

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1+x^2 & x \\ x & 1 \end{pmatrix} \\ \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, U is not unitary. And, $\left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \right\}$ is an invariant subspace since,

$$\forall x \in \mathbb{R}, \\ U(x) \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

Now, let it has a complementary subspace say, $\begin{pmatrix} 0 \\ \beta \end{pmatrix}$

$$\text{then, } U(x) \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} x\beta \\ \beta \end{pmatrix} = \beta \begin{pmatrix} x \\ 1 \end{pmatrix} \notin \left\{ \begin{pmatrix} 0 \\ \beta \end{pmatrix} : \beta \in Y^\perp \right\}.$$

This shows that, $(0, \beta)$ is not invariant.

Definition 1.2.8. A unitary representation U of G on X is called irreducible if and only if the only invariant subspaces of U are $\{0\}$ and X .

The theorem proved earlier implies that,

Corollary 1.2.1. U is an irrep (we will use this for 'irreducible representation') if and only if it cannot be written as a direct sum of non trivial (that is, not zero dimensional) representations.

Proof. Let U is an irrep of G on X . Then the only invariant subspaces of X are $\{0\}$ and X . That means there does not exist any proper subspace of X invariant under U . Hence U cannot be written as direct sum of non trivial representations.

Conversely, let U cannot be written as direct sum of non trivial representations. Then we need to show that U is an irreducible representation.

Let if possible, U is a reducible representation. Then there exist U_1, U_2 corresponding to invariant subspaces Y and Y' of X such that U_1 and U_2 are representations of G on Y and Y' respectively and hence $U \cong U_1 \oplus U_2$ where, U_1 and U_2 are non trivial representations which is a contradiction. Hence U is irrep of G on X . \square

Example 1.2.9. *Orthogonal group $O(n)$ of orthogonal matrices under multiplication has an irrep on \mathbb{R}^n .*

Example 1.2.10. *The representation $\phi : S_3 \rightarrow GL_2\mathbb{C}$ is irreducible.*

Proof. We know that $\dim(\mathbb{C}^2) = 2$. Suppose it is not irreducible. Then, there exists a non zero proper invariant subspace. Now, any non zero proper S_3 invariant subspace is one dimensional. Let $v \in W$ then, $W = \mathbb{C}v$. Let $\sigma \in S_3$ then, $\phi\sigma(v) = \lambda v$ for some $\lambda \in \mathbb{C}$.

So, v must be an eigen vector for all $\phi\sigma$ with $\sigma \in S_3$.

Claim: $\phi(12)$ and $\phi(123)$ do not have a common eigen vector.

On computing, we get $\phi(12)$ has eigen values 1 and -1, with

$V_{-1} = \mathbb{C}e_1$ and $V_1 = \mathbb{C}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Now, e_1 is not an eigen vector of $\phi(123)$ as

$$\phi(123)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \text{ Also, } \phi(123)\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

So, $(-1, 2)$ is not an eigen vector of $\phi(123)$. This implies, $\phi(12)$ and $\phi(123)$ have no eigen vector in common. Hence ϕ is irreducible. \square

Theorem 1.2.11. *Any representation can be written as a direct sum of irreps.*

Proof. We prove this by induction on $\deg(U)$. We know that $\deg(U) = \dim(X)$. If $\deg(U) = 1$, U is irrep (since, any 1-dimensional representation is irreducible).

Let the result is true \forall representations of $\deg < d$.

Let $\deg(U) = d$. If U is irreducible, we are done. If not,

$U = U_1 \oplus U_2$, where, $\deg(U_1) < d$, $\deg(U_2) < d$. Thus by induction, each U_i is direct sum of irreducible representations. That is, U is direct sum of irreps. Therefore any U can be written as $U_1 \oplus U_2 \oplus \dots \oplus U_k$ where U_i is irrep $\forall i=1,2,\dots,k$. \square

Definition 1.2.12. A representation $\phi : G \rightarrow GL(V)$ is said to be completely reducible if $V = V_1 \oplus V_2 \oplus \dots \oplus V_n$ where, V_i are G -invariant subspaces and $\phi|_{V_i}$ is irreducible for all $i=1,2,\dots,n$.

It is equivalent to saying that ϕ is completely reducible if $\phi \sim \phi^1 \oplus \phi^2 \oplus \dots \oplus \phi^n$ where ϕ^i are irreducible representations.

Definition 1.2.13. A representation ϕ is decomposable if $V = V_1 \oplus V_2$ where V_1, V_2 are non zero G -invariant subspaces. Otherwise, V is called indecomposable.

Definition 1.2.14. \hat{G} , known as the dual object, is the set of equivalence classes of irreps, each class consisting of unitarily equivalent irreps.

An explicit matrix realization of each $\alpha \in \hat{G}$, is $D_{ij}^{(\alpha)}(g)$ that is a $d(\alpha) \times d(\alpha)$ matrix, where $d(\alpha)$ is the degree of irrep in α .

1.3 The Group Algebra and the Regular Representations

Let G be a finite group. Consider, $S = \{\sum_{g \in G} a_g \delta_g ; \delta_g \text{ is a symbol, } a_g \in \mathbb{C}\}$. Then, S is a complex vector space of dimension $O(G)$ (i.e., order of group).

Verification :

We need to show that S is a vector space. Let $x, y \in S$. Then $x = \sum_{g \in G} a_g \delta_g$ and $y = \sum_{g \in G} b_g \delta_g$

$$1) x + y = \sum_{g \in G} a_g \delta_g + \sum_{g \in G} b_g \delta_g = \sum_{g \in G} (a_g + b_g) \delta_g ;$$

where, $a_g + b_g \in \mathbb{C}$ (since \mathbb{C} is a vector space).

This implies, $\sum_{g \in G} (a_g + b_g) \delta_g \in S$.

2) $\alpha x = \alpha \sum_{g \in G} a_g \delta_g = \sum_{g \in G} \alpha a_g \delta_g \in S$. since, $\alpha a_g \in \mathbb{C}$ and \mathbb{C} is a vector space.

So, $\alpha x \in S$. Hence, S is a vector space.

Now, since, $a_g \in \mathbb{C}$ and $\dim(S) = O(G)$, (since, summation is over $g \in G$) S is a complex vector space of $\dim = O(G)$.

Note 1.3.1. We consider the sequence $\{a_g\}_{g \in G}$ as a function so we write it as $a(g)$.

Now, we define a product, inherited from the group product for the vector space S such that,

$$\delta_g \delta_h = \delta_{gh} \quad (*)$$

, $\forall g \in G$.

Therefore $(\sum_g a_g \delta_g)(\sum_g b_h \delta_h) = \sum_{g, h \in G} a_g b_h \delta_{gh}$.

Now, By change of variables (i.e., let $gh=x, h=y \Rightarrow g = xy^{-1}$) we get,

$$= \sum_x \left(\sum_y a(xy^{-1})b(y) \delta_x \right).$$

Definition 1.3.2. Let G be a finite group. The complex vector space $\mathcal{A}(G)$ of functions on G is called the group algebra when we are given the product called *CONVOLUTION* as follows,

$$(a * b)(x) = \sum_y a(xy^{-1})b(y); a, b \in \mathcal{A}$$

where support of each function is finite and the conjugate,

$$a^*(g) = \overline{a(g^{-1})}.$$

Remark 1.3.3. $(a * b)^* = b^* * a^*$.

Verification:

$$\begin{aligned}
(a * b)^*(x) &= \overline{(a * b)(x^{-1})} \\
&= \overline{(a * b)(h)} = \overline{\sum_g a(hg^{-1})b(g)} \\
&= \overline{\sum_y a(y)b(y^{-1}h)} \quad (hg^{-1} = y \Rightarrow g = y^{-1}h) \\
&= \overline{\sum_y b(y^{-1}h)a(y)} \\
&= \overline{\sum_y b(y^{-1}h)a(y)} \\
&= \sum_y b^*(h^{-1}y)a^*(y^{-1}) \\
&= \sum_y b^*(xy)a^*(y^{-1}) \\
&= \sum_{y^{-1}} b^*(xy^{-1})a^*(y) \\
&= b^* * a^*.
\end{aligned}$$

Let δ_g be the function defined as,

$$\delta_g(x) = \begin{cases} 1 & : x = g \\ 0 & : x \neq g \end{cases}$$

then, $a = \sum a_g \delta_g$.

Verification of the statement (*) written above :

$$\begin{aligned}
\delta_g(x) &= \begin{cases} 1 & : x = g \\ 0 & : x \neq g \end{cases} \\
\delta_h(x) &= \begin{cases} 1 & : x = h \\ 0 & : x \neq h \end{cases} \\
\delta_{gh}(x) &= \begin{cases} 1 & : x = gh \\ 0 & : x \neq gh \end{cases}
\end{aligned}$$

$$\begin{aligned}
\delta_g \delta_h(x) &= \sum_y \delta_g(xy^{-1})\delta_h(y) \\
&= \begin{cases} 1 & : xh^{-1} = g \\ 0 & : xh^{-1} \neq g \end{cases} \\
(\text{ when, } xy^{-1} = g, y=h) & \\
&= \begin{cases} 1 & : x = gh \\ 0 & : x \neq gh \end{cases} \\
&= \delta_{gh}(x), \forall x \in G.
\end{aligned}$$

$$\Rightarrow \delta_g \delta_h = \delta_{gh}.$$

There is one more property that δ_g has. That is,

$$\delta_{g^*} = \delta_{g^{-1}}.$$

Verification:

$$\begin{aligned}
\delta_{g^*}(x) &= \overline{\delta_g(x^{-1})} \\
&= \overline{\begin{cases} 1 & : x^{-1} = g \\ 0 & : x^{-1} \neq g \end{cases}} \\
&= \begin{cases} 1 & : x = g^{-1} \\ 0 & : x \neq g^{-1} \end{cases} \\
&= \delta_{g^{-1}}(x), \forall x \in G.
\end{aligned}$$

Hence,

$$\delta_{g^*} = \delta_{g^{-1}}.$$

Now, one might wonder, why did we choose such an adjoint. So, following theorem is the reason for the same.

Theorem 1.3.4. *Let G be a finite group. Let U be a representation of G on a Hilbert space X .*

Define for $a \in \mathcal{A}(G)$,

$$U_{\mathcal{A}}(a) = \sum a(g)U(g). \quad (1)$$

Then $U_{\mathcal{A}}$ obeys,

(i) $U_{\mathcal{A}}(a + b) = U_{\mathcal{A}}(a) + U_{\mathcal{A}}(b)$

*(ii) $U_{\mathcal{A}}(a * b) = U_{\mathcal{A}}(a)U_{\mathcal{A}}(b)$*

(iii) $U_{\mathcal{A}}(a^) = U_{\mathcal{A}}(a)^*$*

(iv) $U_{\mathcal{A}}(\delta_e) = \mathbb{I}$; where, e is the identity of the group G .

Conversely, if $U_{\mathcal{A}}$ obeys all the above four conditions then, \exists a unitary representation U of G obeying (1).

Proof. Forward Part:

We have to show, $U_{\mathcal{A}}$ obeys all four properties.

(i)First,

$$\begin{aligned} U_{\mathcal{A}}(a + b) &= \sum (a + b)(g)U(g) \\ &= \sum (a(g) + b(g))U(g) \\ &= \sum a(g)U(g) + \sum b(g)U(g) \\ &= U_{\mathcal{A}}(a) + U_{\mathcal{A}}(b). \end{aligned}$$

(ii) Second,

$$\begin{aligned}
U_{\mathcal{A}}(a * b) &= \sum (a * b)(g)U(g) \\
&= \sum_g \left(\sum_h a(gh^{-1})b(h) \right) U(g) \\
&= \sum_h \left(\sum_g a(gh^{-1})U(g) \right) b(h) \\
&= \sum_h \left(\sum_x a(x)U(xh) \right) b(h) \quad (gh^{-1} = x \Rightarrow h = x^{-1}g) \\
&= \sum_h \left(\sum_x a(x)U(x) \right) (U(h)b(h)) \\
&= \sum_h \left(\sum_x a(x)U(x) \right) (b(h)U(h)) \\
&= \sum_x a(x)U(x) \sum_h b(h)U(h) \\
&= U_{\mathcal{A}}(a)U_{\mathcal{A}}(b).
\end{aligned}$$

(iii) Third,

$$\begin{aligned}
U_{\mathcal{A}}(a^*) &= \sum a^*(g)U(g) \\
&= \sum \overline{a(g^{-1})}U(g) \\
&= \sum_{g^{-1} \in G} \overline{a(g)}U(g^{-1}) \\
&= \sum \overline{a(g)}U(g)^* \\
&= \sum (a(g)U(g))^* \\
&= \left(\sum a(g)U(g) \right)^* \\
&= U_{\mathcal{A}}(a)^*.
\end{aligned}$$

(iv)Fourth,

$$\begin{aligned}
 U_{\mathcal{A}}(\delta_e) &= \sum \delta_e(g)U(g) \\
 &= U(e)(ifg = e) \\
 &= \mathbb{I}(\text{since, } U \text{ is a homomorphism}).
 \end{aligned}$$

Converse Part:

If $U_{\mathcal{A}}$ obeys all four properties then,
define,

$$U(g) = U_{\mathcal{A}}(\delta_g)$$

then, (1) holds .

Now, since, $\delta_g * \delta_h = \delta_{gh}$, Property (ii) implies,

$$U(g)U(h) = U_{\mathcal{A}}(\delta_g)U_{\mathcal{A}}(\delta_h) = U_{\mathcal{A}}(\delta_g * \delta_h) = U_{\mathcal{A}}(\delta_{gh}) = U(gh).$$

And , since, $\delta_{g^*} = \delta_{g^{-1}}$, Property (iii) implies,

$$U(g)^* = U_{\mathcal{A}}(\delta_g)^* = U_{\mathcal{A}}(\delta_g^*) = U_{\mathcal{A}}(\delta_{g^{-1}}) = U(g^{-1}).$$

Hence, we have,

$$\begin{aligned}
 U(g)U(g)^* &= U(g)U(g^{-1}) = U(gg^{-1}) = U_{\mathcal{A}}(\delta_{gg^{-1}}) = U_{\mathcal{A}}(\delta_g * \delta_{g^{-1}}) \\
 &= U_{\mathcal{A}}(\delta_{gg^{-1}}) = U_{\mathcal{A}}(\delta_e) = \mathbb{I}.
 \end{aligned}$$

And, also,

$$\begin{aligned}
 U(g)^*U(g) &= U(g^{-1})U(g) = U_{\mathcal{A}}(\delta_{g^{-1}})U_{\mathcal{A}}(\delta_g) = U_{\mathcal{A}}(\delta_{g^{-1}} * \delta_g) \\
 &= U_{\mathcal{A}}(\delta_{g^{-1}g}) = U_{\mathcal{A}}(\delta_e) = \mathbb{I}.
 \end{aligned}$$

Hence U is unitary. □

A map $U_{\mathcal{A}}$ obeying (i) to (iv) all properties , is called a *-representation of Algebra $\mathcal{A}(G)$.

Hence, we can conclude that there is one one correspondence between representation of G and $*$ -representation of $\mathcal{A}(G)$.

The obvious representation of $\mathcal{A}(G)$ is on itself by the following multiplication:

We define, $L_{\mathcal{A}} : \mathcal{A} \rightarrow \mathbb{U}(\mathcal{A}(G))$ as,

$$L_{\mathcal{A}}(a)b = a * b.$$

Now , we need to define an inner product on $\mathcal{A}(G)$ so that $L_{\mathcal{A}}(a)$ becomes a $*$ -representation.

We can take usual inner product for the same , as

$$\langle f, g \rangle = \frac{1}{O(G)} \sum_{x \in G} \overline{f(x)}g(x).$$

Now, we check that $L_{\mathcal{A}}$ is a $*$ -representation .

For that , we need to prove that,

$$\langle h, f * g \rangle = \langle f^* * h, g \rangle \quad (**)$$

$\forall f, g, h \in \mathcal{A}(G)$

Consider, $L : G \rightarrow \mathbb{U}(G)$. If, $(L(x)f)(y) = f(x^{-1}y)$ then, $L_{\mathcal{A}}$ is the induced map on \mathcal{A} ,
(since, $(\delta_x * f)(y) = \sum_z \delta_x(yz^{-1})f(z) = f(x^{-1}y)$ by letting $yz^{-1} = x$) Then,
 $L_x \delta_y = \delta_{xy}$.

Verification:

$$L_x \delta_y(z) = \delta_y(x^{-1}z) = \begin{cases} 1 & z = x^{-1}y \\ 0 & \text{otherwise} \end{cases} = \delta_{xy}(z)$$

Hence, $L_x \delta_y = \delta_{xy}$. So, L_x is unitary ,

Verification:

We have to show that $\langle L_x f, L_x g \rangle = \langle f, g \rangle$.

$$\begin{aligned}
\langle L_x f, L_x g \rangle &= \frac{1}{O(G)} \sum_{z \in G} \overline{L_x f(z)} L_x g(z) \\
&= \frac{1}{O(G)} \sum_{z \in G} \overline{f(x^{-1}z)} g(x^{-1}z) \\
&= \frac{1}{O(G)} \sum_{xy \in G} \overline{f(y)} g(y) \\
&= \frac{1}{O(G)} \sum_{y \in G} \overline{f(y)} g(y) \\
&= \langle f, g \rangle.
\end{aligned}$$

Now, let $g(z) \in \mathcal{A}(G)$. Then,

$$g(z) = \sum_{y \in G} a(y) \delta_y(z)$$

for any $g \in \mathcal{A}$.

Hence, we can conclude that, $(\sqrt{O(G)}\delta_y)_{y \in G}$ is a basis. L_x is unitarily equivalent to ,

$$\langle h, f * g \rangle = \langle f^* * h, g \rangle.$$

Theorem 1.3.5. $(L_x f)(y) \equiv f(x^{-1}y)$ on $\mathcal{A}(G)$ with inner product ,

$$\langle f, g \rangle = \frac{1}{O(G)} \sum_{x \in G} \overline{f(x)} g(x)$$

is a unitary representation of G . It is called left regular representation.

Similarly, $R_{\mathcal{A}}(f)g = g * \bar{f}^*$, the right multiplication, is also a representation of \mathcal{A} induced by,

$$(R_x f)(y) = f(yx)$$

known as the Right regular representation.

The consequence of existence of this 'L' is the following corollary.

Corollary 1.3.1. *The functions D_{ij}^α separate points that is, for any $x, y \in G$ with $x \neq y, \exists$ a function f of the form:*

$$f = \sum_{k=1}^n C_{kij} D_{ikjk}^{\alpha_k}$$

such that, $f(x)=1, f(y)=0$

Proof. Consider, $f(z) = O(G)\langle \delta_x, L(z)\delta_e \rangle$,
then,

$$\begin{aligned} f(x) &= O(G)\langle \delta_x, L(x)\delta_e \rangle \\ &= O(G) \frac{1}{O(G)} \sum_{y \in G} \overline{\delta_x(y)} L_x \delta_e(y) \\ &= \sum_{y \in G} \overline{\delta_x(y)} \delta_e(x^{-1}y) \\ &= 1. (\text{if } y=x) \end{aligned}$$

$$\begin{aligned} f(y) &= O(G)\langle \delta_x, L(y)\delta_e \rangle \\ &= O(G) \frac{1}{O(G)} \sum_{z \in G} \overline{\delta_x(z)} L(y)\delta_e(z) \\ &= \sum_{z \in G} \overline{\delta_x(z)} \delta_e(y^{-1}z) \\ &= 0. \end{aligned}$$

Now, $L : G \rightarrow \mathbb{U}(\mathcal{A}(G))$ as,

$$z \longmapsto L_z$$

where,

$$L_z : \mathcal{A}(G) \rightarrow \mathcal{A}(G)$$

We know that,

$$\hat{G} = \{ \alpha : \alpha \text{ is irreducible unitary representation of } G \}$$

Let $\alpha_k \in \hat{G}$

$$\alpha_k : G \rightarrow \mathbb{U}(\mathbb{H}_{\alpha_k})$$

as,

$$z \mapsto \alpha_k(z)$$

is a representation, where, $\alpha_k(z) : \mathbb{H}_{\alpha_k} \rightarrow \mathbb{H}_{\alpha_k}$ and $\dim(\mathbb{H}_{\alpha_k}) = t_{\alpha_k}$.
So, the associated matrix is,

$$\left(D_{ij}^{\alpha_k}(z) \right)_{t_{\alpha_k} \times t_{\alpha_k}}$$

therefore, $\bigoplus_{k=1}^n \alpha_k(z) : \bigoplus_{k=1}^n \mathbb{H}_{\alpha_k} \rightarrow \bigoplus_{k=1}^n \mathbb{H}_{\alpha_k}$

Now, let

$$\mathbb{H} = \bigoplus_{k=1}^n \mathbb{H}_{\alpha_k}$$

and,

$$\alpha(z) = \bigoplus_{k=1}^n \alpha_k(z)$$

so, we get,

$$\alpha(z) : G \rightarrow \mathbb{U}(\mathbb{H})$$

defined as,

$$z \mapsto \alpha(z)$$

Now, $L : G \rightarrow \mathbb{U}(\mathcal{A}(G))$ is again a representation of G .

But since, L is a representation, it is unitarily equivalent to a direct sum of irreducibles.

So, given L and α are unitarily equivalent representations of G on $\mathcal{A}(G)$ and \mathbb{H} respectively. Therefore, there exists $U : \mathcal{A}(G) \rightarrow \mathbb{H}$ such that,

$$UL(z)U^{-1} = \alpha(z).$$

So, $\forall z \in G$,

$$L(z) = U^{-1}\alpha(z)U, \forall z \in G.$$

That is, $L(z) = U^{-1} \left(\bigoplus_{k=1}^n \alpha_k(z) \right) U$, $\forall z \in G$ or, $L(z) = \bigoplus_{k=1}^n U^{-1} \alpha_k(z) U$.

Hence $L(z) = \bigoplus_{k=1}^n U^* \alpha_k(z) U$.

Therefore, $(L(z))_{x,y} = \sum_{k,i,j} M_{xi}^{(\alpha_k)} D_{ij}^{(\alpha_k)}(z) \overline{M_{yj}^{(\alpha_k)}}$

for some numbers, $M_{xi}^{(\alpha_k)}$.

Hence, this yields,

$$f = \sum_{k=1}^n C_{kij} D_{ikjk}^{(\alpha_k)}.$$

This completes the proof. \square

Remark 1.3.6. (1) A $*$ -representation of $\mathcal{A}(G)$ on a Hilbert space X is called irreducible iff it leaves no non trivial subspace invariant.

(2) $U_{\mathcal{A}}$ is an irreducible representation of $\mathcal{A}(G)$ iff U is an irreducible representation of G .

1.4 Schur's Lemma

Theorem 1.4.1. First form:

Let $U_{\mathcal{A}}$ be an irreducible representation of $\mathcal{A}(G)$ on X . Let $A \in \text{Hom}(X)$ obeys

$$AU_{\mathcal{A}}(f) = U_{\mathcal{A}}(f)A \quad (1)$$

$\forall f \in \mathcal{A}(G)$.

(equivalently, $AU(x) = U(x)A \forall x \in G$)

Then, $A = c\mathbb{I}$ for some constant c .

Proof. Replacing f by f^* and taking adjoints, we see that ,

$$A^*U_{\mathcal{A}}(f) = U_{\mathcal{A}}(f)A^*$$

Thus, (1) holds if A is replaced by $B = \frac{1}{2}(A + A^*)$ or $C = \frac{1}{2i}(A - A^*)$.

We note that, $A = B + iC$. So, if B, C are constants, so is A .

Hence, it suffices to prove the theorem with $A = A^*$.

Let λ be an eigen value of A and let $(A - \lambda)u = 0$.

Then,

$$(A - \lambda)U_{\mathcal{A}}(f)u = U_{\mathcal{A}}(f)(A - \lambda)u = 0$$

Thus,

$\{u | (A - \lambda)u = 0\}$ is an invariant subspace.

Since, λ is an eigen value, space is not $\{0\}$. So, it must be X .

Therefore, $A = \lambda\mathbb{I}$ where λ is the constant. Hence proved. \square

Theorem 1.4.2. Second Form:

Let U, V be two irreps of a finite group G on spaces X and Y respectively.

Let $T: X \rightarrow Y$ such that,

$$TU(x) = V(x)T \quad (2)$$

$\forall x \in G$.

Then, either, $T=0$ or U, V are unitarily equivalent and T is unique upto constant.

Proof. Replacing x by x^{-1} and taking adjoints to see that

$$T^*V(x) = U(x)T^*$$

Thus,

$$(T^*T)U(x) = U(x)(T^*T)$$

and,

$$(TT^*)V(x) = V(x)(TT^*)$$

Thus, by previous theorem, $T^*T = c\mathbb{I}$ and $TT^* = c\mathbb{I}$. Either, $c=0 \Rightarrow T = 0$ or else, $W = c^{-1/2}T$ is unitary and

$$WU(x) = U(x)W$$

so, U, V are unitarily equivalent.

If they are unitarily equivalent and both T and S obey (2), then, $TS^* = c_1\mathbb{I}$ by same argument. So, $T = TS^*Sc_s^{-1} = c_1c_s^{-1}S$.

For general representations U and V , a map T obeying (2), is called an intertwining map for U, V . \square

Corollary 1.4.1. *Let G be an abelian group. Then, every irrep has degree 1.*

Proof. Let U be an irrep of G on X . By Schur's lemma (first form), each $U(x)$ is a constant multiple of identity. Thus, every subspace of X is invariant. This is consistent with irreducible only if $\dim(X)=1$. \square

1.5 Tensor Products

Let X and Y be finite dimensional Hilbert spaces. Then the tensor product $X \otimes Y$ is defined as the vector space of bi-antilinear maps with a certain inner product. Let $B \in X \otimes Y$. Then, B is a map from $X \times Y \rightarrow \mathbb{C}$ such that,

$$B(u, \alpha v + \beta w) = \bar{\alpha} B(u, v) + \bar{\beta} B(u, w) \quad \forall u \in X, v, w \in Y, \alpha, \beta \in \mathbb{C}$$

$$B(\alpha u + \beta v, w) = \bar{\alpha} B(u, w) + \bar{\beta} B(v, w) \quad \forall u, v \in X, w \in Y, \alpha, \beta \in \mathbb{C}$$

Now, we show that $X \otimes Y$ is a vector space.

Let $A, B \in X \otimes Y$. Then, $A: X \times Y \rightarrow \mathbb{C}$ and $B: X \times Y \rightarrow \mathbb{C}$

$$A(u, \alpha v + \beta w) = \bar{\alpha} A(u, v) + \bar{\beta} A(u, w).$$

$$A(\alpha u + \beta v, w) = \bar{\alpha} A(u, w) + \bar{\beta} A(v, w).$$

$$B(u, \alpha v + \beta w) = \bar{\alpha} B(u, v) + \bar{\beta} B(u, w).$$

$$B(\alpha u + \beta v, w) = \bar{\alpha} B(u, w) + \bar{\beta} B(v, w).$$

Now, Consider, $A+B(u, \alpha v + \beta w)$.

$$\begin{aligned} A+B(u, \alpha v + \beta w) &= A(u, \alpha v + \beta w) + B(u, \alpha v + \beta w) = \bar{\alpha} A(u, v) + \bar{\beta} A(u, w) + \bar{\alpha} B(u, v) + \bar{\beta} B(u, w) \\ &= \bar{\alpha} (A(u, v) + B(u, v)) + \bar{\beta} (A(u, w) + B(u, w)) = \bar{\alpha} (A+B)(u, v) + \bar{\beta} (A+B)(u, w). \end{aligned}$$

Hence $A+B \in X \otimes Y$.

Now, consider,

$$aA(u, \alpha v + \beta w) = a(\bar{\alpha} A(u, v) + \bar{\beta} A(u, w)) = \bar{\alpha} aA(u, v) + \bar{\beta} aA(u, w).$$

Hence $aA \in X \otimes Y$. So, $X \otimes Y$ is a vector space.

Let $u \in X$, $v \in Y$. Then, define $u \otimes v \in X \otimes Y$ by

$$(u \otimes v)(w, z) = \langle w, u \rangle \langle z, v \rangle.$$

Then, $(u, v) \mapsto (u \otimes v)$ is a bilinear map of $X \times Y$ into $X \otimes Y$. Also, $u \otimes v$ spans $X \otimes Y$.

Note 1.5.1. *Loosely speaking, a bilinear map satisfies:*

$B(x+y, z) = B(x, z) + B(y, z)$ that is, additivity in first coordinate.

$B(x, y+z) = B(x, y) + B(x, z)$ that is, additivity in second coordinate.

$B(cx, y) = cB(x, y) = B(x, cy)$ that is, preserves scalar multiplication in each coordinate.

Example 1.5.2. Examples of bilinear maps:

1. *Matrix multiplication is a bilinear map:*

$$M(m, n) \times M(n, p) \rightarrow M(m, p).$$

2. Let V be a vector space over \mathbb{R} carrying an inner product. Then the inner product is a bilinear map $V \times V \rightarrow \mathbb{F}$.

Now, getting back to the tensor products, if we have two finite dimensional Hilbert spaces X and Y and given that $\{e_i\}_{i=1}^n$ is a basis for X and $\{f_j\}_{j=1}^m$ is a basis for Y , then $\{e_i \otimes f_j\}_{i=1, j=1}^{n,m}$ is a basis for $X \otimes Y$.

To prove this, we just need to show that $\{e_i \otimes f_j\}_{i=1, j=1}^{n,m}$ is linearly independent since we already know that it spans the space $X \otimes Y$. So, let us consider,

$$\sum_{i,j} \alpha_{i,j} (e_i \otimes f_j) = 0$$

This implies that,

$$\sum_{i,j} \alpha_{i,j} (e_i \otimes f_j)(x, y) = 0, \forall x \in X, y \in Y$$

i.e.,

$$\sum_{i,j} \alpha_{i,j} \langle x, e_i \rangle \langle y, f_j \rangle = 0.$$

We need to show that $\alpha_{i,j} = 0, \forall i, j$. Suppose this is not true. Then, choose $x \in X$ and $y \in Y$ such that $\langle x, e_i \rangle \neq 0$ if $x \in (\text{span}\{e_j\})^\perp$ where $j \neq i$ and, $\langle y, f_i \rangle \neq 0$ if $y \in (\text{span}\{f_j\})^\perp$ where $j \neq i$. Then, for this choice of x and y ,

$$\sum_{i,j} \alpha_{i,j} \langle x, e_i \rangle \langle y, f_j \rangle \neq 0$$

which is a contradiction.

Hence, $\alpha_{i,j} = 0$. Thus, the set is linearly independent. Therefore, $\{e_i \otimes f_j\}_{i=1, j=1}^{n,m}$ is a basis for $X \otimes Y$.

So, now we can conclude that, $\dim(X \otimes Y) = \dim(X)\dim(Y)$. We can give a unique inner product to $X \otimes Y$ in which,

$$\langle u \otimes Y, B \rangle = B(u, v)$$

where $B \in X \otimes Y$. This inner product is such that if $\{e_i\}$ and $\{f_j\}$ are orthonormal bases for X and Y respectively then, $e_i \otimes f_j$ is an orthonormal basis for $X \otimes Y$.

Now, suppose that $A \in \text{Hom}(X)$ and $B \in \text{Hom}(Y)$ then, $A \otimes B \in \text{Hom}(X \otimes Y)$ is defined as

$$(A \otimes B)(C)(u, v) = C(A * u, B * v)$$

where $C \in X \otimes Y$, $u \in X$, $v \in Y$ so that,

$$(A \otimes B)(u \otimes v) = Au \otimes Bv.$$

Now, we move further to define what is called the **Tensor Product Representation**. Given W and V to be two representations of a group G on spaces X and Y respectively, we define $W \otimes V$ to be a representation of G on $X \otimes Y$ by,

$$(W \otimes V)(x) = W(x) \otimes V(x)$$

known as the tensor product representation. In general, if W^α and V^β are irreps, $W^\alpha \otimes V^\beta$ is not irrep but a direct sum of irreducibles $\bigoplus_{\gamma \in \hat{G}} n_{\alpha\beta}^\gamma U^{(\gamma)}$ given for $\alpha, \beta \in \hat{G}$ that is,

$$W^\alpha \otimes V^\beta \cong \bigoplus_{\gamma \in \hat{G}} n_{\alpha\beta}^\gamma U^{(\gamma)}.$$

Here, $n_{\alpha\beta}^\gamma$ are called **Clebsch-Gordan integers** representing the number of times $U^{(\gamma)}$ appears in the direct sum.

Moreover,

$$D_{ij}^\alpha(x) D_{kl}^\beta(x) = \sum_{m,p,q} C_{ij;kl;pq}^{\alpha\beta m} D_{pq}^{\gamma_m}(x)$$

where D_{ij} is the matrix realization for the irreducible representations, $C_{ij;kl;pq}^{\alpha\beta m}$ are the constants and sum is over $m=1,2,\dots,M$ and for each m , there is associated some $\gamma_m \in \hat{G}$ and $p,q=1,2,\dots,d_{\gamma_m}$.

Chapter 2

Orthogonality Relations

2.1 Morphisms

Definition 2.1.1. Let $\phi : G \rightarrow GL(V)$ and $\psi : G \rightarrow GL(W)$ be representations. A morphism from ϕ to ψ is a linear map $T:V \rightarrow W$ such that $T\phi_g = \psi_g T, \forall g \in G$.

The set of all morphisms from ϕ to ψ is denoted by $Hom_G(\phi, \psi)$ and $Hom_G(\phi, \psi) \subset Hom(V, W)$.

Remark 2.1.2. 1. If $T \in Hom_G(\phi, \psi)$ is invertible, then $\phi \equiv \psi$ and T is isomorphism.

2. $T : V \rightarrow V$ belongs to $Hom_G(\phi, \phi)$ iff $T\phi_g = \phi_g T, \forall g \in G$ that is, T commutes with $\phi(g)$. Therefore, in particular, $I : V \rightarrow V$ is always an element of $Hom_G(\phi, \phi)$.

2.2 The Orthogonality Relations

Let G be a finite group. Consider $\phi : G \rightarrow GL_n(\mathbb{C})$ to be a representation. Let $\phi_g = (\phi_{ij}(g))$ where $(\phi_{ij}(g)) \in \mathbb{C}, \forall 1 \leq i, j \leq n$. That is, there exist n^2 functions $\phi_{ij} : G \rightarrow \mathbb{C}$ associated to degree n representation ϕ . Whenever ϕ is irreducible and unitary, then functions of the form $\phi_{ij} : G \rightarrow \mathbb{C}$ form an orthogonal basis for \mathbb{C}^G .

Proposition 2.2.1. *Suppose we have two representations $\phi : G \rightarrow GL(V)$ and $\psi : G \rightarrow GL(W)$ and $T : V \rightarrow W$ is a linear transformation. Then, the following statements hold:*

- (a). $T^\# = \frac{1}{O(G)} \sum_{g \in G} \psi_{g^{-1}} T \phi_g$ belongs to $Hom_G(\phi, \psi)$.
 (b). If $T \in Hom_G(\phi, \psi)$ then, $T^\# = T$.
 (c). $P : Hom(V, W) \rightarrow Hom(\phi, \psi)$ defined by $P(T) = T^\#$ is an onto linear map.

Proof. To prove statement (a), consider,

$$\begin{aligned} T^\# \phi_h &= \frac{1}{O(G)} \sum_{g \in G} \psi_{g^{-1}} T \phi_g \phi_h \\ &= \frac{1}{O(G)} \sum_{g \in G} \psi_{g^{-1}} T \phi_{gh}. \end{aligned}$$

Suppose $gh=x$. Then, $g^{-1} = hx^{-1}$. Therefore, our expression becomes,

$$\begin{aligned} T^\# \phi_h &= \frac{1}{O(G)} \sum_{x \in G} \psi_{hx^{-1}} T \phi_x \\ &= \psi_h \frac{1}{O(G)} \sum_{x \in G} \psi_{x^{-1}} T \phi_x \\ &= \psi_h T^\#. \end{aligned}$$

Hence, $T^\# \in Hom_G(\phi, \psi)$.

Next, to prove (b), if $T \in Hom_G(\phi, \psi)$, then

$$\begin{aligned} T^\# &= \frac{1}{O(G)} \sum_{g \in G} \psi_{g^{-1}} T \phi_g \\ &= \frac{1}{O(G)} \sum_{g \in G} \psi_{g^{-1}} \psi_g T \\ &= \frac{1}{O(G)} \sum_{g \in G} T = \frac{1}{O(G)} O(G) T \\ &= T. \end{aligned}$$

Now, we prove (c),

Consider,

$P : Hom(V, W) \rightarrow Hom_G(\phi, \psi)$ defined as

$$P(c_1T_1 + c_2T_2) = (c_1T_1 + c_2T_2)^\#$$

$$\begin{aligned} (c_1T_1 + c_2T_2)^\# &= \frac{1}{O(G)} \sum_{g \in G} \psi_{g^{-1}}(c_1T_1 + c_2T_2)\phi_g \\ &= c_1 \frac{1}{O(G)} \sum_{g \in G} \psi_{g^{-1}}T_1\phi_g + c_2 \frac{1}{O(G)} \sum_{g \in G} \psi_{g^{-1}}T_2\phi_g \\ &= c_1T_1^\# + c_2T_2^\# \\ &= c_1P(T_1) + c_2P(T_2). \end{aligned}$$

Hence, the map P is linear. Now, we know from (b) that, if $T \in Hom_G(\phi, \psi)$, $T^\# = P(T) = T$. Thus, it is onto .

□

Before moving further, let us restate the Schur's Lemma:

Let ϕ and ψ be irreps of G and $T \in Hom_G(\phi, \psi)$. Then either T is invertible or T=0. Consequently,

- (a). If $\phi \not\sim \psi$ then, $Hom_G(\phi, \psi) = 0$.
- (b). If $\phi = \psi$ then, $T = \lambda I$, where $\lambda \in \mathbb{C}$.

Proposition 2.2.2. *Let $\phi : G \rightarrow GL(V)$ and $\psi : G \rightarrow GL(W)$ be irreps of G and let $T : V \rightarrow W$ be linear. Then,*

- (a). *If $\phi \not\sim \psi$ then, $T^\# = 0$.*
- (b). *If $\phi \sim \psi$ then, $T^\# = \frac{Trace(T)I}{deg\phi}$.*

Proof. Let $\phi \not\sim \psi$. Then, $Hom_G(\phi, \psi) = 0$ by Schur's lemma since if that is not the case then there exists $0 \neq T \in Hom_G(\phi, \psi)$ invertible and hence $\phi \sim \psi$ that is a contradiction. So, $T^\# = 0$ since $T^\# \in Hom_G(\phi, \psi)$. Next, we suppose $\phi = \psi$. Then, by Schur's lemma, $T^\# = \lambda I$ for some $\lambda \in \mathbb{C}$. Now, as $T^\# : V \rightarrow V$ (since, $\phi = \psi$) we have, $Trace(T^\#) = Trace(\lambda I) = \lambda Trace(I) = \lambda dim V = \lambda deg\phi$. Hence, $\lambda = \frac{Trace(\lambda I)}{deg\phi} = \frac{Trace(T^\#)}{deg\phi}$. Now, we compute the trace. Note that,

(We use 'Tr' for Trace for convenience).

$$Tr(T^\#) = \frac{1}{O(G)} \sum_{g \in G} Tr(\phi_{g^{-1}} T \phi_g) = \frac{1}{O(G)} \sum_{g \in G} Tr(T) = \frac{O(G)}{O(G)} Tr(T) = Tr(T).$$

Thus, $\lambda = \frac{Tr(T)I}{deg\phi}$. Hence proved. □

Suppose we have two representations $\phi : G \rightarrow GL_n(\mathbb{C})$ and $\psi : G \rightarrow GL_m(\mathbb{C})$. Then, $Hom(V, W) = M_{mn}(\mathbb{C})$. $Hom_G(\phi, \psi) \subseteq M_{mn}(\mathbb{C})$. And thus, $P : Hom(V, W) \rightarrow Hom_G(\phi, \psi)$ defined as $P(T) = T^\#$ can be viewed as a linear transformation $P : M_{mn}(\mathbb{C}) \rightarrow M_{mn}(\mathbb{C})$. Note that, the standard basis for $M_{mn}(\mathbb{C})$ is $\{E_{11}, E_{12}, \dots, E_{mn}\}$, where E_{ij} is the $m \times n$ matrix with 1 at ij position and 0 otherwise. Therefore, $a_{ij} = \sum_{i,j} a_{ij} E_{ij}$.

Lemma 2.2.3. *Let $A \in M_{rm}(\mathbb{C})$ and $B \in M_{ns}(\mathbb{C})$ and $E_{ki} \in M_{mn}(\mathbb{C})$. Then, $(AE_{ki}B)_{lj} = a_{lk}b_{ij}$ holds, where $A = (a_{ij})$ and $B = (b_{ij})$.*

Proof.

$$(AE_{ki}B)_{lj} = \sum (a_{lx}(E_{ki})_{xy}b_{yj}) = a_{lk}b_{ij}$$

since, all terms in the above sum are zero except when $x=k, y=i$. □

Lemma 2.2.4. *Let $\phi : G \rightarrow U_n(\mathbb{C})$ and $\psi : G \rightarrow U_m(\mathbb{C})$ be two unitary representations. Let $A = E_{ki} \in M_{mn}(\mathbb{C})$. Then, $A_{lj}^\# = \langle \phi_{ij}, \psi_{kl} \rangle$.*

Proof. Since, ψ is unitary, $\psi_{g^{-1}} = \psi_g^{-1} = \psi_g^*$. This implies, $\psi_{lk}(g^{-1}) = \overline{\psi_{kl}(g)}$.

$$A_{lj}^\# = \frac{1}{O(G)} \sum_{g \in G} (\phi_{g^{-1}} E_{ki} \phi_g)_{lj} = \frac{1}{O(G)} \sum_{g \in G} \psi_{lk}(g^{-1}) \phi_{ij}(g)$$

(by lemma done before.) Therefore,

$$A_{lj}^\# = \frac{1}{O(G)} \sum_{g \in G} \overline{\psi_{kl}(g)} \phi_{ij}(g) = \langle \phi_{ij}, \psi_{kl} \rangle.$$

Hence proved. □

2.3 Schur Orthogonality Relations

Theorem 2.3.1. *Let $\phi : G \rightarrow U_n(\mathbb{C})$ and $\psi : G \rightarrow U_m(\mathbb{C})$ be inequivalent irreducible unitary representations. Then,*

- (a). $\langle \phi_{ij}, \psi_{kl} \rangle = 0$.
 (b). $\langle \phi_{ij}, \phi_{kl} \rangle = \begin{cases} 1/n & ; i = k, j = l \\ 0 & ; \text{otherwise.} \end{cases}$

Proof. To prove first part, note that if $A = E_{ki} \in M_{mn}(\mathbb{C})$ then, $A_{lj}^\# = \langle \phi_{ij}, \psi_{kl} \rangle = 0$ since representations are given to be inequivalent. (We can conclude this using previous proposition and lemma). Now, if $\phi = \psi$ then again by same proposition and lemma, for $A = E_{ki} \in M_n(\mathbb{C})$ (since, $m=n$),

$$A^\# = \frac{\text{Tr}(E_{ki})I}{n}$$

and

$$A_{lj}^\# = \langle \phi_{ij}, \phi_{kl} \rangle$$

First, $j \neq l$, then since $I_{lj} = 0$, this gives $A_{lj}^\# = 0$ and hence, $\langle \phi_{ij}, \phi_{kl} \rangle = 0$. Now, let $i \neq k$, then E_{ki} has only zeros on diagonal. This gives $\text{Tr}(E_{ki}) = 0$. Therefore, again, $0 = A_{lj}^\# = \langle \phi_{ij}, \phi_{kl} \rangle$. Hence, if we consider the case when $i=k$ and $l=j$ then E_{ki} has a single 1 on diagonal and all other elements as zero. Thus, $\text{Tr}(E_{ki}) = 1$ and so, $\frac{1}{n} = A_{lj}^\# = \langle \phi_{ij}, \phi_{kl} \rangle$. Hence, proved. □

Corollary 2.3.1. *For an irreducible unitary representation ϕ of degree d , the d^2 functions,*

$$\{\sqrt{d}\phi_{ij} | 1 \leq i, j \leq d\}$$

form an orthonormal set.

Hence, according to above theorem, the entries of inequivalent unitary representations of G form an orthogonal set of non zero vectors in $\mathcal{A}(G)$, the group algebra. Therefore, G has atmost $O(G)$ equivalence classes of irreps. Now, if $\phi^{(1)}, \dots, \phi^{(s)}$ are a complete set of representatives of equivalence classes of irreps of G and $d_i = \text{deg}\phi^{(i)}$ then, $d_1^2 + d_2^2 + \dots + d_s^2$ functions $\{\sqrt{d_k}\phi_{ij}^{(k)} | 1 \leq i, j \leq d_k\}$ form an orthonormal set of vectors in $\mathcal{A}(G)$ and hence, $s \leq d_1^2 + \dots + d_s^2 \leq O(G)$ since $d_i \geq 1, \forall i$.

2.4 Characters and Class functions

Definition 2.4.1. Let $\phi : G \rightarrow GL(V)$ be a representation. The character $\chi_\phi : G \rightarrow \mathbb{C}$ of ϕ is defined as $\chi_\phi(g) = \text{Tr}(\phi_g)$. We call the character of an irrep as irreducible character. Suppose we have a representation $\phi : G \rightarrow GL_n(\mathbb{C})$ given by $\phi_g = (\phi_{ij}(g))$ then,

$$\chi_\phi(g) = \sum_{i=1}^n \phi_{ii}(g).$$

Remark 2.4.2. For a degree one representation $\phi : G \rightarrow \mathbb{C}^*$, $\chi_\phi = \phi$.

Proposition 2.4.3. Let ϕ be a representation of G . Then, $\chi_\phi(1) = \text{deg}\phi$.

Proof. Note that, $\chi_\phi(1) = \text{Tr}(\phi_1) = \text{Tr}(I) = \text{dim}(V) = \text{deg}\phi$. □

Proposition 2.4.4. Equivalent representations have same characters. In other words, Character depends only on the equivalence class of representation.

Proof. Suppose ϕ and ψ are two equivalent representations. Then, there exists an invertible map T such that, $\phi_g = T\psi_g T^{-1}$. Therefore,

$$\chi_\phi(g) = \text{Tr}(\phi_g) = \text{Tr}(T\psi_g T^{-1}) = \text{Tr}(T T^{-1} \psi_g) = \text{Tr}(\psi_g) = \chi_\psi(g).$$

(Here, we used the property that $\text{Tr}(AB) = \text{Tr}(BA)$. □

Proposition 2.4.5. Let ϕ be a representation of G . Then, $\forall g, h \in G, \chi_\phi(g) = \chi_\phi(hgh^{-1})$.

Proof.

$$\chi_\phi(hgh^{-1}) = \text{Tr}(\phi_{hgh^{-1}}) = \text{Tr}(\phi_h \phi_g \phi_h^{-1}) = \text{Tr}(\phi_h \phi_h^{-1} \phi_g) = \text{Tr}(\phi_g) = \chi_\phi(g).$$

□

Definition 2.4.6. A function $f : G \rightarrow \mathbb{C}$ is called Class Function if $f(g) = f(hgh^{-1}) \forall g, h \in G$ or more precisely, if f is constant on conjugacy classes of G . The space of class functions is denoted by $Z(\mathcal{A}(G))$. In particular, characters are class functions.

Note 2.4.7. If $f : G \rightarrow \mathbb{C}$ is a class function and C is the conjugacy class then, $f(C)$ will denote the constant value that f takes on C .

Proposition 2.4.8. $Z(\mathcal{A}(G))$ is a subspace of $\mathcal{A}(G)$.

Proof. Let $f, g \in Z(\mathcal{A}(G))$. Let $c_1, c_2 \in \mathbb{C}$. Then,

$$\begin{aligned} (c_1f_1 + c_2f_2)(hgh^{-1}) &= c_1f_1(hgh^{-1}) + c_2f_2(hgh^{-1}) \\ &= c_1f_1(g) + c_2f_2(g) \\ &= (c_1f_1 + c_2f_2)(g). \end{aligned}$$

Therefore, $c_1f_1 + c_2f_2 \in Z(\mathcal{A}(G))$ and hence, it is a subspace. \square

Definition 2.4.9. $Cl(G)$ denotes the set of all conjugacy classes of G . For $C \in Cl(G)$, $\delta_C : G \rightarrow \mathbb{C}$ is defined as,

$$\delta_C = \begin{cases} 1 & ; g \in C \\ 0 & ; g \notin C. \end{cases}$$

Proposition 2.4.10. $B = \{\delta_C | C \in Cl(G)\}$ is a basis for $Z(\mathcal{A}(G))$. As a result, $\dim(Z(\mathcal{A}(G))) = |Cl(G)|$.

Proof. Note that each δ_C is constant on conjugacy classes. Therefore, it is a class function. First we show that B spans $Z(\mathcal{A}(G))$. Let $f \in Z(\mathcal{A}(G))$. Then, $f = \sum_{C \in Cl(G)} f(C)\delta_C$ since, $f(g) = \sum f(C)\delta_C(g) = f(C)$ if $g \in C$ and then, $f(C)=f(g)$. Next, we show linear independence. For that, we show orthogonality. For $C, C' \in Cl(G)$,

$$\frac{1}{O(G)} \sum_{g \in G} \delta_C(g)\overline{\delta_{C'}(g)} = \begin{cases} |C|/|G| & ; C = C' \\ 0 & ; C \neq C' \end{cases}$$

where $\frac{1}{O(G)} \sum_{g \in G} \delta_C(g)\overline{\delta_{C'}(g)} = \langle \delta_C, \delta_{C'} \rangle$. This proves the orthogonality and hence the linear independence. Therefore, B is a basis. Moreover, $|B| = |Cl(G)| = \text{number of conjugacy classes} = \dim(Z(\mathcal{A}(G)))$. \square

Theorem 2.4.11. First Orthogonality Relations: Let ϕ and ψ be two representations of a finite group G . Then,

$$\langle \chi_\phi, \chi_\psi \rangle = \begin{cases} 1 & ; \phi \sim \psi \\ 0 & ; \phi \not\sim \psi \end{cases}$$

In other words, the irreducible characters of G form an orthonormal set of class functions.

Proof. Without loss of generality, assume that $\phi : G \rightarrow U_n(\mathbb{C})$ and $\psi : G \rightarrow U_m(\mathbb{C})$ are unitary (since, every representation is equivalent to a unitary one.) Now, if ϕ and ψ are equivalent, then, $\chi_\phi = \chi_\psi$. Consider, $\langle \chi_\phi, \chi_\psi \rangle$.

$$\begin{aligned} \langle \chi_\phi, \chi_\psi \rangle &= \frac{1}{O(G)} \sum_{g \in G} \chi_\phi(g) \overline{\chi_\psi(g)} \\ &= \frac{1}{O(G)} \sum_{g \in G} \sum_{i=1}^n \phi_{ii}(g) \sum_{j=1}^m \overline{\psi_{jj}(g)} \\ &= \sum_{i=1}^n \sum_{j=1}^m \frac{1}{O(G)} \sum_{g \in G} \phi_{ii}(g) \overline{\psi_{jj}(g)} \\ &= \sum_{i=1}^n \sum_{j=1}^m \langle \phi_{ii}(g), \psi_{jj}(g) \rangle. \end{aligned}$$

Now, by Schur's orthogonality Relations, $\langle \phi_{ii}(g), \psi_{jj}(g) \rangle = 0$ if $\phi \not\sim \psi$. This implies, $\langle \chi_\phi, \chi_\psi \rangle = 0$ if $\phi \not\sim \psi$. If $\phi \sim \psi$, assume $\phi = \psi$ then, by Schur's Orthogonality Relation,

$$\langle \phi_{ii}, \psi_{jj} \rangle = \begin{cases} 1/n & ; i \neq j \\ 0 & ; i = j. \end{cases}$$

Therefore,

$$\begin{aligned} \langle \chi_\phi, \chi_\phi \rangle &= \sum_{i=1}^n \langle \phi_{ii}, \phi_{ii} \rangle \\ &= \frac{1}{n} \times n = 1. \end{aligned}$$

Hence, irreducible characters form an orthonormal set of class functions. \square

Corollary 2.4.1. *There are atmost $|Cl(G)|$ equivalence classes of irreps of G .*

Proof. By previous theorem, we know that inequivalent irreps have distinct characters and form an orthonormal set. Therefore, $\dim(Z(\mathcal{A}(G))) = |Cl(G)|$ and the orthonormal sets are linearly independent. Also, we know that, the cardinality of an orthonormal set is less than or equal to the dimension of the space. So there exist atmost $|Cl(G)|$ equivalence classes of irreps of G (if we consider equivalence relation to be conjugation). \square

Notation: $mV = V \oplus V \dots \oplus V$ (m-times) and $m\phi = \phi \oplus \dots \oplus \phi$ (m-times). Let ϕ_i denote the unitary representation of G and d_i denotes the respective degrees.

Definition 2.4.12. If $\psi \sim m_1\phi_1 \oplus m_2\phi_2 \oplus \dots \oplus m_s\phi_s$ then, m_i is the multiplicity of ϕ_i in ψ . If $m_i > 0$ then, ϕ_i is irreducible constituent of ψ .

Note that, since the character depends only on the equivalence class so multiplicity of ϕ_i will be same, no matter what the decomposition of ψ is.

Remark 2.4.13. If $\psi \sim m_1\phi_1 \oplus m_2\phi_2 \oplus \dots \oplus m_s\phi_s$ then, $\text{deg}\psi = m_1d_1 + m_2d_2 + \dots + m_sd_s$.

Lemma 2.4.14. Let $\phi = \rho \oplus \psi$ then, $\chi_\phi = \chi_\rho + \chi_\psi$.

Proof. Proof: Let $\rho : G \rightarrow GL_n(\mathbb{C})$ and $\psi : G \rightarrow GL_m(\mathbb{C})$ then, $\phi : G \rightarrow GL_{n+m}(\mathbb{C})$ has the block form as,

$$\phi_g = \begin{pmatrix} \rho_g & 0 \\ 0 & \psi_g \end{pmatrix}$$

$\chi_\phi(g) = \text{Tr}(\phi_g) = \text{Tr}(\rho_g) + \text{Tr}(\psi_g) = \chi_\rho(g) + \chi_\psi(g)$. This shows that,

$$\chi_\phi = \chi_\rho + \chi_\psi.$$

□

Theorem 2.4.15. Let ϕ_1, \dots, ϕ_s be a complete set of representatives of equivalence classes of irreps of G and let

$$\rho \sim m_1\phi_1 \oplus \dots \oplus m_s\phi_s$$

Then, $m_i = \langle \chi_\rho, \chi_{\phi_i} \rangle$. Consequently, decomposition of ρ into irreducible constituents is unique and ρ is determined upto equivalence by its characters.

Proof. We know that $\chi_\rho = m_1\chi_{\phi_1} + \dots + m_s\chi_{\phi_s}$. $\langle \chi_\rho, \chi_{\phi_i} \rangle = m_1\langle \chi_{\phi_1}, \chi_{\phi_i} \rangle + \dots + m_s\langle \chi_{\phi_s}, \chi_{\phi_i} \rangle = m_i$.

So, decomposition of ρ into irreducible constituents is unique and if representations are equivalent then characters are same.

Hence we conclude that this theorem helps us to check whether a representation is irreducible or not. □

Corollary 2.4.2. *A representation is irreducible iff $\langle \chi_\rho, \chi_\rho \rangle = 1$.*

Proof. Suppose a representation ρ is not irreducible. Then,

$$\rho \sim m_1\phi_1 \oplus \dots \oplus m_s\phi_s.$$

Note that, $\chi_\rho = m_1\chi_{\phi_1} + \dots + m_s\chi_{\phi_s}$. This implies that,

$$\begin{aligned} \langle \chi_\rho, \chi_\rho \rangle &= \langle m_1\chi_{\phi_1} + \dots + m_s\chi_{\phi_s}, m_1\chi_{\phi_1} + \dots + m_s\chi_{\phi_s} \rangle \\ &= m_1^2 + m_2^2 + \dots + m_s^2, \end{aligned}$$

Therefore, $\langle \chi_\rho, \chi_\rho \rangle = 1$ iff there exists j such that $m_j = 1$ and $m_i = 0 \forall i \neq j$ but this is the case only when ρ is irreducible. \square

Example 2.4.16. *Consider $\rho : S_3 \rightarrow GL_2(\mathbb{C})$ as*

$$\rho(12) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \text{ and } \rho(123) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

then, $\chi_\rho(Id) = 2 = \text{deg}\rho$, $\chi_\rho(12) = 0$, and $\chi_\rho(123) = -1$.

$$\langle \chi_\rho, \chi_\rho \rangle = \frac{1}{6} \sum_{g \in G} \chi_\rho(g) \overline{\chi_\rho(g)} = \frac{1}{6} (2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2) = 1.$$

Hence this representation is irreducible.

Example 2.4.17. Characters of S_3 : *Trivial character : $\chi_1 : S_3 \rightarrow \mathbb{C}^*$ defined as , $\chi_1(\sigma) = 1 \forall \sigma \in S_3$. Note that any degree one representation is equal to its character. Since S_3 has three conjugacy classes, this implies there should exist three inequivalent irreps of S_3 . Therefore, by proposition done earlier $\sum d_i^2 \leq O(G)$ which gives $1^2 + d^2 + 2^2 \leq 6$. In this expression '2' represents the degree two irreducible representation of S_3 considered in previous example. This shows that $d=1$.*

So, we can define another degree one representation for S_3 as,

$$\chi_2(\sigma) = \begin{cases} 1 & ; \sigma \text{ is even} \\ -1 & ; \sigma \text{ is odd.} \end{cases}$$

Character Table: Rows of a character table correspond to the irreducible characters and the columns of a character table correspond to conjugacy classes.

Standard Representation of S_3 gives,

$$\phi(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \phi(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

So, for this representation , we have the following character table:

	Id	(1 2)	(1 2 3)
χ_ϕ	3	1	0

Similarly, we can form character table for three inequivalent representations of S_3 . Consider ϕ_1 to be the trivial representation of S_3 . Another, consider ϕ_2 to be the one degree representation whose character is defined as,

$$\chi_2(\sigma) = \begin{cases} 1 & ; \sigma \text{ is even} \\ -1 & ; \sigma \text{ is odd} \end{cases}$$

and lastly, consider ϕ_3 as the two degree irreducible representation considered in the first example. Now the character table is as follows.

	Id	(1 2)	(1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

This shows that $\chi_\phi = \chi_1 + \chi_3$. That is, $\phi \sim \chi_1 \oplus \chi_3$.

We can also conclude the same by computing the following :

$$\langle \chi_\phi, \chi_1 \rangle = \frac{1}{6}(3 + 3.1 + 2.0) = 1 = m_1$$

$$\text{Similarly, } \langle \chi_\phi, \chi_2 \rangle = \frac{1}{6}(3 + 3.(-1) + 2.0) = 0 = m_2$$

$$\text{and also, } \langle \chi_\phi, \chi_3 \rangle = \frac{1}{6}(0 + 3.0 + 2.0) = 1 = m_3.$$

We can note that columns of character table are pairwise orthogonal as can be seen in the table drawn above.

2.5 The Regular Representation

Another way of looking at the Regular Representation is the following: We know that by Cayley's theorem, every finite group G is isomorphic to some subgroup of S_n where n is the order of the group G . Hence, we can restrict the standard representation of S_n that is, $\phi : G \rightarrow GL_n(\mathbb{C})$ to G and get the representation called the Regular Representation. Suppose we have a finite set X . Then,

$$\mathbb{C}X = \left\{ \sum_{x \in X} c_x x \mid c_x \in \mathbb{C} \right\}$$

is the vector space generated with basis X . Note that, $\sum_{x \in X} a_x x = \sum_{x \in X} b_x x$ iff $a_x = b_x \forall x \in X$. Addition here is defined as,

$$\sum a_x x + \sum b_x x = \sum_{x \in X} (a_x + b_x) x.$$

Scalar multiplication is defined as,

$$\sum \alpha a_x x = \alpha \sum a_x x.$$

The inner product is defined as,

$$\left\langle \sum_{x \in X} a_x x, \sum_{x \in X} b_x x \right\rangle = \sum_{x \in X} a_x \bar{b}_x.$$

Definition 2.5.1. Let G be a finite group. The Regular Representation of G is a homomorphism $L : G \rightarrow GL(\mathbb{C}G)$ defined by

$$L_g \sum_{h \in G} c_h h = \sum_{h \in G} c_h gh = \sum_{x \in G} c_{g^{-1}x} x$$

for $g \in G$. (By using change of variable, letting $gh=x$) L_g is a linear operator acting on a linear combination of basis vectors given the action on basis.

Remark 2.5.2. The Regular Representation is never irreducible when G is non trivial but contains all irreps of G as constituents.

Proposition 2.5.3. The Regular Representation is Unitary Representation of G .

Proof. Since L_g is linear so for $g_1, g_2 \in G, h \in G$ is a basis element of $\mathbb{C}(G)$. Hence,

$$(L_{g_1} L_{g_2})(h) = L_{g_1}(g_2 h) = g_1 g_2 h = L_{g_1 g_2} h$$

So, L is a homomorphism.

Now,

$$\begin{aligned} \left\langle L_g \sum c_h h, L_g \sum k_h h \right\rangle &= \left\langle \sum c_{g^{-1}x} x, \sum k_{g^{-1}x} x \right\rangle \\ &= \sum c_{g^{-1}x} \overline{k_{g^{-1}x}}. \end{aligned}$$

Let $y = g^{-1}x$, we get the expression equal to $\sum_{y \in G} c_y \bar{k}_y$ that is further equal to $\left\langle \sum c_y y, \sum k_y y \right\rangle$. This shows that L_g is unitary and hence is invertible since $L_g^{-1} = L_g^*$. \square

Proposition 2.5.4. *The character of Regular Representation is:*

$$\chi_L(g) = \begin{cases} |G| & ; g = 1 \\ 0 & ; g \neq 1. \end{cases}$$

Proof. Let $G = g_1, g_2, \dots, g$ be a basis for $\mathbb{C}(G)$ then, $L_g g_j = g g_j$ where $[L_g]$ is the matrix with respect to g .

We see that

$$\begin{aligned} [L_g]_{ij} &= \begin{cases} 1 & ; g_i = g g_j \\ 0 & ; otherwise \end{cases} \\ &= \begin{cases} 1 & ; g = g_i g_j^{-1} \\ 0 & ; otherwise \end{cases} \end{aligned}$$

In particular,

$$[L_g]_{ii} = \begin{cases} 1 & ; g = 1 \\ 0 & ; otherwise \end{cases}$$

Therefore,

$$\chi_L(g) = Tr(L_g) = \begin{cases} |G| & ; g = 1 \\ 0 & ; g \neq 1 \end{cases}$$

Hence proved. \square

Theorem 2.5.5. *Let L be a regular representation of G . Then, the following decomposition holds:*

$$L \sim d_1 \phi_1 \oplus d_2 \phi_2 \oplus \dots \oplus d_s \phi_s.$$

Proof. We know that, $\psi \sim m_1 \phi_1 \oplus m_2 \phi_2 \oplus \dots \oplus m_s \phi_s$, where $m_i = \langle \chi_\psi, \chi_{\phi_i} \rangle$. Here, $\psi = L$. So,

$$\begin{aligned} \langle \chi_L, \chi_{\phi_i} \rangle &= \frac{1}{O(G)} \sum_{g \in G} \chi_L(g) \overline{\chi_{\phi_i}(g)} \\ &= \frac{1}{O(G)} O(G) \times \overline{\chi_{\phi_i}(1)} \\ &= 1 \times deg \phi_i \\ &= d_i. \end{aligned}$$

Therefore,

$$L \sim d_1 \phi_1 \oplus \dots \oplus d_s \phi_s.$$

\square

Corollary 2.5.1. *The formula $O(G) = d_1^2 + d_2^2 + \dots + d_s^2$ holds.*

Proof. Since by previous theorem,

$$L \sim d_1\phi_1 \oplus \dots \oplus d_s\phi_s$$

This implies that

$$\chi_L = d_1\chi_1 + \dots + d_s\chi_s$$

and hence,

$$\chi_L = d_1\chi_1(1) + \dots + d_s\chi_s(1) = d_1^2 + \dots + d_s^2$$

and we know that $\chi_L(1) = O(G)$ and hence the result. \square

Theorem 2.5.6. *The set $B = \{\sqrt{d_k}\phi_{ij}(k) | 1 \leq k \leq s, 1 \leq i, j \leq d_k\}$ is an orthonormal basis for $\mathcal{A}(G)$.*

Proof. We know that B is an orthonormal set by Schur's Orthogonality Relations.

Now, since $|B| = d_1^2 + \dots + d_s^2 = |G| = \dim(\mathcal{A}(G))$ as $\{\sqrt{d}\phi_{ij} | 1 \leq i, j \leq d\}$ is the set of d^2 functions. Hence, this shows that B is an orthonormal basis for $\mathcal{A}(G)$. \square

Theorem 2.5.7. *The set $\{\chi_1, \chi_2, \dots, \chi_s\}$ is an orthonormal basis for $Z(\mathcal{A}(G))$.*

Proof. We know that the irreducible characters form an orthonormal set of class functions. Now we need to show that this set forms a basis that is, it spans $Z(\mathcal{A}(G))$.

Let $f \in Z(\mathcal{A}(G))$. Now, f can be written as

$$f = \sum_{i,j,k} c_{ij}^{(k)} \phi_{ij}(k)$$

for some $c_{ij}^{(k)} \in \mathbb{C}$; $1 \leq k \leq s, 1 \leq i, j \leq d_k$. Since f is a class function, for

any $x \in G$, we have

$$\begin{aligned}
f(x) &= \frac{1}{O(G)} \sum_{g \in G} f(g^{-1}xg) \\
&= \frac{1}{O(G)} \sum_{g \in G} \sum_{i,j,k} c_{ij}^{(k)} \phi_{ij}^{(k)}(g^{-1}xg) \\
&= \sum_{i,j,k} c_{ij}^{(k)} \left[\frac{1}{O(G)} \sum_{g \in G} \phi_{g^{-1}x}^{(k)} \phi_x^{(k)} \phi_g^{(k)} \right]_{ij} \\
&= \sum_{i,j,k} c_{ij}^{(k)} \left[(\phi_x^{(k)})^\# \right]_{ij} \\
&= \sum_{i,j,k} c_{ij}^{(k)} \frac{Tr(\phi_x^{(k)})}{deg \phi^{(k)}} I_{ij} \\
&= \sum_{i,k} c_{ii}^{(k)} \frac{1}{d_k} \chi_k(x).
\end{aligned}$$

Therefore, $f = \sum_{i,k} c_{ii}^{(k)} \frac{1}{d_k} \chi_k$. This implies, $f \in \text{span}\{\chi_1, \dots, \chi_s\}$ and $\{\chi_1, \dots, \chi_s\}$ is an orthonormal set. Hence, $\{\chi_1, \dots, \chi_s\}$ is an orthonormal basis for $Z(\mathcal{A}(G))$. \square

Corollary 2.5.2. *Number of equivalence classes of irreps of G is equal to the number of conjugacy classes of G .*

Proof. Using above theorem, $s = \dim(Z(\mathcal{A}(G))) = |Cl(G)|$. \square

Theorem 2.5.8. *A finite group G is abelian iff it has $|G|$ equivalence classes of irreps of G .*

Proof. A finite group G is abelian iff $|G| = |Cl(G)| =$ number of equivalence classes of irreps of G . \square

Example 2.5.9. Irreducible Representations of $\frac{\mathbb{Z}}{n\mathbb{Z}}$

Let $\omega_n = \exp 2\pi i/n$. Define $\chi_k : \frac{\mathbb{Z}}{n\mathbb{Z}} \rightarrow \mathbb{C}^*$ by,

$$\chi_k([m]) = \omega_n^{km} ; 0 \leq k \leq n-1$$

Then, $\chi_0, \dots, \chi_{n-1}$ are distinct irreps of $\frac{\mathbb{Z}}{n\mathbb{Z}}$.

Following is the character table of $\frac{\mathbb{Z}}{n\mathbb{Z}}$:

	[0]	[1]	[2]	[3]
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	i	-1	i
χ_4	1	$-i$	-1	i

Theorem 2.5.10. Second Orthogonality Relations

Let C, C' be the conjugacy classes of G and let $g \in C, h \in C'$ then,

$$\sum_{i=1}^s \chi_i(g) \overline{\chi_i(h)} = \begin{cases} |G|/|C| & ; C = C' \\ 0 & ; C \neq C' \end{cases}$$

Consequently, columns of character table are orthogonal and hence character table is invertible.

Proof. Consider $\delta_{C'}$.

$$\begin{aligned} \delta_{C'}(g) &= \sum_{i=1}^s \langle \delta_{C'}, \chi_i \rangle \chi_i(g) \\ &= \sum_{i=1}^s \frac{1}{O(G)} \sum_{x \in G} \delta_{C'}(x) \overline{\chi_i(x)} \chi_i(g) \\ &= \sum_{i=1}^s \frac{1}{O(G)} \sum_{x \in C'} \overline{\chi_i(x)} \chi_i(g) \\ &= \frac{|C|}{|G|} \sum_{i=1}^s \chi_i(g) \overline{\chi_i(h)} \end{aligned}$$

(Note that LHS = 1 if $g \in C'$). So,

$$\sum_{i=1}^s \chi_i(g) \overline{\chi_i(h)} = \begin{cases} |G|/|C| & ; C = C' \\ 0 & ; C \neq C' \end{cases}$$

Therefore, columns of Character table form an orthogonal set of non zero vectors and hence are linearly independent. So, Character table is invertible. \square

2.6 Representation of Finite Abelian Groups

Proposition 2.6.1. *Let G_1, G_2 be abelian groups and suppose ρ_1, \dots, ρ_m and ϕ_1, \dots, ϕ_n be irreps of G_1 and G_2 respectively. Let $m=|G_1|$ and $n=|G_2|$. Then, $\alpha_{ij} : G_1 \times G_2 \rightarrow \mathbb{C}^*$; $1 \leq i \leq m$ and $1 \leq j \leq n$ is given by:*

$$\alpha_{ij}(g_1 g_2) = \rho_i(g_1) \phi_j(g_2)$$

form complete set of irreps of $G_1 \times G_2$.

Proof. First, we need to check that α_{ij} are homomorphisms.

$$\begin{aligned} \alpha_{ij}(g_1, g_2) \alpha_{ij}(g'_1, g'_2) &= \rho_i(g_1) \phi_j(g_2) \rho_i(g'_1) \phi_j(g'_2) \\ &= \rho_i(g_1 g'_1) \phi_j(g_2 g'_2) \\ &= \alpha_{ij}(g_1 g'_1, g_2 g'_2) \\ &= \alpha_{ij}((g_1, g_2)(g'_1, g'_2)). \end{aligned}$$

Next, we check if $\alpha_{ij} = \alpha_{kl}$ then $i=k$ and $j=l$.

Let $\alpha_{ij} = \alpha_{kl}$. Then,

$$\rho_i(g) = \alpha_{ij}(g, 1) = \alpha_{kl}(g, 1) = \rho_k(g).$$

This shows that $i=k$. Similarly, $j=l$.

Now, since, $G_1 \times G_2$ has $|G_1 \times G_2| = mn$ distinct irreps. Therefore, α_{ij} are all of them, where, $1 \leq i \leq m$ and $1 \leq j \leq n$. \square

Example 2.6.2. Character table of Klein's Four group $\mathbb{K}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
Character table of $\mathbb{Z}/2\mathbb{Z}$:

	$[0]$	$[1]$
χ_1	1	1
χ_2	1	-1

Now, character table of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ using above proposition is :

	$([0],[0])$	$([0],[1])$	$([1],[0])$	$([1],[1])$
α_{11}	1	1	1	1
α_{12}	1	-1	1	-1
α_{21}	1	1	-1	-1
α_{22}	1	-1	-1	1

Chapter 3

Fourier Analysis on Groups

3.1 Fourier Analysis on Finite Groups

Definition 3.1.1. A function $f : \mathbb{Z} \rightarrow \mathbb{C}$ is said to be periodic with period n if $f(x) = f(x+n) \forall x \in \mathbb{Z}$.

From the above definition, we can conclude that these functions are in one-one correspondence with the elements of $\mathcal{A}(\mathbb{Z}/n\mathbb{Z})$ that is, the functions $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}$. In other words, these functions are constant on the residue classes modulo n . Now, as we know from previous chapter that the irreducible characters form a basis for $\mathcal{A}(\mathbb{Z}/n\mathbb{Z})$ that is, if f is a function in $\mathcal{A}(\mathbb{Z}/n\mathbb{Z})$ then,

$$f = \langle f, \chi_0 \rangle \chi_0 + \dots + \langle f, \chi_{n-1} \rangle \chi_{n-1}$$

where $\chi_k([m]) = e^{2\pi i k m / n}$.

Definition 3.1.2. Fourier Transform: Let $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}$. Then, the Fourier transform of f is defined by,

$$\hat{f} : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}$$

as

$$\hat{f}([m]) = n \langle f, \chi_m \rangle = \sum_{k=0}^{n-1} f([k]) e^{-2\pi i k m / n}.$$

Proposition 3.1.3. The Fourier transform is invertible. In simple words,

$$f = \frac{1}{n} \sum_{k=0}^{n-1} \hat{f}([k]) \chi_k.$$

Proposition 3.1.4. *The class functions form the center of $\mathcal{A}(G)$.*

Proof. First, Let $f : G \rightarrow \mathbb{C}$ be a class function. Then to show that,

$$a * f = f * a \quad \forall a \in \mathcal{A}(G).$$

Let $a \in \mathcal{A}(G)$. Then,

$$a * f(x) = \sum_{y \in G} a(xy^{-1})f(y) = \sum_{y \in G} a(xy^{-1})f(xy x^{-1})$$

since f is a class function. Now, let $xy^{-1} = z$. This implies that,

$$a * f(x) = \sum_{z \in G} a(z)f(xz^{-1}) = \sum_{z \in G} f(xz^{-1})a(z) = f * a(x).$$

Therefore, $a * f = f * a$. Now we show the converse part. That is, let $f \in Z(\mathcal{A}(G))$. Then, to show that f is a class function. So, the claim is to prove

$$f(gh) = f(hg) \quad \forall g, h \in G.$$

Now

$$\begin{aligned} f(gh) &= \sum_{y \in G} f(gy^{-1})\delta_{h^{-1}}(y) \\ &= f * \delta_{h^{-1}}(g) \\ &= \delta_{h^{-1}} * f(g) \\ &= \sum_{y \in G} \delta_{h^{-1}}(gy^{-1})f(y) \\ &= f(hg) \end{aligned}$$

since, $\delta_{h^{-1}}(gy^{-1})$ is non zero iff $gy^{-1} = h^{-1}$ that is, $y=hg$. Now, by claim we have, $f(ghg^{-1}) = f(h)$. Hence f is a class function. □

3.2 Fourier Analysis o finite abelian groups

Note that if G is abelian then, $\mathcal{A}(G) = Z(\mathcal{A}(G))$. That is, $\mathcal{A}(G)$ is a commutative ring.

Definition 3.2.1. Let G be a finite abelian group. Then, \hat{G} is the set of all irreducible characters $\chi : G \rightarrow \mathbb{C}^*$ known as the **dual group**.

Proposition 3.2.2. Let G be a finite abelian group. We define product on \hat{G} via pointwise multiplication that is, $(\chi.\theta) = \chi(g)\theta(g)$. Then, \hat{G} is an abelian group of order $|G|$ with respect to this binary operation.

Proof. Let $\chi, \theta \in \hat{G}$. Then,

$$\begin{aligned}\chi.\theta(g_1g_2) &= \chi(g_1g_2)\theta(g_1g_2) \\ &= \chi(g_1)\chi(g_2)\theta(g_1)\theta(g_2) \\ &= \chi(g_1)\theta(g_1)\chi(g_2)\theta(g_2) \\ &= (\chi.\theta)(g_1).(\chi.\theta)(g_2).\end{aligned}$$

This shows that \hat{G} is closed under pointwise product. Trivially, this product is associative and commutative. Identity is, $\chi_1(g) = 1 \forall g \in G$. Inverse is, $\chi^{-1}(g) = \chi(g)^{-1} = \overline{\chi(g)}$ since χ is unitary. Trivially, $\chi.\chi^{-1} = \chi_1$. And hence, \hat{G} is an abelian group. The number of irreducible characters of G is equal to $|G|$ where G is abelian. Therefore, $|\hat{G}| = |G|$. \square

Example 3.2.3. Let $G = \mathbb{Z}/n\mathbb{Z}$ and $\hat{G} = \{\chi_0, \dots, \chi_{n-1}\}$. Let $\chi_k([m]) = e^{2\pi i km/n}$. Then, $[k] \rightarrow \chi_k$ is a group isomorphism from $G \rightarrow \hat{G}$. So, $G \cong \hat{G}$.

Definition 3.2.4. Fourier Transform: Let $f : G \rightarrow \mathbb{C}$ be a complex valued function on a finite abelian group G . Then, the fourier transform $\hat{f} : G \rightarrow \mathbb{C}$ is defined by

$$\hat{f}(x) = |G|\langle f, \chi \rangle = \sum_{g \in G} f(g)\overline{\chi(g)}$$

where, $|G|\langle f, \chi \rangle$ are the complex numbers called the fourier coefficients of f .

Example 3.2.5. If $\chi, \theta \in \hat{G}$ then,

$$\hat{\chi}(\theta) = |G|\langle \chi, \theta \rangle = \begin{cases} |G| & ; \chi = \theta \\ 0 & ; \text{otherwise} \end{cases}$$

by orthogonality relations and so, $\hat{\chi} = |G|\delta_\chi$.

Theorem 3.2.6. If $f \in \mathcal{A}(G)$ then,

$$f = \frac{1}{O(G)} \sum_{x \in \hat{G}} \hat{f}(x)\chi.$$

Proof.

$$f = \sum_{\chi \in \hat{G}} \langle f, \chi \rangle \chi = \frac{1}{|G|} \sum_{\chi \in \hat{G}} |G| \langle f, \chi \rangle \chi = \frac{1}{|G|} \sum \hat{f}(\chi) \chi.$$

Hence proved. \square

Proposition 3.2.7. *The map $T : \mathcal{A}(G) \rightarrow \mathcal{A}(\hat{G})$ given as $Tf = \hat{f}$ is invertible linear transformation.*

Proof. Let $|G| = n$. $T(c_1f_1 + c_2f_2) = c_1\hat{f}_1 + c_2\hat{f}_2$. Now, consider

$$\begin{aligned} c_1\hat{f}_1 + c_2\hat{f}_2(\chi) &= n\langle c_1f_1 + c_2f_2, \chi \rangle \\ &= c_1n\langle f_1, \chi \rangle + c_2n\langle f_2, \chi \rangle \\ &= c_1\hat{f}_1(\chi) + c_2\hat{f}_2(\chi). \end{aligned}$$

This shows that $c_1\hat{f}_1 + c_2\hat{f}_2 = \hat{c_1f_1 + c_2f_2}$. Hence, T is linear. By previous theorem, T is injective and thus invertible since $\dim(\mathcal{A}(G)) = n = \dim(\mathcal{A}(\hat{G}))$. \square

Theorem 3.2.8. *The Fourier Transform satisfies $a \hat{*} b = \hat{a} \hat{*} \hat{b}$. Consequently, linear map $T : \mathcal{A}(G) \rightarrow \mathcal{A}(\hat{G})$ given by $Tf = \hat{f}$ provides a ring isomorphism between $(\mathcal{A}(G), +, *)$ and $(\mathcal{A}(\hat{G}), +, \cdot)$.*

Proof. We know by previous theorem that $T : \mathcal{A}(G) \rightarrow \mathcal{A}(\hat{G})$ is an isomorphism of vector spaces. Therefore, it suffices to show that it is a ring homomorphism. That is, we need to show that, $T(a * b) = Ta \cdot Tb$. In other words, we need to show,

$$a \hat{*} b = \hat{a} \hat{*} \hat{b}.$$

Let n be the order of G . Consider,

$$\begin{aligned} a \hat{*} b(\chi) &= n\langle a * b, \chi \rangle \\ &= n \frac{1}{n} \sum_{x \in G} (a * b)(x) \overline{\chi(x)} \\ &= \sum_{x \in G} \overline{\chi(x)} \sum_{y \in G} a(xy^{-1})b(y) \\ &= \sum_{y \in G} b(y) \sum_{x \in G} a(xy^{-1}) \overline{\chi(x)}. \end{aligned}$$

Let $z = xy^{-1}$. Therefore, the expression is now equal to

$$\begin{aligned} \sum_{y \in G} b(y) \sum_{z \in G} a(z) \overline{\chi(z y)} &= \sum_{y \in G} b(y) \overline{\chi(y)} \sum_{y \in G} b(y) \overline{\chi(y)} \\ &= n \langle a, \chi \rangle \cdot n \langle b, \chi \rangle \\ &= \hat{a} \cdot \hat{b}. \end{aligned}$$

Hence proved. □

Example 3.2.9. *The periodic functions on \mathbb{Z} : Let $f, g : \mathbb{Z} \rightarrow \mathbb{C}$ have period n . Their convolution is defined as,*

$$(f * g)(m) = \sum_{k=0}^{n-1} f(m-k)g(k).$$

The fourier transform is then,

$$\hat{f}(m) = \sum_{k=0}^{n-1} f(k) e^{-2\pi i k m / n}.$$

Hence, following is the fourier inversion theorem.

$$f(m) = \frac{1}{n} \sum_{k=0}^{n-1} \hat{f}(k) e^{2\pi i k m / n}.$$

Now, we have a look at abelian case in a different way.

Suppose G is a finite abelian group with irreducible characters $\chi_1, \chi_2, \dots, \chi_n$. Then, to each function $f : G \rightarrow \mathbb{C}$, we associate its vector of fourier coefficients.

Define $T : \mathcal{A}(G) \rightarrow \mathbb{C}^n$ by

$$\begin{aligned} Tf &= (n \langle f, \chi_1 \rangle, n \langle f, \chi_2 \rangle, \dots, n \langle f, \chi_n \rangle) \\ &= (\hat{f}(\chi_1), \hat{f}(\chi_2), \dots, \hat{f}(\chi_n)). \end{aligned}$$

This shows that T is injective by fourier inversion theorem because we can recover \hat{f} and hence f from Tf . Also, T is linear and this implies that, T is a

vector space isomorphism since $\dim(\mathcal{A}(G)) = n$.

In fact, T is a ring isomorphism since

$$\begin{aligned} T(a * b) &= (a \hat{*} b(\chi_1), \dots, a \hat{*} b(\chi_n)) = (\hat{a}(\chi_1)\hat{b}(\chi_1), \dots, \hat{a}(\chi_n)\hat{b}(\chi_n)) \\ &= (\hat{a}(\chi_1), \dots, \hat{a}(\chi_n))(\hat{b}(\chi_1), \dots, \hat{b}(\chi_n)) = Ta.Tb. \end{aligned}$$

Earlier we saw that $a \hat{*} b = \hat{a}.\hat{b}$. Hence, we can conclude that, If G is a finite abelian group of order n . Then,

$$\mathcal{A}(G) \cong \mathbb{C}^n$$

where \mathbb{C}^n has structure of direct product of rings where multiplication is coordinate wise. So, for non abelian groups we must replace \mathbb{C} by matrix rings over \mathbb{C} .

Definition 3.2.10. Define the map

$$T : \mathcal{A}(G) \rightarrow M_{d_1}(\mathbb{C}) \times \dots \times M_{d_s}(\mathbb{C})$$

by

$$Tf = (\hat{f}(\phi^1), \dots, \hat{f}(\phi^s))$$

where

$$\hat{f}(\phi_k)_{ij} = n\langle f, \phi_{ij}^{(k)} \rangle = \sum_{g \in G} f(g)\overline{\phi_{ij}^k(g)}.$$

Here Tf is called the fourier transform of f . Therefore, $\hat{f}(\phi^k) = \sum_{g \in G} f(g)\overline{\phi_g^k}$.

Theorem 3.2.11. Fourier Inversion: Let $f : G \rightarrow \mathbb{C}$ be a complex valued function on G . Then,

$$f = \frac{1}{n} \sum_{i,j,k} d_k \hat{f}(\phi^k)_{ij} \phi_{ij}^k$$

where n is the order of G .

Proof. Since $\{\sqrt{d_k}\phi_{ij}^k\}$ is an orthonormal basis for $\mathcal{A}(G)$ and $f \in \mathcal{A}(G)$. Therefore,

$$\begin{aligned} f &= \sum_{i,j,k} \langle f, \sqrt{d_k}\phi_{ij}^k \rangle \sqrt{d_k}\phi_{ij}^k \\ &= \frac{1}{n} \sum_{i,j,k} d_k n \langle f, \phi_{ij}^k \rangle \phi_{ij}^k \\ &= \frac{1}{n} \sum_{i,j,k} d_k \hat{f}(\phi^k)_{ij} \phi_{ij}^k. \end{aligned}$$

□

Proposition 3.2.12. *The map $T : \mathcal{A}(G) \rightarrow M_{d_1}(\mathbb{C}) \times \dots \times M_{d_s}(\mathbb{C})$ is a vector space isomorphism.*

Proof. First we show that T is linear. That means we need to show that $T(c_1f_1 + c_2f_2) = c_1\hat{f}_1 + c_2\hat{f}_2$, that is, to show $c_1f_1 + c_2f_2(\phi^k) = c_1\hat{f}_1(\phi^k) + c_2\hat{f}_2(\phi^k)$. Now, for $1 \leq k \leq s$,

$$\begin{aligned} c_1f_1 + c_2f_2(\phi^k) &= \sum_{g \in G} (c_1f_1 + c_2f_2)(g)\overline{\phi_g^k} \\ &= c_1 \sum_{g \in G} f_1(g)\overline{\phi_g^k} + c_2 \sum_{g \in G} f_2(g)\overline{\phi_g^k} \\ &= c_1\hat{f}_1(\phi^k) + c_2\hat{f}_2(\phi^k). \end{aligned}$$

By fourier inversion theorem, T is injective since

$$\dim(\mathcal{A}(G)) = |G| = d_1^2 + \dots + d_s^2 = \dim(M_{d_1}(\mathbb{C}) \times \dots \times M_{d_s}(\mathbb{C})).$$

Therefore, T is isomorphism. \square

Theorem 3.2.13. Wedderburn theorem *The Fourier transform*

$$T : \mathcal{A}(G) \rightarrow M_{d_1}(\mathbb{C}) \times \dots \times M_{d_s}(\mathbb{C})$$

is an isomorphism of rings.

Proof. We know that T is an isomorphism of vector spaces. So, to show ring isomorphism we just need to show:

$$T(a * b) = Ta.Tb$$

That is, we need to show,

$$a * b(\phi^k) = \hat{a}(\phi^k) \cdot \hat{b}(\phi^k) \text{ for } 1 \leq k \leq s$$

Consider,

$$\begin{aligned} a * b(\phi^k) &= \sum_{x \in G} (a * b)(x)\overline{\phi_x^k} \\ &= \sum_{x \in G} \overline{\phi_x^k} \sum_{y \in G} a(xy^{-1})b(y) \\ &= \sum_{y \in G} b(y) \sum_{x \in G} a(xy^{-1})\overline{\phi_x^k}. \end{aligned}$$

Let $z = xy^{-1} \Rightarrow x = zy$. Therefore,

$$\begin{aligned}
 a \hat{*} b(\phi^k) &= \sum_{y \in G} b(y) \sum_{z \in G} a(z) \overline{\phi_{zy}^k} \\
 &= \sum_{y \in G} b(y) \sum_{z \in G} a(z) \overline{\phi_z^k} \cdot \overline{\phi_y^k} \\
 &= \sum_{z \in G} a(z) \overline{\phi_z^k} \sum_{y \in G} b(y) \overline{\phi_y^k} \\
 &= \hat{a}(\phi^k) \cdot \hat{b}(\phi^k).
 \end{aligned}$$

Hence it is a ring isomorphism. □

Example 3.2.14. Representation theory of S_n can be used to analyse voting. Here is an example of Diaconis. Suppose in an election, each voter needs to rank n candidates on a ballot. Let the candidates be $\{1, 2, \dots, n\}$. Then to each ballot, we correspond a permutation $\sigma \in S_n$. For example for $n=3$, let the ballot ranks the candidates in 3 1 2 order then the corresponding permutation is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

then election corresponds to the function $f : S_n \rightarrow \mathbb{N}$ where $f(\sigma)$ is the number of people whose ballot corresponds to the permutation σ .

Chapter 4

Burnside's theorem

4.1 Number Theory

Definition 4.1.1. Algebraic number: A complex number 'k' is called algebraic number if it is a root of a polynomial with integer coefficients. The numbers that are not algebraic are called transcendental.

Example 4.1.2. Consider the polynomial $2z-1$. We see that, $1/2$ is a root of this polynomial. So, $1/2$ is algebraic number. Similarly, consider z^2 to be another polynomial. Note that $\sqrt{2}$ is a root of this polynomial and hence is algebraic. On the other hand, π and e are transcendental numbers since they are not the roots of any polynomial with integer coefficients.

Definition 4.1.3. Algebraic integer: A complex number 'c' is called an algebraic integer if it is a root of a monic polynomial with integer coefficients. In other words, c is an algebraic integer if there exists a polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$$

where $a_0, a_2, \dots, a_{n-1} \in \mathbb{Z}$ and $p(c)=0$.

Example 4.1.4. Following the above definition, we can say that any n th root of unity is an algebraic integer. Consider the polynomial $p(z) = z^n - \alpha$ then n th root of α is an algebraic integer. Also, the characteristic polynomial of any square matrix A is a monic polynomial. Hence, each eigen value of the matrix A is an algebraic integer.

Proposition 4.1.5. A rational number is an algebraic integer iff it is an integer.

Proof. Let r be a rational number. Then, $r=p/q$ where $p, q \in \mathbb{Z}$, $q \neq 0$, $\gcd(p,q)=1$. Let r is an algebraic integer. Then, it is root of a polynomial with integer coefficients say

$$z^k + a_{k-1}z^{k-1} + \dots + a_0.$$

Then,

$$0 = (p/q)^k + a_{k-1}(p/q)^{k-1} + \dots + a_0.$$

This implies,

$$0 = p^k + na_{k-1}p^{k-1} + \dots + a_0q^k$$

or,

$$m^k = -n(a_{k-1}m^{k-1} + \dots + a_0q^{k-1}).$$

So, $q|p^k$. Now, as $\gcd(p,q)=1$. This shows that n has to be 1 or -1. Therefore, $r = p \in \mathbb{Z}$. \square

Lemma 4.1.6. *An element $y \in \mathbb{C}$ is an algebraic integer iff there exists $y_1, y_2, \dots, y_t \in \mathbb{C}$ not all zero such that*

$$yy_i = \sum_{j=1}^t a_{ij}y_j$$

where $a_{ij} \in \mathbb{Z} \forall 1 \leq i \leq t$.

Proof. To prove the forward part, first let y be an algebraic integer. Then, there exists a polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$ such that y is root of $p(z)$. Let $y_i = y^{i-1}$ for $1 \leq i \leq n$. Then, for $1 \leq i \leq n-2$, we have,

$$yy_i = yy^{i-1} = y^i = y_{i+1}$$

and

$$yy_n = y_{n+1} = y^n = -a_0 - \dots - a_{n-1}y^{n-1}.$$

Hence, the forward part is satisfied. Next, we prove the converse. Let $A = (a_{ij})$ and $Y = [y_1, y_2, \dots, y_t]' \in \mathbb{C}^t$ then, $[AY]_i = \sum_{j=1}^t a_{ij}y_j = yy_i = y[Y]_i$. This implies that $AY=yY$ since $Y \neq 0$ (by assumption). Hence, y is an eigen value of $t \times t$ integer matrix A . Hence, y is an algebraic integer. \square

Corollary 4.1.1. *The set \mathbb{A} of algebraic integers form a subring of \mathbb{C} . In particular, sum and product of algebraic integers is algebraic.*

Proof. Note that if $a \in \mathbb{A}$ then, $-a \in \mathbb{A}$. Let $x, x' \in \mathbb{A}$. We choose $x_1, x_2, \dots, x_t \in \mathbb{C}^t$ (not all zero) and $x'_1, \dots, x'_t \in \mathbb{C}^t$ (not all zero) such that

$$xx_i = \sum_{j=1}^t a_{ij}x_j, x'x'_k = \sum_{j=1}^s b_{kj}y'_j$$

then,

$$(x + x')x_i x'_k = xx_i x'_k + x'x'_k x_i = \sum_{j=1}^t a_{ij}x_j x'_k + \sum_{j=1}^s b_{kj}x'_j x_i$$

which is an integral linear combination of $x_j x'_l$. This shows that $x + x' \in \mathbb{A}$. Similarly, $xx'x_i x'_k = xx_i x'x'_k$ is integral combination of $x_j x'_l$. So, $xx' \in \mathbb{A}$. Therefore, \mathbb{A} is a subring of \mathbb{C} . \square

Note 4.1.7. If k is an algebraic integer then \bar{k} is also an algebraic integer.

4.2 The Dimension Theorem

Corollary 4.2.1. Let χ be a character of finite group G . Then, $\chi(g)$ is an algebraic integer $\forall g \in G$.

Proof. Proof: Let $\phi : G \rightarrow GL_m(\mathbb{C})$ be a representation with character χ . Let n be order of G . Then, $g^n = 1$. This implies $\phi_g^n = \mathbb{I}$. Now ϕ_g is diagonalizable since every representation is equivalent to a unitary representation and ϕ_g has eigen values $\lambda_1, \lambda_2, \dots, \lambda_m$ which are n th roots of unity. So, eigen values of ϕ_g are algebraic integers. Now, $\chi_g = \text{Tr}(\phi_g) = \lambda_1 + \dots + \lambda_m$ and set of all algebraic integers form a ring. Therefore, χ_g is an algebraic integer. Hence, we can conclude that $\chi_\phi(g)$ is a sum of m n th roots of unity. \square

Theorem 4.2.1. Let ϕ be an irrep of finite group G of degree d . Let $g \in G$ and let s be the size of conjugacy class of g . Then, $s\chi_\phi(g)/d$ is an algebraic integer.

Proof. Let C_1, C_2, \dots, C_n be the conjugacy classes of G . Let $s_i = |C_i|$ and let χ_i denote χ_ϕ on C_i . We need to show that $s_i \chi_i / d$ is an algebraic integer for all i . For that, we consider the operator

$$T_i = \sum_{x \in C_i} \phi_x.$$

Our first claim is, $T_i = \frac{s_i}{d}\chi_i \cdot I$.

The proof of this claim goes as follows. We make use of Schur's lemma to prove this claim. First, consider $\phi_g T_i \phi_{g^{-1}}$.

$$\begin{aligned} \phi_g T_i \phi_{g^{-1}} &= \sum_{x \in C_i} \phi_g \phi_x \phi_{g^{-1}} \\ &= \sum_{x \in C_i} \phi_{g x g^{-1}} \\ &= \sum_{y \in C_i} \phi_y \\ &= T_i \end{aligned}$$

since C_i is conjugacy class of G . Now, by Schur's lemma, $T_i = \lambda I$ for some $\lambda \in \mathbb{C}$. Now,

$$\text{Tr}(\lambda I) = \text{Tr}(T_i) = \sum_{x \in C_i} \text{Tr}(\phi_x) = \sum_{x \in C_i} \chi_\phi(x) = |C_i| \chi_i = s_i \chi_i.$$

Also, we know that, $\text{Tr}(\lambda I) = d\lambda$. So, we get $\lambda = s_i \chi_i / d$. Hence, the claim. Our next claim is to show

$$T_i T_j = \sum_{k=1}^n a_{ijk} T_k \text{ for some } a_{ijk} \in \mathbb{Z}.$$

Proof goes as follows:

$$T_i T_j = \sum_{x \in C_i} \phi_x \sum_{y \in C_j} \phi_y = \sum_{x \in C_i, y \in C_j} \phi_{xy} = \sum_{g \in G} a_{ijg} \phi_g$$

where $a_{ijg} \in \mathbb{Z}$ is the number of ways to write $g=xy$ with $x \in C_i$ and $y \in C_j$. Now, the subclaim is to show that a_{ijg} depends only on conjugacy class of g . So, to prove subclaim, we start by considering $X_g = \{(x, y) \in C_i \times C_j | xy = g\}$. From this, we can conclude that $a_{ijg} = |X_g|$. Let g' be conjugate to g then we show that number of ways to write $g=xy$ is same as number of ways of writing $g'=xy$ with $x \in C_i$ and $y \in C_j$ that is, we show, $|X_g| = |X_{g'}|$. Let $g' = k g k^{-1}$ since, g' is conjugate to g . Now define the bijection $\psi : X_g \rightarrow X_{g'}$ by $\psi(x, y) = (k x k^{-1}, k y k^{-1})$ where $k x k^{-1} \in C_i$ and $k y k^{-1} \in C_j$ and

$kxk^{-1}kyk^{-1} = kxyk^{-1} = k g k^{-1} = g'$ that is, $(kxk^{-1}, kyk^{-1}) \in X_{g'}$. This implies that, $\psi(x, y) \in X_{g'}$. Hence, ψ has inverse $\rho : X_{g'} \rightarrow X_g$ given by

$$\rho(x', y') = (k^{-1}x'k, k^{-1}y'k)$$

and hence ψ is a bijection and so, $|X_g| = |X_{g'}|$. Therefore, the subclaim is proved. Now getting back to proof of second claim, let a_{ijk} be value of a_{ijg} with $g \in C_k$ then,

$$\begin{aligned} \sum_{g \in G} a_{ijg} \phi_g &= \sum_{k=1}^n \sum_{g \in C_k} a_{ijg} \phi_g \\ &= \sum_{k=1}^n a_{ijk} \sum_{g \in C_k} \phi_g \\ &= \sum_{k=1}^n a_{ijk} T_k. \end{aligned}$$

So, the second claim is done. Now, as $T_i T_j = \sum a_{ijk} T_k$. This implies,

$$\left(\frac{s_i}{d} \chi_i\right) \left(\frac{s_j}{d} \chi_j\right) = \sum_{k=1}^n a_{ijk} \left(\frac{s_k}{d} \chi_k\right).$$

So, $s_i \chi_i / d$ is an algebraic integer by the equivalent condition for a complex number to be an algebraic integer stated in the lemma done before. \square

Theorem 4.2.2. The Dimension Theorem: *Let ϕ be an irrep of G of degree d . Then, $d|O(G)$.*

Proof. By first Orthogonality Relations,

$$1 = \langle \chi_\phi, \chi_\phi \rangle = \frac{1}{O(G)} \sum_{x \in G} \chi_\phi(x) \overline{\chi_\phi(x)}.$$

This implies that,

$$\frac{O(G)}{d} = \sum_{x \in G} \frac{\chi_\phi(x) \overline{\chi_\phi(x)}}{d}.$$

Now let C_1, C_2, \dots, C_n be conjugacy classes of G and let χ_i denote value of χ_ϕ on C_i . Let $s_i = |C_i|$.

$$\begin{aligned}
\frac{|G|}{d} &= \sum_{i=1}^n \sum_{g \in C_i} \frac{\chi_\phi(g)}{d} \overline{\chi_\phi(g)} \\
&= \sum_{i=1}^n \sum_{g \in C_i} \left(\frac{1}{d} \chi_i\right) \overline{\chi_i} \\
&= \sum_{i=1}^n \left(\frac{s_i}{d} \chi_i\right) \overline{\chi_i}.
\end{aligned}$$

But by previous theorem, we know that $s_i \chi_i / d$ is an algebraic integer. And also, $\overline{\chi_i}$ is an algebraic integer. Now since, algebraic integer forms ring, this shows that $|G|/d$ is algebraic integer. Thus, it is an integer. So, $d|O(G)$. Hence proved. \square

Corollary 4.2.2. *Let p be prime and let G be a group of order p^2 . Then, G is abelian.*

Proof. Let d_1, d_2, \dots, d_n be degrees of irreps of G . Then, d_i can be 1, p or p^2 . Since trivial representation has degree 1 and

$$p^2 = |G| = d_1^2 + \dots + d_n^2.$$

This implies that $d_i = 1 \forall i$. Hence, G is abelian. \square

Definition 4.2.3. *Let G be a group. Let G' be the commutator subgroup of G . Then, G' is generated by all elements of the form $g^{-1}h^{-1}gh$ where $g, h \in G$. The commutator subgroup of G is a normal subgroup and G/G' is abelian and if N is some other normal subgroup of G such that G/N is abelian then, $G' \subseteq N$.*

Lemma 4.2.4. *Let G be a finite group. Then, the number of degree one representations of G divides $|G|$. More precisely, if G' is the commutator subgroup of G , there is a bijection between the degree one representations of G and the irreps of abelian group G/G' . Hence, G has $|G/G'| = [G : G']$ degree one representations.*

Proof. Proof: Consider the canonical projection $\pi : G \rightarrow G/G'$ defined by $\pi(g) = gG'$. If $\psi : G/G' \rightarrow \mathbb{C}^*$ is an irreducible representation then, $\psi\pi : G \rightarrow \mathbb{C}^*$ is a degree one representation. We need to show that every degree

one representation is of this form. Let $\rho : G \rightarrow \mathbb{C}^*$ be degree one representation. Then, $\text{Im}\rho \cong G/\text{Ker}\rho$ is abelian. Therefore, $G' \subseteq \text{Ker}\rho$. We define $\psi : G/G' \rightarrow \mathbb{C}^*$ by $\psi(gG') = \rho(g)$. If we take $gG' = hG'$ then, $h^{-1}g \in G' \subseteq \text{Ker}\rho$ and so, $\rho(h^{-1}g) = 1$. Thus, $\rho(h) = \rho(g)$. This implies $\psi(gG') = \psi(hG')$. So, this map is well defined. Next, we show map to be homomorphism as follows. $\psi(gG'hG') = \psi(ghG') = \rho(gh) = \rho(g)\rho(h) = \psi(gG')\psi(hG')$. Hence, map is homomorphism. Now, note that $\rho(g) = \psi\pi(g) = \psi(gG') \forall g \in G$. This shows $\rho = \psi\pi$. So, there exists bijection between degree one representations of G and irreps of abelian group G/G' . Now, irreducible representations of abelian group $G/G' = |G/G'| = |G|/|G'| = \text{degree one representations}$. Hence, proved. \square

Corollary 4.2.3. *Let p and q be primes with $p < q$ and $p \nmid q - 1$ (or $q \not\equiv 1 \pmod{p}$). Then, any group G of order pq is abelian.*

Proof. Let d_1, d_2, \dots, d_s be degrees of irreps of G . Since, $d_i | O(G)$ and $p < q$ and $pq = |G| = d_1^2 + \dots + d_s^2$. This implies, $d_i = 1$ or $p \forall i$. Let n be the number of degree p representations of G and let m be the number of degree one representations of G . Then, $pq = m + np^2$. Now, since by previous lemma, number of degree one representations of G divides order of G . So, $m | O(G)$. Since, there always exists a trivial representation therefore, $m \geq 1$. Now note that, $p | pq$ that is, $p | m + np^2 \Rightarrow m + np^2 = kp$ or $m = kp - np^2$ that is, $m = p(k - np)$. Hence, $p | m$. This shows that $m = p$ or $m = pq$. Note that $m \neq q$ since, $p \nmid q$. If $m = p$, $pq = p + np^2 = p(1 + np^2)$ that is, $q = 1 + np^2$. This means, $p | q - 1$ which is a contradiction. Therefore, $m = pq$. Hence, all irreducible representations of G have degree one. So, G is abelian. \square

4.3 The Burnside's Theorem

Recall that if we have a group of order n and a representation $\phi : G \rightarrow GL_d(\mathbb{C})$. Then, $\chi_\phi(g)$ is sum of d n th roots of unity.

Lemma 4.3.1. *Let $\lambda_1, \lambda_2, \dots, \lambda_d$ be n th roots of unity. Then,*

$$|\lambda_1 + \dots + \lambda_d| \leq d$$

and equality holds iff all λ_i 's are equal.

Proof. Let v, w be two vectors in \mathbb{R}^2 . Then,

$$\|v + w\|^2 = \|v\|^2 + 2\langle v, w \rangle + \|w\|^2 = \|v\|^2 + 2\|v\|\|w\|\cos\theta + \|w\|^2$$

where θ is the angle between v and w . Now $\cos\theta = 1$ iff $\theta = 0$. That is, $\|v + w\| \leq \|v\| + \|w\|$ and $\|v + w\| = \|v\| + \|w\|$ iff $v = \lambda w$ or $w = \lambda v$ for some $\lambda \geq 0$ (since, $\cos\theta = 1$ iff $\theta = 0$ that is, v and w are scalar multiples of each other). Therefore, by induction,

$$\|v_1 + v_2 + \dots + v_n\| \leq \|v_1\| + \dots + \|v_n\|$$

and holds equality iff v_i are non negative scalar multiples of a common vector. If we identify complex numbers with vectors in \mathbb{R}^2 then, we have,

$$|\lambda_1 + \dots + \lambda_d| \leq |\lambda_1| + \dots + |\lambda_d|$$

with equality iff λ_i are non negative scalar multiples of some complex number z . But $|\lambda_1| = \dots = |\lambda_d| = 1$. This shows that $\lambda_1 = \lambda_2 = \dots = \lambda_d$ and $|\lambda_1 + \dots + \lambda_d| \leq d$. \square

Let $\omega_n = e^{2\pi i/n}$. Then, $Q[\omega_n]$ denotes smallest subfield \mathbb{F} of \mathbb{C} containing ω_n .

Lemma 4.3.2. *The field $Q[\omega_n]$ has dimension $\phi(n)$ as a Q -vector space.*

Notation: Let $\Gamma = \text{Gal}(Q[\omega_n] : Q)$ be the Galois group of $Q[\omega_n]$ over Q . That is, Γ is the group of all field automorphisms $\sigma : Q[\omega_n] \rightarrow Q[\omega_n]$ where $Q[\omega_n] = \{f(\omega_n) | f(x) \in Q[x]\}$ such that $\sigma(r) = r \forall r \in Q$. Note that $|\Gamma| = \phi(n)$ since, $\dim Q[\omega_n] = \phi(n)$ as a Q -vector space.

Fact: If $p(z)$ is a polynomial with rational coefficients then, Γ permutes the roots of p in $Q[\omega_n]$.

Lemma 4.3.3. *Let $p(z)$ be a polynomial with rational coefficients and suppose that $\alpha \in Q[\omega_n]$ is a root of p . Then, $\sigma(\alpha)$ is also root of $p \forall \sigma \in \Gamma$.*

Proof. Proof: Consider the polynomial $p(z) = a_k z^k + a_{k-1} z^{k-1} + \dots + a_0$ with $a_i \in Q$. Then,

$$\begin{aligned} p(\sigma(\alpha)) &= a_k \sigma(\alpha)^k + a_{k-1} \sigma(\alpha)^{k-1} + \dots + a_0 \\ &= \sigma(a_k \alpha^k + a_{k-1} \alpha^{k-1} + \dots + a_0) \\ &= \sigma(0) \\ &= 0 \text{ since, } \sigma(a_i) \\ &= a_i \forall i. \end{aligned}$$

□

Corollary 4.3.1. *Let α be an n th root of unity. Then, $\sigma(\alpha)$ is also an n th root of unity $\forall \sigma \in \Gamma$.*

Remark 4.3.4. Γ is isomorphic to $\mathbb{Z}/n\mathbb{Z}^*$.

Corollary 4.3.2. *Let $\alpha \in Q[\omega_n]$ be an algebraic integer and suppose that $\sigma \in \Gamma$. Then, $\sigma(\alpha)$ is an algebraic integer.*

Proof. Let α be the root of monic polynomial p with integer coefficients. Then, so is $\sigma(\alpha)$ by previous lemma.

□

Theorem 4.3.5. *Let $\alpha \in Q[\omega_n]$. Then, $\sigma(\alpha) = \alpha \forall \alpha \in \Gamma$ if and only if $\alpha \in Q$.*

Corollary 4.3.3. *Let $\alpha \in Q[\omega_n]$. Then, $\prod_{\sigma \in \Gamma} \sigma(\alpha) \in Q$.*

Proof: Let $\tau \in \Gamma$, Then,

$$\tau \left(\prod_{\sigma \in \Gamma} \sigma(\alpha) \right) = \prod_{\sigma \in \Gamma} (\tau\sigma(\alpha)) = \prod_{\rho \in \Gamma} \rho(\alpha)$$

where $\rho = \tau\sigma$. Now, by previous theorem, $\prod_{\sigma \in \Gamma} \sigma(\alpha) \in Q$.

Theorem 4.3.6. *Let G be a group of order n . Let C be conjugacy class of $g \in G$. Let h be the order of C . Let $\phi : G \rightarrow GL_d(\mathbb{C})$ be irreducible representation of G and $\gcd(h,d)=1$. Then, either, (i) there exists $\lambda \in \mathbb{C}^*$ such that $\phi_g = \lambda I \forall g \in C$ or, (ii) $\chi_\phi(g) = 0 \forall g \in C$*

Proof. If $\phi_g = \lambda I$ for some $g \in C$ then, $\phi_x = \lambda I \forall x \in C$ since conjugating a scalar matrix does not change it. So, it suffices to prove that if $\phi_g \neq \lambda I$ for some $g \in C$ then, $\chi_\phi(g) = 0$. We know that $h\chi(g)/d$ is an algebraic integer and $\chi(g)$ is an algebraic integer. Also, $\gcd(h,d)=1$ so there exists integers k, j such that $kh+jd=1$. Now,

$$k \left(\frac{h\chi(g)}{d} \right) + j\chi(g) = \frac{\chi(g)}{d} = \alpha(\text{say})$$

is an algebraic integer since set of all algebraic integers form a ring.

Now, ϕ_g is diagonalizable with eigen values $\lambda_1, \lambda_2, \dots, \lambda_d$. And, $\text{chi}(g) =$

$\lambda_1 + \lambda_2 + \dots + \lambda_d$ that is, $\chi(g)$ is the sum of d n th roots of unity. Now $|\lambda_1 + \dots + \lambda_d| < d$ since all eigen values are not same as we assumed that $\phi_g \neq \lambda I$. Thus, $|\chi(g)| < d$ that is, $|\frac{\chi(g)}{d}| < 1$ that is, $|\alpha| < 1$. And hence, $|\sigma(\alpha)| = |\sigma(\frac{\chi(g)}{d})| < 1$.

Note that $q = \prod_{\sigma \in \Gamma} \sigma(\alpha)$ is an algebraic integer since α is an algebraic integer. It follows that

$$|q| = \left| \prod_{\sigma \in \Gamma} \sigma(\alpha) \right| = \prod_{\sigma \in \Gamma} |\sigma(\alpha)| < 1.$$

This shows that $q \in \mathbb{Q}$ since it is an algebraic integer less than 1. We know that a rational number is an algebraic integer iff it is an integer. This implies that $q \in \mathbb{Z}$ and $|q| < 1$. Thus, $q=0$. Therefore, $\sigma(\alpha) = 0$ and hence, $\alpha = 0$. Finally, we conclude that $\chi(g) = 0$. Hence, proved. \square

Lemma 4.3.7. *Let G be a finite non abelian group. Let $C \neq \{1\}$ be the conjugacy class of G such that $|C| = p^t$ where p is prime and $t \geq 0$. Then, G is not simple.*

Proof. Suppose G is not simple. Then, let ϕ^1, \dots, ϕ^s be the complete set of representatives of irreducible representations of G . Let χ_1, \dots, χ_s be their corresponding characters and d_1, \dots, d_s be their degrees. Without loss of generality, let ϕ^1 be the trivial representation. Since G is simple, $\text{Ker} \phi^k = \{1\} \forall k > 1$ since if $\text{Ker} \phi^k = G$ then we have trivial representation. This shows that ϕ^k is one one for all $k > 1$. Now, G is non abelian and \mathbb{C}^* is abelian. This implies that $d_k > 1 \forall k > 1$. Since G is simple and non abelian this shows that $Z(G) = \{1\}$. Hence, $t > 0$. Let $g \in C$ and $k > 1$. Define

$$Z_k = \{x \in G \mid \phi_x^k \text{ is a scalar matrix}\}$$

and

$$H = \{\lambda I_{d_k} \mid \lambda \in \mathbb{C}^*\}.$$

Note that H is a subgroup of $GL_{d_k}(\mathbb{C})$. In fact, $H=Z(G)$. Hence, H is normal. Now as Z_k is inverse image of H under ϕ^k which is a homomorphism, Z_k is a normal subgroup of G . Since, $d_k > 1$ and $Z_k \neq G$, $Z_k = \{1\} \forall k > 1$.

Suppose, $p \nmid d_k$ then, $\text{gcd}(p, d_k) = 1$. That is, $\chi_{\phi^k}(g) = 0$. Let L be regular representation then,

$$L \equiv d_1 \phi^1 \oplus \dots \oplus d_s \phi^s.$$

Since, $g \neq 1$, $\chi_L(g) = d_1\chi_1 + \dots + d_s\chi_s = 1 + \sum_{k=2}^s d_k\chi_k = 1 + \sum_{p|d_k} d_k\chi_k = 1 + pz$.

Also, $\chi_L(g) = 0$. This implies that $1+pz=0$. Thus, $1/p=-z$ where z is algebraic integer. Hence, we get a contradiction since z has to be an integer. Therefore, our supposition was wrong. Thus, G is not simple. \square

Theorem 4.3.8. Burnside's theorem: *Let G be a group of order $p^a q^b$ with primes p, q . Then, G is not simple unless it is cyclic of prime order.*

Proof. An abelian group is simple iff it is cyclic of prime order. Assume that G is non abelian. We know that a group of prime power order has non trivial centres. So, it is not simple. That is, here, in this case, if one of a or b is zero then, G is not simple. So, let $a, b \geq 1$.

By Sylow's theorem, there exists a subgroup H of G of order q^b . Let $1 \neq g \in Z(H)$.

Let $N_G(g) = \{x \in G | gx = xg\}$ is the normalizer of g in G . Now, since $g \in Z(H)$, g commutes with every element of H . That is, every element of H commutes with g . Therefore, every element of H is in $N_G(g)$. Hence, $H \subseteq N_G(g)$.

Now $p^a = [G : H] = [G : N_G(g)][N_G(g) : H]$. This shows that $[G : N_G(g)] = p^t$ where $t > 0$ but $|Cl(g)| = \text{conjugacy class of } g = [G : N_G(g)] = p^t$ where $t > 0$. Hence, G is not simple by previous lemma. \square

Chapter 5

Group Actions and the Permutation Representations

5.1 Group Actions

Definition 5.1.1. *An action of a finite group G on a finite set X is a homomorphism $\sigma : G \rightarrow S_X$ where S_X is the set of all bijections from X to X . Moreover, $|X|$ is the degree of the action.*

Definition 5.1.2. *Let the action λ be defined on G itself that is, $\lambda : G \rightarrow S_G$ be an action. Then, λ is called as regular action. And, the permutation $\tilde{\lambda} : G \rightarrow GL(\mathbb{C}G)$ is called the Regular Representation of G . We call the regular representation to be L .*

Example 5.1.3. *Let G be a group and let $\lambda : G \rightarrow S_G$ be an action of G on G itself defined as, $\lambda_g(x) = gx$. Then, this action is called the Regular action of G on G .*

Definition 5.1.4. *The subset $Y \subseteq X$ is called G -invariant if $\sigma_g(y) \in Y \forall y \in Y$ and $g \in G$.*

The set X partitions into disjoint union of minimal G -invariant subsets called Orbits.

Definition 5.1.5. Orbit: *Let $\sigma : G \rightarrow S_X$ be a group action. The orbit of $x \in X$ under G is the set $G.x = \{\sigma_g(x) | g \in G\}$.*

An orbit is a G -invariant subset of X .

Definition 5.1.6. Transitive Action: A group action $\sigma : G \rightarrow S_X$ is said to be transitive if $\forall x, y \in X$, there exists $g \in G$ such that $\sigma_g(x) = y$.

In other words, an action is said to be transitive if there exists one orbit of G on X .

Example 5.1.7. Let H is a subgroup of G . Then, there exists an action $\sigma : G \rightarrow S_{G/H}$ by $\sigma_g(xH) = gxH$ known as the Coset Action of G . We note that this coset action is transitive.

Definition 5.1.8. 2-Transitive Action: Let $\sigma : G \rightarrow S_X$ be action of G on X . Then it is said to be 2-transitive if for any given two pairs of distinct elements $x, y \in X$, $x', y' \in X$, there exists a group element g such that $\sigma_g(x) = x'$ and $\sigma_g(y) = y'$.

Note: The property of an action to be 2-transitive is a stronger property than an action to be transitive.

Example 5.1.9. Let $n \geq 2$. Then, the action of the symmetric group S_n is 2-transitive on $\{1, 2, \dots, n\}$. Consider two pairs of distinct elements say $i \neq j$ and $k \neq l$. Now, let $X = \{1, 2, \dots, n\} \setminus \{i, j\}$ and $Y = \{1, 2, \dots, n\} \setminus \{k, l\}$. Then, $|X| = n - 2 = |Y|$. Now, we choose bijection $\alpha : X \rightarrow Y$. Define $\tau \in S_n$ by

$$\tau(m) = \begin{cases} k & ; m = i \\ l & ; m = j \\ \alpha(m) & ; \text{otherwise} \end{cases}$$

Then, $\tau(i) = k$ and $\tau(j) = l$. Hence, S_n is 2-transitive.

Definition 5.1.10. Orbital: Let $\sigma : G \rightarrow S_X$ be a transitive group action. Define $\sigma^2 : G \rightarrow S_{X \times X}$ defined by

$$\sigma_g^2(x, x) = (\sigma_g(x_1), \sigma_g(x_2))$$

An orbit of σ^2 is known as the orbital of σ . The number of orbitals gives the **rank** of σ .

Consider the set $\Delta = \{(x, x) : x \in X\}$. We note that $\sigma_g^2(x, x) = (\sigma_g(x), \sigma_g(x))$. Now, if we consider σ to be transitive then we conclude that Δ is an orbital of σ called the trivial orbital or the diagonal orbital.

Proposition 5.1.11. *Let $\sigma : G \rightarrow S_X$ be group action where $(|X| \geq 2)$ then, σ is 2-transitive iff σ is transitive and $\text{rank}(\sigma) = 2$.*

Proof. If σ is 2-transitive and $x, y \in X$ then we can choose $x' \neq x$ and $y' \neq y$ such that $\sigma_g(x) = x'$ and $\sigma_g(y) = y'$. So, σ is transitive. Next, $(X \times X)$ other than Δ or the complement of Δ in $X \times X$ is the following set.

$$\{(x, y) | x \neq y\}$$

Therefore, Δ^c is an orbital iff for any two pairs $x \neq y$ and $x' \neq y'$ of distinct elements, there exists $g \in G$ such that $\sigma_g(x) = x'$ and $\sigma_g(y) = y'$ that is, σ is 2-transitive. Consequently, $\text{Rank}(\sigma) = 2$ since there are two orbitals, Δ and Δ^c . \square

Definition 5.1.12. *Let $\sigma : G \rightarrow S_X$ be a group action. Then, for $g \in G$, we define,*

$$\text{Fix}(g) = \{x \in X | \sigma_g(x) = x\}$$

to be the set of all fixed points with respect to G .

Similarly, $\text{Fix}^2(g)$ is the set of fixed points with respect to g on $X \times X$.

Proposition 5.1.13. *Let $\sigma : G \rightarrow S_X$ be the group action. Then, $\text{Fix}^2(g) = \text{Fix}(g) \times \text{Fix}(g)$ and hence, $|\text{Fix}^2(g)| = |\text{Fix}(g)|^2$.*

Proof. Let $(x, y) \in X \times X$. Then, $\sigma_g^2(x, y) = (\sigma_g(x), \sigma_g(y))$. This implies that $(x, y) = \sigma_g^2(x, y)$ if and only if $\sigma_g(x) = x$ and $\sigma_g(y) = y$. We know that,

$$\text{Fix}^2(g) = \{(x, y) | \sigma_g(x) = x, \sigma_g(y) = y\}$$

Hence, $\text{Fix}^2(g) = \text{Fix}(g) \times \text{Fix}(g)$. \square

Suppose we have an action $\sigma : G \rightarrow S_n$ and the standard representation of S_n , $\alpha : S_n \rightarrow GL_n(\mathbb{C})$ then, the composition $\alpha \circ \sigma : G \rightarrow GL_n(\mathbb{C})$ gives us the representation of G .

Definition 5.1.14. Let $\sigma : G \rightarrow S_X$ be a group action and hence $\tilde{\sigma} : G \rightarrow GL(\mathbb{C}X)$ is the representation defined as,

$$\tilde{\sigma}_g \left(\sum_{x \in X} c_x x \right) = \sum_{x \in X} c_x \sigma_g(x) = \sum_{y \in X} c_{\sigma_{g^{-1}}(y)} y$$

This representation $\tilde{\sigma}$ is called the permutation representation associated to σ . In this representation, x is mapped to $\sigma_g(x)$ and hence, the degree of representation $\tilde{\sigma}$ is equal to the degree of action σ .

Definition 5.1.15. Let $\lambda : G \rightarrow S_G$ be the regular action of G . Then,

$$\tilde{\lambda} : G \rightarrow GL(\mathbb{C}G)$$

is the regular representation of G .

Proposition 5.1.16. Let $\sigma : G \rightarrow S_X$ be a group action. Then, the representation $\tilde{\sigma} : G \rightarrow GL(\mathbb{C}X)$ is the permutation representation of G and is a unitary representation.

Proposition 5.1.17. Let $\sigma : G \rightarrow S_X$ be a group action. Then, $\chi_{\tilde{\sigma}(g)} = |Fix(g)|$.

Proof. Consider the set $X = \{x_1, x_2, \dots, x_n\}$. Now, corresponding to this given action, we will have the representation $\tilde{\sigma}$ as defined in the proposition. Hence, $[\tilde{\sigma}_g]$ will be the matrix of $\tilde{\sigma}$ with respect to the basis X . We know that, $\tilde{\sigma}_g(x_j) = \sigma_g(x_j)$ and so,

$$[\tilde{\sigma}_g]_{ij} = \begin{cases} 1 & ; x_i = \sigma_g(x_j) \\ 0 & ; otherwise \end{cases}$$

In particular,

$$[\tilde{\sigma}_g]_{ii} = \begin{cases} 1 & ; x_i = \sigma_g(x_i) \\ 0 & ; otherwise \end{cases}$$

That is,

$$[\tilde{\sigma}_g]_{ii} = \begin{cases} 1 & ; x_i \in Fix(g) \\ 0 & ; otherwise \end{cases}$$

Hence, $\chi_{\tilde{\sigma}(g)} = tr([\tilde{\sigma}_g]) = |Fix(g)|$. \square

Definition 5.1.18. Fixed Subspace: Let $\phi : G \rightarrow GL(V)$ be a representation. Then,

$$V^G = \{v \in V \mid \phi_g(v) = v \ \forall g \in G\}$$

is called as the fixed subspace of V .

We note that V^G is the G -invariant subspace.

Note: The subrepresentation $\phi \upharpoonright V^G$ is equivalent to $\dim V^G$ copies of the trivial representation.

Proposition 5.1.19. Let $\phi : G \rightarrow GL(V)$ be a representation. Let χ_1 be the trivial character of G . Then, $\langle \chi_\phi, \chi_1 \rangle = \dim V^G$.

Proof. Proof: Let $V = m_1V_1 \oplus \dots \oplus m_sV_s$ where V_1, V_2, \dots, V_s are the irreducible G -invariant subspaces of V .

Without loss of Generality, assume that V_1 is equivalent to the trivial representation. Let $\phi^i = \phi \upharpoonright V_i$.

Let $v \in V$. Then, $v = v_1 + v_2 + \dots + v_s$ with $v_i \in m_iV_i$ and

$$\phi_g(v) = (m_1\phi^1)_g(v_1) + \dots + (m_s\phi^s)_g(v_s)$$

and so, we conclude that,

$$v \in V^G \text{ iff } v_i \in m_iV_i^G \ \forall 2 \leq i \leq s. \text{ or we can say that,}$$

$$V^G = m_1V_1 \oplus m_2V_2^G \oplus \dots \oplus m_sV_s^G$$

Let $i \geq 2$. Since, each V_i is irreducible and not equivalent to the trivial representation and V_i^G is G -invariant, $V_i^G = 0$ and thus, $V^G = m_1V_1$.

Therefore, $\dim V^G = m_1 \dim V_1$ and hence, m_1 is the dimension of V_g where m_1 is the multiplicity of trivial representation in ϕ \square

Proposition 5.1.20. Let $\sigma : G \rightarrow S_X$ be a group action. Let O_1, O_2, \dots, O_m be the orbits of G on X and define $v_i = \sum_{x \in O_i} x$. Then, v_1, v_2, \dots, v_m is the basis for $\mathbb{C}X^G$ and hence, $\dim \mathbb{C}X^G$ is the number of orbits of G on X .

Proof. We need to show that v_1, v_2, \dots, v_m is the linearly independent set and spans $\mathbb{C}X^G$. First to show linear independence, we observe that,

$$\tilde{\sigma}_g(v_i) = \sum_{x \in O_i} \sigma_g(x) = \sum_{y \in O_i} y = v_i$$

From this we conclude that v_1, v_2, \dots, v_m belongs to the fixed subspace X^G and hence, $v_1, v_2, \dots, v_m \in \mathbb{C}X^G$. Now, since the orbits are disjoint and we know that if $v_i = \sum_{x \in O_i} c_i x$ and $v_j = \sum_{x \in O_j} c_j x$ are the two formal sums where c_i and c_j are the coefficients then

$$\langle v_i, v_j \rangle = \sum c_i \bar{c}_j$$

As here, we have all coefficients in the formal sum to be 1, therefore,

$$\langle v_i, v_j \rangle = \begin{cases} |O_i| & ; i = j \\ 0 & ; i \neq j \end{cases}$$

which shows that $\{v_1, v_2, \dots, v_m\}$ is an orthogonal set of non zero vectors and hence linearly independent. Now it remains to prove that this spans $\mathbb{C}X^G$. For this, suppose, $v = \sum_{x \in X} c_x x \in \mathbb{C}X^G$. We first show that if some element $z \in G.y$ then, $c_y = c_z$. Let $z \in G.y$ then, $z = \sigma_g(y)$ then,

$$\sum_{x \in X} c_x x = v = \tilde{\sigma}_g(v) = \sum_{x \in X} c_x \sigma_g(x)$$

Now the coefficient of z in left hand side is c_z and the coefficient of z in right hand side is c_y since $z = \sigma_g(y)$. Therefore, $c_z = c_y$.

This implies that there exist $c_1, c_2, \dots, c_m \in \mathbb{C}$ such that $c_x = c_i \forall x \in O_i$. Thus,

$$v = \sum_{x \in X} c_x x = \sum_{i=1}^m \sum_{x \in O_i} c_x x = \sum_{i=1}^m c_i \sum_{x \in O_i} x = \sum_{i=1}^m c_i v_i$$

Hence, v_1, v_2, \dots, v_m spans $\mathbb{C}X^G$ and hence is the basis for the same. \square

Corollary 5.1.1. Burnside's lemma: Let $\sigma : G \rightarrow S_X$ be a group action and let m be the number of orbits of G on X . Then, $m = \frac{1}{O(G)} \sum_{g \in G} |Fix(g)|$

Proof. Let χ_1 be the trivial character of G . We know that the multiplicity of the trivial representation in $\tilde{\sigma}$, $m = \langle \chi_{\tilde{\sigma}}, \chi_1 \rangle = \dim V^G$.

And also, we know that

$$\langle \chi_{\tilde{\sigma}}, \chi_1 \rangle = \frac{1}{O(G)} \sum_{g \in G} \chi_{\tilde{\sigma}}(g) \overline{\chi_1(g)} = \frac{1}{O(G)} \sum_{g \in G} |Fix(g)|$$

Hence, $m = \frac{1}{O(G)} \sum_{g \in G} |Fix(g)|$. \square

Corollary 5.1.2. *Let $\sigma : G \rightarrow S_X$ be a transitive group action. Then,*

$$\text{rank}(\sigma) = \frac{1}{O(G)} \sum_{g \in G} |\text{Fix}(g)|^2 = \langle \chi_{\tilde{\sigma}}, \chi_{\tilde{\sigma}} \rangle$$

Proof. We know that the rank of the action σ is equal to the number of orbits of σ^2 on $X \times X$. And, the number of fixed points of g on $X \times X$ is $|\text{Fix}(g)|^2$.

From the consequence of Burnside's lemma, we can conclude the first equality. For the second one,

$$\begin{aligned} \langle \chi_{\tilde{\sigma}}, \chi_{\tilde{\sigma}} \rangle &= \frac{1}{O(G)} \sum_{g \in G} |\text{Fix}(g)| \overline{|\text{Fix}(g)|} \\ &= \frac{1}{O(G)} \sum_{g \in G} |\text{Fix}(g)|^2. \end{aligned}$$

Hence, by this we get the second equality. \square

Suppose $\sigma : G \rightarrow S_X$ be a transitive group action. Let $v_0 = \sum_{x \in X} x$. Then, we prove that $\mathbb{C}X^G = \mathbb{C}v_0$.

We know that $\mathbb{C}X^G = \{x \in X \mid \sigma_g(x) = x\}$ and $\mathbb{C}v_0 = \text{span}\{\sum_{x \in X} x\}$. Now, let $v \in \mathbb{C}v_0$ that is, $v = \sum_{x \in X} cx$. now,

$$\tilde{\sigma}_g\left(\sum_{x \in X} cx\right) = \sum_{x \in X} \sigma_g(cx) = \sum_{x \in X} c\sigma_g(x) = \sum_{y \in X} cy = v$$

since the action is transitive and the representation $\tilde{\sigma}$ is a unitary representation. This shows that $v \in \mathbb{C}X^G$ and hence $\mathbb{C}v_0 \subseteq \mathbb{C}X^G$.

Now, we show the other part. We know that the orbit of x is,

$$G.x = \{\sigma_g(x) \mid g \in G\}$$

now, since the action is given to be transitive, we have $G.x = X$. Hence, by previous theorem, it follows that v_0 is a basis for $\mathbb{C}X^G$.

Hence, $\mathbb{C}X^G = \mathbb{C}v_0$. Now let $V_0 = \mathbb{C}v_0^\perp$ is a G -invariant subspace.

We define $\mathbb{C}v_0$ to be trace of σ and V_0 to be augmentation of σ . Let $\tilde{\sigma}'$ be the restriction of $\tilde{\sigma}$ to V_0 . Hence, $\tilde{\sigma}'$ is the augmentation representation associated to σ .

As $\mathbb{C}X = V_0 \oplus \mathbb{C}v_0$, we have,

$$\chi_{\tilde{\sigma}} = \chi'_{\tilde{\sigma}} + \chi_1$$

where χ_1 is the trivial character.

Theorem 5.1.21. *Let $\sigma : G \rightarrow S_X$ be a transitive group action. Then, the augmentation representation $\tilde{\sigma}'$ is irreducible iff σ is 2-transitive on X .*

Proof. We know that σ is 2-transitive iff $\text{rank}(\sigma) = 2$. Let χ_1 be the trivial character of G . Then,

$$\langle \chi_{\tilde{\sigma}'}, \chi_{\tilde{\sigma}'} \rangle = \langle \chi_{\tilde{\sigma}} - \chi_1, \chi_{\tilde{\sigma}'} - \chi_1 \rangle = \langle \chi_{\tilde{\sigma}}, \chi_{\tilde{\sigma}'} \rangle - \langle \chi_{\tilde{\sigma}}, \chi_1 \rangle - \langle \chi_1, \chi_{\tilde{\sigma}'} \rangle + \langle \chi_1, \chi_1 \rangle$$

Now, since σ is transitive, $\langle \chi_{\tilde{\sigma}}, \chi_1 \rangle = 1$ and $\langle \chi_1, \chi_{\tilde{\sigma}'} \rangle = 1$. Also, $\langle \chi_1, \chi_1 \rangle = 1$. This implies that,

$$\langle \chi_{\tilde{\sigma}'}, \chi_{\tilde{\sigma}'} \rangle = \text{rank}(\sigma) - 1$$

and so, $\chi_{\tilde{\sigma}'}$ is irreducible character iff $\text{rank}(\sigma) = 2$ that is, iff σ is 2-transitive on X . □

5.2 The Centralizer Algebra

Some Notations:

Let $\sigma : G \rightarrow S_X$ is a transitive group action. Our aim is to study the set $\text{Hom}_G(\tilde{\sigma}, \tilde{\sigma})$ which is the set of all morphisms from $\tilde{\sigma}$ to itself. That is, it is the set of all invertible maps T such that $T\tilde{\sigma}_g = \tilde{\sigma}_g T$.

Let us consider the set, $X = \{x_1, x_2, \dots, x_n\}$. Let $\phi : G \rightarrow GL_n(\mathbb{C})$ be defined as, $\phi_g = [\tilde{\sigma}_g]_X$.

Then, $\phi \sim \tilde{\sigma}$ and so, $\text{Hom}_G(\tilde{\sigma}, \tilde{\sigma}) \equiv \text{Hom}_G(\phi, \phi)$.

$\text{Hom}_G(\phi, \phi) = \{A \in M_n(\mathbb{C}) \mid A\phi_g = \phi_g A \forall g \in G\}$ that is exactly the set

$$\{A \in M_n(\mathbb{C}) \mid \phi_g A \phi_g^{-1} = A \forall g \in G\}$$

We denote $\text{Hom}_G(\phi, \phi)$ as $C(\sigma)$ known as the Centralizer algebra of σ .

Proposition 5.2.1. *$C(\sigma)$ is a unital subring of $M_n(\mathbb{C})$.*

Proof. We note that, $\phi_g I_n \phi_g^{-1} = I_n \forall g \in G$. Let $A, B \in C(\sigma)$ then,

$$\phi_g(A + B)\phi_g^{-1} = \phi_g A \phi_g^{-1} + \phi_g B \phi_g^{-1} = A + B \forall g \in G$$

Similarly, $\phi_g(AB)\phi_g^{-1} = \phi_g A \phi_g^{-1} \phi_g B \phi_g^{-1} = AB$.

This shows that $C(\sigma)$ is a unital subring of $M_n(\mathbb{C})$. □

Next, we show that, $\dim(C(\sigma)) = \text{rank}(\sigma)$. The proof goes as follows.

Let $V = M_n(\mathbb{C})$ and $\tau : G \rightarrow GL(V)$ by $\tau_g(A) = \phi_g A \phi_g^{-1}$ be a representation. Then,

$$V^G = \{A \in M_n(\mathbb{C}) \mid \phi_g A \phi_g^{-1} = A \forall g \in G\} = C(\sigma)$$

For the next proposition we retain the same notations described right now. And also, consider $\sigma^2 : G \rightarrow S_{X \times x}$

Proposition 5.2.2. *Define a map $T : M_n(\mathbb{C}) \rightarrow \mathbb{C}(X \times X)$ by*

$$T(a_{ij}) = \sum_{i,j=1}^n a_{ij}(x_i, x_j)$$

Then, T is an equivalence between τ and σ^2 .

Proof. The defined map T is clearly linear and invertible with the inverse,

$$\sum_{i,j=1}^n a_{ij}(x_i, x_j) \longmapsto (a_{ij})$$

Let $g \in G$ and let $A = (a_{ij}) \in M_n(\mathbb{C})$ and $B = (b_{ij}) = \tau_g A = \phi_g A \phi_g^{-1}$. Next, we define action $\gamma : G \rightarrow S_n$ as

$$\sigma_g(x_i) = x_{\gamma_g(i)} \text{ for } g \in G$$

As,

$$\phi(g)_{ik} = \begin{cases} 1 & ; x_i = \sigma_g(x_k) \\ 0 & ; \text{otherwise} \end{cases}$$

and,

$$\phi(g^{-1})_{lj} = \begin{cases} 1 & ; x_l = \sigma_{g^{-1}}(x_j) \\ 0 & ; \text{otherwise} \end{cases}$$

we have,

$$b_{ij} \sum_{k=1, l=1}^n \phi(g)_{ik} a_{kl} \phi(g^{-1})_{lj} = a_{\gamma_{g^{-1}(i), \gamma_{g^{-1}}(j)}}$$

Therefore,

$$\begin{aligned} T\tau_g A &= \sum_{i,j=1}^n b_{ij}(x_i, x_j) \\ &= \sum_{i,j=1}^n a_{\gamma_{g^{-1}(i), \gamma_{g^{-1}}(j)}}(x_i, x_j) \\ &= \sum_{i,j=1}^n a_{kl}(\sigma_g(x_k), \sigma_g(x_l)) \\ &= \sum_{i,j=1}^n a_{ij}(\sigma_g(x_i), \sigma_g(x_j)) \\ &= \sum_{i,j=1}^n a_{ij} \sigma_g^2(x_i, x_j) \\ &= \tilde{\sigma}_g^2 T A. \end{aligned}$$

Hence, T is an equivalence between τ and σ^2 .

Now, as our motive was to determine the dimension of $C(\sigma)$, so, must be thinking that what can be the basis for $C(\sigma)$.

Let O be an orbital of σ that is, orbital of σ^2 .

Define $A(O) \in M_n(\mathbb{C})$ by

$$A(O)_{ij} = \begin{cases} 1 & ; (x_i, x_j) \in O \\ 0 & ; \text{otherwise.} \end{cases}$$

Following corollary describes the basis for $C(\sigma)$.

Corollary 5.2.1. *Let $\sigma : G \rightarrow S_X$ be a transitive group action. Let O_1, O_2, \dots, O_r be the orbitals of σ where r is the rank of σ . Then, the set $\{A(O_1), A(O_2), \dots, A(O_r)\}$ is a basis for $C(\sigma)$ and consequently $\dim(C(\sigma)) = \text{rank}(\sigma)$*

Proof. Proof: By the proposition done earlier, we know that,

$$v_k = \sum_{(x_i, x_j) \in (k)} (x_i, x_j)$$

is a basis for $\mathbb{C}(X \times X)^G$.

Clearly, $A(O_k) = T^{-1}v_k$ where, $T : M_n(\mathbb{C}) \rightarrow \mathbb{C}(X \times X)$ defined as,

$$T(a_{ij}) = \sum_{i,j=1}^n a_{ij}(x_i, x_j)$$

Now, as T is an equivalence from $M_n(\mathbb{C})$ and $\mathbb{C}(X \times X)$, so it restricts its equivalence to $M_n(\mathbb{C}) = C(\sigma)$ and $\mathbb{C}(X \times X)$.

Therefore, $\{A(O_1), \dots, A(O_r)\}$ is a basis for $C(\sigma)$.

Hence, $\dim(C(\sigma)) = \text{rank}(\sigma)$. □

□

5.3 Gelfand Pair

Definition 5.3.1. Let G be a group. Let H be the subgroup of G . Let $\sigma : G \rightarrow S_{G/H}$ be the coset action. Then, (G, H) is said to be Gelfand Pair if the centralizer algebra $C(\sigma)$ is commutative.

Example 5.3.2. Let G be a group. Let H be its subgroup such that $H = \{1\}$. Let $\lambda : G \rightarrow S_G$ be the regular action of G and hence, $\lambda : G \rightarrow GL(\mathbb{C}G)$ is the regular representation of G say, L .

Our claim is $C(\lambda)$ is isomorphic to $L(G)$.

Let $T \in C(\lambda)$ and define $f_T : G \rightarrow \mathbb{C}$ by,

$$T.1 = \sum_{x \in G} f_T(x^{-1})x$$

We show that the map $\psi : C(\lambda) \rightarrow L(G)$ defined as,

$$T \longmapsto f_T$$

is an isomorphism.

First, note that, since $T \in C(\lambda)$ that is, $T \in \text{Hom}_G(L, L)$ so, $L_g T = T L_g$.

Now, for $g \in G$,

$$T_g = T L_g.1 = L_g T.1 = L_g \sum_{x \in G} f_T(x^{-1})x.$$

Therefore, T is determined by f_T and hence, this shows that, ψ is one one. Next, we need to show that ψ is onto.

Let $f : G \rightarrow \mathbb{C}$ be a function in $L(G)$. Then, we can define $T \in \text{End}(\mathbb{C}G)$ on the basis as,

$$T_g = L_g \sum_{x \in G} f(x^{-1})x$$

Now, if $g, y \in G$ then,

$$TL_yg = T_{yg} = L_{yg} \sum_{x \in G} f(x^{-1})x = L_yT_g.$$

So, $T \in C(\lambda)$. Also, $T.1 = \sum_{x \in G} f(x^{-1})x$ and so, $f_T = f$. Hence, ψ is onto. Next, we show that ψ is homomorphism.

Let $T_1, T_2 \in C(\lambda)$.

$$\begin{aligned} T_1T_2.1 &= T_1 \sum_{x \in G} f_{T_2}(x^{-1})x \\ &= \sum_{x \in G} f_{T_2}(x^{-1})T_1L_x1 \\ &= \sum_{x \in G} f_{T_2}(x^{-1})L_x \sum_{y \in G} f_{T_1}(y^{-1})y \\ &= \sum_{x, y \in G} f_{T_1}(y^{-1})f_{T_2}(x^{-1})xy \end{aligned}$$

Let $g=xy$ and $u = x^{-1}$ then, $y^{-1} = g^{-1}u^{-1}$. This shows that,

$$\begin{aligned} T_1T_2.1 &= \sum_{g \in G} \sum_{u \in G} f_{T_1}(g^{-1}u^{-1})f_{T_2}(u)g \\ &= \sum_{g \in G} f_{T_1} * f_{T_2}(g^{-1})g. \end{aligned}$$

This implies that, $f_{T_1T_2} = f_{T_1} * f_{T_2}$.

Hence, ψ is homomorphism. Therefore, we conclude that, ψ is an isomorphism.

Now, as a consequence,

$(G, \{1\})$ is a Gelfand Pair

$\iff C(\lambda)$, the centralizer algebra is commuattive that is,

$\iff L(G)$ is commutative.

$\iff L(G) = Z(L(G))$.

We know that, $\dim(Z(L(G))) = |Cl(G)|$ and $\dim(L(G)) = |G|$.

So, $(G, \{1\})$ is a Gelfand Pair.

$\iff |Cl(G)| = |G|$.

$\iff G$ is abelian.

Therefore, $(G, \{1\})$ is a Gelfand Pair $\iff G$ is abelian.

Note: (G, H) is a Gelfand Pair iff $\tilde{\sigma}$ is multiplicity free that is, each irreducible constituent of $\tilde{\sigma}$ has multiplicity one.

Definition 5.3.3. Let $\sigma : G \rightarrow S_X$ be a transitive action. Then, to each orbital O of σ , we can associate its transpose,

$$O^T = \{(x_1, x_2) \in X \times X \mid (x_2, x_1) \in O\}$$

which is also an orbital since the action is transitive.

Definition 5.3.4. O is symmetric orbital if $O = O^T$.

Example 5.3.5. The diagonal orbital Δ is symmetric.

Note that, $A(O^T) = A(O)^T$.

Hence, O is symmetric iff the matrix $A(O)$ is symmetric.

Definition 5.3.6. Symmetric Gelfand Pair: Let G be a group and H be its subgroup with corresponding group action $\sigma : G \rightarrow S_{G/H}$. Then, (G, H) is called a symmetric Gelfand Pair if each orbital of σ is symmetric.

Example 5.3.7. Let H be a subgroup of a group G . Let action of G on G/H is 2-transitive. Then, orbitals are, Δ and Δ^c in $G/H \times G/H$. Clearly, each one of them is symmetric.

Example 5.3.8. Let $n \geq 2$. Let N be the set of all 2-element subsets of $\{1, 2, \dots, n\}$. Then, S_n acts on N as follows.

Define, $\tau : S_n \rightarrow S_N$ by,

$$\tau_\sigma(\{i, j\}) = \{\sigma(i), \sigma(j)\}.$$

Note that this action is clearly transitive since S_n is 2-transitive on $\{1, 2, \dots, n\}$. Let H be the stabilizer of $\{n-1, n\}$ in S_n .

So, H is the internal direct product of S_{n-2} and $S_{\{n-1,n\}}$ and hence, $H \cong S_{n-2} \times S_2$.

Let us recall the definition of internal direct product.

A group G is the internal product of two subgroups N_1, N_2 if the following conditions hold:

- (i) N_1, N_2 are both normal subgroups and,
- (ii) $N_1 \cap N_2 = \{0\}$ and $N_1 N_2 = G$.

Equivalently, G is the internal direct product of N_1, N_2 if every element of N_1 commutes with every element of N_2 that is, $N_1 \subset \text{centralizer}(N_2)$.

Getting back to the example, we find that the action of S_n on N can be identified with the action of S_n on S_n/H .

Now, let O be a non trivial orbital then, any element of O is of the form, $(\{i, j\}, \{k, l\})$ where $\{i, j\} \neq \{k, l\}$. Now we have two cases:

Case(i) : If all are distinct, then, the permutation of S_n , $(i k)(j l)$ takes this element to $(\{k, l\}, \{i, j\})$ so, O is symmetric.

Case(ii) : Suppose $i=k$. Then, the permutation $(i k)(j l)$ becomes $(j l)$ which takes this element to $(\{i, l\}, \{i, j\}, \{i, j\})$. Therefore, O is again symmetric since $i=k$.

So, (S_n, H) is a Symmetric Gelfand Pair.

Lemma 5.3.9. Let R be a subring of $M_n(\mathbb{C})$ consisting of symmetric matrices. Then R is commutative.

Proof. Let $A, B \in R$. Then, $AB \in R$ and hence,

$$AB = (AB)^T = B^T A^T = BA.$$

Hence, R is commutative. □

Theorem 5.3.10. Let (G, H) be symmetric Gelfand Pair. Then, (G, H) is a Gelfand Pair.

Proof. Let $\sigma : G \rightarrow S_{G/H}$ be an action. Let O_1, O_2, \dots, O_r be the orbitals of σ .

Since, each O_i is symmetric, each $A(O_i)$ is symmetric for $i = 1, 2, \dots, r$. Since, symmetric matrices form a vector subspace of $M_n(\mathbb{C})$ and

$$\{A(O_1), A(O_2), \dots, A(O_r)\}$$

is a basis for $C(\sigma)$ (by corollary done before).

This shows that $C(\sigma)$ consists of symmetric matrices. So, $C(\sigma)$ is commutative by previous lemma. Hence, (G,H) is a Gelfand Pair. \square

Bibliography

- [1] B. Simon, *Representations of finite and compact groups*, Graduate Studies in Mathematics, 10, American Mathematical Society, Providence, RI, 1996. MR1363490
- [2] B. Steinberg, *Representation theory of finite groups*, Universitext, Springer, New York, 2012. MR2867444
- [3] J.-P. Serre, *Représentations linéaires des groupes finis*, third revised edition, Hermann, Paris, 1978. MR0543841