SPECTRAL THEORY OF ABSOLUTELY MINIMUM ATTAINING POSITIVE OPERATORS

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A Thesis Submitted to
Indian Institute of Technology Hyderabad
In Partial Fulfillment of the Requirements for
The Degree of Doctor of Philosophy

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

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March 15, 2018

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Acknowledgements

This thesis would not have been completed without the support of many people around me, whom I owe a debt of gratitude.

First and foremost, I would like to express my sincere gratitude to my advisors Dr. G. Ramesh and Dr. D. Sukumar for their continuous support in my Ph.D research, for their patience and encouragement. This work was not possible without their help. I am very much thankful for their research inputs and mathematical discussions, which made my job easy and helped me grow in this field. I am greatly indebted to them for their teaching and guidance. Dr. G. Ramesh was always active and ready to give his valuable time for my research discussions whenever I needed without any prior appointments. His strict deadlines on completing the given tasks, motivated me to work hard and that lead to the present research output. Dr . D. Sukumar's friendly attitude made my research life easy and smooth, his encouraging words always boosted my confidence and helped me to keep spirits high.

I am grateful to all the faculty members in the Department of Mathematics, IIT Hyderabad for the training I received from them, for their valuable suggestions and help. My special thanks are due to Dr. C. S. Sastry, former Head of the Department for his encouragement and support. He used to enquire about my research updates and future plans every now and then. His words asking about family well being, health and festivals provided me with so much of emotional and moral support. I would like to thank Dr. Balasubramaniam Jayaram and Dr. P. Anantha Lakshmi Narayana for their moral support during my critical times.

Next, I would like to thank my doctoral committee members Dr. Tanmoy Paul and Dr. Suryanarayana for their encouragement, useful comments and suggestions.

I would like to express my sincere gratitude to the faculty at Stat Math Unit, ISI-Bangalore and Kolkata, especially to Prof. Gadadhar Misra, Prof. Thangavelu, Prof. TSSRK Rao, Prof. Rajarama Bhat and Prof. Pradipta Bandyopadhyay for teaching me various aspects of functional analysis and operator theory. Their lectures helped me to improve my subject knowledge to a great extent.

I am grateful to the faculty members in the Department of Mathematics, University of Hyderabad, especially `Prof. R. Tandon`, `Prof. V. Kannan`, `Prof. S.K. Ray` and `Prof. T. Amaranath`, who's teachings inspired me a lot to take up the research career in mathematics.

I would like to thank all my teachers from my school through university who taught me mathematics and influenced me in some or the other way to reach the present position.

I would like to thank the Department of Mathematics, IIT Hyderabad for providing the necessary facilities for carrying out my research.

I wish to acknowledge NBHM (National Board for Higher Mathematics) for the financial support, otherwise, it would have been very difficult for me to complete my thesis.

I would like to thank all my colleagues from juniors to seniors in the Department of Mathematics for their love and support. Everyone was so helpful and understanding. You all have made my stay at IIT Hyderabad so enjoyable and memorable. Thank you for letting me be real and allowing me to show my frustrations whenever I was going through some rough phase of life. If I try to list out the names, I know I would miss some. So, thank you to every one.

Also, I would like to thank all those people whose names are not mentioned here but who helped me in some or the other way during my thesis work and I express my apology that I could not mention them individually.

Finally, I owe the greatest debt of gratitude to my family members for their unconditional love and support. My parents always have given me freedom to do what I wanted to do. They always stood with me in difficult times. Thank you `Varsha`, you have been a fantastic wife during this period of work. Thank you for bearing with me. Sweetheart, without your endless patience and tremendous support I would not have made it this far.

Hyderabad   Jadav Ganesh
Date

Dedication

In the loving memory of my grand mother

LATE SMT. RATNA BAI JADAV

Abstract

Keywords: minimum modulus, essential minimum modulus, minimum attaining operator, absolutely minimum attaining operator, diagonalizable operator, compact operator, spectral theorem, spectrum, essential spectrum, compact perturbation.

In this thesis our primary goal is to study the structure of absolutely minimum attaining operators. First we begin with studying spectral properties of absolutely minimum attaining positive operators and with the help of them we prove a spectral theorem for this class. Using the polar decomposition theorem we try to give a structure for general absolutely minimum attaining operators. Apart from this we also consider the minimum attaining operators and investigate for their perturbation properties.

This thesis contains three chapters. In **Chapter 1**, we discuss about the class of minimum attaining operators and some of their basic properties. Using this we define absolutely minimum attaining operators, discuss some examples and list out some important basic properties of this class. We motivate our study of the structure of absolutely minimum attaining positive operators by the classical spectral theory of compact operators. We record some of the basic results and terminology from operator theory which will be useful for the further chapters.

In **Chapter 2**, we study the spectral properties of absolutely minimum attaining positive operators defined on infinite dimensional complex Hilbert spaces. Using this we derive a spectral theorem for this class. We construct several examples and establish some important basic properties of this class such as the closed range property and finite dimensionality of the null space or the range space etc. Moreover, with the help of the polar decomposition theorem we give a possible structure for absolutely minimum attaining operators.

Chapter 3 deals with the perturbation properties of minimum attaining operators. First we focus on the compact perturbations and prove that the minimum attaining property of a bounded operator whose minimum modulus lies in the discrete spectrum is stable under small compact perturbations. We observe that given a bounded operator with strictly positive essential minimum modulus, the set of compact perturbations which fail to produce a minimum attaining operator is a very small set, in fact a porous set in the ideal of all compact operators on the given Hilbert space. Finally, we discuss the stability of minimum attaining

property under perturbations by all bounded operators with small norm and obtain related results. At the end of the chapter we list a few problems based on our work.

Mathematics Subject Classification: 47A10, 47A53, 47A55, 47A65, 47A75, 47B07.

Contents

List of Notations

$\mathbb{C}^n$	cartesian product of n copies of the complex field $\mathbb{C}$ (page 1)
H, H_1, H_2	Hilbert spaces (page 3)
$\langle , \rangle$	inner product (page 3)
$\|.\|$	norm (page 3)
$\oplus$	direct sum (page 3)
$\overset{\perp}{\oplus}$	orthogonal direct sum (page 3)
$\overline{M}$	closure of a subspace M (page 3)
$M^{\perp}$	orthogonal complement of M (page 3)
S_M	unit sphere in M (page 3)
B_M	open unit ball in M (page 3)
$[N]$	closed linear span of a subset N (page 3)
$N(T)$	null space of an operator T (page 3)
$R(T)$	range space of an operator T (page 3)
$\mathcal{B}(H_1, H_2)$	set of all bounded operators from H_1 to H_2(page 3)
$\mathcal{B}(H)$	set of all bounded operators on H (page 3)
T^*	adjoint of T (page 4)
$\mathcal{B}^s(H)$	set of all self-adjoint operators on H (page 4)
$\mathcal{B}^+(H)$	set of all positive operators on H (page 4)
P_M	orthogonal projection onto the closed subspace M (page 5)
$\sigma(T)$	spectrum of T (page 6)
$\sigma_p(T)$	point spectrum of T (page 6)
$\sigma_c(T)$	continuous spectrum of T (page 6)
$\sigma_r(T)$	residual spectrum of T (page 6)
$T^{\frac{1}{2}}$	square root of T (page 6)

$\|T\|$	modulus of T (page 6)
$\mathbb{N}$	set of all natural numbers (page 9)
$\mathcal{K}(H_1, H_2)$	set of all compact operators from H_1 to H_2 (page 9)
$\mathcal{K}(H)$	set of all compact operators on H (page 9)
$\mathcal{K}^s(H)$	set of all self-adjoint compact operators on H (page 9)
$\mathcal{K}^+(H)$	set of all positive compact operators on H (page 9)
$\sigma_{ess}(T)$	essential spectrum of T (page 12)
$C(H)$	Calkin algebra on H (page 12)
$\sigma_{disc}(T)$	discrete spectrum of T (page 13)
$\sigma_w(T)$	Weyl spectrum of T (page 13)
$\overline{\lambda}$	conjugate of the complex scalar λ (page 15)
$\mathcal{N}(H_1, H_2)$	set of all norm attaining operators from H_1 to H_2(page 15)
$\mathcal{N}(H)$	set of all norm attaining operators on H (page 15)
$\mathcal{AN}(H_1, H_2)$	set of all absolutely norm attaining operators from H_1 to H_2(page 16)
$\mathcal{AN}(H)$	set of all absolutely norm attaining operators on H (page 16)
$\mathcal{AN}^+(H)$	set of all absolutely norm attaining positive operators on H (page 16)
$m(T)$	minimum modulus of T (page 17)
T^{-1}	inverse of T (page 17)
$\mathcal{M}(H_1, H_2)$	set of all minimum attaining operators from H_1 to H_2 (page 19)
$\mathcal{M}(H)$	set of all minimum attaining operators on H (page 19)
$\mathcal{M}^s(H)$	set of all self-adjoint minimum attaining operators on H (page 19)
$\mathcal{M}^+(H)$	set of all positive minimum attaining operators on H (page 19)
$\mathcal{AM}(H_1, H_2)$	set of all absolutely minimum attaining operators from H_1 to H_2(page 20)
$\mathcal{AM}(H)$	set of all absolutely minimum attaining operators on H(page 20)
$\mathcal{AM}^s(H)$	set of all absolutely minimum attaining self-adjoint operators on H(page 20)
$\mathcal{AM}^+(H)$	set of all absolutely minimum attaining positive operators on H(page 20)
$m_e(T)$	essential minimum modulus of T (page 49)

Chapter 1

Introduction

The Spectral theorem is a milestone in the theory of Hilbert space operators. It is well known that every self-adjoint matrix can be diagonalized. More precisely, every self-adjoint matrix is unitarily equivalent to a real diagonal matrix. In other words, all the eigenvalues of a self-adjoint matrix are real and there is an orthonormal basis for $\mathbb{C}^n$ entirely consisting of eigenvectors of that matrix.

Eigenvalues and diagonalization were introduced by Augustine Louis Cauchy in 1826, while he was working on finding the normal forms for quadratic functions. Cauchy proved the finite dimensional spectral theorem which says that every hermitian matrix is diagonalizable [30]. The generalizations of this basic result to the operators on infinite dimensional Hilbert spaces are called the Spectral theorems.

A bounded linear operator on an infinite dimensional separable Hilbert space is said to be diagonalizable if there exists an infinite diagonal matrix that represents the operator with respect to some orthonormal basis. More generally, a bounded linear operator on a Hilbert space (not necessarily separable) is said to be diagonalizable if the Hilbert space has an orthonormal basis which is made up of all eigenvectors of that operator [15, page 54, Proposition 7.4]. Diagonalizable operators are of interest because they are easy to handle, their eigenvalues and eigenvectors are completely known, spectrum is just the closure of the set of all its diagonal entries. Thus the task of finding the classes of diagonalizable operators on a Hilbert space is an extremely important one. The spectral theorem for self-adjoint compact operators says that every such operator is diagonalizable. There are several versions of the spectral theorem depending on the kind of operator considered for instance positive, self-adjoint, normal, etc [42, 17, 55, 6] and the form of the statement for instance spectral measure version or integral form, multiplication operator form, functional calculus version (For details see, [7]). More detailed information on spectral theorems can be found in the Lecture Notes by Henry Helson [33].

A bounded linear operator T on a Banach space is said to be norm attaining if there exists a unit vector x such that $\|Tx\| = \|T\|$. Investigation for the properties of norm attaining operators began by Lindenstrauss in 1963 [44], where he proved that the set of norm attaining operators defined on a reflexive Banach space is dense in the operator norm. Recently, Carvajal and Neves studied some of the properties of norm attaining operators defined between Hilbert spaces in [13], where they have introduced a special class of norm attaining operators, namely the class of absolutely norm attaining operators. A norm attaining operator is called absolutely norm attaining if it remains norm attaining on every nonzero closed subspace of the given Hilbert space. Absolutely norm attaining operators behave in a similar fashion with that of the compact operators. Spectral characterization of this class of operators is discussed in [46, 50].

A related idea to norm attaining operators is the minimum attaining operators. These operators are studied by Xavier Carvajal and Wladimir Neves in [14]. Though the properties of minimum attaining operators have some similarities with that of norm attaining operators, they differ in characteristics, for instance the injectivity and closed range properties play an important role for the class of minimum attaining operators which leads to significant changes in their study.

Absolutely minimum attaining operators are a special kind of minimum attaining operators. More precisely, an operator on a Hilbert space is called absolutely minimum attaining if it is minimum attaining on the whole space and also it remains minimum attaining on every nonzero closed subspace of it.

The primary goal of this thesis is to study the spectral properties of the absolutely minimum attaining operators and derive a spectral theorem for this class. Apart from this we also discuss the perturbation properties of minimum attaining operators. This thesis is based on the published/preprint articles [20, 21, 22].

This chapter contains four sections. In the first section, we record some of the basic results and terminology from operator theory which will be useful for the chapters that come later. In the second section we discuss about compact and Fredholm operators, mention some of their basic properties. In the third section we define the absolutely norm attaining operators and give some examples. In the fourth section we define the absolutely minimum attaining operators and list out some examples.

1.1 Basic terminology

In this section, we list out some basic terminology and notations that we will use later. More details can be found in [7, 15, 24, 40, 48].

Throughout the thesis we will assume all the Hilbert spaces under consideration are complex and infinite dimensional, which we denote by H, H_1, H_2 etc., the inner product and the norm induced by this inner product are denoted by $\langle , \rangle$ and $\|.\|$ respectively. The unit sphere of a closed subspace M of H is denoted by S_M that is $S_M := \{x \in M\colon \|x\| = 1\}$ and the open unit ball is denoted by B_M that is $B_M := \{x \in M\colon \|x\| < 1\}$. If N is a subset of H, then the closed linear span of N is denoted by $[N]$. If M is a subspace of a Hilbert space, the closure of M in H and the orthogonal complement of M in H are denoted by $\overline{M}$ and $M^{\perp}$ respectively. The direct sum and the orthogonal direct sum of subspaces M, N of H are denoted by $M \oplus N$ and $M \overset{\perp}{\oplus} N$ respectively. Throughout the thesis we consider linear operators from H_1 to H_2 and we call a linear operator shortly as an operator. If $T\colon H_1 \to H_2$ is an operator then $N(T)$ and $R(T)$ denote the null space and range space of T respectively. Let M be a subspace of H then $T|_M$ denotes the restriction map of T to M. We write I to denote the identity operator on a Hilbert space. The sequence $\{e_n\}_{n\geq 1}$ denotes the standard orthonormal basis of the Hilbert space ℓ^2.

Definition 1.1.1. *[24, page 51] An operator $T : H_1 \to H_2$ is said to be bounded if there exists a positive scalar α such that $\|T(x)\| \leq \alpha \|x\|$, for all $x \in H_1$. The quantity $\|T\| = \sup\{\|T(x)\| : x \in S_{H_1}\}$ is called the the norm of T.*

Notation 1.1.2. *We denote the set of all bounded operators from H_1 to H_2 by $\mathcal{B}(H_1, H_2)$ and $\mathcal{B}(H, H) = \mathcal{B}(H)$.*

A simple example of a bounded operator is given below.

Example 1.1.3. *Let $R : \ell^2 \to \ell^2$ be defined as $R(x_1, x_2, x_3...) = (0, x_1, x_2, x_3, ...)$, for all $(x_1, x_2, x_3...) \in \ell^2$. Then $R \in \mathcal{B}(\ell^2)$ and $\|R\| = 1$. The operator R is called the right shift operator.*

Definition 1.1.4. *[1, Definition 2.1, page 69] An operator $T : H_1 \to H_2$ is said to be bounded below if there exists a positive scalar γ such that $\|T(x)\| \geq \gamma \|x\|$, for all $x \in H_1$.*

For example, the right shift operator is bounded below. Note that a bounded operator need not be bounded below.

Note 1.1.5. *Let $T \in \mathcal{B}(H)$. Then it is easy to prove that T is bounded below if and only if it is one to one and $R(T)$ is closed (For details see, [1, Theorem 2.5, page 70]).*

Definition 1.1.6. *[48, page 32] Let $T \in \mathcal{B}(H)$. The unique element $S \in \mathcal{B}(H)$ which satisfies*

$$\langle T(x), y\rangle = \langle x, S(y)\rangle, \text{ for all } x, y \in H$$

is called the adjoint of T and we denote it by T^*.

Remark 1.1.7.

1. *Note that $(T^*)^* = T$.*
2. *Note that $N(T)^{\perp} = \overline{R(T^*)}$ and $R(T)^{\perp} = N(T^*)$.*

Below is an example of a bounded operator and its adjoint.

Example 1.1.8. *Let R be the right shift operator and let $L\colon \ell^2 \to \ell^2$ be defined by*

$$L(x_1, x_2, x_3, ...) = (x_2, x_3, x_4, ...), \text{ for all } (x_1, x_2, x_3, ...) \in \ell^2.$$

The operator L is called the left shift operator. It is easy to observe that $R^* = L$

Definition 1.1.9. *[36, page 103] Let $T \in \mathcal{B}(H)$. Then T is said to be,*

1. *normal if $T^*T = TT^*$.*
2. *unitary if $T^*T = TT^* = I$.*
3. *self-adjoint if $T^* = T$.*
4. *positive if $T^* = T$ and $\langle T(x), x\rangle \geq 0$, for all $x \in H$.*

Notation 1.1.10. *We denote the set of all self-adjoint operators and the set of all positive operators on a Hilbert space H by $\mathcal{B}^s(H)$ and $\mathcal{B}^+(H)$ respectively.*

Example 1.1.11. *(Multiplication operator) [40, Problem 6.4, page 497]) For a given $\varphi \in L^\infty[0, 1]$, let $M_\varphi\colon L^2[0, 1] \to L^2[0, 1]$ be defined as,*

$$M_\varphi f(t) = \varphi(t)f(t), \text{ for all } t \in [0, 1] \text{ and } f \in L^2[0, 1].$$

The operator M_φ is called the multiplication operator.

1. *M_φ is always normal.*
2. *M_φ is unitary if we take $\varphi(t) = e^{2\pi it}$, for all $t \in [0, 1]$.*

3. M_φ *is self-adjoint if we take* $\varphi(t) = -t$, *for all* $t \in [0,1]$

4. M_φ *is positive if we take* $\varphi(t) = t$, *for all* $t \in [0,1]$.

Definition 1.1.12. *[49, Theorem 12.14, page 314] Let* $T \in \mathcal{B}(H)$. *Then* T *is said to be a projection if* $T^2 = T$. *A projection is said to be an orthogonal projection if* $N(T)^\perp = R(T)$ *or* $T^* = T$.

Notation 1.1.13. *We usually denote the orthogonal projection onto a closed subspace* M *of* H *by* P_M, *that is* P_M *is an orthogonal projection and* $R(P_M) = M$.

Example 1.1.14. *Let* $P : \ell^2 \to \ell^2$ *be defined as,*

$$P(x_1, x_2, x_3, \dots) = (x_1, x_2, x_3, x_4, x_5, 0, 0, 0, \dots), \text{ for all } (x_1, x_2, x_3, ...) \in \ell^2.$$

Clearly, $N(P) = [e_6, e_7, \dots]$, $R(P) = [e_1, e_2, \dots, e_5]$ *and* $N(P)^\perp = R(P)$. *Therefore* P *is an orthogonal projection onto* $[e_1, e_2, \dots, e_5]$.

Definition 1.1.15. *[40, page 404] Let* $T \in \mathcal{B}(H_1, H_2)$. *Then,*

1. T *is said to be an isometry if* $\|T(x)\| = \|x\|$, *for all* $x \in H_1$.

2. *partial isometry if* $T|_{N(T)^\perp}$ *is an isometry. Here* $T|M$ *is the restriction of the operator* T *to the closed subspace* M *of* H_1.

3. *It is said to be co-isometry if* T^* *is an isometry.*

Note 1.1.16. 1. *Let* $T \in \mathcal{B}(H_1, H_2)$. *Then* T *is an isometry if and only if* $T^*T = I_{H_1}$.

2. *Let* $T \in \mathcal{B}(H_1, H_2)$. *Then* T *is a partial isometry if and only if* $T^*T = P_{N(T)^\perp}$.

3. *Let* $T \in \mathcal{B}(H_1, H_2)$. *Then* T *is an co-isometry if and only if* $TT^* = I_{H_2}$.

Example 1.1.17. 1. *The operator* R *in Example 1.1.8 is an isometry.*

2. *The operator* P *in Example 1.1.14 is a partial isometry.*

3. *The operator* L *in Example 1.1.8 is a co-isometry.*

Definition 1.1.18. *(Diagonalizable operator) [15, Proposition 7.4, page 54] Let* $T \in \mathcal{B}(H)$. *Then* T *is said to be diagonalizable if there exists an orthonormal basis for* H *consisting entirely of eigenvectors of* T.

Example 1.1.19. *The operator* P *defined in Example 1.1.14 is a diagonalizable operator.*

We use the following theorem frequently in the later part of this thesis.

Theorem 1.1.20. *(Projection theorem)[40, page 339, Theorem 5.20] If M is a closed subspace of the Hilbert space H, we have the direct sum decomposition of H, $H = M \oplus M^{\perp}$.*

The concept of the spectrum of a bounded operator is a generalization of the concept of the set of eigenvalues of a matrix. Most of the important properties related to the behaviour and the internal structure of a bounded operator can be investigated by studying its spectrum.The term "spectrum" was first introduced by David Hilbert, who made major contributions to functional analysis, while he was developing a theory of finitely many variables without having any idea that it would later find applications in the real spectra of Physics. Later it was found that there is a close relation between the spectrum in Mathematics with that of Physics [31, page 297].

Definition 1.1.21. *(Spectrum) Let $T \in \mathcal{B}(H)$. Then the set*

$$\sigma(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is not invertible in } \mathcal{B}(H)\}$$

is called the spectrum of T.

We list out below some important basic properties of the spectrum.

1. The spectrum of every bounded operator on a complex Hilbert space is non empty [43, Theorem 2.8, page 209] and compact [43, Theorem 12.6, page 202].

2. Let $T \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}$, then $\sigma(T^*) = \{\overline{\lambda} \colon \lambda \in \sigma(T)\}$.

According to various conditions which affects the invertibility of an operator the spectrum can be subdivided into different subsets.

Definition 1.1.22. *[18, page 580] Let $T \in \mathcal{B}(H)$. Then, the point spectrum $\sigma_p(T)$, the continuous spectrum $\sigma_c(T)$ and the residual spectrum $\sigma_r(T)$ of T are defined by*

1. $\sigma_p(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is not injective}\}$.

2. $\sigma_c(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is injective}, \overline{R(T - \lambda I)} = H \text{ and } R(T - \lambda I) \neq H\}$.

3. $\sigma_r(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is injective}, \overline{R(T - \lambda I)} \neq H\}$.

Note 1.1.23. *Note that the sets $\sigma_p(T), \sigma_c(T)$ and $\sigma_r(T)$ are mutually disjoint. Moreover, $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$.*

It is important to note that many of the basic properties of the spectrum do not hold true for the point spectrum, continuous spectrum and residual spectrum. We will illustrate it in the following examples.

Example 1.1.24. *Let $M_t\colon L^2[0,1] \to L^2[0,1]$ be defined as, $M_t f(t) = tf(t)$, for all $t \in [0,1]$ and $f \in L^2[0,1]$. Next, $\lambda \in \mathbb{C}$ and $M_t f(t) = \lambda f(t)$ implies $f(t) = 0$ for almost all $t \in [0,1]$, that is $f = 0$. Consequently, λ is not an eigenvalue. It is easy to observe that $M_t - \lambda I$ is not onto for all $\lambda \in [0,1]$ and it is invertible for all complex scalars $\lambda \notin [0,1]$. Therefore we have the following.*

1. $\sigma_p(M_t) = \emptyset$.
2. $\sigma_c(M_t) = [0,1]$.
3. $\sigma_r(M_t) = \emptyset$.
4. $\sigma(M_t) = [0,1]$.

Example 1.1.25. *Consider the left shift, right shift operators L and R on ℓ^2. Let $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C}\colon |z| = 1\}$. By direct calculations we can prove that $\sigma_p(L) = \mathbb{D}$, $\sigma_p(R) = \emptyset$ and $\sigma(L) = \sigma(R) = \overline{\mathbb{D}}$. Using the equality, $\sigma_r(T) = \{\lambda \in \mathbb{C}\colon R(T-\lambda I)^{\perp} \neq 0\} \setminus \sigma_p(T)$ and Note 1.1.23, it is easy to prove that,*

1. *$\sigma_r(L) = \emptyset$ and $\sigma_c(L) = \mathbb{T}$.*
2. *$\sigma_r(R) = \mathbb{D}$ and $\sigma_c(R) = \mathbb{T}$.*

Definition 1.1.26. *[53, page 275] Let $T \in \mathcal{B}(H)$. A closed linear subspace M of H is said to be invariant under T if $T(M) \subseteq M$ and reducing if both M and $M^{\perp}$ are invariant under T.*

Example 1.1.27. *Let $P \in \mathcal{B}(\ell^2)$ be the operator in Example 1.1.14. Then,*

1. *The closed subspaces both $N(P)$ and $R(P)$ are invariant under P.*
2. *Since $N(P)^{\perp} = R(P)$, we have both $N(P)$ and $R(P)$ are reducing subspace for P.*

Remark 1.1.28. *Note that M is reducing for T if and only if $M^{\perp}$ is reducing for T.*

Theorem 1.1.29. *Let P_M be the projection on a closed linear subspace M of H. Then,*

1. *M is invariant under T if and only if $TP_M = P_M T P_M$. [53, page 275, Theorem D].*
2. *M reduces T if and only if $TP_M = P_M T$. [53, page 275, Theorem E].*

Definition 1.1.30. *(Completely reducing pair)[54, page 287] Let $T \in \mathcal{B}(H)$. Then a pair of closed subspaces M_1, M_2 is said to be* completely reducing *for T if both M_1 and M_2 are reducing subspaces for T and $H = M_1 \oplus M_2$.*

Definition 1.1.31. *(Direct sum of operators)[15, page 30] Let $\{H_i\}_{i=1}^{\infty}$ be a family of Hilbert spaces such that $H = \overset{\infty}{\underset{i=1}{\oplus}} H_i$. Let $\{T_i\}_{i=1}^{\infty}$ be a family of bounded operators such that $T_i \in \mathcal{B}(H_i)$, for all $i = 1, 2, 3, \ldots$. Then their direct sum, $T = \overset{\infty}{\underset{i=1}{\oplus}} T_i$ is a bounded operator from H to H defined as,*

$$\left(\overset{\infty}{\underset{i=1}{\oplus}} T_i\right)(x_1, x_2, x_3, \ldots) = (T_1x_1, T_2x_2, T_3x_3, \ldots), \text{ for all } (x_1, x_2, x_3, \ldots) \in H.$$

Remark 1.1.32. *In the above definition, we can observe that $T_i = T|_{H_i}$, for all $i = 1, 2, \ldots$ and $\|T\| = \sup\{\|T_i\|\}_{i=1}^{\infty}$.*

Definition 1.1.33. *(Square root of a positive operator). [40, Theorem 5.85, page 402] Let $T \in \mathcal{B}^+(H)$. Then there exists a unique operator $S \in \mathcal{B}(H)$ such that $S \geq 0$ and $T = S^2$. The operator $T^{\frac{1}{2}} := S$ is called the square root of T. The operator S commutes with every operator in $\mathcal{B}(H)$ that commutes with T.*

Definition 1.1.34. *(Modulus). [16, page 306] Let $T \in \mathcal{B}(H_1, H_2)$. Then we have $T^*T \in \mathcal{B}^+(H_1)$ and the operator $|T| := (T^*T)^{\frac{1}{2}}$ is called the modulus of T.*

Example 1.1.35. *Let $R \in \mathcal{B}(\ell^2)$ be the operator in Example 1.1.8. Then $R^*R = I$ and $|R| = (R^*R)^{\frac{1}{2}} = I$.*

We use the Polar decomposition theorem stated below frequently in the next coming chapters.

Theorem 1.1.36. *(Polar decomposition)[40, Theorem 5.89, page 406] If $T \in \mathcal{B}(H_1, H_2)$, then there exists a partial isometry $V \in \mathcal{B}(H_1, H_2)$ such that $T = V|T|$ and $N(V) = N(|T|)$. Moreover, this decomposition is unique. That is, if $W \in \mathcal{B}(H_1, H_2)$ is a partial isometry and $Q \in \mathcal{B}(H_1)$ is a positive operator such that $T = WQ$ and $N(W) = N(Q)$, then $W = V$ and $Q = |T|$.*

Example 1.1.37.

1. *Let R be the right shift operator. Then $|R| = I$. Clearly, $R = RI$ is the polar decomposition of R. Note that $N(R) = N(I) = \{0\}$.*

2. *Let L be the left shift operator. Then $|L| = P_{[e_1]^\perp}$. Clearly, $L = LP_{[e_1]^\perp}$ is the polar decomposition of L. Note that $N(L) = N(P_{[e_1]^\perp}) = [e_1]$.*

We will also use the following version of the spectral mapping theorem in the forthcoming sections.

Theorem 1.1.38. *([28, page 42]) Let $T \in \mathcal{B}(H)$ be an operator and 'p' be a polynomial with complex coefficients. Then,*

$$\sigma\left(p(T)\right) = p\left(\sigma(T)\right) = \{p(\lambda) \colon \lambda \in \sigma(T)\}.$$

1.2 Compact and Fredholm operators

1.2.1 Spectral theorem

The compact operators are originated while studying the integral equations. Typical examples of compact operators come from integral equations. They are the natural generalization of finite rank operators to infinite dimensional setting. Every compact operator on a Hilbert space can be expressed as a norm limit of a sequence of finite rank operators [15, Theorem 4.4, page 41].

Definition 1.2.1. *A bounded operator $T \colon H_1 \to H_2$ is said to be compact if $\overline{T(B_{H_1})}$ is compact in H_2.*

Notation 1.2.2. *We denote the set of all compact operators from H_1 to H_2 by $\mathcal{K}(H_1, H_2)$ and $\mathcal{K}(H, H) = \mathcal{K}(H)$. Similarly, $\mathcal{K}^s(H)$ and $\mathcal{K}^+(H)$ denotes the set of all self-adjoint compact and positive compact operators on H respectively.*

It is easy to observe that all bounded finite rank operators are compact. A non trivial example is given below.

Example 1.2.3. *Let $D : \ell^2 \to \ell^2$ be defined by*

$$D(x_1, x_2, x_3, ...) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_n}{n}, \dots\right), \textit{for all } (x_1, x_2, x_3, ...) \in \ell^2.$$

Since the Hilbert cube is compact it follows that $\overline{D(B_{\ell^2})}$ is compact. This is an example of a diagonal operator. (For details see, [35, Example 9.19, page 230]).

We list out some of the basic properties of the compact operators below.

1. Sum and product of two compact operators is always compact.

2. Range of a compact operator is closed if and only if it is of finite rank.

3. An operator is compact if and only if its adjoint is compact.

Following version of the spectral theorem will be used frequently in the forthcoming sections.

Theorem 1.2.4. *([32, 24]) Let K be a self-adjoint compact operator on H. Then there exists an orthonormal sequence $\{e_n\}_{n=1}^k$ ($k < \infty$ or $k = \infty$) of eigenvectors of K and corresponding sequence of eigenvalues $\{\lambda_n\}_{n=1}^k$ such that the following are true;*

1. $Kx = \sum_{n=1}^{k} \lambda_n \langle x, e_n \rangle e_n$, *for all* $x \in H$.
2. $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots$.
3. *If* $k = \infty$, *then* $\lim_{n\to\infty} \lambda_n = 0$. *Moreover, the above series converges in the strong operator topology of* $\mathcal{B}(H)$.

1.2.2 Fredholm operators

We know that the class of compact operators is a natural generalization of the class of finite-rank operators in an infinite-dimensional setting. A property of finite rank operators that does not generalize to this setting is the following :

If $T\colon X \to Y$ is a linear transformation where X, Y are finite dimensional vector spaces, then

$$\text{dim } (N(T)) - \text{dim } (Y/R(T)) = \text{dim}(X) - \text{dim}(Y).$$

Here dim M denotes the dimension of the vector space M.

In case if X or Y is infinite dimensional then the right hand side of the above equality does not make sense. However, the property that the equality implies could be generalized. The abstract idea of Fredholm operator is derived from this connection. It turns out that many of the operators arising naturally in geometry, the Laplacian, the Dirac operator etc give rise to Fredholm operators. Fredholm operators appear in a natural way in the theory of Toeplitz operators[25, page 347].

A good amount of theory and related results of Fredholm operators can be found in [51] and also in some recent books [15, 25].

We list out below two different kinds of definitions for a Fredholm operator. By Atkinson's Theorem [2, Theorem 3.3.2, page 93] both of these definitions are equivalent.

Definition 1.2.5. *(Fredholm operator)[25, page 184] A bounded linear operator $T\colon H_1 \to H_2$, is called a* Fredholm operator *if its range, $R(T)$, is closed and the numbers*

$$n(T) = dim\ N(T),\ d(T) = dim\, R(T)^{\perp}$$

are finite. In this case ind $(T) = n(T) - d(T)$ is said to be the index *of T.*

Definition 1.2.6. *[15, page 349] Let $T \in \mathcal{B}(H_1, H_2)$. Then T is called right semi-Fredholm , if there exists a $A \in \mathcal{B}(H_2, H_1)$ and $K' \in \mathcal{K}(H_2)$ such that $TA = I + K'$ and left semi-Fredholm , if there exists a $B \in \mathcal{B}(H_2, H_1)$ and $K \in \mathcal{K}(H_1)$ such that $BT = I + K$.*

If T is both left semi-Fredholm and right semi-Fredholm, then T is called Fredholm.

Here we give some examples of Fredholm operators.

Example 1.2.7.

1. *All invertible operators in $\mathcal{B}(H)$ are Fredholm.*
2. *An example of a noninvertible Fredholm operator is the right shift operator.*
3. *$\lambda I + K$ is Fredholm for all $\lambda \in \mathbb{C} \setminus \{0\}$ and $K \in \mathcal{K}(H)$.*

Below we list out some basic properties of Fredholm operators that directly follow from its definition.

1. If $T \in \mathcal{B}(H)$ is Fredholm then $T + K$ is also Fredholm for all $K \in \mathcal{K}(H)$ that is, Fredholm property of an operator is stable under compact perturbations.
2. $T \in \mathcal{B}(H)$ is Fredholm if and only if T^* is Fredholm.
3. The product of two Fredholm operators is Fredholm.

Below we have an example to show that the sum of two Fredholm operators need not be Fredholm.

Example 1.2.8. *Let $T \in \mathcal{B}(H)$ be a Fredholm operator. Then, $-T$ is also Fredholm but their sum is '0', which is not a Fredholm operator because H is infinite dimensional.*

An extension of a Fredholm operator into a bijective operator is defined in [25]. Let $T\colon H_1 \to H_2$ be a Fredholm operator. Then the $N(T)$ and $R(T)$ are complemented in H_1 and H_2 by subspaces W_1 and W_2, respectively, where W_2 is finite dimensional. Then we can define a bijection $\widetilde{T}\colon W_1 \times W_2 \to H_2 = R(T) \times W_2$ by

$$\widetilde{T}(x_0, y_0) = (Tx_0, y_0).$$

This is called the bijection *associated with the Fredholm operator* T.

Theorem 1.2.9. *[25] Suppose $T\colon H_1 \to H_2$ is a Fredholm operator, and let $\widetilde{T}$ be the bijection associated with T. If $S\colon H_1 \to H_2$ is a bounded linear operator with $\|S\| < \|\widetilde{T}^{-1}\|^{-1}$, then $S+T$ is Fredholm and*

1. $n(S+T) \leq n(T)$
2. $d(S+T) \leq d(T)$
3. *ind* $(S+T) =$ *ind* T.

1.2.3 Essential and Weyl spectrum

Definition 1.2.10. *(Essential spectrum) [45, page 30] Let $T \in \mathcal{B}(H)$. Then the essential spectrum $\sigma_{ess}(T)$ of T is defined as,*

$$\sigma_{ess}(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is not a Fredholm operator}\}.$$

Below we list out some important basic properties of the essential spectrum.

1. Since every invertible operator is Fredholm, it follows that $\sigma_{ess}(T) \subseteq \sigma(T)$.
2. The essential spectrum of $T \in \mathcal{B}(H)$ is invariant under all compact perturbation of T.
3. In case of finite dimensional Hilbert spaces all the operators are Fredholm and hence $\sigma_{ess}(T) = \emptyset$.
4. In infinite dimensional Hilbert space case, let $C(H)$ denote the quotient algebra $\mathcal{B}(H)/\mathcal{K}(H)$. This algebra is called the Calkin algebra on H. If π is the quotient map from $\mathcal{B}(H)$ to $C(H)$. It follows from the Atkinson's theorem [45, Theorem 1.4.6, page 28] that $\sigma_{ess}(T) = \sigma(\pi(T))$. Therefore $\sigma_{ess}(T)$ is always a non empty compact set.

As an example we calculate the essential spectrum of a diagonal operator below.

Example 1.2.11. *Let D be the operator defined in Example 1.2.3 and $n \in \mathbb{N}$. Since $(D - \frac{1}{n}I)e_n = 0$, we have $D - \frac{1}{n}I$ is not one-one. Hence it is not invertible in $\mathcal{B}(\ell^2)$. Next, we have $D - \frac{1}{n}I$ is self-adjoint and $N(D - \frac{1}{n}I) = R(D - \frac{1}{n}I)^{\perp} = [e_n]$. Also, $R(D - \frac{1}{n}I)$ is closed because D is compact. Hence $D - \frac{1}{n}I$ is a Fredholm operator.*

Since D is self-adjoint, we have $\overline{R(D)} = N(D)^{\perp} = H$. Next, we observe that $(1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots) \in \ell^2$ but $(1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots) \notin R(D)$. Hence $R(D)$ is not closed. Therefore D is not invertible and also not Fredholm.

Suppose $\lambda \notin \{\frac{1}{n} \colon n \in \mathbb{N}\} \cup \{0\}$, then it is easy to observe that $D - \lambda I$ is invertible in $\mathcal{B}(\ell^2)$ (For details see, [24, page 111]). Consequently, it is Fredholm.

Using all the above details, we can conclude the following.

1. $\sigma(T) = \{\frac{1}{n} \colon n \in \mathbb{N}\} \cup \{0\}$.

2. $\sigma_{ess}(T) = \{0\}$.

The following theorem is a summary of the results present in Reed and Simon [47, page 236]. It gives an alternative description of the essential spectrum and also provides a definition for the discrete spectrum of an operator.

Theorem 1.2.12. *Let $T \in \mathcal{B}(H)$ be self-adjoint. Then the spectrum $\sigma(T)$ of T decomposes as the disjoint union of the discrete spectrum $\sigma_{disc}(T)$ of T and the essential spectrum $\sigma_{ess}(T)$ of T, where,*

The discrete spectrum *is the set of all eigenvalues with finite multiplicity which are isolated from the rest of the spectrum of T.*

The essential spectrum *is the set of all $\lambda \in \sigma(T)$ that satisfy at least one of the following.*

1. *λ is an eigenvalue with infinite multiplicity,*

2. *λ is a limit point of $\sigma_p(T)$,*

3. *$\lambda \in \sigma_c(T)$, the continuous spectrum of T. That is $T - \lambda I$ is one to one but not onto.*

We know that for the case of operators on infinite dimensional Hilbert space the essential spectrum is non empty but this is not true with the discrete spectrum. We illustrate this in the following example.

Example 1.2.13. *Let M_t be the multiplication operator defined in Example 1.1.24. Then by using Theorem 1.2.12, we can conclude that $\sigma_{ess}(M_t) = [0, 1] = \sigma(M_t)$ and $\sigma_{disc}(M_t) = \emptyset$.*

The following version of the Weyl's theorem will be used frequently in the sequel.

Theorem 1.2.14 (Weyl's theorem)**.** *[38, Theorem 2] Let $S, T \in \mathcal{B}(H)$ be self-adjoint. Then*

1. *$\sigma_{ess}(S) = \sigma_{ess}(T)$ if and only if $S - T$ is compact.*

2. *$\lambda \in \sigma(T)$ if and only if there exists a sequence of unit vectors $\{u_n\}_{n=1}^{\infty}$ such that $\|(T - \lambda I)u_n\| \to 0$ as $n \to \infty$.*

3. *$\lambda \in \sigma_{ess}(T)$ if and only if there exists an orthonormal sequence of vectors $\{u_n\}_{n=1}^{\infty}$ such that $\|(T - \lambda I)u_n\| \to 0$ as $n \to \infty$.*

Definition 1.2.15. *(Weyl spectrum).[5, page 530] Let $T \in \mathcal{B}(H)$. Then the* Weyl spectrum *$\sigma_w(T)$ of T is defined as,*

$$\sigma_w(T) = \{\lambda \in \mathbb{C} \colon T - \lambda I \text{ is not a Fredholm operator of index } 0\}.$$

Remark 1.2.16. *Let $T \in \mathcal{B}(H)$. Then, we have $\sigma_{ess}(T) \subseteq \sigma_w(T)$. We already know that $\sigma_{ess}(T) \neq \emptyset$. Hence $\sigma_w(T) \neq \emptyset$.*

Note that the index of a self-adjoint Fredholm operator is always zero. But this is not true in general. We illustrate it in the examples below.

Example 1.2.17. *Let $W \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be defined as $W(\ldots, x_{-2}, x_{-1}, (x_0), x_1, x_2, \ldots) = (\ldots, x_{-2}, (x_{-1}), x_0, x_1, x_2, \ldots)$, for all $(\ldots, x_{-2}, x_{-1}, (x_0), x_1, x_2, \ldots) \in \ell^2(\mathbb{Z})$. This operator is called the bilateral forward shift on $\ell^2(\mathbb{Z})$. Note that W^* is the bilateral backward shift on $\ell^2(\mathbb{Z})$ and W is not self-adjoint. Since $W^*W = WW^* = I$, W is unitary (see [24, page 56], for details). Hence it is a non self-adjoint Fredholm operator of index zero.*

Example 1.2.18. *Let R be the right shift operator. Then, we have $R^* = L$, the left shift operator. Since R is an isometry R is one one and range of R is closed. We have $N(R) = 0$ and $N(R^*) = N(L) = [e_1]$. Therefore, R is a non self-adjoint Fredholm operator of index '-1'.*

The Weyl spectrum of a linear operator is the set of elements in the spectrum which are not eigenvalues of finite multiplicity. For self-adjoint operators the Weyl spectrum is the remainder of the spectrum once the isolated eigenvalues of finite multiplicity are removed (see [39], for details).

Remark 1.2.19. *For a self-adjoint operator the Weyl spectrum and the essential spectrum are the same.*

Remark 1.2.19 is not valid in general for all operators on H. For instance we have an example below.

Example 1.2.20. *Let R be the right shift operator. Consider the operator $R_\lambda := R - \lambda I$, for all $\lambda \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Then we have*

$$\|R_\lambda x\| \geq \|Rx\| - |\lambda|\|x\| \geq (1 - |\lambda|)\,\|x\|, \text{ for all } x \in \ell^2.$$

Suppose $\lambda \in \mathbb{D}$. Then, we have R_λ is bounded below and hence its range is closed. By direct calculations we can prove that $N(R_\lambda) = \{0\}$ and $N(R_\lambda^*) = N(L - \overline{\lambda} I) = [e_{\overline{\lambda}}]$ where $e_{\overline{\lambda}} = (1, \overline{\lambda}, \overline{\lambda}^2, \overline{\lambda}^3, \dots)$. So R_λ is a Fredholm operator of index -1.

Let $\lambda \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. In this case, we have $N(R_\lambda) = \{0\}$ and $N(R_\lambda^*) = N(L - \overline{\lambda} I) = \{0\}$. Suppose, $R(R_\lambda)$ is closed. Then, we have $R(R_\lambda) = N(R_\lambda^*)^\perp = H$. Consequently, R_λ is bijective. So it has to be invertible. But this is not possible because $\mathbb{T} \subseteq \sigma(R)$. Therefore $R(T_\lambda)$ is not closed. So T_λ is not a Fredholm operator.

Now we can conclude the following.

1. *$\sigma_{ess}(R) = \mathbb{T}$.*

2. *$\sigma_w(R) = \overline{\mathbb{D}}$.*

1.3 $\mathcal{AN}$-operators

1.3.1 Norm attaining operators

The study of norm attaining operators began by Lindenstrauss in 1963. He proved that the set of norm attaining operators defined on a reflexive Banach space is dense in the operator norm [44]. Recently, Carvajal and Neves studied the properties of norm attaining operators in [13], where they have introduced the class of absolutely norm attaining operators.

Definition 1.3.1. *(norm attaining operator)) [13, Definition 1.1, page 181] Let $T \in \mathcal{B}(H_1, H_2)$. Then, T is said to be norm attaining if there exists an element $x_0 \in S_H$ such that*

$$\|Tx_0\| = \|T\|.$$

Notation 1.3.2. *We denote the set of all norm attaining operators from H_1 to H_2 by $\mathcal{N}(H_1, H_2)$ and $\mathcal{N}(H, H) = \mathcal{N}(H)$.*

We list out some basic examples of norm attaining operators below.

Example 1.3.3. *1. The Identity operator.*

2. *Every compact operator.*

3. *Every isometry.*

1.3.2 $\mathcal{AN}$-operators

The restriction of a compact operator to a subspace is a compact operator. Therefore a compact operator remains norm attaining on every closed subspace. This suggests the following definition.

Definition 1.3.4. *(Absolutely norm attaining operator) [13, Definition 1.2, page 181] Let $T \in \mathcal{B}(H_1, H_2)$. Then, T is said to be absolutely norm attaining if for every non zero closed subspace M of H_1, $T|M_1$ is norm attaining.*

Notation 1.3.5. *We denote the set of all absolutely norm attaining operators from H_1 to H_2 by $\mathcal{AN}(H_1, H_2)$ and $\mathcal{AN}(H, H) = \mathcal{AN}(H)$. Similarly, $\mathcal{AN}^+(H)$ denotes the set of all absolutely norm attaining positive operators on H.*

We list out some basic examples of absolutely norm attaining operators below.

Example 1.3.6. 1. *Every compact operator.*

2. *Every isometry.*

Not every norm attaining operator is absolutely norm attaining. For instance, we have an example below.

Example 1.3.7. *Let $T\colon \ell^2 \to \ell^2$ be defined as*

$$T(x_1, x_2, x_3, ...) = \left(x_1, \frac{x_2}{2}, \frac{2x_3}{3}, \frac{3x_4}{4}, \dots, \frac{(n-1)x_n}{n}, \dots\right), \textit{ for all } (x_1, x_2, x_3 ...) \in \ell^2.$$

Then $T \in \mathcal{N}(\ell^2)$ but $T|_M \notin \mathcal{N}(M)$ where $M = [e_1]^\perp$ and hence $T \notin \mathcal{AN}(\ell^2)$.

The following characterization of absolutely norm attaining operators is proved in [50].

Theorem 1.3.8. *[50, Theorem 5.1, page 19] Let $T \in \mathcal{B}^+(H)$. Then $T \in \mathcal{AN}^+(H)$ if and only if $T := \alpha I + K + F$, where $\alpha \geq 0, K \in \mathcal{K}^+(H)$ and $F \in \mathcal{B}^s(H)$ is a finite rank operator.*

1.3.3 Minimum modulus

The term *minimum modulus* is first introduced by Gindler and Taylor in [23]. They mention that minimum modulus of an operator is a useful tool in studying the spectrum of that operator. It is a non negative quantity which is analogous to the norm of a bounded linear operator on a Hilbert space. We give the formal definition below.

Definition 1.3.9. *(minimum modulus)[37, page 542] Let $T \in \mathcal{B}(H_1, H_2)$. Then the quantity,*

$$m(T) := \inf\{\|Tx\| : x \in S_{H_1}\},$$

is called the minimum modulus *of T.*

The following basic properties can be easily verified using the definition of minimum modulus .

Note 1.3.10.
1. *Let $T \in \mathcal{B}(H)$. Then T is bounded below if and only if $m(T) > 0$ [40, Definition 4.19, page 223].*
2. *Let $T \in \mathcal{B}(H)$ be invertible. Then $\|T^{-1}\| = \frac{1}{m(T)}$ [23, Lemma 2.1].*
3. *Let $T_1, T_2 \in \mathcal{B}(H)$. Then $m(T_1T_2) \geq m(T_1)m(T_2)$ [23, page 21].*

The following inequality regarding the minimum modulus will be used frequently later.

Lemma 1.3.11. *[23, Lemma 2.2] Let $T_1, T_2 \in \mathcal{B}(H)$. Then,*

$$|m(T_1) - m(T_2)| \leq \|T_1 - T_2\|.$$

The next proposition provides a formula to compute the minimum modulus of a positive operator using inner product.

Proposition 1.3.12. *[14, Proposition 2.2] Let $T \in \mathcal{B}^+(H)$. Then,*

$$m(T) = \inf\{\langle Tx, x\rangle : x \in S_H\}.$$

Note 1.3.13. *Let $T \in \mathcal{B}(H)$. Then by Proposition 1.3.12, we have $m(T) = m(T^*T)^{\frac{1}{2}}$.*

We know that the operator norm satisfies the triangle inequality for the class of all bounded linear operators on a Hilbert space. So it is natural to ask whether the same inequality is also satisfied by minimum modulus.

First, we observe that the triangle inequality is not valid for the minimum modulus, not even for the sub class of positive operators.

Example 1.3.14. *Let $H = \ell^2(\mathbb{N})$. Consider the subspaces $M_1 = [e_{2n-1} : n \in \mathbb{N}]$ and $M_2 = [e_{2n} : n \in \mathbb{N}]$. Now, define $T_1 := P_{M_1}$ and $T_2 := P_{M_2}$. Note that $m(T_1) = 0 = m(T_2)$ but $m(T_1 + T_2) = m(I) = 1$.*

But the following inequality is true for the class of positive operators $\mathcal{B}^+(H)$.

Proposition 1.3.15. *Let $T_1, T_2 \in \mathcal{B}^+(H)$. Then,*

$$m(T_1 + T_2) \geq m(T_1) + m(T_2).$$

Proof. Since $T_1, T_2 \geq 0$, we have $T_1 + T_2 \geq 0$. Now the proof follows directly from Proposition 1.3.12. □

Remark 1.3.16. *The above Proposition is not valid in general for the class of all bounded operators $\mathcal{B}(H)$, for instance consider $T_1 = I$ and $T_2 = -I$ where I is the Identity operator on ℓ^2.*

Given $T \in \mathcal{B}^s(H)$, it is well known that, $\|T\| = \sup\{|\langle Tx, x\rangle| : x \in S_H\}$. Along the similar lines it is natural to expect, $m(T) = \inf\{|\langle Tx, x\rangle| : x \in S_H\}$. But this is not the case. We have the following example.

Example 1.3.17. *Let us consider the operator $T : \ell^2 \to \ell^2$ defined by*

$$T(x_1, x_2, x_3, ...) = (-x_1, x_2, -x_3, x_4, -x_5, x_6, \dots), \text{ for all } (x_1, x_2, x_3 ...) \in \ell^2.$$

Clearly, $T \in \mathcal{B}^s(\ell^2)$ and T is an isometry. By the definition $m(T) = 1$. On the other side, we have $\langle Te_1, e_1\rangle = 1$ and $\langle Te_2, e_2\rangle = -1$ and by the Toeplitz-Hausdorff Theorem [26, Theorem 1.1-2, page 4], it follows that $\inf\{|\langle Tx, x\rangle| : x \in S_H\} = 0$.

Remark 1.3.18. *Let $T \in \mathcal{B}(H)$. Then, we have $\|T\| = \sup\{|\langle Tx, y\rangle| : x, y \in S_H\}$. By Example 1.3.17, $m(T) \neq \inf\{|\langle Tx, y\rangle| : x, y \in S_H\}$.*

Proposition 1.3.19. *[46, Proposition 2.2(1)] Let $T \in \mathcal{B}(H)$ be normal. Then,*

$$m(T) = d(0, \sigma(T)) = \inf\{|\lambda| : \lambda \in \sigma(T)\}.$$

Remark 1.3.20. *Let $T \in \mathcal{B}(H)$ be normal. It is easy to verify that $m(T^*) = m(T)$.*

The Remark 1.3.20 is not valid in general for all operators on H. Below is an example.

Example 1.3.21. *Let L and R be the left and right shift operators. Then we have $L^* = R$ and $m(L) = 0$ but $m(R) = 1$.*

Remark 1.3.22. *If $T \in \mathcal{B}^+(H)$ then by the compactness of $\sigma(T)$ we have,*

$$m(T) = \min \{\lambda : \lambda \in \sigma(T)\}.$$

1.3.4 Minimum attaining operators

The class of minimum attaining operators is studied by Carvajal and Neves in [14] where they have introduced the class of absolutely minimum attaining operators. We begin with the following definition.

Definition 1.3.23. *(minimum attaining operator)[14, Definition 1.1, page 294]*

Let $T \in \mathcal{B}(H_1, H_2)$ and let $m(T)$ be the minimum modulus of T. Then, T is said to be minimum attaining if there exists an element $x_0 \in S_H$ such that

$$\|Tx_0\| = m(T).$$

Notation 1.3.24. *We denote the set of all minimum attaining operators from H_1 to H_2 by $\mathcal{M}(H_1, H_2)$ and $\mathcal{M}(H, H) = \mathcal{M}(H)$. Similarly, $\mathcal{M}^s(H)$ and $\mathcal{M}^+(H)$ denotes the set of self-adjoint minimum attaining operators and positive minimum attaining operators respectively.*

We list out some basic examples of minimum attaining operators below.

Example 1.3.25.

1. *Every non injective operator.*
2. *Every finite rank operator.*
3. *Every isometry and in particular every unitary operator.*

Every compact operator is norm attaining but it need not be minimum attaining. We have that only the non injective compact operators are minimum attaining, the following result provides a complete characterization for minimum attaining compact operators on infinite dimensional spaces.

Proposition 1.3.26. *[14, Proposition 1.3, page 295] Let $T \in \mathcal{B}(H)$ be a compact operator. Then $T \in \mathcal{M}(H)$ iff T is non injective.*

The sum of two minimum attaining operators need not be minimum attaining. Below example illustrates this.

Example 1.3.27. *Let $U\colon \ell^2 \to \ell^2$ be defined as,*

$$U(x_1, x_2, x_3, ...) = \left(x_1, \left(\sqrt{\frac{1}{2}} + i\sqrt{1-\frac{1}{2}}\right) x_2, \left(\sqrt{\frac{1}{3}} + i\sqrt{1-\frac{1}{3}}\right) x_3, \dots\right)$$

for all $(x_1, x_2, x_3...) \in \ell^2$. Then $U \in \mathcal{M}(\ell^2)$ because it is unitary. Obviously, the identity operator $I \in \mathcal{M}(\ell^2)$. But $I + U \notin \mathcal{M}(\ell^2)$. In fact, for each $x \in S_{\ell^2}$ we have,

$$\|(I+U)x\|^2 = \sum_{n=1}^{\infty} \left[\left(1+\sqrt{\frac{1}{n}}\right)^2 + \left(1-\frac{1}{n}\right)\right] |x_n|^2 = \sum_{n=1}^{\infty} \left(2+2\sqrt{\frac{1}{n}}\right) |x_n|^2 > 2.$$

On the other hand, we have $\inf \|(I+U)e_n\| = \sqrt{2}$. Therefore, $m(I+U) = \sqrt{2}$. But $\|(I+U)x\| > \sqrt{2}$, for all $x \in S_{\ell^2}$. Hence $I + U \notin \mathcal{M}(\ell^2)$.

Remark 1.3.28.

1. *The set of minimum attaining operators on H, $\mathcal{M}(H)$ is not a vector space because it is not closed under addition.*
2. *Consider I, U as in the above example. Then, we have $I, U \in \mathcal{M}$ but $\frac{I+U}{2} \notin \mathcal{M}(\ell^2)$. We can conclude that the convex combination of two minimum attaining operators need not be minimum attaining.*

1.3.5 $\mathcal{AM}$-operators

The absolutely minimum attaining operators are a special kind of minimum attaining operators. This class of operators was first introduced and studied by Carvajal and Neves in [14]. We give the formal definition of for an absolutely minimum attaining operator below.

Definition 1.3.29. *(Absolutely minimum attaining operator)[14, Definition 1.4, page 296] Let $T \in \mathcal{B}(H_1, H_2)$. Then, T is said to be an absolutely minimum attaining operator if for every non zero closed subspace M of H_1, $T|_M$ is minimum attaining. Shortly, we say that T is an $\mathcal{AM}$-operator.*

Notation 1.3.30. *We denote the set of all absolutely minimum attaining operators from H_1 to H_2 by $\mathcal{AM}(H_1, H_2)$ and $\mathcal{AM}(H, H) = \mathcal{AM}(H)$. Similarly, $\mathcal{AM}^s(H), \mathcal{AM}^+(H)$ denotes the set of all absolutely minimum attaining self-adjoint operators and absolutely minimum attaining positive operators on H, respectively.*

Some basic examples of absolutely minimum attaining operators are listed below.

Example 1.3.31. *1. Every isometry.*

2. Every finite rank operator.

Not every minimum attaining operator is absolutely minimum attaining. For instance, we have an example below.

Example 1.3.32. *Let $T\colon \ell^2 \to \ell^2$ be defined as*

$$T(x_1, x_2, x_3, ...) = (x_1, (1 + \frac{1}{2})x_2, (1 + \frac{1}{3})x_3, \dots), \textit{ for all } (x_1, x_2, x_3...) \in \ell^2.$$

Then $T \in \mathcal{M}(\ell^2)$ but $T|_M \notin \mathcal{M}(M)$ where $M = [e_1]^\perp$ and hence $T \notin \mathcal{AM}(\ell^2)$.

The sum of two absolutely minimum attaining operators need not be absolutely minimum attaining. Below example illustrates this.

Example 1.3.33. *Let $U\colon \ell^2 \to \ell^2$ be defined as,*

$$U(x_1, x_2, x_3, ...) = (x_1, \left(\sqrt{\frac{1}{2}} + i\sqrt{1 - \frac{1}{2}}\right) x_2, \left(\sqrt{\frac{1}{3}} + i\sqrt{1 - \frac{1}{3}}\right) x_3, \dots)$$

for all $(x_1, x_2, x_3...) \in \ell^2$. Then $U \in \mathcal{AM}(\ell^2)$ because it is unitary. Obviously, the identity operator $I \in \mathcal{AM}(\ell^2)$. But $I + U \notin \mathcal{M}(\ell^2)$. In fact, for each $x \in S_{\ell^2}$ we have,

$$\|(I + U)x\|^2 = \sum_{n=1}^{\infty} \left[\left(1 + \sqrt{\frac{1}{n}}\right)^2 + \left(1 - \frac{1}{n}\right) \right] |x_n|^2 = \sum_{n=1}^{\infty} \left(2 + 2\sqrt{\frac{1}{n}}\right) |x_n|^2 > 2.$$

On the other hand, we have $\inf \|(I + U)e_n\| = \sqrt{2}$. Therefore, $m(I + U) = \sqrt{2}$. But $\|(I+U)x\| > \sqrt{2}$, for all $x \in S_{\ell^2}$. Hence $I+U \notin \mathcal{M}(\ell^2)$. Consequently, $I+U \notin \mathcal{AM}(\ell^2)$.

Remark 1.3.34. *1. The set of absolutely minimum attaining operators on H, $\mathcal{AM}(H)$ is not a vector space because it is not closed under addition.*

2. Consider I, U as in the above example. Then, we have $I, U \in \mathcal{AM}$ but $\frac{I+U}{2} \notin \mathcal{AM}(\ell^2)$. We can conclude that the convex combination of two absolutely minimum attaining operators need not be absolutely minimum attaining.

Chapter 2

Structure of absolutely minimum attaining operators

The spectral theorem for positive compact operators assures its diagonalizability. The class of absolutely norm attaining operators (shortly $\mathcal{AN}$- operators) is introduced by Carvajal and Neves [14]. The spectral theory or diagonalizability of positive $\mathcal{AN}$-operators is discussed in [46, 50]. This class includes the set of all positive operators as a subclass. Analogously, absolutely minimum attaining operators (or $\mathcal{AM}$ operators) are introduced by Carvajal and Neves in [14].

The main goal of this chapter is to prove a characterization of absolutely minimum attaining operators. For this purpose we first study a few spectral properties of positive minimum attaining operators in detail and using these we deduce a characterization theorem for positive $\mathcal{AM}$-operators. Finally, we prove a characterization of $\mathcal{AM}$-operators without assuming positivity.

This chapter contains four sections. In the first section we prove a few basic properties of minimum attaining operators. The second section is devoted for the study of the necessary conditions of positive $\mathcal{AM}$-operators and in the third section we present some sufficient conditions to be satisfied by this class of operators. In the final section we prove the main theorem. All these results appeared in [20, 21].

2.1 Properties of minimum attaining operators

Recall that $T \in \mathcal{B}(H_1, H_2)$ is said to be minimum attaining if there exists a $x_0 \in S_{H_1}$ such that

$$\|Tx_0\| = m(T) = \inf\{\|Tx\| : x \in S_{H_1}\}.$$

[14, Definition 1.1, page 294].

We prove an important property regarding the self-adjoint minimum attaining operators in the following proposition.

Proposition 2.1.1. *Let $T \in \mathcal{B}^s(H)$. Then $T \in \mathcal{M}(H)$ iff $m(T)$ or $-m(T)$ is an eigenvalue of T.*

Proof. Suppose $T \in \mathcal{M}(H)$. Then there exists a $x_0 \in S_H$ such that $\|Tx_0\| = m(T)$. So we get

$$\begin{aligned}
\|Tx_0\|^2 = m(T)^2 &\implies \langle T^2x_0, x_0\rangle = m(T)^2 = m(T)^2\langle x_0, x_0\rangle \\
&\implies \left\langle \left(T^2 - m(T)^2 I\right)(x_0), x_0\right\rangle = 0 \\
&\implies \left(T^2 - m(T)^2 I\right)(x_0) = 0 \; \left[\because \left(T^2 - m(T)^2 I\right) \geq 0\right] \\
&\implies (T + m(T)I)\,(T - m(T)I)\,(x_0) = 0.
\end{aligned}$$

Now if $(T - m(T))(x_0) = 0$, then $Tx_0 = m(T)x_0$, which means $m(T)$ is an eigenvalue of T. In the other case, if $z = (T - m(T))(x_0) \neq 0$, then $z_0 = \frac{z}{\|z\|}$ satisfies $Tz_0 = -m(T)z_0$, which means $-m(T)$ is an eigenvalue of T.

Conversely, let $m(T)$ be an eigenvalue of T. Then there exists a $x_0 \in S_H$ such that $Tx_0 = m(T)x_0$. This implies $m(T) = \|Tx_0\|$ and hence $T \in \mathcal{M}^s(H)$. Similarly, if $-m(T)$ is an eigenvalue of T, then $T \in \mathcal{M}^s(H)$. $\square$

Corollary 2.1.2. *Let $T \in \mathcal{B}^+(H)$. Then $T \in \mathcal{M}^+(H)$ if and only if $m(T)$ is an eigenvalue of T.*

Proof. The proof follows immediately from Proposition 2.1.1. $\square$

Proposition 2.1.3. *Let $T \in \mathcal{B}(H_1, H_2)$. Then the following statements are equivalent:*

1. $T \in \mathcal{M}(H_1, H_2)$
2. $|T| \in \mathcal{M}^+(H_1)$
3. $T^*T \in \mathcal{M}^+(H_1)$.

Proof. (1) $\Leftrightarrow$ (2):
We have, $\|Tx\|_{H_2}^2 = \langle Tx, Tx\rangle_{H_2} = \langle |T|x, |T|x\rangle_{H_1} = \||T|x\|_{H_1}^2$, for all $x \in H_1$. So it follows that $T \in \mathcal{M}(H_1, H_2)$ iff $|T| \in \mathcal{M}^+(H_1)$.

Note that $T^*T \geq 0$. By [14, Proposition 2.2], we have the following;

$$\begin{aligned} m(T^*T) &= \inf\{\langle T^*Tx, x\rangle : x \in S_{H_1}\} \\ &= \inf\left\{\langle |T|^2 x, x\rangle : x \in S_{H_1}\right\} \\ &= \inf\left\{\||Tx|\|^2 : x \in S_{H_1}\right\} \\ &= [m(|T|)]^2. \end{aligned}$$

(2) $\Rightarrow$ (3): Let $|T| \in \mathcal{M}^+(H_1)$. We have $|T| \geq 0$, so by Proposition 2.1.1, $\exists\, x_0 \in S_{H_1}$ such that $|T|x_0 = m(|T|)x_0$. This implies, $|T|^2 x_0 = [m(|T|)]^2 x_0$ and so $T^*Tx_0 = m(T^*T)x_0$. Consequently, $T^*T \in \mathcal{M}^+(H_1)$.

(3) $\Rightarrow$ (2): Let $T^*T \in \mathcal{M}^+(H_1)$. Then by Proposition 2.1.1, $\exists\, x_0 \in S_{H_1}$ such that

$$T^*Tx_0 = m(T^*T)x_0. \tag{2.1}$$

We have the following two cases,

Case(I): $m(T^*T) = 0$

By (2.1), $T^*Tx_0 = 0$. Since $N(T^*T) = N(T) = N(|T|)$, we get $|T|x_0 = 0$ and so $|T| \in \mathcal{M}^+(H_1)$.

Case(II): $m(T^*T) = [m(|T|)]^2 > 0$

By (2.1), we have $|T|^2 x_0 = [m(|T|)]^2 x_0$. Consequently,

$$[(|T| + m(|T|)I)\,(|T| - m(|T|)I)]\,(x_0) = 0.$$

Since $|T| + m(|T|)I$ is invertible, we get $|T|x_0 = m(|T|)x_0$ and so $|T| \in \mathcal{M}^+(H_1)$. $\square$

Corollary 2.1.4. *Let $T \in \mathcal{B}(H)$ be normal. Then $T \in \mathcal{M}(H)$ if and only if $T^* \in \mathcal{M}(H)$.*

Proof. We have T is normal and so $|T^*| = |T|$. Now the proof is immediate from Proposition 2.1.3. $\square$

Remark 2.1.5. *Note that Corollary 2.1.4 is not valid in general for all minimum attaining operators. We have the following example.*

Example 2.1.6. *Let $T : \ell^2 \to \ell^2$ be defined by,*

$$T(x_1, x_2, x_3, ...) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots), \textit{ for all } (x_1, x_2, x_3...) \in \ell^2.$$

Then $m(T) = 0 = \|Te_1\|$, where $e_1 = (1, 0, 0, 0, \dots) \in S_{\ell^2}$. Hence $T \in \mathcal{M}(\ell^2)$. The adjoint of T is given by $T^* : \ell^2 \to \ell^2$ which satisfies,

$$T^*(x_1, x_2, x_3, ...) = (x_2, \frac{x_3}{2}, \frac{x_4}{3}, \dots), \textit{ for all } (x_1, x_2, x_3...) \in \ell^2.$$

We have,

$$0 \leq m(T^*) \leq \|T^* e_n\| = \frac{1}{n}, \textit{ for all } n \in \mathbb{N},$$

consequently, $m(T^*) = 0$. But $\|T^*x\| = 0$ implies that $x = 0$. Hence $T^* \notin \mathcal{M}(\ell^2)$. Here we notice that T is not normal.

2.2 Necessary conditions for $\mathcal{AM}$-operators

2.2.1 Basic properties of $\mathcal{AM}$-operators

Below we discuss some important basic properties of $\mathcal{AM}$-operators.

Lemma 2.2.1. *Let $T \in \mathcal{AM}^s(H)$. Then $R(T)$ is closed.*

Proof. Using the projection theorem and self-adjointness of T we decompose H as $H = N(T) \oplus \overline{R(T)}$. Note that $N(T)$ is a reducing subspace for T. Let $T_1 = T|_{\overline{R(T)}}$. Suppose $m(T_1) = 0$. We know that $T \in \mathcal{AM}^s(H)$ and so $T_1 \in \mathcal{M}^s(\overline{R(T)})$. By Proposition 2.1.1, there exists $x_1 \in \overline{R(T)}$ such that $T_1 x_1 = m(T_1)x_1$ or $T_1 x_1 = -m(T_1)x_1$ and $x_1 \neq 0$. Then $x_1 \in N(T) \cap \overline{R(T)}$ and we get $x_1 = 0$, which is a contradiction. Therefore we must have $m(T_1) > 0$. Now self-adjointness of T_1 implies that it is invertible and so $R(T_1) = \overline{R(T)}$. But $R(T_1) \subseteq R(T)$ as T_1 is a restriction map of T. This implies $\overline{R(T)} \subseteq R(T)$ and hence $R(T)$ is closed. □

Next we can drop the self-adjointness of T in Lemma 2.2.1 and prove the following.

Proposition 2.2.2. *Let $T \in \mathcal{AM}(H)$. Then $R(T)$ is closed.*

Proof. Let $T = V|T|$ be the polar decomposition of T. We have $\|Tx\| = \| \, |T|x\|$, for all $x \in H$ and so $|T| \in \mathcal{AM}(H)$. Since $|T|$ is positive, by Lemma 2.2.1, $R(|T|)$ is closed. Then $R(T) = V\left(R(|T|)\right) = R\left(V|_{N(|T|)^\perp}\right)$ is closed because $V|_{N(|T|)^\perp}$ is an isometry and its domain $R(|T|)$ is closed. □

Remark 2.2.3. *Let $T \in \mathcal{AM}(H)$ be a compact operator. Then T must be a finite rank operator.*

Proposition 2.2.4. *Let M be a closed subspace of H and $\lambda, \mu \in \mathbb{C}$ such that $|\lambda| \neq |\mu|$. Then the following are equivalent:*

1. $\lambda P_M + \mu P_{M^\perp} \in \mathcal{AM}(H)$
2. *either M or $M^\perp$ is finite dimensional*
3. $P_M \in \mathcal{AM}^+(H)$
4. $P_{M^\perp} \in \mathcal{AM}^+(H)$.

Proof. $(1) \Longleftrightarrow (2)$:

We have the following two cases.

Case(1) :$|\lambda| > |\mu|$

Since $P_{M^\perp} = I - P_M$, we have $\lambda P_M + \mu P_{M^\perp} = (\lambda - \mu)P_M + \mu I$. Now for all $x \in H$, we get

$$\begin{aligned} \|(\lambda P_M + \mu P_{M^\perp})(x)\|^2 &= \|((\lambda-\mu)P_M + \mu I)(x)\|^2 \\ &= \big\langle ((\lambda-\mu)P_M + \mu I)(x), ((\lambda-\mu)P_M + \mu I)(x) \big\rangle \\ &= |\mu|^2 \|x\|^2 + \left(|\lambda|^2 - |\mu|^2\right) \|P_M x\|^2 . \end{aligned}$$

Let N be a non zero closed subspace of H. Taking infimum over all $x \in S_N$ both sides we get that,

$$m\left((\lambda P_M + P_{M^\perp})|_N\right)^2 = |\mu|^2 + \left(|\lambda|^2 - |\mu|^2\right) m(P_M|_N)^2.$$

Moreover, there exists $x_0 \in S_N$ such that $m\left((\lambda P_M + P_{M^\perp})|_N\right) = \|(\lambda P_M + P_{M^\perp})|_N(x_0)\|$ iff $m\left(P_M|_N\right) = \|P_M|_N(x_0)\|$. Hence $\lambda P_M + \mu P_{M^\perp} \in \mathcal{AM}(H)$ iff $P_M \in \mathcal{AM}(H)$.

Case(2) : $|\mu| > |\lambda|$

The result follows by interchanging the roles of λ and μ in case(1).

By the above two observations and by [13, Theorem 3.10], we have $\lambda P_M + \mu P_{M^\perp} \in \mathcal{AM}(H)$ iff either M or $M^\perp$ is finite dimensional. The equivalence of $(2), (3)$ and (4) follows directly from [13, Theorem 3.10]. ☐

Remark 2.2.5. *If $|\mu| = |\lambda|$, then $\lambda P_M + \mu P_{M^\perp} \in \mathcal{AM}(H)$, because it is a scalar multiple of an isometry. But it is not necessary that M or $M^\perp$ is finite dimensional. For example, let $H = \ell^2$ and $M = [e_{2n-1} : n \in \mathbb{N}]$. Then $M^\perp = [e_{2n} : n \in \mathbb{N}]$ and $T = P_M - P_{M^\perp} \in \mathcal{AM}(\ell^2)$ because it is an isometry. But both M and $M^\perp$ are infinite dimensional.*

2.2.2 Spectral properties of positive $\mathcal{AM}$-operators

In this subsection we investigate some of the important properties satisfied by the spectrum of positive $\mathcal{AM}$-operators. Recall that $T \in \mathcal{B}(H_1, H_2)$ is said to be absolutely minimum attaining if $T|_M$ is minimum attaining for any non zero closed subspace M of H_1 [14, Definition 1.4, page 296].

Theorem 2.2.6. *Let $T \in \mathcal{AM}^+(H)$. Then T is diagonalizable.*

Proof. The proof follows in the similar lines to that of [50, Theorem 3.8]. Let $\mathcal{B}$ be the collection of all orthonormal sets of eigenvectors of T. Since $T \geq 0$ and $T \in \mathcal{M}(H)$, by Proposition 2.1.1, we have $m(T)$ is an eigenvalue of T and there exists a corresponding eigenvector for T, so $\mathcal{B} \neq \emptyset$. The elements of $\mathcal{B}$ can be ordered by inclusion, and every chain $\mathcal{C}$ in $\mathcal{B}$ has an upper bound, given by the union of all elements of $\mathcal{C}$. Thus, Zorns Lemma [24, page 267] assures the existence of a maximal element B in $\mathcal{B}$. Let $B = \{u_\lambda \colon \lambda \in \Lambda\}$ be a maximal orthonormal set of eigenvectors of T. We claim that B is an orthonormal basis for H.

Let $H_0 := [B]$. We claim that $H_0 = H$. It is enough to prove that ${H_0}^\perp = \{0\}$. Firstly, we observe that ${H_0}^\perp$ is invariant under T. Let $\mathcal{F} := \{F \subseteq \Lambda \colon F \text{ is finite}\}$. Then given $x \in H_0$, we have

$$x = \sum_{\lambda \in \Lambda} \langle x, u_\lambda \rangle u_\lambda = \lim_{F \in \mathcal{F}} \sum_{\lambda \in F} \langle x, u_\lambda \rangle u_\lambda.$$

Since the above net converges in the norm topology and T is bounded, it follows that

$$\begin{aligned} Tx = T\left(\lim_{F \in \mathcal{F}} \sum_{\lambda \in F} \langle x, u_\lambda \rangle u_\lambda\right) &= \lim_{F \in \mathcal{F}} T\left(\sum_{\lambda \in F} \langle x, u_\lambda \rangle u_\lambda\right) \\ &= \lim_{F \in \mathcal{F}} \sum_{\lambda \in F} \langle x, u_\lambda \rangle T u_\lambda = \sum_{\lambda \in \Lambda} \langle x, u_\lambda \rangle \alpha_\lambda u_\lambda \in H_0, \end{aligned}$$

considering $Tu_\lambda = \alpha_\lambda u_\lambda$, where $\alpha_\lambda \geq 0$ for every $\lambda \in \Lambda$. This shows that H_0 is an invariant subspace for T. Since $T^* = T$, it follows that $H_0^\perp$ is also invariant under T.

It remains to show that ${H_0}^\perp = \{0\}$. Suppose, on the contrary that ${H_0}^\perp \neq \{0\}$. Then ${H_0}^\perp$ is a nontrivial closed subspace of H. As $T \in \mathcal{AM}^+(H)$, $T|_{{H_0}^\perp}$ is also positive and attains its minimum on ${H_0}^\perp$. Consequently, $m(T|_{{H_0}^\perp})$ is an eigenvalue of $T|_{{H_0}^\perp}$. Let u be a unit eigenvector of $T|_{{H_0}^\perp}$ corresponding to the eigenvalue

$m(T|_{{H_0}^\perp})$. Clearly, $u \in {H_0}^\perp$ and so $u \in B^\perp$, a contradiction to the maximality of B and we conclude that ${H_0}^\perp = \{0\}$. This shows that H has an orthonormal basis consisting of eigenvectors of T. Hence T is diagonalizable. $\square$

Corollary 2.2.7. *Let $T \in \mathcal{AM}^+(H)$. Then we have,*

$$Tx = \sum_{\lambda \in \Lambda} \alpha_\lambda \langle x, u_\lambda \rangle u_\lambda, \text{for all } x \in H,$$

where $\{u_\lambda \colon \lambda \in \Lambda\}$ is an orthonormal basis for H such that $Tu_\lambda = \alpha_\lambda u_\lambda$, for all $\lambda \in \Lambda$. Moreover, for every nonempty subset Γ of Λ, we have $\inf\{\alpha_\lambda \colon \ \lambda \in \Gamma\} = \min\{\alpha_\lambda \colon \ \lambda \in \Gamma\}$.

Proof. By Theorem 2.2.6, there exists an orthonormal basis $\{u_\lambda \colon \lambda \in \Lambda\}$ for H such that $Tu_\lambda = \alpha_\lambda u_\lambda$, for all $\lambda \in \Lambda$ with $\alpha_\lambda \geq 0$. Consequently, we have

$$Tx = \sum_{\lambda \in \Lambda} \alpha_\lambda \langle x, u_\lambda \rangle u_\lambda, \text{for all } x \in H.$$

To prove the next part, assume the contrary. That is for some nonempty subset Γ_0 of Λ,

$$\inf\{\alpha_\lambda \colon \lambda \in \Gamma_0\} \neq \min\{\alpha_\lambda \colon \lambda \in \Gamma_0\}.$$

Let $\alpha := \inf\{\alpha_\lambda \colon \lambda \in \Gamma_0\}$ and $H_0 := [u_\lambda \colon \lambda \in \Gamma_0]$. Then for every $x \in S_{H_0}$, we have

$$\begin{aligned}
\|T|_{H_0} x\|^2 &= \left\| \sum_{\lambda \in \Gamma_0} \alpha_\lambda \langle x, u_\lambda \rangle u_\lambda \right\|^2 \\
&= \sum_{\lambda \in \Gamma_0} |\alpha_\lambda|^2 |\langle x, u_\lambda \rangle u_\lambda|^2 \\
&> \sum_{\lambda \in \Gamma_0} |\alpha|^2 |\langle x, u_\lambda \rangle u_\lambda|^2 \\
&= |\alpha|^2 \sum_{\lambda \in \Gamma_0} |\langle x, u_\lambda \rangle u_\lambda|^2 \\
&= |\alpha|^2 \|x\|^2 \\
&= |\alpha|^2 \\
&= (\inf\{\alpha_\lambda \colon \ \lambda \in \Gamma_0\})^2 \\
&= [m(T|_{H_0})]^2.
\end{aligned}$$

This implies that $\|T|_{H_0} x\| > m(T|_{H_0})$ for every $x \in S_{H_0}$, which means that T is not minimum attaining on H_0, a contradiction to $T \in \mathcal{AM}(H)$. This proves the assertion. □

Using techniques from [50], we prove the following lemma.

Lemma 2.2.8. *Let $T \in \mathcal{AM}^+(H)$. Suppose there exists two increasing sequences of eigenvalues $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ of T with corresponding orthonormal sequences of eigenvectors $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}}$ such that $a_n \to a$ and $b_n \to b$. Then we must have $a = b$.*

Proof. Suppose $a \neq b$, then we have either $a < b$ or $b < a$. Let us consider the case $a < b$ and the other case can be dealt similarly. Without loss of generality we may assume that $a < b_1$ so that $a_n < b_n$ for each $n \in \mathbb{N}$ (otherwise, we can choose a natural number m such that $a < b_m$, redefine the sequence $(b_n)_{n=m}^{\infty}$ by $(\widetilde{b}_n)_{n=1}^{\infty}$ and proceed). Note that $Tf_n = a_n f_n$ and $Tg_n = b_n g_n$ for each $n \in \mathbb{N}$. Define $H_0 := \left[t_n f_n + \sqrt{1-t_n^2}g_n \colon n \in \mathbb{N}\right]$, where $t_n \in [0,1]$ are yet to be determined. Observe that H_0 is a closed subspace of H and hence a Hilbert space by itself. Moreover, the set $e_n := t_n f_n + \sqrt{1-t_n^2}g_n$ serves to be an orthonormal basis of H_0. Now let us define a sequence $(c_n)_{n\in\mathbb{N}}$ by $c_n = a + \frac{(b_1-a)}{2n}$ for each $n \in \mathbb{N}$. Then $(c_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence such that for every $n \in \mathbb{N}, {a_n}^2 < {c_n}^2 < {b_n}^2$ and $\lim_{n\to\infty} c_n = a$. Notice that ${t_n}^2{a_n}^2 + (1-{t_n}^2){b_n}^2$ is a convex combination of ${a_n}^2$ and ${b_n}^2$, and hence it follows that ${t_n}^2{a_n}^2 + (1-{t_n}^2){b_n}^2 \in [{a_n}^2, {b_n}^2]$ for each $n \in \mathbb{N}$. In fact, by choosing the right value of ${t_n}^2 \in [0,1]$, ${t_n}^2{a_n}^2 + (1-{t_n}^2){b_n}^2$ can give any point in the interval $[{a_n}^2, {b_n}^2]$. Let us then choose a sequence $(t_n)_{n\in\mathbb{N}}$ such that ${t_n}^2{a_n}^2 + (1-{t_n}^2){b_n}^2 = {c_n}^2$. Now $H_0 = [e_n \colon n \in \mathbb{N}]$ gives that,

$$\begin{aligned}
[m(T|_{H_0})]^2 &= \inf\{\|Tx\|^2 \colon x \in H_0, \|x\| = 1\} \\
&\leq \inf\{\|Te_n\|^2 \colon n \in \mathbb{N}\} \\
&\leq \inf\{{t_n}^2{a_n}^2 + (1-t_n^2){b_n}^2 \colon n \in \mathbb{N}\} \\
&\leq \inf\{{c_n}^2 \colon n \in \mathbb{N}\} \\
&\leq a^2.
\end{aligned}$$

However, any $x \in H_0$ with $\|x\| = 1$, can be written as,

$$x = \sum_{n=1}^{\infty} s_n e_n, \text{ where} \sum_{n=1}^{\infty} |s_n|^2 = 1.$$

Consequently,

$$\begin{aligned} \|T|_{H_0}x\|^2 &= \left\| T\left(\sum_{n=1}^{\infty} s_n e_n\right)\right\|^2 \\ &= \sum_{n=1}^{\infty} |s_n|^2 \left({t_n}^2{a_n}^2 + (1-{t_n}^2){b_n}^2\right) \\ &> \sum_{n=1}^{\infty} |s_n|^2 {c_n}^2 \\ &> a^2. \end{aligned}$$

This implies that for every element $x \in H_0$ with $\|x\| = 1$ we have $\|T|_{H_0}x\| > a \geq m(T|_{H_0})$. Which means that T is not minimum attaining on H_0, a contradiction to $T \in \mathcal{AM}(H)$. Hence our assumption $a < b$ is wrong. Similarly, by changing the roles of a and b, we prove that $b < a$ cannot be true. So we must have $a = b$. This completes the proof. □

Proposition 2.2.9. *Let $T \in \mathcal{AM}^+(H)$. Then $\sigma(T)$ has at most one limit point. Moreover, this unique limit point (if it exists) can only be the limit of an increasing sequence in the spectrum.*

Proof. Since $T \in \mathcal{AM}^+(H)$, by Corollary 2.2.7, we have,

$$Tx = \sum_{\lambda \in \Lambda} \alpha_\lambda \langle x, u_\lambda \rangle u_\lambda, \text{ for all } x \in H,$$

where $\{u_\lambda \colon \lambda \in \Lambda\}$ is an orthonormal basis for H such that $Tu_\lambda = \alpha_\lambda u_\lambda$, for all $\lambda \in \Lambda$. Hence the spectrum $\sigma(T)$ of T, is the closure of $\{\alpha_\lambda\}_{\lambda \in \Lambda}$, (see [28, Problem 63, page 34]).

Let a be a limit point of $\sigma(T)$. We prove that there exists an increasing sequence $(a_n)_{n\in\mathbb{N}} \subseteq \{\alpha_\lambda \colon \lambda \in \Lambda\}$ such that $a_n \to a$. It is enough to prove that there are at most only finitely many terms of the sequence $(a_n)_{n\in\mathbb{N}}$ that are strictly greater than a. Suppose not, for a moment, let us assume that there are infinitely many such terms. This implies, there exists a decreasing subsequence (a_{n_k}) such that $a_{n_k} \to a$ and for each $n_k \in \mathbb{N}$, $a_{n_k} > a$. Let $H_0 := [u_{n_k}]$, where $\{u_{n_k}\} \subseteq \{u_\lambda \colon \lambda \in \Lambda\}$. Then H_0 is a closed subspace of H and hence a Hilbert space by itself. We have $T|_{H_0}$ is positive and by [46, Proposition 2.1] we get $m(T|_{H_0}) = \inf\{|a_{n_k}|\} = a$. However, for every $x = \sum_{n_k} s_{n_k} u_{n_k} \in H_0$ with $\|x\|^2 = \sum_k |s_{n_k}|^2 = 1$, we have

$$\|T|_{H_0} x\|^2 = \left\| \sum_{n_k} s_{n_k} a_{n_k} u_{n_k} \right\|^2 = \sum_{n_k} |s_{n_k}|^2 |a_{n_k}|^2 > a^2 \sum_{n_k} |s_{n_k}|^2 = a^2.$$

This implies that $\|T|_{H_0} x\| > m(T|_{H_0}) = a$ for every $x \in S_{H_0}$, which means that T is not minimum attaining on H_0, a contradiction to $T \in \mathcal{AM}(H)$. This proves our claim.

Next, let a and b be any two limit points (if exist) of the spectrum $\sigma(T)$. By the discussion in the above paragraph, there exist two increasing sequences $(a_n)_{n\in\mathbb{N}} \subseteq \{\alpha_\lambda\}_{\lambda\in\Lambda}$, $(b_n)_{n\in\mathbb{N}} \subseteq \{\alpha_\lambda\}_{\lambda\in\Lambda}$ with corresponding orthonormal sequences of eigenvectors $\{f_n\}_{n\in\mathbb{N}} \subseteq \{u_\lambda\}_{\lambda\in\Lambda}$, $\{g_n\}_{n\in\mathbb{N}} \subseteq \{u_\lambda\}_{\lambda\in\Lambda}$ respectively, such that $a_n \to a$ and $b_n \to b$. Then by applying Lemma 2.2.8, we get $a = b$. This shows that the limit point (if it exists) of the spectrum $\sigma(T)$ of T is unique. □

Corollary 2.2.10. *Let $T \in \mathcal{AM}^+(H)$. Then $\sigma_p(T)$ is a countable set.*

Proof. Suppose our claim is not true. Then $\sigma(T)$ will be an uncountable subset of $\mathbb{R}$. So by the fact that every uncountable subset of real numbers must have at least two limit points, $\sigma(T)$ will have two limit points, which is a contradiction to Proposition 2.2.9. Therefore $\sigma_p(T)$ must be countable. □

Corollary 2.2.11. *Let $T \in \mathcal{AM}^+(H)$. Then T can have at most one eigenvalue with infinite multiplicity.*

Proof. Let a and b be two eigenvalues(if exist) of T with infinite multiplicity. Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two infinite sequences of orthonormal eigenvectors corresponding to the eigenvalues a and b, respectively. Consider the two sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that $a_n = a$, for all $n \in \mathbb{N}$ and $b_n = b$, for all $n \in \mathbb{N}$. Then $\{a_n\}_{n\in\mathbb{N}}$ increases to a and $\{b_n\}_{n\in\mathbb{N}}$ increases to b. By applying Lemma 2.2.8, we must have $a = b$. Therefore T can have at most one eigenvalue with infinite multiplicity. □

Corollary 2.2.12. *Let $T \in \mathcal{AM}^+(H)$. If $\sigma(T) = \overline{\{\alpha_\lambda\}}_{\lambda\in\Lambda}$ has a limit point α and an eigenvalue with infinite multiplicity $\hat{\alpha}$ then, $\alpha = \hat{\alpha}$.*

Proof. Since α is a limit point of $\sigma(T)$, by Proposition 2.2.9, there exists an increasing sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq \{\alpha_\lambda\}_{\lambda\in\Lambda}$ such that $a_n \to \alpha$. Let $\{b_n\}_{n\in\mathbb{N}} \subseteq \{\alpha_\lambda\}_{\lambda\in\Lambda}$ be the constant sequence such that $b_n = \hat{\alpha}$, for all $n \in \mathbb{N}$. Let us denote by $\{f_n\}_{n\in\mathbb{N}}$ and

$\{g_n\}_{n\in\mathbb{N}}$ the orthonormal sequence of eigenvectors corresponding to the eigenvalues $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ respectively. Clearly, b_n is increasing to $\hat{\alpha}$. Now, by applying Lemma 2.2.8, we get $\alpha = \hat{\alpha}$. □

All the results discussed above concerning the spectrum of a positive $\mathcal{AM}$-operators can be summarized as follows.

Theorem 2.2.13. *(compare with [50, Theorem 3.8]) Let $T \in \mathcal{AM}^+(H)$. Then we have,*

$$Tx = \sum_{\lambda\in\Lambda} \alpha_\lambda \langle x, u_\lambda\rangle u_\lambda, \text{ for all } x \in H,$$

where $\{u_\lambda \colon \lambda \in \Lambda\}$ is an orthonormal basis for H such that $Tu_\lambda = \alpha_\lambda u_\lambda$, for all $\lambda \in \Lambda$ and the following hold true:

1. *for every nonempty subset Γ of Λ, we have $\inf\{\alpha_\lambda \colon \lambda \in \Gamma\} = \min\{\alpha_\lambda \colon \lambda \in \Gamma\}$;*
2. *$\sigma(T) = \overline{\{\alpha_\lambda\}}_{\lambda\in\Lambda}$ has at most one limit point. Moreover, this unique limit point (if it exists) can only be the limit of an increasing sequence in $\sigma(T)$;*
3. *$\sigma_p(T)$ is countable and there can exist at most one eigenvalue for T with infinite multiplicity;*
4. *if $\sigma(T)$ has both, a limit point α and an eigenvalue $\hat{\alpha}$ with infinite multiplicity, then $\alpha = \hat{\alpha}$.*

Remark 2.2.14. *The result (3) of Theorem 2.2.13 is not valid for self-adjoint operators that are absolutely minimum attaining. For example, let $T : \ell^2 \to \ell^2$ be defined by*

$$T(x_1, x_2, x_3, \ldots) = (-x_1, x_2, -x_3, x_4, -x_5, \ldots), \text{ for all } (x_1, x_2, x_3, \ldots) \in \ell^2.$$

Then $T = T^*$ and $T \in \mathcal{AM}(H)$, since it is an isometry. Moreover, we have $\sigma_p(T) = \{1, -1\}$ and both eigenvalues have infinite multiplicity.

The following is a necessary condition that has to be satisfied by a positive $\mathcal{AM}$-operator.

Theorem 2.2.15. *Let $T \in \mathcal{AM}^+(H)$. Then there exists a positive scalar α, a positive compact operator K and a positive finite rank operator F such that the following is true:*

1. *$T = \alpha I - K + F$;*
2. *$\|K\| \leq \alpha$ and $KF = FK = 0$.*

Proof. We have $T \in \mathcal{AM}^+(H)$. Then by Theorem 2.2.13(3), there can exist at most one eigenvalue for T with infinite multiplicity. We prove the theorem in the following two cases separately.

Case(I): T has no eigenvalue with infinite multiplicity.

To prove this case, we follow the approach used in [20]. Let $H_1 := H$ and $T_1 := T$. Since $T \in \mathcal{AM}(H)$ and $T \geq 0$ we get $T_1 \in \mathcal{M}(H_1)$ and $T_1 \geq 0$. Then by Proposition 2.1.1, there exists a $u_1 \in S_{H_1}$ such that $T_1 u_1 = m(T_1)u_1$. Let $\alpha_1 = m(T_1)$. Then $\alpha_1 \geq 0$.

Let $H_2 := [u_1]^{\perp}$. Note that $H_1 \supseteq H_2$ and H_2 reduces T. Let $T_2 := T|_{H_2}$. Since $T \in \mathcal{AM}(H)$ and $T \geq 0$, we get $T_2 \in \mathcal{M}(H_2)$ and $T_2 \geq 0$. Then by Proposition 2.1.1, there exists a $u_2 \in S_{H_2}$ such that $T_2 u_2 = m(T_2)u_2$. Let $\alpha_2 = m(T_2)$. Then $\alpha_2 \geq \alpha_1 \geq 0$ and $u_1 \perp u_2$.

Let $H_3 := [u_1, u_2]^{\perp}$. Note that $H_1 \supseteq H_2 \supseteq H_3$ and H_3 reduces T. Let $T_3 := T|_{H_3}$. Since $T \in \mathcal{AM}(H)$ and $T \geq 0$ we get $T_3 \in \mathcal{M}(H_3)$ and $T_3 \geq 0$. Then by Proposition 2.1.1, there exists a $u_3 \in S_{H_3}$ such that $T_3 u_3 = m(T_3)u_3$. Let $\alpha_3 = m(T_3)$. Then $\alpha_3 \geq \alpha_2 \geq \alpha_1 \geq 0$ and $u_3 \perp u_i, i = 1, 2$.

Proceeding this way after n steps we get a sequence of subspaces $\{H_i\}_{i=1}^n$ of H such that $H_1 \supseteq H_2 \supseteq H_3 \cdots \supseteq H_n$ where $H_i = [u_1, u_2, u_3, \ldots, u_{i-1}]^{\perp}$, for all $1 \leq i \leq n$ and also a sequence of scalars $\{\alpha_i\}_{i=1}^n$ such that $0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots \leq \alpha_n$, where $\alpha_i = m(T_i)$, for all $1 \leq i \leq n$.

Next, we claim that $H_n \neq \{0\}$, for all $n \in \mathbb{N}$. If not, then there exists a $n \in \mathbb{N}$ such that $H_n = \{0\}$. By the projection theorem we have,

$$H = H_n \oplus {H_n}^{\perp} = {H_n}^{\perp} = [u_1, u_2, u_3, \ldots, u_{n-1}],$$

a contradiction to H is infinite dimensional. Therefore $H_n \neq \{0\}$, for all $n \in \mathbb{N}$. So there exists an infinite sequence of scalars $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $0 \leq \alpha_n \leq \alpha_{n+1} \leq \|T\|$, for all $n \in \mathbb{N}$. By the monotonic convergence theorem $\alpha_n \to \alpha$ for some $\alpha \leq \|T\|$.

Let $M_1 = [u_n : n \in \mathbb{N}]$. Denote by $M_2 = M_1^{\perp}$. We must have dim $M_2 < \infty$, if not then by applying the same procedure as above, we can find an increasing sequence of eigenvalues of T that converges to a scalar which is greater than α, but this is a contradiction to Theorem 2.2.13(2), that is $\sigma(T)$ can have at most one limit point.

Denote by, $K := \alpha P_{M_1} - TP_{M_1}$. Then we have, $Kx := \sum\limits_{n=1}^{\infty}(\alpha - \alpha_n)\langle x, u_n\rangle u_n$, for all $x \in H$. Now the converse of spectral theorem [24, Theorem 6.2, page 181] gives that K

is a positive compact operator. Clearly, $\|K\| \leq \alpha$ and $\overline{R(K)} = M_1$.

Denote by, $F := TP_{M_2} - \alpha P_{M_2}$. Note that F is a finite rank operator and $R(F) \subseteq M_2$. Next, M_2 is a reducing subspace for T implies that $TP_{M_2} = P_{M_2}T$ [15, Proposition 3.7, page 39] and so F is self-adjoint. Now, $m\left(T|_{M_2}\right) \geq \alpha_n$, for all $n \in \mathbb{N}$ implies that $m\left(T|_{M_2}\right) \geq \alpha$. Therefore $\sigma_p(F) = \{\lambda - \alpha \colon \lambda \in \sigma_p\left(T|_{M_2}\right)\} \subseteq [0, \infty)$. Consequently, F is positive.

Clearly, $H = M_1 \oplus M_2$. Then $T = TP_{M_1} + TP_{M_2} = (\alpha P_{M_1} - K) + (F + \alpha P_{M_2}) = \alpha I - K + F$. Obviously, $KF = FK = 0$.

Case(II): T has exactly one eigenvalue with infinite multiplicity α (say).

Let $M_1 := N(T - \alpha I)$. By the Projection theorem, we have $H = M_1 \oplus M_1^{\perp}$.

Suppose dim $M_1^{\perp} < \infty$. Since M_1 is a reducing subspace for T, $T|_{M_1^{\perp}}$ is a positive finite rank operator. By the Spectral theorem, there exists an orthonormal eigenbasis for $M_1^{\perp}$. We have $\alpha \notin \sigma_p\left(T|_{M^{\perp}}\right)$ since $T \geq 0$. Let M_2 be the subspace of $M_1^{\perp}$ which is spanned by all eigenvectors corresponding to the eigenvalues of $T|_{M_1^{\perp}}$ that are less than α (if no eigenvalues is smaller than α then we take $M_2 = \{0\}$). Similarly, define M_3 to be the subspace of $M_1^{\perp}$ that is spanned by all eigenvectors corresponding to the eigenvalues of $T|_{M_1^{\perp}}$ that are greater than α (if no eigenvalues is greater than α then we take $M_3 = \{0\}$). Clearly, $M_1^{\perp} = M_2 \oplus M_3$ and $H = M_1 \oplus M_2 \oplus M_3$. Then we have,

$$
\begin{aligned}
T &= TP_{M_1} + TP_{M_2} + TP_{M_3} \\
&= \alpha P_{M_1} + TP_{M_2} + TP_{M_3} \\
&= \alpha I - (\alpha P_{M_2} - TP_{M_2}) + (TP_{M_3} - \alpha P_{M_3}) \left[\because I = P_{M_1} + P_{M_2} + P_{M_3}\right].
\end{aligned}
$$

This implies that $T = \alpha I - K + F$ where $K = \alpha P_{M_2} - TP_{M_2}$ and $F = TP_{M_3} - \alpha P_{M_3}$ are both positive finite rank operators such that $KF = FK = 0$ and $\|K\| \leq \alpha$.

In case dim $M_1^{\perp} = \infty$, let $H_1 := M_1^{\perp}$. Note that H_1 reduces T. Denote by $T_1 = T|_{H_1}$. Since $T \in \mathcal{AM}(H)$ and $T \geq 0$, we get $T_1 \in \mathcal{AM}(H_1)$ and $T_1 \geq 0$. Moreover, T_1 has no eigenvalue with infinite multiplicity. Then by applying same procedure to T_1, like in Case(I), we can get an infinite orthonormal sequence of eigenvectors $\{u_n\}_{n=1}^{\infty}$ and a corresponding sequence of eigenvalues $\{\alpha_n\}_{n=1}^{\infty}$ of T such that $0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots \leq \|T_1\| \leq \|T\|$ and $\alpha_n \to \beta$. From Theorem 2.2.13(4), we must have $\alpha = \beta$. Let us denote by, $M_2 = [u_n \colon n \in \mathbb{N}]$. Let M_3 be the orthogonal compliment of M_2 in H_1. Then we must have dim $M_3 < \infty$, otherwise

$\sigma(T)$ will have two distinct limit points, which is not possible by Theorem 2.2.13(2).

Denote by, $K := \alpha P_{M_2} - TP_{M_2}$. Then we have, $Kx := \sum_{n=1}^{\infty}(\alpha - \alpha_n)\langle x, u_n\rangle u_n$, for all $x \in H$. Now the converse of spectral theorem [24, Theorem 6.2, page 181] gives that K is a positive compact operator. Clearly, $\|K\| \leq \alpha$ and $\overline{R(K)} = M_2$.

Denote by, $F := TP_{M_3} - \alpha P_{M_3}$. Note that F is a finite rank operator and $R(F) \subseteq M_3$. Next, M_3 is a reducing subspace for T implies that $TP_{M_3} = P_{M_3}T$ [15, Proposition 3.7, page 39] and so F is self-adjoint. Now, $m\left(T|_{M_3}\right) \geq \alpha_n$, for all $n \in \mathbb{N}$ implies that $m\left(T|_{M_3}\right) \geq \alpha$. Therefore $\sigma_p(F) = \{\lambda - \alpha \colon \lambda \in \sigma_p\left(T|_{M_3}\right)\} \subseteq [0, \infty)$. Consequently, F is positive.

Finally, we have, $T = TP_{M_1} + TP_{M_2} + TP_{M_3} = \alpha I - (\alpha P_{M_2} - TP_{M_2}) + (TP_{M_3} - \alpha P_{M_3})$. It follows that $T = \alpha I - K + F$. ☐

2.3 Sufficient conditions for $\mathcal{AM}$-operators

In this section we discuss some sufficient conditions to be satisfied by $\mathcal{AM}$-operators.

Lemma 2.3.1. *Let $F \in \mathcal{B}^s(H)$ be a finite rank operator. Then for every $\alpha \geq 0$, we have $\alpha I - F \in \mathcal{M}^s(H)$.*

Proof. Let the range of F be k-dimensional. Since F is self-adjoint, by the spectral theorem there exists an orthonormal basis $B = \{u_\lambda \colon \lambda \in \Lambda\}$ for H corresponding to which the matrix of F is diagonal with k non zero real diagonal entries, say $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. This implies that the matrix of $T := \alpha I - F$ with respect to B is also diagonal and consequently, $\sigma(T) = \{\alpha - \alpha_1, \alpha - \alpha_2, \alpha - \alpha_3, \ldots, \alpha - \alpha_k, \alpha\}$. Note that T is self-adjoint. Now by using [46, Proposition 2.1] we get,

$$\begin{aligned} m(T) &= d\left(0, \sigma(T)\right) \\ &= \inf\{|\alpha - \alpha_1|, |\alpha - \alpha_2|, |\alpha - \alpha_3|, \ldots, |\alpha - \alpha_k|, \alpha\} \\ &= \min\{|\alpha - \alpha_1|, |\alpha - \alpha_2|, |\alpha - \alpha_3|, \ldots, |\alpha - \alpha_k|, \alpha\}. \end{aligned}$$

It immediately follows that T attains its minimum at u_{λ_0} for some $\lambda_0 \in \Lambda$. ☐

Let M be any closed subspace of H and $i_M \colon M \to H$ be the inclusion map from M to H, which is defined as $i_M x = x$, for all $x \in M$. Then it is easy to observe that the adjoint of i_M is the map $i_M^* \colon H \to M$, which is defined as,

$$i_M^*(x) = \begin{cases} x\,, & \text{if } x \in M, \\ 0\,, & \text{if } x \in M^{\perp}. \end{cases}$$

Proposition 2.3.2. *Let $T \in \mathcal{B}(H)$. Then the following are equivalent;*

1. *$T \in \mathcal{AM}(H)$*
2. *$Ti_M \in \mathcal{M}(M, H)$ for every nonzero closed subspace M of H*
3. *$i_M^*(T^*T)i_M \in \mathcal{M}(M)$ for every nonzero closed subspace M of H.*

Proof. $(1) \Leftrightarrow (2)$: The proof is direct from the definition of absolutely minimum attaining operator, if we observe that $T|_M = Ti_M$ for every closed subspace of M of H.
$(2) \Leftrightarrow (3)$: It is a direct consequence of Proposition 2.1.3 □

Theorem 2.3.3. *Let $F \in \mathcal{B}(H)$ be a finite rank operator. Then for every $\alpha \geq 0$ we have $\alpha I - F \in \mathcal{AM}(H)$.*

Proof. Let $T := \alpha I - F$. Then we have $T^* = \alpha I - F^*$ and $T^*T = \beta I - \widetilde{F}$ where $\beta = \alpha^2$ and $\widetilde{F} = \alpha(F + F^*) - F^*F$ is a self-adjoint finite rank operator. Using Proposition 2.3.2, it suffices to show that for every closed subspace M of H, $i_M^*(\beta I - \widetilde{F})i_M \in \mathcal{M}(M)$. But $i_M^*(\beta I - \widetilde{F})i_M$ is a operator from the Hilbert space M to itself and $i_M^*(\beta I - \widetilde{F})i_M = \beta(i_M^* I i_M) - i_M^*\widetilde{F}i_M = \beta I_M - \widetilde{F}_M$, where $\beta \geq 0$, I_M is the identity operator on M and $i_M^*\widetilde{F}i_M = \widetilde{F}_M$ is a self-adjoint finite rank operator on M. Now, Lemma 2.3.1 implies that $\beta I_M - \widetilde{F}_M \in \mathcal{M}(M)$. Hence the theorem. □

Remark 2.3.4. *As a particular case of the above theorem, it follows that $\alpha I - F \in \mathcal{M}(H)$, where $\alpha \geq 0$ and F is any finite rank operator not necessarily self-adjoint.*

We know that finite rank operators, unitary operators and isometries are absolutely minimum attaining and the modulus of these operators is either a positive finite rank operator or the identity operator. In the first case, 0 is the eigenvalue with infinite multiplicity and in the second case, 1 is the eigenvalue with infinite multiplicity. Let $T \in \mathcal{AM}^+(H)$ and λ be the eigenvalue of T with infinite multiplicity. In general it is not true that, always either $\lambda = m(T)$ or $\lambda = \|T\|$. We have the following example to illustrate this.

Example 2.3.5. *Let $F : \ell^2 \to \ell^2$ be defined by*

$$F(x_1, x_2, x_3, ...) = (x_1, -x_2, x_3, -x_4, x_5, 0, 0, 0, \dots), \textit{ for all } (x_1, x_2, x_3, ...) \in \ell^2.$$

Consider the operator $T := I - F$. Then we have $T \geq 0$ and $T \in \mathcal{AM}(H)$ by Theorem 2.3.3. In this case, 1 is the eigenvalue for T with infinite multiplicity, which is different from $m(T) = 0$ and $\|T\| = 2$.

Lemma 2.3.6. *Let $K \in \mathcal{K}^+(H)$ and $F \in \mathcal{B}^s(H)$ be a finite rank operator. Then for every $\alpha > 0$, we have $\alpha I - K + F \in \mathcal{M}^s(H)$.*

Proof. Firstly, if K is of finite rank then from Lemma 2.3.1, $T := \alpha I - K + F \in \mathcal{M}^s(H)$.

Next, assume that K is of infinite rank. By the spectral theorem, there exists an orthonormal system of eigenvectors $\{u_n\}_{n\geq 1}$ and corresponding eigenvalues $\{\alpha_n\}_{n\geq 1}$ such that for all $x \in H$,

$$(K - F)x = \sum_{n=1}^{\infty} \alpha_n \langle x, u_n \rangle u_n. \tag{2.2}$$

Moreover, $\alpha_n \in \mathbb{R}$, for all $n \in \mathbb{N}$ and $\{|\alpha_n|\}_{n\geq 1}$ is decreasing to 0. Therefore for each $x \in H$, we have

$$\langle (K - F)x, x \rangle = \sum_{n=1}^{\infty} \alpha_n |\langle x, u_n \rangle|^2. \tag{2.3}$$

We claim that, there exists a n_1 such that $\alpha_{n_1} > 0$. Suppose not, then by (2.3), we have $0 \leq K \leq F$. But F is of finite rank and K is positive, so it follows that K is also of finite rank. In fact, for every $x \in R(F)^{\perp} = N(F)$ we have $0 \leq \langle Kx, x \rangle \leq \langle Fx, x \rangle = 0$ and so $\langle Kx, x \rangle = 0$. Next, $K \geq 0$ implies that $Kx = 0$, for all $x \in R(F)^{\perp} = N(F)$. Therefore we have $N(F) \subseteq N(K)$ and consequently $R(K) \subseteq R(F)$, which is a contradiction because $R(K)$ is infinite dimensional. Hence our claim is true. From Equation(2.2), we have $\sigma(K - F) = \{\alpha_n\}_{n=1}^{\infty} \cup \{0\}$ and the spectral mapping theorem gives that $\sigma(T) = \{\alpha - \alpha_n\}_{n=1}^{\infty} \cup \{\alpha\}$. Now, [46, Proposition 2.1] implies that $m(T) = d\,(0, \sigma(T)) = \inf\{\alpha, |\alpha - \alpha_n|\}_{n=1}^{\infty}$. But we know that $\{|\alpha_n|\}_{n\geq 1}$ is decreasing and $\alpha_{n_1} \geq 0$. This implies that $\alpha - \alpha_{n_1} \leq \alpha - \alpha_n$, for all $n \geq n_1$. Next, $|\alpha_n| \to 0$ implies that there exists a n_2 such that $|\alpha_n| \leq \alpha$, for all $n \geq n_2$. Consequently, $\alpha - \alpha_n \geq 0$, for all $n \geq n_2$. Let $n_3 = \max\{n_1, n_2\}$. Then we have $|\alpha - \alpha_n| \geq |\alpha - \alpha_{n_3}|$, for all $n \geq n_3$ and so $m(T) = \min\{\alpha, |\alpha - \alpha_n|\}_{n=1}^{n_3}$. Clearly, T attains its minimum either at u_k for some $k \in \{1, 2, 3, \ldots, n_3\}$ or at a unit vector in $N(K - F)$. □

Theorem 2.3.7. *Let $K \in \mathcal{K}^+(H)$ and $F \in \mathcal{B}(H)$ be a finite rank operator. Then for every $\alpha \geq \frac{\|K\|}{2}$ we have $\alpha I - K + F \in \mathcal{AM}(H)$.*

Proof. Let $T := \alpha I - K + F$. We prove the theorem in two cases as below.

Case(I): $\alpha = 0$

In this case, $\alpha \geq \frac{\|K\|}{2}$ implies that $K = 0$ and so T is a finite rank operator. Therefore $T \in \mathcal{AM}(H)$.

Case(II): $\alpha > 0$

We have $T^*T = \beta I - \widetilde{K} + \widetilde{F}$ where $\beta = \alpha^2$, $\widetilde{K} = 2\alpha K - K^2$ is a compact operator which is positive because $\alpha \geq \frac{\|K\|}{2}$ and $\widetilde{F} = \alpha(F + F^*) - (KF + F^*K) + F^*F$ is a self-adjoint finite rank operator. Using Proposition 2.3.2, it suffices to show that for every closed subspace M of H, $i_M{}^*(\beta I - \widetilde{K} + \widetilde{F})i_M \in \mathcal{M}(M)$. But $i_M{}^*(\beta I - \widetilde{K} + \widetilde{F})i_M$ is an operator from the Hilbert space M to itself and $i_M^*(\beta I - \widetilde{K} + \widetilde{F})i_M = \beta(i_M^* I i_M) - i_M^*\widetilde{K}i_M + i_M^*\widetilde{F}i_M = \beta I_M - \widetilde{K}_M + \widetilde{F}_M$, where, $\beta > 0$, I_M is the identity operator on M, $\widetilde{K}_M = i_M^*\widetilde{K}i_M$ is a positive compact operator on M and $\widetilde{F}_M = i_M^*\widetilde{F}i_M$ is a self-adjoint finite rank operator on M. Now, Lemma 2.3.6 implies that $\beta I_M - \widetilde{K}_M + \widetilde{F}_M \in \mathcal{M}(M)$. Hence the theorem. $\square$

2.4 Characterization of $\mathcal{AM}$-operators

Using the necessary and sufficient conditions that we have proved in the previous sections, we prove a characterization theorem for positive $\mathcal{AM}$-operators.

Theorem 2.4.1. *The following are equivalent:*

1. $T \in \mathcal{AM}^+(H)$
2. *There exists a decomposition for T of the form $T := \alpha I - K + F$ where K is a positive compact operator with $\|K\| \leq \alpha$ and F is a positive finite rank operator satisfying $KF = FK = 0$. Moreover, this decomposition is unique.*

Proof. (1) $\Rightarrow$ (2): We have $T \in \mathcal{AM}^+(H)$. Then from Theorem 2.2.15, T is of the form, $T = \alpha I - K + F$ where K is a positive compact operator with $\|K\| \leq \alpha$ and F is a positive finite rank operator satisfying $KF = FK = 0$.

It remains to prove the uniqueness part:

Suppose, if possible T has another decomposition of the form $T := \hat{\alpha} I - \hat{K} + \hat{F}$ where $\hat{K}$ is a positive compact operator with $\|\hat{K}\| \leq \alpha$ and $\hat{F}$ is a positive finite rank operator satisfying $\hat{K}\hat{F} = \hat{F}\hat{K} = 0$.

By the spectral mapping theorem, we have $\sigma(T) = \alpha - \sigma(K - F)$. Since $K - F$ is a self-adjoint compact operator and $\dim H = \infty$, by applying the spectral theorem we get that α is either the limit point of $\sigma(T)$ or the eigenvalue of T with infinite multiplicity. By the similar arguments, we get $\hat{\alpha}$ is also, either the limit point of $\sigma(T)$ or the eigenvalue of T with infinite multiplicity. Now, Theorem 2.2.13 implies that $\alpha = \hat{\alpha}$. Next, $\alpha I - K + F = \hat{\alpha} I - \hat{K} + \hat{F}$ implies that,

$$K - F = \hat{K} - \hat{F}. \tag{2.4}$$

We also have, $(K + F)^2 = (\hat{K} + \hat{F})^2$ because $KF = FK = \hat{K}\hat{F} = \hat{F}\hat{K} = 0$, but every positive operator has a unique positive square root[47, Theorem VI.9, page 196], so we must have,

$$K + F = \hat{K} + \hat{F}. \tag{2.5}$$

Now, combining the Equations (2.4) and (2.5), we get $K = \hat{K}$ and $F = \hat{F}$.

(2) $\Rightarrow$ (1): We have $T \in \mathcal{B}(H)$ and T is of the form $T := \alpha I - K + F$ where K is a positive compact operator such that $\|K\| \leq \alpha$ and F is a finite rank operator. Then, $T \geq 0$ because $\alpha I - K \geq 0$ and $F \geq 0$. From Theorem 2.3.7, it follows that $T \in \mathcal{AM}^+(H)$. □

Let $T \in \mathcal{AM}^+(H)$. Then, according to [20, Definition 3.9], T is said to be of *first type* if it has no eigenvalue of infinite multiplicity, otherwise it is of *second type*.

The next theorem completely characterizes the absolutely minimum attaining positive operators of both the types.

Theorem 2.4.2. *The following are equivalent:*

1. $T \in \mathcal{AM}^+(H)$
2. *There exists a unique decomposition for T of the form $T := \alpha I - K + F$ where K is a positive compact operator with $\|K\| \leq \alpha$ and F is a positive finite rank operator satisfying $KF = FK = 0$. Moreover,*
   (a) *T is of* first type *whenever $N(K - F)$ is finite dimensional,*
   (b) *T is of* second type *whenever $N(K - F)$ is infinite dimensional.*

Proof. First part of the proof follows directly from Theorem 2.4.1. Next, by spectral mapping theorem, we have $\sigma(T) = \alpha - \sigma(K - F)$. Clearly, T has an eigenvalue of infinite multiplicity if and only if $N(K - F)$ is infinite dimensional. Hence the theorem. □

Remark 2.4.3. *If $F = 0$, in Theorem 2.4.2 2(a), then $T = \alpha I - K$. In this case, $\alpha = \|T\|$. This is exactly the structure theorem obtained for* first type *absolutely minimum attaining* *positive operators [20, Theorem 4.6] . But this is not the case always, for instance, we have an example below.*

Example 2.4.4. *Let $D : \ell^2 \to \ell^2$ be defined by,*

$$D(x_1, x_2, x_3, ...) = (0, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \dots), \textit{ for all } (x_1, x_2, x_3, ...) \in \ell^2.$$

Let P be the orthogonal projection onto $[e_1]$. Consider the operator, $T := I - D + P$. We have $T \in \mathcal{AM}^+(\ell^2)$ by Theorem 2.3.7. Clearly, $\alpha = 1, \|T\| = 2$ and $\alpha < \|T\|$.

Corollary 2.4.5. *The class, $\mathcal{AM}^+(H)$ is closed under addition. In fact, $\mathcal{AM}^+(H)$ is a cone in the real Banach space of self-adjoint operators.*

Proof. Let $T_1, T_2 \in \mathcal{AM}^+(H)$. Then $T_1 + T_2$ is positive. Next, by Theorem 2.4.1, there exists positive scalars α_1, α_2, positive compact operators K_1, K_2 and positive finite rank operators F_1, F_2 such that $T_1 = \alpha_1 I - K_1 + F_1$ and $T_2 = \alpha_2 I - K_2 + F_2$ where $\|K_1\| \le \alpha_1$ and $\|K_2\| \le \alpha_2$. Moreover, $K_1F_1 = F_1K_1 = 0$ and $K_2F_2 = F_2K_2 = 0$. Then $T_1 + T_2 = (\alpha_1 + \alpha_2)I - (K_1 + K_2) + (F_1 + F_2)$. Now, by Theorem 2.3.7, $T_1 + T_2 \in \mathcal{AM}^+(H)$. Suppose T and $-T$ both are in $\mathcal{AM}^+(H)$, then $T = 0$. This shows that $\mathcal{AM}^+(H)$ is a proper cone in the real Banach space of self-adjoint operators. □

Unlike the minimum attaining operators, the direct sum of two absolutely minimum attaining operators need not be absolutely minimum attaining. Below example illustrates this.

Example 2.4.6. *Let $H = \ell^2$ and denote by $M_1 = [e_{2n-1} \colon n \in \mathbb{N}]$ and $M_2 = [e_{2n} \colon n \in \mathbb{N}$. Clearly, $H = M_1 \oplus M_2$. Let $0_{M_1} \in \mathcal{B}(M_1)$ be the zero operator on M_1 and I_{M_2} be the identity operator on M_2. Obviously, $0_{M_1} \in \mathcal{AM}(M_1)$ and $I_{M_2} \in \mathcal{AM}(M_2)$. We will prove that $0_{M_1} \oplus I_{M_2} \notin \mathcal{AM}(\ell^2)$.*

Let us denote by $T := 0_{M_1} \oplus I_{M_2}$. Then, we have $Te_{2n-1} = 0$ and $Te_{2n} = e_{2n}$ for each $n \in \mathbb{N}$. Define $H_0 := \left[\frac{\sqrt{n^2-1}}{n} e_{2n-1} + \frac{1}{n} e_{2n} \colon n \in \mathbb{N}\right]$. Observe that H_0 is a closed subspace of H and hence a Hilbert space by itself. Moreover, the set $f_n = \frac{\sqrt{n^2-1}}{n} e_{2n-1} + \frac{1}{n} e_{2n}$ serves to be an orthonormal basis of H_0. By the definition of minimum modulus, we have

$$\begin{aligned}
[m(T|_{H_0})]^2 &= \textit{inf}\{\|Tx\|^2 \colon x \in H_0, \|x\| = 1\} \\
&\le \textit{inf}\{\|Te_{2n}\|^2 \colon n \in \mathbb{N}\} \\
&\le \textit{inf}\{\frac{1}{n^2} \colon n \in \mathbb{N}\} \\
&\le 0.
\end{aligned}$$

It follows that, $m(T|_{H_0}) = 0$. However, any $x \in H_0$ with $\|x\| = 1$, can be written as,

$$x = \sum_{n=1}^{\infty} s_n \left(\frac{\sqrt{n^2-1}}{n} e_{2n-1} + \frac{1}{n} e_{2n} \right), \text{where} \sum_{n=1}^{\infty} |s_n|^2 = 1.$$

Consequently,

$$\begin{aligned} \|T|_{H_0}x\|^2 &= \left\| T \left(\sum_{n=1}^{\infty} s_n \left(\frac{\sqrt{n^2-1}}{n} e_{2n-1} + \frac{1}{n} e_{2n} \right) \right) \right\|^2 \\ &= \sum_{n=1}^{\infty} |s_n|^2 (\frac{1}{n^2}) \\ &> 0. \end{aligned}$$

This implies that for every element $x \in H_0$ with $\|x\| = 1$ we have $\|T|_{H_0}x\| > 0 = m(T|_{H_0})$. Which means that T is not minimum attaining on H_0, a contradiction to $T \in \mathcal{AM}(\ell^2)$.

Theorem 2.4.7. *Let $T \in \mathcal{B}(H_1, H_2)$. Then the following statements are equivalent:*

1. $T \in \mathcal{AM}(H_1, H_2)$
2. $|T| \in \mathcal{AM}^+(H_1)$
3. $T^*T \in \mathcal{AM}^+(H_1)$.

Proof. (1) $\Leftrightarrow$ (2): We have, $\|Tx\|_{H_2}^2 = \langle Tx, Tx \rangle_{H_2} = \langle |T|x, |T|x \rangle_{H_1} = \||T|x\|_{H_1}^2$, for all $x \in H_1$. Clearly, $T \geq 0$. It follows that $T \in \mathcal{AM}(H_1, H_2)$ iff $|T| \in \mathcal{AM}^+(H_1)$.

(2) $\Rightarrow$ (3): Let $|T| \in \mathcal{AM}^+(H_1)$. Then by Theorem 2.4.1, there exists a decomposition for $|T|$ of the form $|T| := \alpha I - K + F$ where K is a positive compact operator with $\|K\| \leq \alpha$ and F is a positive finite rank operator satisfying $KF = FK = 0$. This implies, $T^*T = |T|^2 = \beta I - \widetilde{K} + \widetilde{F}$, where $\beta = \alpha^2$, $\widetilde{K} = 2\alpha K - K^2$ is a compact operator which is positive because $\alpha \geq \|K\|$ and $\widetilde{F} = 2\alpha F + F^2$ is a positive finite rank operator. Next, by the Spectral radius formula for normal operators [6, Theorem 1] , $\|\widetilde{K}\| = \sup\{2\alpha\lambda - \lambda^2 \colon \lambda \in \sigma(K)\} \leq \alpha^2 = \beta$. Since $R(\widetilde{K}) \subseteq R(K)$ and $R(\widetilde{F}) \subseteq R(F)$, we have $\widetilde{K}\widetilde{F} = \widetilde{F}\widetilde{K} = 0$. Then by applying Theorem 2.4.1 once again, we conclude that $T^*T \in \mathcal{AM}^+(H)$.

(3) $\Rightarrow$ (2): Let $T^*T \in \mathcal{AM}^+(H_1)$. Then by Theorem 2.4.1, there exists a decomposition for T^*T of the form $T^*T := \alpha I - K + F$ where K is a positive compact operator with $\|K\| \leq \alpha$ and F is a positive finite rank operator satisfying $KF = FK = 0$.

By the spectral theorem, there exists an finite or infinite orthonormal sequence of eigenvectors $\{u_n\}_{n\geq 1}$ corresponding to the eigenvalues $\{\alpha_n\}_{n\geq 1}$ of K such that,

$$Kx = \sum_{n\geq 1} \alpha_n \langle x, u_n\rangle u_n, \text{ for all } x \in H.$$

Moreover, $\alpha_n \geq \alpha_{n+1} \geq 0$, for all $n \geq 1$ and in case if, $\{u_n\}_{n\geq 1}$ is an infinite sequence then $\alpha_n \to 0$. Let us denote by $M_1 = [u_n \colon n \geq 1]$. Similarly, by applying the spectral theorem to F, we get a finite orthonormal sequence of eigenvectors $\{v_n\}_{n=1}^k$ corresponding to the eigenvalues $\{\beta_n\}_{n=1}^k$ of F such that,

$$Fx = \sum_{n=1}^{k} \beta_n \langle x, v_n\rangle v_n, \text{ for all } x \in H.$$

Moreover, $\beta_k \geq \beta_{k-1} \geq \ldots \beta_2 \geq \beta_1 \geq 0$. Let us denote by $M_2 = [v_n \colon 1 \leq n \leq k]$. We have, $M_1 \perp M_2$. Let M_3 be the orthogonal compliment of $M_1 \oplus M_2$ in H. Clearly, $H = M_1 \oplus M_2 \oplus M_3$. Let $\{w_\lambda\}_{\lambda\in\Lambda}$ be an orthonormal basis for M_3. Using the decomposition for T^*T, we have,

$$T^*Tx = \alpha \sum_{\lambda\in\Lambda} \langle x, w_\lambda\rangle w_\lambda + \sum_{n\geq 1} (\alpha - \alpha_n)\langle x, u_n\rangle u_n + \sum_{n=1}^{k} (\alpha + \beta_n)\langle x, v_n\rangle v_n, \text{ for all } x \in H.$$

Let us consider the operator $S : H \to H$ defined as,

$$Sx = \alpha^{\frac{1}{2}} \sum_{\lambda\in\Lambda} \langle x, w_\lambda\rangle w_\lambda + \sum_{n\geq 1} (\alpha - \alpha_n)^{\frac{1}{2}} \langle x, u_n\rangle u_n + \sum_{n=1}^{k} (\alpha + \beta_n)^{\frac{1}{2}} \langle x, v_n\rangle v_n, \text{ for all } x \in H.$$

Then $S \geq 0$ because we have $\alpha_n \leq \alpha$, for all $n \geq 1$. By the definition, S is positive square root of $T^*T = |T|^2$, but positive square root is unique. Therefore we must have $S = |T|$. Let us define $K_1 : H \to H$ by,

$$K_1 x := \sum_{n\geq 1} \left((\alpha - \alpha_n)^{\frac{1}{2}} - \alpha^{\frac{1}{2}} \right) \langle x, u_n\rangle u_n, \text{ for all } x \in H.$$

If the set,$\{u_n\}_{n\geq 1}$ is finite then K_1 is a positive finite rank operator. In case if the set,$\{u_n\}_{n\geq 1}$ is infinite, we have the sequence $\{(\alpha-\alpha_n)^{\frac{1}{2}}-\alpha^{\frac{1}{2}}\}_{n\geq 1}$ is monotonically decreasing to 0. Now, the converse of Spectral theorem [24, Theorem 6.2, page 181] gives that K_1 is a positive compact operator. Clearly, $\|K_1\|\leq \alpha^{\frac{1}{2}}$ and $R(K_1)=M_1$. Let us define $F_1: H\to H$ by,

$$Fx=\sum_{n=1}^{k}\left((\alpha+\beta_n)^{\frac{1}{2}}-\alpha^{\frac{1}{2}}\right)\langle x, v_n\rangle v_n, \text{ for all } x\in H.$$

Clearly, F_1 is a positive finite rank operator and $R(F_1)=M_2$. Moreover, we have $K_1F_1=F_1K_1=0$. Now, it is easy to observe that $|T|=\alpha^{\frac{1}{2}}I-K_1+F_1$. Then by Theorem 2.4.1, we have $|T|\in\mathcal{AM}^+(H)$. □

Corollary 2.4.8. *Let $T\in\mathcal{B}(H)$ be positive. Then $T\in\mathcal{AM}^+(H)$ if and only if $T^{\frac{1}{2}}\in\mathcal{AM}^+(H)$.*

Proof. Since $T\geq 0$, Theorem 2.4.7 implies that $T\in\mathcal{AM}^+(H)$ if and only if $T^2\in\mathcal{AM}^+(H)$. Consequently, $T\in\mathcal{AM}^+(H)$ if and only if $T^{\frac{1}{2}}\in\mathcal{AM}^+(H)$. □

Corollary 2.4.9. *Let $T\in\mathcal{B}(H)$ be normal. Then $T\in\mathcal{AM}(H)$ if and only if $T^*\in\mathcal{AM}(H)$.*

Proof. We have T is normal and so $|T^*|=|T|$. Now the proof is immediate from Theorem 2.4.7. □

Remark 2.4.10. *Note that Corollary 2.4.9 is not valid in general for all absolutely minimum attaining operators. We have the following example.*

Example 2.4.11. *Let $V:\ell^2\to\ell^2$ be defined by,*

$$V(x_1,x_2,x_3,...)=(0,x_1,0,x_2,0,x_3,\dots), \textit{for all } (x_1,x_2,x_3,...)\in\ell^2.$$

Then, we have $V\in\mathcal{AM}(\ell^2)$ because it is an isometry. Now, the adjoint $V^*:\ell^2\to\ell^2$ is given by,

$$V^*(x_1,x_2,x_3,...)=(x_2,x_4,x_6,\dots), \textit{for all } (x_1,x_2,x_3,...)\in\ell^2.$$

Then V^* is a partial isometry such that both $N(V^*)$ and $R(V^*)$ are infinite dimensional, hence by Proposition 2.4.14(which is proved later), $V^*\notin\mathcal{AM}(\ell^2)$. Notice that V is not normal.

Proposition 2.4.12. *Let $T\in\mathcal{AM}(H)$. Then either $N(T)$ or $R(T)$ is finite dimensional.*

Proof. From Theorem 2.4.7, $T \in \mathcal{AM}(H)$ implies that $T^*T \in \mathcal{AM}^+(H)$. Then by Theorem 2.4.1, T^*T has a decomposition of the form,

$$T^*T = \alpha I - K + F$$

where K is a positive compact operator with $\|K\| \leq \alpha$ and F is a positive finite rank operator such that $KF = FK = 0$.

Next, $\sigma(T^*T) = \alpha - \sigma(K - F)$ implies that if there exists an eigenvalue with infinite multiplicity for T^*T then it must be 'α'. We have the following two cases,

Case(I): $\alpha = 0$

Since $\|K\| \leq \alpha$, we have $K = 0$ and $T^*T = F$ is a finite rank operator. Now, $\overline{R(T^*T)} = \overline{R(T^*)}$ implies that dim $R(T^*) < \infty$. From the singular value decomposition theorem [24, Theorem 4.1, page 248], it is easy to observe that an operator is of finite rank if and only if its adjoint is of finite rank. So we must have dim $R(T) < \infty$.

Case(II): $\alpha > 0$

In this case, we have dim $N(T)$= dim $N(T^*T) < \infty$. Otherwise, '0' has to be the eigenvalue for T^*T with infinite multiplicity and $\alpha = 0$, which is not true. □

Proposition 2.4.13. *Let $P \in \mathcal{B}(H)$ be an orthogonal projection. Then $P \in \mathcal{AM}^+(H)$ iff either $N(P)$ or $R(P)$ is finite dimensional.*

Proof. We have $P \geq 0$. Suppose $R(P)$ is finite dimensional. Then obviously, $P \in \mathcal{AM}^+(H)$. In case $N(P)$ is finite dimensional, then $P = I - P_{N(P)}$ and $P_{N(P)} \geq 0$. Therefore by Theorem 2.4.1, $P \in \mathcal{AM}^+(H)$. □

Proposition 2.4.14. *Let $V \in \mathcal{B}(H)$ be partial isometry. Then $V \in \mathcal{AM}(H)$ iff either $N(V)$ or $R(V)$ is finite dimensional.*

Proof. The proof is immediate from Theorem 2.4.7 and Proposition 2.4.13, if we observe that $|V| = P_{R(V^*)}$. □

Proposition 2.4.15. *Let $V \in \mathcal{AM}(H)$ be a partial isometry and F be a finite rank operator. Then for all $\alpha \geq 0$, we have $\alpha V - F \in \mathcal{AM}(H)$.*

Proof. Let us denote by $T := \alpha V - F$. Since $V \in \mathcal{AM}(H)$, we have either $N(V)$ or $R(V)$ is finite dimensional. Firstly, if $R(V)$ is finite dimensional, we are already done because T is a finite rank operator. In the other case, let $N(V)$ be finite dimensional. Then we have $T^*T = \alpha^2 P_{R(V^*)} - [\alpha V^*F + \alpha F^*V + F^*F] = \alpha^2 I - \widetilde{F}$ where $\widetilde{F} = \alpha^2 P_{N(V)} + \alpha V^*F + \alpha F^*V + F^*F$ is a finite rank operator. Now, the proof follows directly from Theorem 2.3.3 and Theorem 2.4.7. □

Remark 2.4.16. *The above Proposition is valid, in particular if we allow V to be an isometry, projection or a co-isometry.*

Using the polar decomposition theorem, Theorem 2.4.1 can be extended to a more general case, as below.

Theorem 2.4.17. *The following are equivalent:*

1. $T \in \mathcal{AM}(H_1, H_2)$;

2. *There exists a decomposition for T of the form $T := V(\alpha I - K + F)$ where $K \in \mathcal{B}(H_1)$ is a positive compact operator with $\|K\| \leq \alpha$, $F \in \mathcal{B}(H_1)$ is a positive finite rank operator satisfying $KF = FK = 0$ and $V \in \mathcal{B}(H_1, H_2)$ is a partial isometry such that $N(V) = N(\alpha I - K + F)$. Moreover, this decomposition is unique.*

Proof. $(1) \Rightarrow (2)$:

We have $T \in \mathcal{B}(H_1, H_2)$, then by the polar decomposition theorem there exists a unique partial isometry $V \in \mathcal{B}(H_1, H_2)$ such that $T = V|T|$ and $N(V) = N(|T|)$. From Theorem 2.4.7, $T \in \mathcal{AM}(H_1, H_2)$ implies that $|T| \in \mathcal{AM}^+(H_1)$. Next, by Theorem 2.4.1, there exists a decomposition for $|T|$ of the form $T := \alpha I - K + F$ where $K \in \mathcal{B}(H_1)$ is a positive compact operator with $\|K\| \leq \alpha$ and $F \in \mathcal{B}(H_1)$ is a positive finite rank operator satisfying $KF = FK = 0$. Clearly, we have $N(V) = N(|T|) = N(\alpha I - K + F)$. Next, the uniqueness of the V is clear and the uniqueness of α, K, F comes from the uniqueness of the decomposition for $|T|$.

$(2) \Rightarrow (1)$:

Let $T \in \mathcal{B}(H_1, H_2)$ and has the decomposition of the form given in (2). Let us, denote by $P := \alpha I - K + F$. Since $\alpha I - K \geq 0$ and $F \geq 0$, we have $P \geq 0$. We also, have V is a partial isometry with $N(V) = N(\alpha I - K + F) = N(P)$. Therefore by the uniqueness of the polar decomposition theorem, we must have $|T| = P$. That is, $|T| = \alpha I - K + F$. By applying Theorem 2.4.7 and Theorem 2.4.1, it follows that $T \in \mathcal{AM}(H_1, H_2)$. □

Chapter 3

Perturbation of minimum attaining operators

In this chapter we are concerned with the perturbation theory of minimum attaining operators. The perturbation properties of norm attaining operators are discussed by J. Kover in [38, 39]. Analogous to the norm attaining bounded operators on H, minimum attaining operators are defined and studied by Carvajal and Neves in [14].

This chapter has two sections. In the first one we discuss the compact perturbations of minimum attaining operators and in the second one we discuss the stability results for minimum attaining operators. All the results in this chapter are written based on the published article [22].

3.1 Compact perturbations

Our goal in this section is to study stability of the minimum attaining property under compact perturbations. In other words, we try to answer the question that which compact perturbations of minimum attaining operators on a Hilbert space are again minimum attaining. We also observe that for any fixed bounded linear operator T on H with $m(T) > 0$, the set of compact perturbations of T that fail to produce a minimum attaining operator is very small in size, in fact it is a porous set.

Lemma 3.1.1. *[38, Lemma 3] Let $T \in \mathcal{B}(H)$. Then,*

$$\lambda \in \sigma_{ess}(|T|) \textit{ if and only if } \lambda^2 \in \sigma_{ess}(|T|^2).$$

Let H be a complex separable Hilbert space and $T \in \mathcal{B}(H)$. For $n \in \mathbb{N} \cup \{0, \infty\}$ we define,

$$\rho_n(T) := \inf\{\|T - S\| : S \in \mathcal{B}(H), \dim N(S) = n\}.$$

Theorem 3.1.2. *[9, Theorem 2] Assume that $n < \dim N(T)$. We have,*

1. *If $n \geq \text{ind}\, T$ then $\rho_n(T) = 0$*
2. *If $n < \text{ind}\, T$ then $\rho_n(T) = m(T^*)$*

The following theorem is crucial in proving the stability of minimum attaining property under small compact perturbations.

Theorem 3.1.3. *Let $T \in \mathcal{B}(H)$ and $K \in \mathcal{K}(H)$ be such that $m(T+K) \notin \sigma_{ess}(|T|)$. Then $m(T+K) \in \sigma_{disc}(|T+K|)$ and $T+K \in \mathcal{M}(H)$.*

Proof. Let us consider the operator,

$$|T+K|^2 = (T+K)^*(T+K) = T^*T + T^*K + K^*T + K^*K = |T|^2 + C,$$

where $C = T^*K + K^*T + K^*K \in \mathcal{K}(H)$. Since $|T+K|^2$ and $|T|^2$ are self-adjoint, by Theorem 1.2.14 it follows that,

$$\sigma_{ess}(|T+K|^2) = \sigma_{ess}(|T|^2).$$

Now, Lemma 3.1.1 gives that, $\sigma_{ess}(|T+K|) = \sigma_{ess}(|T|)$. Hence, $m(T+K) \notin \sigma_{ess}(|T+K|)$. As $|T+K| \geq 0$, by Lemma 1.3.22, we can conclude that $m(T+K) \in \sigma_{disc}(|T+K|)$. Consequently, $T+K \in \mathcal{M}(H)$, by Proposition 2.1.1. $\square$

The above theorem yields the following stability result.

Corollary 3.1.4. *Let $T \in \mathcal{B}(H)$ and $m(T) \in \sigma_{disc}(|T|)$. Then there exists an $\epsilon > 0$ such that for all $K \in \mathcal{K}(H)$ with $\|K\| < \epsilon$, we have $T+K \in \mathcal{M}(H)$.*

Proof. By the definition of the discrete spectrum,

$$d = d\left(m(T), \sigma_{ess}(|T|)\right) = \inf\left\{|\lambda - m(T)| : \lambda \in \sigma_{ess}\left(|T|\right)\right\} > 0.$$

Now choose an $\epsilon \in (0, d)$. By Lemma 1.3.11, for any $K \in \mathcal{K}(H)$ with $\|K\| < \epsilon$ we have,

$$|m(T+K) - m(T)| \leq \|T+K-T\| = \|K\| < \epsilon.$$

This implies that $m(T+K) \notin \sigma_{ess}\left(|T|\right) = \sigma_{ess}\left(|T+K|\right)$. By Theorem 3.1.3, we have $T+K \in \mathcal{M}(H)$. $\square$

Remark 3.1.5. *Note that the condition $m(T) \in \sigma_{disc}(|T|)$ cannot be dropped in Corollary 3.1.4, by the following example.*

Example 3.1.6. *For every $n \in \mathbb{N}$, let $D_n : \ell^2 \to \ell^2$ be defined by,*

$$D_n(x_1, x_2, x_3, ...) = (\frac{x_1}{n}, \frac{x_2}{n+1}, \frac{x_3}{n+2}, \dots), \textit{ for all } (x_1, x_2, x_3, ...) \in \ell^2.$$

Clearly, $D_n \in \mathcal{K}(\ell^2)$ and $\|D_n\| = \frac{1}{n}$, for all $n \in \mathbb{N}$. Next, we have $I + D_n \geq 0$ and $I + D_n \notin \mathcal{M}(\ell^2)$, for all $n \in \mathbb{N}$. Note that $m(I) = 1 \in \sigma_{ess}(I)$.

We denote by, $\mathcal{M}_d(H) := \{T \in \mathcal{B}(H) \colon m(T) \in \sigma_{disc}(|T|)\}$. Similarly, $\mathcal{M}_d^s(H) := \{T \in \mathcal{B}^s(H) \colon m(T) \in \sigma_{disc}(|T|)\}$ and $\mathcal{M}_d^+(H) := \{T \in \mathcal{B}^+(H) \colon m(T) \in \sigma_{disc}(|T|)\}$.

The following corollary will be used frequently in proving many compact perturbation results that are coming later.

Corollary 3.1.7. *Let $T \in \mathcal{B}(H)$ and $K \in \mathcal{K}(H)$. If $m(T+K) < m(T)$, then $m(T+K) \in \sigma_{disc}(|T+K|)$ and $T+K \in \mathcal{M}_d(H)$.*

Proof. Lemma 1.3.22 and the spectral radius formula [6, Theorem1] implies that $\sigma(|T|) \subseteq [m(T), \|T\|]$ and so $m(T+K) \notin \sigma_{ess}(|T|) = \sigma_{ess}(|T+K|)$. Consequently, $m(T+K) \in \sigma_{disc}(|T+K|)$ and $T+K \in \mathcal{M}_d(H)$. □

Remark 3.1.8. *Note that Corollary 3.1.7 is meaningful only for all $T \in \mathcal{B}(H)$ with $m(T) > 0$ because $m(T+K) < 0$ does not hold true for any $K \in \mathcal{K}(H)$.*

Remark 3.1.9. *Let $T \in \mathcal{B}(H)$ and $K \in \mathcal{K}(H)$. Suppose, $m(T+K) \geq m(T)$ then $T+K$ may or may not be minimum attaining. The following example illustrates this.*

Example 3.1.10. *Let $T : \ell^2 \to \ell^2$ be defined by,*

$$T(x_1, x_2, x_3, ...) = (0, \frac{2x_1}{3}, \frac{3x_2}{2}, \frac{4x_3}{3}, \dots), \textit{ for all } (x_1, x_2, x_3, ...) \in \ell^2.$$

Clearly, $m(T) = 0$. Let $P_{[e_1]}$ be the orthogonal projection onto $[e_1]$. Then, $P_{[e_1]} \in \mathcal{K}(\ell^2)$. For every $\mu \in \mathbb{C}$, consider the operator, $T + \mu P_{[e_1]} \in \mathcal{B}(H)$. We have always, $m(T + \mu P_{[e_1]}) \geq m(T)$. But, $T + \mu P_{[e_1]} \in \mathcal{M}(\ell^2)$ whenever $|\mu| \leq 1$ and in the other case, $T + \mu P_{[e_1]} \notin \mathcal{M}(\ell^2)$.

The following lemma will be used frequently in proving the Theorems coming later and the proof is essentially contained in the proof of [41, Theorem 3.4]. We provide the details for the sake of completeness.

Lemma 3.1.11. *Let $T \in \mathcal{B}^+(H)$ and $m(T) > 0$. Then there exists a sequence of positive finite rank operators $\{R_n\}_{n \geq 1}$ with $\|R_n\| = \frac{1}{n}$ such that $T_n := T - R_n \in \mathcal{M}_d^+(H)$, for all $n \in \mathbb{N}$ and $T_n := T - R_n \to T$ in norm as $n \to \infty$.*

Proof. We prove the Lemma in the following two cases separately.

Case(I) $T \in \mathcal{M}_d^+(H)$: In this case, the result follows trivially by taking $T_n = T$ and $R_n = 0$, for all $n \in \mathbb{N}$.

Case(II) $T \notin \mathcal{M}_d^+(H)$: We have $m_e(T) = m(T) > 0$. By Proposition 1.3.12, for every $n \in \mathbb{N}$, there exists a $x_n \in S_H$ such that,

$$m(T) \leq \langle Tx_n, x_n \rangle < m(T) + \frac{1}{2n}. \tag{3.1}$$

For a fixed $n \in \mathbb{N}$, denote by, $R_n x := \frac{1}{n}\langle x, x_n \rangle x_n$, for all $x \in H$. Clearly, R_n is a positive rank one operator for all $n \in \mathbb{N}$ and $\|R_n\| = \frac{1}{n}$. Without loss of generality, we can assume that $\frac{1}{n} < m(T)$, for all $n \in \mathbb{N}$. Let $T_n := T - R_n$, for all $n \in \mathbb{N}$. Then for every $x \in S_H$, we have,

$$\begin{aligned}
\langle T_n x, x \rangle &= \langle Tx, x \rangle - \frac{1}{n}|\langle x, x_n \rangle|^2 \\
&\geq \langle Tx, x \rangle - \frac{1}{n} \quad \text{(by Cauchy-Schwarz inequality)} \\
&\geq m(T) - \frac{1}{n}.
\end{aligned}$$

Consequently, $T_n \in \mathcal{B}^+(H)$, for all $n \in \mathbb{N}$. Again by Proposition 1.3.12, we have,

$$\begin{aligned}
m(T_n) &\leq \langle T_n x_n, x_n \rangle \\
&\leq \langle Tx_n, x_n \rangle - \frac{1}{n} \\
&< \left(m(T) + \frac{1}{2n} \right) - \frac{1}{n} \quad \text{(by Equation 3.1)} \\
&< m(T) - \frac{1}{2n} \\
&< m(T).
\end{aligned}$$

Next, by Corollary 3.1.7, we have $T_n \in \mathcal{M}_d^+(H)$, for all $n \in \mathbb{N}$. Clearly, $T_n \to T$ in norm as $n \to \infty$. □

Definition 3.1.12. *(Essential minimum modulus)[8, 10] Let $T \in \mathcal{B}(H)$. Then the quantity,*

$$m_e(T) = \inf\{\lambda \colon \lambda \in \sigma_{ess}(|T|)\},$$

is called the essential minimum modulus *of* T.

For a fixed $T \in \mathcal{B}(H)$, let us define, $A_T = \{K \in \mathcal{K}(H) \colon T + K \notin \mathcal{M}(H)\}$ and $S_T = \{K \in \mathcal{K}(H) \colon m(T+K) = m_e(T)\}$. Now we prove a relation among these sets.

Lemma 3.1.13. *Let $T \in \mathcal{B}(H)$. Then, $A_T \subseteq S_T$.*

Proof. First, note that $\sigma_{ess}(|T+K|) = \sigma_{ess}(|T|)$. Therefore,

$$m_e(T) = \inf\{\lambda \colon \lambda \in \sigma_{ess}(|T+K|)\}, \text{ for all } K \in \mathcal{K}(H).$$

That is, for a fixed T, $m_e(T)$ is constant under all compact perturbations of T.

Suppose $A_T = \emptyset$, the result is trivial. Assume $A_T \neq \emptyset$. Let $K \in A_T$. Since $T + K \notin \mathcal{M}(H)$ we know that $m(T+K) \notin \sigma_{disc}(|T+K|)$. Since, $m(|T+K|) = m(T+K)$ and $|T+K| \geq 0$, we have $m(T+K) \in \sigma(|T+K|)$. It follows that, $m(T+K) \in \sigma_{ess}(|T+K|) = \sigma_{ess}(|T|)$. By Remark 1.3.22, we can conclude that $m_e(T) = m(T+K)$ and $A_T \subseteq S_T$. □

Remark 3.1.14. *Note that for $T \in \mathcal{B}(H)$ such that $m(T) \in \sigma_{ess}(|T|)$, we have, $A_T \subseteq S_T$.*

Recall that a subset of a topological space is *nowhere dense* if its closure has empty interior. Equivalently, a subset is nowhere dense if and only if the complement of its closure is dense (see, [12, page 132]).

We are now ready to prove a theorem, which is one of our main goals of this chapter. We use the previous lemma to characterize the size of A_T.

Theorem 3.1.15. *Let $T \in \mathcal{B}(H)$ and $m_e(T) > 0$. Then A_T is nowhere dense in $\mathcal{K}(H)$.*

Proof. By Lemma 3.1.13, we have $A_T \subseteq S_T$. To conclude that A_T is nowhere dense, it is sufficient to show that S_T is nowhere dense or equivalently it suffices to prove that $\overline{S_T}^c = \mathcal{K}(H) \setminus \overline{S_T}$ is dense in $\mathcal{K}(H)$. Using Lemma 1.3.11, it is easy to observe that $\overline{S_T} = S_T$.

Suppose $S_T = \emptyset$, the result is trivial. Assume $S_T \neq \emptyset$. Let $K \in S_T$. Then $m(T+K) = m_e(T)$. Let $T + K = V|T+K|$ be the Polar decomposition of $T+K$. Note that V is an isometry because $m(T+K) > 0$. Since, $|T+K| \in \mathcal{B}^+(H)$ and $m(|T+K|) = m(T+K) > 0$, by Lemma 3.1.11, there exists a sequence of positive rank one operators $\{R_n\}_{n\geq 1}$ such that $S_n := |T+K| - R_n \in \mathcal{M}_d^+(H)$, for all $n \in \mathbb{N}$ and $S_n \to |T+K|$ in norm as $n \to \infty$. Denote by $T_n := VS_n$ and $K_n := K - VR_n$, for all $n \in \mathbb{N}$. Then, we have $T_n = T + K_n$ and $|T_n| = (S_nV^*VS_n)^{\frac{1}{2}} = S_n$, for all $n \in \mathbb{N}$.

Consequently, $T_n \in \mathcal{M}_d(H)$ and $m(T+K_n) \notin \sigma_{ess}(|T+K_n|)$, for all $n \in \mathbb{N}$. By Lemma 3.1.1, we have $\sigma_{ess}(|T+K_n|) = \sigma_{ess}(|T|)$. Therefore, $m(T+K_n) \notin \sigma_{ess}(|T|)$ and $m(T+K_n) < m_e(T)$, for all $n \in \mathbb{N}$. Consequently, $K_n \notin S_T$, for all $n \in \mathbb{N}$. Next, we have $K_n \to K$, since $R_n \to 0$ as $n \to \infty$. Therefore, we can conclude that $S_T^c = \mathcal{K}(H) \setminus S_T$ is dense and S_T is nowhere dense in $\mathcal{K}(H)$. □

Remark 3.1.16. *In case $m_e(T) = 0$, we have $S_T = \mathcal{K}(H)$ and it cannot be nowhere dense. But A_T may be nowhere dense or may not be. Below we illustrate this.*

Firstly, by the Spectral theorem it is easy to observe that every compact operator on a non separable Hilbert space has non trivial kernel and hence not injective.

Example 3.1.17. *Let H be a non separable complex Hilbert space and $T \equiv 0$. Then, $A_0 = \{K \in \mathcal{K}(H) \colon K \text{ is injective}\} = \emptyset$ and hence nowhere dense in $\mathcal{K}(H)$.*

We need to prove the following basic tools to provide an example for $T \in \mathcal{B}(H)$ such that $m_e(T) = 0$ and A_T is not nowhere dense.

Lemma 3.1.18. *Let H be separable and $F \in \mathcal{K}^+(H)$ be a finite rank operator. Then there exists a sequence $\{K_n\} \subseteq \mathcal{K}(H)$ such that K_n is injective for all $n \in \mathbb{N}$ and $K_n \to F$ in norm as $n \to \infty$.*

Proof. Note that $N(F)$ is an infinite dimensional Hilbert space. Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal set such that $N(F) = [e_j : j \in \mathbb{N}]$. Let $\{f_j : 1 \leq j \leq n\}$ be another orthormal set such that $R(F) = [f_j \colon 1 \leq j \leq n]$. Since, F is positive, by Projection theorem we have, $H = N(F) \oplus R(F)$. Define a linear map $D \colon H \to H$ such that $De_i := \frac{e_i}{i}$, for all $i \in \mathbb{N}$ and $Df_j = 0$, for all $1 \leq j \leq n$. Now, it is easy to verify that $K_n := F + \frac{D}{n}$ is an injective compact operator for all $n \in \mathbb{N}$ and $K_n \to F$ as $n \to \infty$ in norm. □

The above lemma leads to the following result on the denseness of injective compact operators.

Theorem 3.1.19. *Let H be separable. Then the set of all injective compact operators is dense in $\mathcal{K}(H)$.*

Proof. Since the set of finite rank operators is dense in $\mathcal{K}(H)$, it is enough to prove that given any finite rank operator F, there exists a sequence $\{C_n\} \subseteq \mathcal{K}(H)$ such that C_n is injective for all $n \in \mathbb{N}$ and $C_n \to F$ in norm as $n \to \infty$.

Both the subspaces $N(F)$ and $N(F^*)$ are infinite dimensional and separable. Therefore, dim $N(F)$= dim $N(F^*)$. Consequently, there exists an isometry V such that $F = V|F|$ (For details, see [28, Problem 135]). Since $|F|$ is a positive finite rank operator, by Lemma 3.1.18, there exists a sequence $\{K_n\} \subseteq \mathcal{K}(H)$ such that K_n is injective for all $n \in \mathbb{N}$ and $K_n \to |F|$ in norm as $n \to \infty$. Now, consider $C_n := VK_n$, for all $n \in \mathbb{N}$. Since V is an isometry, C_n is injective for all $n \in \mathbb{N}$ and $C_n \to F$ in norm as $n \to \infty$. $\square$

Now, we are in a position to construct many examples of T such that the set A_T is not nowhere dense.

Example 3.1.20. *Let H be separable and $T \equiv 0$. Then, from Theorem 3.1.19, $A_0 = \{K \in \mathcal{K}(H)/K \text{is injective}\}$ is a dense set and so it cannot be a nowhere dense set. In fact, for every $C \in \mathcal{K}(H)$, we have $A_C = \{K \in \mathcal{K}(H)/C + K \text{is injective}\}$ is a dense set and so it cannot be a nowhere dense set. This is because $-C + A_0 \subseteq A_C$. Note that $m_e(C) = 0$ for every $C \in \mathcal{K}(H)$.*

From many equivalent definitions of *porosity* that can be found in the literature (See, [56, 38, 29]), we choose the following one which is used by J. Kover in [38].

Let X be a Banach space. An open ball with center x and radius r will be denoted by $B(x, r)$. That is $B(x, r) = \{y \in X : \|y - x\| < r\}$.

Definition 3.1.21. *(Porous set)[38, Definition 11] A set E in a Banach space X is called* porous *if there is a number $0 < \lambda < 1$ with the following property: For every $x \in E$ and for every $r > 0$ there is a $y \in B(x, r)$ such that $B(y, \lambda\|x - y\|) \cap E = \emptyset$.*

It is easy to observe that every porous set is nowhere dense. In [56], Zajíček, showed that a porous set is smaller than a nowhere dense set. He proved that even in $\mathbb{R}^n$ there exists a closed nowhere dense set which is not porous.

Next we prove that the set A_T is porous in $\mathcal{K}(H)$.

Theorem 3.1.22. *Let $T \in \mathcal{B}(H)$ and $m_e(T) > 0$. Then, A_T is porous in $\mathcal{K}(H)$.*

Proof. Let $X = \mathcal{K}(H)$ and $E = A_T$ be as in the definition of porous set. We prove that $\lambda = \frac{1}{2}$ is one such scalar that satisfies all the requirements for A_T to be porous. Let $K \in A_T$. From Lemma 3.1.1, we have $m(T + K) = m_e(T)$. Let $T + K = V|T + K|$ be the Polar decomposition of $T + K$. Note that V is an isometry because $m(T + K) > 0$.

Let $r > 0$ be arbitrary. We can choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \min\{m_e(T), \frac{r}{2}\}$. Since $|T + K| \in \mathcal{B}^+(H)$ and $m(|T + K|) > 0$, by proceeding similarly like in Lemma 3.1.11, we can find a positive rank one operator R_n with $\|R_n\| = \frac{1}{n}$ such that,

$$m(|T + K| - R_n) \leq m_e(T) - \frac{1}{2n}. \tag{3.2}$$

Let $K_n := K - VR_n$. Then, $\|K - K_n\| = \frac{1}{n}$. Hence, $K_n \in B(K, r)$. Next, it remains to prove that $B(K_n, \frac{1}{2n}) \cap A_T = \emptyset$. Let $C \in B(K_n, \frac{1}{2n})$. Then, by Lemma 1.3.11, we have, $|m(T+C) - m(T+K_n)| \leq \|T + C - T - K_n\| < \frac{1}{2n}$. Next, it follows that,

$$\begin{aligned}
m(T+C) &< m(T+K_n) + \frac{1}{2n} \\
&< m(V|T+K| - VR_n)) + \frac{1}{2n} \\
&< m(|T+K| - R_n) + \frac{1}{2n} \quad (\because V \text{ is an isometry}) \\
&< m_e(T) - \frac{1}{2n} + \frac{1}{2n} \quad (\text{From Equation } 3.2) \\
&< m_e(T).
\end{aligned}$$

Therefore, $C \notin S_T$. It follows that $C \notin A_T$ because $A_T \subseteq S_T$. Consequently, $B(K_n, \frac{1}{2n}) \cap A_T = \emptyset$. □

3.2 Stability

After proving the compact perturbation results in the previous section, it is natural to ask to what extent those results can be generalized. In this section answering this question will be our main concern. For this purpose, we build upon the ideas of compact perturbations used in the last section.

Firstly, we will try to extend the stability results for minimum attaining operators under compact perturbations to a more general setting, by making use of the connection between Fredholm operators and the essential spectrum.

Theorem 3.2.1. *Let $m(T) \in \sigma_{disc}(|T|)$. Then there exists an $\epsilon > 0$ such that for all $S \in \mathcal{B}(H)$ if $\|S - T\| < \epsilon$ then $m(S) \in \sigma_{disc}(|S|)$. In particular, if $A \in \mathcal{B}(H)$ with $\|A\| < \epsilon$ then $m(A+T) \in \sigma_{disc}(|A+T|)$.*

Proof. Suppose this is not true. Then there exists $\{T_n\} \subseteq \mathcal{B}(H)$ such that $T_n \to T$ and $m(T_n) \notin \sigma_{disc}(|T_n|)$, for all $n \in \mathbb{N}$. Now $m(T) \in \sigma_{disc}(|T|)$, so $m(T)I - |T|$ must be a Fredholm operator of index 0, by the definition of the Weyl spectrum. Let $\widetilde{T}$ be the bijection associated with the Fredholm operator $m(T)I - |T|$. Let $\epsilon = \|\widetilde{T}^{-1}\|^{-1}$. Since $T_n \to T$ we know that $|T_n| \to |T|$ [47, Problem 15(a), page 217]. Fix n_0 large enough so that $\||T_n| - |T|\| < \frac{\epsilon}{2}$ and $\|T_n - T\| < \frac{\epsilon}{2}$, for all $n \geq n_0$. By using Lemma

1.3.11,

$$\begin{aligned}\|(m(T_n)I - |T_n|) - (m(T)I - |T|)\| &\leq |m(T_n) - m(T)| + \||T_n| - |T|\| \\ &\leq \|T_n - T\| + \||T_n| - |T|\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon.\end{aligned}$$

Now, by Theorem 1.2.9, it follows that $m(T_n)I - |T_n|$ is a Fredholm operator of index 0, for all $n \geq n_0$. Since, Weyl's spectrum and essential spectrum are same for the case of self-adjoint operators, we have $m(T_n) \notin \sigma_{ess}(|T_n|)$, for all $n \geq n_0$. Consequently, $m(T_n) \in \sigma_{disc}(|T_n|)$, for all $n \geq n_0$. This contradicts our assumption and hence the theorem. The particular case holds true if we consider $S := A + T$ in the main theorem. □

Remark 3.2.2. *Note that for $T \in \mathcal{B}(H)$ and ϵ as above in the Theorem 3.2.1, we have $B(T, \epsilon) \cap \mathcal{K}(H) = \emptyset$.*

The following result extends the Corollary 3.1.4 from small compact perturbations to perturbations by all bounded linear operators of small norm.

Corollary 3.2.3. *Let $m(T) \in \sigma_{disc}(|T|)$. Then there exists an $\epsilon > 0$ such that for all $S \in \mathcal{B}(H)$ if $\|S - T\| < \epsilon$ then $S \in \mathcal{M}(H)$. In particular, if $\|A\| < \epsilon$ then $T + A \in \mathcal{M}(H)$.*

Proof. The proof follows directly from Theorem 3.2.1, once we observe that for any $S \in \mathcal{B}(H)$ we have $\sigma_{disc}(|S|) \subseteq \sigma_{pt}(|S|)$ and $|S| \in \mathcal{M}(H)$ implies $S \in \mathcal{M}(H)$. □

Next, we measure the size of the set $\mathcal{M}_d^+(H)$ in $\mathcal{B}^+(H)$.

Theorem 3.2.4. *Let $T \in \mathcal{B}^+(H)$. Then there exists a sequence $\{T_n\}_{n\geq 1}$ such that $T_n \in \mathcal{M}_d^+(H)$, for all $n \in \mathbb{N}$ and $T_n \to T$ in norm as $n \to \infty$. In particular, $\overline{\mathcal{M}_d^+(H)} = \mathcal{B}^+(H)$.*

Proof. We prove the result in two cases separately.

Case(I) $m(T) > 0$: In this case, the result follows directly from Lemma 3.1.11.

Case(II) $m(T) = 0$: For each $n \in \mathbb{N}$, let us consider $S_n = T + \frac{1}{n}I$. Clearly, $m(S_n) > 0$, for all $n \in \mathbb{N}$. Then by Lemma 3.1.11, there exists a $T_n \in \mathcal{M}_d^+(H)$ such that $\|S_n - T_n\| < \frac{1}{n}$. Next, for every $n \in \mathbb{N}$, we have,

$$\|T_n - T\| \leq \|T_n - S_n\| + \|S_n - T\| \leq \frac{2}{n}.$$

Therefore, $T_n \to T$ in norm as $n \to \infty$.

Combining both the cases, we can conclude that $\overline{\mathcal{M}_d^+(H)} = \mathcal{B}^+(H)$. □

The following result is an easy consequence of the above theorem.

Corollary 3.2.5. *Let $T \in \mathcal{B}^+(H)$. Then, the set of positive minimum attaining operators is dense in $\mathcal{B}^+(H)$. That is,* $\overline{\mathcal{M}^+(H)} = \mathcal{B}^+(H)$.

Proof. Since $\mathcal{M}_d^+(H) \subseteq \mathcal{M}^+(H)$, the proof follows directly from Theorem 3.2.4. □

To prove similar results for the class of self-adjoint operators, we need the following lemma.

Lemma 3.2.6. *Let (M_1, M_2) be a completely reducing pair for $T \in \mathcal{B}(H)$. Let $T_1 = T|_{M_1}$ and $T_2 = T|_{M_2}$ and $m(T_1) < m(T_2)$. Then, $T \in \mathcal{M}_d(H)$ iff $T_1 \in \mathcal{M}_d(M_1)$.*

Proof. We have, $T = T_1 \oplus T_2$. Then, $T^* = T_1^* \oplus T_2^*$. It follows that $|T| = |T_1| \oplus |T_2|$. From [54, Theorem 5.4, page 289], we have,

$$\sigma(|T|) = \sigma(|T_1|) \cup \sigma(|T_2|). \tag{3.3}$$

Using Remark 1.3.22, we can conclude that,

$$m(T) = \min\{m(T_1), m(T_2)\}. \tag{3.4}$$

Therefore, $m(T) = m(T_1)$. Let $T \in \mathcal{M}_d(H)$. Then, $m(T) \in \sigma_{disc}(|T|)$. That means, $m(T)$ is an eigenvalue for $|T|$ with finite multiplicity, which is also an isolated point of $\sigma(|T|)$. By Remark 1.3.22, $m(T) \notin \sigma(|T_2|)$. From [54, Theorem 5.4, page 289], we have,

$$\sigma_p(|T|) = \sigma_p(|T_1|) \cup \sigma_p(|T_2|). \tag{3.5}$$

Therefore, we can conclude that $m(T_1) = m(T) \in \sigma_p(|T_1|)$. Clearly, $m(T_1)$ is an isolated point of $\sigma(|T_1|)$, since it is isolated in a bigger set $\sigma(|T|)$. Next, the multiplicity of $m(T_1)$ is finite because M_2 does not contribute anything to the multiplicity of $m(T_1)$ as $m(T_1) \notin \sigma_p(|T_2|)$. So, we can conclude that $m(T_1) \in \sigma_{disc}(|T_1|)$. Consequently, $T_1 \in \mathcal{M}_d(M_1)$.

Conversely, let $T_1 \in \mathcal{M}_d(M_1)$. Then, $m(T_1)$ is an eigenvalue for $|T_1|$ with finite multiplicity, which is also an isolated point of $\sigma(|T_1|)$. Now, Equation 3.4 implies that $m(T) = m(T_1)$. From Equation 3.5, we have $m(T) \in \sigma_p(|T|)$. By Remark 1.3.22, $m(T) \notin \sigma(|T_2|)$. Now, $\sigma(|T_2|)$ is a closed set implies that $m(T)$ is not a limit point of $\sigma(|T_2|)$. Already, it is not a limit point of $\sigma(|T_1|)$. Consequently, it is an isolated point of $\sigma(|T|)$. Its multiplicity is finite, because $m(T) \notin \sigma_p(|T_2|)$. Therefore, $m(T) \in \sigma_{disc}(|T|)$ and $T \in \mathcal{M}_d(H)$. □

Remark 3.2.7. *Suppose $m(T_2) < m(T_1)$ in Lemma 3.2.6, then it still holds true with the roles of T_1 and T_2 interchanged.*

Remark 3.2.8. *Suppose $m(T_1) = m(T_2)$ in Lemma 3.2.6, then it need not hold true. For instance, we have an example below.*

Example 3.2.9. *Let $M_1 = [e_n : 1 \leq n \leq 5, n \in \mathbb{N}]$ and $M_2 = [e_n : n > 5, n \in \mathbb{N}]$. Let $T \in \mathcal{B}(\ell^2)$ be defined by,*

$$T(x_1, x_2, x_3, ...) = (0, x_2, 2x_3, 3x_4, 4x_5, \frac{x_6}{6}, \frac{x_7}{7}, \ldots), \text{ for all } (x_1, x_2, x_3, ...) \in \ell^2.$$

Clearly, (M_1, M_2) is a completely reducing pair for T. Let $T_1 = T|_{M_1}$ and $T_2 = T|_{M_2}$. We have $T_1 \in \mathcal{M}_d(M_1)$ but $T \notin \mathcal{M}_d(H)$. Notice that $m(T_1) = m(T_2) = 0$.

The following theorem proves that $\mathcal{M}_d^s(H)$ is a very large set in $\mathcal{B}^s(H)$, in fact it is dense.

Theorem 3.2.10. *Let $T \in \mathcal{B}^s(H)$. Then there exists a sequence $\{T_n\}_{n\geq 1}$ such that $T_n \in \mathcal{M}_d^s(H)$, for all $n \in \mathbb{N}$ and $T_n \to T$ in norm as $n \to \infty$. In particular, $\overline{\mathcal{M}_d^s(H)} = \mathcal{B}^s(H)$.*

Proof. We prove the result in two cases separately.

Case(I): $m(T) > 0$: Let $T = V|T|$ be the polar decomposition of T. Since $T^* = T$, we have $V^* = V$. Next, H has the decomposition

$$H = H_+ \oplus H_-,$$

where $H_+ = N(I - V)$ and $H_- = N(I + V)$. Also, (H_+, H_-) is an completely reducing pair for T and $|T|$. Let $T_1 = T|_{H_+} = |T|\big|_{H_+}$ and $T_2 = T|_{H_-} = -|T|\big|_{H_-}$. Then we have,

$$T = T_1 \oplus T_2. \tag{3.6}$$

Moreover, T_1 is strictly positive and T_2 is strictly negative (for details, see [52, Example 7.1, page 139]).

First we consider, the case $m(T_1) \leq m(T_2)$. Then, the Equation 3.4, gives that $m(T) = m(T_1)$. So, $m(T_1) > 0$. Since $T_1 \in \mathcal{B}^+(H)$, from Lemma 3.1.11, there exists a sequence $\{S_n\}_{n\geq 1}$ such that $S_n \in \mathcal{M}_d^+(H)$, for all $n \in \mathbb{N}$ and $S_n \to T_1$ in norm as $n \to \infty$. Moreover, $m(S_n) < m(T_1)$, for all $n \in \mathbb{N}$. Next, consider the sequence of operators $\{T_n\}_{n\geq 1}$ where $T_n := S_n \oplus T_2$, for all $n \in \mathbb{N}$. Being the direct sum of two self-adjoint operators, T_n is self-adjoint for all $n \in \mathbb{N}$. By applying Lemma 3.2.6, we can conclude that $T_n \in \mathcal{M}_d^+(H)$, for all $n \in \mathbb{N}$. Clearly, $T_n \to T$ in norm as $n \to \infty$.

Next consider, the case $m(T_2) < m(T_1)$. Then, $m(T) = m(T_2) > 0$. Now, T_2 is strictly negative implies, $-T_2$ is strictly positive. Also, we have $m(-T_2) = m(T_2) > 0$. From Lemma 3.1.11, there exists a sequence $\{S_n\}_{n\geq 1}$ such that $S_n \in \mathcal{M}_d^+(H)$, for all $n \in \mathbb{N}$ and $S_n \to -T_2$ in norm as $n \to \infty$. Moreover, $m(S_n) < m(-T_2)$, for all $n \in \mathbb{N}$. Next, consider the sequence of operators $\{T_n\}_{n\geq 1}$ where $T_n := T_1 \oplus -S_n$, for all $n \in \mathbb{N}$. Rest of the proof is same as above.

Case(II) $m(T) = 0$: For each $n \in \mathbb{N}$, let us consider $S_n = T + \frac{1}{n}P_{N(T)}$. Then, $m(S_n) > 0$, for all $n \in \mathbb{N}$. By the Case (I) above, there exists a $T_n \in \mathcal{M}_d^s(H)$ such that $\|S_n - T_n\| < \frac{1}{n}$. Next, for every $n \in \mathbb{N}$, we have,

$$\|T_n - T\| \leq \|T_n - S_n\| + \|S_n - T\| \leq \frac{2}{n}.$$

Therefore, $T_n \to T$ in norm as $n \to \infty$.

Combining both the cases, we can conclude that $\overline{\mathcal{M}_d^s(H)} = \mathcal{B}^s(H)$. □

Corollary 3.2.11. *Let $T \in \mathcal{B}^s(H)$. Then, the set of self-adjoint minimum attaining operators is dense in $\mathcal{B}^s(H)$. That is, $\overline{\mathcal{M}^s(H)} = \mathcal{B}^s(H)$.*

Proof. Since $\mathcal{M}_d^s(H) \subseteq \mathcal{M}^s(H)$, the proof follows directly from Theorem 3.2.10. □

It follows from [41, Theorem 3.5] that $\mathcal{M}(H)$ is dense in $\mathcal{B}(H)$. Along the similar lines, one expects $\overline{\mathcal{M}_d(H)} = \mathcal{B}(H)$. But it is not the case, we will observe this in the following results.

First we prove that specific operators in $\mathcal{B}(H)$ can be approximated by the operators in $\mathcal{M}_d(H)$.

Theorem 3.2.12. *Let $T \in \mathcal{B}(H)$ and $m_e(T) > 0$. Then there exists a sequence $\{T_n\}_{n\geq 1}$ such that $T_n \in \mathcal{M}_d(H)$, for all $n \in \mathbb{N}$ and $T_n \to T$ in norm as $n \to \infty$.*

Proof. We prove the theorem in the following two cases separately.

Case(I) $m(T) = 0$: Since $m_e(T) > 0$, we have $m(T) = m(|T|) = 0 \in \sigma_{disc}(|T|)$ and $T \in \mathcal{M}_d(H)$. Now, the result follows trivially by taking $T_n = T$, for all $n \in \mathbb{N}$.

Case (II) $m(T) > 0$: Let $T = V|T|$ be the Polar decomposition of T. Since $m(T) > 0$, we have T is injective and V is an isometry. We have $|T| \in \mathcal{B}^+(H)$ and $m(|T|) = m(T) > 0$. By Lemma 3.1.11, there exists a sequence $\{S_n\}_{n\geq 1}$ such that $S_n \in \mathcal{M}_d^+(H)$, for all $n \in \mathbb{N}$ and $S_n \to |T|$ in norm as $n \to \infty$. Let us denote by $T_n := VS_n$, for all $n \in \mathbb{N}$. Next, $|T_n| = (S_nV^*VS_n)^{\frac{1}{2}} = S_n$ implies that $T_n \in \mathcal{M}_d(H)$, for all $n \in \mathbb{N}$. Clearly, $T_n \to T$ in norm as $n \to \infty$. □

The lemma below is an important tool in proving the next theorem.

Lemma 3.2.13. *Let $T \in \mathcal{B}(H)$. Then there exists a sequence of closed range operators $\{T_n\}_{n\geq 1} \subseteq \mathcal{B}(H)$ such that $N(T_n) = N(T)$, for all $n \in \mathbb{N}$ and $T_n \to T$ in norm as $n \to \infty$. In particular, the set of all closed range operators are dense in $\mathcal{B}(H)$.*

Proof. Let $T = V|T|$ be the Polar decomposition of T. Next, let us denote by $T_n := V(|T| + \frac{1}{n}P_{\overline{R(|T|)}})$, for all $n \in \mathbb{N}$. Then, for every $x \in H$ we have,

$$\begin{aligned}
\|T_n x\|^2 &= \left\| V\left(|T| + \frac{1}{n}P_{\overline{R(T)}}\right)x \right\|^2 \\
&= \left\langle V\left(|T|x + \frac{1}{n}P_{\overline{R(|T|)}}\right)x, V\left(|T|x + \frac{1}{n}P_{\overline{R(|T|)}}x\right)\right\rangle \\
&= \left\langle \left(|T|x + \frac{1}{n}P_{\overline{R(|T|)}}\right)x, \left(|T|x + \frac{1}{n}P_{\overline{R(|T|)}}x\right)\right\rangle \ (\because V^*V = I) \\
&= \|Tx\|^2 + \frac{2}{n}\langle |T|x, x\rangle + \frac{1}{n^2}\left\| P_{\overline{R(|T|)}}x \right\|^2.
\end{aligned}$$

From this equation it follows that $N(T_n) = N(T)$, for all $n \in \mathbb{N}$. Next, for every $x \in N(T_n)^\perp = N(T)^\perp = N(|T|)^\perp = \overline{R(|T|)}$, we have,

$$\|T_n x\| \geq \frac{1}{n}\|x\|. \tag{3.7}$$

Therefore, the reduced minimum modulus, $\gamma(T_n) = \inf\{\|T_n x\| \colon x \in N(T_n)^\perp\} > 0$ and hence $R(T_n)$ is closed[15, page 363, Proposition 6.1], for all $n \in \mathbb{N}$. □

The next result is again about approximation of specific kind of operators on a separable Hilbert space by the operators in $\mathcal{M}_d(H)$.

Theorem 3.2.14. *Let H be a separable infinite dimensional complex Hilbert space and $T \in \mathcal{B}(H)$ be such that $m_e(T) = m_e(T^*) = 0$. Then there exists a sequence $\{T_n\}_{n\geq 1}$ such that $T_n \in \mathcal{M}_d(H)$, for all $n \in \mathbb{N}$ and $T_n \to T$ in norm as $n \to \infty$.*

Proof. We prove the theorem in the following three cases separately.

Case(I) dim $N(T) = 0$: Since, $m_e(T) = 0$, we have $m(T) = 0$ and T is injective. Let $T = V|T|$ be the Polar decomposition of T. Now, T is injective implies V is an isometry. We have $|T| \in \mathcal{B}^+(H)$ and $m(|T|) = m(T) = 0$. By Case(II) of Theorem 3.2.4, there exists a sequence $\{S_n\}_{n\geq 1}$ such that $S_n \in \mathcal{M}_d^+(H)$, for all $n \in \mathbb{N}$ and $S_n \to |T|$ in norm as $n \to \infty$. Let us denote by $T_n := VS_n$, for all $n \in \mathbb{N}$. Next, $|T_n| = (S_nV^*VS_n)^{\frac{1}{2}} = S_n$ implies that $T_n \in \mathcal{M}_d(H)$, for all $n \in \mathbb{N}$. Clearly, $T_n \to T$ in norm as $n \to \infty$.

Case(II) $0 < \dim N(T) < \infty$: We have $N(T) \neq 0$. Since $m_e(T) = 0$, $R(|T|)$ is not closed. From Lemma 3.2.13, there exists a sequence of closed range operators $\{T_n\}_{n\geq 1} \subseteq \mathcal{B}(H)$ such that $N(T_n) = N(T) \neq 0$ and $T_n \to T$ in norm as $n \to \infty$. Now, $R(T_n)$ is closed implies that $R(|T_n|)$ is closed and dim $N(|T_n|)$= dim $N(T_n) < \infty$. Therefore, $|T_n|$ is a Fredholm operator of index '0'. By Remark 1.2.19, $0 \notin \sigma_{ess}(|T_n|)$. But $0 \in \sigma_p(|T_n|) \subseteq \sigma(|T_n|)$. So, $0 \in \sigma_{disc}(|T_n|)$. Since $m(|T_n|) = 0$, we have $T_n \in \mathcal{M}_d(H)$, for all $n \in \mathbb{N}$.

Case(III) dim $N(T) = \infty$: Suppose, dim $N(T^*) = \infty$. Then, Ind $T = \dim N(T) - \dim N(T^*) = 0$. From Theorem 3.1.2(i), we have $\rho_1(T) = \inf\{\|T - S\| : S \in \mathcal{B}(H) \text{ and } \dim N(S) = 1\} = 0$. Then for every $n \in \mathbb{N}$, we can find a S_n with dim $N(S_n) = 1$ such that $\|T - S_n\| \leq \frac{1}{n}$ and using Lemma 3.2.13, there exists a T_n such that $\|S_n - T_n\| \leq \frac{1}{n}$. We have, dim $N(|T_n|)$ = dim $N(S_n) = 1$ and $R(|T_n|)$ is closed. Therefore, $|T_n|$ is a Fredholm operator of index '0'. By Remark 1.2.19, $0 \notin \sigma_{ess}(|T_n|)$. But $0 \in \sigma_p(|T_n|) \subseteq \sigma(|T_n|)$. So, $0 \in \sigma_{disc}(|T_n|)$. Since $m(|T_n|) = 0$, we have $T_n \in \mathcal{M}_d(H)$, for all $n \in \mathbb{N}$. Obviously, $T_n \to T$ in norm as $n \to \infty$.

In the case, if dim $N(T^*) < \infty$. Then, ind T = dim $N(T)$ − dim $N(T^*) = \infty$. From Theorem 3.1.2(ii), we have $\rho_1(T) = \inf\{\|T - S\| : S \in \mathcal{B}(H) \text{ and } \dim N(S) = 1\} = 0$. Rest of the proof is same as above, when dim $N(T^*) = \infty$. □

Now, we observe that the set $\mathcal{M}_d(H)$ is not dense in $\mathcal{B}(H)$, for the case of separable infinite dimensional complex Hilbert spaces H.

Remark 3.2.15. *Let H be a separable infinite dimensional complex Hilbert space and $T \in \mathcal{B}(H)$ be such that $m_e(T) = 0$ and $m_e(T^*) > 0$. Then, $R(T)$ is closed and dim $N(T) = \infty$. Denote by, $r := m(T^*)$. From Theorem 3.1.2(ii), we have dim $N(S) = \infty$, for all $S \in B(T, r)$ and $\mathcal{M}_d(H) \cap B(T, r) = \emptyset$. Therefore, Theorem 3.2.12 is not valid in this case.*

For a fixed $T \in \mathcal{B}(H)$, define

$$B_T^d = \{S \in \mathcal{M}_d(H) \colon S + T \in \mathcal{M}(H)\}.$$

Next theorem measures the size of the set B_T^d in $\mathcal{M}_d(H)$.

Theorem 3.2.16. *Let $T \in \mathcal{B}(H)$. Then B_T^d is dense in $\mathcal{M}_d(H)$.*

Proof. Let $S \in \mathcal{M}_d(H)$ and $S \notin B_T^d$. From Theorem 3.5 of [41], there exists a sequence of bounded operators $\{R_n\}_{n\geq 1}$ such that $T_n := S + T + R_n \in \mathcal{M}(H)$, for all

$n \in \mathbb{N}$ and $T_n := S+T+R_n \to S+T$ in norm as $n \to \infty$. Since $S \in \mathcal{M}_d(H)$, by Theorem 3.2.1, it follows that $S + R_n \in \mathcal{M}_d(H)$, for all large n. Therefore, $S + R_n \in B_T^d$ and B_T^d is dense in $\mathcal{M}_d(H)$. □

We know that the set of minimum attaining operators is dense in $\mathcal{B}(H)$. On the other hand we observe that the set of non minimum attaining operators is very small in $\mathcal{B}(H)$.

Theorem 3.2.17. *Let H be a separable Hilbert space. Then the set of all non minimum attaining operators is nowhere dense in $\mathcal{B}(H)$.*

Proof. Let $E = \{T \in \mathcal{B}(H) \colon T \notin \mathcal{M}(H)\}$ and let $S := \{T \in \mathcal{B}(H) \colon m(T) \in \sigma_{ess}(|T|)\}$. Let $\{A_n\}$ be a Cauchy sequence in S. Then by the completeness of $\mathcal{B}(H)$, there exists a $A \in \mathcal{B}(H)$ such that $A_n \to A$ in norm. Suppose $m(A) \notin \sigma_{ess}(|A|)$. Then $m(A) \in \sigma_{disc}(|A|)$. Now, by Theorem 3.2.1, $m(A_n) \in \sigma_{disc}(|A|)$ for large n. This contradicts $\{A_n\} \subseteq S$. Therefore, $A \in S$ and S is a closed set.

To prove E is nowhere dense, it is enough to prove that $\overline{E}^{\complement}$ is dense in $\mathcal{B}(H)$. If $m(T) = m(|T|) \notin \sigma_{ess}(|T|)\}$. Then, $m(T) \in \sigma_{disc}(|T|)\}$ and Proposition 2.1.3 implies that $T \in \mathcal{M}(H)$. Consequently, $E \subseteq S$ and so $\overline{E} \subseteq S$. In the view of Remark 3.2.15, we also have $\overline{E} \subseteq \mathcal{B}(H) \setminus \{T \in \mathcal{B}(H) \colon m(T) = 0 \,\text{and}\, m(T^*) > 0\}$. It follows that, $\mathcal{M}_d(H) \cup \{T \in \mathcal{B}(H) \colon m(T) = 0 \,\text{and}\, m(T^*) > 0\} \subseteq \overline{E}^{\complement}$. Next, by applying Theorem 3.2.12 and Theorem 3.2.14, we can conclude that $\overline{E}^{\complement}$ is dense in $\mathcal{B}(H)$. Hence the result. □

For a fixed $T \in \mathcal{B}(H)$, define

$$C_T^d = \{S \in \mathcal{M}_d(H) \colon S + T \notin \mathcal{M}(H)\}.$$

Next theorem measures the size of the set C_T^d in $\mathcal{M}_d(H)$.

Theorem 3.2.18. *Let $T \in \mathcal{B}(H)$. Then C_T^d is nowhere dense in $\mathcal{M}_d(H)$.*

Proof. Let us define the set F by $F := \{S \in \mathcal{B}(H) \colon S + T \notin \mathcal{M}(H)\}$ and the set R by $R := \{S \in \mathcal{B}(H) \colon m(S+T) \in \sigma_{ess}(|S+T|)\}$. Note that we can prove R is closed by the similar arguments as given in Theorem 3.2.17.

Next, we observe that $F \subseteq R$. Let $m(S+T) = m(|S+T|) \notin \sigma_{ess}(|S+T|)\}$. Then, $m(S+T) \in \sigma_{disc}(|S+T|)\}$ and Proposition 2.1.3 implies that $S+T \in \mathcal{M}(H)$. Consequently, $F \subseteq R$ and so $\overline{F} \subseteq R$. From Theorem 3.2.16, we have $R^{\complement} = B_T^d$ is dense in $\mathcal{M}_d(H)$. It follows that R is nowhere dense in $\mathcal{M}_d(H)$. Since $F \subseteq R$, we can conclude that F is also nowhere dense in $\mathcal{M}_d(H)$. □

Questions and future work :

The perturbations results discussed in the previous sections gives rise to the following questions that we want to focus in our future work.

Question 3.2.19. *Are Theorem 3.2.14 and Remark 3.2.15 still valid, if H is a non separable complex Hilbert space ?*

Question 3.2.20. *Is Theorem 3.2.17 still valid, if H is a non separable complex Hilbert space ?*

Question 3.2.21. *Let $T \in \mathcal{B}(H)$. Then, is the set C_T^d porous in $\mathcal{M}_d(H)$?*

After having studied the perturbation properties of minimum attaining operators, we are also interested to investigate for the similar results concerning the absolutely minimum attaining operators.

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PUBLICATIONS BASED ON THIS WORK

Journal Publications

1. J. Ganesh, G. Ramesh and D. Sukumar, A characterization of absolutely minimum attaining operators, J. Math. Anal. Appl. **468** (2018), no. 1, 567–583. MR3849004

2. J. Ganesh, G. Ramesh and D. Sukumar, Perturbation of minimum attaining operators, Adv. Oper. Theory **3** (2018), no. 3, 473–490. MR3795095

3. J. Ganesh, G. Ramesh and D. Sukumar, On the structure of absolutely minimum attaining operators, J. Math. Anal. Appl. **428** (2015), no. 1, 457–470. MR3326997

Presentations in Conferences/Workshops

1. Title : A characterization of absolutely minimum attaining operators
   Venue : **International Workshop on Operator Theory and its Appications-2016 (IWOTA-2016)**, held at **Washington University, St. Louis, USA** during 18-22, July 2016.

2. Title: On the structure of absolutely minimum attaining operators
   Venue : **International Conference on Linear Algebra and its Applications-2014 (ICLAA-2014)**, held at **Manipal University, Manipal, India** during 18-20, Dec 2014.

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