A study of Banach algebras and maps on it through invertible elements

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Abstract

This thesis is divided into two parts.

In the first part of the thesis we define a new class of elements within the invertible group of a complex unital Banach algebra, namely the B class. An invertible element a of a Banach algebra A is said to satisfy condition B or belong to the B class if the boundary of the open ball centered at a, with radius $\frac{1}{\|a^{-1}\|}$, necessarily intersects the set of non invertible elements. A Banach algebra is said to satisfy condition B if all its invertible elements satisfy condition B. We investigate the requirements for a Banach algebra to satisfy condition B and completely characterize commutative Banach algebras satisfying the same. In the process we prove that: A is a commmutative unital Banach algebra satisfying condition B iff $\|a^2\| = \|a\|^2$ for every invertible element a in A and A is isomorphic to a uniform algebra. We also construct a commutative unital Banach algebra, in which the property: $\|a^2\| = \|a\|^2$, is true for the invertible elements but not true for the whole algebra.

On the whole we discuss, how Banach algebras satisfying condition B can be characterized by observing the nature of the invertible elements. In the next part of the thesis, we study maps on Banach algebras, which can be studied by their behaviour on the invertible elements.

Gleason, Kahane and Żelazko([11], [13] and [36]), through the famous Gleason-Kahane-Żelazko (GKZ) theorem, showed how the multiplicativity of a linear functional on a Banach algebra, is totally dependent on what value the functional takes at the invertible elements. Mashregi and Ransford [22] generalized the GKZ theorem to modules and later used it to show that every linear functional on a Hardy space that is non-zero on the outer functions, is a constant multiple of a point evaluation. Kowalski and Słodkowski in [15] dropped the condition of linearity in the hypothesis of the GKZ theorem, they also replaced the assumption of preservation of invertibility, by a single weaker assumption, to give the same conclusion.

The second part of the thesis involves us working along the same lines as Kowalski and Słodkowski. In the hypothesis of the GKZ theorem for modules, we remove the assumption of the map being linear, tailor the hypothesis, and get a weaker Gleason-Kahane-Żelazko Theorem for Banach modules. With this, we further give applications to functionals on Hardy spaces.

Contents

Chapter 1

Preliminaries

1.1 Basic definitions and properties

In this section we introduce definitions and concepts that are used in the thesis. Whenever required, we refer to well known text books for a proofs of results that are discussed.

Let $\mathbb{F}$ denote a field that is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

Definition 1.1.1 (Algebra)**.** A vector space A over $\mathbb{F}$ is said to be an *algebra* if it is equipped with a binary operation, referred to as multiplication and denoted by juxtaposition, from $A \times A$ to A such that

(i) $(ab)c = a(bc)$

(ii) $a(b + c) = ab + bc$; $(b + c)a = ba + ca$

(iii) $\lambda(ab) = (\lambda a)b = a(\lambda b) \qquad (a, b, c \in A,\ \lambda \in \mathbb{F})$

It is said that A is a *commutative algebra* if A is an algebra and

(iv) $ab = ba \qquad (a, b \in A)$,

whereas A is said to be *unital* if it possesses a (multiplicative) unit, denoted by 1 such that

(v) $1a = a1 = a \qquad (a \in A)$.

Note that, if A is a unital algebra, then the unit element 1 is unique. Because if 1 and $1'$ are unit elements, then $1 = 11' = 1'$.

Definition 1.1.2 (Normed algerba)**.** A normed linear space $(A, \|.\|)$ over $\mathbb{F}$ is said to be a *normed algebra* if A is an algebra and

$$\|ab\| \leq \|a\|\|b\| \qquad (a, b \in A).$$

$(A, \|.\|)$ is said to be a unital normed algebra if A has a unit element and $\|1\| = 1$. A normed algebra A is said to be a *Banach algebra* if the normed linear space $(A, \|.\|)$ is a Banach space.

Note that multiplication in a normed algebra A is a continuous mapping of $A \times A$ into A (Proposition 2.4 [3]). The standard examples of normed algebras can be broadly divided into three classes: function algebras, algebras of linear transformations, and convolution algebras. Some of the commonly used algebras from each class are listed below.

Example 1.1.3. Let K be a compact Hausdorff space. By $C(K)$ we denote the algebra of all continuous complex-valued functions on K. The algebra operations are the usual ones of point-wise addition, multiplication, and scalar multiplication. Consider the uniform norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \qquad (f \in C(K)).$$

Then $(C(K), \|.\|_\infty)$ is a commutative unital Banach algebra with the unit element being the constant function 1.

Example 1.1.4. Let X be a non zero Banach space. $B(X)$ will denote the algebra of all bounded linear transformations from X to itself. With the usual notion of composition of linear transformations as multiplication, and the operator norm defined as

$$\|T\|_{op} = \sup_{\|x\| \leq 1} \|Tx\| \qquad (T \in B(X)),$$

$B(X)$ is a unital Banach algebra which is non-commutative (except when X is of dimension 1). The unit element in $B(X)$ is the identity map.

Example 1.1.5. Let G be a locally compact Abelian group and let m_G be the Haar measure on G. $L^1(G)$ will denote the set of all Borel functions on G such that

$$\|f\|_1 = \int_G |f| dm_G \qquad (f \in L^1(G))$$

is finite. $L^1(G)$ is a commutative Banach algebra, if multiplication is defined by convolution ([31],Theorem 1.1.7). And has a unit element iff G is discrete ([31], 1.1.8).

For a detailed study of further examples, see [25] Section 1.5-1.9 , Appendix in [27] and [19] Section 1.2 .

1.2 The group of invertible elements in a Banach Algebra

Let A be a unital Banach algebra. An element $a \in A$ is said to have an inverse if there exists $b \in A$ such that

$$ab = ba = 1.$$

If $a \in A$ has an inverse, then a is said to be invertible and the inverse is denoted by a^{-1}. The element a is said to be singular if it is not invertible. It is easy to check that an element can have at most one inverse. The family of invertible elements in A is denoted by A^{-1} and the set of singular elements is denoted as $\mathrm{Sing}(A)$. It is clear that A^{-1} is a multiplicative group.

Example 1.2.1. Let $A = C(K)$ where K is a compact Hausdorff space. Then

$$A^{-1} = \{f \in A \mid f(t) \neq 0 \text{ for every } t \in K\}$$

Example 1.2.2. Let $A = B(X)$ where X is a non zero Banach space. Then by the *Bounded Inverse Theorem*, an element of A is invertible in A iff it is bijective.

Example 1.2.3. Let $A = L^1(G)$ where G be a discrete (hence locally compact) Abelian group. Let us write $\int_G f(x)dx$ in place of $\int_G f dm_G$, hence dx will denote the integration with respect to Haar measure. Let Γ denote the dual group of G ([31], Section 1.2). If G is discrete, Γ is compact ([31], Theorem 1.2.5). Let $A(\Gamma)$ be the set of all functions $\hat{f}$, where $\hat{f}$ is defined on Γ as

$$\hat{f}(\gamma) = \sum_G f(x)\gamma(-x)dx \qquad (\gamma \in \Gamma, f \in L^1(G)). \tag{1.1}$$

We will see later (Example 1.5.4 and Theorem 1.5.2) that f is invertible in $L^1(G)$ iff $\hat{f}$ is invertible in $C(\Gamma)$.

Algebraically A^{-1} is a group and topologically it is an open set. We get this by the following series and estimate.

Theorem 1.2.4 (**Neumann series**). *Let A be a unital Banach algebra. If a is an element of A with $\|a\| < 1$, then $1 - a$ is invertible with inverse*

$$(1-a)^{-1} = 1 + \sum_{n=1}^{\infty} a^n$$

Proof. As $\|a\| < 1$, the series $1 + \sum_{n=1}^{\infty} \|a^n\|$ converges. Therefore $1 + \sum_{n=1}^{\infty} a^n$ converges as A is a Banach space, with sum $s \in A$. Let $s_n = 1 + a + \cdots + a^{n-1}$. Then $s_n \to s$ and $\|a^n\| \to 0$ as $n \to \infty$. And

$$(1-a)s_n = s_n(1-a) = 1 - a^n.$$

Then by continuity of multiplication we have

$$(1-a)s = s(1-a) = 1.$$

Hence proved. □

Theorem 1.2.5. *If A is a unital Banach algebra, then A^{-1} is an open subset of A.*

Proof. We will fix the notation for an open ball, centered at a with radius $r > 0$, as

$$B(a, r) := \{x \in A : \|x - a\| < r\}.$$

To show that A^{-1} is open in A, we need to show that given $a \in A^{-1}$ there exists a $r > 0$ such that $B(a, r) \subset A^{-1}$. We choose $r = \frac{1}{\|a^{-1}\|} > 0$. Let $x \in B\left(a, \frac{1}{\|a^{-1}\|}\right)$. Then

$$\|1 - a^{-1}x\| = \|a^{-1}(a - x)\| \leq \|a^{-1}\|\|a - x\| < 1.$$

From Theorem 1.2.4 we have $a^{-1}x \in A^{-1}$. And hence $x = aa^{-1}x \in A^{-1}$. Thus

$$B\left(a, \frac{1}{\|a^{-1}\|}\right) \subseteq A^{-1}.$$

Hence proved. □

1.3 The Spectrum

Throughout this section A will denote a unital algebra over $\mathbb{C}$.

Definition 1.3.1. Let A be a algebra. The spectrum of an element a of A is the set $\sigma(a)$ of complex numbers defined as follows:

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda 1 - a \notin A^{-1}\}.$$

Example 1.3.2. Let $A = C(K)$ where K is a compact Hausdorff space. Then $\sigma(f) = f(K) =$ Range of f, for every f in A.

Example 1.3.3. For a detailed account of the spectrum of an element in $B(X)$ where X is a non zero Banach space, see [1], Section 4.6.

Example 1.3.4. Let $A = L^1(G)$ where G be a discrete (hence locally compact) Abelian group. Then $\sigma(f)_{L^1(G)} = \sigma(\hat{f})_{C(\Gamma)}$, where $\hat{f}$ is defined as in (1.1). See Example 1.5.4 and Theorem 1.5.2.

Theorem 1.3.5. *Let A be a Banach algebra, and let $a \in A$. Then $\sigma(a)$ is a non-empty, compact subset of the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\}$.*

Proof. See [37], Theorem 3.1 and 3.3. □

Theorem 1.3.6 (Spectral mapping theorem). *Let A be a Banach algebra, let $a \in A$, and let f be analytic on $|z| \leq \|a\|$. Then*

$$\sigma(f(a)) = \{f(\lambda) : \lambda \in \sigma(a)\}.$$

Proof. See [37], Theorem 5.4. □

Definition 1.3.7. Let A be a Banach algebra, and let $a \in A$. The spectral radius $r(a)$ is defined as

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Theorem 1.3.8 (Spectral radius formula). *Let A be a Banach algebra, and let $a \in A$. Then*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Proof. See [37], Theorem 5.5. □

Proposition 1.3.9. *Let A be a Banach algebra and $a, b \in A$. Then*

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}.$$

Proof. See [37], Proposition 3.4. $\square$

Definition 1.3.10. Let $a \in A$. We define

$$\exp(a) = e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!},$$

where $a^0 = 1$. And

$$\exp(A) = \{\exp(a) : a \in A\}.$$

The convergence of the above series is a consequence of A being a Banach space and the inequality $\|a^n\| \leq \|a\|^n$.

Theorem 1.3.11. *Let $a, b \in A$ and $ab = ba$. Then*

(i) $\exp(a + b) = \exp(a)\exp(b)$

(ii) $\exp(a) \in A^{-1}$ *and* $\exp(a)^{-1} = \exp(-a)$.

Proof. See [1], Proposition 4.98. $\square$

Now we see a condition for the commutativity of a Banach algebra which is based on results from [2] and [23] .

Theorem 1.3.12. *Let A be a Banach algebra such that for some $\kappa > 0$, $\|a\|^2 \leq \kappa\|a^2\|$ for every $a \in A$. Then A is commutative.*

Proof. Given $\|a\|^2 \leq \kappa\|a^2\|$ for every $a \in A$, by induction we get that

$$\|a\| \leq \kappa^{1-2^{-n}} \|a^{2^n}\|^{2^{-n}} \qquad (n \in \mathbb{N}).$$

Using the *spectral radius formula* (Theorem 1.3.8), we get that $\|a\| \leq \kappa r(a)$ for every $a \in A$. Using Proposition 1.3.9, we get for every $a, b \in A$

$$\|ab\| \leq \kappa\|ba\|. \tag{1.2}$$

Let $a, b \in A$, $f \in X'$ (the space of all continuous linear functionals on X) and define F on $\mathbb{C}$ as

$$F(z) = f(\exp(za)b\exp(-za)) \qquad (z \in \mathbb{C}).$$

Then F is an entire function and for $z \in \mathbb{C}$,

$$|F(z)| \leq \kappa \|f\| \|b\| \qquad \text{(using (1.2))}.$$

So by *Liouville's Theorem* F is an entire function and the coefficient of z in the power series expansion of F is zero. And hence $f(ab - ba) = 0$. Since $f \in X'$ was arbitary, we have the result. □

As an immediate consequence of this theorem we get the following.

Corollary 1.3.13. *Let A be a Banach algebra such that $\|a\|^2 = \|a^2\|$ for every $a \in A$. Then A is commutative.*

Note that a Banach algebra may be commutative but $\|a^2\| \neq \|a^2\|$ for some $a \in A$, as seen in this example.

Example 1.3.14. Consider the commutative unital Banach algebra $(L^1(\mathbb{Z}), \|.\|_1)$. Let $f \in (L^1(\mathbb{Z}), \|.\|_1)$ be defined as

$$f(n) = \begin{cases} 1, & \text{if } n = 1 \\ \text{-1}, & \text{if } n = 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Check that $\|f\|^2 = 9$ but $\|f^2\| = 7$.

1.4 Multiplicative linear functionals and Maximal ideals

Let A denote a complex unital Banach algebra. The proof of results in this section can be found in [37], Chapter 4.

Definition 1.4.1. A multiplicative linear functional (or a character) on A is a non zero linear functional φ on A such that

$$\varphi(ab) = \varphi(a)\varphi(b) \qquad (a, b \in A),$$

i.e a non- zero algebra homomorphism of A onto $\mathbb{C}$.

Theorem 1.4.2. *Let φ be a multiplicative linear functional on A. Then φ is continuous, and $\|\varphi\| \leq 1$. If A has a unit element, then $\varphi(1) = 1$.*

Definition 1.4.3. Let A be a Banach algebra and let $\mathcal{M}_A$ denote the set of all multiplicative linear functionals on A. $\mathcal{M}_A$ will be called the *maximal ideal space* (or *character space*) of A because of the theorem stated below.

Theorem 1.4.4. *Let A be a commutative unital Banach algebra. Then $\mathcal{M}_A \neq \emptyset$ and the mapping $\varphi \longrightarrow \ker \varphi$ is a bijection from $\mathcal{M}_A$ onto the set of all maximal ideals of A.*

Theorem 1.4.5. *$\mathcal{M}_A$ is a compact Hausdorff space with the weak star topology inherited from the dual space of A.*

A characterization of multiplicative linear functionals is given by the Gleason-Kahane-Żelazko (GKZ) Theorem

Theorem 1.4.6 (GKZ Theorem). *If $\varphi : A \longrightarrow \mathbb{C}$ is a non zero linear functional, then the following are equivalent.*

(i) $\varphi(1) = 1$ *and* $\varphi(a) \neq 0$ *for every* $a \in A^{-1}$.

(ii) $\varphi(a)1 - a \notin A^{-1}$ *for all* $a \in A$.

(iii) φ *is multiplicative.*

Example 1.4.7. Let $A = C(K)$ where K is a compact Hausdorff space. For every $x \in K$, there exists a multiplicative linear functional φ_x on A defined as

$$\varphi_x(f) = f(x) \qquad (f \in A).$$

In [37], Theorem 6.2 one can see that these are the only multiplicative linear functionals on A. Thus $\mathcal{M}_A = K$.

Example 1.4.8. Consider $A = B(X)$ where $X = \mathbb{C}^n$. Then $\mathcal{M}_A = \{0\}$. Because if there exists a multiplicative linear functional φ on A, then by the uniqueness of the trace function, φ has to be a constant multiple of the trace function. But the trace function is multiplicative only when $n = 1$. Now if $X = H$ where H is a Hilbert space with $dim H > 1$ then $B(H)$ has an embedded copy of $M_2(\mathbb{C})$. A multiplicative linear functional will have to restrict to a multiplicative linear functional on $M_2(\mathbb{C})$, but that does not exist.

Example 1.4.9. Let $A = L^1(G)$ where G is a discrete (hence locally compact) Abelian group. Let Γ be the dual group of G. For a $\gamma \in \Gamma$, the map $f \longrightarrow \hat{f}(\gamma)$ $(f \in A)$ is a multiplicative linear functional on A and is not identically 0. Also every multiplicative linear functional on A is obtained in this way. See [31] Theorem 1.2.2. Therefore here $\mathcal{M}_A = \Gamma$.

1.5 The Gelfand transform

Definition 1.5.1. Let A be a commutative Banach algebra. The map $a \longrightarrow \hat{a}$, of A into $C(\mathcal{M}_A)$ defined as

$$\hat{a}(\varphi) = \varphi(a) \qquad (a \in A, \varphi \in \mathcal{M}_A)$$

is called the *Gelfand Transform* of A. The image of A under the *Gelfand Transform* will be denoted by $\hat{A}$.

Theorem 1.5.2. *Let A be a commutative unital Banach algebra.*

(i) *The Gelfand Transform is a contractive Banach algebra homomorphism from A into $C(\mathcal{M}_A)$.*

(ii) *$\hat{A}$ separates the points of $\mathcal{M}_A$.*

(iii) *a is invertible in A iff $\hat{a}$ is invertible in $C(\mathcal{M}_A)$.*

(iv) *$\sigma(a)$=$\sigma(\hat{a})$= Range of $\hat{a}$* $\qquad (a \in A)$.

(v) $r(a) = \|\hat{a}\|_\infty = \sup\{|\varphi(a)| : \varphi \in \mathcal{M}_A\} \qquad (a \in A)$.

Proof. See [3], Theorem 17.4. $\square$

Example 1.5.3. If $A = C(K)$, then we identified the maximal ideal space of A with K. And hence the Gelfand Transform of A is the identity mapping.

Example 1.5.4. Let $A = L^1(G)$ where G be a discrete (hence locally compact) Abelian group. Then the map $f \longrightarrow \hat{f}$ is actually the Gelfand transform of A into $C(\Gamma)$.

Definition 1.5.5. The *radical* of A is the subset of A denoted as rad(A) and defined as

$$\text{rad}(A) = \cap\{\ker\varphi : \varphi \in \mathcal{M}_A\}.$$

A Banach algebra is called *semi-simple* if rad$(A) = \{0\}$.

Theorem 1.5.6. *Let A be a commutative Banach algebra. Then the following are equivalent*

(i) *A is semi-simple.*

(ii) *$\mathcal{M}_A$ separates the points of A.*

(iii) *The Gelfand Transform is a monomorphism of* A *into* $C(\mathcal{M}_A)$.

(iv) *The spectral radius is a norm on* A.

Proof. See [3], Corollary 17.7. $\square$

Theorem 1.5.7 ([37], Section 5.5)**.** *Let* A *be a commutative unital Banach algebra. The Gelfand Transform is an isometry iff* $\|a^2\| = \|a\|^2$ *for every* $a \in A$.

Proof. If the Gelfand Transform is an isometry, then from Theorem 1.5.2 (v.) we have $r(a) = \|a\|$ for every $a \in A$. Hence by the *spectral mapping theorem* we have

$$\|a^2\| = r(a^2) = r(a)^2 = \|a\|^2.$$

Now suppose $\|a^2\| = \|a\|^2$ for every $a \in A$, then

$$\|a^4\| = \|a\|^4, \ldots, \|a^{2^n}\| = \|a\|^{2^n}, \qquad (n \in \mathbb{N}).$$

Hence by Theorem 1.5.2 (v.) and the *spectral radius formula*

$$\|a\| = \lim_{n\to\infty} \|a^n\|^{1/n} = r(a) = \|\hat{a}\|_\infty.$$

Hence proved. $\square$

Chapter 2

Open ball centered at an invertible element

In this chapter we will first introduce what would be called the condition (B) for a complex unital Banach algerba. Next we briefly make ourselves familiar to the theory of uniform algebra. Later we explore the Banach algebras which satisfy condition (B) and show that a commutative unital Banach algebras which satisfies condition (B) is isomorphic to a uniform algebras. Finally, we completely characterize commutative Banach algebras satisfying the same by constructing a commutative unital Banach algebra that satisfies condition (B) and is isomorphic to a uniform algebra via the Gelfand transform, but is not isometrically isomorphic.

2.1 Introduction

Let A be a complex unital Banach algebra. Let $a \in A^{-1}$, then from Theorem 1.2.5 we have A^{-1} is open in A with

$$B\left(a, \frac{1}{\|a^{-1}\|}\right) \subseteq A^{-1} \qquad (a \in A^{-1}).$$

We also have

$$\begin{aligned} r(a^{-1}) = \sup\left\{\left|\frac{1}{\lambda}\right| : \lambda \in \sigma(a)\right\} &= \frac{1}{\inf\{|\lambda| : \lambda \in \sigma(a)\}} \\ &= \frac{1}{\inf\{\|a - (a - \lambda 1)\| : \lambda \in \sigma(a)\}} \\ &= \frac{1}{\operatorname{dist}(a, S')}, \end{aligned}$$

where $S' = \{a - \lambda 1 : \lambda \in \sigma(a)\} \subset \mathrm{Sing}(A)$). Hence combining them we get

$$\frac{1}{\|a^{-1}\|} \leq \mathrm{dist}(a, \mathrm{Sing}(A) \leq \frac{1}{r(a^{-1})}.$$

We say that an element $a \in A^{-1}$ satisfies condition (B) (or belongs to the B class) if the boundary of the ball $B\left(a, \frac{1}{\|a^{-1}\|}\right)$ necessarily intersects the set of singular elements. A Banach algebra A satisfies condition (B) if every $a \in A^{-1}$ satisfies condition (B).

Before discussing the motivation behind studying Banach algebras that satisfy condition (B), we will familiarize ourselves with the notion of the ϵ-*condition spectrum*. Let $a \in A^{-1}$, the *condition number* of a is defined as $\|a\|\|a^{-1}\|$ and denoted by $\kappa(a)$. Note that $\kappa(a) \geq 1$ for every $a \in A^{-1}$. We make a convention that $\kappa(a) = \infty$ if $a \in \mathrm{Sing}(A)$. For a fixed $0 < \epsilon < 1$, define

$$\Omega_\epsilon = \left\{a \in A^{-1} : \kappa(a) < \frac{1}{\epsilon}\right\}.$$

Now for that fixed ϵ, the ϵ-*condition spectrum* of a is defined as

$$\sigma_\epsilon(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin \Omega_\epsilon\}.$$

Condition (B) was first encountered by Kulkarni and Sukumar in [18] where Corollary 2.21, as stated below, says that in a Banach algebra A satisfying condition (B), every member of the ϵ-condition spectrum of an element a in A, is the spectral value of a perturbed a.

Result 2.1.1. Let A be a complex Banach algebra with the following property:

$$\forall a \in A^{-1}, \exists b \in \mathrm{Sing}(A) \text{ such that } \|a - b\| = \frac{1}{\|a^{-1}\|}.$$

Let $a \in A$ such that A is not a scalar multiple of identity, then $\lambda \in \sigma_\epsilon(a)$ iff $\exists b \in A$ with $\|b\| \leq \epsilon\|\lambda - a\|$ such that $\lambda \in \sigma(a + b)$.

Theorem 3.3 in [16] further states that if A is Banach algebra satisfying condition (B), and $a \in A$, then for every open set Ω containing $\sigma(a)$, there exists $0 < \epsilon < 1$ such that $\sigma_\epsilon(a) \subset \Omega$. For examples and properties of ϵ-condition spectrum see [18]. The ϵ-condition spectrum is a handy tool in the numerical solutions of operator equations. One of the applications is as follows: Suppose X is a Banach space and $T : X \to X$ is a bounded linear map. If $\lambda \notin \sigma_\epsilon(T)$, then the operator equation $Tx - \lambda x = y$ has a stable solution for every $y \in X$.

A similar (to the B class) but a more particular class of operators on a Hilbert space, called the G_1 class of operators have received considerable attention in literature earlier.

Definition 2.1.2 ([26])**.** Let H be a Hilbert space. An operator $T \in B(H)$ is said to be G_1(or to satisfy G_1 condition or to be of class G_1) if

$$\|(T-\lambda 1)^{-1}\|_{op} = \frac{1}{\mathrm{dist}(\lambda, \sigma(T))} \qquad \lambda \notin \sigma(T).$$

Essentially, normal operators ($TT^* = T^*T$) and more generally hyponormal operators ($TT^* - T^*T \geq 0$) satisfy G_1 condition. Analogous results concerning elements in a C*-algebra can be made using the fact that every C*-algebra can be embedded in $B(H)$ for some Hilbert space H (See [37], Section 14.3). Some other interesting results about G_1 operators are as follows.

Result 2.1.3 ([24])**.** If $T \in B(H)$ satisfies G_1 condition, then $Co(\sigma(T)) = W(T)$. Where $Co(\sigma(T))$ is the convex hull of $\sigma(T)$ and $W(T)$ is the closure of the numerical range of T, where the numerical range of T is the set:

$$\{\langle Tx, x\rangle : x \in H, \|x\| = 1\}.$$

Result 2.1.4 ([33])**.** If T satisfies G_1 condition and the underlying space is finite dimensional, then T is normal.

2.2 Uniform Algebras

In this subsection we recall the basics of the theory of uniform algebras.

Definition 2.2.1. Let K be a compact Hausdorff space. A *uniform algebra* on K is a subalgebra $\mathfrak{A}$ of the complex algebra $C(K)$ satisfying the following

(i) $\mathfrak{A}$ is closed in $C(K)$ under the uniform norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \qquad (f \in \mathfrak{A}).$$

(ii) $\mathfrak{A}$ contains the constant functions.

(iii) the functions of $\mathfrak{A}$ separate the points of K, i.e for every $x \neq y \in K$, there exists $f \in \mathfrak{A}$ such that $f(x) \neq f(y)$.

Every uniform algebra on K is a commutative unital Banach algebra with respect to the uniform norm. By *Urysohn's Lemma*, $C(K)$ separates the points of K and hence $C(K)$ itself is a uniform algebra on K.

If $\mathfrak{A}$ is a uniform algebra, then clearly

$$\|f^2\|_\infty = \|f\|_\infty^2 \qquad (f \in \mathfrak{A}).$$

Conversely if $(A, \|.\|)$ is a Banach algebra satisfying

$$\|a^2\| = \|a\|^2 \text{ for every } a \in A, \tag{2.1}$$

then from Theorem 1.5.7 the *Gelfand Transform* is an isometry from A onto $\hat{A}$. Note that in this case $\hat{A}$ is a uniformly closed point separating sub algebra of $C(\mathcal{M}_A)$. That is, condition 2.1 is a necessary and sufficient condition for the *Gelfand Transform* to carry A isometrically onto a uniform algebra on the maximal ideal space of A. One should note that we might have that $\|a\|^2 = \|a^2\|$ for every $a \in A$ but still A may not be a uniform algebra, as we demonstrate in the following example.

Example 2.2.2. Let $A = \{f \in C[0,1] : f(0) = f(1)\}$. Note that $\|f^2\|_\infty = \|f\|_\infty{}^2$ for every $f \in A$, but $(A, \|.\|_\infty)$ is not a uniform algebra on $[0,1]$ as it does not separate the points 0 and 1 of $[0,1]$.

A model example of a proper uniform algebra is the Disc algebra $(A(\bar{\mathbb{D}}), \|.\|_\infty)$,

Example 2.2.3. Let $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $A(\bar{\mathbb{D}})$ be the sub algebra of those functions in $C(\bar{\mathbb{D}})$ which are analytic on $\mathbb{D}$. Note that $A(\bar{\mathbb{D}})$ is closed in $(C(\bar{D}), \|.\|_\infty)$. Because if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $A(\bar{\mathbb{D}})$ with $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$ for some $f \in C(\bar{D})$, then by *Cauchy's Theorem*,

$$\int_\gamma f_n(z)dz = 0 \qquad (n \in \mathbb{N})$$

for every simple closed contour γ in $\mathbb{D}$. Using *Morera's theorem* we get that f is analytic on $\mathbb{D}$ and hence $f \in A(\bar{\mathbb{D}})$. Clearly we have that $(A(\bar{\mathbb{D}}), \|.\|_\infty)$ is a uniform algebra on $\bar{\mathbb{D}}$. Also each function in $A(\bar{\mathbb{D}})$ is the uniform limit on $\bar{\mathbb{D}}$ of some sequence of polynomials (See [1], Theorem 2.86). The maximal ideal space of $A(\bar{\mathbb{D}})$ is $\bar{\mathbb{D}}$ and the Gelfand transform is basically the inclusion map ([37], Theorem 6.1).

Example 2.2.4. Some other standard uniform algebras for any non- empty compact subset K of $\mathbb{C}$ are $P(K)$ and $R(K)$ where

(i) $P(K)$ is the set of functions on K which can be uniformly approximated on K by polynomials.

(ii) $R(K)$ is the set of functions on K which can be uniformly approximated on K by rational functions with poles off K.

We have

$$P(K) \subseteq R(K) \subseteq A(K) \subseteq C(K)$$

where $A(K)$ is the set of those functions in $C(K)$ which are analytic on the interior of K. From Example 2.2.3 we see that $A(\bar{\mathbb{D}}) = P(\bar{\mathbb{D}})$. From Runge's theorem it follows that $P(K) = R(K)$ iff $\mathbb{C} \setminus K$ is connected (see [1], Corollary 4.85). Clearly we have $A(K) = C(K)$ iff interior of $K = \emptyset$.

For further reference to theory of uniform algebras see [9], [4] and [34]. Using Theorem 2.4.1, we will see that every uniform algebra satisfies condition (B). A (unital) *Banach function algebra* satisfies the same axioms as a uniform algebra except that the complete norm on the algebra need not be the uniform norm. A Banach function algebra may not satisfy condition (B) (using Theorem 2.4.2) we demonstrate in the this example.

Example 2.2.5. Let $C^{(1)}[0,1]$ be the space of all complex valued functions on [0,1] with continuous first order derivative equipped with the norm

$$\|f\| = \|f\|_\infty + \|f^{(1)}\|_\infty \quad \text{for all } f \in C^{(1)}[0,1].$$

Then $(C^{(1)}[0,1], \|.\|)$ is a commutative semi simple Banach function algebra. Consider the function $f(x) = e^x$ for all $x \in [0,1]$ and notice that $\|(f^{-1})^2\| \neq \|f^{-1}\|^2$.

2.3 Condition (B)

In this section we study one of the main concepts of the thesis (Condition (B)), hence we formally define it and develop the results obtained regarding the same.

Definition 2.3.1. An element $a \in A^{-1}$ is said to satisfy condition (B) if

$$\overline{B\left(a, \frac{1}{\|a^{-1}\|}\right)} \cap \operatorname{Sing}(A) \neq \phi. \tag{2.2}$$

We say a Banach algebra A satisfies condition (B) if every $a \in A^{-1}$ satisfies condition (B).

We can see that if $a \in A^{-1}$ satisfies condition (B), then for every $\epsilon > 0$

$$B\left(a, \frac{1}{\|a^{-1}\|} + \epsilon\right) \cap \text{Sing}(A) \neq \phi,$$

thus

$$B\left(a, \frac{1}{\|a^{-1}\|} + \epsilon\right) \not\subseteq A^{-1},$$

hence,

$$\textit{the largest open ball centered at } a\textit{, contained in } A^{-1}\textit{, is of radius } \frac{1}{\|a^{-1}\|}. \tag{2.3}$$

Hence (2.2) implies (2.3). Note that (2.3) implies (2.2) in some special case like when

(i) $\text{dist}(a, \text{Sing}A) = \frac{1}{r(a^{-1})}$.

(ii) The Banach algebra A is finite dimensional, where the closed balls are compact.

Note: *One can also explore these Banach algebras where the largest open ball centered at an invertible a, contained in A^{-1}, is of radius $\frac{1}{\|a^{-1}\|}$.*
But for this thesis, we stick to examining the ones which satisfy (2.2). We attempt on understanding as well as characterizing all Banach Algebras which satisfy condition (B) and which do not. To actualize the above mentioned objective, we discuss the observations made in some classical Banach Algebras. We start by inspecting some examples and progress further. The basic approach adopted to show if $a \in A^{-1}$ satisfies condition (B) is calculating $\|a^{-1}\|$ and producing an element $s \in \text{Sing}(A)$ such that $\|a - s\| = \frac{1}{\|a^{-1}\|}$. We obtain the following examples that come immediately.

Example 2.3.2. Consider the space of all complex numbers with respect to usual addition and multiplication. Let $z = a + ib, z \neq 0$ then,

$$\left|\frac{1}{z^{-1}}\right| = \frac{1}{|z^{-1}|} = \sqrt{a^2 + b^2} = |z|.$$

Hence we have

$$B\left(z, \frac{1}{|z^{-1}|}\right) = B(z, |z|)$$

So we have the singular element zero on the boundary.

Example 2.3.3. Consider $C(K)$ for a compact Hausdorff space K. Let $f \in C(K)$ be invertible. As K is compact, $\inf|f|$ is attained at some point say x_0 in K. From Example 1.2.1 we have

$$f^{-1}(x) = \frac{1}{f(x)} \qquad (x \in X).$$

Hence,

$$\|f^{-1}\|_\infty = \sup\left\{\left|\frac{1}{f(x)}\right| : x \in X\right\} = \frac{1}{\inf\{|f(x)| : x \in X\}} = \frac{1}{|f(x_0)|}.$$

Now consider the function $g = f - f(x_0) \in C(K)$. Then $g(x_0) = 0$. And

$$\|f - g\|_\infty = |f(x_0)| = \frac{1}{\|f^{-1}\|_\infty}.$$

So, we have found a singular function g on the boundary.

Next we give an example of a Banach algebra that does not satisfy condition B.

Example 2.3.4 ([17], Ex 2.10)**.** Let $A = \left\{a \in \mathcal{M}_2(\mathbb{C}) : a = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}\right\}$, equipped with the norm

$$\|a\| = |\alpha| + |\beta|.$$

Then $(A, \|.\|)$ is a Banach algebra. Note that any $b = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \in A$ is invertible iff $\alpha \neq 0$. Let $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then $a^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $\|a^{-1}\| = 2$. Suppose there exists an element $b = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \in \mathrm{Sing}(A)$ such that $\|a - b\| = \frac{1}{\|a^{-1}\|} = \frac{1}{2}$. We have $|1| + |1 - \beta| = \frac{1}{2}$, which is not possible.

Remark 2.3.5. A Banach algebra in which every $a \in A^{-1}$ has condition number $\kappa(a)$ equal to 1 satisfies condition (B), with the element 0 on the boundary. But such algebras are actually isometrically isomorphic to $\mathbb{C}$ ([19], Theorem 1.5.2).

Now we move to a more general class of Banach algebras to check if they satisfy condition (B). Using the polar decomposition of invertible elements in a C* algebra we easily prove the following result.

Theorem 2.3.6. *Let A be a unital C^*-algebra, then A satisfies condition (B).*

Proof. If $a \in A^{-1}$, then a has a unique decomposition $a = bu$ where $b \geq 0$ and u is a unitary element in A. Moreover, $b = (aa^*)^{\frac{1}{2}}$ (by Corollary 6.40 in [1]). Applying continuous functional calculus we get the invertibility of b from the invertibility of a, also $b^{-1} \geq 0$ and

$$\|b^{-1}\|^2 = \|b^{-1}b^{-1*}\| = \|a^{-1}a^{-1*}\| = \|a^{-1}\|^2.$$

Hence $\|b^{-1}\| = \|a^{-1}\|$. Given that b^{-1} is self adjoint,

$$\frac{1}{\|b^{-1}\|} = \frac{1}{\sup\{|\lambda| : \lambda \in \sigma(b^{-1})\}} = \inf\{|\lambda| : \lambda \in \sigma(b)\} = |\lambda_0| \text{ say.}$$

As $a - \lambda_0 u = bu - \lambda_0 u = (b - \lambda_0 1)u$, non invertibility of $b - \lambda_0 1$ implies non invertibility of $a - \lambda_0 u$. Moreover,

$$\|a - (a - \lambda_0 u)\| = \|\lambda_0 u\| = |\lambda_0| = \frac{1}{\|b^{-1}\|} = \frac{1}{\|a^{-1}\|}.$$

Thus $(a - \lambda_0 u)$ is the desired singular element. $\square$

If we take H to be a Hilbert space then $B(H)$ is a C*-algebra and hence by the above theorem it satisfies condition (B). On considering a Banach space instead of a Hilbert space, we have obtained a sufficient condition to satisfy condition (B). First we recall the concept of a norm attaining operator.

Definition 2.3.7. An operator $T \in B(X)$ is called *norm attaining* if there exists an element $x \in X$ with $\|x\| = 1$, such that

$$\|Tx\| = \|T\|_{op}.$$

A common example of a norm attaining operator is any $T \in B(X)$ such that $\|T\|_{op}$ or $-\|T\|_{op}$ is an eigenvalue of T. For more information on such operators see [5]. Now we prove the following result.

Theorem 2.3.8. *Let $T \in B(X)$ be such that T^{-1} is norm attaining, then T satisfies condition (B).*

Proof. We have that T^{-1} is norm attaining, hence there exist $x, y \in X$, $\|x\| = \|y\| = 1$ such that

$$T^{-1}x = (\|T^{-1}\|_{op})y.$$

Using Hahn Banach theorem there exists $f \in X'$ for which $\|f\| = f(y) = 1$. Consider $A \in B(X)$ defined by

$$Au = -(\|T^{-1}\|_{op}{}^{-1})f(u)x \text{ for } u \in X.$$

In particular $Ay = -(\|T^{-1}\|_{op}{}^{-1})x$. Using the fact that $T^{-1}x = (\|T^{-1}\|_{op})y$, we get $(\|T^{-1}\|_{op}{}^{-1})x = Ty$ and hence $Ay = -Ty$, implying $0 \in \sigma(T + A)$. Observe that $\|A\|_{op} \leq \|T^{-1}\|_{op}{}^{-1}$. Suppose $\|A\|_{op} < \|T^{-1}\|_{op}{}^{-1}$. Then

$$\|A - T + T\|_{op} = \|T - (T + A)\|_{op} < \|T^{-1}\|_{op}{}^{-1},$$

which implies $(T + A)$ is invertible, a contradiction. Hence $\|A\|_{op} = \|T^{-1}\|_{op}{}^{-1}$. □

Remark 2.3.9. The converse of Theorem 2.3.8 may not be true. Let H denote the Hilbert space $(\ell^2, \|.\|_2)$ and $\{e_n\}_{n\in\mathbb{N}}$ be the standard orthonormal basis for $(\ell^2, \|.\|_2)$. Consider $T \in B(H)$ defined by

$$T(e_n) = \left(1 + \frac{1}{(n+1)}\right) e_n \qquad (n \geq 1.)$$

Then T is invertible and satisfies condition (B) as H is a Hilbert space, but T^{-1} is not norm attaining.

Remark 2.3.10. If X is finite dimensional, then any $T \in B(X)$ attains its norm, and hence from Theorem 2.3.8, $B(X)$ satisfies condition (B).

Recall that if $1 \leq p < \infty$ and $\{X_\alpha\}_{\alpha\in\Lambda}$ is a family of Banach spaces, then their ℓ_p-direct sum is the space

$$X = \left\{ x \in \prod_{\alpha\in\Lambda} X_\alpha : \sum_{\alpha\in\Lambda} \|x_\alpha\|^p < \infty \right\}$$

endowed with the norm

$$\|x\| = \left(\sum_{\alpha\in\Lambda} \|x_\alpha\|^p \right)^{\frac{1}{p}}.$$

With this information, we prove another result.

Theorem 2.3.11. *Let X be the ℓ_p direct sum of the family $\{X_\alpha : \alpha \in \Lambda\}$ of finite dimensional Banach spaces, $1 < p < \infty$. Then $B(X)$ satisfies condition (B).*

Proof. Let $T \in B(X)^{-1}$. Then

$$\frac{1}{\|T^{-1}\|_{op}} = \inf\{\|Tx\| : \|x\| = 1\}.$$

From ([32], Lemma 3.4) we have that there exists a $S \in B(X)$ such that $\|S\|_{op} \leq \frac{1}{\|T^{-1}\|_{op}}$ and $\inf\{\|(T+S)x\| : \|x\| = 1\} = 0$ which gives us that $T+S$ is singular, and hence acts as a required singular element on the boundary. □

2.4 Characterization of commutative Banach Algebras satisfying condition (B)

Let A be a complex unital Banach algebra with unit 1. We begin by obtaining a sufficient condition for an element $a \in A^{-1}$ to satisfy condition (B).

Theorem 2.4.1. *Let $a \in A^{-1}$ such that $\|(a^{-1})^2\| = \|a^{-1}\|^2$, then a satisfies condition (B).*

Proof. Since $\|(a^{-1})^2\| = \|a^{-1}\|^2$, by the compactness of spectrum there exists $\lambda_0 \in \sigma(a)$ such that

$$\frac{1}{\|a^{-1}\|} = \frac{1}{r(a^{-1})} = \inf\{|\lambda| : \lambda \in \sigma(a)\} = |\lambda_0|.$$

The element $s = a - \lambda_0 \in A$ can be taken as a singular element in the boundary of $B\left(a, \frac{1}{\|a^{-1}\|}\right)$ with the required property. □

Now we will show that the sufficient condition in Theorem 2.4.1 turns out to be necessary for commutative Banach algebras.

Theorem 2.4.2. *Let A be a commutative Banach algebra. Then $a \in A^{-1}$ satisfies condition (B) if and only if $\|(a^{-1})^2\| = \|a^{-1}\|^2$.*

Proof. If a satisfies (B), there exists $s \in \mathrm{Sing}(A)$ such that

$$\|a^{-1}\|^2 = \frac{1}{\|a-s\|^2} \leq \frac{1}{\|(a-s)^2\|} = \frac{1}{\|a^2 - (sa + as - s^2)\|} \leq \|(a^{-1})^2\|.$$

The last inequality holds true because a^2 is invertible and $sa + as - s^2 \in \mathrm{Sing}(A)$ as A is commutative. And also $(a^{-1})^2 = (a^2)^{-1}$. Thus we have $\|a^{-1}\|^2 = \|(a^{-1})^2\|$. □

Corollary 2.4.3. *Let A be a finite dimensional Banach algebra that satisfies condition (B). Then A is commutative if and only if $\|a^2\| = \|a\|^2$ for every $a \in A$.*

Proof. The proof follows from the fact that invertible elements are dense in a finite dimensional Banach algebra and Theorem 1.3.12. $\square$

Remark 2.4.4. The converse of Theorem 2.4.1 may not be true if A is non-commutative. We saw in Theorem 2.3.6 that any invertible operator on a Hilbert space satisfies condition (B), but if we take J to be a complex valued invertible matrix such that J^{-1} is a Jordan matrix with $r(J) < 1$, then $r(J^{-1}) \neq \|J^{-1}\|$.

Remark 2.4.5. In the commutative case, as (B) is solely dependent on the spectral radius of $a^{-1} \in A$, a will satisfy (B) even in the smallest Banach subalgebra containing 1, a and a^{-1}.

Remark 2.4.6. It can be shown that in a commutative unital Banach algebra, the elements satisfying condition (B) form a monoid under multiplication.

Before proceeding further to characterize commutative Banach algebras that satisfy condition (B), we will make some observations using Theorem 2.4.2.
The mere definition of condition (B) results into the following theorem.

Theorem 2.4.7. *Let A and B be unital Banach algebras. Let $\phi : A \to B$ be an isometric Banach algebra isomorphism. Then ϕ preserves condition (B).*

In the following example we show that Theorem 2.4.7 may not hold if ϕ is not onto.

Example 2.4.8. Consider the complex field $\mathbb{C}$. Then the identity map ($\lambda \longrightarrow \lambda 1$ for every $\lambda \in \mathbb{C}$, where 1 denotes the constant function 1.) is an isometric homomorphism from $\mathbb{C}$ into $(C^{(1)}[0,1], \|.\|)$ but $(C^{(1)}[0,1], \|.\|)$ does not satisfy (B) (See Example 2.2.5).

Further if we drop the isometry condition, we show that Theorem 2.4.7 may not work.

Example 2.4.9. Let X be a locally compact Hausdorff space and X^∞ denote the one point compactification of X. Then X^∞ is a compact Hausdorff space (See [14], Chapter 5, Theorem 21). $C(X^\infty)$, being a uniform algebra satisfies condition (B). Let $C_0(X)$ denote the vector space of all continuous functions on X that vanish at infinity. Then $C_0(X)$ is a Banach algebra with point wise multiplication and the uniform norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$. Since it is not unital, consider the standard *unitization* of $C_0(X)$ which is

$$C_0(X)^e := C_0(X) \times \mathbb{C}$$

with multiplication

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$$

and norm defined as

$$\|(a, \lambda)\| = \|a\| + |\lambda|.$$

$(C_0(X)^e, \|.\|)$ is a Banach algebra with unit element $(0, 1)$. Now let X be the interval $[1, \infty)$. Then by Proposition 16.5 in [3], $\left(\frac{1}{x^2}, 1\right)$ has the inverse $\left(\frac{-1}{1+x^2}, 1\right)$ in $C_0([1, \infty))^e$. But in view of Theorem 2.4.2, $\left(\frac{1}{x^2}, 1\right)$ does not satisfy condition (B), as $\left\|\left(\frac{-1}{1+x^2}, 1\right)\right\|^2 \neq \left\|\left(\frac{-1}{1+x^2}, 1\right)^2\right\|$. Define the map $\psi : C_0([1, \infty))^e \to C([1, \infty)^\infty)$ by

$$\psi(f, \lambda) = f + \lambda e,$$

where $e(x) = 1$ for every $x \in [1, \infty)^\infty$ and each $f \in C_0([1, \infty))$ is extended by assigning zero to the point ∞. It can be proved that ψ is a Banach algebra isomorphism, but not an isometry (See [7] Lemma 2.3.2).

From the next example we show that even finite dimensional Banach algebras may fail to satisfy condition (B).

Example 2.4.10. Consider $L^1(\mathbb{Z}_2) = \{f | f : \mathbb{Z}_2 \longrightarrow \mathbb{C}\}$ with the norm $\|f\| = |f(0)| + |f(1)|$ and multiplication defined as

$$(f * g)(0) = f(0)g(0) + f(1)g(1)$$

$$(f * g)(1) = f(0)g(1) + f(1)g(0).$$

Here the identity element being e defined by $e(0) = 1, e(1) = 0$. From Theorem 2.4.2 it is easy to verify that $f = (1, 0)$ and $g = (0, i)$ satisfy condition (B) but $f + g$ does not. In particular, as in this case, $\alpha f + \beta e$ may not satisfy condition (B) for some $\alpha, \beta \in \mathbb{C}$.

Next we will show how a commutative unital Banach algebra that satisfies condition (B), is ismorphic to a uniform algebra. Before that we will state the following result given by Arundathi and Kulkarni ([17], Corollary 3.13).

Result 2.4.11. Let A be a Banach algebra and $a \in A$ be such that

$$\|(a - \lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(a))} \qquad (\lambda \in \rho(a)).$$

Then $V(a) = Co(\sigma(a))$ and $\|a\| \leq exp(1)r(a)$, where

$$V(a) := \{f(a) : f \in A', f(1) = 1, \|f\| = 1\},$$

A' is the space of all continuous linear functionals on A and $Co(\sigma(a))$ is the convex hull of $\sigma(a)$.

Theorem 2.4.12. *Let A be a commutative Banach algebra that satisfies condition (B), then A is isomorphic to a uniform algebra.*

Proof. As A satisfies condition (B), by Theorem 2.4.2 we have $r(a) = \|a\|$ for every $a \in A^{-1}$, and hence for every $a \in A$

$$r((a-\lambda)^{-1}) = \|(a-\lambda)^{-1}\| \qquad (\lambda \in \rho(a)).$$

Therefore,

$$\|(a-\lambda)^{-1}\| = \frac{1}{dist(\lambda, \sigma(a))} \qquad (\lambda \in \rho(a)).$$

Hence from Result 2.4.11 we get

$$\|a\| \leq exp(1)r(a) \text{ for every } a \in A. \tag{2.4}$$

Let $\mathcal{M}_A$ denote the character space of A, which is a compact Hausdorff space. From Equation (2.4), and by using the Gelfand transform (see Theorem 1.5.2) we have

$$\|a\| \leq exp(1)\|\hat{a}\|_\infty \text{ for every } a \in A. \tag{2.5}$$

Equation 2.5 actually says that the Gelfand transform is injective on A, and its range is closed in $C(\mathcal{M}_A)$. Thus A is isomorphic onto a subalgebra (basically $\hat{A}$) of $C(\mathcal{M}_A)$ that is closed, point separating and contains the constants . $\square$

2.5 Characterizing property of commutativity is not enough on invertible elements

In this section, while attempting to check if a commutative unital Banach algebra which satisfies condition (B), is isometrically isomorphic (via the Gelfand transform) to a uniform algebra or not, we end up constructing a commutative unital Banach algebra, in which the property: $\|a^2\| = \|a\|^2$; is true for the invertible elements but cannot be extended to the whole algebra.

In Theorem 2.4.12 we showed that if A is a commutative unital Banach algebra that satisfies condition (B) i.e $\|a\|^2 = \|a^2\|$ for every $a \in A^{-1}$(by Theorem 2.4.2), then A is isomorphic to a uniform algebra (which is basically the range of the Gelfand transform). One should note that, a Banach algebra A may be isomorphic to a uniform algebra but $r(a)$ may not be equal to $\|a\|$ or equivalently $\|a^2\| \neq \|a\|^2$, for some $a \in A^{-1}$, as shown in the following example

Example 2.5.1. Let $A = \mathbb{C}^2$, with coordinate-wise multiplication. A with the uniform norm $\|(a,b)\|_\infty = \max(|a|,|b|)$, is a uniform algebra. A is also a Banach algebra with the norm $\|(a,b)\|_1 = |a| + |b|$ and $(A, \|.\|_1)$ is isomorphic to $(A, \|.\|_\infty)$. But $(A, \|.\|_1)$ does not satisfy condition (B). In fact $r(a,b) < \|(a,b)\|_1$ if and only if (a,b) is invertible in $(A, \|.\|_1)$.

In place of the above example one could also consider the space $(C_0([1,\infty))^e, \|.\|)$ (refer Example 2.4.9) and see that it is isomorphic to the uniform algebra $(C([1,\infty)^\infty.\|.\|_\infty)$. But $\left\|\left(\frac{-1}{1+x^2}, 1\right)\right\|^2 \neq \left\|\left(\frac{-1}{1+x^2}, 1\right)^2\right\|$, where $\left(\frac{-1}{1+x^2}, 1\right)$ is invertible in $(C_0([1,\infty))^e, \|.\|)$ with the inverse $\left(\frac{1}{x^2}, 1\right)$.
Let us recall Theorem 1.5.7, which states that: *Let A be a commutative unital Banach algebra. Then the Gelfand Transform is an isometry iff $\|a^2\| = \|a\|^2$ for every $a \in A$*. Now with regard to Theorem 2.4.12 and Theorem 1.5.7, we asked the following question:

Question 2.5.2. *Suppose A is a commutative unital Banach algebra satisfying condition (B). Then is $\|a\|^2 = \|a^2\|$ for every $a \in A$?*

Note: Recall Corollary 1.3.13, which states that: *Let A be a Banach algebra such that $\|a\|^2 = \|a^2\|$ for every $a \in A$. Then A is commutative.*
Considering Corollary 2.4.3, Corollary 1.3.13 and Question 2.5.2, we can also ask:
Can Corollary 2.4.3 be generalized to the following: Let $(A, \|.\|)$ be a complex unital Banach algebra satifying condition(B), is it true that A is commutative iff $\|a^2\| = \|a\|^2$ for every $a \in A$?

Note that the answer to the above question is positive if A^{-1} is dense in $(A, \|.\|)$. Regarding denseness there is a complete characterization for commutative Banach algebras ([28] Proposition 3.1): Let $(A, \|.\|)$ be a commutative Banach algebra, then A has dense invertible group if and only if the topological stable rank of A is one. Recently, Dawson and Feinstein [6] also investigated the condition that a complex commutative Banach algebra has dense invertible group.

2.5.1 Counter example

The answer to both the above questions is negative in general. We give an example of a commutative unital Banach algebra that satisfies condition (B) (and hence is isomorphic to a uniform algebra with $\|a^2\| = \|a\|^2$ for every $a \in A^{-1}$ but still there exists an $a \in A$ such that $\|a^2\| \neq \|a\|^2$. Now let us construct the example which answers question 2.5.2 negatively.

Example 2.5.3. Let $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $A(\bar{\mathbb{D}})$ be the set of those functions in $C(\bar{\mathbb{D}})$ which are analytic on $\mathbb{D}$. Consider $U = (A(\bar{\mathbb{D}}), \|.\|_\infty)$, the disc algebra on $\bar{\mathbb{D}}$, where $\|.\|_\infty$ denotes the uniform norm,

$$\|f\|_\infty = sup\{|f(t)| : t \in \bar{\mathbb{D}}\}$$

and multiplication is point-wise. U is a uniform algebra and $r(f) = \|f\|_\infty$ for every $f \in U$. Note that in this case U^{-1} is not dense in U, as any element in $\overline{U^{-1}}$ is either identically zero or has all its zeros contained in the unit circle (see [29]).

For any $f \in U$, let

$$p(f) := inf\left\{ \sum_{i=1}^{n} \|f_i\|_\infty : f = \sum_{k=1}^{n} f_i,\ f_i \in U^{-1} \right\}$$

where the infimum is taken over all representations of f as a finite combination of elements of U^{-1}. The set of such representations is non empty as $f = (f - \lambda 1) + \lambda 1$ for $f \in U$ and $\lambda \in \rho(f)$. Now $(A(\bar{\mathbb{D}}), p(.))$ is a normed algebra with the following properties.

(i) $p(f) = \|f\|_\infty = r(f) \qquad (f \in U^{-1})$

(ii) $\|f\|_\infty \leq p(f) \leq 3\|f\|_\infty \quad (f \in A(\bar{\mathbb{D}}))$. We have $p(f) \leq 3\|f\|_\infty$, as every $0 \neq f \in A(\bar{\mathbb{D}})$ can be expressed as $f + (1+\epsilon)\|f\|_\infty - (1+\epsilon)\|f\|_\infty$ for any $\epsilon > 0$.

Clearly $(A(\bar{\mathbb{D}}), p(.))$ is isomorphic to the disc algebra, but not isometrically isomorphic. That is, $\|f\|_\infty \neq p(f)$ for some $f \in A(\bar{\mathbb{D}})$.

In fact, we have $p(f) \geq \frac{e}{2}|f'(0)|$ for every f(where $e = exp(1)$). [1] Basically we show that $\|f\|_\infty \geq \frac{e}{2}|f'(0)|$ for every $f \in U^{-1}$. To show this, consider a non constant function $f \in U^{-1}$. Note here that if $f'(0) = 0$, then $\|f\|_\infty \geq \frac{e}{2}|f'(0)|$ holds trivially. Now since $f(z) \neq 0$ for every $z \in \bar{\mathbb{D}}$, there exists an analytic function $g : \bar{\mathbb{D}} \longrightarrow \mathbb{C}$

[1] The authors thank Fedor Nazarov whose contribution over `mathoverflow.net` was helpful in developing ideas for this argument.

such that $f = e^g$. We can scale f such that $f(0) = 1$ (since we are estimating $\frac{\|f\|_\infty}{|f'(0)|}$). We may assume $g(0) = 0$. Therefore $g(z) = \alpha z + \dots$, where $\alpha = f'(0)$. Now $\|f\|_\infty = e^{\sup(\Re g)}$. Let $\beta = \sup(\Re g)$, and define the conformal map h from D onto the region $\Re z < \beta$ as $h(z) = \frac{2\beta z}{1+z}$. Note that $\beta > 0$ by the open mapping theorem. Applying *Schwarz Lemma* to $h^{-1} \circ g$, we get

$$\begin{aligned} |(h^{-1} \circ g)'(0)| &= |(h^{-1})'(g(0))g'(0)| \\ &= \frac{|g'(0)|}{|h'(0)|} \leq 1. \end{aligned}$$

Therefore $|g'(0)| \leq |h'(0)|$ i.e $\frac{|\alpha|}{2} \leq \beta$. Hence

$$e^{\frac{|\alpha|}{2}} \leq e^\beta = e^{\sup \Re g} = \|f\|_\infty.$$

Thus

$$\min_{|\alpha| \in \mathbb{R}} \frac{e^{\frac{|\alpha|}{2}}}{|\alpha|} \leq \min_{f \in U^{-1}} \frac{\|f\|_\infty}{|f'(0)|}.$$

Since $\min_{|\alpha| \in \mathbb{R}} \frac{e^{\frac{|\alpha|}{2}}}{|\alpha|} = \frac{e}{2}$, we have $\|f\|_\infty \geq \frac{e}{2}|f'(0)|$ for every $f \in U^{-1}$.

Now let $f \in A(\bar{\mathbb{D}})$ and let $f = \sum_{i=1}^n f_i$ where $f_i \in U^{-1}$. Then we have

$$\sum_{i=1}^n \|f_i\|_\infty \geq \frac{e}{2} \sum_{i=1}^n |f_i{}'(0)| \geq \frac{e}{2} |f'(0)|.$$

and hence $p(f) \geq \frac{e}{2}|f'(0)|$. Considering the function $f(z) = z$ for all $z \in \bar{\mathbb{D}}$, we get $p(f) \geq \frac{e}{2}|f'(0)| = \frac{e}{2}\|f\|_\infty$ i.e $p(f) \neq r(f)$. □

The above example hence proves that a characterizing property of commutative Banach algebras may not be sufficient on the invertible elements.

Chapter 3

A weaker Gleason Kahane Żelazko theorem

3.1 Introduction

Let A be a complex unital Banach algebra. In [11], [13] and [36], Gleason, Kahane and Żelazko gave a characterization of multiplicative linear functionals on a Banach algebra (it will be referred to as the GKZ theorem). The classical proofs used tools from complex function theory. An elementary proof was given by Roitman and Sternfeld in [30]. For a survey of the GKZ theorem and its early generalizations, see [12].

Theorem 3.1.1 (**Gleason- Kahane- Zelazko Theorem**). *If $\varphi : A \longrightarrow \mathbb{C}$ is a non zero linear functional, then the following are equivalent.*

(i) $\varphi(1) = 1$ *and* $\varphi(a) \neq 0$ *for every* $a \in A^{-1}$.

(ii) $\varphi(a)1 - a \notin A^{-1}$ *for all* $a \in A$.

(iii) φ *is multiplicative i.e.* $\varphi(ab) = \varphi(a)\varphi(b)$ *for all* $a, b \in A$

Proof. See [3], Theorem 16.7. □

In 2015, Mashregi and Ransford [22], generalized the GKZ theorem to modules. Before stating it, we will revise the concept of A- modules.

Definition 3.1.2. Let A be an algebra over a field $\mathbb{F}$, and $\mathcal{X}$ be a vector space over $\mathbb{F}$. The vector space $\mathcal{X}$ is said to be a *left A-module* if the map from

$$A \times \mathcal{X} \longrightarrow \mathcal{X}$$

defined as

$$(a, x) \longrightarrow ax$$

satisfies the following:

(i) given any fixed $a \in A$ the map $x \longrightarrow ax$ is linear on $\mathcal{X}$.

(ii) given any fixed $x \in \mathcal{X}$ the map $a \longrightarrow ax$ is linear on A.

(iii) $a(bx) = (ab)x \qquad (a, b \in A, x \in \mathcal{X})$.

Here the map $(a, x) \longrightarrow ax$ is called the *module multiplication*.

A *right A-module* can also be defined analogously. We call a left A-module to be *unit linked*, if A has a unit element 1 and $1x = x$ $(x \in \mathcal{X})$. We can similarly define a *unit linked right A-module.*

Definition 3.1.3. Let A be a normed algebra over a field $\mathbb{F}$, and $\mathcal{X}$ be a normed linear space over $\mathbb{F}$. Then $\mathcal{X}$ is said to be a *normed left A-module* if $\mathcal{X}$ is a left A-module and there exists a $K > 0$ such that

$$\|ax\| \leq K\|a\|\|x\| \qquad (a \in A, x \in \mathcal{X}).$$

A normed left A-module is called a *Banach left A-module* if it is complete as a normed linear space. We can similary define *Banach right A-modules.*

If A is a normed algebra, then we can immediately see that A is a left/right normed A-module, with the module multiplication given by the algebra product itself. Also any left/right ideal of A is a normed left/right A-module. Note that every A-module considered in this section is *unit linked.*
Now we state the result by Mashregi and Ransford.

Theorem 3.1.4. *Let A be a complex unital Banach algebra, let $\mathcal{X}$ be a left A-module and let S be a non empty subset of $\mathcal{X}$ satisfying the following conditions:*

$(P1)$ *S generates $\mathcal{X}$ as a A-module. That is, for every $x \in \mathcal{X}$ there exists $a_1, a_2, \cdots a_n \in A$ and $s_1, s_2, \cdots s_n \in S$ such that $x = \sum_{i=1}^{n} a_i s_i$;*

$(P2)$ *if $a \in A^{-1}$ and $s \in S$, then $as \in S$;*

$(P3)$ *for all $s_1, s_2 \in S$, there exists $a_1, a_2 \in A$ such that $a_j s_j \in S$ $(j = 1, 2)$ and $a_1 s_1 = a_2 s_2$.*

Let $\xi : \mathcal{X} \longrightarrow \mathbb{C}$ be a linear functional such that $\xi(s) \neq 0$ for all $s \in S$. Then there exists a unique character $\eta : A \longrightarrow \mathbb{C}$ such that

$$\xi(ax) = \eta(a)\xi(x) \qquad (a \in A, x \in \mathcal{X}).$$

Remark 3.1.5. Theorem 3.1.4 holds even if we replace the condition (P2) by the following: *If $a \in exp(A)$ and $s \in S$, then $as \in S$.*

Remark 3.1.6. If we replace $\mathcal{X}$ by A and S by A^{-1} in Theorem 3.1.4, then $\frac{\xi}{\xi(1)} : A \longrightarrow \mathbb{C}$ is multiplicative.

Remark 3.1.7. If $\xi : \mathcal{X} \longrightarrow \mathbb{C}$ be a linear functional, then, like (i) and (ii) of Theorem 3.1.1 the following are equivalent.

(i) $\xi(s) \neq 0$ for every $s \in S$.

(ii) $\xi(x)s - \xi(s)x \notin S$ for every $x \in \mathcal{X}$ and $s \in S$.

Very recently, Mashregi et al. extended their work from [22] to give a Dirichlet space analogue of the GKZ theorem [21]. Also in 2017, Ghodrat and Sady [10], introduced zero sets and spectrum-like sets for an element of a Banach module. They also introduced a subset which behaves as the set of invertible elements of commutative unital Banach algebra, and gave some GKZ-type theorems for Banach left modules.

An interesting development to the GKZ Theorem 3.1.1 was obtained by Kowalski and Słodkowski [15], where they replaced the two assumptions of linearity and preservation of invertibility by a single weaker assumption and showed that under this weaker condition, the same conclusion of Theorem 3.1.1 holds. Additionally, they also proved linearity. More precisely, they proved the following:

Theorem 3.1.8. *Let A be a complex unital Banach algebra and let $\varphi : A \longrightarrow \mathbb{C}$ satisfy*

$$\varphi(0) = 0,$$

and

$$(\varphi(x) - \varphi(y))1 - (x - y) \notin A^{-1} \text{ for every } x, y \in A.$$

Then φ is multiplicative and linear.

In the following section we consider complex valued maps on Banach modules. We gradually drop linearity in the hypothesis of Theorem 3.1.4, alter it and see to what extent can we obtain its exact conclusion along with linearity. In Proposition 3.2.3

we assume ξ to be only $\mathbb{R}$-linear. In Theorem 3.2.4 we even drop $\mathbb{R}$-linearity of ξ, replace it by a weaker assumption on S and consequently get a Kowalski-Słodkowski type Theorem for modules. Explicitly speaking, in place of assuming ξ to be linear, we put the following conditions:

$$\xi(0) = 0$$

and

$$(\xi(x_1) - \xi(x_2))s - (x_1 - x_2)\xi(s) \notin S \text{ for every } x_1, x_2 \in \mathcal{X} \textit{ and } s \in S;$$

and obtain a similar conclusion as Theorem 3.1.4. At the end we give some applications to linear functionals on Hardy spaces.

Of late some problems, along similar lines have been studied. Like in the case of Banach algebras [35], it is investigated, as to when a continuous multiplicative map, with values belonging spectrum, is automatically linear.

3.2 A weaker Gleason Kahane Żelazko theorem for modules

Definition 3.2.1. Let $\mathcal{X}$ be a complex linear space. We say that a map $\phi : \mathcal{X} \longrightarrow \mathbb{C}$ is complex (real) linear (shortly $\mathbb{C}$-linear or $\mathbb{R}$-linear) if it is additive and homogeneous with respect to complex (real) scalars.

In the next proposition, we use the following lemma from [15].

Lemma 3.2.2. *Let A be a complex Banach algebra with unit element 1 and let f be an $\mathbb{R}$- linear map on A such that*

$$f(a)1 - a \notin A^{-1} \textit{ for every } a \in A.$$

Then f is $\mathbb{C}$-linear.

Proposition 3.2.3. *Let A be a complex unital Banach algebra, let $\mathcal{X}$ be a left A-module and let S be a non empty subset of $\mathcal{X}$ satisfying $(P1), (P2)$ and $(P3)$. Let ξ be a non zero $\mathbb{R}$-linear functional on $\mathcal{X}$ such that*

$$\xi(x)s - \xi(s)x \notin S \tag{3.1}$$

for every $x \in \mathcal{X}$ and $s \in S$. Then ξ is $\mathbb{C}$-linear and there exists a unique character $\eta : A \longrightarrow \mathbb{C}$ such that

$$\xi(ax) = \eta(a)\xi(x) \qquad (a \in A, x \in \mathcal{X}).$$

Proof. By the use of (3.1) and (P2), we observe that for every $s \in S$, $\xi(s) \neq 0$. Given $s \in S$, define $\eta_s : A \longrightarrow \mathbb{C}$ by

$$\eta_s(a) := \frac{\xi(as)}{\xi(s)} \qquad (a \in A).$$

Clearly, η_s is an $\mathbb{R}$-linear map on A. Also $\eta_s(a)1 - a \notin A^{-1}$ for every $a \in A$ because if there exist an $a' \in A$ such that $\eta_s(a')1 - a' \in A^{-1}$, then $\xi(a's)s - (a's)\xi(s) \in S$ by (P2), but this is a contradiction to (3.1). Hence from Lemma 3.2.2 we get that η_s is $\mathbb{C}$-linear and from Theorem 3.1.1, η_s is multiplicative on A.

Given $s_1, s_2 \in S$, from property $(P3)$ we get $a_1, a_2 \in A$ such that $a_j s_j \in S$ $(j = 1, 2)$ and $a_1 s_1 = a_2 s_2$. Then,

$$\xi(a_1 s_1) = \xi(a_2 s_2). \tag{3.2}$$

And for each $a \in A$, we have $aa_1 s_1 = aa_2 s_2$, whence

$$\eta_{s_1}(a)\xi(a_1 s_1) = \eta_{s_2}(a)\xi(a_2 s_2). \tag{3.3}$$

As $a_j s_j \in S$, both sides of (3.2) are non zero. Thus we may divide (3.3) by (3.2) and get

$$\eta_{s_1}(a) = \eta_{s_2}(a).$$

Thus η_s is independent of s. We denote it simply by η and we have

$$\xi(as) = \eta(a)\xi(s) \qquad (a \in A, s \in S).$$

Now let $a \in A$ and $x \in \mathcal{X}$. By condition $(P1)$, there exists $a_1, a_2 \cdots a_n \in A$ and $s_1, s_2 \cdots s_n \in S$ such that $x = \sum_{j=1}^{n} a_j s_j$. Then we have

$$\xi(ax) = \sum_{j=1}^{n} \xi(aa_j s_j) = \sum_{j=1}^{n} \eta(aa_j)\xi(s_j) = \eta(a) \sum_{j=1}^{n} \eta(a_j)\xi(s_j) = \eta(a)\xi(x). \tag{3.4}$$

Now as η is $\mathbb{C}$-linear, we get that ξ is $\mathbb{C}$-linear. $\square$

Next we drop the linearity condition and replace condition (3.1) by (3.6).

Theorem 3.2.4. *Let A be a complex unital Banach algebra, let $\mathcal{X}$ be a left A-module and let S be a non empty subset of $\mathcal{X}$ satisfying $(P1), (P2)$ and $(P3)$. Let $\xi : \mathcal{X} \longrightarrow \mathbb{C}$ be non zero and satisfies,*

$$\xi(0) = 0, \tag{3.5}$$

$$(\xi(x_1) - \xi(x_2))s - (x_1 - x_2)\xi(s) \notin S \text{ for every } x_1, x_2 \in \mathcal{X} \text{ and } s \in S. \tag{3.6}$$

Then there exist a unique character $\eta : A \longrightarrow \mathbb{C}$ such that

$$\xi(as) = \eta(a)\xi(s) \qquad (a \in A, s \in S).$$

Proof. By the use of (3.5), (3.6) and (P2), we observe that for every $s \in S$, $\xi(s) \neq 0$. Given $s \in S$, define $\eta_s : A \longrightarrow \mathbb{C}$ by

$$\eta_s(a) := \frac{\xi(as)}{\xi(s)} \qquad (a \in A).$$

Observe that $\eta_s(0) = 0$ and using (3.6) and $(P2)$ we get

$$(\eta_s(a_1) - \eta_s(a_2))1 - (a_1 - a_2) \notin A^{-1} \qquad (a_1, a_2 \in A).$$

Hence from Theorem 3.1.8 η_s is a multiplicative linear functional on A. Proceeding as in the proof of Proposition 3.2.3 we can show that η_s is independent of s and can be denoted as η with

$$\xi(as) = \eta(a)\xi(s) \qquad (a \in A, s \in S)).$$

□

Remark 3.2.5. If we replace $\mathcal{X}$ by A and S by A^{-1} in Theorem 3.2.4, then $\frac{\xi}{\xi(1)} : A \longrightarrow \mathbb{C}$ is multiplicative and linear. Thus Theorem 3.2.4 contains the Kowalski and Słodkowski theorem (Theorem 3.1.8) as a special case if we also add $\xi(1) = 1$. Note that Theorem 3.1.8 was used in the proof of Theorem 3.2.4.

Corollary 3.2.6. *Under the assumption of Theorem 3.2.4, if $\mathcal{X}$ is generated as a Banach A-module by an element s_0 of S, then ξ is linear and there exist a unique character $\eta : A \longrightarrow \mathbb{C}$ such that*

$$\xi(ax) = \eta(a)\xi(x) \qquad (a \in A, x \in \mathcal{X}).$$

Proof. Assume that ξ satisfies the stated property. Then, as in the proof of the above theorem, exists a unique character $\eta : A \longrightarrow \mathbb{C}$ such that

$$\xi(as) = \eta(a)\xi(s) \qquad (a \in A, s \in S)).$$

Let $x_1, x_2 \in \mathcal{X}$. Then by hypothesis, there exists $a_1, a_2 \in A$ such that $x_1 = a_1 s_0$ and $x_2 = a_2 s_0$. Then

$$\begin{aligned} \xi(x_1 + x_2) &= \xi(a_1 s_0 + a_2 s_0) \\ &= \eta(a_1 + a_2)\xi(s_0) \\ &= \eta(a_1)\xi(s_0) + \eta(a_2)\xi(s_0) \\ &= \xi(x_1) + \xi(x_2) \end{aligned}$$

Now like in equation (3.2.2), using condition $(P1)$ we can show that

$$\xi(ax) = \eta(a)\xi(x) \qquad (a \in A, x \in \mathcal{X}),$$

and as η is homogeneous, we get that ξ is also homogeneous. □

Corollary 3.2.7. *Under the assumption of Theorem 3.2.4, if ξ satisfies the following property*

$$\sum_{j=1}^{n} \xi(s_j) = \xi\left(\sum_{j=1}^{n} s_j\right), \quad n \in \mathbb{N},\ s_j \in S. \tag{3.7}$$

Then ξ is linear and there exists a unique character $\eta : A \longrightarrow \mathbb{C}$ such that

$$\xi(ax) = \eta(a)\xi(x) \qquad (a \in A, x \in \mathcal{X}).$$

Proof. From the proof of Theorem 3.2.4 there exists a unique character $\eta : A \longrightarrow \mathbb{C}$ such that

$$\xi(as) = \eta(a)\xi(s) \qquad (a \in A, s \in S).$$

Given $x \in \mathcal{X}$, by property $(P1)$, there exists $a_1, a_2 \cdots a_n \in A$ and $s_1, s_2 \cdots s_n \in S$ such that $x = \sum_{j=1}^{n} a_j s_j$.

$$\begin{aligned} \xi(x) &= \xi\left(\sum_{j=1}^{n} a_j s_j\right) \\ &= \xi\left(\sum_{j=1}^{n} (a_j - t_j)s_j + (t_j)s_j\right) \quad t_j \in \mathbb{C} \setminus \sigma(a_j) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \xi((a_j - t_j)s_j) + \sum_{j=1}^{n} \xi(t_j s_j) \;\; \textit{from } (3.7) \textit{ and } (P2) \\
&= \sum_{j=1}^{n} \eta(a_j - t_j)\xi(s_j) + \sum_{j=1}^{n} \eta(t_j)\xi(s_j) \\
&= \sum_{j=1}^{n} \eta(a_j)\xi(s_j) = \sum_{j=1}^{n} \xi(a_j s_j)
\end{aligned}
$$

Using the above argument we can show that given $x_1, x_2 \in \mathcal{X}$

$$\xi(x_1 + x_2) = \xi(x_1) + \xi(x_2).$$

Now as in the proof of Proposition (3.2.3), we can show that, ξ is linear and

$$\xi(ax) = \eta(a)\xi(x) \qquad (a \in A,\ x \in \mathcal{X}).$$

Hence proved. $\square$

A related question

GKZ theorem answers when a linear functional is multiplicative. Now we consider another question. That is, under what condition can a multiplicative kind of map on a Banach module, be linear? It turns out that, it depends on the analytic condition of the associated map, as given in [35].

Theorem 3.2.8. *Let A be a complex unital Banach algebra, let $\mathcal{X}$ be a left A-module and let S be a non empty subset of $\mathcal{X}$ satisfying $(P1), (P2)$ and $(P3)$. Let $\eta : A \longrightarrow \mathbb{C}$ and $\xi : \mathcal{X} \longrightarrow \mathbb{C}$ be such that*

$$\xi(ax) = \eta(a)\xi(x) \qquad (a \in A, x \in \mathcal{X})$$

and

$$\xi(x)s - \xi(s)x \notin S \qquad (s \in S, x \in \mathcal{X}).$$

Then η is linear iff for each $a \in A$ the map $\tau_a : \mathbb{C} \longrightarrow \mathbb{C}$ defined as

$$\tau_a(\lambda) = \eta(\lambda 1 - a)$$

is an entire function on $\mathbb{C}$.

Proof. Let $s \in S$. As $\xi(s) \neq 0$,

$$\eta(ab) = \frac{\xi(abs)}{\xi(s)} = \frac{\eta(a)\xi(bs)}{\xi(s)} = \frac{\eta(a)\eta(b)\xi(s)}{\xi(s)} = \eta(a)\eta(b).$$

Thus η is multiplicative. And like in Proposition 3.2.3 we can show that $\eta(a)-a \notin A^{-1}$ for every $a \in A$. Now the result follows from Theorem 2.1 in [35] which states that: *Let A be a Banach algebra. Then a multiplicative functional ϕ on A satisfying*

$$\phi(x) \in \sigma(x) \textit{ for every } x \in A$$

is linear if and only if for each $x \in A$ the map

$$\lambda \mapsto \phi(\lambda 1 - x)$$

is an entire function on $\mathbb{C}$. □

In the following example we see that under the hypothesis of the above theorem, the condition that for each $a \in A$, τ_a is entire is sufficient for η to be linear, but the same condition may not be not sufficient for ξ to be linear.

Example 3.2.9. Let $\mathcal{X} = \mathbb{C}^2$ with usual pointwise operations and any algebra norm. Let S be the set of invertible elements in $\mathcal{X}$. We take A to be the complex field $\mathbb{C}$. Let $\eta : A \longrightarrow \mathbb{C}$ be the identity map. Define $\xi : \mathcal{X} \longrightarrow \mathbb{C}$ as follows:

$$\xi(a,b) = \begin{cases} b, & a \neq 0, \\ 0, & a = 0. \end{cases}$$

Note that $\xi(\alpha(a,b)) = \eta(\alpha)\xi(a,b)$ for every $\alpha \in A$ and $(a,b) \in \mathcal{X}$ but ξ is not linear. Moreover, $\xi(x)s - \xi(s)x$ is not invertible, for any $x \in \mathcal{X}$ and any invertible element s in S.

3.3 Applications to Hardy spaces on the unit disc

In this section, we apply Theorem 3.2.4 to the Hardy spaces. Initially let us give a quick revision of the Hardy spaces on the open unit disc, and also the concept of inner and outer functions.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let

$$Hol(\mathbb{D}) := \{f : \mathbb{D} \longrightarrow \mathbb{C} \mid f \text{ is holomorphic on } \mathbb{D}\}.$$

The Hardy spaces on $\mathbb{D}$ are defined as:

$$H^p(\mathbb{D}) := \left\{ f \in Hol\,(\mathbb{D}) : \sup_{r<1} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^p d\theta < \infty \right\} \qquad (0 < p < \infty),$$

$$H^\infty(\mathbb{D}) := \left\{ f \in Hol\,(\mathbb{D}) : \sup_{z \in D} |f\,(z)| < \infty \right\}.$$

Using the *Hölder's inequality*, one can prove that

$$H^\infty(\mathbb{D}) \subset H^q(\mathbb{D}) \subset H^p(\mathbb{D})$$

if $0 < p < q < \infty$. Note that $H^\infty(\mathbb{D})$ with the norm defined as

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| \qquad (f \in H^\infty(\mathbb{D})),$$

is a commutative unital Banach algebra with point-wise multiplication and the unit element as the constant function one.
Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $0 < p \leq \infty$. Then $g \in H^p(\mathbb{D})$ is **outer** if there exists $G : \mathbb{T} \longrightarrow [0, \infty)$ with $G \in L^1(\mathbb{T})$ such that

$$g\,(z) = \alpha exp \left(\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} G\left(e^{i\theta}\right) \frac{d\theta}{2\pi} \right) \qquad (z \in \mathbb{D})$$

and $|\alpha| = 1$.
Let $F \in H^\infty(\mathbb{D})$. Then

$$f(e^{i\theta}) = \lim_{r \longrightarrow 1} F(re^{i\theta})$$

exists for almost all $e^{i\theta} \in \mathbb{T}$ (See [20], Lemma 3.10). If

$$|f(e^{i\theta})| = 1,$$

for almost all $e^{i\theta} \in \mathbb{T}$ then F is an **inner** function. A commom example of an inner function is the Blashke product (See [20] section 7.2).

Theorem 3.3.1 (Canonical factorization)**.** *Let f be a non zero function in $H^p(\mathbb{D})$,*

$0 < p \le \infty$. Then f has a factorization $f = I_f O_f$, where I_f is inner and O_f is outer. This factorization is unique upto a constant of modulus one.

Proof. See [20], Section 7.6 or [8], Theorem 2.8. $\square$

Now we will apply Theorem 3.2.4 to Hardy spaces on the unit disc.

Theorem 3.3.2. *Let $0 < p \le \infty$ and let $\xi : H^p \longrightarrow \mathbb{C}$ satisfy*

$$\xi(0) = 0, \quad \xi(1) \neq 0, \tag{3.8}$$

$$(\xi(f_1) - \xi(f_2))1 - (f_1 - f_2)\xi(1) \notin S \text{ for every } f_1, f_2 \in H^p \tag{3.9}$$

where S is the set of all outer functions in H^p. Then there exists $c \in \mathbb{C} \setminus \{0\}$ and $z_0 \in \mathbb{D}$ such that

$$\xi(g) = cg(z_0) \qquad (g \in H^\infty).$$

Proof. We will take $\mathcal{X} = H^p$, $A = H^\infty$ and S to be the set of all outer functions in H^p, in Theorem 3.2.4. S satisfies condition $(P1)$ because of the canonical factorization theorem. Condition $(P2)$ holds because every invertible function $g \in H^\infty$ is outer (if we multiply the inner-outer factorizations of h and $1/h$, then the factorization of 1 is obtained. Since the factorization is unique, we get that the inner factorization of h and $1/h$ are both 1). As every outer function can be expressed as the quotient of two outer bounded functions ([8], Proof of Theorem 1.2), condition $(P3)$ holds.

Using (3.8) and (3.9) we get that $\xi(s) \neq 0$ for every outer functions s in H^p and proceeding as in the proof of Theorem 3.2.4 we get that there a exists a unique character $\eta : A \longrightarrow \mathbb{C}$ such that

$$\xi(gs) = \eta(g)\xi(s) \qquad (g \in H^\infty, s \in S).$$

By the canonical factorization theorem, every function in H^p can be expressed uniquely as the product of an inner function (which belongs to H^∞) and an outer function in H^p. And thus we actually have

$$\xi(gf) = \eta(g)\xi(f) \qquad (g \in H^\infty, f \in H^p).$$

In particular

$$\frac{\xi(g)}{\xi(1)} = \eta(g) \qquad (g \in H^\infty).$$

Let $c = \xi(1)$ and $z_0 = \eta(v)$ where v is the function $v(z) = z$ for every $z \in \mathbb{D}$. As 1 is an outer function, we have $c \in \mathbb{C} \setminus \{0\}$. For $\lambda \in \mathbb{C} \setminus \mathbb{D}$, the function $(v - \lambda 1)$ is outer, so we have $\xi(v - \lambda 1) \neq 0$ and hence $\eta(v - \lambda 1) \neq 0$, thus $z_0 \neq \lambda$. And we have $z_0 \in \mathbb{D}$. Let $g \in H^\infty$. Define

$$k(z) = \frac{g(z) - g(z_0)}{z - z_0}.$$

Then $k \in H^\infty$. Also $g = g(z_0)1 + (v - z_0 1)k$. Thus

$$\eta(g) = g(z_0)\eta(1) + \eta(v - z_0 1)\eta(k) = g(z_0) + 0$$

i.e

$$\frac{\xi(g)}{\xi(1)} = g(z_0).$$

Hence we get $\xi(g) = cg(z_0)$ as desired. $\square$

Corollary 3.3.3. *Under the assumption of Theorem 3.3.2 if ξ is taken to be continuous, then there exists $c \in \mathbb{C} \setminus \{0\}$ and $z_0 \in \mathbb{D}$ such that*

$$\xi(f) = cf(z_0) \qquad (f \in H^p).$$

Proof. The proof follows from the fact that polynomials are dense in H^p for $0 < p < \infty$ ([8], Theorem 3.3). $\square$

Theorem 3.3.4. *Let $0 < p \leq \infty$ and let $T : H^p \longrightarrow Hol(\mathbb{D})$ satisfy*

$$T(0)(z) = 0 \qquad (z \in \mathbb{D}),$$

$$(T1)(z) \neq 0 \qquad (z \in \mathbb{D}),$$

$$((Tf_1)(z) - (Tf_2)(z))1 - (f_1 - f_2)(T1)(z) \notin S \qquad (f_1, f_2 \in H^p, z \in \mathbb{D}).$$

Then there exist holomorphic functions $\psi : \mathbb{D} \longrightarrow \mathbb{D}$ and $\phi : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0\}$ such that

$$Tg = \phi \cdot (g \circ \psi) \qquad (g \in H^\infty).$$

Proof. Define $\phi = T1$, then $\phi \in Hol(\mathbb{D})$ and is nowhere zero. Define $\psi = (Tv)/\phi$, where $v(z) := z$, then $\psi \in Hol(\mathbb{D})$. For $z \in \mathbb{D}$, the map $g \longrightarrow (Tg)(z)$ satisfies condition (3.8) and (3.9) of Theorem 3.3.2. Hence there exists $z_0 \in \mathbb{D}$ and $c \in \mathbb{C} \setminus \{0\}$ such that $(Tg)(z) = cg(z_0)$ for all $g \in H^\infty$. Taking $g = 1$ we see that $c = \phi(z)$.

Taking $g = v$ we get $z_0 = \psi(z)$. Then $\psi(z) \in \mathbb{D}$ and $(Tg)(z) = \phi(z)g(\psi(z))$ for all $g \in H^\infty$. $\square$

Corollary 3.3.5. *Under the assumption of Theorem 3.3.4 if T is taken to be continuous. Then there exist holomorphic functions $\psi : \mathbb{D} \longrightarrow \mathbb{D}$ and $\phi : \mathbb{D} \longrightarrow \mathbb{C} \setminus \{0\}$ such that*

$$Tf = \phi \cdot (f \circ \psi) \qquad (f \in H^p).$$

Concluding Remarks and Future work

Conclusion

- Commutative unital Banach algebras satisfying condition (B) have been characterized fully. For the non- commutative case we have investigated C^*-Algebra and some cases of $B(X)$.
- Working along the same lines as Kowalski and Słodkowski, we removed the assumption of linearity in the hypothesis of GKZ theorem for modules, modified it, and got a weaker GKZ Theorem for modules. We further gave applications to functionals on Hardy spaces.

Future Work

- We need to characterize **non commutative** Banach algebras for condition (B).
- Given the hypothesis in Theorem 3.3.2, we get that ξ is of multiplicative kind, but we still aim to get linearity. This could be achieved by assuming that ξ is continuous, as we did in Corollary 3.3.3. The question remains that does condition (3.9) in Theorem 3.3.2 imply that ξ is continuous? One possible approach could be by proving that ξ is Lipschitz continuous or using the sequential criteria. But one should note that the set of all outer functions is neither open nor closed in H^p, $0 < p < \infty$.

  Does hypothesis in Theorem 3.3.2 imply that ξ is **continuous**?

Publications Based on this Thesis

Journal Publications:

1. Geethika Sebastian and Sukumar Daniel, *On the open ball centered at an invertible element of a Banach Algebra*, Oper. Matrices. **12**, 1, (2018), 19–25
Doi: `http://dx.doi.org/10.7153/oam-2018-12-02`

2. Geethika Sebastian and Sukumar Daniel, *A characterizing property of commutative Banach algebras may not be sufficient only on the invertible elements*, Comptes Rendus Mathematique **356**, 6, (2018), 594–596
Doi: `https://doi.org/10.1016/j.crma.2018.05.002`

3. Geethika Sebastian and Sukumar Daniel, *A weaker Gleason Kahane Żelazko theorem for modules and applications to Hardy spaces.* (Communicated).

Symbols

$A(\bar{\mathbb{D}})$ The Disc algebra. 14

A^{-1} Set of all invertible elements in A. 3

$B(X)$ space of bounded linear maps on a vector space X. 2

$B(a, r)$ Open ball centered at a with radius r. 4

$C(K)$ space of continuous functions on topological space K. 2

$L^1(G)$ space of integrable functions on a LCA group G. 2

Γ Dual of a LCA group G. 3

$\bar{\mathbb{D}}$ The closed unit disc on the complex plane. 14

$\exp(a)$ Exponential of a Banach algebra element a. 6

$\hat{A}$ Image of Banach algebra A under the Gelfand transform. 9

$\kappa(a)$ Condition number. 12

$\mathbb{C}$ The field of all Complex Numbers. 1

$\mathcal{M}_A$ The maximal ideal space of a Banach algebra A. 8

$\mathfrak{A}$ Uniform algebra. 13

$\sigma(a)$ Spectrum of an element a. 5

σ_ϵ Condition spectrum. 12

$\mathrm{Sing}(A)$ Set of all singular elements in A. 3

$\mathrm{dist}(a, S')$ Distance from λ to the set S', i.e. $inf\{|\lambda - \mu| : \mu \in S'\}$. 11

$r(a)$ Spectral radius of an element a. 5

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