TOPOLOGICAL VECTOR SPACE AND ITS PROPERTIES

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under the supervision of
Dr. Venku Naidu Dogga

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DECLARATION

This thesis entitled **TOPOLOGICAL VECTOR SPACE AND ITS PROPERTIES** submitted by me to the Indian Institute of Technology, Hyderabad for the award of the degree in Master of Science in Mathematics contains a literature survey of the work done by some authors in this area. The work presented in this thesis has been carried out under the supervision of **Dr. Venku Naidu Dogga**, Department of Mathematics, Indian Institute of Technology, Hyderabad, Telangana.

I hereby declare that, to the best of my knowledge, the work included in this thesis has been taken from the books, "Functional Analysis" by Walter Rudin, and "Convexity and Optimization in Banach Spaces" by Viorel Barbu and Teodor Precupanu. No new results have been created in this thesis. The definitions, notations and results in Topological vector space and its properties are learnt from the above mentioned sources and are presented here. I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

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This Thesis entitled **TOPOLOGICAL VECTOR SPACE AND ITS PROPERTIES** by **Kamal Harwani** is approved for the degree in Master of Science from IIT Hyderabad.

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ABSTRACT

The main aim of this project is to learn a branch of Mathematics that studies vector spaces endowed with some kind of limit-related structure (e.g. inner product, norm, topology, etc.) and the linear functions defined on these spaces and respecting these structures in a suitable sense.

Specifically we will learn vector space with some topology on it (called topological vector space). A topological vector space (also called a linear topological space) is one of the basic structures investigated in functional analysis. A topological vector space is a vector space (an algebraic structure) which is also a topological space, thereby admitting a notion of continuity. More specifically, its topological space has a uniform topological structure, allowing a notion of uniform convergence.

In chapter 1 we will learn some Separation properties in TVS (Topological Vector Space) in the sense that two disjoint closed and compact sets can be separate by finding suitable disjoint neighborhoods. And when TVS is metrizable in the sense that is there any metric which is compatible with vector topology having some condition on TVS. And we will learn relation between Seminorms and Local Convexity. Finally we will conclude that there are some TVS which are not normed spaces.

In chapter 2, we will learn some definitions like Banach space, dual of Spaces, convex (affine) set and convex (affine) hull of sets, algebraic relative interior, algebraic closure of set, and radial boundary etc. Then Separation properties on convex sets in the sense that there exists a hyperplane (linear functional) that separates two disjoint compact convex and closed convex sets.

In chapter 3, we will see Banach- Steinhaus theorem, open mapping theorem and closed graph theorem. **Banach–Steinhaus theorem** In its basic form, it asserts that for a family of continuous linear operators (and thus bounded operators) whose domain is a TVS (secound category), pointwise boundedness is equivalent to uniform boundedness in vector topology.

In chapter 4, we will learn notion of Weak and $Weak^*$ topology. And

most important theorems Banach-Alaoglu theorem and Krien-Milman theorem which have many application. **Krein–Milman theorem** is a proposition about convex sets in topological vector spaces. Let X be a locally convex topological vector space and let K be a compact convex subset of X. Then, the theorem states that K is the closed convex hull of its extreme points.

Contents

Chapter 1

Topological Vector Space

1.1 Introduction

Definition 1.1.1. ***Topological Vector Space*** *: Suppose τ is a topology on a vector space X such that*

(a) every point of X is a closed set, and

(b) the vector space operations are continuous with respect to τ.

Then X is said to be Topological Vector Space.

Note 1.1.2. *To say that addition is continuous means, by definition, that the mapping*

$$(x, y) \to x + y$$

of the cartesian product $X \times X$ into X is continuous: If $x_i \in X$ for $i = 1, 2$ and if V is a neighborhood of $x_1 + x_2$, there should exist neighborhoods V_i of x_i such that

$$V_1 + V_2 \subset V.$$

Similarly, the assumption that scalar multiplication is continuous means that the mapping

$$(\alpha, x) \to \alpha x$$

of $K \times X$ into X is continuous : If $x \in X$, α is scalar, and V is a neighborhood of αx, then for some $r > 0$ and some neighborhood W of x we have $\beta W \subset V$ whenever $|\beta - \alpha| < r$.

Definition 1.1.3. *A subset E of a topological vector space is said to be* ***bounded*** *if for every neighborhood V of 0 in X corresponds a number $s > 0$ such that $E \subset tV$ for every $t > s$.*

Definition 1.1.4. ***Invariance*** *: Let X be a topological vector space. Associate to each $a \in X$ and to each scalar $\lambda \neq 0$ the translation operator T_a and the multiplication operator M_λ by the formulas*

$$T_a(x) = a + x, \quad M_\lambda(x) = \lambda x \quad (x \in X).$$

Proposition 1.1.5. *T_a and M_λ are homeomorphisms of X onto X.*

Definition 1.1.6. *A* ***local base*** *of a topological vector space X is a collection $\mathscr{B}$ of neighborhoods of 0 such that every neighborhood of 0 contains a member of $\mathscr{B}$. The open sets of X are then precisely those that are unions of translates of members of $\mathscr{B}$.*

Note 1.1.7. *In the vector space context, the term* ***local base*** *will always mean a local base at 0.*

Definition 1.1.8. *A metric d on a vector space X will be called* ***invariant*** *if*

$$d(x + z, y + z) = d(x, y)$$

for all x, y, z in X.

Types of topological vector spaces: In the following definitions, X always denotes a topological vector space, with topology τ.

(a) X is *locally convex* if there is a local base $\mathscr{B}$ whose members are convex.

(b) X is *locally bounded* if 0 has a bounded neighborhood.

(c) X is *locally compact* if 0 has a neighborhood whose closure is compact.

(d) X is metrizable if τ is compatible with some metric d.

(e) X is an *F-space* if its topology τ is induced by a complete invariant metric d.

(f) X is a *Frechet* space if X is a locally convex *F-space*.

1.2 Separation properties

Lemma 1.2.1. *: If W is a neighborhood of 0 in X, then there is a neighborhood U which is symmetric (in the sense that $U = -U$) and which satisfies $U + U \subset W$.*

Proof. Since $0+0 = 0$ and addition is continuous, there exist neighborhoods $V_1, V_2 \in \mathcal{N}_0$ such that

$$V_1, V_2 \subset W.$$

Set

$$U = V_1 \cap (-V_1) \cap V_2 \cap (-V_2).$$

U is symmetric; it is an intersection of four open sets that contain zero, hence it is a non-empty neighborhood of zero. Since $U \subset V_1$ and $U \subset V_2$ it follows that

$$U + U \subset W.$$

□

Note 1.2.2. *Similarly we can prove $U + U + U + U \subset W$.*

Theorem 1.2.3. *Suppose K and C are subsets of a topological vector space X, K is compact, C is closed, and $K \cap C = \phi$. Then 0 has a neighborhood V such that*

$$(K + V) \cap (C + V) = \phi.$$

Proof. If $K = \phi$, then $K + V = \phi$ and the conclusion of the theorem is obvious. We therefore assume that $K \neq \phi$ and consider a point $x \in K$. Since C is closed, since x is not in C, and since the topology of X is invariant under translations, the preceding proposition shows that 0 has a symmetric neighborhood V_x such that $x+V_x+V_x+V_x$ does not intersect C, the symmetry of V_x shows then that

$$(x + V_x + V_x) \cap (C + V_x) = \phi.$$

Since K is compact, there are finitely many points $x_1, x_2, \ldots, x_n$ in K such that

$$K \subset (x_1 + V_{x_1}) \cup \cdots \cup (x_n + V_{x_n}).$$

Put $V = V_{x_1} \cap \cdots \cap V_{x_n}$. Then

$$K + V \subset \bigcup_{i=1}^{n}(x_i + V_{x_i} + V) \subset \bigcup_{i=1}^{n}(x_i + V_{x_i} + V_{x_i}),$$

and no term in this last union intersects C + V. □

Theorem 1.2.4. *If $\mathscr{B}$ is a local base for a topological vector space X, then every member of $\mathscr{B}$ contains the closure of some member of $\mathscr{B}$.*

Proof. Let $U \in \mathscr{B}$. Let $K = \{0\}$ (compact) and $C = U^c$ (closed).

Implies there exists a $V \in \mathcal{N}_0^{sym}$, such that

$$V \cap (U^c + V) = \phi.$$

It follows that

$$V \subset (U^c + V)^c \subset U.$$

By the definition of a local base there exists a neighborhood $W \in \mathscr{B}$ s.t.

$$W \subset V \subset (U^c + V)^c \subset U.$$

Since $(U^c + V)^c$ is closed,

$$\overline{W} \subset (U^c + V)^c \subset U.$$

□

Proposition 1.2.5. *: Every topological vector space is a Hausdorff space.*

Theorem 1.2.6. *: Let X be a topological vector space.*

(a) If $A \subset X$ then $\overline{A} = \bigcap(A + V)$, where V runs through all neighborhoods of 0.
(b) If $A \subset X$ and $B \subset X$ then $\overline{A} + \overline{B} \subset \overline{A + B}$.
(c) If Y is a subspace of X, so is $\overline{Y}$.
(d) If C is a convex subset of X, so is $\overline{C}$ and C^0.
(e) If B is a balanced subset of X, so is $\overline{B}$; if also $0 \in B^0$ then B^0 is balanced.
(f) If E is a bounded subset of X, so is $\overline{E}$.

Theorem 1.2.7. : *In a topological vector space X,*

(a) every neighborhood of 0 contains a balanced neighborhood of 0, and

(b) every convex neighborhood of 0 contains a balanced convex neighborhood of 0.

Corolloary 1.2.8. :

(a) Every topological vector space has a balanced local base.

(b) Every locally convex space has a balanced convex local base.

Theorem 1.2.9. : *Suppose V is a neighborhood of 0 in a topological vector space X.*

(a) If $0 < r_1 < r_2 < \dots$ and $r_n \to \infty$, then

$$X = \bigcup_{i=1}^{\infty} r_n V.$$

(b) Every compact subset K of X is bounded.

(c) If $a_1 > a_2 > \dots$ and $a_n \to 0$ as $n \to \infty$, and if V is bounded, then the collection

$$\{a_n V : n = 1, 2, 3, ...\}$$

is a local base for X.

Proof. :

(a) Let $x \in X$ and consider the sequence x/r_n. This sequence converge to zero by the continuity of the scalar multiplication. Thus, for sufficiently large n,

$$x/r_n \in V \text{ i.e. } \quad x \in r_n V.$$

(b) Let $K \subset X$ be compact. We need to prove that it is bounded, namely, that for every $V \in \mathcal{N}_0$,

$$K \subset tV \quad \text{for sufficiently large t.}$$

Let $V \in \mathcal{N}_0$ be given, implies $\exists\ W \in \mathcal{N}_0^{bal}$ s.t. $W \subset V$. We have

$$K \subset \bigcup_{i=1}^{\infty}(nW).$$

Since K is compact, there are integers $n_1 < n_2 < \cdots < n_s$ such that

$$K \subset n_1 W \cup \cdots \cup n_s W = n_s W.$$

The equality holds because W is balanced. If $t > n_s$, it follows that $K \subset tW \subset tV$.

(c) Let U be a neighborhood of 0 in X. If V is bounded, there exists $s > 0$ such that $V \subset tU$, $\forall\ t > s$. If n is so large that $sa_n < 1$, it follows that $V \subset (1/a_n)U$. Hence U actually contains all but finitely many of the sets $a_n V$.

□

1.3 Linear Mapping

Definition 1.3.1. *: Suppose that X and Y are vector spaces over the same scalar field. A mapping $T : X \longrightarrow Y$ is said to be* ***linear*** *if*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall\ x, y \in X \quad \textit{and} \quad \forall\ \alpha, \beta \in K.$$

Linear mappings of X into its scalar field are called ***linear functionals****.*

Theorem 1.3.2. *: Let X and Y be topological vector spaces. If $T : X \longrightarrow Y$ is linear and continuous at 0, then T is continuous. In fact, T is uniformly continuous, in the following sense : To each neighborhood W of 0 in Y corresponds a neighborhood V of 0 in X such that*

$$y - x \in V \implies Ty - Tx \in W.$$

Theorem 1.3.3. *: Let T be a linear functional on a topological vector space X. Assume $Tx \neq 0$ for some $x \in X$. Then each of the following four properties implies the other three :*

(a) T is continuous.

(b) The null space $\mathscr{N}(T)$ is closed.
(c) $\mathscr{N}(T)$ is not dense in X.
(d) T is bounded in some neighborhood V of 0.

Proof. :

- (a) $\Longrightarrow$ (b) : Since $\mathscr{N}(T) = T^{-1}(\{0\})$ and $\{0\}$ is a closed subset of the scalar field K.

- (b) $\Longrightarrow$ (c) : By hypothesis, $\mathscr{N}(T) \neq X$ and $\mathscr{N}(T)$ is closed.

- (c) $\Longrightarrow$ (d) : Suppose that $\mathscr{N}(T)$ is not dense in X. That is, its complement has a non-empty interior. There exists an $x \in X$ and a $V \in \mathcal{N}_0$ such that
$$(x+V)\bigcap \mathscr{N}(T) = \phi.$$
This means that
$$0 \notin T(x+V)$$
i.e.
$$\forall y \in V, \quad Ty \neq -Tx.$$
we know that every neighborhood contains a balanced neighborhood, we may assume that V is balanced.
Implies T(V) is balanced as well.
Balanced set in $\mathbb{C}$ are either bounded, in which case we are done, of equal to the whole of $\mathbb{C}$, which contradicts the requirement that $Ty \neq -Tx$ for all $y \in V$.

- (d) $\Longrightarrow$ (a) : Suppose that T(V) is bounded for some $V \in \mathcal{N}_0$ i.e., there exists an M such that
$$\forall x \in V, \quad | Tx | \leq M.$$
Let $\epsilon > 0$ be given. Set W=$(\epsilon/M)V$. Then for all $y \in W$
$$| Ty | \leq \frac{\epsilon}{M} \sup_{x \in V} | Tx | \leq \epsilon,$$
which proves that T is continuous at zero (and hence everywhere).

□

1.4 Finite Dimensional Spaces

Lemma 1.4.1. *: If X is a complex topological vector space and* $f : \mathbb{C} \longrightarrow X$ *is linear, then* f *is continuous.*

Theorem 1.4.2. *: If* n *is a positive integer and* Y *is an* n*-dimensional subspace of a complex topological vector space* X*, then*

(a) every isomorphism of $\mathbb{C}^n$ *onto* Y *is a homeomorphism, and*

(b) Y *is closed.*

Theorem 1.4.3. *: Every locally compact topological vector space* X *has finite dimension.*

Proof. : Let X be a locally compact topological vector space: it has some neighborhood V whose closure $\overline{V}$ is compact. Being a compact set $\overline{V}$ is bounded so is V. Moreover,

$$\mathscr{B} \;=\; \{V/2^n : n \in \mathbb{N}\}$$

is a local base for X.
Let $y \in \overline{V}$. Implies $y - \frac{1}{2}V$ intersects V i.e. y $\in V + \frac{1}{2}V$,
So $\overline{V} \subset V + \frac{1}{2}V = \bigcup\limits_{x \in V} (x + \frac{1}{2}V)$.
Since $\overline{V}$ is compact, it can be covered by finite union:

$$\overline{V} \subset (x_1 + \frac{1}{2}V) \cup \cdots \cup (x_m + \frac{1}{2}V).$$

Let Y = Span$\{x_1, \ldots, x_m\}$. It is a finite dimension subspace of X, hence closed. Since V $\subset Y + \frac{1}{2}V$,
It follows that

$$\frac{1}{2}V \subset \frac{1}{2}Y + \frac{1}{4}V = Y + \frac{1}{4}V$$

and hence

$$V \subset Y + \frac{1}{2}V \subset Y + Y + \frac{1}{4}V = Y + \frac{1}{4}V.$$

By same procedure

$$V \subset Y + \frac{1}{8}V$$

Which implies,

$$V \subset \bigcap_{n=1}^{\infty}(Y + \frac{1}{2^n}V).$$

Since $\mathscr{B}$ is a local base at zero which implies that V is a subset of every neighborhood of Y

$$V \subset \overline{Y} = Y.$$

But we know that

$$X = \bigcup_{n=1}^{\infty} nV \subset Y$$

which implies that X=Y i.e. X is finite dimensional. □

Theorem 1.4.4. *: If X is a locally bounded topological vector space with the Heine-Borel property, then X has finite dimension.*

1.5 Metrization

Theorem 1.5.1. *: If X is a topological vector space with a countable local base, then there is a metric d on X such that*

(a) d is compatible with the topology of X,

(b) the open balls centered at 0 are balanced, and

(c) d is invariant: $d(x + z, y + z) = d(x, y)$ for x, y, $z \in X$.

If, in addition, X is locally convex, then d can be chosen so as to satisfy (a), (b), (c), and also

(d) all open balls are convex.

Proof. : Without loss of generality we can choose a countable loacl base whose member are balanced $\mathscr{B} = \{V_n\}$, and futhermore,

$$V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subset V_n.$$

(If X is locally convex, the local base can be chosen to include convex sets.) This implies that for all n and k:

$$V_{n+1} + V_{n+2} + \cdots + V_{n+k} \subset V_n.$$

Let D be the set of dyadic rational numbers:

$$D = \left\{ \sum_{n=1}^{\infty} \tfrac{c_n}{2^n} : c_n \in \{0,1\},\ c_n = 0\ for\ n > N,\ N \in \mathbb{N} \right\}$$

D is dense in [0,1]. Define the function $\varphi : D \cup \{r \geqslant 1\} \longrightarrow 2^X$:

$$\varphi(r) = \begin{cases} X & r \geq 1 \\ c_1(r)V_1 + c_2(r)V_2 + \dots & r \in D \end{cases}$$

The sum in this definition is always finite.For example, $\varphi(1.2) = X$ and $\varphi(0.75) = V_1 + V_2$. By property of $\mathscr{B}$,

$$\varphi\left(\sum_{n=N_1}^{N_2} \tfrac{c_n}{2^n} \right) = \sum_{n=N_1}^{N_2} c_n V_n \subset V_{N_1 - 1}$$

Then define the functional $f : X \longrightarrow \mathbb{R}$:

$$f(x) = \inf\{r : x \in \varphi(r)\}$$

and define

$$d(x,y) = f(y-x)$$

Claim 1: for s, r $\in D$

$$\varphi(r) + \varphi(s) \subseteq \varphi(r+s)$$

If $r + s \geq 1$ *then this is obvious as* $\varphi(r+s) = X$.

Suppose then that r, s $\in D$ and r+s $\in D$. The first possibility is that $c_n(r) + c_n(s) = c_n(r+s)$ for all n. This happens if $c_n(r)$ *and* $c_n(s)$ are never both equal to one. Then,

$\varphi(r+s) = \sum_{n=1}^{\infty} c_n(r+s)V_n = \sum_{n=1}^{\infty} c_n(r)V_n + \sum_{n=1}^{\infty} c_n(s)V_n = \varphi(r) + \varphi(s)$.

Otherwise, there exists an n for which

$$c_n(r) + c_n(s) \neq c_n(r+s)$$

Let N is the smallest n: then

$$c_N(r) = c_N(s) = 0 \text{ and } c_N(r+s) = 1$$

which implies

$$\begin{aligned}
\varphi(r) &\subseteq c_1(r)V_1 + \cdots + c_{N-1}(r)V_{N-1} + V_{N+1} + V_{N+2} + \ldots \\
&\subseteq c_1(r)V_1 + \cdots + c_{N-1}(r)V_{N-1} + V_{N+1} + V_{N+1} \\
\varphi(s) &\subseteq c_1(s)V_1 + \cdots + c_{N-1}(s)V_{N-1} + V_{N+1} + V_{N+1}
\end{aligned}$$

Implies

$$\begin{aligned}
\varphi(r) + \varphi(s) &\subseteq c_1(r+s)V_1 + \cdots + c_{N-1}(r+s)V_{N-1} + V_{N+1} + V_{N+1} + V_{N+1} + V_{N+1} \\
&\subseteq c_1(r+s)V_1 + \cdots + c_{N-1}(r+s)V_{N-1} + V_N \\
&\subseteq \varphi(r+s)
\end{aligned}$$

Observation : For all r $\in D \cup [1, \infty)$:

$$0 \in \varphi(r).$$

Claim 2 :

$$\text{if } r < t \text{ implies } \varphi(r) \subseteq \varphi(t)$$

For $r < t$:

$$\varphi(r) \subseteq \varphi(r) + \varphi(t-r) \subseteq \varphi(t).$$

Claim 3 : For all x, y $\in X$:

$$f(x+y) \leqslant f(x) + f(y)$$

Let x, y $\in X$ be given. Note that the range of f is [0,1], hence we can limit ourselves to the case where the right hand side is less than 1. Fix $\varepsilon > 0$. There are r, s $\in D$ such that

$$f(x) < r, \quad f(y) < s \text{ and } r + s < f(x) + f(y) + \varepsilon.$$

Implies $x \in \varphi(r),\ y \in \varphi(s)$, hence

$$x + y \in \varphi(r) + \varphi(s) \subseteq \varphi(r+s)$$

Thus,

$$f(x+y) \leqslant r + s < f(x) + f(y) + \varepsilon$$

true for all $\varepsilon > 0$.
Claim 4 :
(a) $f(x) = f(-x)$.
(b) $f(0) = 0$.
(c) $f(x) > 0\ for\ x \neq 0$.

Since the $\varphi(r)$ are unions of balanced sets they are balanced, from which follows that $f(x) = f(-x)$.
Since $0 \in \varphi(r) for\ all\ r \in D.\ f(0) = 0$.

Finally, if $x \neq 0$ it does no belong to some V_n (by separation) i.e., to some $\varphi(s)$, and since the $\{\varphi(r)\}$ form an ordered set, it does not belong to $\varphi(r)$ for all $r < s$, from which follows that $f(x) > 0$.

Now finally d is metric:
(a) $d(x,y) \geqslant 0$ as $f(y-x) \geqslant 0$.
(b) $d(x,y) = 0$ iff $f(y-x) = 0$ iff $y - x = 0$ iff $y = x$.
(c) $d(x,y) = d(y,x)$ as $f(x) = f(-x)$.
(d)

$$\begin{aligned} d(x,y) &= f(x-y) = f(x-z-(y-z)) \\ &\leqslant f(x-z) + f(y-z) \\ &= d(x,z) + d(z,y). \end{aligned}$$

We next want to show that this metric is compatible with the topology. Consider the d-open balls,

$$\mathcal{B}(0,t) = \{x : d(x,0) < t\} = \{x : f(x) < t\} = \bigcup_{r<t} \varphi(r).$$

Now if $t < \frac{1}{2^n}$ then $\mathcal{B}(0,t) \subseteq V_n$ which proves that the open balls, $\mathcal{B}(0, \frac{1}{2^n})$ forms a local base.
If the V_n are convex then $\varphi(r)$ are convex. $\square$

Definition 1.5.2. *: **Cauchy Sequences***

(a) *Suppose d is a metric on a set X. A sequence $\{x_n\}$ in X is a Cauchy sequence if to every $\varepsilon > 0$ there corresponds an integer N such that $d(x_n, x_m) < \varepsilon$ whenever $m > N$ and $n > N$. If every Cauchy sequence in X converges to a point of X, then d is said to be a complete metric on X (say d-cauchy sequence).*

(b) *Let τ be the topology of a topological vector space X.Fix a local base $\mathscr{B}$ for τ. A sequence $\{x_n\}$ in X is then said to be a Cauchy sequence if to every $V \in \mathscr{B}$ corresponds an N such that $x_n - x_m \in V$ if $n > N$ and $m > N$ (say τ-cauchy sequence).*

Note 1.5.3. *Since*

$$d(x_n, x_m) = d(x_n - x_m, 0) \qquad \textit{(since d is invariant)}$$

and since the d-balls centered at the origin form a local base for τ, we conclude,
A sequence x_n in X is a d-Cauchy sequence if and only if it is a τ-Cauchy sequence.

Note 1.5.4. *: If d_1 and d_2 are invariant metrics on a vector space X which induce the same topology on X, then*

(a) *d_1 and d_2 have the same cauchy sequence, and*
(b) *d_1 is complete iff d_2 is complete.*

Theorem 1.5.5. *: Suppose that (X, d_1) and (Y, d_2) are metric spaces, and (X, d_1) is complete. If E is a closed set in X, $f : E \longrightarrow Y$ is continuous, and*

$$d_2(f(x^{'}), f(x^{''}) \geqslant d_1(x^{'}, x^{''})$$

for all $x^{'}$, $x^{''} \in E$, then $f(E)$ is closed.

Theorem 1.5.6. *: Suppose Y is a subspace of a topological vector space X, and Y is an F-space (in the topology inherited from X). Then Y is a closed subspace of X.*

Proof. : Let d be such an invariant metric, and set

$$\mathscr{B}_{1/n} = \{y \in Y : d(y,0) < 1/n\}$$

which are open balls in Y. Let $U_n \in \mathscr{N}_0$ be neighborhood of zero in X s.t. $U_n \cap Y = \mathscr{B}_{1/n}$. Let then $V_n \in \mathscr{N}_0^{sym}$ s.t. $V_{n+1} \subset V_n$ and $V_n + V_n \subset U_n$. Let x$\in \overline{Y}$. Define

$$E_n = (x + V_n) \cap Y.$$

Suppose that $y_1, y_2 \in E_n$, then

$$y_1 - y_2 \in Y \text{ and } y_1 - y_2 \in V_n + V_n \subset U_n \textit{ implies } y_1 - y_2 \in \mathscr{B}_{1/n}.$$

The sets E_n are non-empty and their diameter tends to zero. Since Y is complete, intersection of the Y-closure of the sets E_n contains exactly one point, which we denote by y_0. y_0 is only point, s.t. for every $U' \in \mathscr{N}_0$ in X and every n,

$$(y_0 + U') \cap [(x + V_n) \cap Y] \neq \phi.$$

Since open neighborhoods in Y are intersections of open neighborhoods in X with Y, it follows that for every $U \in \mathscr{N}_0$ in X and every n

$$(y_0 + U) \cap (x + V_n) \cap Y \neq \phi.$$

Take now any neighborhood W$\in \mathscr{N}_0$ in X and define

$$F_n^W = (x + W \cap V_n) \cap Y.$$

By the exact same argument the intersection of the Y closure of the sets F_n^W contains exactly one point. But since $F_n^W \subset E_n$ this point has to be y_0. Thus, there exists a unique point y_0, such that for every U, W$\in \mathscr{N}_0$ in X and every n,

$$(y_0 + U) \cap (x + W \cap V_n) \cap Y \neq \phi.$$

Since the space is Hausdorff, $x = y_0 \in Y$ i.e. $\overline{Y} = Y$. □

Theorem 1.5.7. *:*
(a) If d is a translation-invariant metric on a vector space X then

$$d(nx, 0) \leqslant nd(x, 0)$$

for every $x \in X$ and for n=1,2,3,...
(b) If x_n is a sequence in a metrizable topological vector space X and if

$x_n \to 0$ as $n \to \infty$, then there are positive scalars γ_n such that $\gamma_n \to \infty$ and $\gamma_n x_n \to 0$.

1.6 Boundedness and Continuity

Definition 1.6.1. ***Bounded sets****:*
The notion of a bounded subset of a Topological vector space X already defined in ***Definition 1.1.3****. Now notion of boundedness w.r.t, metric d as follows: If d is a metric on a set X, a set $E \subseteq X$ is said to be d-bounded if there is a number $M < \infty$ such that $d(z, y) \leqslant M$ for all x and y in E.*

Note 1.6.2. *If X is a topological vector space with a compatible metric d, the bounded sets and the d-bounded ones need not be the same, even if d is invariant.*

Example 1.6.3. *: If d is a metric such as the one constructed in* ***Theorem 1.5.1***

$$d(x, y) = f(y - x)$$

$$where\ f(x) = inf\{r : x \in \varphi(r)\}\ and$$

$$\varphi(r) = \begin{cases} X & r \geq 1 \\ c_1(r)V_1 + c_2(r)V_2 + \dots & r \in D \end{cases} \quad and\ D\ is\ dyadic\ number$$

then X is itself d-bounded (with M=1) but X cannot be bounded unless X=$\{0\}$.

Note 1.6.4. *: (1) If X is a normed space and d is the metric induced by the norm, then the two notions of boundedness coincide ; but if d is replaced by $d_1 = d/(1 + d)$ (an invariant metric which induces the same topology) they do not.*
(2) Cauchy sequences are bounded and hence convergent sequences are bounded. Also closures of bounded sets are bounded.

Example 1.6.5. *: $E = \{nx : n = 1, 2, 3 \dots\}$ then E is not bounded because there is a neighborhood V of 0 that does not contain x; hence nx is not in nV; it follows that no nV contains E.*

Theorem 1.6.6. *: The following two properties of a set E in a topological vector space are equivalent :*

(a) E is bounded.

(b) If $\{x_n\}$ is a sequence in E and $\{\alpha_n\}$ is a sequence of scalers s.t. $\alpha_n \to 0$ as $n \to \infty$, then $\alpha_n x_n \to 0$ as $n \to \infty$.

Definition 1.6.7. ***Bounded Linear Transformations*** *: Suppose X and Y are topological vector spaces and $T : X \to Y$ is linear. T is said to be bounded if T maps bounded sets into bounded sets, i.e., if $T(E)$ is a bounded subset of Y for every bounded set $E \subseteq X$.*

Theorem 1.6.8. *: Suppose X and Y are topological vector spaces and $T : X \to Y$ is linear. Among the following four properties of T, the implications*

$$(1) \longrightarrow (2) \longrightarrow (3)$$

holds. If X is metrizable, then also

$$(3) \longrightarrow (4) \longrightarrow (1)$$

so that all four properties are equivalent.

(1) T is continuous.
(2) T is bounded.
(3) If $x_n \to 0$ then $\{Tx_n : n = 1, 2, 3...\}$ is bounded.
(4) If $x_n \to 0$ then $Tx_n \to 0$.

Proof. : (1) $\to$ (2). Let T is continuous. Let $V \in X$ be bounded. Let $W \in \mathscr{N}_0$ in Y. Since $T(0) = 0$, $\exists\ U \in \mathscr{N}_0,\ T(U) \in W$.
Because V is bounded there exists an s s.t. for all $t > s$,

$$V \subset tU.$$

By linearity, for all $t > s$,

$$T(V) \subset T(tU) = tT(U) \subset tW,$$

hence $T(V)$ is bounded, which implies that T is bounded.
(2) $\to$ (3). Let T is bounded and let $x_n \to 0$. Since convergent sequences are bounded,

$$E = \{x_n \mid n \in \mathbb{N}\}$$

is bounded,then

$$T(E) = \{T(x_n) \mid n \in \mathbb{N}\}$$

is bounded.
(3) $\to$ (4). Let X is metrizable and let d be a compatible and invariant metric. Let $x_n \to 0$ and $\{T(x_n) \mid n \in \mathbb{N}\}$ is bounded. As we have metrizability, there exists a sequence $\alpha_n \to \infty$, s.t. $\alpha_n x_n \to 0$.Then,

$$T(x_n) = \tfrac{1}{\alpha_n} T(\alpha_n x_n).$$

As $T(\alpha_n x_n)$ is bounded and by previous theorem $T(x_n) \to 0$.
(4) $\to$ (1). As we have given space is metrizable implies space is Hausdorff hence sequential continuity implies T is continuous.

□

1.7 Seminorms and Local Convexity

Definition 1.7.1. *A **seminorm** on a vector space X is a real-valued function p on X such that*
(a) $p(x+y) \leqslant p(x) + p(y)$ and
(b) $p(\alpha x) = |\alpha| p(x)$
for all x and y in X and all scalers α.

Definition 1.7.2. *A family $\mathscr{P}$ of seminorms on X is said to be **separating** if to each $x \neq 0$ corresponds at least one $p \in \mathscr{P}$ with $p(x) \neq 0$.*

Definition 1.7.3. *Consider a convex set $A \subseteq X$ called **absorbing** if for every $x \in X$ lies in tA for some $t > 0$. Every absorbing set contain 0.*

Definition 1.7.4. *For convex absorbing sets A we defined the **Minkowski functional** μ_A by*

$$\mu_A(x) = inf\{t > 0 : x \in tA\} \qquad (x \in X).$$

Note that $\mu_A(x) < \infty$ for all $x \in X$, since A is absorbing.

Properties 1.7.5. *If A is an absorbing set then*

(1) $\mu_A(x) \geq 0$, $\forall x \in X$ *and* $\mu_A(0) = 0$.

(2) $\mu_A(\lambda x) = \lambda \mu_A(x), \quad \forall \lambda \geq 0, \quad \forall x \in X.$

(3) $A \subset \{x \in X : \mu_A(x) \leq 1\}$

(4) $\mu_{A_1}(x) \leq \mu_{A_2}(x)$, $\forall x \in X$, *if* $A_2 \subset A_1$.

(5) Moreover, If the set A is convex then $\{x \in X : \mu_A(x) < 1\} \subset A$.

Theorem 1.7.6. *: Suppose p is a seminorm on a vector space X. Then*

(a) $p(0) = 0$.

(b) $|p(x) - p(y)| \leqslant p(x - y)$.

(c) $p(x) \geqslant 0$.

(d) $\{x : p(x) = 0\}$ *is a subspace of X.*

(e) The set $B = \{x : p(x) < 1\}$ *is convex, balanced, absorbing, and* $p = \mu_B$.

Proof. :

(a) $p(0) = p(0.x) = 0.p(x) = 0$

(b)
$$p(x) \leqslant p(y) + p(x - y)$$
and
$$p(y) \leqslant p(x) + p(y - x) = p(x) + p(x - y).$$

(c) for every x:
$$0 \leqslant |p(x) - p(0)| = |p(x)| \leqslant p(x).$$

(d) If $p(x) = p(y) = 0$ and α, β are scalers then,
$$0 \leqslant p(\alpha x + \beta y) \leqslant |\alpha| p(x) + |\beta| p(y) = 0.$$

(e) If $p(x) < 1$ then for every $|\alpha| \leqslant 1$, $p(\alpha x) = |\alpha| p(x) < 1$, implies B is balanced. If $x, y \in B$ then for every $0 \leqslant t \leqslant 1$

$$p(tx + (1-t)y) \leqslant tp(x) + (1-t)p(y) < 1$$

hence B is convex. For every $x \in X$ and $s > p(x)$, $p(x/s) < 1$, i.e., $x \in sB$, so that B is absorbing.

$$\begin{aligned} \mu_B(x) =& \inf\{s > 0 : x \in sB\} \\ =& \inf\{s > 0 : p(x/s) < 1\} \\ =& \inf\{s > 0 : p(x) < s\}. \end{aligned}$$

If $p(x) < r$ then $\mu_B(x) < r$ i.e. $\mu_B(x) \leqslant p$. If $\mu_B(x) < r$, then there exists an $s < \mu_B(x)$ s.t. $p(x) = s$ i.e. $p \leqslant \mu_B(x)$.

□

Theorem 1.7.7. *: Suppose A is a convex absorbing set in vector space X. Then*

(a) $\mu_A(x+y) \leqslant \mu_A(x) + \mu_A(y)$.

(b) $\mu_A(tx) = t\mu_A(x)$ *if* $t \geqslant 0$.

(c) μ_A *is seminorm if A is balanced.*

(d) If $B = \{x : \mu_A(x) < 1\}$ *and* $C = \{X : \mu_A(x) \leqslant 1\}$, *then* $B \subset A \subset C$ *and* $\mu_B = \mu_A = \mu_C$.

Proof. :
(a) If $t = \mu_A(x) + \varepsilon$ and $s = \mu_A(y) + \varepsilon$, for some $\varepsilon > 0$, then x/t and y/s are in A, hence so is their convex combination

$$\tfrac{x+y}{s+t} = \tfrac{t}{s+t} \cdot \tfrac{x}{t} + \tfrac{s}{s+t} \cdot \tfrac{y}{s}$$

This implies that $\mu_A(x+y) \leqslant s + t = \mu_A(x) + \mu_A(y) + 2\varepsilon$.
(b) From definition.
(c) Follows from (a) and (b).
(d) Let $x \in B$

There exists $r \in (0,1)$ *s.t.* $x \in rA$
imlies $a = x/r \in A$

$ra + (1-r)0 = x \in A$.
Let $a \in A$ $1 \in \{s > 0 : a \in sA\}$
$\Rightarrow a \in C$
$\Rightarrow B \subset A \subset C$.
This shows that $\mu_C \leqslant \mu_A \leqslant \mu_B$. To prove equality, fix $x \in X$, and choose s,t so that $\mu_C(x) < s < t$. Then $x/s \in C$, $\mu_A(x/s) \leqslant 1, \mu_A(x/t) \leqslant s/t < 1$, hence $x/t \in B$ so that $\mu_B(x) \leqslant t$. This holds for every $t > \mu_C(x)$. Hence $\mu_B(x) \leqslant \mu_C(x)$.

□

Theorem 1.7.8. *: Suppose $\mathscr{B}$ is a convex balanced local base in a topological vector space X. Associate to every $V \in \mathscr{B}$ its Minkowski functional μ_V. Then*

(a) $V = \{x \in X : \mu_V(x) < 1\}$, for every $V \in \mathscr{B}$, and

(b) $\{\mu_V : V \in \mathscr{B}\}$ is a separating family of continuous seminorms on X.

Proof. :

(a) If $x \in V$ then $x \in tV$ *or* $x/t \in V,\ t < 1$ because V is open.
Hence $\mu_V(x) < 1$.
If $x \notin V$ then $x \in tV$ *or* $x/t \in V,\ t \geqslant 1$ as V is balanced.
Hence $\mu_V(x) \geqslant 1$.

(b) By **Theorem 1.7.5 (e)** μ_V is seminorm
also

$$| \mu_V(x) - \mu_V(y) | \leqslant \mu_V(x-y) < r$$

if $x - y \in rV$ then μ_V is continuous. If $x \in X,\ x \neq 0$, then $x \notin V$ for some $V \in \mathscr{B}$. For this V, $\mu_V(x) \geqslant 1$. Thus $\{\mu_V\}$ is separating.

□

Theorem 1.7.9. *Suppose $\mathscr{P}$ is a separating family of seminorms on a vector space X. Associate to each $p \in \mathscr{P}$ and to each positive integer n the set*

$$V(p,n) = \left\{ x : p(x) < \frac{1}{n} \right\}$$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $V(p,n)$. Then $\mathscr{B}$ is a convex balanced local base for a topology τ on X, which turns X into a

locally convex space such that
(a) every $p \in \mathscr{P}$ is continuous, and
(b) a set $E \subset X$ is bounded iff every $p \in \mathscr{P}$ is bounded on E.

Proof. We define

$$\mathscr{B} = \left\{ \bigcap_{(p,n)\in I} V(p,n) : I \subset P \times \mathbb{N},\ I\ is\ finite\ set \right\}$$

where all the intersection are finite. Clearly $\mathscr{B}$ is a collection of set that contain the origin and that are closed under finite intersection.
Set $A \subset X$ be open if it is a union of translate of member of $\mathscr{B}$ i.e.,
A is open iff

$$A = \bigcup_{B\in\mathscr{B}, x\in A} (x + B)$$

or for $x \in A$, $\exists N_x \in \mathscr{B}\ s.t.\ x + N_x \subset A$. We denote the collection of all such set by τ. τ is translation invariant topology on X because
(1) $\phi \in \tau$ by taking empty union.
(2) $X \in \tau$.
(3) τ is closed under arbitrary union. Obvious as open set are defined in such a way.
(4) τ is closed under finite intersection.
$\bigcap\limits_{i\in I} A_i$, take $x \in \bigcap\limits_{i\in I} A_i$ where I is finite.
implies

$$x \in A_i, \quad \forall i \in I$$

implies

$$\exists N_x \in \mathscr{B}\ s.t.\ x + N_x^i \subset A_i$$

$$\cap N_x^i \in \mathscr{B}\ also\ x + \cap N_x^i \subset A_i, \quad \forall i$$

implies

$$x + \bigcap_{i\in I} N_x^i \subset \bigcap_{i\in I} A_i$$

Now each member of $\mathscr{B}$ is convex and balanced and $\mathscr{B}$ is local base for τ.
Let $0 \neq x \in X$. We have family of separating seminorm so

$$\exists p \in \mathscr{P}, \quad p(x) > 0$$

$$\exists n \in \mathbb{N}\ s.t.\ np(x) \geqslant 1, \quad p(x) \geqslant 1/n$$

$$x \notin V(p,n) \textit{ or } 0 \notin x - V(p,n)$$

$\{0\}$ is closed set as Hence $x \notin \{0\}$.
$\{x\}$ is closed by translation invariant.

Let $U \in \mathscr{N}$, by property of local base, there exists

$$V(p_1, n_1) \cap \cdots \cap V(p_k, n_k) \subset U.$$

Take

$$V = V(p_1, 2n_1) \cap \cdots \cap V(p_k, 2n_k).$$

But we have,

$$V(p_i, 2n_i) + V(p_i, 2n_i) = V(p_i, n_i).$$

Hence $V + V \subset U$ gives addition is continuous.

Suppose now that $x \in X$, α is a scaler, and U and V are as above. Then $x \in sV$ for some $s > 0$. Put $t = s/(1+|\alpha|s)$. If $y \in x+tV$ and $|\beta-\alpha| < 1/s$, then

$$\beta y - \alpha x = \beta(y - x) + (\beta - \alpha)x$$

which lies in

$$|\beta|tV + |\beta - \alpha|sV \subset V + V \subset U$$

since $|\beta|t \leqslant 1$ and V is balanced.

$$\beta y \in \alpha x + U.$$

Hence scaler multiplication is continuous.
Thus X is locally convex topological vector space.

(a) Also every $p \in \mathscr{P}$ is continuous at 0 because for every $\varepsilon > 0$ set $n > \frac{1}{\varepsilon}$ and $p(V(p,n)) < 1/n$

$$\text{Take } x \in V(p,n), \quad |p(x) - p(0)| < 1/n < \varepsilon.$$

Hence it is continuous everywhere by

$$|p(x) - p(y)| \leqslant p(x - y).$$

(b) Let E is bounded. Take $p \in \mathscr{P}$ then

$$V(p,1) = \{x \in X : p(x) < 1\}$$

implies $E \subset nV(p,1)$ for some n (by definition).
Conversly, take U neighbourhood of 0 s.t.,

$$V(p_1,n_1) \cap \cdots \cap V(p_m,n_m) \subset U$$

implies $\exists M_i < \infty \; s.t. \; p_i(E) < M_i, \quad \forall i = [1:m]$
If $n > M_i n_i$ for $i = [1:m]$ then
$nV(p_i,n_i) = \{nx : p(nx) < n/n_i\} = \{y : p(y) < n/n_i\}$.
Hence $E \subset nU$ and E is bounded.

□

Note 1.7.10. *If $\mathscr{B}$ is a convex balanced local base for topology τ of Locally convex space X then $\mathscr{B}$ generates a separating family $\mathscr{P}$ of continuous seminorm on X. This $\mathscr{P}$ induces a topology τ_1 in X then we have $\tau = \tau_1$. As every $p \in \mathscr{P}$ is continuous w.r.t, τ topology implies $V(p,n) \in \tau$. Hence $\tau_1 \subset \tau$.
Now take $W \subset \mathscr{B}$ and $p = \mu_W$ then*

$$W = \{x : \mu_W(x) < 1\} = V(p,1)$$

implies $W \in \tau_1$ for every $W \in \mathscr{B}$. Hence $\tau \subset \tau_1$.

Theorem 1.7.11. *A topological vector space X is normable iff its origin has a convex bounded neighbourhood.*

Proof. Let if X is normable and $\|.\|$ is norm which is compatible with our topology on X then the open unit ball $\{x : \|x\| < 1\}$ is convex and bounded.
Let V be a convex bounded neighbourhood of 0 then $\exists$ a convex balanced neighbourhood of 0, say U (by **Theorem 1.2.7**) s.t., $U \subset V$, clearly U is bounded.
Now define

$$\|x\| = \mu_U(x), \qquad x \in X \tag{1.1}$$

This is a seminorm by **Theorem 1.7.6**. Now αU $(\alpha > 0)$ form a local base of (X,τ). Now take $x \neq 0$ implies $\exists\; r > 0 \; s.t., \; x \notin rU$ as U is absorbing.

Hence (1.1) is norm.
As U is open

$$\begin{aligned}U =&\{x \in X : \mu_U < 1\}\\ U =&\{x \in X : \|x\| < 1\}\\ rU =&\{x \in X : \|x\| < r\},\ \forall r > 0.\end{aligned}$$

Hence the norm topology coincide with τ. □

1.8 Quotient Space

Definition 1.8.1. *Let N be a subspace of X. For every $x \in X$ define $\{x+n : n \in N\} = x + N$ called coset. Take all collection of coset with opertion:*

$$(x + N) + (y + N) = x + y + N$$

$$\text{and } \alpha(x + N) = \alpha x + N.$$

This gives a vector space called ***Quotient Space****. Denote by X/N.*

Definition 1.8.2. *Define $\pi : X \to X/N$ by*

$$\pi(x) = x + N$$

Here π is linear map where N is kernel of π and this map is called ***Quotient Map****.*

Definition 1.8.3. *Now let (X, τ) be a TVS (Topological vector space) and N be a closed subspace of X. Define collection of subset of X/N (say τ_N) to be*

$$E \in \tau_N \text{ if } \pi^{-1}(E) \in \tau, \text{ where } E \subset X/N$$

Now τ_N is a topology on X/N called ***Quotient Topology****.*

Theorem 1.8.4. *Let N be a closed subspace of a topological vector space X. Let τ be the topology of X and define τ_N as above.*

(a) *τ_N is a vector topology on X/N; the quotient map $\pi : X \to X/N$ is linear, continuous, and open.*

(b) If $\mathscr{B}$ is a local base for τ, then the collection of all sets $\pi(V)$ with $V \in \mathscr{B}$ is a local base for τ_N.

(c) Each of the following properties of X is inherited by X/N: local convexity, local boundedness, metrizability, normability.

(d) If X is an F-space, or a Frechet space, or a Banach space, so is X/N.

Proof. :

(a) Here π is an onto map.
And we know that:

$$\pi^{-1}\left(\bigcup_{\alpha\in\Lambda} E_\alpha\right) = \bigcup_{\alpha\in\Lambda} \pi^{-1}(E_\alpha)$$

Also

$$\pi^{-1}\left(\bigcap_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} \pi^{-1}(E_i)$$

Hence from this information we can say τ_N is a topology on X/N.
Now it remains to prove that its a vector topology.

Observe that A set $F \subset X/N$ is τ_N-close iff $\pi^{-1}(F)$ *is* τ-closed. Also

$$\pi^{-1}(\pi(x)) = \bigcup_{n\in N} n + x = N + x.$$

Hence $N + x$ is $\tau-$closed as N is closed.
$\Rightarrow \pi(x)$ *is* τ_N-closed.
$\Rightarrow N + x$ *is* τ_N-closed. Hence singleton set are closed in X/N.
By definition of τ_N, π it is continuous.

Now let $V \in \tau$. As

$$\pi^{-1}(\pi(V)) = \bigcup_{n\in N} n + V = N + V$$

and $N + V \in \tau$ (union of translation of open sets)
Implies $\pi(V) \in \tau_N$

Implies π is an open mapping.
Now

$$\pi(x+y) = x+y+N = x+N+y+N = \pi(x)+(y)$$

Also

$$\pi(\alpha x) = \alpha x + N = \alpha(x+N) = \alpha\pi(x)$$

Hence π is linear.
Now let W be neighbourhood of 0 in τ_N. $0 \in \pi^{-1}(W) \in \tau$
Implies $\exists \quad (0 \in) V \in \tau \ s.t.$

$$V + V \subset \pi^{-1}(W)$$

Hence $\pi(V+V) \subset W \ \Rightarrow \pi(V) + \pi(V) \subset W$.
Since π is open, $\pi(V)$ is neighbourhood of 0 in X/N. Hence addition is continuous.
Now let W be a neighbourhood of 0 in X/N. So $\pi^{-1}(W)$ be a neighbourhood of 0 in X.
$\Rightarrow \exists \quad$ V nbd of 0 in X and $|\beta - \alpha| < \varepsilon$
$\Rightarrow \beta V \subset \pi^{-1}(W)$
$\Rightarrow \beta\pi(V) \subset W$
$\pi \ is\ open \Rightarrow \pi(V) \in \mathscr{N}_0^{X/N}$
Hence scaler multiplication is continuous.

(b) By definition of τ_N, (b) is true by (a).

(c) Comes by (b).

(d) Let X be a F-space and let d is invariant metric on X compatible with τ.
Define

$$\lambda(\pi(x), \pi(y)) = \inf\{d(x-y, z) : z \in N\}.$$

Here λ is well defined and is invariant metric on X/N.

$$\pi(\{x : d(x,0) < r\}) = \{u : \lambda(u,0) < r\}.$$

By (b) λ is compatible with τ_N.
If X is NLS, define

$$\|\pi(x)\| = \inf\{\|x - z\| : z \in N\}$$

called **Quotient Norm.**

Now to prove λ is complete metric whenever d is complete:
Suppose $\{u_n\}$ is a Cauchy sequence in X/N, relative to λ. There is a subsequence $\{u_{n_i}\}$ with $\lambda(u_{n_i}, u_{n_{i+1}}) < 2^{-i}$. One can then inductively choose $x_i \in X$ such that $\pi(x_i) = u_{n_i}$, and $d(x_i, d_{i+1}) < 2^{-i}$. If d is complete, the Cauchy sequence $\{x_i\}$ converges to some $x \in X$. The continuity of π implies that $u_{n_i} \to \pi(x)$ as $i \to \infty$. But if a Cauchy sequence has a convergent subsequence then the full sequence must converge. Hence λ is complete.

□

Theorem 1.8.5. *Suppose N and F are subspace of TVS X. N is closed and F has finite dimension. Then N+F is closed.*

Proof. Let π be a quotient map from X to X/N. Then $\pi(F)$ is finite dim space of X/N as F is finite dim. But X/N is TVS and finite dim subspace of TVS is closed in X/N.
Also

$$\pi^{-1}(\pi(F)) = \bigcup_{n \in N} n + F = N + F$$

and π is continuous hence N+F is closed. □

1.9 Examples

Example 1.9.1. ***The space*** $C(\Omega)$ *: Let Ω be open set in $(\mathbb{R}^n, \|.\|_2)$. We consider space $C(\Omega)$ a vector space of all comples valued continuous function. Here sup norm won't work as there exists unbounded continuous function on open sets.*
We know open set Ω can be written as

$$\Omega = \bigcup_{n=1}^{\infty} K_n$$

where K_n's are compact set s.t., K_n is in the interior of K_{n+1}.

We topologize $C(\Omega)$ with separating family of seminorms,

$$p_n(f) = sup\{|f(x)| : x \in K_n\}$$

*by **Theorem 1.7.8**.*
Since $p_1 \leqslant p_2 \leqslant \dots$ the sets

$$V_n = \{f \in C(\Omega) : p_n(f) < 1/n\}$$

forms a convex local base for $C(\Omega)$.
Now topology of $C(\Omega)$ is compatible with the metric,

$$d(f, g) = \max_{\boldsymbol{n}} \frac{p_n(f-g)}{2^n(1 + p_n(f-g))}$$

Now let $\{f_i\}$ be cauchy sequence relative to this metric d, an easy computation shows that $p_n(f_i - f_j) \to 0$, $\forall n$ as $i, j \to \infty$.
Hence $\{f_i\}$ converges uniformly on K_n to a function $f(n) \in C(K_n)$ but $K_n^0 \subset K_{n+1}$ so restriction of $f(n+1)$ on K_n would be $f(n)$ so we conclude $\{f_i\}$ converges to $f \in C(\Omega)$.
Now for given $\varepsilon > 0$ let $2^{-K} < \varepsilon$ then

$$\max_{k>K} \frac{p_k(f_n - f)}{2^k(1 + p_k(f_n - f))} < \varepsilon$$

also $\exists N$ s.t., $\forall n > N$

$$\max_{k \leqslant K} \frac{p_k(f_n - f)}{2^k(1 + p_k(f_n - f))} < \varepsilon.$$

Hence $d(f, f_i) \to 0$. Thus d is a complete metric. Hence $C(\Omega)$ is a Frechet space.
A set $E \subset C(\Omega)$ is bounded iff there are numbers $M_n < \infty$ s.t., $p_n(f) \leqslant M_n$, $\forall f \in E$ ie, $|f(x)| \leqslant M_n$ if $f \in E$ and $x \in K_n$.

Choose $K_n \subsetneq K_{n+1}$ then $\overline{K_{n+1} \setminus K_n} \subset K_{n+1}$.
Let $f_j \in V_n$

$$\text{Define } f_j : \Omega \longrightarrow \mathbb{R} \text{ by } f_j(x) = \frac{jd(x, K_n)}{d(x, K_n) + d(x, \overline{K_{n+1} \setminus K_n})}$$

Hence every V_n contains f for which p_{n+1} is large as we please. Hence V_n is unbounded true for all n. Hence $C(\Omega)$ is not normable as it is not locally bounded.

Example 1.9.2. ***The spaces*** $C^\infty(\Omega)$ *and* $\mathscr{D}_K$ *: A complex function* f *defined in some nonempty open set* $\Omega \subset \mathbb{R}^n$ *is said to belong to* $C^\infty(\Omega)$ *if* $D^\alpha f \in C(\Omega)$ *for every multi-index* α*. If* K *is a compact set in* $\mathbb{R}^n$*, then* $\mathscr{D}_K$ *denotes the space of all* $f \in C^\infty(\mathbb{R}^n)$ *whose support lies in* K*. If* $K \subset \Omega$ *then* $\mathscr{D}_K$ *is subspace of* $C^\infty(\Omega)$*.*

Choose compact set $K_i (i = 1, 2, 3 \dots)$ *s.t.* K_i *lies in the interior of* K_{i+1} *and* $\Omega = \cup K_i$*. Define seminorms* p_N *on* $C^\infty(\Omega)$*,* $N = 1, 2, 3 \dots$ *by*

$$p_N(f) = max\{|D^\alpha f(x)| : x \in K_N, |\alpha| \leqslant N\}.$$

With these collection of seminorms we have metrizable locally convex topology on $C^\infty(\Omega)$*.*
Now for $x \in \Omega$ *define* $J_x : C^\infty(\Omega) \to \mathbb{R}$ *by* $J_x(f) = f(x)$ *which is a continuous function in our topology.* $\mathscr{D}_K$ *is the intersection of kernel of these function where* x *ranges in* K^c*, hence* $\mathscr{D}_K$ *is closed in* $C^\infty(\Omega)$*.*
Here $p_1 \leqslant p_2 \leqslant \dots$
Hence a local base is given by the sets

$$V_N = \left\{ f \in C^\infty(\Omega) : p_N(f) < 1/N \right\} \qquad (N = 1, 2, \dots)$$

as $V_1 \subset V_2 \subset \dots$
Now if $\{f_i\}$ *is cauchy sequence in* $C^\infty(\Omega)$ *and if* N *is fixed then* $f_i - f_j \in V_N$ $for\ sufficiently\ large\ i, j$*.*
Thus $|D^\alpha f_i - D^\alpha f_j| < 1/N$ *on* K_N *if* $|\alpha| \leqslant N$
Implies $D^\alpha f_i \longrightarrow g_\alpha^{K_N}$ *uniformly on* K_N *but* $g_\alpha^{K_N}\big|_{K_{N-1}} = g_\alpha^{K_{N-1}}$
So $D^\alpha f_i \longrightarrow g_\alpha$ *uniformly on compact subset of* Ω*.*
In particular $f_i(x) \to g_0$*. Hence* $g_0 \in C^\infty(\Omega)$ *and* $g_\alpha = D^\alpha g_0$ *and* $f_i \to g$ *in topology of* $C^\infty(\Omega)$*.*
Thus $C^\infty(\Omega)$ *is a Frechet space. So is closed subspace* $\mathscr{D}_K$*.*

Now the metric constructed by these seminorms which is compatible with the topology on $C^\infty(\Omega)$ *is a bounded metric but no norm is bounded, so the metric is not induced by any norm. Hence* $C^\infty(\Omega)$ *is not normable.*

Chapter 2

Convexity in Topological Vector Space

2.1 Introduction

Definition 2.1.1. *: A* ***topological vector space*** *X is a vector space over a topological field K that is endowed with topology s.t. vector addition $+ : X \times X \to X$ and scaler multiplication $\cdot : K \times X \to X$ are continuous function.*

Example 2.1.2. *: Every normed linear space has a natural topology such that (the norm induced metric and metric induces a topology.) :*
1. $+$ is continuous with this topology by tringle inequality.
2. $\cdot$ is continuous with this topology by tringle inequality and homogenity of norm.

Theorem 2.1.3. *: The mapping $x \to x + x_0$ and $x \to \lambda x$ where $\lambda \neq 0$, $\lambda \in K$ are homeomorphic of X onto itself.*

Theorem 2.1.4. *: If f is a linear functional on a topological linear space, then the following statements are equivalent:*

(i) f is continuous.

(ii) The kernel of f, $kerf = \{x : f(x) = 0\}$ is closed.

(iii) There is a neighborhood of the origin on which f is bounded.

Note 2.1.5. *From now on we consider X, Y to be normed linear space and $\boldsymbol{K}$ to be field (either $\mathbb{R}$ or $\mathbb{C}$) unless stated.*

Definition 2.1.6. *: Let X, Y be normed linear space of same nature. A linear map $T : X \to Y$ is said to be* ***bounded*** *if $T(M)$ is bounded in Y for every bounded set $M \subset X$. In other words,there exists $N > 0$ s.t.*

$$\|Tx\| \leqslant N \|x\|, \qquad \forall x \in X$$

In normed linear space linear map $T : X \to Y$ is continuous iff it is bounded.

Definition 2.1.7. *: $L(X, Y)$ The set of all linear continuous map from X with values in Y which becomes a normed linear space by*

$$\|T\| = sup\{\|Tx\| : \|x\| \leqslant 1\} = inf\{K : \|Tx\| \leqslant K \|x\|, \forall x \in X\} \tag{2.1}$$

Definition 2.1.8. *A complete normed linear space is called* ***Banach space****.*

Lemma 2.1.9. *If Y is a Banach space, the $L(X, Y)$ is also a Banch space.*

Definition 2.1.10. *If we consider $Y = \boldsymbol{K}$ then $L(X, \boldsymbol{K})$ is called* ***dual*** *of X i.e, set of all continuous linear functional on X. Denoted as X^*. It becomes a banach space by introducing norm on functional given in equation (2.1)*

$$\|x^*\| = sup\{|x^*(x)| : \|x\| \leqslant 1\}.$$

Hence if $x^* \in X^*$ then

$$|x^*(x)| \leq \|x^*\| \|x\| \qquad \forall x \in X.$$

Definition 2.1.11. *A family $\mathscr{A} \subset L(X, Y)$ is called* ***uniformly bounded*** *if*

$$\sup_{T \in \mathscr{A}} \|T\| < \infty$$

Definition 2.1.12. *A family $\mathscr{A} \subset L(X, Y)$ is called* ***pointwise bounded*** *if for fixed $x \in X$, $\mathscr{A}_x = \{Tx : T \in \mathscr{A}\}$ is bounded set in Y.*

Theorem 2.1.13. *If X is a Banach space, then every pointwise bounded family of linear continuous maps from $L(X, Y)$ is uniformly bounded.*

Definition 2.1.14. *A mapping $\langle \cdot, \cdot \rangle : X \times X \to \boldsymbol{K}$ is said to be an* ***inner product*** *if it has the following properties :*

(1) $\langle x,x\rangle \geq 0, \forall x \in X$ *and* $\langle x,x\rangle = 0$ *implies* $x = 0$.

(2) $\langle x,y\rangle = \langle \overline{y,x}\rangle\ \forall x,y \in X$.

(3) $\langle ax+by,z\rangle = a\langle x,z\rangle + b\langle y,z\rangle \quad \forall a,b \in \boldsymbol{K},\ \forall x,y \in X$.

Definition 2.1.15. *A linear space endowed with an inner product is called a **pre-Hilbert space**. A pre-Hilbert space is also considered as a linear normed space by the norm induced by inner product.*

$$\|x\| = \langle x,x\rangle^{\frac{1}{2}}, \quad \forall x \in X$$

Definition 2.1.16. *Two elements x and y in pre Hilbert space are said to be **orthogonal** if* $\langle x,y\rangle = 0$. *Denoted by* $x \perp y$.
If $x \perp y = 0\ \forall y \in X$ *then* $x = 0$.

Proposition 2.1.17. *The elements x, y are orthogonal iff*

$$\|x+\lambda y\| \geq \|x\|, \qquad \forall \lambda \in \boldsymbol{K}$$

Definition 2.1.18. *If a pre-Hilbert space is complete in the norm associated to the given inner product, then it is called a **Hilbert space**.*

Theorem 2.1.19. ***Riesz*** *: If f is a continuous linear functional on the Hilbert space X, then there exists a unique element* $a \in X$ *such that*

$$f(x) = \langle x,a\rangle, \quad \forall x \in X$$

$$\|f\| = \|a\|.$$

Conversly, for every $a \in X$*, the linear functional* $f_a : X \to \boldsymbol{K}$ *defined by*

$$f_a(x) = \langle x,a\rangle, \quad \forall x \in X$$

is continuous, hence $f_a \in X^*$*, also* $\|f_a\| = \|a\|, \forall a \in X$.

2.2 Convex Sets

Let X be a real linear space.

Definition 2.2.1. *A subset C of linear space X is said to be convex if, for all x and y in C and all t in the interval $(0, 1)$, the point $(1-t)x + ty$ also belongs to C. In other words, every point on the line segment connecting x and y is in C. We denote*

$$[x, y] = \{\lambda_1 x + \lambda_2 y;\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 = 1\}$$

called the segment generated by elements x, y.

Definition 2.2.2. *A subset $\boldsymbol{A}$ of a linear space X is called affine set if for all $x, y \in A$ implies $\lambda_1 x + \lambda_2 y \in A$, $\forall \lambda_1, \lambda_2 \in \mathbb{R}$ where $\lambda_1 + \lambda_2 = 1$.*

If $x_1, x_2, \ldots, x_n \in X$, every element of the form $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$ where $\lambda_i \in \mathbb{R}$ and $\sum_{i=1}^{n} \lambda_i = 1$ is called an affine combination of $x_1, x_2, \ldots, x_n$. If $\lambda_i \geq 0$, then affine combination is called a convex combination.

Proposition 2.2.3. *Any convex (affine) set contains all the convex (affine) combinations formed with its elements.*

Proof. Let C is a convex(affine) subset of X.
We prove by mathematical induction, as for $k = 2$ the result is obvious by definition. Let the hypothesis is true for $k = n - 1$.
Take convex (affine)combination of n elements $x_1, x_2, \ldots, x_n \in C$,

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \lambda_1 x_1 + \cdots + \lambda_{n-2} x_{n-2} + (\lambda_{n-1} + \lambda_n)\bar{x}_{n-1} \in C,$$

where

$$\bar{x}_{n-1} = \frac{\lambda_{n-1}}{\lambda_{n-1} + \lambda_n} x_{n-1} + \frac{\lambda_n}{\lambda_{n-1} + \lambda_n} x_n$$

as $\bar{x}_{n-1} \in C$ whenever $\lambda_{n-1} + \lambda_n \neq 0$ (As $\lambda_i + \lambda_j \neq 0$ for some $i, j \in \{1, \ldots, n\}$ *and* $i \neq j$ otherwise all $\lambda_i = 0$).
Hence true for all finite $n \in \mathbb{N}$. □

Properties 2.2.4.

(1) *The intersection of many arbitrary convex (affine) sets is again a convex (affine) set.*

(2) The union of a directed by inclusion family of convex (affine) sets is a convex (affine) set.

(3) If $A_1, A_2, \ldots, A_n$ are convex(affine) sets and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, then $\lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n$ is a convex (affine) set.

(4) The linear image and the linear inverse image of a convex (affine) set are again convex (affine) sets.

(5) If X is a linear topological space, then the closure and the interior of a convex (affine) set is again convex (affine).

Definition 2.2.5. *Let C be any arbitrary subset of X, then intersection of all convex(affine) sets containing C is called* ***convex (affine) hull****,i.e. the smallest convex (affine) set which contain C. Denoted by conv C (aff C).*

$$\text{conv } C = \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid n \in \mathbb{N}, \lambda_i \geq 0, x_i \in C, \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

$$\text{aff } C = \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid n \in \mathbb{N}, \lambda_i \in \mathbb{R}, x_i \in C, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Proposition 2.2.6. *In a real linear space, a set is affine iff it is a translation of a linear subspace.*

Definition 2.2.7. *A point $a_0 \in X$ (real linear space) is said to be* ***algebraic relative interior*** *of $A \subset X$ if, for every straight line through a_0 which lies in affA, there exists an open segment contained in A which contains a_0. If affA $= X$, the point a_0 is called the* ***algebraic interior*** *of A. The set of all the algebraic (relative) interior points of A is called the algebraic (relative) interior of the set A and we denote it by $(A^{ri})A^{i}$.*

$$A^{ri} = \{a_0 \in A : \forall x \in X, \exists\ t_x > 0,\ \forall t \in [0, t_x],\ a_0 + tx \in A\}$$

Example 2.2.8. *Let $X = \mathbb{R}^2$ and $A = (0,1) \times \{0\}$, then every point of A is an algebraic relative interior.*
Now if $A = (0,1) \times (0,1)$ then every point of A is an algebraic interior.

Definition 2.2.9. *If X is a topological vector space, then a point $a_0 \in X$ is said to be a* ***relative interior*** *of $A \subset X$ if a_0 is contained in an open subset of affA (induced topology) which is completely contained in A. The set of all relative interior points of A is called the relative interior of A, and denoted by riA. And interior of A by intA.*

Note 2.2.10. *If aff$A = X$ then $riA = intA$. Also if $intA \neq \phi$ or $A^i \neq \phi$ then aff$A = X$.*

Definition 2.2.11. *The set of all points $x \in X$ for which there exists $u \in A$ s.t. $[u, x[\subset A$, where $[u, x[$ is the segment joining u and x, including u and excluding x, is called **algebraic closure**. Denoted by A^{ac}.*

Definition 2.2.12. *The set of all elements $x \in X$ for which $[u, x] \cap A \neq \phi$ for every $u \in]0, x[$ and $\lambda x \notin A$ for every $\lambda > 1$ is called **radial boundary** of a set A.*

Result 2.2.13. *:*

(1) $\mu_A = \mu_{\{0\} \cup A^{rb}}$.

(2) $A^{rb} = \{x \in X : \mu_A(x) = 1\}$

Proposition 2.2.14. *Let X be a finite-dimensional separated topological linear space and let A be a convex set of X. A point $x_0 \in A$ is algebraic interior of A if and only if x_0 is an interior point (in the topological sense) of A.*

Corolloary 2.2.15. *A point $x_0 \in A$ where A is a convex set from a finite-dimensional separated topological linear space, is an algebraic relative interior point of A if and only if it is a relative interior point of A.*

Result 2.2.16. *If X is a separated topological linear space, then every (relative) interior point of a set is again an algebraic (relative) interior point of this set, that is,*

$$intA \subseteq A^i \qquad riA \subseteq A^{ri} \tag{2.2}$$

Proposition 2.2.17. *If A is a convex set for which the origin is an algebraic relative interior point, then*

$$A^{ri} = \{x \in X : \mu_A(x) < 1\} \text{ and } A^{ac} = \{x \in X : \mu_A(x) \leq 1\}.$$

Corolloary 2.2.18. *The interior of a convex set is either an empty set or it coincides with its algebraic interior.*

Corolloary 2.2.19. *The Minkowski functional of a convex, absorbent set A of a topological linear space is continuous iff $intA \neq \phi$. In this case, we have*

$$intA = A^i, \quad \bar{A} = A^{ac}, \quad FrA = A^{rb},$$

where $FrA = \bar{A} \cap \bar{cA}$.

Definition 2.2.20. *A maximal affine set is called a* ***hyperplane****. We say that the hyperplane is homogeneous (hyperspace) if it contains the origin. Equivalently, any subspace of X having co-dimension equal to 1 is* ***hyperspace****.*

A set is a hyperplane if and only if it is the translation of a maximal linear subspace. Hence hyperspace is a maximal linear subspace of X.

Proposition 2.2.21. *In a real topological linear space X, any homogeneous hyperplane is either closed or dense in X.*

Proof. Let H is homogeneous hyperplane.
Take $x, y \in \overline{H} \quad and\ \lambda_1, \lambda_2 \in \mathbb{R}$
Implies $\exists\ nets\ (x_i)_{i\in I}$ and $(y_i)_{j\in J}$ in H which converges to x and y respectively.
As TVS is a hausdorff space and $+$, $\cdot$ are continuous.
We have $\lambda_1 x + \lambda_2 y \in \overline{H}$
So $\overline{H}$ is a linear subspace of X. By maximality of H, $H \subseteq \overline{H}$, either $\overline{H} = H\ or\ \overline{H} = X$. □

Theorem 2.2.22. *The kernel of a nontrivial linear functional is a homogeneous hyperplane. Conversely, for every homogeneous hyperplane H there exists a functional, uniquely determined up to a nonzero multiplicative constant, with the kernel H.*

Proof. Since $f \neq 0$, $Kerf = f^{-1}(\{0\})$, $Kerf$ is proper subspace of X. Let $a \in X$ s.t. $f(a) \neq 0$.
For every $x \in X$, take $z = x - \frac{f(x)}{f(a)}a$.
Hence $z \in Kerf$ so that span$(Kerf \cup \{a\}) = X$. Hence $Kerf$ is homogeneous hyperplane in X.

Conversely, Let H be hyperspace in X and $a \notin H$.
Now for every $x \in X$ $\exists!\ z \in H\ and\ k \in \mathbf{K}$ s.t. $x = z + ka$. Define

$$f(x) = k$$

Then $Kerf = H$.
Uniqueness Let $f_1\ and\ f_2$ be two non trivial linear functional s.t. $Kerf_1 = Kerf_2$.

If $x_0 \notin Ker f_1$ we have $x - \frac{f_1(x)}{f_1(x_0)} x_0 \in Ker f_1 \ \ \forall x \in X$

$$f_2(x - \frac{f_1(x)}{f_1(x_0)} x_0) = 0 \Rightarrow f_2(x) = k f_1 \ \ \forall x \in X$$

where $k = \frac{f_2(x_0)}{f_1(x_0)}$.

□

Corolloary 2.2.23. *If f is a nontrivial linear functional on the linear space X, then $\{x \in X : f(x) = k\}$ is a hyperplane of X, for every $k \in \mathbb{R}$. Conversely, for every hyperplane H, there exists a linear functional f and $k \in \mathbb{R}$, such that $H = \{x \in X : f(x) = k\}$.*

Corolloary 2.2.24. *A hyperplane is closed iff it is determined by a nonidentically zero continuous linear functional.*

2.3 Separation of Convex Sets

If $f(x) = k, \ k \in \mathbb{R}$, is the equation of hyperplane in a real linear space X, we have two **open half-spaces** $\{x \in X : f(x) < k\}, \{x \in X : f(x) > k\}$ and two **closed half-spaces** $\{x \in X : f(x) \leq k\}, \{x \in X : f(x) \geq k\}$.

Result 2.3.1. *A convex set which contains no point of a hyperplane is contained in one of the two open half-spaces determined by that hyperplane.*
Indeed, if C is convex set and x_1, x_2 be s.t. $f(x_1) > k$ and $f(x_2) < k$ but $\lambda x_1 + (1-\lambda) x_2 \in C$ and f is continuous so $\exists \lambda_1 \in (0,1)$ s.t. $f(\lambda_1 x_1 + (1 - \lambda_1) x_2) = k$, hence x_1 and x_2 cannot be contained in a convex set which is dsjoint from hyperplane $f(x) = k$.

Remark 2.3.2. *If X is a topological linear space, then the open half-spaces are open sets and the closed half-spaces are closed sets if and only if the linear functional f which generated them is continuous, or, equivalently, the hyperplane $\{x \in X : f(x) = k\}$ is closed.*

Definition 2.3.3. *A function $f : X \to (-\infty, \infty)$ is called* ***convex*** *if*

$$f(\lambda_1 x + \lambda_2 y) \leq \lambda_1 f(x) + \lambda_2 f(y) \tag{2.3}$$

for all $x, y \in X$ and $\lambda_1 \geq 0, \lambda_2 \geq 0$, with $\lambda_1 + \lambda_2 = 1$. If inequality is strict for $x \neq y$ in $Dom(f)$ and $\lambda_1 > 0, \lambda_2 > 0$, then the function f is called ***strictly convex***.

Equivalent to inequality (2.3)

$$(a_1 + a_2)f\left(\frac{a_1x_1 + a_2x_2}{a_1 + a_2}\right) \leq a_1f(x_1) + a_2f(x_2) \tag{2.4}$$

for all $x_1, x_2 \in X$ and $a_1 > 0, a_2 > 0$.

Theorem 2.3.4. ***(Hahn-Banach)*** *Let X be a real linear space, let p be a real convex function on X and let Y be a linear subspace of X. If a linear functional f_0 defined on Y satisfies*

$$f_0(y) \leq p(y), \qquad \forall y \in Y, \tag{2.5}$$

then f_0 can be extended to a linear functional f defined on all of X, satisfying

$$f(x) \leq p(x), \qquad \forall x \in X. \tag{2.6}$$

Proof. If $u, v \in Y$, $x_0 \in X \setminus Y$ and $\alpha > 0$ $\beta < 0$ we have

$$\begin{aligned}
\alpha f_0(u) - \beta f_0(v) = f_0(\alpha u - \beta v) = (\alpha - \beta)f_0&\left[\frac{\alpha(u + \frac{1}{\alpha}x_0)}{\alpha - \beta} + \frac{-\beta(v + \frac{1}{\beta}x_0)}{(\alpha - \beta)}\right] \\
&\leq (\alpha - \beta)p\left[\frac{\alpha(u + \frac{1}{\alpha}x_0)}{\alpha - \beta} + \frac{-\beta(v + \frac{1}{\beta}x_0)}{(\alpha - \beta)}\right] \qquad (by\ 2.5) \\
&\leq \alpha p\left(u + \frac{1}{\alpha}x_0\right) - \beta p\left(v + \frac{1}{\beta}x_0\right). \qquad (by\ 2.4)
\end{aligned}$$

So we have,

$$\beta p(v + \frac{1}{\beta}x_0) - \beta f_0(v) \leq \alpha p(u + \frac{1}{\alpha}x_0) - \alpha f_0(u)\ true\ for\ all\ u, v \in Y.$$

Hence, $\forall u, v \in Y$ *and* $\alpha > 0, \beta < 0$, $\exists c \in \mathbb{R}$ such that

$$sup\left\{\beta p(v + \frac{1}{\beta}x_0) - \beta f_0(v)\right\} \leq c \leq \inf\left\{\alpha p(u + \frac{1}{\alpha}x_0) - \alpha f_0(u)\right\} \tag{2.7}$$

Now consider subspace $X_1 = span(Y \cup \{x_0\})$. For each element $x_1 \in X_1$ we have $x_1 = y + \lambda x_0$ with $y \in Y$ and $\lambda \in \mathbb{R}$ uniquely determined.
Define f_1 *on* X_1 by

$$f_1(x_1) = f_1(y + \lambda x_0) = f_0(y) + \lambda c\ where\ c\ is\ from\ (2.7)$$

Hence f_1 is linear and $f|_Y = f_0$. Now for $\lambda \neq 0$

$$f_1(x_1) = f_0(y) + \lambda c \leq f_0(y) + \lambda \left[\frac{1}{\lambda} p(y + \lambda x_0) - \frac{1}{\lambda} f_0(y) \right] = p(x_1) \quad (by\ 2.7)$$

Let $\mathscr{P} = \{(Z, f_z) : Y \subseteq Z\ and\ \ f_z|_Y = f_0\ and\ f_z(y) \leq p(y)\ \forall y \in Z\}$. So $\mathscr{P}$ is non empty.
Define $\prec$ on $\mathscr{P}$ by

$$(Z, f_z) \prec (W, f_w)\ if\ Z \subseteq W\ and\ \ f_w|_Z = f_z$$

Then $(\mathscr{P}, \prec)$ is poset.
Let (Z_α, f_{z_α}) be chain in $(\mathscr{P}, \prec)$. Define $\bigcup\limits_{\alpha \in \Lambda} Z_\alpha$. Define f *on* $\bigcup\limits_{\alpha \in \Lambda} Z_\alpha$ by $f(z) = f_{z_\alpha}(z)\ for\ z \in Z_\alpha$. Hence $\bigcup\limits_{\alpha \in \Lambda} Z_\alpha$ is an upper bound. By Zorn's lemma there exists maximal (W, f_w) in $(\mathscr{P}, \prec)$. Hence $W = X$, if not maximality of W will controdict.

□

Theorem 2.3.5. *If A is a convex set with $A^{ri} \neq \phi$ and M is an affine set such that $M \cap A^{ri} = \phi$, then there exists a hyperplane containing M, which is disjoint from A^{ri}.*

Theorem 2.3.6. *If A is a convex set with a nonempty interior and if M is an affine set which contains no interior point of A, then there exists a closed hyperplane which contains M and which again contains no interior point of A.*

Corolloary 2.3.7. *On a topological linear space there exist nontrivial continuous linear functionals (or closed hyperplanes) if and only if there exist proper convex sets with nonempty interior. On any proper locally convex space there exist nontrivial continuous functionals and closed hyperplanes.*

Definition 2.3.8. *A hyperplane H is called a* ***Supporting hyperplane*** *of a set A if H contains at least one point of A and A lies in one of the two closed half-spaces determined by H. A point of A through which a supporting hyperplane passes is called* ***support point*** *of A.*

Note 2.3.9. *In a linear topological space, any supporting hyperplane of a set with a nonempty interior is closed. Algebraic interior point cannot be a support point. Hence, any support point is necessarily an algebraic boundary point.*

Theorem 2.3.10. *If the interior of a convex set is nonempty, then all the boundary points are support points.*

Theorem 2.3.11. *If A_1 and A_2 are two nonempty convex sets and if at least one of them has a nonempty interior and is disjoint from the other set, then there exists a separating hyperplane. Moreover, if A_1 and A_2 are open, the separation is strict.*

Corolloary 2.3.12. *If A_1 and A_2 are two nonempty disjoint convex set of $\mathbb{R}^n$, there exists a nonzero element $c = (c_1, \ldots, c_n) \in \mathbb{R}^n \setminus \{0\}$, such that*

$$\sum_{i=1}^{n} c_i u_i \leq \sum_{i=1}^{n} c_i v_i, \quad \forall u = (u_i) \in A_1,\ \forall v = (v_i) \in A_2.$$

Example 2.3.13. *Counter example to Theorem 2.3.11 that if we drop condition A_1 and A_2 to be open: Consider disjoint convex sets*

$$A_1 = \{(x_1, x_2) : x_1 \leq 0\} \text{ and } A_2 = \{(x_1, x_2) : x_1 x_2 \geq 1, x_1 \geq 0, x_2 \geq 0\}$$

has nonempty interiors in $\mathbb{R}^2$ but cannot be strictly separated, only separating hyperplane is $x_1 = 0$.

Theorem 2.3.14. *If F_1 and F_2 are two disjoint nonempty closed convex sets in a separated locally convex space such that at least one of them is compact, then there exists a hyperplane strictly separating F_1 and F_2. Moreover, there exists a continuous linear functional f such that*

$$sup\{f(x) : x \in F_1\} < inf\{f(x) : x \in F_2\}. \tag{2.8}$$

Corolloary 2.3.15. *If $x_0 \notin F$, where F is a nonempty closed convex set of a separated locally convex space, then there exists a closed hyperplane strictly separating F and x_0, that is, there is a nontrivial continuous linear functional such that*

$$sup\{f(x) : x \in F\} < f(x_0).$$

Theorem 2.3.16. *A proper convex set of a separated locally convex space is closed if and only if it coincides with an intersection of closed half-spaces.*

Corolloary 2.3.17. *A closed convex set with nonempty interior of a separated locally convex space coincides with the intersection of all half-spaces generated by its supporting hyperplanes.*

Chapter 3

Completeness

3.1 Baire Category

Definition 3.1.1. *Let S be a Topological Space. The sets of the **first category** in S are those that are countable unions of nowhere dense sets. Any subset of S that is not of the first category is said to be of the **second category** in S.*
Example *: $\mathbb{Q} \subset (\mathbb{R}, \tau_{usual})$ is of first category. But $\mathbb{Q}^c \subset (\mathbb{R}, \tau_{usual})$ is of second category.*

Note 3.1.2. *In general, host space and its topology plays fundamental role in determing category. Example - $\mathbb{Z} \subset \mathbb{R}$. $\mathbb{Z}$ as subspace of $\mathbb{R}$ is of second category in itself but as $\mathbb{Z} \subset \mathbb{R}$ is of first category.*

Properties 3.1.3. *Let S be a Topological Space.*

(1) If $A \subset B$ and B is of the first category in S, so is A.

(2) Any countable union of sets of the first category is of the first category.

(3) Any closed set $E \subset S$ whose interior is empty is of the first category in S.

(4) If h is a homeomorphism of S onto S and if $E \subset S$, then E and $h(E)$ have the same category in S.

Theorem 3.1.4. *If S is either*

(1) a complete metric space, or

(2) a locally compact Hausdorff space,

then the intersection of every countable collection of dense open subsets of S is dense in S.

Proof. (1) Let $\{D_n\}_{n\in\mathbb{N}}$ is any countable collection of open dense set.
Let $x_0 \in S$, $r > 0$. And let $B(x_0, r)$ be the open ball centered at x_0 with radius r.
To prove : $\bigcap\limits_{n=1}^{\infty} D_n$ and $B(x_0, r)$ has non empty intersection.
Since D_1 is dense in S, we have

$$D_1 \cap B(x_0, r) \neq \phi.$$

So let $x_1 \in D_1 \cap B(x_0, r)$. Both are open sets $\exists\, r_1 > 0$ with $r_1 < 1$ such that,

$$\overline{B(x_1, r_1)} \subset D_1 \cap B(x_0, r)$$

Since D_2 is dense in S, we have

$$D_2 \cap B(x_1, r_1) \neq \phi$$

So let $x_2 \in D_2 \cap B(x_1, r_1)$. Both are open sets $\exists\, r_2 > 0$ with $r_2 < \frac{1}{2}$ such that,

$$\overline{B(x_2, r_2)} \subset D_2 \cap B(x_1, r_1)$$

Processing in this way we get,
D_{n+1} is dense in S, we have

$$D_{n+1} \cap B(x_n, r_n) \neq \phi$$

So let $x_{n+1} \in D_{n+1} \cap B(x_n, r_n)$. Both are open sets $\exists\, r_{n+1} > 0$ with $r_{n+1} < \frac{1}{n+1}$ such that,

$$\overline{B(x_{n+1}, r_{n+1})} \subset D_{n+1} \cap B(x_n, r_n)$$

We obtain a sequence $(x_n) \in S$ and $(r_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} r_n = 0$

$$\overline{B(x_{n+1}, r_{n+1})} \subset D_{n+1} \cap B(x_n, r_n) \subset B(x_n, r_n)$$

By Cantor Intersection Theorem we have

$$\bigcap_{n=1}^{\infty} \overline{B(x_{n+1}, r_{n+1})} \neq \phi$$

$$\phi \neq \bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)} \subset \bigcap_{n=1}^{\infty} D_n \cap B(x_n, r_n) \subset (\bigcap_{n=1}^{\infty} D_n) \cap B(x_0, r)$$

(2) Let B_0 be an arbitary non empty open set in S. Similarly by part (1) if $n \geq 1$ and an open set $B_{n-1} \neq \phi$ has been choosen, then $\exists$ an open set $B_n \neq \phi$ with

$$\overline{B_n} \subset D_n \cap B_{n-1}$$

Since S is locally compact Hausdorff Space, $\overline{B_n}$ can be chosen compact. Put $K = \bigcap\limits_{n=1}^{\infty} \overline{B_n}$.

$K \neq \phi$ by compactness. By our construction $K \subset B_0$ and $K \subset D_n$ for each n. Hence B_0 intersects $\bigcap\limits_{n \in \mathbb{N}} D_n$.

□

Note 3.1.5. *If $\{E_i\}$ is a countable collection of nowhere dense subsets of S, and if V_i is the complement of $\overline{E}_i$, then each V_i is dense, and the conclusion of Baire's theorem is that $\cap V_i \neq \phi\cdot$ Hence $S \neq \cup E_i$. Therefore, complete metric spaces, as well as locally compact Hausdorff spaces, are of the second category in themselves.*

Definition 3.1.6. ***Equicontinuity*** *: Suppose X and Y are topological vector spaces and Γ is a collection of linear mappings from X into Y. We say that Γ is equicontinuous if to every neighborhood W of 0 in Y there corresponds a neighborhood V of 0 in X such that $T(V) \subset W$ for all $T \in \Gamma$.*

Theorem 3.1.7. *Suppose X and Y are topological vector spaces, Γ is an equicontinuous collection of linear mappings from X into Y, and E is a bounded subset of X. Then Y has a bounded subset F such that $T(E) \subset F$ for every $T \in \Gamma$.*

Proof. Let $F = \bigcup\limits_{T \in \Gamma} T(E)$. Let W be a neighborhood of 0 in Y. Since Γ is equicontinuous there is a neighborhood V of 0 in X s.t. $T(V) \subset W$, $\forall T \in \Gamma$. Since E is bounded, $E \subset tV$ for large t,

$$T(E) \subset T(tV) = tT(V) \subset tW.$$

Hence $\bigcup\limits_{T\in\Gamma} T(E) \subset tW$ implies $F \subset tW$. Hence F is bounded. □

Theorem 3.1.8. ***(Banach-Steinhaus)*** *: Suppose X and Y are topological vector spaces, Γ is a collection of continuous linear mappings from X into Y, and B is the set of all $x \in X$ whose orbits*

$$\Gamma(x) = \{T(x) : T \in \Gamma\}$$

are bounded in Y.
If B is of secound category in X, then B=X and Γ is equicontinuous.

Proof. Pick balanced neighborhood W and U of 0 in Y s.t. $\overline{U} + \overline{U} = W$. Put

$$E = \bigcap_{T\in\Gamma} T^{-1}(\overline{U}).$$

If $x \in B$, then $\Gamma(x) \subset nU$ for some n, so that $x \in nE$. Hence $B \subset \bigcup\limits_{n=1}^{\infty} nE$. Now atleast one nE is of the second category of X, since this is true of B. But $x \to nx$ is homeomorphism of X onto X, implies E itself of second category in X. $\overline{E}$ has nonempty interior but E is closed since T is continuous. Let E has an interior point x. Then $x - E$ contain a neighborhood V of 0 in X.

$$V \subset x - E$$

$$T(V) \subset Tx - T(E) \subset \overline{U} - \overline{U} \subset W \quad \forall T \in \Gamma.$$

Implies Γ is equicontinuous and by previous theorem Γ is uniformly bounded. Each Γ is bounded in Y, Hence $B = X$. □

Corolloary 3.1.9. *If Γ is a collection of continuous linear mappings from an F-space X into a topological vector space Y, and if the sets*

$$\Gamma(x) = \{Tx : T \in \Gamma\}$$

are bounded in Y, for every $x \in X$, then Γ is equicontinuous.

Proposition 3.1.10. *Suppose X and Y are topological vector spaces, and $\{T_n\}$ is a sequence of continuous linear mappings of X into Y.*

(1) If C is the set of all $x \in X$ for which $\{T_n(x)\}$ is a Cauchy sequence in Y, and if C is of the second category in X, then $C = X$.

(2) If L is the set of all $x \in X$ at which

$$T(x) = \lim_{n\to\infty} T_n(x)$$

exists, if L is of the second category in X, and if Y is an F-space, then $L = X$ and $T : X \to Y$ is continuous.

Corolloary 3.1.11. *If T_n is a sequence of continuous linear mappings from an F-space X into a topological vector space Y, and if*

$$T(x) = \lim_{n\to\infty} T_n(x)$$

exists for every $x \in X$, then T is continuous.

Theorem 3.1.12. *Suppose X and Y are topological vector spaces, K is a compact convex set in X, Γ is a collection of continuous linear mappings of X into Y, and the orbits*

$$\Gamma(x) = \{Tx : T \in \Gamma\}$$

are bounded subset of Y, for every $x \in K$.
Then there is bounded set $B \subset Y$ s.t. $T(K) \subset B$ for every $T \in \Gamma$.

Proof. Let B be the union of all sets $\Gamma(x)$, for $x \in K$. Pick balanced neighborhood W and U of 0 in Y s.t. $\overline{U} + \overline{U} \subset W$. Put

$$E = \bigcap_{T\in\Gamma} T^{-1}(\overline{U})$$

If $x \in K$, then $\Gamma(x) \subset nU$ for some n, so that $x \in nE$. We have

$$K = \bigcup_{n=1}^{\infty} (K \cap nE).$$

Since E is closed, Baire's theorem shows that $K \cap nE$ has nonempty interior (relative to K) for at least one n (say n_0).
We fix an interior point x_0 of $K \cap nE$, we fix a balanced neighborhood V of 0 in X s.t.

$$K \cap (x_0 + V) \subset nE$$

and we fix a $p > 1$ s.t.

$$K \subset x_0 + pV.$$

Such p exists since K is compact.
If now x is any point of K and

$$z = (1 - p^{-1})x_0 + p^{-1}x$$

then $z \in K$, since K is convex. Also,

$$z - x_0 = p^{-1}(x - x_0) \in V.$$

Hence $z \in nE$. Since $T(nE) \subset n\overline{U}$ for every $T \in \Gamma$ and since $x = pz - (p - 1)x_0$,and since $\overline{U}$ is balanced we have

$$Tx \in pn\overline{U} - (p-1)n\overline{U} \subset pn(\overline{U} + \overline{U}) \subset pnW.$$

Thus $B \subset pnW$, hence B is bounded.

□

3.2 The Open Mapping Theorem

Theorem 3.2.1. *Suppose X is an F-space and Y is a topological vector space. Also suppose $T : X \to Y$ is continuous and linear and $T(X)$ is of second category in Y. Then*

(1) T(X)=Y,

(2) T is open mapping, and

(3) Y is an F-space.

Proof. Obviously (2) implies (1), $T(X)$ is open in Y and only open subspace of Y is Y itself, so T(X)=Y.
To prove (2), let V be a neighborhood of 0 in X. We have to show that $T(V)$ contains a neighborhood of 0 in Y (since our topologies are invariant so enough to show nbd around 0).
Let d be invariant metric on X that is compatible with the topology of X. Define

$$V_n = \{x : d(x, 0) < 2^{-n}r\} \qquad (n \in \mathbb{N})$$

where $r > 0$ is so small that $V_0 \subset V$. We proceed by proving that for some neighborhood W of 0 in Y satisfies

$$W \subset \overline{T(V_1)} \subset T(V) \tag{3.1}$$

Since $V_2 - V_2 \subset V_1$, by Theorem 1.2.6 (b)

$$\overline{T(V_2)} - \overline{T(V_2)} \subset \overline{T(V_2) - T(V_2)} \subset \overline{T(V_1)}.$$

Also we know that $T(X) = \bigcup_{k=1}^{\infty} kT(V_2)$, because V_2 is neighborhood of 0. At least one $kT(V_2)$ is therefor of second category in Y. Because of homeomorphism $y \to ky$ of Y onto Y, $T(V_2)$ is of second category in Y. Its closure therefore has nonempty interior. Now take $x \in int(\overline{V_2})$

$$0 \in x - int(\overline{V_2}) \subset \overline{T(V_2)} - \overline{T(V_2)}.$$

First inclusion of equation (3.1) is done. To prove second inclusion in equation (3.1), fix $y_1 \in \overline{T(V_1)}$. Assume $n \geqslant 1$ *and* y_n has been chosen in $\overline{T(V_n)}$.

What was just proved for V_1 is true for V_{n+1}, so that $\overline{T(V_{n+1})}$ contains a neighborhood of 0. Hence

$$(y_n - \overline{T(V_{n+1})}) \cap T(V_n) \neq \phi.$$

This says that $\exists x_n \in V_n$ s.t.

$$T(x_n) \in y_n - \overline{T(V_{n+1})}.$$

Put $y_{n+1} = y_n - T(x_n)$. Then $y_{n+1} \in \overline{T(V_{n+1})}$ and construction proceeds. Since $d(x_n, 0) < 2^{-n}r \ for \ n \in \mathbb{N}$, the sums $x_1 + \cdots + x_n$ forms a cauchy sequence which converges (by the completeness of X) to some $x \in X$, with $d(x, 0) < r$. Hence $x \in V$. Since

$$\sum_{n=1}^{m} T(x_n) = \sum_{n=1}^{m} (y_n - y_{n+1}) = y_1 - y_{m+1}$$

and since $y_{m+1} \to 0 \ as \ m \to \infty$, hence $y_1 = T(x) \in T(V)$. Hence (2) is proved.

By Theorem 1.8.4 X/N is an F-space, if N is the null space of T. Define

$$f(x+N) = T(x) \qquad (x \in X).$$

So f is isomorphism and that $T(x) = f(\pi(x))$, where π is quotient map. If V is open in Y,then

$$\begin{aligned} f^{-1}(V) &= \{x+N \in X/N : f(x+N) \in V\} \\ &= \{x+N \in X/N : T(x) \in V\} \\ &= \{x+N \in X/N : x \in T^{-1}(V)\} \\ &= \pi(T^{-1}(V)) \end{aligned}$$

is open, since T is continuous and π is open. Hence f is continuous. If E is open in X/N, then

$$f(E) = T(\pi^{-1}(E))$$

is open, π is continuous and T is open. Hence F is homeomorphism. As X/N is an F-space so is Y. □

Corolloary 3.2.2.

(a) *If T is a continuous linear mapping of an F-space X onto an F-space Y, then T is open.*

(b) *If T satisfies (a) and is one-to-one, then $T^{-1} : Y \to X$ is continuous.*

(c) *If X and Y are Banach spaces, and if $T : X \to Y$ is continuous, linear, one-to-one, and onto, then there exist positive real numbers a and b such that*

$$a\|x\| \leqslant \|T(x)\| \leqslant b\|x\|$$

for every $x \in X$.

(d) *If $\tau_1 \subset \tau_2$ are vector topologies on a vector space X and if both (X, τ_1) and (X, τ_2) are F-spaces, then $\tau_1 = \tau_2$.*

3.3 The Closed Graph Theorem

Definition 3.3.1. ***Graphs*** *: If X and Y are sets and f maps X into Y, the graph of f is the set of all points $(x, f(x))$ in the cartesian product $X \times Y$.*

Proposition 3.3.2. *If X is a topological space, Y is a Hausdorff space, and $f : X \to Y$ is continuous, then the graph G of f is closed.*

Theorem 3.3.3. ***The closed graph theorem*** *: Suppose X and Y are F-space and $T : X \to Y$ is linear map. Further $G = \{(x, T(x)) : x \in X\}$ is closed in $X \times Y$. Then T is continuous.*

Proof. $X \times Y$ is a vector space if addition and scaler multiplication are defined componentwise:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1).$$

There are complete invariant metrics d_X *and* d_Y on X and Y, respectively, which induce their topologies. If

$$d((x_1, , y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

then d is an invariant metric on $X \times Y$ which is compatible with its product topology and which makes $X \times Y$ into an F-space.
Since T is linear map hence G is subspace of $X \times Y$. Closed subset of a complete metric space is complete. Therefore G is F-space.

Define $\pi_1 : G \to X$ and $\pi_2 : X \times Y \to Y$ by

$$\pi_1(x, T(x)) = x, \qquad \pi_2(x, y) = y.$$

So π_1 is continuous beijective linear map. So from open mapping theorem

$$\pi_1^{-1} : X \to G$$

is continuous. Also $T = \pi_2 \circ \pi_1^{-1}$ and π_2 is continuous. Hence T is continuous.

□

Definition 3.3.4. ***Bilinear Map*** *: Suppose X, Y, Z are vector spaces and B maps $X \times Y$ into Z. Associate to each $x \in X$ and to each $y \in Y$ the mapping*

$$B_x : Y \to Z \quad \text{and} \quad B^y : X \to Z$$

by defining

$$B_x(y) = B(x, y) = B^y(x).$$

B is said to be bilinear if every B_x and every B^y are linear.

Definition 3.3.5. *If X, Y, Z are topological vector spaces and if every B_x and every B^y is continuous, then B is said to be **separately continuous.***

Proposition 3.3.6. *Suppose $B : X \times Y \to Z$ is bilinear and separately continuous, X is an F-space, and Y and Z are topological vector spaces. Then*

$$B(x_n, y_n) \to B(x_0, y_0) \quad \text{in } Z$$

whenever $x_n \to x_0$ in X and $y_n \to y_0$ in Y. If Y is metrizable, it follows that B is continuous.

Chapter 4

Weak and Weak*-Topology

4.1 Prerequisites

Proposition 4.1.1. *Suppose M is a subspace of a vector space X, p is a seminorm on X, and f is a linear functional on M such that*

$$|f(x)| \leqslant p(x) \qquad (x \in M).$$

Then f extends to a linear functional T on X that satisfies

$$|T(x)| \leqslant p(x) \qquad (x \in X).$$

Proposition 4.1.2. *Suppose A and B are disjoint, nonempty and convex sets in a topological vector space X.*

(1) If A is open there exists $T \in X^*$ and $\gamma \in \mathbb{R}$ s.t.

$$Re\ Tx < \gamma \leqslant Re\ Ty$$

$\forall x \in A$ and $\forall y \in B$.

(2) If A is compact and B is closed and X is locally convex space, then there exist $T \in X^*$, $\gamma_1, \gamma_2 \in \mathbb{R}$ s.t.

$$Re\ Tx < \gamma_1 < \gamma_2 < Re\ Ty$$

$\forall x \in A$ and $\forall y \in B$.

Proposition 4.1.3. *Suppose M is a subspace of a locally convex space X, and $x_0 \in X$. If x_0 is not in the closure of M, then there exists $T \in X^*$ such that $T(x_0) = 1$ but $Tx = 0$ for every $x \in M$.*

Proposition 4.1.4. *Suppose B is a convex, balanced, closed set in a locally convex space X, $x_0 \in X$, but $x_0 \notin B$. Then there exists $T \in X^*$ such that $|T(x)| \leqslant 1$ for all $x \in B$, but $T(x_0) > 1$.*

4.2 Weak Topologies

Lemma 4.2.1. *If $\tau_1 \subset \tau_2$ are topologies on a set X, if τ_1 is a Hausdorff topology, and if τ_2 is compact, then $\tau_1 = \tau_2$.*

Proof. Let $F \subset X$ be $\tau_2 - closed$. Since X is τ_2-compact, so is F. Since $\tau_1 \subset \tau_2$ implies F is τ_1-compact. Since τ_1 is a Hausdorff topology, implies that F is τ_1-closed. Conclude that $\tau_2 \subset \tau_1$. Hence $\tau_1 = \tau_2$.

□

Definition 4.2.2. *Suppose that X is a set and $\mathscr{F}$ is a nonempty family of mappings $f : X \to Y_f$, where each Y_f is a topological space. Let τ be the collection of all unions of finite intersections of sets $f^{-1}(V)$, with $f \in \mathscr{F}$ and V open in Y_f . Then τ is a topology on X, and it is in fact the weakest topology on X that makes every $f \in \mathscr{F}$ continuous. This τ is called the* ***weak topology*** *on X induced by $\mathscr{F}$, or, more succinctly, the $\mathscr{F}$-topology of X.*

Example 4.2.3. *Let X be the cartesian product of a collection of topological spaces X_α. If $\pi_\alpha(x)$ denotes the αth coordinate of a point $x \in X$, then π_α maps X onto X_α, and the product topology τ of X is its $\{\pi_\alpha\}$-topology, the weakest one that makes every π_α continuous.*

Lemma 4.2.4. *If $\mathscr{F}$ is a family of mappings $f : X \to Y_f$, where X is a set and each Y_f is a Hausdorff space, and if $\mathscr{F}$ separates points on X, then the $\mathscr{F}$-topology of X is a Hausdorff topology.*

Lemma 4.2.5. *If X is a compact topological space and if some sequence $\{f_n\}$ of continuous real-valued functions separates points on X, then X is metrizable.*

Proof. Let τ be the given topology on X. Suppose, without loss of generality, that $|f_n| \leqslant 1$ for all n, and let τ_d be the topolgy induced on X by the metric

$$d(p,q) = \sum_{n=1}^{\infty} 2^{-n} |f_n(p) - f_n(q)|.$$

Since each f_n is τ-continuous and the series converges uniformly on $X \times X$, d is τ-continuous function on $X \times X$. The balls

$$B(p,r) = \{q \in X : d(p,q) < r\}$$

are therefore τ-open. Thus $\tau_d \subset \tau$. Since τ_d is induced by a metric, τ_d is a Hausdorff topology, and now Lemma 4.2.1 implies that $\tau = \tau_d$. □

Lemma 4.2.6. *Suppose $T_1, T_2, \ldots, T_n$ and T are linear functionals on a vector space X. Let*

$$N = \{x \in X : T_1 x = \cdots = T_n x = 0\}.$$

The following are equivalent:

(1) Ther are scalars $\beta_1, \ldots, \beta_n$ such that

$$T = \beta_1 T_1 + \cdots + \beta_n T_n.$$

(2) There exists $\gamma < \infty$ such that

$$|Tx| \leq \gamma \; max_{1 \leq i \leq n} \; |T_i x| \qquad (x \in X).$$

(3) $Tx = 0$ for every $x \in N$.

Theorem 4.2.7. *Suppose X is a vector space and X' is a separating vector space of linear functionals on X. Then the $X'-$topology τ' makes X into a locally convex space whose dual space is X'.*

Proof. We know that $\mathbb{R}$ *and* $\mathbb{C}$ are Hausdorff space so by Lemma 4.2.3 τ' is a Hausdorff topology. The elements of X' are linear which gives τ' translational-invariant. If $T_1, \ldots, T_n \in X'$ and if $r_i > 0$ and if

$$V = \{x : |T_i x| < r_i \; for \; 1 \leq i \leq n\},$$

then V is balanced, convex and $V \in \tau^{'}$. And collection of all V form local base for $\tau^{'}$.
Here $\frac{1}{2}V + \frac{1}{2}V = V$, hence addition is continuous. Suppose $x \in X$ and α is scalar. Then $x \in sV$ for some $s > 0$ as V is absorbing set too. If $|\beta - \alpha| < r$ and $y \in x + rV$ then

$$\begin{aligned}\beta y - \alpha x &= (\beta - \alpha)y + \alpha(y - x)\\ &\in (rx + r^2V) + (|\alpha|rV)\\ &\subset rsV + r^2V + |\alpha|rV\\ &\subset V.\end{aligned}$$

provided $r(s + r) + |\alpha|r < 1$. Hence scalar multiplication is continuous.
So we proved $\tau^{'}$ is locally convex vector topology. Also every $T \in X^{'}$ is $\tau^{'}$ continuous. Conversely, suppose T is $\tau^{'}$ continuous linear functional on X. Then $|Tx| < 1$ for all x in some set V constucted earlier. By Lemma 4.2.6 $T = \sum \beta_i T_i$. Since $T_i \in X^{'}$ and $X^{'}$ is vector space, $T \in X^{'}$.

□

4.2.8 The weak topology of a topological vector space

Suppose X is a topological vector space (τ) whose dual X^* separates points on X. The X^*-topology of X is called weak topology of X.
Let X be topologized by weak topology τ_w and denote it by X_w.
τ_w is weakest topology on X means $\tau_w \subset \tau$, So τ will often called original topology.

Let $\{x_n\}$ be sequence in X. $x_n \to 0$ originally means every original neighborhood of 0 contains all x_n except finite. $x_n \to 0$ weakly means every weak neighborhood of 0 contains all x_n except finite.

Note 4.2.9.

(1) *Since every weak neighborhood of 0 contains a neighborhood of the form*

$$V = \{x : |T_i x| < r_i \ for\ 1 \le i \le n\}$$

where $T_i \in X^*$ and $r_i > 0$, so $x_n \to 0$ weakly iff $Tx_n \to 0$ for every $T \in X^*$.

(2) Every originally convergent sequence converges weakly.

(3) A set $E \subset X$ is weakly bounded iff for every $T \in X^*$ is bounded function on E.

(4) Consider V construct earlier and $N = \{x : T_1 x = \cdots = T_n = 0\}$. Since $x \to (T_1 x, \ldots, T_n x)$ maps X into $\mathbb{C}$ with null space N, so $dim X \leq n + dim N$. Since $N \subset V$ so if X is infinite dimensional then every weak neighborhood of 0 contains an infinite dimensional subspace; hence X_w is not locally bounded.

Theorem 4.2.10. *Suppose E is a convex subset of locally convex space X. Then the weak closure $\overline{E}_w$ of E is equal to its original closure $\overline{E}$.*

Proof. $\overline{E}_w$ is weakly closed, hence originally closed, so $\overline{E} \subset \overline{E}_w$. Choose $x_0 \in X$, $x_0 \notin \overline{E}$. By Theorem 4.1.2 there exist $T \in X^*$ and $\alpha \in \mathbb{R}$ such that, for every $x \in \overline{E}$,

$$Re\ Tx_0 < \alpha < Re\ Tx.$$

The set $\{x : Re\ Tx < \alpha\}$ is weak neighborhood of x_0 that does not intersect E. Thus $x_0 \notin \overline{E}_w$. Hence $\overline{E}_w \subset \overline{E}$. $\square$

Corolloary 4.2.11. *For convex subsets of a locally convex space,*

(1) originally closed equals weakly closed and

(2) originally dense equals weakly dense.

Theorem 4.2.12. *Suppose X is a metrizable locally convex space. If $\{x_n\}$ is a sequence in X that converges weakly to some $x \in X$, then there is a sequence $\{y_i\}$ in X such that*

(1) each $\{y_i\}$ is a convex combination of finitely many x_n , and

(2) $y_i \to x$ originally.

4.2.13 The weak*-topology of a dual space

Let X be topological vector space whose dual is X^*. Now every $x \in X$ induces a linear functional f_x on X^* defined by

$$f_x T = Tx$$

and that $\{f_x : x \in X\}$ separates points on X^*. The X-topology of X^* is called *weak**-topology of X^*.

4.3 Compact Convex Sets

Theorem 4.3.1. ***The Banach-Alaoglu theorem*** *: If V is neighborhood of 0 in a topological vector space X and if*

$$K = \{T \in X^* : |Tx| \leq 1 \ for \ every \ x \in V\}$$

then K is weak**-compact.*

Proof. Since neighborhoods of 0 are absorbing, so corresponds to every $x \in X$ there is number $\gamma_x < \infty$ such that $x \in \gamma_x V$. Hence

$$|Tx| \leq \gamma_x \qquad (x \in X, \ T \in K).$$

Let $D_x = \{\alpha \in \mathbb{R} : |\alpha| \leq \gamma_x\}$. Let τ be the product topology on P, cartesian product of all D_x, one for each $x \in X$. Set P is compact set being cartesian product of compact sets (Tychonoff's theorem). $\pi_x : D \to D_x$ defined by $\pi_x(\alpha) = \alpha_x$, x coordinate of α. So π_x is continuous. Elements of P are the functions on X (linear or not) that satisfy

$$|f(x)| \leq \gamma_x \qquad (x \in X)$$

Thus $K \subset X^* \cap P$. So K inherits two topologies: one from X^* $(weak^* - topology)$ and τ from P.
Claim (a) : These two topologies coincide on K.
Fix some $T_0 \in K$. Choose $x_i \in X$, for $1 \leq i \leq n$; choose $\delta > 0$. Take

$$\begin{aligned} W_1 &= \{T \in X^* : |f_{x_i}T - f_{x_i}T_0| < \delta 1 \leq i \leq n\} \\ &= \{T \in X^* : |Tx_i - T_0 x_i| < \delta 1 \leq i \leq n\} \end{aligned}$$

$$W_2 = \{f \in P : |f(x_i) - T_0 x_i| < \delta 1 \leq i \leq n\}.$$

Let $n, x_i, \ and \ \delta$ range over all possible values. So W_1 then form a local base for the *weak**-topology of X^* at T_0 and the sets W_2 form a local base for product topology τ of P at T_0. As $K \subset X^* \cap P$, we have

$$W_1 \cap K = W_2 \cap K.$$

Cliam (b) : K is closed subset of P.
Suppose f_0 is in τ-closure of K. Choose $x, y \in X$, scalers $\alpha, \beta \ and \ \epsilon > 0$. The

set of all $f \in P$ suct that $|f - f_0| < \epsilon$ at $x, y,$ *and* $\alpha x + \beta y$ is a τ-neighborhood of f_0. Therefore K contains such a f. Since f is linear,

$$f_0(\alpha x+\beta y)-\alpha f_0(x)-\beta f_0(y) = (f_0-f)(\alpha x+\beta y)+\alpha(f-f_0)(x)+\beta(f-f_0)(y)$$

so that

$$|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| = (1 + |\alpha| + |\beta|)\epsilon.$$

Since ϵ was arbitrary, so f_0 is linear. Now if $x \in V$ and $\epsilon > 0$, by same argument shows $f \in K$ such that $|f(x) - f_0(x)| < \epsilon$. Since $|f(x)| \leq 1$, by definition of K, hence $|f_0(x)| \leq 1$. Hence $f_0 \in K$.
Now since P is compact, (b) implies that K is τ-compact and then (a) implies that K is *weak**-compact.

□

Theorem 4.3.2. *If X is separable topological vector space, if $K \subset X^*$ and if K is weak*-compact, then K is metrizable, in the weak*-topology.*

Proof. Let $\{x_n\}$ be countable dense set in X. Define $f_n(T) = Tx_n$, for $T \in X^*$. Each f_n is *weak**-coontinuous. If $f_n(T_1) = f_n(T_2)\quad \forall n$, then $T_1 x_n = T_2 x_n, \forall n$, which implies that $T_1 = T_2$, since both are continuous on X and coincide on a dense set.
Thus $\{f_n\}$ is a countable family of continuous function that separates points on X^*. So by Lemma 4.2.5 K is metrizable. □

Proposition 4.3.3. *If V is neighborhood of 0 in a separable topological vector space X, and if $\{T_n\}$ is a sequence in X^* such that*

$$|T_n x| \leq 1 \qquad (x \in V, n = 1, 2, \dots),$$

there is a subsequence $\{T_{n_i}\}$ and there is a $T \in X^*$ such that

$$Tx = \lim_{n\to\infty} T_{n_i} x \qquad (x \in X).$$

In other words, the polar of V is sequentially compact in the weak*-topology.

Theorem 4.3.4. *In a locally convex space X, every weakly bounded set is originally bounded and vice versa.*

Proof. Every originally bounded set is weakly bounded because every weak neighborhood of 0 in X is an original neighborhood of 0.

Conversely, assume that $E \subset X$ is weakly bounded and U is an original neighborhood of 0 in X. We have to show there exists $t > 0$ such that $E \subset tU$. Since X is locally convex, there is a convex, balanced, original neighborhood V of 0 in X such that $\overline{V} \subset U$. Let $K \subset X^*$ be the polar of V :

$$K = \{T \in X^* : |Tx| \leq 1 \quad \forall x \in V\}.$$

Claim : $\overline{V} = \{x \in X : |Tx| \leq 1,\ T \in K\}$ (1)
$V \subset \{x \in X : |Tx| \leq 1,\ T \in K\}$ by definition of K, implies $\overline{V} \subset \{x \in X : |Tx| \leq 1,\ T \in K\}$ as the set in right side is closed. Suppose $a \in X$ *but* $a \notin \overline{V}$. Proposition 4.1.4 shows that $T(a) > 1$ for some $T \in K$. Hence claim holds. Since E is weakly bounded, there corresponds to each $T \in X^*$ a number $\beta_T < \infty$ such that

$$|Tx| < \beta_T \quad \forall x \in E. \qquad (2)$$

Since K is convex and *weak**-compact and since the function $T \to Tx$ are *weak**-continuous, by Theorem 3.1.13 we conclude from equation (2) that there is constant $\beta < \infty$ such that

$$|Tx| \leq \beta \qquad (x \in E,\ T \in K).$$

Now from equation (1) and (2), $\beta^{-1}x \in \overline{V} \subset U$ for all $x \in E$. Since E is balanced,

$$E \subset t\overline{V} \subset tU \qquad (t > \beta).$$

Hence E is originally bounded.

□

Corolloary 4.3.5. *If X is normed space, if $E \subset X$ and if*

$$\sup_{x \in E} |Tx| < \infty \qquad (T \in X^*)$$

then there exists $\beta < \infty$ such that

$$\|x\| \leq \beta \qquad (x \in E).$$

Definition 4.3.6.

(1) *If X is a topological vector space and $E \subset X$, the closed convex hull of E, written $\overline{co}(E)$, is the closure of $co(E)$.*

(2) *A subset E of a metric space X is said to be totally bounded if E lies in the union of finitely many open balls of radius ϵ, for every $\epsilon > 0$.*

(3) *A set E in a topological vector space X is said to be totally bounded if to every neighborhood V of 0 in X corresponds a finite set F such that $E \subset F + V$.*

Proposition 4.3.7. *If $E \subset \mathbb{R}^n$ and $x \in co(E)$, then x lies in the convex hull of some subset of E which contains at most $n+1$ points.*

Theorem 4.3.8.

(1) *If $A_1, \ldots, A_n$ are compact convex sets in a topological vector space X, then $co(A_1 \cup \cdots \cup A_n)$ is compact.*

(2) *If X is a locally convex topological vector space and $E \subset X$ is totally bounded, then $co(E)$ is totally bounded.*

(3) *If X is a Frechet space and $K \subset X$ is compact, then $\overline{co}(K)$ is compact.*

(4) *If K is a compact set in $\mathbb{R}^n$, then $co(K)$ is compact.*

Proof. (1) Let S be the simplex in $\mathbb{R}^n$ consisting of all $s = (s_1, \ldots, s_n)$ with $s_i \geq 0, \sum_{i=1}^{n} s_i = 1$. Put $A = A_1 \times \cdots \times A_n$. Define $f : S \times A \to X$ by

$$f(s,a) = \sum_{i=1}^{n} s_i a_i$$

and put $K = f(S \times A)$.
Since f is continuous and $S \times A$ is compact so K is compact and $K \subset co(A_1 \cup \cdots \cup A_n)$.
If (s,a) *and* (t,b) are in $S \times A$ *and if* $\alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta = 1$, then

$$\alpha f(s,a) + \beta f(t,b) = f(u,c),$$

where $u = \alpha s + \beta t \in S$ *and* $c \in A$, because

$$c_i = \frac{\alpha s_i a_i + \beta t_i b_i}{\alpha s_i + \beta t_i} \in A_i \qquad 1 \leq i \leq n$$

So K is convex. Since $A_i \subset K$ for each i $[take\ s_i = 1,\ s_j = 0\ for\ j \neq i]$, the convexity of K implies that $co(A_i \cup \cdots \cup A_n) \subset K$.
So $K = co(A_i \cup \cdots \cup A_n)$, hence conclusion hold.

(2) Let U be a neighborhood of 0 in X. Choose a convex neighborhood V of 0 in X such that $V + V \subset U$. Then $E \subset F + V$ for some finite set $F \subset X$. Hence $E \subset co(F) + V$. The set $co(F) + V$ is convex. It follows that

$$co(E) \subset co(F) + V.$$

But $co(F)$ is compact [by (1)], and therefore $co(F) \subset F_1 + V$ for some finite set $F_1 \subset X$. Thus

$$co(E) \subset F_1 + V + V \subset F_1 + U.$$

Since U was arbitrary, co(E) is totally bounded.

(3) Closures of totally bounded sets are totally bounded in every metric space, and hence are compact in every complete metric space. So if K is compact in a Frechet space, then K is obviously totally bounded ; hence co(K) is totally bounded, by (2), and therefore $\overline{co}(K)$ is compact.

(4) Let S be simplex in $\mathbb{R}^{n+1}$ consisting of all $t = (t_1, \ldots, t_{n+1})$ *with* $t_i \geq 0$ and $\sum t_i = 1$. Let K be compact, $K \subset \mathbb{R}^n$. By Proposition 4.3.7, $x \in co(K)$ iff

$$x = \sum_{i=1}^{n+1} t_i x_i$$

for some $t \in S$ and $x_i \in K$ $(1 \leq i \leq n+1)$. In other words, $co(K)$ is the image of $S \times K^{n+1}$ under the continuous mapping

$$(t, x_1, \ldots, x_{n+1}) \to \sum_{i=1}^{n+1} t_i x_i.$$

Hence co(K) is compact.

$\square$

Proposition 4.3.9. *Suppose X is a topological vector space on which X^* separates points. Suppose A and B are disjoint, nonempty, compact, convex sets in X. Then there exists $T \in X^*$ such that*

$$\sup_{x\in A} Re\ Tx < \inf_{y\in B} Re\ Ty.$$

Definition 4.3.10. ***Extreme points*** *: Let K be a subset of a vector space X. A nonempty set $S \subset K$ is called an* ***extreme set*** *of K if no point of S is an internal point of any line interval whose end points are in K, except when both end points are in S. i.e. If $x \in K,\ y \in K,\ 0 < t < 1,$ and*

$$(1-t)x + ty \in S$$

then $x, y \in S$.
The ***extreme points*** *of K are the extreme sets that consist of just one point. And the set of all extreme points are denoted by E(K).*

Theorem 4.3.11. ***The Krein-Milman Theorem*** *: Suppose X is a topological vector space on which X^* separates points. If K is a nonempty compact convex set in X, then K is the closed convex hull of the set of its extreme points. In symbols, $K = \overline{co}(E(K))$.*

Proof. Let $\mathscr{P}$ be the collection of all compact extreme sets of K. Since $K \in \mathscr{P},\ \mathscr{P} \neq \phi$.
Claim(1) : The intersection S of any nonempty subcollection of $\mathscr{P}$ is a member of $\mathscr{P}$, unless $S = \phi$.
Assume $S = \bigcap\limits_{i\in I} E_i, \quad E_i \in \mathscr{P}$. Being the intersection of closed sets, S is closed and $S \subset K$ so S is compact. Let $x \in S$ and $x = \lambda u + (1-\lambda)v$ for some $u, v \in K$ *and* $\lambda \in [0, 1]$. Now $x \in E_i\ \forall i \in I$ and E_i is an extreme subset of K so we have $u, v \in E_i\ \forall i \in I$. Hence $u, v \in S$. Thus claim hold.
Claim(2) : If $S \in \mathscr{P}, T \in X^*$, μ is the maximum of $Re\ T$ on S, and

$$S_T = \{x \in S : Re\ Tx = \mu\},$$

then $S_T \in \mathscr{P}$.
Suppose $tx + (1-t)y = z \in S_T, x, y \in K, 0 < t < 1$. Since $z \in S$ *and* $S \in \mathscr{P}$ we have $x, y \in S$. Hence $Re\ Tx \leq \mu, Re\ Ty \leq \mu$. Since $Re\ Tz = \mu$ and T is linear we have $Re\ Tx = \mu = Re\ Ty$. Hence $x, y \in S_T$. Hence claim hold.
Choose some $S \in \mathscr{P}$. Let $\mathscr{P}' = \{E : E \subset S, E \in \mathscr{P}\}$. Since $S \in \mathscr{P}'$, $\mathscr{P}'$ is

not empty. Partially order $\mathscr{P}'$ by set inclusion, let ω be a maximal totally ordered subcollection of $\mathscr{P}'$, and let M be the intersection of all members of ω. Since ω is a collection of compact sets with the finite intersection property, $M \neq \phi$. By (1), $M \in \mathscr{P}'$. The maximality of ω implies that no proper subset of M belongs to $\mathscr{P}$. It now follows from (2) that every $T \in X^*$ is constant on M. Since X^* separates points on X, M has only one point. Therefore M is an extreme point of K.
We proved

$$E(K) \cap S \neq \phi \qquad (*)$$

for every $S \in \mathscr{P}$. In other words, every compact extreme set of K contains an extreme point of K.
Since K is compact and convex, we have

$$\overline{co}(E(K)) \subset K$$

and hence $\overline{co}(E(K))$ is compact.

Assume that some $x_0 \in K$ is not in $\overline{co}(E(K))$. By Proposition 4.3.9, there is $T \in X^*$ such that $Re\ Tx < Re\ Tx_0$ for every $x \in \overline{co}(E(K))$. If K_T is defined as in (2), then $K_T \in \mathscr{P}$. Choice of T shows that K_T is disjoint from $\overline{co}(E(K))$ and controdicts equation (*). Hence $K = \overline{co}(E(K))$. □

Theorem 4.3.12. *If K is compact subset of locally convex space X then* $K \subset \overline{co}(E(K))$. *Equivalently,* $\overline{co}(K) = \overline{co}(E(K))$.

Theorem 4.3.13. *If K is a compact set in a locally convex space X, and if* $\overline{co}(K)$ *is also compact, then every extreme point of* $\overline{co}(K)$ *lies in K.*

Proof. Assume that some extreme point p of $\overline{co}(K)$ is not in K i.e. $p \in K^c$, K^c is open. Then there is a convex balanced neighborhood V of 0 in X such that

$$(p + \overline{V}) \cap K = \phi \qquad (1)$$

$K \subset \bigcup_{x \in K} (x+V)$ and since K is compact every open cover have finite subcover,

So there are $x_1, \ldots, x_n$ in K such that $K \subset \bigcup_{i=1}^{n} (x_i + V)$. Each set

$$A_i = \overline{co}(K \cap (x_i + V)) \qquad (1 \leq i \leq n)$$

is convex and also compact, since $A_i \subset \overline{co}(K)$. Also $K \subset A_1 \cup \cdots \cup A_n$. By Theorem 4.3.8 (1) we have

$$\overline{co}(K) \subset \overline{co}(A_1 \cup \cdots \cup A_n) = co(A_1 \cup \cdots \cup A_n).$$

Since $A_i \subset \overline{co}(K)$ for each i, we get $\overline{co}(K) \supset co(A_1 \cup \cdots \cup A_n)$. Thus

$$\overline{co}(K) = co(A_1 \cup \cdots \cup A_n). \qquad (2)$$

So $p \in co(A_1 \cup \cdots \cup A_n)$, in particular $p = t_1 y_1 + \cdots + t_N y_N$, where each y_j lies in some A_i, each t_j is positive and $\sum t_j = 1$. The grouping

$$p = t_1 y_1 + (1 - t_1) \frac{t_2 y_2 + \cdots + t_N y_N}{t_2 + \cdots + t_N} \qquad (3)$$

exhibits p as a convex combination of two points of $\overline{co}(K)$, by (2). Since p is an extreme point of $\overline{co}(K)$, we conclude from (3) that $y_1 = p$. Thus, for some i,

$$p \in A_i \subset x_i + \overline{V} \subset K + \overline{V},$$

which contradicts (1). □

Bibliography

[1] Rudin, Walter. Functional analysis. Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991. xviii+424 pp. ISBN: 0-07-054236-8 MR1157815

[2] Barbu, Viorel; Precupanu, Teodor. Convexity and optimization in Banach spaces. Fourth edition. Springer Monographs in Mathematics. Springer, Dordrecht, 2012. xii+368 pp. ISBN: 978-94-007-2246-0; 978-94-007-2247-7 MR3025420