A STUDY OF THE WEIBEL INSTABILITY

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DECLARATION

This is to certify that the project titled 'A STUDY OF THE WEIBEL INSTABILITY' is a bona fide record of work done by J Sanjay towards the partial fulfilment of the requirements of the Master of Science degree in Physics at the Indian Institute of Technology, Hyderabad.

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ABSTRACT

Plasma (an electrified gas with atoms dissociated into positive ions and negative electrons) is often said to be the most abundant form of matter in the universe. The density of a plasma can vary over 28 orders of magnitude – lower density plasmas behaving like alternating gradient synchrotrons (where single particle trajectories need to be considered) while higher density plasmas tend to behave like fluids (motions of individual particles are unimportant) – thus encouraging us to think of plasmas as a 'fourth state of matter'. In this report, we analyse the basic parameters of a plasma, briefly looking at the equations governing its behaviour to start with. We then proceed to study the Weibel Instability, and explore its evolution with time using numerical simulations.

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INTRODUCTION

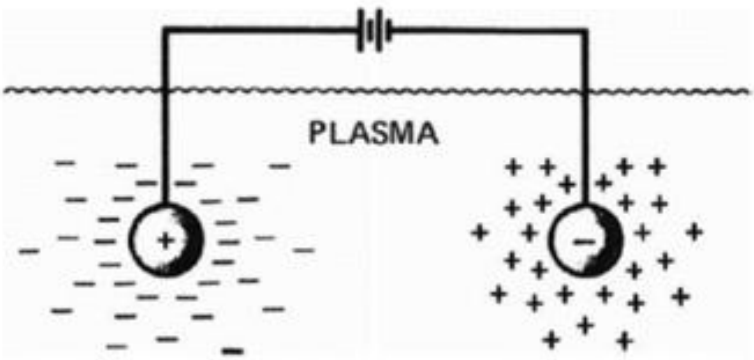
Most of the matter in the universe is said to exist in the form of a plasma – an electrified gas with positively charged ions and negatively charged electrons. Although this hypothesis is based on the existence of dark matter, the estimate is not an entirely unreasonable one. Consider the Saha equation for a moment, which predicts the amount of ionization in a gas in thermal equilibrium:

$$\frac{n_i}{n_n} \approx 2.4 \times 10^{21} \frac{T^{3/2}}{n_i} e^{-U_i/KT}$$

The left hand side is the ratio of the number density of the ionized atoms to the neutral atoms in the gas; T is the temperature of the gas in question, Ui the ionization energy, and K the Boltzmann constant. Plugging in the values corresponding to our immediate surroundings, we obtain an extremely low value for the fractional ionization of the gas, roughly 10^{-122}. This explains why we don't encounter plasmas on a daily basis. At the same time, however, it is evident that for very high values of T (say millions of degrees), the fraction takes a more

appreciable value – thus giving credibility to the hypothesis stated earlier (most astronomical bodies boast of ridiculously high temperatures).

Not all ionized gases are classified as plasmas though. Strictly speaking, plasmas are defined as 'quasineutral' gases that exhibit 'collective behaviour'. The latter simply indicates that the motion of particles in one part of a plasma is influenced not only by local conditions, but also by the conditions of the plasma in remote regions as well, owing to long range electrodynamic forces. To understand what the former term means, we need to examine the phenomenon of 'Debye shielding'. Consider the diagram below:



We insert two oppositely charged spheres into a plasma. Ideally, the two spheres would attract opposite charges to themselves, and the potential produced by the spheres in question would be shielded out perfectly by the charge clouds surrounding them. However, if the plasma has a finite temperature, some electrons near the 'edge' of the cloud would possess enough thermal energy to cross the potential barrier, thereby rendering the shielding incomplete. The thickness of the charge cloud in a plasma, called the 'Debye length', is given by:

$$\lambda_D \equiv \left(\frac{\epsilon_0 K T_e}{n e^2}\right)^{1/2}$$

By 'quasineutral', we mean that the electron and ion densities in the plasma are roughly equal (so we may refer to a common plasma density n, which appears in the equation above), but not so equal that the electromagnetic forces of interest vanish. For an ionized gas to qualify as a plasma, we require the density to be high enough for the Debye length to be negligible in comparison to the dimensions of the system L, so that any potentials in the plasma are shielded out in a distance short compared to L, leaving a majority of the plasma free of any potentials or fields. We also require enough particles in the

'Debye sphere' for the shielding to be a statistically valid concept, that is:

$$N_D = n\tfrac{4}{3}\pi\lambda_D^3 \ggg 1$$

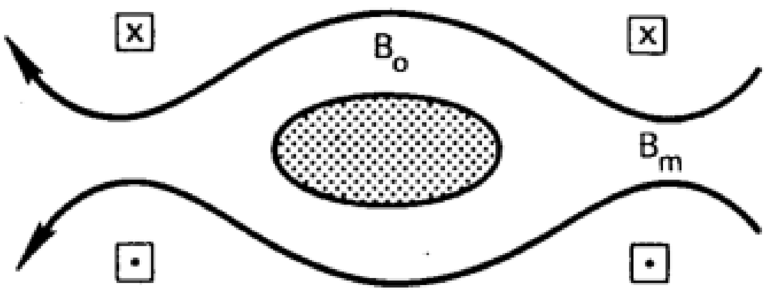
Finally, for a plasma to behave differently from a neutral gas governed by ordinary hydrodynamic forces, we require:

$$\omega\tau > 1$$

Omega being the frequency of plasma oscillations, and tau representing the mean time between collisions with neutral atoms.

MAGNETIC MIRRORS

One of the key topics of interest in the study of plasma physics is the confinement of plasmas. A magnetic 'mirror' is a way of achieving this. Consider the following field for example:



Here, we are considering a field whose gradient is parallel to the field itself. For such a configuration, we find that the magnetic moment of a charged particle, given by

$$\mu \equiv \tfrac{1}{2} m v_\perp^2 / B$$

happens to be conserved, where the component of the particle's velocity perpendicular to the field appears on the right hand side

of the equation. Applying energy conservation at the two points indicated in the diagram (B_O corresponds to the region of minimum magnetic field; B_m corresponds to the maximum value of the field, at the 'throat' of the mirror configuration), and taking into account the invariance of the magnetic moment of the particle, we find:

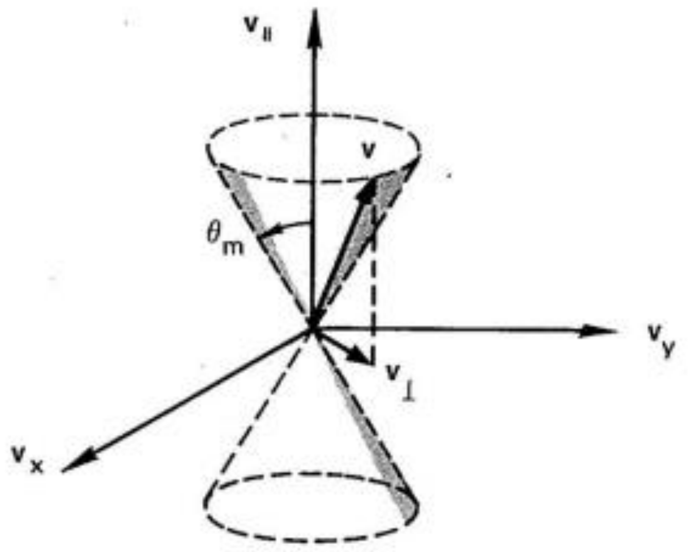
$$\frac{B_0}{B'} = \frac{v_{\perp 0}^2}{v_\perp'^2} = \frac{v_{\perp 0}^2}{v_0^2} \equiv \sin^2\theta$$

where theta is the pitch angle of the particle's orbit in the weak field region (B' is the field at the point where the particle 'reflects', that is, its velocity parallel to the field is zero at this point). Now, B' must be less than B_m, the maximum value of the field. This gives:

$$\sin^2\theta_m = B_0/B_m \equiv 1/R_m$$

where R_m is called the 'mirror ratio'. The relation above says that any particle with a pitch angle greater than $1/R_m$ will

reflect, implying that all particles having a pitch angle less than the value above will be 'lost'. This inspires the concept of a 'loss cone'. Given below is the pictorial representation of the same:



A naturally occurring example of a magnetic mirror is in the Van Allen belts. As the magnetic field of the Earth happens to be strong at the poles and weak at the equator, a natural mirror is formed, with a rather large value of R_m.

PLASMAS AS FLUIDS

If we were to try to solve the equations corresponding to plasmas exactly, it would be a ridiculously difficult job. Even assuming the electric and magnetic fields beforehand, we find it a herculean task to evaluate the trajectories of the particles exactly, and approximate methods are usually employed. To take into account the fields generated by the particles, and to then solve for the trajectories of the particles, in a time varying case no less, represents a near impossible task. Fortunately, a majority (close to 80%) of the plasma phenomena that we observe can be explained by a rather simple mathematical model. We employ the model used in fluid dynamics, where the identity of the individual particles is ignored, and only the motion of the fluid elements is taken into consideration. Only in the case of plasma, the 'fluid' contains electrical charges.

A more refined method applied to study plasmas is via the kinetic theory, but in some problems, neither model is good enough to describe the plasma's behaviour. In such cases, we are forced to follow individual trajectories of particles – this is often done with the help of computer simulations. Modern computers have enough memory to store the position and velocity components of close to 10^4 particles, but this too, is only enough to solve problems in one or two dimensions. Regardless, computer simulations are playing an important role in bridging the gap between theory and experiment, and help explain plasma behaviour in cases where kinetic theory fails miserably.

We find that the equation of motion of the plasma 'fluid' can be written down as follows:

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla \cdot \mathbf{P} - \frac{mn(\mathbf{u} - \mathbf{u}_0)}{\tau}$$

where n is the number density, **u** is the fluid velocity, **P** the stress tensor, and $\mathbf{u}_o$ the velocity of the neutral gas with which the charged fluid exchanges momentum upon collision (the last term on the right hand side simply indicates that the momentum lost is proportional to the relative velocity).

Comparing the equation above with ordinary fluids that obey the Navier Stokes equation,

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = -\nabla p + \rho\nu\, \nabla^2 \mathbf{u}$$

we see that the two equations are very similar. Electromagnetic forces (and the collision term) do not appear in the equation above (to be expected), whereas the viscosity term (the last term in the right hand side of the equation above) represents the collisional part of the difference of the gradient of the stress tensor and the gradient of the pressure, in the absence of any magnetic field.

One reason why the fluid model seems to work for plasmas is that, the magnetic field can simulate collisions, in a sense – it limits the free streaming of particles by forcing them to gyrate along Larmor orbits. Free streaming does occur, however, along the magnetic field, which indicates that the fluid model is not exactly suitable for motions in that direction. For motions perpendicular to the field though, the fluid theory serves as a good approximation.

The complete set of fluid equations for a plasma (neglecting collisions and viscosity) can be described by:

$$\sigma = n_i q_i + n_e q_e$$

$$\mathbf{j} = n_i q_i \mathbf{v}_i + n_e q_e \mathbf{v}_e$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = n_i q_i + n_e q_e$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$$

$$\nabla \cdot \mathbf{B} = 0$$

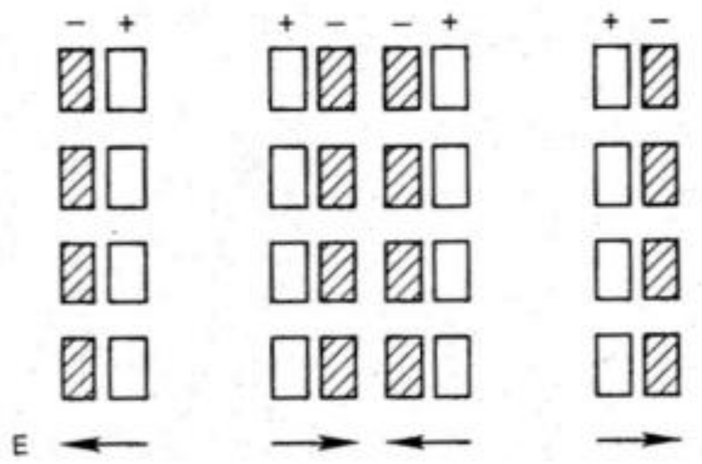
$$\mu_0^{-1} \nabla \times \mathbf{B} = n_i q_i \mathbf{v}_i + n_e q_e \mathbf{v}_e + \epsilon_0 \dot{\mathbf{E}}$$

$$m_j n_j \left[\frac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j \cdot \nabla) \mathbf{v}_j \right] = q_j n_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \nabla p_j$$

where the j corresponds to i or e (ion or electron), and the other quantities have their usual meanings.

PLASMA OSCILLATIONS

Another interesting phenomenon occurring in plasmas is plasma oscillations. If the electrons in a plasma are slightly displaced from a uniform background of positive ions, electric fields will be so established as to restore the electrons to their original positions, in a bid to maintain neutrality. The electrons, however, shoot past their original positions because of their inertia, and begin oscillating about their equilibrium positions. We can show this pictorially as follows:



The open rectangles correspond to the ion fluid elements, while the darkened ones represent the electron fluid elements. The resulting grouping of charges causes the development of a

periodic E field in space, which tries to restore the electrons to their original positions.

These oscillations are referred to as plasma oscillations, and there is a characteristic frequency associated with them, which we can show (using fluid equations) to be:

$$\omega_p = \left(\frac{n_0 e^2}{\epsilon_0 m}\right)^{1/2}$$

Note that these frequencies will have very large values, given how small m is. For instance, a plasma with a density of around 10^{18} m $^{-3}$, the frequency works out to be close to 9 GHz.

The relation above also tells us that the plasma frequency depends only on the plasma density, and not k. The group velocity, given by the derivative of the frequency with respect to k, is therefore zero.

THE WEIBEL INSTABILITY – A QUALITATIVE DESCRIPTION

Consider a simple situation – an electron beam launched into a plasma chamber. As the beam enters the chamber, it creates an azimuthal magnetic field around itself, leading to the rise of a counter electromotive force (CEMF). The CEMF interacts with the plasma electrons, resulting in a plasma current in the direction opposite to that of the beam. If the beam current and the plasma current overlap, the net current is zero, and there is no effective magnetic field.

However, if we consider a small magnetic field – a perturbation – we begin to see an increasing filamentation of the beam. This happens because the variation in the magnetic field causes electrons to "bunch" at specific pathways, creating regions of varying densities. The initial filamentation of the beam (and the plasma current) reinforces the perturbation, resulting in more filamentation, and so on. This is the primary mechanism of the Weibel instability.

The above interpretation of the Weibel instability was presented by Burton Fried, who proposed that the mechanism could be understood in a simple fashion as the superposition of two or more counter streaming electron beams. The Weibel instability differs from what is called the 'two stream instability' – the perturbations are electromagnetic instead of electrostatic, and the result is current filamentation instead of charge bunching.

MATHEMATICAL PARAMETERS – WEIBEL INSTABILITY

The growth of the instability is characterized by the growth rate γ, given by:

$$\gamma = \frac{\omega_p k v_0}{(\omega_p^2 + k^2 c^2)^{1/2}} = \omega_p \frac{v_0}{c} \frac{1}{(1 + \frac{\omega_p^2}{k^2 c^2})^{1/2}}$$

where v_0 is the unperturbed velocity of the beam, k is the wave number, c is the speed of light, and ω_p is the 'effective plasma frequency'.

The electric and magnetic fields are then calculated to be:

$$\mathbf{E}_1 = A\hat{\mathbf{z}} e^{\gamma t} e^{ikx}$$
$$\mathbf{B}_1 = \hat{\mathbf{y}} \frac{k}{\omega} E_1 = \hat{\mathbf{y}} \frac{k}{i\gamma} A e^{\gamma t} e^{ikx}$$

where A is the amplitude of the EM wave.

The relative magnitude of the fields works out to be:

$$\frac{|B_1|}{|E_1|} = \frac{k}{\gamma} \propto \frac{c}{v_0} >> 1$$

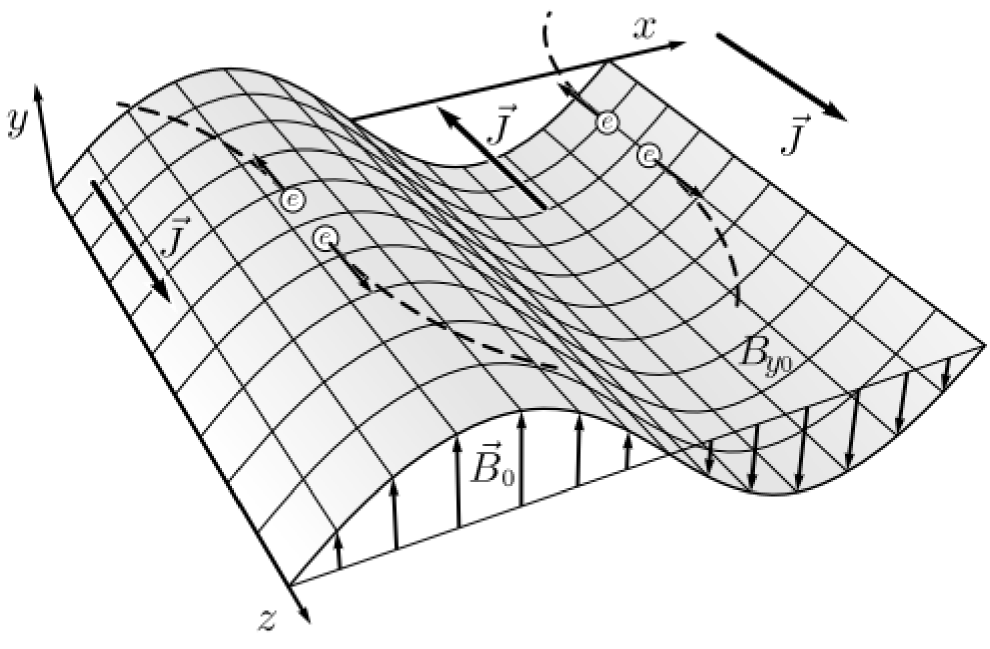
showing that it is a primarily magnetic perturbation.

SIMULATING THE INSTABILITY

As mentioned earlier, analytically solving for the solutions of the equations in plasma physics can represent a near impossible task, and numerical methods have to be employed from time to time to obtain greater insight into the problems at hand. Attempts were made to replicate the results produced in L. V. Borodachev and D.O. Kolomiets' work ('Single Species Weibel Instability of Radiationless Plasma'). In what follows, a basic understanding of the logic employed (and an interpretation of some of the results) is discussed.

The paper employs a Particle–in–Cell (PIC) numerical simulation of the electron Weibel Instability (WI) under the radiationless approximation of self consistent fields. The motivation behind the work is to supplement the classical picture of the instability by exploring the evolution of the initial (thermal) anisotropy with time, and also understand the dependency of important instability parameters on the initial degree of anisotropy.

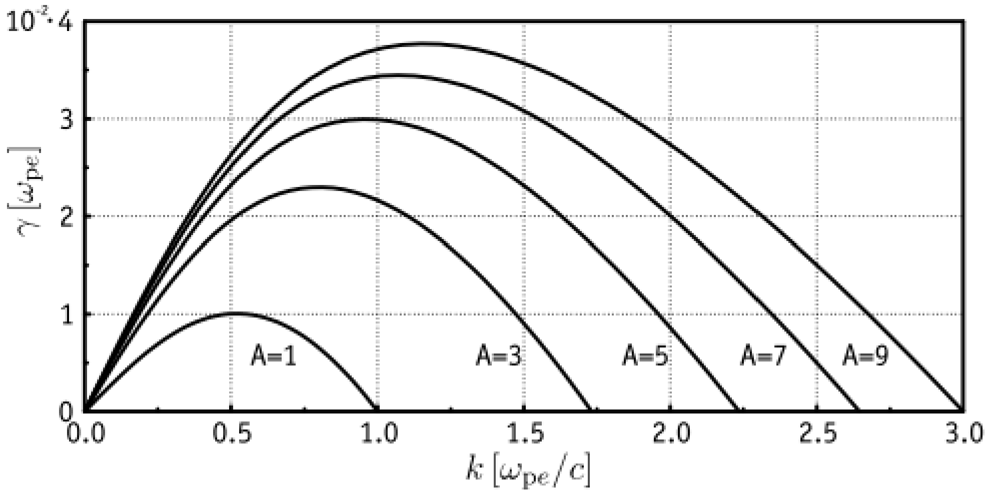
Without loss of generality, we consider an initial velocity distribution u_x, u_y, u_z such that u_z is greater than the other two components, which are assumed to be equal. We then consider a perturbation in the y direction, which leads to a Lorentz force that results in a change in the direction of a particle moving along the z axis, thereby bunching the currents into spatially separated sheets (as explained in a previous section). We expect to see two current sheets forming for each wavelength of the magnetic perturbation, as shown in the figure below:



We assume the perturbations to be exponential in nature, as we usually do in cases of linear analyses. This leads to a dispersion relation given by:

$$k_x^2 c^2 - \omega^2 = \omega_{\mathrm{pe}}^2 \left(A + \frac{\omega \left(A + 1 \right)}{k_x u_x} \, Z \left(\frac{\omega}{k_x u_x} \right) \right)$$

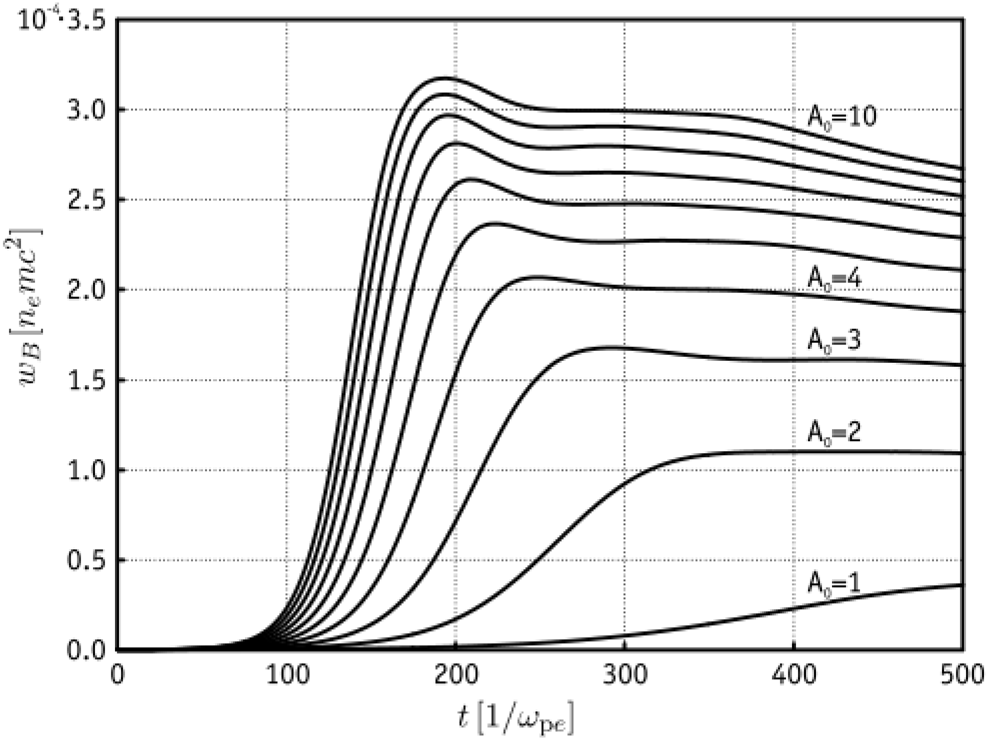
This equation is then numerically solved to see the dependence of the growth rate gamma with wave number k, which can be graphically represented as follows:



Next, we employ the Maxwell's equations, only with the elimination of the transverse displacement current, which corresponds to us neglecting radiation. Inductive effects associated with Faraday's laws are partially retained by the system, which implies that the continuity equations still hold.

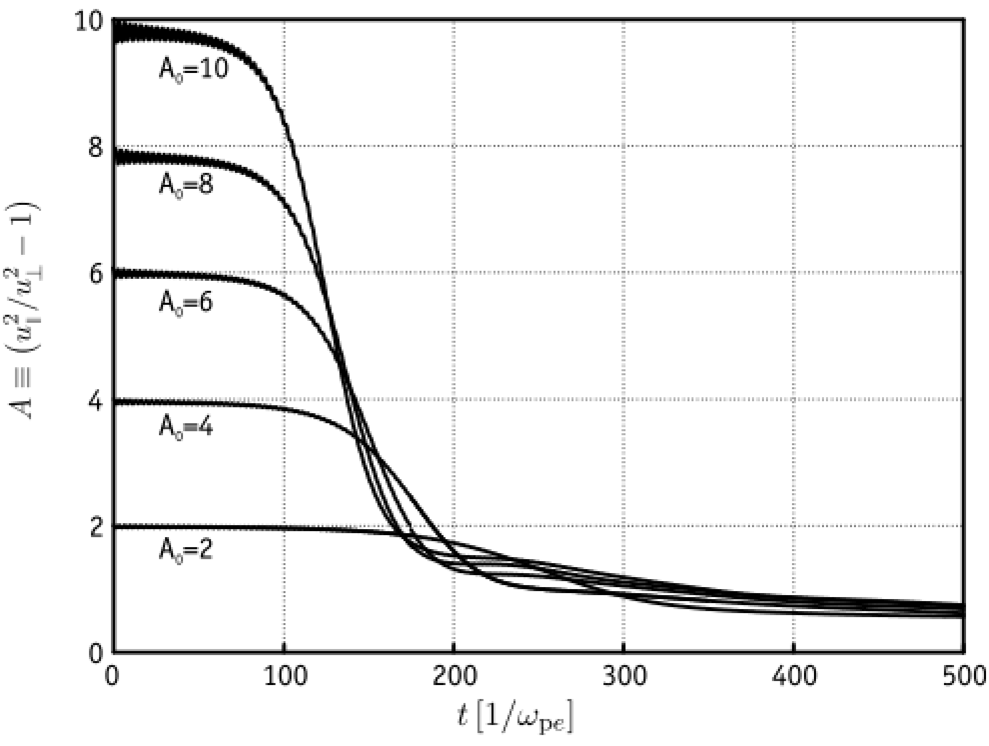
The characteristic linear size of the computational domain is chosen such that the initial values of anisotropy correspond to wavelengths that maximize the growth rate. The total number of the particles has to be such that the time frame is sufficient for the instability to develop – however, the computational cost per simulation must also be kept in mind (This works out to be roughly 10^6, considering some test runs and theoretical estimates for collisionless time for large particles).

We then proceed to plot the average magnetic field energy density against time for different values of initial anisotropy:



Notice that the initial value of the energy density is close to zero, which then increases sharply – this corresponds to different areas of current localization. Also note that the increase is larger for larger values of initial anisotropy. The peak corresponds to the end of the linear stage, when the particles become significantly magnetized on average. The non linear regime that follows corresponds to a phase where current filaments in similar directions merge to form larger structures (This overlaps with a stabilization of the magnetic field energy density).

Another plot of interest is the development of the anisotropy parameter A with time:



Observe that irrespective of the initial degree of anisotropy, the system saturates to a non zero threshold value. Our original problem described with appropriate boundary conditions mandates that waves longer than the linear size of the domain cannot exist, which explains the residual anisotropy.

Finally, we proceed to calculate (through theoretical estimates) the dependence of the magnetic energy density on the anisotropy parameter (A). This coupled with the calculation of the maximum value of A gives us the maximum value of the magnetic energy density, given by:

$$w_{B\max}^{\lim} = \frac{n_0 T_{z0}}{12} = \frac{w_{z0}^{\mathrm{kin}}}{6}$$

Simply put, this expression means that the fraction of the kinetic energy (associated with the z direction) which is eventually lost to the creation of the Weibel Instability, cannot exceed 1/6. Furthermore, using the dependency of the magnetic energy density on time, and the anisotropy parameter A, one can deduce the characteristic time of the instability, which could be

thought of as the time required to reach the maxima of the magnetic energy density.

CONCLUSION

The study of plasma physics covers a range of applications – from the development of fusion power to our understanding of astrophysical phenomena. Other important features of plasmas – such as plasma instabilities (that is, regions of space which see turbulence due to change in plasma characteristics) have important applications in space physics. An in depth study of Weibel instability in particular was undertaken.

Attempts were made to replicate the results produced in [5]. The paper was chosen as it adds to the classical understanding of the instability, by exploring the time evolutions of key parameters such as the anisotropy and the magnetic field energy density using PIC simulations. It also explores the non linear regime, which would be an extremely difficult task through analytical methods alone.

REFERENCES

[1] Introduction to Plasma Physics and Controlled Fusion, Francis F. Chen

[2] http://www.ccpp.ac.uk/index.html, CCP Plasma – Collaborative Computational Project in Plasma Physics

[3] http://boutproject.github.io/, BOUT++ Project

[4] B.D. Fried, Phys. Fluids 2, 337 (1959); Mechanism for Instability of Transverse Plasma Waves

[5] https://arxiv.org/abs/0910.0361