भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Gauge Theories

Subham Koley

PH17MSCST11014

Advisor: Dr.Raghavendra Srikanth Hundi

Indian Institute Of Technology Hyderabad

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Certificate

This is to certify that subham koley(PH17MSCST11014) has satisfactorily completed the project entitled Gauge Theories as partial fulfillment for the award of Master Of Science(physics) degree at the Indian Institute Of Technology Hyderabad during the year 2017-2019

(Dr.Raghavendra Srikanth Hundi ,project supervisor)

Author's Declaration

I declare that the work in this thesis was carried out in accordance with the requirements of the Institute's Regulations and for the partial fulfillment of Master of Physics and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text,the work is the candidate's own work. Any views expressed in the thesis are those of the author.

Subham Koley

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Abstract

This term paper explores The phenomenon of self interaction among the vector gauge bossons.starting form the Abelian symmetries we modified the transformation laws for local invariance to non-abelian cases,where it includes non homogeneous part due to it's non commutative nature.Besides that it includes the chiral symmetry for fermions that says in chiral basis we cannot put mass term to the fermions.Finally we give some examples of Yang-Mills theories which have been applied in physics.

Introduction

In this report i begin with classical field theory introducing the equation of motion,symmetry transformations that leave the equation invariant would give conserved current. Apart from that it is shown that complex scalar field is nothing but two non interaction fields.Then starting the statistics followed by particles i have gone through all fundamental interaction and exchange particles associated with each type of interaction.It is alsho shown in the scalar QED and spinor QED that bossons interact with charge particles then i begin with non abelian gauge theory where i showed that in general case gauge field transform in a different way than U(1) symmetry as generators of gauge group does not commute with each other due to this non commutative nature we found some interesting phenomenon like Apart form that i showed that in chiral basis expansion of dirac lagrangian we can't put mass term to fermions due to non-invariance and found that the gauge field transformation of each basis is completely differnt

Contents

1 classical field theory

For a system of particles the equation of motion of classical mechanics is the Euler-Lagrange equation for extremizing the action integral. For a conserved system Lagrangian of the system of particles depends only on the generalized coordinates and generalized velocity $L(q_i, \dot{q}_i)$. In the Classical field theory we turned the dynamical variables into fields $\phi(x)$, here x is space time coordinate.

1.1 real scalar field theory

so considering real scalar field $\phi(x)$ we can construct the action to be

$$I[\phi(x)] = \int d^4x\mathcal{L}$$

where $\mathcal{L}$ is the Lagrangian density. $\mathcal{L}$ is a function of scalar field $\phi(x)$ and it's space time derivatives $\partial_\mu\phi(x)$ analogous to dynamical system, Here x is the space-time coordinate.

$$\mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu\phi(x))$$

the equation of motion we can get by using the variation principle $\phi(x) \to \phi(x) + \delta\phi(x)$ and using the boundary condition $\delta\phi(t_i, \vec{x}) = \delta\phi(t_f, \vec{x}) = 0$ $\delta(\partial_\mu\phi)$

$$\delta S = \int d^4x[\frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi)]$$

upon integrating by parts we get

$$\delta S = \int d^4x\partial_\mu(\frac{\delta L}{\delta\partial_\mu\phi}\delta\phi) + \int d^4x\delta\phi[\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi})]$$

the first term is the surface term so it vanishes at the boundary and what we are left with is that

$$\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}) = 0 \quad (1)$$

Considering the kinetic term to be quadratic and potential to be $V(\phi(x)) = \frac{1}{2}m^2\phi^2$ we can get the equation of motion of the Lagrangian

$$\mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu\phi(x)) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

putting this Lagrangian back in the equation (1) we get

$$(\partial^2 + m^2)\phi = 0$$

This is the klein-gordon equation

1.2 complex scalar field theory

Here we consider a complex field ϕ to make the action real the action is considered to be the following

$$I[\phi(x), \phi^\dagger(x)] = \int d^4x[(\partial_\mu\phi)^\dagger)(\partial_\mu\phi) - V(\phi^\dagger\phi)]$$

for simplicity take $V(\phi^\dagger\phi) = m^2\phi^\dagger\phi$, it should not be considered that ϕ and $\phi^\dagger$ they are same though they are complex conjugate of each other but in the field theory they are independent of each other.
so here $\mathcal{L} = \mathcal{L}(\phi, \phi^\dagger, \partial_\mu\phi, \partial_\mu\phi^\dagger)$ taking variation of each field we have two sets of equation

$$(\partial^2 + m^2)\phi = 0 \tag{2}$$

$$(\partial^2 + m^2)\phi^\dagger = 0 \tag{3}$$

again rewriting ϕ and $\phi^\dagger$ in terms of real and imaginary quantity we have

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}, \phi^\dagger = \frac{\phi_1 - i\phi_2}{\sqrt{2}}$$

putting them back in the action

$$\begin{aligned} I[\phi(x), \phi^\dagger(x)] &= \int d^4x[(\partial_\mu\phi)^\dagger)(\partial_\mu\phi) - m^2\phi^\dagger\phi)] \\ &= \int d^4x(\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2) + \int d^4x(\frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi^2) \end{aligned}$$

as we can see there is no cross term in the kinetic part and the potential part so it implies that complex scalar filed theory is theory of two non interacting real scalar field theory.

1.3 Noether's Theorem And conserved current

it deal with conserved current associated with the fields, it says that conserved charges only appear for continuous global symmetries. For translation symmetry the conserved quantity is Stress-Energy momentum tensor.The basic idea is that make the symmetry parameter to be space time dependent then make change in the action and that would be zero. This will give us conserved current.

In case of translation $x^\mu \rightarrow x^\mu + a^\mu$ we make the continuous symmetry parametera^μ) to be space time dependent and varying the action one can intuitively say that the variation in action should have derivative of a^μ because in case of global transformation it would be zero.

$$\delta I = -\int d^4x (\partial_\mu a^\nu)\theta^\mu_\nu$$

if $a^\nu = constant$ then $\partial_\mu a^\nu = 0$ that will make $\delta I = 0$ but the other way around,

$$\delta I = -\int d^4x \partial_\mu (a^\mu \theta^\mu_\nu) + \int d^4x (\partial_\mu \theta^\mu_\nu) a^\nu$$

the first term got cancelled as it will give us surface term so what we are left with

$$\delta I = \int d^4x (\partial_\mu \theta^\mu_\nu) a^\nu$$

implies $\partial_\mu \theta^\mu_\nu = 0$, thus θ^μ_ν is a conserved quantity for translation symmetry it can be shown that

$$\theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$

taking the $\mu = 0$ and $\nu = 0$

$$\begin{aligned} \theta^{00} &= \frac{\partial L}{\partial(\partial_0 \phi)} -^{00} L \\ &= (\partial_o \phi)^2 - L \\ &= (\partial_o \phi)^2 - \frac{1}{2}(\partial_o \phi)^2 + \frac{1}{2}(\bigtriangledown \phi)^2 + V(\phi) \\ &= \frac{1}{2}(\partial_o \phi)^2 + \frac{1}{2}(\bigtriangledown \phi)^2 + V(\phi) \end{aligned}$$

this is the energy density of the field. It can be shown that other components gives us pressure,stress tensor, momentum flux,momentum density.

2 Fundamental forces and particle classification

so far i have discussed about equation of motion from Lagrangian, conserved current associated with fields, non interacting fields. Before going into Abelian gauge theory where i will discuss about how the particles interacts with each other. Now lets start with information about particles and fundamental forces in nature, so far interaction in nature are found to be of four kind,
1) Strong interaction
2)weak interaction
3)electromagnetic interaction
4)gravitational interaction
Particles in nature observed to follow two kind of statistics Fermi statistics and Boson

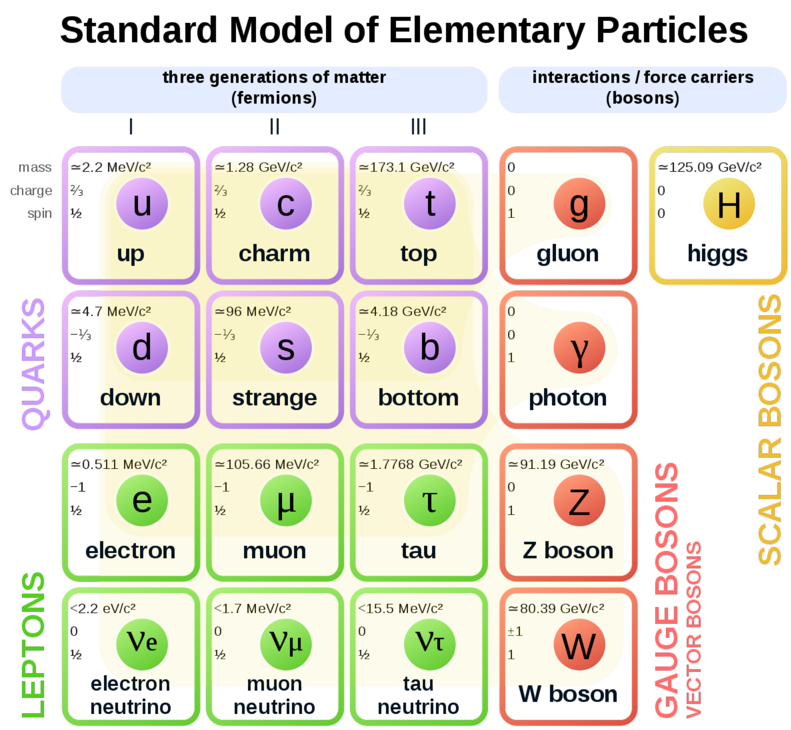

Figure 1: Standard Model

statistics, 3 generation of quark and leptons follows Fermi statistics called fermions,Gauge bosons and scalar boson follows Boson statistics.Gauge bosons are spin 1 particle whereas all fermions are spin 1/2 particles besides that recently discovered particle Higgs is spin 0 scalar particle.For each type of interaction we have fundamental particles associated with it.

gluons	exchange particles for strong interaction
photons	exchange particles for electromagnetic interaction
W boson and Z boson	exchange particles for Weak interaction

Table 1: Fundamental interactions

W bosons have either a positive or a negetive charge, Z bosons have no charge and are electrically neutral.quark interacts through gluons that is also strong interaction and forms baryons(qqq) and mesons($q\bar{q}$) all these particles have their anti particles they have same mass but opposite charge.

Starting with the gauge theory we will see later that the fermions interact with each other through photon these are not real photon it is virtual.

3 Abelian Gauge Theory

so what is gauge theory ? A gauge theory is a theory where the action is invariant under continuous symmetry transformation where the symmetry parameter depends on the space time parameter. It is different from Noether's theorem where the symmetry parameter was global that is independent of space time.
consider the maxwell field equation the field equation is written as follows

$$\vec{E} = -\bigtriangledown\phi - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{B} = \bigtriangledown \times \vec{A}$$

the most general transformation which leave $\vec{E}$ and $\vec{B}$ unchanged is

$$\phi \to \phi + \frac{\partial\lambda}{\partial t}$$

$$\vec{A} \to \vec{A} - \bigtriangledown\lambda$$

In four vector notation $A_\mu \to A_\mu + \partial_\mu\lambda(x)$ where $A_\mu = (\phi, \vec{A})$ under this transformation Maxwell source free equation remain invariant but as far we consider the source term it does not remain invariant

3.1 Spinor Electrodynamics

Let us start with the Dirac fermion action

$$S_{dirac} = \int d^4x\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

A global U(1) symmetry $\psi(x) \to e^{-ie\alpha}\psi(x)$ where α is a constant. Before going further i should mention that what is U(1) symmetry is,The elements of the group U(1) are points on the unit circle, which can be labeled by a unit complex number $e^{i\theta}$. U(1) group is a commutative group that's why the term arise Abelian. Now following Noether's procedure the corresponding Noether's current is found to be

$$j^\mu = e\bar{\psi}\gamma^\mu\psi \quad and \quad \partial_\mu j^\mu = 0$$

now for global symmetry the Lagrangian is invariant but taking the local symmetry something strange happen.

$$\begin{aligned}\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi &= \bar{\psi}e^{ie\alpha(x)}(i\gamma^\mu\partial_\mu - m)e^{-ie\alpha(x)}\psi \\ &= \bar{\psi}(i\gamma^\mu(\partial_\mu - ie\partial_\mu\alpha(x)) - m)\psi \\ &= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + [\partial_\mu\alpha(\end{aligned}$$

so it is not remain invariant however if we add a term $A_\mu j^\mu$ to the dirac lagrangian

$$\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - A_\mu j^\mu$$

and do these following gauge transformation

$$\psi(x) \to e^{-ie\alpha(x)}\psi(x) \;\;,\;\; A_\mu \to A_\mu + \partial_\mu\alpha(x)$$

Then the lagrangian remain invariant, one can rewrite the interaction in terms of covarient derivative

$$D_\mu = \partial_\mu + ieA_\mu$$

so under gauge transformation

$$D_\mu\psi(x) \to e^{-ie\alpha(x)}D_\mu\psi(x)$$

thus we have coupled fermions to the photons because we certainly have $A_\mu\psi$ term from gauge invariance,that is the interaction term so fermions always interacts with photons.

3.2 scalar QED

Conserved vector currents are available only for complex scalars. Put differently, real scalars are uncharged. Let us therefore consider a complex scalar theory, whose lagrangian prior to coupling to the Maxwell field reads

$$L = (\partial_\mu\phi)^\dagger)(\partial_\mu\phi) - V(\phi^\dagger\phi)$$

The noether current of the free theory associated with the global $U(1)$ symmetry $\phi \to e^{-ie\alpha\phi}$ with α as real is

$$j^\mu_{free} = ie(\phi \dagger \partial^\mu\phi - (\partial^\mu\phi) \dagger \phi)$$

The naive guess for the coupling to the gauge would be

$$\mathcal{L}^{naive}_{int} = -A_\mu j^\mu_{free}$$

However for the local gauge invarience $\phi \to e^{-ie\alpha(x)}\phi$ and $A-\mu \to A_\mu+\partial_\mu\alpha(x)$ let us instead follow the general route ot replacing usual derivative $\partial_\mu\psi$ by the covarient derivative

$$D_\mu\phi(x) = \partial_\mu + ieA_\mu)\phi$$

so the interaction which follows by expanding $D_\mu = \partial_\mu + ieA_\mu$ is

$$S_int[A,\phi] = -\int dx^4[ie(\phi\dagger\partial^\mu\phi - (\partial^\mu\phi)\dagger\phi)A_\mu - e^2A_\mu A^\mu\phi\dagger\phi]$$

the last term quadratic in A^μ is required for the gauge invarience and was missed in the naive guess.
The point is that in general the Noether current may itself depend on A_μ once the coupling to the gauge field is taken into account because $S_int[A,\phi]$ may depend not just linearly on A.therefor merely writing $-A_\mu J^\mu_{free}$ with j^μ_{free} the conserved current associated with S^{rest}_{matter} only is not the correct way to do.

4 Chiral Transformation

In this section i discussed about two kind of electrons that interacts completely different way, left handed and right handed electron.They transform in a different way under chiral transformation.

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

in terms of spinors

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

here

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma^\mu} & 0 \end{pmatrix} \ \bar{\psi} = \psi^\dagger\gamma^0 and \ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

putting these in the lagrangian density we get

$$\mathcal{L} = i\psi_L^\dagger\bar{\sigma^\mu}\partial_\mu\psi_L + i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R - m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R)$$

now for global U(1) transformation lagrangian density remain invariant but for chiral transformation $\psi \to exp^{i\gamma^5\theta}\psi$ it is a different case.

$$P_L\psi = \psi_L \ where \ P_L = \frac{1}{2}(1-\gamma^5)$$

$$and \ P_R\psi = \psi_R \ where \ P_R = \frac{1}{2}(1+\gamma^5)$$

so

$$
\begin{aligned}
\psi_L = P_L\psi &\to P_L exp^{i\gamma^5\theta}\psi \\
&\to \frac{1}{2}(1-\gamma^5)(1+i\gamma^5\theta-\frac{\theta^2}{2}-i\frac{\gamma^5\theta^3}{3!}+\cdots)\psi \quad [using(\gamma^5)^2=1] \\
&\to \frac{1}{2}(1-\gamma^5)(cos\theta+i\gamma^5 sin\theta)\psi \\
&\to exp^{-i\theta}P_L\psi \\
&\to exp^{-i\theta}\psi_L
\end{aligned}
$$

For ψ_R this transformation is completely different in similar way as above it is found out to be,

$$\psi_R = exp^{i\theta}\psi_R$$

so these transformations leads to a invariance problem associated with mass term in the lagrangian which can be removed by putting the mass term to be zero. so the lagrangian density without mass term is invariant under chiral transformation ,

$$\mathcal{L} = i\psi_L^\dagger\bar{\sigma^\mu}\partial_\mu\psi_L + i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R$$

At local transformation where θ is a space time variable we get another different thing that in order to hold invariance we need to include two differnt gauge fields A_μ and B_μ because of different transformation of ψ_L and ψ_R.These transformation rules for gauge fields are,

$$A_\mu \to A_\mu + \partial_\mu\theta \;\; and \;\; B_\mu \to B_\mu - \partial_\mu\theta$$

5 Non Abelian Gauge Theory

Let us consider a theory in which the degrees of freedom are N complex scalar fields $\phi^i, i = 1, 2, 3, ...N$. if they are all independent.Then the action would be,

$$I[\phi^i(x)] = \int d^4x \sum[(\partial^\mu\phi^i)^\star\partial_\mu\phi^i - m_i\phi^{\star i}\phi^i - V(\phi^{\star i}\phi^i)]$$

as there are no interaction they do not talk to each other there is no new physics compared to the single complex scalar field.we can couple this decoupled N scalar system to a single maxwell field by turning the global symmetry into a gauge symmetry

$$\phi^i \to (\phi^{'})^i = e^{-ig\alpha(x)}\phi$$

$$A_\mu \to A_\mu^{'} = A_\mu + \partial_\mu\alpha(x)$$

Alternatively we can couple each scalar to a different maxwell field

$$\phi^i \to (\phi^{'})^i = e^{-ig_i\alpha(x)^i}\phi$$

$$A_\mu \to (A_\mu^{'})^i = A_\mu^i + \partial_\mu\alpha^i(x)$$

here we get N copies of non interacting photons.this is also not give rise to any new physics however some new interesting physics emerges when these fields can be gathered together in a column vector

$$\phi = \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ . \\ . \\ . \\ \phi^N \end{pmatrix}$$

and the action be made invarient under a larger group/matrix transformation

$$\phi \rightarrow \phi^{'=U\phi}$$

here $U \in SU(N)$ i.e N unitary matrix of unit determinant. the action we are talking about is

$$I[\phi] = \int d^4x[(\partial^\mu \phi^)^\dagger(\partial_\mu \phi) - m^2\phi^\dagger\phi - V(\phi^\dagger\phi)]$$

Compared to the previous case we see that coefficient of the mass terms and the potential terms are identical for all i in the present case. Now, as we have seen in the scalar electrodynamics case i.e. for U(1), turning the global symmetry U(1) into a local/gauge symmetry leads to coupling of the complex scalar to some form of massless vector field (Maxwell).

5.1 Motivation

The near degeneracy of the neutron and proton masses, the charge- independence of nuclear forces, and many subsequent observations support the notion of isospin conservation in the strong interactions. What is meant by isospin conservation is that the laws of physics should be invariant under rotations in isospin space, and that the proton and neutron should appear symmetrically in all equations. This means that if electromagnetism can be neglected, the isospin orientation is of no significance. The distinction between proton and neutron thus becomes entirely a matter of arbitrary convention.

5.2 Construction

All we have done so far is spinor electrodynamics, dirac spinors and how it couples to gauge bossons. Down there are those following transformation so far,

$$S_{dirac} = \int d^4x\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

A local U(1) symmetry that is gauge symmetry $\psi(x) \rightarrow e^{-ie\alpha(x)}\psi(x)$ where α is a space time dependent function.we find that to make the lagrangian invariant we hove to do these following gauge transformation.

$$\psi(x) \rightarrow e^{-ie\alpha(x)}\psi(x) \ , \ A_\mu \rightarrow A_\mu + \partial_\mu\alpha(x)$$

Then the lagrangian remain invariant, one can rewrite the interaction in terms of covarient derivative

$$D_\mu = \partial_\mu + ieA_\mu$$

so under gauge transformation

$$D_\mu\psi(x) \rightarrow e^{-ie\alpha(x)}D_\mu\psi(x)$$

in more general case these transformations are

$$\phi(x) \rightarrow G\phi(x)$$

$$D_\mu \rightarrow \partial_\mu - igA_\mu$$

$$A_\mu^{'} = GA_\mu G^{-1} + \frac{i}{g}(\partial_\mu G)G^{-1}$$

under these transformation $D_\mu\phi$ behaves as a same way as ϕ like

$$D_\mu\phi \rightarrow GD_\mu\phi$$

The above discussion can be generalized to more complicated gauge groups than $U(1)$. All Lie groups can be represented by matrices, and except$U(1)$ they are all non-Abelian, meaning that the commutator of two elements of the group is non-zero. The gauge groups of the Standard Model are $U(1)$, $SU(2)$ and $SU(3)$,and the gauge groups used in theories beyond the Standard Model are mostly $SU(N)$ and $SO(N)$.
A general group element of SU(N) is an N N-matrix with determinant equal to one.It can be written as $U = exp(i\alpha^a T_a)$ where α^a is a set of N^2-1 parameter and T^a is a set of N^2-1 generators.These generators are matrices and fulll the commutation relation

$$[T^a, T^b] = if_c^{ab}T^c \qquad [f_c^{ab} = structure\ \ constant]$$

if $\alpha^a << 1$ then

$$U \approx 1 + i\alpha^a T_a$$

so the transformation of gauge field is

$$A_\mu^{a'} = UA_\mu^a U^{-1} + \frac{i}{T_a}(\partial_\mu U)U^{-1}$$

This is a crucial equation establishing two thing:
1. A_μ is now a $N \times N$ matrix. This is clear from the second term in the transformation equation.This term is a being a product of two$N \times N$ matrices, $\partial_\mu U$ and U^{-1} is an $N \times N$ matrix. Thus the lhs i.e. $A^{'}_\mu$ must be so as wel.

2. A_μ is itself a non-tensor i.e. it transforms under a local U(N) or SU(N) transformation inhomogeneously due to the second term on the rhs. This does not involve A while the first term is linear in A. This is a bit ironic, because we introduced this gauge field in the first place to make the derivative of scalar transform homogeneously under local/gauge symmetry,

however in the process of achieving that, it itself ends up failing to be a homogeneously transforming object.

$$\begin{aligned}
T_a A_\mu^{a'} &= UT_a A_\mu^a U^{-1} + i(\partial_\mu U)U^{-1} \\
&= (1 + i\alpha^b T_b)T_a A_\mu^a (1 - i\alpha^c T_c) + i\partial_\mu (1 + i\alpha^a T_a)(1 - i\alpha^b T_b) \\
&= (T_a A_\mu^a + i\alpha^b T_b T_a A_\mu^a)(1 - i\alpha^c T_c) + i\partial_\mu (i\alpha^a T_a)(1 - i\alpha^b T_b) \\
&= T_a A_\mu^a - iA_\mu^a \alpha^c T_a T_c + iA_\mu^a \alpha^c T_c T_a - \partial_\mu(\alpha^a T_a) + O(\alpha^2) \\
&= T_a A^a - \partial_\mu(\alpha^a T_a) + iA_\mu^a \alpha^c [T_c, T_a] \qquad [\textit{neglecting higher orders in } \alpha] \\
&= T_a A^a - \partial_\mu(\alpha^a T_a) + iA_\mu^a \alpha^c i f_{ac}^b T_b
\end{aligned}$$

That's implies

$$A_\mu^{\prime a} = A_\mu^a - \partial_\mu \alpha^a - A_\mu^b \alpha^c f_{bc}^a$$

here something different is coming up ,the third term it was never before in the abelian gauge symmetry as it is commutative.

5.3 Kinetic Term for the Gauge field

To this point in the construction of the isospin gauge theory of nucieons, we have a Lagrangian given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

$$\mathcal{L} = \mathcal{L}_\wr - g\bar{\psi}\gamma^\mu A_\mu \psi$$

namely a free Dirac Lagrangian plus an interaction term that couples the isovector gauge fields to the conserved isospin current of the nucieons. The structure of the interaction between the gauge fields and matter is precisely analogous to that found in the case of QED. To proceed further, we must construct a field-strength tensor and hence a Recall that in the case of the U(1) gauge field, the kinetic term was given by, $L_{EM} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Then one might suggest that a similar construction be carried out for the nonabelian gauge field. The first step in this direction to construct the field strength tensor, $F^{\mu\nu}$, and this field strength tensor should transform as a tensor, unlike the non-tensor transformation of the potential in equation A_μ. In the Maxwell case, $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$. Let's try to define a matrix valued field strength for the nonabelian case analogously and check it's transformation properties kinetic term for the gauge fields.

$$\begin{aligned}
F'_{\mu\nu} &= \partial_\nu A'_\mu - \partial_\mu A'_\nu && (4)\\
&= \partial_\mu (G A_\nu G^{-1} + \frac{i}{g}(\partial_\nu G) G^{-1}) - \mu \leftrightarrow \nu && (5)\\
&= G\partial_\mu A_\nu G^{-1} + \partial_\mu G A_\nu G^{-1} + G A_\nu \partial \mu G^{-1} + \frac{i}{g}(\partial_\nu G)(\partial_\mu G^{-1}) - \mu \leftrightarrow \nu && (6)\\
&= G(\partial_\nu A_\mu - \partial_\mu A_\nu)G^{-1} + [(\partial_\nu G)A_\mu - (\partial_\mu G)A_\nu]G^{-1} + G[A_\mu(\partial_\nu G^{-1}) - A_\nu(\partial_\mu G^{-1})] && (7)\\
&+ \frac{i}{g}[(\partial_\mu G)(\partial_\nu G^{-1}) - (\partial_\nu G)(\partial_\mu G^{-1})] && (8)\\
&\neq G(\partial_\nu A_\mu - \partial_\mu A_\nu)G^{-1} && (9)\\
& && (10)
\end{aligned}$$

This result may be cast in slightly more symmetric form by recalling that

$$G^{-1}G = GG^{-}1 = 1$$

so $(\partial_\mu G^{-1})G = G^{-1}(\partial_{mu} G)$ taking two terms form line 7 and 8 and using the above equation we get ,

$$\begin{aligned}
&(\partial_\nu G)A_\mu G^{-1} + G A_\mu (\partial_\nu G^{-1})\\
&\qquad = GG^{-}1(\partial_\nu G)A_\mu G^{-1} + GA_\mu(\partial_\nu G^{-1})GG^{-1}\\
&\qquad = GG^{-1}(\partial_\nu G)A_\mu G^{-1} + GA_\mu G^{-1}(\partial_\nu G)G^{-1}\\
&\qquad = G[G^{-1}(\partial_\nu)G, A_\mu]G^{-1}
\end{aligned}$$

similarly for $-(\partial_\mu G)A_\nu G^{-1} - GA_\nu(\partial_\mu G^{-1})$ we get $-G[G^{-1}(\partial_\mu)G, A_\nu]G^{-1}$ from the last part of the line (8) we have

$$\begin{aligned}
&\frac{i}{g}[(\partial_\mu G)(\partial_\nu G^{-1}) - (\partial_\nu G)(\partial_\mu G^{-1})]\\
&\qquad = \frac{i}{g}GG^{-1}[(\partial_\mu G)(\partial_\nu G^{-1}) - (\partial_\nu G)(\partial_\mu G^{-1})]\\
&\qquad = \frac{i}{g}G[-(\partial_\mu G^{-1})G(\partial_\nu G^{-1}) + (\partial_\nu G^{-1})G(\partial_\mu G^{-1})]\\
&\qquad = \frac{1}{ig}G[(\partial_\nu G^{-1})(\partial_\mu G) - (\partial_\mu G^{-1})(\partial_\nu G)]G^{-1}
\end{aligned}$$

so finally we have a compact form of the stress tensor in terms of commutator

$$\begin{aligned}
F'_{\mu\nu} &= G(\partial_\nu A_\mu - \partial_\mu A_\nu)G^{-1}\\
&+ G\{[G^{-1}(\partial_\nu)G, A_\mu] - [G^{-1}(\partial_\mu)G, A_\nu]\\
&+ \frac{1}{ig}G[(\partial_\nu G^{-1})(\partial_\mu G) - (\partial_\mu G^{-1})(\partial_\nu G)]G^{-1}
\end{aligned}$$

so this does not transform homogeneously it contains non homgeneous parts due to non abelian group structure. now If we recall that the minimal coupling prescription instructs replacing the partial derivative by a covariant derivative, then can do this change in the definition of the field strengt to

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$$

so it contains an additional term due to non abelian part

$$\begin{aligned} F_{\mu\nu} &= D_\nu A_\mu - D_\mu A_\nu && (11)\\ &= (\partial_\nu + igA_\nu)A_\mu - (\partial_\mu + igA_\mu)A_\nu && (12)\\ &= \partial_\nu A_\mu - \partial_\mu A_\nu + ig[A_\nu, A_\mu] && (13)\\ & && (14) \end{aligned}$$

we have already done the gauge transformation of the first two terms of 14 and got some aditional non homogeneous part with it that does not transform in usual way of gauge transformation. so we will now looking for the additional non abelian term of (14) and lets see how it transform under gauge transformation.

$$\begin{aligned} &ig[A'_\nu, A'_\mu]\\ &= ig[(GA_\nu G^{-1} + \frac{i}{g}(\partial_\nu G)G^{-1}), (GA_\mu G^{-1} + \frac{i}{g}(\partial_\mu G)G^{-1})]\\ &= ig[GA_\nu G^{-1}, GA_\mu G^{-1}] + ig[GA_\nu G^{-}1, \frac{i}{g}(\partial_\mu G)G^{-1}] + ig[\frac{i}{g}(\partial_\nu G)G^{-1}, GA_\mu G^{-1}]\\ &+ ig[\frac{i}{g}(\partial_\nu G)G^{-1}, \frac{i}{g}(\partial_\mu G)G^{-1}] \end{aligned}$$

so from the first term

$$\begin{aligned} &ig[GA_\nu G^{-1}, GA_\mu G^{-1}]\\ &= igGA_\nu G^{-1}gA_\mu G^{-1} - igGA_\mu G^{-1}gA_\nu G^{-1}\\ &= igG[A_\nu, A_\mu]G^{-1} \end{aligned}$$

and form the 2nd term we have

$$\begin{aligned} &ig[GA_\nu G^{-}1, \frac{i}{g}(\partial_\mu G)G^{-1}]\\ &= igGA_\nu G^{-1}\frac{i}{g}(\partial_\mu G)G^{-1} - igGA_\mu G^{-1}\frac{i}{g}(\partial_\nu G)G^{-1}\\ &= -GA_\nu G^{-1}(\partial_\mu G)G^{-1} + (\partial_\mu G)G^{-1}GA_\nu G^{-1}\\ &= GG^{-1}(\partial_\mu G)A_\nu G^{-1} - GA_\nu G^{1}(\partial_\mu G)G^{-1}\\ &= G[G^{-1}(\partial_\mu G), A_\nu]G^{-1} \end{aligned}$$

similarly from the third term we get

$$-G[G^{-1}(\partial_\nu G), A_\mu]G^{-1}$$

and the final part

$$\begin{aligned}
& ig[\frac{i}{g}(\partial_\nu G)G^{-1}, \frac{i}{g}(\partial_\mu G)G^{-1}] \\
& \qquad = \frac{i}{g}(\partial_\mu G)G^{-1}(\partial_\nu G)G^{-1}) - \frac{i}{g}(\partial_\nu G)G^{-1}(\partial_\mu G)G^{-1}) \\
& \qquad = \frac{i}{g}G(\partial_\nu G^{-1}(\partial_\mu G) - \frac{i}{g}G(\partial_\mu G^{-1}(\partial_\nu G) \\
& \qquad = \frac{-1}{ig}G[(\partial_\nu G^{-1})(\partial_\mu G) - (\partial_\mu G^{-1})(\partial_\nu G)]G^{-1}
\end{aligned}$$

so putting all these together back in the equation (14) we get all the non homogeneous terms cancels each other the only term that survive is $F'_{\mu\nu} = G[\partial_\nu A_\mu - \partial_\mu A_\nu + ig[A_\nu, A_\mu]]G^{-1}$.Now it is easy to write down a gauge invariant kinetic term for the gauge field in terms of the field strength tensor similar to the Maxwell case,

$$I[A_\mu] = -\frac{1}{4}Tr(F_{\mu\nu}F^{\mu\nu})$$

The trace is crucial for the term to be gauge invariant

$$Tr(F'_{\mu\nu}F'^{\mu\nu}) = Tr(GF_{\mu\nu}F^{\mu\nu}G^{-1}) = Tr(F_{\mu\nu}F^{\mu\nu}G^{-1}G) = Tr(F_{\mu\nu}F^{\mu\nu})$$

This part explain something new that was never before in the abelian case.this gauge field part of the Lagrangian can be expanded to

$$-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2}gf^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)A^{b\mu}A^{c\nu} - \frac{1}{4}g^2(f^{abc}A^b_\mu A^c_\nu)^2$$

This Lagrangian has quartic and cubic terms in the A^a_μ so unlike the maxwell Lagrangian, it contains interactions whose forms are fixed by symmetry. There is no mass term of the form $m^2(A^a_\mu)2$ so the gauge fields are massless and a term of such a form would violate gauge invariance just like in the U(1) case.

6 conclution

We have construct the clasic non abelian gauge theory by introducing the non commutative part through lie algebra.It is already shown that U(1) symmetry explain the electromagnetic interaction,weak interaction is given by su(2) symmetry for this the 3 gauge fields are W^+, W^-, z^0.The Unification of $U(1) \times SU(2)$ we have electroweak interaction.The strong interaction can be explained by SU(3) symmetry where there will be 8 gauge fields,the gluons. There are several other interesting Yang-Mills theories. For example, it has been suggested that the standard model, based on the group $SU(3) \times SU(2) \times U(1)$, is a subgroup of a larger simple group, such as SU(5). Theories of this kind, which attempt to unify interactions are sometimes known as grand unified theories (GUTs). However, in order to obtain a proper description of Nature we have to quantize these theories.

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