Introduction to AdS Space

A project report
submitted in partial fulfillment for the award of the degree of
Master of Science in Physics

by

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April 2019

DECLARATION

I hereby declare that the project work entitled "Introduction to AdS space" is a work carried out by me under the supervision of Dr.Shubho Roy, Indian Institute Of Technology, Hyderabad.

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CERTIFICATE

This is to certify that the project work titled "Introduction to AdS space" submitted to the Indian Institute Of Technology, Hyderabad, in partial fulfillment of requirements for the award of the degree of Masters in Science in Physics, is a work done by Kanhu Kishore Nanda, during the period of his study, Aug 2018 - April 2018, in the Indian Institute Of Technology, Hyderabad, under my supervision and guidance. This thesis has not been submitted for the award of any other Degree/Diploma/Fellowship or any other similar title.

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ABSTRACT

In the present work we start with a basic discussion of CFT that only focuses on two point operator in CFT and then we move to basics of AdS/CFT including different coordinate charts, solution of scalar wave equation in different coordinate systems, normalizability and Breitenlohner-Freedman (BF) bounds.

Contents

Chapter 1

Background

1.1 Introduction and aim of the project

AdS/CFT duality offers a fully non-perturbative formulation of quantum gravity in asymptotically anti de Sitter spacetime in terms of a conformal field theory. A conformal field theory (CFT) is a quantum field theory with some extra symmetries. Any quantum theory of gravity must resolve the black hole information paradox - the apparent loss of unitarity during Hawking evaporation. AdS/CFT postulates the Hawking process is just another unitary process in quantum field theory. Thus, in principle, AdS/CFT implies no information paradox. But this is only in principle, and attempts are underway to understand this in a lot more detail. The aim of this project is to understand the basics of AdS space and CFT.

1.2 CFT and conformal algebra

Conformal field theory is a quantum field theory that's invariant under conformal transformations. The conformal group is group of the transformations that leave the metric invariant up to a scaling factor, $g_{\mu\nu} \to \zeta^2 g_{\mu\nu}$ where ζ^2 is the scaling factor. Those transformations are, the Lorentz transformation ($M_{\mu\nu}$), scaling transformation (D), translation (P_μ) and special conformal transformation (K_μ). The scaling transformation is defined as, $x^\mu \to \lambda x^\mu$ and special conformal transformation is given by,

$$y^\mu = \frac{x^\mu - b^\mu x^2}{1 - (2b \cdot x) - b^2 x^2}. \tag{1.1}$$

This group also follows the conformal algebra.

$$[M_{\mu\nu}, P_\rho] = -i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right); \qquad [M_{\mu\nu}, K_\rho] = -i\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right);$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} \pm permutations; \quad [M_{\mu\nu}, D] = 0; \quad [D, K_\mu] = iK_\mu;$$

$$[D, P_\mu] = -iK_\mu; \qquad [P_\mu, K_\nu] = 2i\left[M_{\mu\nu} - \eta_{\mu\nu}D\right],$$

with all other commutators being zero. In the next section we compute the two point function in CFT.

1.3 Operators in CFT

In a conformal transformation, $x \to y$, a quasi-primary transforms as,

$$\tilde{O}(y) = \left|\frac{y}{x}\right|^{-\frac{\Delta}{d}} O(x), \tag{1.2}$$

where Δ is the scaling dimension of the O and d is the spacetime dimension of the space. Then, for a two point function, we have,

$$\langle O_1(x_1)O_2(x_2)\rangle = \left|\frac{y_1}{x_1}\right|^{\frac{\Delta_1}{d}} \left|\frac{y_2}{x_2}\right|^{\frac{\Delta_2}{d}} \left\langle \tilde{O}_1(y_1)\tilde{O}_2(y_2)\right\rangle. \tag{1.3}$$

If we take the scaling transformation $x \to \lambda x$, we have,

$$\langle O_1(x_1)O_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2} \left\langle \tilde{O}_1(y_1)\tilde{O}_2(y_2)\right\rangle$$

Lorentz and translational invariance require that the correlation function must be a function of the separation between the points. And the above condition constraints it to be, for some constant C_{12},

$$\langle O_1(x_1)O_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}. \tag{1.4}$$

It also has to be invariant under special conformal transformation which is given by (1.7).

Under this transformation, the separation distance transforms as,

$$|y_2 - y_1| = \frac{|x_2 - x_1|}{(1 - (2b\cdot x_1) - b^2x_1^2)(1 - (2b\cdot x_2) - b^2x_2^2)}.$$

Then the two point function transforms as, with $\beta = 1 - (2b\cdot x) - b^2x^2$,

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \frac{(\beta_1\beta_2)^{\frac{\Delta_1+\Delta_2}{2}}}{\beta_1^{\Delta_1}\beta_2^{\Delta_2}}. \tag{1.5}$$

Only if $\Delta_1 = \Delta_2$, is this constraint satisfied, otherwise it vanishes. So, we finally have,

$$\langle O_1(x_1)O_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{2\Delta}}. \tag{1.6}$$

Chapter 2

Introduction to AdS

AdS_{d+1} space is a solution to vacuum Einstein equations with negative cosmological constant. It is maximally symmetric and has a negative curvature. AdS space is commonly introduced as the covering space of a related object called the AdS hyperboloid. The AdS hyperboloid is a hypersurface embedded inside of a higher dimensional Minkowski space. This preserves the number of isometries or killing vectors of the original AdS space and also gives us the embedding coordinates to describe this hyperboloid. Then, with l defined as the AdS radius and X's as the embedding coordinates, one can write,

$$-X_{-1}^2 - X_0^2 + \sum_{i=1}^{d} X_i^2 = -l^2. \tag{2.1}$$

Next we introduce global and local coordinate charts to parameterize this hyperboloid.

2.1 Coordinate charts

This section and the next section follows [3].

Global coordinates

Global coordinates cover the entire AdS space. These are defined as, for simplicity when $d = 2$,

$$X_{-1} = l \cosh u \sin t; X_0 = l \cosh u \cos t$$
$$X_1 = l \sinh u \cos\theta; X_2 = l \sinh u \sin\theta.$$

So the metric is written as,

$$ds^2 = l^2\left[-\cosh^2 u\, dt^2 + \sinh^2\rho\, d\theta^2 + du^2\right]. \tag{2.2}$$

Here $0 \leq u \leq \infty$, $0 \leq \theta, t \leq 2\pi$. Further defining $\sinh u = \tan\rho$, one can write, with $0 \leq \rho \leq \frac{\pi}{2}$,

$$ds^2 = l^2\left[-\sec^2\rho\, dt^2 + \tan^2\rho\, d\theta^2 + \sec^2\rho\, d\rho^2\right].$$

It can then be generalized to any higher dimension with arbitrary d, so,

$$ds^2 = l^2\left[-\sec^2\rho\, dt^2 + \tan^2\rho\, d\Omega^2_{d-1} + \sec^2\rho\, d\rho^2\right]. \tag{2.3}$$

Here notice that 2π and 0 are the same point for time t. This is a closed timelike curve and hence it violates causality. So we need to unwrap the t circle to preserve causality, as a result t then $\in (-\infty, \infty)$.

Poincare Coordinates

Poincare coordinates cover only half of the AdS hyperboloid. These are defined as, for simplicity when $d = 2$,

$$X_{-1} = \frac{1}{2r}(l^2 + x^2 + r^2 - t^2); X_0 = l\frac{t}{r}$$
$$X_1 = -\frac{1}{2r}(-l^2 + x^2 + r^2 - t^2); X_2 = l\frac{x}{r}.$$

So the metric becomes,

$$ds^2 = \frac{l^2}{r^2}(-dt^2 + dx^2 + dr^2). \tag{2.4}$$

Here t, x range between $(-\infty, \infty)$ and $0 \leq r \leq \infty$. we can easily generalize this to any arbitrary d with the following substitution; dx^2 becomes $\mathbf{dx}^2$, and $\mathbf{dx}^2 = \mathbf{dx}\cdot\mathbf{dx}$ is the flat metric on R^{d-1}.

BTZ coordinates

Again we consider $d = 2$. We divide the hyperboloid into three regions. Then each region is parameterized.
For region I,

$$X_{-1} = \pm s\cosh\theta; X_1 = s\sinh\theta$$
$$X_0 = \sqrt{s^2 - l^2}\sinh\frac{t'}{l}; X_2 = \pm\sqrt{s^2 - l^2}\cosh\frac{t'}{l}.$$

Then the metric is,

$$ds_I^2 = -\left(\frac{s^2}{l^2} - 1\right) dt'^2 + \frac{ds^2}{\frac{s^2}{l^2} - 1} + s^2 d\theta^2. \tag{2.5}$$

For region 2,

$$X_{-1} = \pm s \cosh\theta; X_1 = s \sinh\theta$$
$$X_0 = \pm\sqrt{l^2 - s^2} \cosh\frac{t'}{l}; X_2 = \sqrt{l^2 - s^2} \sinh\frac{t'}{l}.$$

For region 3,

$$X_{-1} = r \sinh\theta; X_0 = \pm s \cosh\theta$$
$$X_0 = \pm\sqrt{s^2 + l^2} \cosh\frac{t'}{l}; X_2 = \sqrt{s^2 + l^2} \sinh\frac{t'}{l}.$$

The metric then becomes,

$$ds_{II/III}^2 = \left(\pm\frac{s^2}{l^2} - 1\right) dt'^2 - \frac{ds^2}{\pm\frac{s^2}{l^2} - 1} + s^2 d\theta^2. \tag{2.6}$$

So one has 12 patches to cover the entire AdS_3 hyperboloid.

2.2 Trajectories of particles

For a radial null geodesic one has, $ds^2 = 0$. Then from (2.2) we get the following relation,

$$dt = \frac{du}{\cosh u}.$$

Integrating both sides we have,

$$t = \tan^{-1}(\sinh u) + c. \tag{2.7}$$

Putting the limits of u, that is 0 to ∞, we have, $t = \frac{\pi}{2}$. With reflecting boundary conditions, it will reflect and will take equal time to return. So it takes then total $\triangle t = \pi$ time to come back to the original position. From [5], the massive particle also shows an analogous trajectory, that is, it also has periodic solution. However it never reaches the spatial infinity and turn back and after a time of π, it reaches its origin.

2.3 Scalar field in AdS

As particles, and light are considered in the previous section, in this section, scalar field is considered. The equation describing scalar field is given by,

$$(\Box - m^2)\Phi = 0. \tag{2.8}$$

The operator $\Box$ in global coordinates is,

$$l^2\Box = -\cos^2\rho\,\partial_t^2 + \cos^2\rho\,\partial_\rho^2 + (d-1)\cot\rho\,\partial_\rho + \cot^2\rho\,\nabla^2_{S^{d-1}}. \tag{2.9}$$

In Poincare coordinates, the expression is,

$$l^2\Box = -r^2\partial_t^2 + r^2\partial_r^2 - (d-1)r\,\partial_r + r^2\,\nabla^2_{R^{d-1}}. \tag{2.10}$$

$\nabla^2_{S^{d-1}}$ is the Laplacian operator for S^{d-1}, where S means the sphere. Similarly $\nabla^2_{R^{d-1}}$ is the Laplacian for a $(d-1)$ dimensional flat space. The solution of Klein$-$Gordon equation in these coordinates is discussed in the following subsections.

2.3.1 Solution in Poincare coordinates

For a scalar of mass squared $\frac{m^2}{l^2}$, we can write the following form for Φ,

$$\Phi = e^{-i\omega t + i\vec{k}.\vec{x}} r^{\frac{d}{2}}\chi(r), \tag{2.11}$$

where $\chi(r)$ satisfies the following property,

$$r^2\partial_r^2\chi + r\partial_r\chi - \left((m^2 + \frac{d^2}{4}) + (\vec{k}^2 - \omega^2)\right)\chi = 0.$$

Then for $q^2 = \vec{k}^2 - \omega^2 > 0$, we have the following solution,

$$\Phi = e^{-i\omega t + i\vec{k}.\vec{x}} r^{\frac{d}{2}} K_\nu(qr), \tag{2.12}$$

for $\nu = \sqrt{d^2 + 4m^2}$. For $q^2 < 0$, and ν is not an integral, we get two independent solutions,

$$\Phi^\pm = e^{-i\omega t + i\vec{k}.\vec{x}} r^{\frac{d}{2}} J_{\pm\nu}(|q|r). \tag{2.13}$$

For an integral ν, the above solution is only for Φ^+ and the other is given by,

$$\Phi^- = e^{-i\omega t + i\vec{k}.\vec{x}} r^{\frac{d}{2}} Y_\nu(|q|r). \tag{2.14}$$

For $\nu = 0$, we have,

$$\Phi^- \sim r^{\frac{d}{2}} \ln r. \tag{2.15}$$

2.3.2 Solution in global coordinates

For global coordinates, with the same scalar mass, we have the following solution,

$$\Phi = e^{-i\omega t} Y_{l,(m)}(\Omega)\chi(\rho), \tag{2.16}$$

where $Y_{l,(m)}(\Omega)$ is determined from the following relation,

$$\nabla^2_{S^{d-1}} Y_l = -l(l+d-2)Y_l, \tag{2.17}$$

with $l \geq 0$. Using this, we get the following equation for χ,

$$\frac{1}{(\tan\rho)^{d-1}}\partial_\rho\left((\tan\rho)^{d-1}\partial_\rho\right)\chi + \left[\omega^2 - l(l+d-2)\csc^2\rho - m^2\sec^2\rho\right]\chi = 0.$$

To solve this, we substitute χ as,

$$\chi(\rho) = (\cos\rho)^{2p}(\sin\rho)^{2q} f(\rho),$$

with $x = sin^2\rho$, we get,

$$x(1-x)\partial_x^2 f + \left[2q + \frac{d}{2} - (2p+2q+1)x\right]\partial_x f - \left[(p+q)^2 - \frac{\omega^2}{4}\right] f = 0, \tag{2.18}$$

and with $x' = \cos^2\rho$, we have,

$$x'(1-x')\partial_{x'}^2 f + \left[2p + 1 - \frac{d}{2} - (2p+2q+1)x'\right]\partial_{x'} f + \left[(p+q)^2 - \frac{\omega^2}{4}\right] f = 0, \tag{2.19}$$

where p and q satisfy the following conditions,

$$p\left(p - \frac{d}{2}\right) = \frac{m^2}{4}; \quad 2q(2q+d-2) = l(l+d-2).$$

Both p and q have two solutions,

$$p_\pm = \frac{d \pm \sqrt{d^2+4m^2}}{4},$$
$$q = \frac{l}{2}, \frac{1}{2}(2-d-l).$$

(2.17) will have two independent solutions that correspond to two solutions for indicial equation of q. For even d, one gets a logarithmic solution. Similarly, (2.18) will have two independent solutions corresponding to the indicial equation of p. One solution will be logarithmic if ν is an integer.

2.4 Normalizable and non-nomralizable modes

Norm of a scalar field is defined as,

$$(\phi_1, \phi_2) = i \int_\Omega \sqrt{g} g^{tt} \phi_1(x) \overleftrightarrow{\partial_t} \phi_2(x) d^d x, \tag{2.20}$$

where t is orthogonal to spacelike surface Ω. We found the solutions in Poincare coordinates as, for $q^2 = \vec{k}^2 - \omega^2 > 0$,

$$\Phi = e^{-i\omega t + i\vec{k}.\vec{x}} z^{\frac{d}{2}} K_\nu(qz),$$

for $\nu = \frac{1}{2}\sqrt{d^2 + 4m^2}$. For $q^2 < 0$, and ν is not an integral, solutions are,

$$\Phi^\pm = e^{-i\omega t + i\vec{k}.\vec{x}} z^{\frac{d}{2}} J_{\pm\nu}(|q|z).$$

For an integral ν, the above solution is only for Φ^+ and the other is given by,

$$\Phi^- = e^{-i\omega t + i\vec{k}.\vec{x}} z^{\frac{d}{2}} Y_\nu(|q|z).$$

For integral ν, and $\nu > 1$, the expansion of Φ^+ is.

$$\Phi^+ = e^{(ik\cdot x)} z^{\frac{d}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu + m + 1)} \left(\frac{z}{2}\right)^{\nu+2m}. \tag{2.21}$$

Using the formula (2.20) and using only the first term in the above expansion, we have, for some constant a,

$$\begin{aligned}(\Phi^+, \Phi^+) &= a \int z^{2\nu-3} dz \\ &= \frac{1}{2\nu - 2} z^{2\nu-2}\end{aligned}$$

So as $z \to 0$, this norm remains finite. Second term in the expansion is 2 orders higher than the first and the third, 4 and so on. So they go to zero as $z \to 0$. However Y_ν doesn't have a finite norm. For $z \to 0$, Y_ν goes as, up to some constant $z^{-\nu}$. Then doing the same calculation, we obtain, for some constant b

$$(\Phi^-, \Phi^-) = b z^{-2-2\nu},$$

which blows up as $z \to 0$. So, norm is not finite. Same conclusion is also obtained in global coordinates as well. Normalizable modes can be quantized straightforwardly. So to move to quantum description, we need normalizable modes. Non-normalizable modes on the other hand are described as non-fluctuating classical value of the field.

2.5 Breitenlohner-Freedman bound

In the above section, the solution for non-integral ν is $\Phi^{\pm}$, and it's expansion around $z=0$ is

$$\Phi^{\pm} = e^{(ik\cdot x)} z^{\frac{d}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{\nu+2m}$$

Expanding it, we have the first term proportional to, with $\nu = \Delta - \frac{d}{2}$, z^{Δ}, the second term, $z^{\Delta+2}$, third term, $z^{\Delta+4}$, and so on. The action in Poincare coordinates is,

$$S = \frac{1}{2}\int d^{d+1}x\sqrt{g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right] \tag{2.22}$$

$$= \frac{1}{2}\int d^d x dz z^{-(d+1)}\left[-z^2(\partial_t\phi)^2 + z^2(\partial_{x_i}\phi)^2 + z^2(\partial_z\phi)^2 + m^2\phi^2\right] \tag{2.23}$$

Now we put the solution, term by term, so we take the first term which is, $\phi = e^{-i\omega t + i\vec{k}.\vec{x}} z^{\Delta}$, and noting that $q^2 = k^2 - \omega^2$, we have the following form for action,

$$S^{(1)} = \frac{1}{2}\int d^{d-1}x dt dz z^{-d-1}\left[z^2\omega^2\phi^2 - z^2k^2\phi^2 + z^2(\partial_z\phi)^2 + m^2\phi^2\right] \tag{2.24}$$

$$= \frac{1}{2}\int d^{d-1}x dt dz z^{-d-1}\left[z^2(\partial_z\phi)^2 + (m^2 - z^2q^2)\phi^2\right] \tag{2.25}$$

$$= \frac{1}{2}\int d^{d-1}x dt dz (\Delta^2 + m^2 - z^2q^2) z^{2\Delta-d-1}. \tag{2.26}$$

We can then integrate z and in the limit $z \to 0$, we would have, S proportional to $z^{2\Delta-d}$. This is the leading order divergence, the next order goes like, $z^{2\Delta-d+2}$. If we put higher orders of z, in the solution of ϕ, we get even higher orders of divergence, such as $z^{2\Delta+4-d}$ and so on and so forth. So if we demand that $2\Delta > d$, then action remains finite for all orders. But this condition renders Δ_- as a non-normalizable solution. To avoid this, we consider a modified action of the form,

$$S_m = \frac{1}{2}\int d^{d+1}x\sqrt{g}\phi(-\Box + m^2)\phi. \tag{2.27}$$

In the chosen coordintes, $\Box = -z^2\partial_t^2 + z^2\partial_z^2 - (d-1)z\,\partial_z + z^2\,\nabla^2_{R^{d-1}}$, and then substituting the solution of ϕ, we have

$$S_m^{(1)} = \frac{1}{2}\int d^{d+1}x z^{-(d+1)}\left[(m^2 + z^2q^2)\phi^2 - z^2\phi\partial_z^2\phi + (d-1)z\phi\partial_z\phi\right] \tag{2.28}$$

$$= \frac{1}{2}\int d^{d+1}x\left[\Delta(d-\Delta) + m^2 + z^2q^2\right] z^{2\Delta-d-1}. \tag{2.29}$$

Since, $\Delta = \frac{1}{2}[d \pm +\sqrt{d^2+4m^2}]$,the first term term vanishes and leading divergent term is proportional to $z^{2\Delta-d+2}$. For higher order z in ϕ, the next divergent goes as

$z^{2\Delta-d+6}$ and so on. So if we take $\Delta = \frac{d-2}{2}$, then action remains finite to all order in z. In order to satisfy this condition, mass squared must satisfy,

$$-\frac{d^2}{4} < m^2 < 1 - \frac{d^2}{4}. \tag{2.30}$$

We see that both $\Delta_{\pm}$ satisfy the bound. The bound for the negative mass squared is the BF bound. As action remains finite, so does the generating functional. Generating functional is defined as, in Lorentzian picture,

$$Z_{eff}(\Phi) = \int D\Phi e^{iS[\phi]}.$$

In Euclidean picture this changes to,

$$Z_{eff}(\Phi) = \int D\Phi e^{-S[\phi]}.$$

So when S becomes infinite, this generating function goes to zero and thus the theory is unstable. So to have a stable theory we must have a finite action. And from the above analysis we can conclude that provided negative mass square remain within the bound, tachyonic fields can be stable in AdS space.

2.6 Conclusion

In this final section we briefly summarize the contents of this thesis.

We started by introducing conformal algebra and two point operator in CFT. We then turned our attention on AdS and started the chapter by introducing various coordinate systems. We then looked at the behaviour of particles in AdS. Then next object of our attention was scalar fields and solution of Klein−Gordon equation in different coordinate systems. We then discussed the normalizability conditions where we found two independent solutions, namely normalizable and non-normalizable. Normalizable solutions can be quantized while non-normalizable solutions are non-fluctuating classical value of the field. We then discussed how AdS space admits tachynoic solutions, that is fields with negative mass squared, an inherently unstable theory in Minkowski space, provided the mass squared remain within a bound called BF bounds.

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