# Some topics on the Fourier coefficients of modular forms

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# Dedication

**This thesis is dedicated to**

my beloved parents and Sonu,
for their Belief, Encouragement and Love.

# Abstract

This thesis consists of three parts. In the first part, we study the gaps between non-zero Fourier coefficients of cuspdial CM eigenforms in the short intervals. In the second part, we study the sign changes for the Fourier coefficients of Hilbert modular forms of half-integral weight. In the third part, we study the simultaneous behaviour of Fourier coefficients of two different Hilbert modular cusp forms of integral weight.

In **Chapter 1**, we present the definitions and some preliminaries on classical modular forms. We shall also recall some relevant results from the literature, which are useful in the subsequent chapters.

In **Chapter 2**, we show that for an elliptic curve $E$ over $\mathbb{Q}$ of conductor $N$ with complex multiplication (CM) by $\mathbb{Q}(i)$, the $n$-th Fourier coefficient of $f_E$ is non-zero in the short interval $(X, X + cX^{\frac{1}{4}})$ for all $X \gg 0$ and for some $c > 0$, where $f_E$ is the corresponding cuspidal Hecke eigenform in $S_2(\Gamma_0(N))^{\mathrm{new}}$, by the modularity theorem. As a consequence, we produce infinitely many cuspidal CM eigenforms $f$ level $N > 1$ and weight $k > 2$ for which $i_f(n) \ll n^{\frac{1}{4}}$ holds, for all $n \gg 0$. This is a generalization of the result of Das and Ganguly [14] to the weight $k > 2$ situation.

In **Chapter 3**, we prove a result concerning the sign changes for the Fourier coefficients of Hilbert modular forms of half-integral weight. Our study focuses on certain subfamilies of coefficients that are accessible via the Shimura correspondence. This is a generalization of the results of Inam and Wiese [20] to the setting of totally real number fields.

In **Chapter 4**, we prove that for two Hilbert cusp forms, say $\mathbf{f}$ and $\mathbf{g}$, of same level and different integral weights, there exists infinitely Fourier coefficients of $\mathbf{f}$ and $\mathbf{g}$ having the same sign (resp., having the opposite sign). We show that the simultaneous non-vanishing of the Fourier coefficients, of two non-zero distinct primitive Hilbert cuspidal non-CM eigenforms, at the powers of a fixed prime ideal has positive density. These are generalizations of some results of Gun, Kohnen and Rath [17] and Gun, Kumar and Paul [16] to the setting of totally real number fields.

**Keywords:** Elliptic curves, CM eigenforms, Fourier coefficients, Hilbert modular forms of integral and half-integral weights, Sign changes, non-vanishing.

**MSC 2010:** 11F03, 11F30, 11F31, 11F33, 11F37, 11F41, 11G05.

# Contents

# Chapter 1

# Introduction

This thesis broadly comes under the area of Number Theory, and in particular in the sub-area of modular forms. In this thesis, the basic objects of our interests are elliptic curves and modular forms.

In number theory, the Fourier coefficients of integral or half-integral weight modular forms over number fields have been extensively studied because of their rich algebraic and arithmetic properties that they encompass. In this thesis, we are interested in studying the algebraic and arithmetic properties of the Fourier coefficients attached to modular forms.

In this chapter, we shall recall the basic definitions of modular forms and its properties briefly. We shall also recall the results that we need for the next chapters. Finally, we recall the concept of old forms, newforms and the notion of complex multiplication (CM). There is nothing new in this chapter that we have contributed and we closely follow the exposition in [15].

## 1.1 Modular forms

**Definition 1.1.1.** *The modular group is the group of* $2 \times 2$ *matrices with integer entries and with determinant* $1$*. The modular group is denoted by* $\mathrm{SL}_2(\mathbb{Z})$*.*

Each element of modular group is also viewed as an automorphism of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, via the linear fractional transformation

$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right].z = \frac{az+b}{cz+d},\ z \in \hat{\mathbb{C}}.$$

If $c \neq 0$, then $\frac{-d}{c}$ maps to $\infty$ and $\infty$ maps to $\frac{a}{c}$, and if $c = 0$ then $\infty$ maps to $\infty$.

The identity matrix $I$ and its negative matrix $-I$ both give the identity transformation.

Observe that the modular group $\mathrm{SL}_2(\mathbb{Z})$ is generated by $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Hence, the group of transformations defined by the modular group is generated by the two matrix generators,

$$z \to z+1 \text{ and } z \to -1/z.$$

We define the upper half-plane $\mathcal{H}$ as

$$\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

### 1.1.2 Congruence subgroups:

Now, let us recall the definition of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. For any $N \in \mathbb{N}$, the principle congruence subgroup of level $N$ is defined by

$$\Gamma(N) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}) : \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \pmod N \right\}.$$

**Definition 1.1.3.** *A subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subseteq \Gamma$ for some $N \in \mathbb{Z}^+$. We say that $\Gamma$ is a congruence subgroup of level $N$, when $N$ is minimal such that $\Gamma(N) \subseteq \Gamma$.*

By definition, every congruence subgroup $\Gamma$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$, since $\Gamma(N)$ has finite index. Now, we define some important congruence subgroups:

$$\Gamma_0(N) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}) : \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right] \pmod N \right\}$$

and

$$\Gamma_1(N) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z}) : \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \equiv \left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right] \pmod N \right\}.$$

### 1.1.4 Modular forms:

For any matrix $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$, we define the factor of automorphy $j(\gamma, z) \in \mathbb{C}$ for $z \in \mathcal{H}$, as

$$j(\gamma, z) = cz + d.$$

For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and any integer $k \geq 1$, we define weight-$k$ operator $|_k\gamma$ on function $f : \mathcal{H} \to \mathbb{C}$ by

$$(f|_k\gamma)(z) = j(\gamma, z)^{-k} f(\gamma z), \; z \in \mathcal{H}.$$

Since the factor of automorphy is never zero or infinity, if $f$ is meromorphic then $f|_k\gamma$ is also meromorphic and has the same zeros and poles as of $f(\gamma z)$.

**Definition 1.1.5.** *A function $f : \mathcal{H} \to \mathbb{C}$ is weakly modular form of weight $k$ with respect to a congruence subgroup $\Gamma$, if $f$ meromorphic and weight-$k$ invariant under $\Gamma$, i.e., if $(f|_k\gamma) = f$ for all $\gamma \in \Gamma$.*

Every congruence subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$ contains a translation matrix of the form

$$\left[\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right] : z \to z + h,$$

for some minimal $h \in \mathbb{Z}^+$. This is because $\Gamma$ contains $\Gamma(N)$ for some $N$, but $h$ may properly divide $N$.

Every weakly modular form $f : \mathcal{H} \to \mathbb{C}$ of weight $k$ with respect to $\Gamma$ is therefore $h\mathbb{Z}$-periodic. Such a form $f$ has the corresponding function $g : D' \to \mathbb{C}$ where $D'$ is the punctured disk and $f(z) = g(q_h)$ where $q_h = e^{2\pi i z/h}$. If $f$ is holomorphic on upper half plane then $g$ is holomorphic on the punctured disk and so it has a Laurent expansion at $q = 0$. We define such a $f$ to be holomorphic at $\infty$, if $g$ extends holomorphically to open complex unit disc. Thus $f$ has a Fourier expansion

$$f(z) = \Sigma_{n=0}^{\infty} a_f(n) q_h^n, \; q_h = e^{2\pi i z/h} \text{ where } a_f(n) \in \mathbb{C}.$$

We write Fourier coefficients of $f$ by $a_n(f)$ or $a_f(n)$ depending on the context.

To keep the space of modular forms to be finite-dimensional, modular forms need to be holomorphic not only on $\mathcal{H}$ but also at limit points. For a congruence subgroup $\Gamma$ the idea is to adjoin not only $\infty$ but also the rational numbers $\mathbb{Q}$ to $\mathcal{H}$, and then identify adjoin points under $\Gamma$-equivalence. A $\Gamma$-equivalence class of points $\mathbb{Q} \bigcup \{\infty\}$ is called a cusp of $\Gamma$.

When $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, all rational numbers are $\Gamma$-equivalent to $\infty$ and so $\mathrm{SL}_2(\mathbb{Z})$ has only one cusp, represented by $\infty$. But when $\Gamma$ is a proper subgroup of $\mathrm{SL}_2(\mathbb{Z})$ fewer points are $\Gamma$-equivalent and so $\Gamma$ will have other cusps as well, represented by rational numbers. Since each $s \in \mathbb{Q}$ takes the form $s = \alpha(\infty)$ for some $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, the number of cusps is at most the number of cosets $\Gamma\alpha$ in $\mathrm{SL}_2(\mathbb{Z})$, but possibly fewer, a finite number since the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ is finite.

A modular form with respect to a congruence subgroup $\Gamma$ should be holomorphic at the cusps. Writing any $s \in \mathbb{Q}\bigcup\{\infty\}$ as $s = \alpha(\infty)$, holomorphy at $s$ is naturally defined in terms of holomorphy at $\infty$ via the operator $|_k\alpha$. Since $f|_k\alpha$ is holomorphic on $\mathcal{H}$ and weakly modular with respect to $\alpha^{-1}\Gamma\alpha$, again a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the notion of its holomorphy at $\infty$ makes sense.

**Definition 1.1.6.** *Let $\Gamma$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let $k \geq 1$ be an integer. A function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight $k$ with respect to $\Gamma$ if*

1. *$f$ is holomorphic on $\mathcal{H}$,*
2. *$f$ is weight-$k$ invariant under $\Gamma$, i.e., $f|_k\gamma = f$ for all $\gamma \in \Gamma$,*
3. *$f|_k\alpha$ is holomorphic at $\infty$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.*

*In addition,*

4. *if the constant term in the Fourier expansion of $f|_k\alpha$ is zero, for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, then $f$ is a cusp form of weight $k$ with respect to $\Gamma$.*

The modular forms of weight $k$ with respect to $\Gamma$ are denoted by $M_k(\Gamma)$ and the cusp forms weight $k$ with respect to $\Gamma$ are denoted by $S_k(\Gamma)$.

**Definition 1.1.7.** *For any positive integer $N$, a Dirichlet character modulo $N$ is a homomorphism of multiplicative groups*

$$\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*.$$

*The trivial Dirichlet character is denoted by $\chi_{\mathrm{triv}}$.*

The group $\Gamma_0(N)/\Gamma_1(N)$ acts on the space of modular forms $M_k(\Gamma_1(N))$, and the space decomposes into a direct sum of subspaces, as

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi:(\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*} M_k(N, \chi),$$

where for any Dirichlet character $\chi$ modulo $N$, the $\chi$-eigenspace of $M_k(\Gamma_1(N))$ defined as $M_k(N, \chi) := \{f \in M_k(\Gamma_1(N)) : f|_k\gamma = \chi(d_\gamma)f \text{ for all } \gamma \in \Gamma_0(N)\}$, where $d_\gamma$ denote the lower right entry of $\gamma$. In particular, the eigenspace $M_k(N, \chi_{\mathrm{triv}})$ is $M_k(\Gamma_0(N))$, which we denote by $M_k(N)$, for simplicity. Similarly, for cusp forms, we have the following decomposition:

$$S_k(\Gamma_1(N)) = \bigoplus_{\chi:(\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*} S_k(N, \chi),$$

where $S_k(N, \chi) := S_k(\Gamma_1(N)) \cap M_k(N, \chi)$. In particular, the eigenspace $S_k(N, \chi_{\text{triv}})$ is $S_k(\Gamma_0(N))$, which we denote by $S_k(N)$, for simplicity.

In the next section, we shall define a set of linear transformations between the spaces of modular forms, which respects the subspace of cusp forms.

## 1.2 Hecke Operators:

Let $\Gamma_1$ and $\Gamma_2$ be two congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Let $\mathrm{GL}_2^+(\mathbb{Q})$ denote the group of $2 \times 2$ matrices with rational entries and positive determinant.

**Definition 1.2.1.** *For each $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, the set*

$$\Gamma_1 \alpha \Gamma_2 = \{\gamma_1 \alpha \gamma_2 : \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\}$$

*is a double coset in $\mathrm{GL}_2^+(\mathbb{Q})$.*

Basically, the action of the double coset $\Gamma_1 \alpha \Gamma_2$ transform the space of modular forms of level $\Gamma_1$ to modular forms of level $\Gamma_2$, which we will explain now.

The group $\Gamma_1$ acts on the double coset $\Gamma_1 \alpha \Gamma_2$ by left multiplication, partitioning it into orbits. A typical orbit is $\Gamma_1 \beta$ with representative $\beta = \gamma_1 \alpha \gamma_2$, and the orbit space $\Gamma_1 \setminus \Gamma_1 \alpha \Gamma_2$ is thus a finite disjoint union $\bigcup \Gamma_1 \beta_j$ for some choice of representative $\beta_j$ of the form $\alpha \gamma_j$.

**Definition 1.2.2.** *For congruence subgroups $\Gamma_1$ and $\Gamma_2$ of $\mathrm{SL}_2(\mathbb{Z})$ and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, the weight-$k$ operator $|_k(\Gamma_1 \alpha \Gamma_2)$ maps the modular forms $f \in M_k(\Gamma_1)$ to*

$$f|_k(\Gamma_1 \alpha \Gamma_2) = \Sigma_j f|_k \beta_j,$$

*where $\beta_j$ are orbit representative, i.e., $\Gamma_1 \alpha \Gamma_2 = \cup_j \Gamma_1 \beta_j$ is a disjoint union.*

The double coset operator is well-defined, i.e., it is independent of how the representatives $\beta_j$ are chosen, and it maps modular forms with respect to $\Gamma_1$ to modular forms with respect to $\Gamma_2$,

$$|_k(\Gamma_1 \alpha \Gamma_2) : M_k(\Gamma_1) \to M_k(\Gamma_2).$$

The double coset operator preserves the subspace of cusp forms, i.e.,

$$|_k(\Gamma_1 \alpha \Gamma_2) : S_k(\Gamma_1) \to S_k(\Gamma_2).$$

An inclusion of congruence subgroups $\Gamma_1(N) \subseteq \Gamma_0(N)$ induces the reverse inclusion

$$M_k(\Gamma_1(N)) \supseteq M_k(\Gamma_0(N)).$$

Now, we shall introduce two types of operators on $M_k(\Gamma_1(N))$, which preserves the subspace of cusp forms.

### 1.2.3 First type of Hecke operators:

The map $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^*$ taking $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to $d \pmod N$ is surjective homomorphism with kernel $\Gamma_1(N)$. This shows that $\Gamma_1(N)$ is normal in $\Gamma_0(N)$ and induces an isomorphism

$$\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^* \text{ where } \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \to d \pmod N.$$

For any $\alpha \in \Gamma_0(N)$, consider the weight-$k$ operator $|_k(\Gamma_1(N)\alpha\Gamma_1(N))$ for modular forms $f \in M_k(\Gamma_1(N))$ to

$$f|_k(\Gamma_1(N)\alpha\Gamma_1(N)) = f|_k\alpha,\ \alpha \in \Gamma_0(N),$$

again in $M_k(\Gamma_1(N))$, since $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$. Thus, the group $\Gamma_0(N)$ acts on $M_k(\Gamma_1(N))$, and since its subgroup $\Gamma_1(N)$ acts trivially, hence this is really an action of the quotient $(\mathbb{Z}/N\mathbb{Z})^*$.

**Definition 1.2.4.** *The action of $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, determined by $d \pmod N$, and denoted by $\langle d \rangle$, is*

$$\langle d \rangle : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$$

*given by*

$$\langle d \rangle f = f|_k\alpha \text{ for any } \alpha = \left[\begin{smallmatrix} a & b \\ c & \delta \end{smallmatrix}\right] \in \Gamma_0(N) \text{ with } \delta \equiv d \pmod N.$$

*These are the first type of Hecke operators and also called as diamond operators. These operators respect the subspace of cusp forms.*

Now we can re-interpret the definition of $S_k(N, \chi)$ as follows. For any character $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$, the space $S_k(N, \chi)$ is precisely the $\chi$-eigenspace of the diamond operators,

$$S_k(N, \chi) = \{f \in S_k(\Gamma_1(N)) : \langle d \rangle f = \chi(d) f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^*\}.$$

This also means that the diamond operators $\langle d \rangle$ respects the decomposition

$$S_k(\Gamma_1(N)) = \bigoplus_{\chi:(\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*} S_k(N, \chi).$$

### 1.2.5 Second type of Hecke operators:

For every prime $p$, we shall define the second type of Hecke operators, denoted by $T_p$, on the space of modular forms $M_k(\Gamma_1(N))$.

**Definition 1.2.6.** *For any prime $p$, the weight $k$-operator $|_k(\Gamma_1(N)\left[\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right]\Gamma_1(N))$ on the space of modular forms $M_k(\Gamma_1(N))$ is denoted by $T_p$. These operators respect the subspace of cusp forms.*

Observe that, the double coset

$$\Gamma_1(N)\left[\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right]\Gamma_1(N) = \left\{\gamma \in M_2(\mathbb{Z}) : \gamma \equiv \left[\begin{smallmatrix}1 & *\\ 0 & p\end{smallmatrix}\right] \pmod N, \det\gamma = p\right\}.$$

So, in fact $\left[\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right]$ can be replaced by any matrix in this double coset to define the operator $T_p$. For $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$, we have the following proposition

**Proposition 1.2.7.** ([15, Proposition 5.2.1]) Let $N \in \mathbb{N}$ and $p$ be a prime. The operator $T_p = |_k(\Gamma_1(N)\left[\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right]\Gamma_1(N))$ on $M_k(\Gamma_1(N))$ is given by

$$T_p f = \begin{cases} \Sigma_{j=0}^{p-1} f|_k(\left[\begin{smallmatrix}1 & j\\ 0 & p\end{smallmatrix}\right]) & \text{if } p \mid N,\\ \Sigma_{j=0}^{p-1} f|_k(\left[\begin{smallmatrix}1 & j\\ 0 & p\end{smallmatrix}\right]) + f|_k(\left[\begin{smallmatrix}m & n\\ N & p\end{smallmatrix}\right]\left[\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right]) & \text{if } p \nmid N, \text{where } mp - nN = 1.\end{cases}$$

Letting $\Gamma_1 = \Gamma_2 = \Gamma_0(N)$ instead and keeping $\alpha = \left[\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}\right]$ gives the same orbit representative for $\Gamma_1 \backslash \Gamma_1 \alpha \Gamma_2$, but in this case the last representative can be replaced by $\beta_\infty = \left[\begin{smallmatrix}p & 0\\ 0 & 1\end{smallmatrix}\right]$, since $\left[\begin{smallmatrix}m & n\\ N & p\end{smallmatrix}\right] \in \Gamma_0(N)$. One can explicitly write down the Fourier expansion of $T_p f$ with respect to the Fourier coefficients of $f$ (cf. [15, Proposition 5.2.2]). The following proposition says that the Hecke operators commute.

**Proposition 1.2.8.** ([15, Proposition 5.2.4]) Let $d$ and $e$ be elements of $(\mathbb{Z}/N\mathbb{Z})^*$, and let $p$ and $q$ be prime. Then

(a) $\langle d \rangle T_p = T_p \langle d \rangle$,

(b) $\langle d \rangle \langle e \rangle = \langle e \rangle \langle d \rangle$,

(c) $T_p T_q = T_q T_p$.

So far, we have defined the Hecke operators $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^*$ and $T_p$ operators for any prime $p$. Now, we will generalize these operators to $\langle n \rangle$ for $n \in \mathbb{Z}^+, (n, N) = 1$ and $T_n$ for all $n \in \mathbb{Z}^+$.

For $n \in \mathbb{Z}^+$ with $(n, N) = 1$, $\langle n \rangle$ is determined by $n \pmod N$. For $n \in \mathbb{Z}^+$ with $(n, N) > 1$, define $\langle n \rangle = 0$, the zero operator on $M_k(\Gamma_1(N))$. The mapping $n \to \langle n \rangle$ is totally multiplicative. To define $T_n$, set $T_1 = 1$; for prime powers, the operator is defined inductively

$$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} \langle p \rangle T_{p^{r-2}}, \text{ for } r \geq 2, \tag{1.1}$$

and as $T_{p^r} T_{q^s} = T_{q^s} T_{p^r}$ for distinct primes $p$ and $q$, one can extend the multiplicatively of Hecke operators to $T_n$ for all $n$,

$$T_n = \prod T_{p_i^{r_i}} \text{ where } n = \prod p_i^{r_i}. \tag{1.2}$$

By Proposition 1.2.8, we get that $T_{nm} = T_n T_m$ if $(n, m) = 1$. Now, we shall recall a proposition about the Fourier expansion of $T_n(f)$ with respect to the Fourier coefficients of $f$.

**Proposition 1.2.9.** ([15, Proposition 5.3.1]) Suppose $f \in M_k(\Gamma_1(N))$ has the Fourier expansion

$$f(z) = \Sigma_{m=0}^{\infty} a_m(f) q^m, \text{ where } q = e^{2\pi i z}.$$

Then, for all $n \in \mathbb{Z}^+$, $T_n(f)$ has Fourier expansion

$$(T_n f)(z) = \Sigma_{n=0}^{\infty} a_m(T_n f) q^m,$$

where

$$a_m(T_n f) = \sum_{d|(m,n)} d^{k-1} a_{mn/d^2}(\langle d \rangle f). \tag{1.3}$$

In particular, if $f \in M_k(N, \chi)$ then

$$a_m(T_n f) = \sum_{d|(m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f). \tag{1.4}$$

## 1.3 Old forms and Newforms

In this section, we shall recall the basic theory of old forms and newforms. The space of cusp forms $S_k(\Gamma_1(N))$ equipped with an inner product, which is known as the Petersson inner product (cf. [15, Section 5.4]).

By the Spectral theorem for finite-dimensional inner product space $V$, given a commuting family of normal operators $\mathcal{F}$ on $V$, the space $V$ has an orthogonal basis of eigenvectors for all the operators $T \in \mathcal{F}$. In our context, we refer to such eigenvectors by eigenforms.

**Theorem 1.3.1.** *The space $S_k(\Gamma_1(N))$ has an orthogonal basis of simultaneous eigenforms for the Hecke operators $\{\langle n\rangle, T_n : (n, N) = 1\}$*

There is a way to move between levels, i.e., taking modular forms of lower levels $M$ to higher level $N$. For example, if $M \mid N$ then $S_k(\Gamma_1(M)) \subseteq S_k(\Gamma_1(N))$. There is another way to embed $S_k(\Gamma_1(M))$ into $S_k(\Gamma_1(N))$ is by composing with multiply-by-$d$ map where $d$ is any factor of $N/M$. For any such $d$, let

$$\alpha_d = \left[\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right].$$

So $f|_k\alpha_d(z) = d^{k-1}f(dz)$ for any modular form $f$. The map $|_k\alpha_d$ takes $S_k(\Gamma_1(M))$ to $S_k(\Gamma_1(N))$, lifting the level from $M$ to $N$.

**Definition 1.3.2.** *For each divisor $d$ of $N$, let $i_d$ be the map*

$$i_d : (S_k(\Gamma_1(Nd^{-1})))^2 \to S_k(\Gamma_1(N))$$

*given by*

$$(f, g) \to f + g|_k\alpha_d.$$

*The subspaces of old forms at level $N$ is*

$$S_k(\Gamma_1(N))^{\text{old}} = \sum_{p|N,\ \text{prime}} i_p((S_k(\Gamma_1(Np^{-1})))^2)$$

*and the subspace of newforms at level $N$ is the orthogonal complement with respect to the Petersson inner product,*

$$S_k(\Gamma_1(N))^{\text{new}} = (S_k(\Gamma_1(N))^{\text{old}})^{\perp}.$$

Since the Hecke operators preserve the spaces $S_k(\Gamma_1(N))^{\text{old}}$ and $S_k(\Gamma_1(N))^{\text{new}}$, the following result is a consequence of the spectral theorem.

**Proposition 1.3.3.** ([15, Corollary 5.6.3]) The spaces $S_k(\Gamma_1(N))^{\text{old}}$ and $S_k(\Gamma_1(N))^{\text{new}}$ have orthogonal bases of eigenforms for the Hecke operators $\{T_n, \langle n\rangle : (n, N) = 1\}$.

**Definition 1.3.4.** *A non-zero modular form $f \in M_k(\Gamma_1(N))$ that is an eigenform for the Hecke operators $T_n$ and $\langle n \rangle$ for all $n \in \mathbb{Z}^+$ is a Hecke eigenform or simply an eigenform. The eigenform $f(z) = \sum_{n=0}^{\infty} a_n(f) q^n$ is normalized when $a_1(f) = 1$. A newform, or a primitive form, is a normalized eigenform in $S_k(\Gamma_1(N))^{\text{new}}$.*

**Theorem 1.3.5.** *Let $f \in S_k(\Gamma_1(N))^{\text{new}}$ be a nonzero eigenform for the Hecke operators $T_n$ and $\langle n \rangle$ for all $n$ with $(n, N) = 1$. Then*

*(a) $f$ is a Hecke eigenform, i.e., an eigenform for $T_n$ and $\langle n \rangle$ for all $n \in \mathbb{N}$. A suitable scalar multiple of $f$ is a newform.*

*(b) If $\tilde{f}$ satisfy the same condition as $f$ and has the same $T_n$-eigenvalues, then $\tilde{f} = cf$ for some constant $c$.*

*The set of newforms is an orthogonal basis of $S_k(\Gamma_1(N))^{\text{new}}$. Each such newform lies in an eigenspace $S_k(N, \chi)$ and satisfies $T_n f = a_n(f) f$ for all $n \in \mathbb{Z}^+$, i.e., the $n$-th Fourier coefficient coincides with the $T_n$-eigenvalues, and $\langle d \rangle f = \chi(d) f$ for any $d \in \mathbb{N}$.*

### 1.3.6 Elliptic Curves:

**Definition 1.3.7.** *An elliptic curve $E$ over a field $K$ is a non-singular projective plane curve over $k$ defined by the equation*

$$y^2 z + a_1 xyz + a_3 yz^2 = x^3 + a_2 x^2 z + a_4 xz^2 + a_6 z^3,$$

*where $a_1, \ldots, a_6 \in K$.*

For a number field $K$, the group of rational points on elliptic curve $E$ defined over $K$, denoted by $E(K)$, is a finitely generated abelian group, by the Mordell-Weil theorem. Hence, one can talk about the rank and torsion group of $E(K)$.

There is a correspondence between the elliptic curves over $\mathbb{Q}$ with weight $2$ modular forms with rational Fourier coefficients, which is known as the modularity theorem.

**Theorem 1.3.8.** ([44], [8]) Let $E$ be an elliptic curve over $\mathbb{Q}$ with conductor $N_E$. Then there exist a primitive form $f \in S_2(N_E)$ such that

$$a_p(f) = a_p(E) \text{ for all primes } p,$$

where $a_p(E)$ is $p + 1$ minus the number of points on the reduced elliptic curve $E$ modulo $p$.

### 1.3.9 CM forms

In this section, we shall recall the notion of CM modular forms (cf. [19, page 717]). Let $K$ be an imaginary quadratic field and $\mathcal{O}_K$ be its integral closure. Let $\mathfrak{m}$ be an integral ideal of $K$ and let $I(\mathfrak{m})$ be the group of fractional ideals of $K$ co-prime to $\mathfrak{m}$. By definition, a Hecke character $\Psi$ of $K$ is a homomorphism

$$\Psi : I(\mathfrak{m}) \to \mathbb{C}^*,$$

such that $\Psi((\alpha)) = \alpha^r$ for all $\alpha \in K^*$ with $\alpha \equiv 1 \pmod{\mathfrak{m}}$. For such a Hecke character $\Psi$, one can associate the function $f_\Psi$ defined by

$$f_\Psi(z) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K, (\mathfrak{a},\mathfrak{m})=1} \Psi(\mathfrak{a}) e^{2\pi i (N\mathfrak{a}) z}, \tag{1.5}$$

where $N\mathfrak{a}$ denotes the norm of $\mathfrak{a}$. We can re-write the function as

$$f_\Psi(z) = \sum_{n \geq 1} a_\Psi(n) e^{2\pi i n z},$$

where

$$a_\Psi(n) = \sum_{(\mathfrak{a},\mathfrak{m})=1, N\mathfrak{a}=n} \Psi(\mathfrak{a}). \tag{1.6}$$

We see that $a_\Psi(p) = 0$, if $p$ is does not split in $K$. By Hecke's theorem, the modular form $f_\Psi \in S_{r+1}(|d_K|N\mathfrak{m}, \epsilon)$ is an eigenform, where $d_K$ is discriminant of field $K$, $\epsilon$ is a character modulo $|d_K|N\mathfrak{m}$.

**Definition 1.3.10.** *We say that a cuspidal eigenform $f$ is CM if $f = f_\Psi$ for some Hecke character $\Psi$ of some imaginary quadratic field $K$.*

# Chapter 2

# On the gaps between non-zero Fourier coefficients of cuspidal CM eigenforms

## 2.1 Introduction

Let $f$ be a primitive form of weight $k \geq 2$ and level $N \geq 1$. By definition, every such eigenform $f$ has a Fourier expansion, say $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n$, where $q = e^{2\pi iz}$.

A famous conjecture of Lehmer predicts that $\tau(n) \neq 0$ for any $n \geq 1$, where

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n$$

is the unique normalized cuspidal eigenform of weight $12$ and level $1$. This conjecture motivated many mathematicians to study the vanishing or non-vanishing of the Fourier coefficients of modular forms.

In order to prove the non-vanishing of Fourier coefficients, a natural way is to bound the the size of possible gaps between the non-zero Fourier coefficients, and show that they are arbitrarily small. This problem has been extensively studied by many mathematicians in several different directions through various approaches (cf. [5], [7], [34]). Typically, the approaches are either by using Rankin estimates, or Chebotarev density theorem, or distribution of $\mathcal{B}$-free numbers, etc.

In [41], Serre initiated a general study of estimating the size of possible gaps between the non-zero Fourier coefficients of modular cusp forms. In fact, for any

$n \in \mathbb{N}$, he defined a function

$$i_f(n) := \max\{i : a_f(n+j) = 0 \text{ for all } 0 \le j \le i\}.$$

So, in order to estimate the size of possible gaps between the non-zero Fourier coefficients, it is enough to find bounds for the function $i_f(n)$.

Recall that, for two functions $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{N} \to \mathbb{R}$ such that $g(n) > 0$ for all $n$ sufficiently large. One writes $f(n) \ll g(n)$ as $n \to \infty$ if there exists an $M \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that $|f(n)| \le Mg(n)$ for all $n \ge n_0$.

In next section, we shall discuss the history of $i_f(n)$.

## 2.2 History

Recall that $\tau(n)$ denote the Fourier coefficient of $q^n$ in the series

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n,$$

where $\Delta(z)$ is the unique normalized cusp form of weight $12$ and level $1$. In [41], Serre showed that $\tau(n)$ is non-zero for the vast majority of $n$. However, Lehmer's speculation that $\tau(n) \neq 0$ for every positive integer $n$ remains open till today. In the same article, Serre proposes the study of the non-vanishing of Fourier coefficients in short intervals. In particular, if

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k(N, \chi),$$

then the study reduces to finding an upper bound for the function $i_f(n)$. In fact, in [41], he proved that if $f(z)$ is a cusp form of integral weight $k \ge 2$ such that $f(z)$ is not a linear combination of forms with CM, then

$$i_f(n) \ll n. \tag{2.1}$$

In view of this estimation, Serre poses the following question:

**Question 2.2.1.** *If $f(z)$ is a non-zero cusp form of integral weight $k \ge 2$ such that $f(z)$ is not a linear combination of forms with CM, then can the estimate* (2.1) *be improved to* *an estimate of the form*

$$i_f(n) \ll_f n^{\delta} \text{ for some } 0 < \delta < 1?,$$

*where the symbol* $\ll_f$ *means that the implied constant depends only on* $f$*.*

More generally, are there analogous results for the forms with non-integral weights? Such questions are directly related to some examples founds by Knopp and Lehner in [25, Theorem 5.3].

### 2.2.2 By Rankin's Method:

The first response to Serre's question occurs in an article of V. Kumar Murty (cf. [35]), where he observed that the Question 2.2.1 can be answered by using a classical result of Rankin in [39]. Rankin showed that there is a positive constant $A_f$ for which

$$\sum_{n \leq X} |a_f(n)|^2 n^{1-k} = A_f X + O(X^{3/5})$$

holds. As a consequence, we get that

$$i_f(n) \ll_f n^{3/5}.$$

This answers Question 2.2.1 affirmatively and now the question is to improve the exponent $\delta$.

### 2.2.3 Theory of $\mathcal{B}$-free Numbers:

In [12], Balog and Ono considered the stronger forms of the Serre's question. In fact, they realized that the theory of $\mathcal{B}$-free numbers shall be useful to answer the questions related to $i_f(n)$. Let us recall the definition of $\mathcal{B}$-free numbers.

**Definition 2.2.4.** *Let* $\mathcal{B} := \{b_1, b_2, \dots\}$ *be such that*

$$(b_i, b_j) = 1 \textit{ for } i \neq j \textit{ and } \sum_{i=1}^{\infty} \frac{1}{b_i} < \infty.$$

*A number* $n \in \mathbb{N}$ *is said to be* $\mathcal{B}$*-free if it is not divisible by any element of the set* $\mathcal{B}$*.*

- The set $\{p^2 : p \in \mathbb{P}\}$ can be taken as $\mathcal{B}$, because $\sum_{p \in \mathbb{P}} \frac{1}{p^2} < \infty$. For this choice of $\mathcal{B}$, a number $n \in \mathbb{N}$ is $\mathcal{B}$-free if and only if it is square-free.

- Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n$ be a primitive eigenform of level $N \geq 1$, weight $k \geq 2$ and without CM. The set $C = \{p \in \mathbb{P} \,|a_f(p) = 0\}$ can be taken as $\mathcal{B}$. This is because
$$\sum_{p \in C} \frac{1}{p} < \infty,$$
which follows from a result of Serre ( [41]). This is one of main reason why many authors consider only non-CM forms, especially when one uses the theory of $\mathcal{B}$-free numbers.

In [12], Balog and Ono proved that:

**Theorem 2.2.5.** ([12]) Suppose that $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(N,\chi)$ is a non-zero cusp form of weight $k \geq 2$ such that $f(z)$ is not a linear combination of forms with CM. For every $\varepsilon > 0$ and $X^{\frac{17}{41}} \leq Y$, we get

$$Y \ll_{f,\varepsilon} \#\{X < n < X+Y : a_f(n) \neq 0\}, \tag{2.2}$$

the symbol $\ll_{f,\varepsilon}$ means that the implied constant depends on $f$ and $\varepsilon$. In particular, we have
$$i_f(n) \ll_{f,\varepsilon} n^{\frac{17}{41}+\varepsilon}.$$

In *loc.cit.*, they have also considered similar questions for weight $1$ forms, half-integral cusp forms of weight $\geq \frac{3}{2}$, that are not a linear combinations of forms with CM and obtained similar results for the Fourier coefficients in short intervals.

In [1], Alkan has improved the results of [12] and proved the following theorem for elliptic curves:

**Theorem 2.2.6.** ([1, Theorem 2]) Let $E$ be an elliptic curve over $\mathbb{Q}$ without CM. If $f_E(z) = \sum_{n=1}^{\infty} a_E(n)q^n$ is the associated weight $2$ newform, then for any $\varepsilon > 0$ and $X^{\frac{69}{169}+\varepsilon} \leq Y$, we have

$$\#\{X < n < X+Y : a_E(n) \neq 0\} \gg_{E,\varepsilon} Y.$$

In particular, we have
$$i_{f_E}(n) \ll_{E,\varepsilon} n^{\frac{69}{169}+\varepsilon}$$
for every $\varepsilon > 0$.

In the same article, Alkan proved the following conditional result for cusp forms of integral weight $\geq 2$:

**Theorem 2.2.7.** ([1, Theorem 3]) Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(N, \chi)$ be a non-zero cusp form of integral weight $k \geq 2$ such that $f(z)$ is not a linear combination of forms with CM. Assuming the GRH for Dedekind Zeta functions, for every $\varepsilon > 0$ and $X^{\frac{69}{169}+\varepsilon} \leq Y$, we have

$$\#\{X < n < X + Y : a_f(n) \neq 0\} \gg_{f,\varepsilon} Y.$$

In particular, we have

$$i_f(n) \ll_{f,\varepsilon} n^{\frac{69}{169}+\varepsilon}$$

for every $\varepsilon > 0$.

In [2], Alkan improved the above results further and proved the following:

**Theorem 2.2.8.** ([2, Theorem 4])

(a) Let $E$ be an elliptic curve over $\mathbb{Q}$ without CM. For every $\varepsilon > 0$ and $x^{51/134+\varepsilon} \leq y$, we have

$$\#\{x - y < n \leq x : a_E(n) \neq \ 0\} \gg_{E,\varepsilon} y.$$

In particular,

$$i_{f_E}(n) \ll_{E,\varepsilon} n^{51/134+\varepsilon}.$$

(b) Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(N, \chi)$ be a non-zero cusp form with integral weight $k \geq 2$ such that $f(z)$ is not a linear combination of forms with CM. Assuming the GRH for Dedekind Zeta functions, for every $\varepsilon > 0$ and $x^{51/134+\varepsilon} \leq y$, we have

$$\#\{x - y < n \leq x : a_f(n) \neq \ 0\} \gg_{f,\varepsilon} y.$$

In particular,

$$i_f(n) \ll_{f,\varepsilon} n^{51/134+\varepsilon}.$$

If we drop the assumption of GRH for Dedekind Zeta functions in the above theorem, one gets a weaker bound for $i_f(n)$.

**Theorem 2.2.9.** ([2, Theorem 5]) Let $f(z) \in S_k(N, \chi)$ be a non-zero cusp form with integral weight $k \geq 2$ such that $f(z)$ is not a linear combination of forms with CM. For every $\varepsilon > 0$ and $x^{40/97+\varepsilon} \leq y$, we have

$$\#\{x - y < n < x : a_f(n) \neq 0\} \gg_{f,\epsilon} y.$$

In particular,

$$i_f(n) \ll_{f,\epsilon} n^{40/97+\varepsilon}.$$

Currently, the best bound for $i_f(n)$ is available due to Kowalski, Robert, and Wu and they proved that for any holomorphic non-CM cuspidal eigenform $f$ on general congruence groups

$$i_f(n) \ll_f n^{7/17+\epsilon} \tag{2.3}$$

holds for all $n \in \mathbb{N}$ (cf. [27]). They obtained some new results in theory of exponential sums and in theory of $\mathcal{B}$-free numbers to prove the bound (2.3).

There is some interesting theorem in the literature, which is due to Alkan, to show that most of the times the possible gaps are extremely short. More precisely:

**Theorem 2.2.10.** ([1, Theorem 1]) Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(N, \chi)$ be a non-zero cusp form of integral weight $k \geq 2$ such that $f(z)$ is not a linear combination of forms with CM. If $\phi(x)$ is function which tends monotonically to infinity with $x$ and satisfy $\phi(2x) \ll \phi(x)$ for all large $x$, then for almost all $n$, we have

$$\#\{n < m < n + \phi(n) : a_f(m) \neq 0\} \gg_{f,\phi} \phi(n).$$

In particular, we have

$$i_f(n) \ll_{f,\phi} \phi(n)$$

for almost all $n$.

### 2.2.11 By Congruences:

For the first time, Alkan and Zaharescu, in [4], have exploited the idea of using congruences to study $i_f(n)$. There it was shown that $i_\Delta(n) \ll n^{\frac{1}{4}}$, where $\Delta$ is the unique normalized cuspidal eigenform of weight 12. The proof relies on the existence of sums of squares in short intervals of the form $(x, x + x^{\frac{1}{4}})$.

**For level $N = 1$:**

Das and Ganguly extended the above idea to show that for any non-zero eigenform $f \in S_k(1)$ with $k \geq 12$, one has $i_f(n) \ll n^{1/4} \quad \forall\, n \gg 0$, where the implied constant depends only on $k$. More precisely, they proved that:

**Theorem 2.2.12.** ([13, Theorem 1]) Given any even positive integer $k \geq 12$, there is positive constant $c$ that depends only on $k$ such that for all non-zero Hecke eigenforms $f \in S_k(1)$, $a_f(n) \neq 0$ for some integer $n \in (X, X + cX^{\frac{1}{4}})$ for all $n \geq 1154$. In particular, we have

$$i_f(n) \ll n^{1/4} \quad \forall\, n \gg 0, \tag{2.4}$$

where the implied constant depends on $k$.

In this proof, they exploit some congruence relations established by Hatada, in [18], for eigenvalues of Hecke operators acting on the space $S_k(1)$. In the case of level 1, this result is sharp when compared with the result of [27]. The implied constant in the above theorem can be made absolute, but this comes at the cost of larger exponent. In *loc. cit.*, the authors showed that

**Theorem 2.2.13.** ([13, Theorem 2]) For any fixed $\varepsilon > 0$, there exist constant $c$ and $X_0$ that depend only on $\varepsilon$ such that for any cusp form $f$ of level one, we have that $a_f(n) \neq 0$ for some $n \in (X, X + cX^{131/416+\varepsilon})$, whenever $X > X_0$.

**For level $N > 1$:**

If the level $N > 1$, then there are no similar results are available with $i_f(n) \ll n^{\frac{1}{4}}$ for all $n \gg 0$. However, in [14], Das and Ganguly were able to produce infinitely many non-isogenous elliptic curves for which (2.4) holds.

For weight $k > 2$, in [30], Kumar proved that either if a modular cuspidal eigenform $f$ of weight $2k$ is 2-adically close to an elliptic curve $E$ over $\mathbb{Q}$, which has a cyclic rational 4-isogeny or if there is a higher congruence for the prime above 2 holds between them, then (2.4) holds for $f$ as well.

He uses this fact to construct examples of non-CM, as well as CM, cuspidal eigenforms $f$ of level $N > 1$ and weight $k > 2$ for which (2.4) holds for $f$. To state these results, we need to recall some definitions.

**Higher congruences:**

For $a \in \mathbb{N}$, a commutative ring $R$ and a formal power series $f = \sum_{n=0}^{\infty} c_n q^n \in R[[q]]$, we define

$$\mathrm{ord}_{\mathfrak{q}^a} f = \inf\{n \in \mathbb{N} \cup 0 | \mathfrak{q}^a \nmid (c_n)\},$$

where $\mathfrak{q}$ is a prime ideal of $R$. Here, the convention that $\mathrm{ord}_{\mathfrak{q}^a} f = \infty$ if $\mathfrak{q}^a | (c_n)$ for all $n$.

**Definition 2.2.14.** *We say that formal power series $f_1$ and $f_2$ in $R[[q]]$ are congruent modulo $\mathfrak{q}^a$, if $\mathrm{ord}_{\mathfrak{q}^a}(f_1 - f_2) = \infty$, and we denote this by $f_1 \equiv f_2 \pmod{\mathfrak{q}^a}$.*

We need to a state a lemma before we go on to state the main results of [30]. This lemma shall also be useful in proving our main theorem of this chapter.

**Lemma 2.2.15.** ([27, Lemma 2.1]) If $f = \sum_{n=1}^{\infty} a_f(n)q^n$ is a normalized cuspidal eigenform in $S_{2k}(N, \chi)$, then there exists a natural number $M_f \geq 1$ such that for any prime $p \nmid M_f$, either $a_f(p) = 0$ or $a_f(p^r) \neq 0$ for all $r \geq 1$. If $\chi$ is trivial and $f$ has integer coefficients, then one can take $M_f = N$.

Now, we are in a position to state the main results of [30]. Let $f \in S_{2k}(N)$ be a normalized cuspidal eigenform of weight $2k$ with $k > 1$ and level $N$ with coefficient field $K$ and ring of integer $\mathcal{O}_K$. Let $N_f := \mathrm{lcm}(N, M_f)$, where $M_f$ is a natural number corresponding to $f$ as in Lemma 2.2.15. Let $\mathfrak{q}$ be a prime ideal of $\mathcal{O}_K$ lying above $2$ and let $e(\mathfrak{q}/2)$ denote the ramification index of $\mathfrak{q}$.

Let $f_E$ be the primitive form corresponding to the elliptic curve $E$ over $\mathbb{Q}$ of conductor $N_E$, which has a cyclic rational $4$-isogeny (cf. Theorem 1.3.8).

**Theorem 2.2.16.** ([30, Theorem 3.2]) Let $f$ and $f_E$ be as above. If $f \equiv f_E \pmod{\mathfrak{q}^m}$ for some $m > e(\mathfrak{q}/2)$, then

$$i_{f_E}(n) \ll n^{1/4}$$

for $n \gg 0$ and implied constant depends only on $N_f N_E$.

$2$**-adically close:**

Define a function $\alpha : \mathbb{Z} \to \mathbb{Z}$ as follows:

$$\alpha(n) = \begin{cases} 0 & \text{if } n \leq 1, \\ 1 & \text{if } n = 2, \\ n-2 & \text{if } n > 2. \end{cases}$$

**Definition 2.2.17.** *Let $k_1$ and $k_2$ be positive integers such that $2k_1 \equiv 2k_2 \pmod{2^s}$ for some integer $s \geq 1$. For $i = 1, 2$, suppose $f_i$ are cuspidal eigenforms on $\Gamma_0(N_i)$ of level $N_i$ and weight $k_i$ with coefficients in $\mathcal{O}_K$.*

*We say that $f_1$ and $f_2$ are $2$-adically close, if there exist a prime ideal $\mathfrak{q}$ over $2$ in $\mathcal{O}_K$ with ramification index $e(\mathfrak{q}/2)$ and an integer $m$ with $s \geq \alpha\left(\lceil \frac{m}{e(\mathfrak{p})/2} \rceil\right) \geq 1$ such that*

$$a_{f_1}(p) \equiv a_{f_2}(p) \pmod{\mathfrak{q}^m},$$

*for all primes $p \nmid 2N_1N_2$.*

There is a slight difference between the notion of higher congruence and the notion of 2-adically close between two modular forms. In the former, we require that the congruences hold between all the Fourier coefficients of modular forms, where as in the latter, we require that the congruences hold between the Fourier coefficients at prime numbers, which are away from the levels. In [30], the author prove the following theorem:

**Theorem 2.2.18.** ([30, Theorem 4.3]) Let $f$ and $f_E$ be as in Theorem 2.2.16. If $f$ and $f_E$ are 2-identically close, then

$$i_{f_E}(n) \ll n^{1/4}$$

for $n \gg 0$ and implied constant depends only on $N_f N_E$.

By using Theorem 2.2.16 and Theorem 2.2.18, the author has constructed examples of CM and non-CM cuspidal eigenforms of weight $k > 2$ and level $N > 1$ for which (2.4) holds. A priori, one does not know how to construct infinitely many cuspidal eigenforms of level $N > 1$ and weight $k > 2$ for which (2.4) holds, i.e., generalization of results of [14] to higher weights. In this chapter, we shall answer this question.

## 2.3 Main theorem

The main theorem of this chapter is

**Theorem 2.3.1.** *Let $E$ be an elliptic curve over $\mathbb{Q}$ of conductor $N$ with CM by $\mathbb{Q}(i)$ and $f = f_E \in S_2(N)^{\mathrm{new}}$ be the corresponding cuspidal Hecke eigenform. Then,*

$$i_f(n) \ll n^{\frac{1}{4}}$$

*for $n \gg 0$, where the implied constant depends on $E$.*

This result is in fact the first result in the direction of elliptic curves $E$ over $\mathbb{Q}$ with CM. In the case of elliptic curves $E$ over $\mathbb{Q}$ without CM, Alkan showed that the bound $i_f(n) \ll_{f,\epsilon} n^{\frac{51}{134}+\epsilon}$ holds for any $\epsilon > 0$ (cf. Theorem 2.2.8 in the text). In comparison with Alkan's result, our bound is even sharper (i.e., $\frac{1}{4} < \frac{51}{134}$). Before moving to the proof of Theorem 2.3.1, we shall recall the main ingredients of the proof.

The following theorem tells us about the existence of a number in a small interval, which can be written as sum of two squares and co-prime to a given number. More precisely:

**Theorem 2.3.2.** ([13, Theorem 1]) Given any integer $N \in \mathbb{N}$, there exists $X_0 \in \mathbb{R}^+$ and $c > 0$ (depending only on $N$) such that there exists an integer $n$ which is a sum of two squares and co-prime to $N$ in intervals of type $(X, X + cX^{\frac{1}{4}})$ for all $X \gg X_0$.

The following theorem is due to Deuring, which plays a crucial role in the proof of Theorem 2.3.1.

**Theorem 2.3.3.** ([32, Chapter 3, Theorem 12]) Let $E$ be an elliptic curve over $\mathbb{Q}$ with CM by $K$. Let $\Psi$ be the corresponding Hecke character. If $p \geq 5$, then the number $a_E(p)$ is zero if and only if either $p$ is inert or ramified in $K$ or divides the conductor of $\Psi$ (equivalently, the elliptic curve has bad reduction).

Let $f$ be an eigenform of weight $2$ with trivial nebentypus with CM by $K$. Let $\Psi$ be the Hecke character corresponding to $f$, i.e., $f = f_\Psi = \sum_{n=1}^{\infty} a_\Psi(n) q^n$. Consider the Hecke character $\Psi^m$, for some odd $m$, then the corresponding $f_{\Psi^m} = \sum_{n=1}^{\infty} a_{\Psi^m}(n) q^n$ is an eigenform with CM of weight $m + 1$ with trivial character. However, the eigenform $f_{\Psi^m}$ may not be a newform unless $\Psi^m$ is primitive. Now, we state a result of Laptyeva and Kumar Murty from [31], which provides an information about the vanishing of Fourier coefficients between the eigenforms $f_\Psi$ and $f_{\Psi^m}$.

**Proposition 2.3.4.** ([31, Proposition 5.1]) Let $p$ be a prime number. For any odd natural number $m \in \mathbb{N}$, the following statements are equivalent:

1. $a_{\Psi^m}(p) \equiv 0 \pmod{p}$,

2. $a_\Psi(p) \equiv 0 \pmod{p}$.

Moreover, if $p \geq 5$, the above statements are equivalent to the following:

1. $a_{\Psi^m}(p) = 0$,

2. $a_\Psi(p) = 0$,

3. $p$ is inert or ramified in $K$ or divides the conductor of $\Psi$.

Now, we are in a position to prove Theorem 2.3.1.

### 2.3.5 Proof of Theorem 2.3.1

*Proof.* By the modularity theorem, Theorem 1.3.8 in the text, the elliptic curve $E$ over $\mathbb{Q}$ with CM corresponds to a CM eigenform $f$ with trivial nebentypus . Let $\Psi$ be the Hecke character corresponding to $f$.

Take $N = 6M_f\mathrm{cond}(\Psi)$, where $M_f$ as in Lemma 2.2.15. By Theorem 2.3.2, there exists $X_0 \in \mathbb{R}$ and $c > 0$ (depending only on $N$) such that there exists an integer, say $m$, which is a sum of two squares and co-prime to $N$ in intervals of type $(X, X + cX^{\frac{1}{4}})$ for all $X \gg X_0$. So, to prove the theorem, it is sufficient to show that $a_f(m)$ is non-zero.

Since $m$ is a sum of squares and $(m, 6) = 1$, $m$ can be written as

$$m = \prod_{p_i \equiv 1 \pmod 4} p_i^{r_i} \prod_{q_i \equiv 3 \pmod 4} q_i^{2s_i}. \tag{2.5}$$

By (1.6), we see that for inert primes $q$ of $\mathbb{Q}(i)$, the Fourier coefficients $a_f(q)$ are zero because there are no ideals of norm $q$. By the quadratic reciprocity law, the odd primes $q$ which remain inert in $\mathbb{Q}(i)$ are exactly the primes $q \equiv 3 \pmod 4$. However, the power of $q_i$ in $m$ are even and we show that $a_f(q_i^{2s_i})$ is non-zero. This is because, the Hecke relations would imply that

$$a_f(q_i^r) = a_f(q_i)a_f(q_i^{r-1}) - q_i a_f(q_i^{r-2}), \tag{2.6}$$

as $a_f(q_i) = 0$, (2.6) would imply that $a_f(q_i^{2r}) = (-q_i)^r a_f(q_i^{2r-2})$. Since $a_f(q_i^2)$ is non-zero, we see that $a_f(q_i^{2r})$'s are also non-zero, for all $r \geq 1, i \geq 1$.

For any split prime $p$ of $\mathbb{Q}(i)$, the Fourier coefficient $a_f(p)$ is non-zero, by Theorem 2.3.3 and $(m, 6\mathrm{cond}(\Psi)) = 1$. This implies that $a_f(p^r) \neq 0$ for all $r \geq 1$, since $(m, M_f) = 1$ and by Lemma 2.2.15. This shows that, for $p \equiv 1 \pmod 4$, we have $a_f(p^r) \neq 0$ for all $r \geq 1$, since the odd primes $p \equiv 1 \pmod 4$ are exactly the split primes of $\mathbb{Q}(i)$. Hence

$$a_f(m) = \prod_{p_i \equiv 1 \pmod 4} a_f(p_i^{r_i}) \prod_{q_i \equiv 3 \pmod 4} a_f(q_i^{2s_i}) \neq 0.$$

This finishes the proof of the theorem. $\square$

**Remark 2.3.6.** *The crux in the proof of above theorem is Deuring's theorem. We could have a theorem similar to that of the above theorem for eigenforms of weight $2k$ with $k > 1$, if we had an analogous result of Deuring for higher weights. But, we are not aware of such results.*

**Remark 2.3.7.** *One might wonder the reason for working with* $\mathbb{Q}(i)$*, but not with any other imaginary quadratic fields. This is because, the split (resp., inert) primes of* $\mathbb{Q}(i)$ *are exactly the primes* $p$ *which are* $\equiv 1 \pmod 4$ *(resp.,* $p \equiv 3 \pmod 4$*). This fact is what enabled us to use the Deuring's theorem in the above proof.*

**Remark 2.3.8.** *In the above proof, we have chosen* $m \in \mathbb{N}$ *in such a way that* $m$ *and* $6M_f\mathrm{cond}(\Psi)$ *are relatively prime. We need the condition* $(m, 6\mathrm{cond}(\Psi)) = 1$ *to make an effective use of Deuring's theorem. To apply Lemma 2.2.15, we need* $(m, M_f) = 1$.

There exists at least one elliptic curve $E$ over $\mathbb{Q}$ satisfying the statement of Theorem 2.3.1. For example, the one parameter family of elliptic curves $y^2 = x^3 + ax$, with $a$ varies over $\mathbb{Q}^*$, are defined over $\mathbb{Q}$ and they are with CM by $\mathbb{Q}(i)$. However, all these curves are isogenous, since any two CM elliptic curves with the same endomorphism algebra are isogenous. Hence, all of them corresponding to the same weight $2$ eigenform with CM.

We shall finish this chapter with the following proposition in which we construct infinitely many cuspidal CM eigenforms level $N > 1$ and weight $k > 2$ for which the (2.4) holds. This result is a generalization of the result in [14] to the higher weight case.

**Proposition 2.3.9.** *There exists infinitely many eigenforms* $f$ *with CM of weight* $k > 2$ *and level* $N > 1$ *for which*

$$i_f(n) \ll n^{\frac{1}{4}}$$

*holds for all* $n \gg 0$.

*Proof.* Take any elliptic curve $E$ over $\mathbb{Q}$ with CM by $\mathbb{Q}(i)$ and this corresponds to a CM eigenform $f$ of weight $2$ with trivial nebentypus (cf. [8]). Let $\Psi$ be the Hecke character corresponding to $f$.

Consider the Hecke character $\Psi^m$, for some odd $m$. By Hecke's Theorem (cf. [19, page 717]), the corresponding $f_{\Psi^m}$ is an eigenform with CM of weight $m+1$ with trivial character. However, the eigenform $f_{\Psi^m}$ may not be a newform unless $\Psi^m$ is primitive.

By [40, Cor. 3.5], corresponding to the eigenform $f_{\Psi^m}$, there exists a unique newform with CM, which we denote with $g_m = \sum_{n=1}^{\infty} a_{g_m}(n)q^n$. Then, the newform $g_m$ of weight $m+1$ with trivial character and level dividing the level of $f_{\Psi^m}$. Moreover, the newform $g_m(z)$ has the property that $a_{g_m}(p) = a_{\Psi^m}(p)$ for primes $p$ away from the level of $f_{\Psi^m}$. Therefore, $a_{g_m}(p) = 0$ if and only if $a_{\Psi^m}(p) = 0$ for all but finitely prime $p$. By Proposition 2.3.4, if $p \geq 5$, we see that $a_{\Psi^m}(p) = 0$ if and only if $a_\Psi(p) = 0$. Hence, $a_{g_m}(p) = 0$ if and only if $a_\Psi(p) = 0$, for all but finitely many primes $p$. In fact, by Lemma 2.2.15, we have

$$a_{g_m}(p^k) = 0 \iff a_\Psi(p^k) = 0, \tag{2.7}$$

for all but finitely many primes $p$ and for all $k \geq 1$. Now, arguing as in the proof of Theorem 2.3.1, i.e., if we take $N = 6M_f \mathrm{cond}(\Psi) \prod_{p \in B} p$, where $B$ is the set of primes for which (2.7) does not hold and $M_f$ is as in Lemma 2.2.15. By Theorem 2.3.2, there exists $X_0 \in \mathbb{R}$ and $c > 0$ (depending only on $N$) such that there exists an integer, say $n$, which is a sum of two squares and co-prime to $N$ in intervals of type $(X, X + cX^{\frac{1}{4}})$ for all $X \gg X_0$. Then, we have $a_\Psi(n) \neq 0$ and hence

$$i_{g_m}(n) \ll n^{\frac{1}{4}}$$

holds for all $n \gg 0$.

As a consequence, we see that there exists infinitely many eigenforms with CM of different weights for which (2.4) holds. The levels of the eigenforms $g_m$'s are $> 1$ because the eigenforms in $S_k(SL_2(\mathbb{Z}))(k \in \mathbb{N})$ are without CM. $\square$

The proof of the above proposition would also imply that, for every integer $k \geq 1$, there exists an integer $N > 1$ (may depend on $k$) and an eigenform $f$ with CM of weight $2k$ and level $N$ for which

$$i_f(n) \ll n^{\frac{1}{4}}$$

holds for all $n \gg 0$.

# Chapter 3

# Sign changes for the Fourier coefficients of Hilbert modular forms of half-integral weight

## 3.1 Introduction

The Fourier coefficients of integral or half-integral weight modular forms over number fields have been extensively studied because of rich arithmetic and algebraic properties that they encompass. In recent years, many problems addressing the sign changes of these Fourier coefficients have been studied by various authors. In this chapter, we are interested in the sign changes of these Fourier coefficients.

In [20], Inam and Wiese showed the equi-distribution of signs for certain subfamilies of coefficients for half-integral modular forms that are accessible via the Shimura correspondence. It is natural to ask similar questions for modular forms defined over number fields, in particular, over totally real number fields, say $F$. There are not many results available in this setting as compared to the classical case (over $\mathbb{Q}$).

This chapter is a modest attempt to show that the ideas of Inam and Wiese in [20] generalize to the case of Hilbert modular cusp forms of half-integral weight. We show that sign change results holds for certain subfamilies of Fourier coefficients that are accessible via the Shimura correspondence. The proof uses the Sato-Tate equidistribution theorem for non-CM primitive Hilbert modular forms. As a consequence, we see that the Fourier coefficients change signs infinitely often.

As we mentioned above, the study of the Fourier coefficients of modular forms over number fields has a long history. In the next section, we shall recall the known results in the literature briefly.

## 3.2 History

In literature, the study of sign changes for the Fourier coefficients of integral weight cusp forms over $\mathbb{Q}$ has been initiated by Ram Murty in [36]. There, he proved that the Fourier coefficients of a cusp form of integral weight has infinitely many sign changes.

**Theorem 3.2.1.** ([36, Theorem 5]) If

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n$$

is a non-zero cusp form belonging to any congruence subgroup, then either $\operatorname{Re} a_f(p)$ or $\operatorname{Im} a_f(p)$ change signs infinitely often.

For primitive forms of integral weight without CM, the equidistribution of signs is a consequence of the Sato-Tate equidistribution theorem due to Barnet-Lamb, Geraghty, Harris, and Taylor. They proved that:

**Theorem 3.2.2.** ([10, Theorem B]) Let $f$ be a primitive form of weight $k \geq 2$ and level $N$ and nebentypus character $\psi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$. Write

$$f(z) = q + \sum_{n=2}^{\infty} a_f(n) q^n.$$

Suppose that $f$ is non-CM. If $\zeta$ is a root of unity with $\zeta^2$ in the image of $\psi$, then as $p$ varies over primes with $\psi(p) = \zeta^2$, the numbers $a_f(p)/(2p^{k-\frac{1}{2}}\zeta) \in \mathbb{R}$ are equidistributed in $[-1, 1]$ with respect to the measure $\mu = (2/\pi)\sqrt{1-t^2}dt$. In other words, for any subinterval $I$ of $[-1, 1]$, we have

$$\lim_{x \to \infty} \frac{\#\{p \in \mathbf{P} : p \leq x, a_f(p)/(2p^{k-\frac{1}{2}}\zeta) \in I\}}{\#\{p \in \mathbf{P} : p \leq x\}} = \mu(I) = \frac{2}{\pi} \int_I \sqrt{1-t^2}dt,$$

where $\mathbf{P}$ denotes the set of all prime numbers.

Recently, the study of sign changes has been extended to the Fourier coefficients of cusp forms of half-integral weight over $\mathbb{Q}$ (cf. [29], [11]). To state the result of [11], we need to recall some notations.

Let $\mathbb{D}$ denote the set of square-free positive integers. Let $N$ be a positive integer divisible by $4$, and let $\chi$ be a Dirichlet character modulo $N$. Let $S_{k+\frac{1}{2}}(N,\chi)$ be the space of cusp form of weight $k+\frac{1}{2}$ for the group $\Gamma_0(N)$ with character $\chi$. Let $f=\sum_{n=0}^{\infty} a_f(n)q^n \in S_{k+\frac{1}{2}}(N,\chi)$ be a non-zero cusp form with Fourier coefficients $a_f(n)\in\mathbb{R}$.

**Theorem 3.2.3.** ([11, Theorem 2.1]) Let $t\in\mathbb{D}$ such that $a_f(t)\neq 0$, and write $\chi_{t,N}$ for the quadratic character $\chi_{t,N}=(\frac{(-1)^k N^2 t}{\cdot})$. Assume that Dirichlet $L$-function $L(s,\chi_{t,N},N)$ has no zeros in the interval $(0,1)$. Then the sequence $(a_f(tn^2))_{n\in\mathbb{N}}$ has infinitely many sign changes.

In the case of Hecke eigenforms, they proved a stronger theorem. More precisely:

**Theorem 3.2.4.** ([11, Theorem 2.2]) Suppose that the character $\chi$ of $f$ is real, and suppose that $f$ is an eigenform of all Hecke operators $T(p^2)$. Let $t\in\mathbb{D}$ such that $a_f(t)=1$. Then, for all but finitely many primes $p$ such that $(p,N)=1$, the sequence $(a_f(tp^{2m}))_{m\in\mathbb{N}}$ has infinitely many sign changes.

In *loc. cit.*, they conjectured an equidistribution of signs for Fourier coefficients of half-integral weight modular forms over $\mathbb{Q}$, i.e.,

$$\lim_{x\to\infty}\frac{\#\{n\le x: a_f(n)\lesseqgtr 0\}}{\#\{n\le x: a_f(n)\neq 0\}}=\frac{1}{2},$$

(cf. [26] for more details). In [20], Inam and Wiese showed an equidistribution of signs for certain sub-families of coefficients for half-integral modular forms that are accessible via the Shimura correspondence. Before, we state the results of [20], we shall recall the definition of natural density. Let $\mathbf{P}$ denote the set of all prime numbers.

**Definition 3.2.5.** *Let $S\subseteq\mathbf{P}$ be a subset. We define the natural density of $S$ to be*

$$\lim_{x\to\infty}\frac{\#\{p: p\le x, p\in S\}}{\#\{p: p\le x, p\in\mathbf{P}\}}.$$

*If the limit exists, then we denote it by $d(S)$.*

Let $k\geq 2$, $N$ be an integer divisible by $4$, $\chi$ be a Dirichlet character modulo $N$ such that $\chi^2=1$. Let $f(z)=\sum_{n=1}^{\infty}a_f(n)q^n\in S_{k+\frac{1}{2}}(N,\chi)$ be a non-zero cuspidal Hecke eigenform of weight $k+\frac{1}{2}$ with real coefficients. Let $t$ be a square-free integer such that $a_f(t)\neq 0$. Let $F_t(z)=\sum_{n=1}^{\infty}A_t(n)q^n\in S_{2k}(N/2,\chi^2)$ be the Hecke eigenform of weight $2k$ corresponding to $f$ under the Shimura lift (cf. [20, §3]). Define the set of primes

$$\mathbf{P}_{>0} = \{p \in \mathbf{P} | a_f(tp^2) > 0\}$$

and similarly $\mathbf{P}_{<0}$, $\mathbf{P}_{\geq 0}$, $\mathbf{P}_{\leq 0}$ and $\mathbf{P}_{=0}$.

**Theorem 3.2.6.** ([20, Theorem 4.1]) If $F_t$ is without CM, then $\mathbf{P}_{>0}$, $\mathbf{P}_{<0}$, $\mathbf{P}_{\geq 0}$, $\mathbf{P}_{\leq 0}$ have natural density $1/2$ and the set $\mathbf{P}_{=0}$ has natural density $0$.

For non-zero Hilbert cusp forms of integral weight, in [33], Meher and Tanabe proved the following theorem:

**Theorem 3.2.7.** ([33, Theorem 1.1]) Let $\mathbf{f}$ be a Hilbert cusp form of weight $k = (k_1, \ldots, k_n)$ and level $\mathfrak{n}$, and let $C(\mathfrak{m}, \mathbf{f})$ be a Fourier coefficient of $\mathbf{f}$ at each integral ideal $\mathfrak{m}$ (as defined in (3.7)). If $\{C(\mathfrak{m}, \mathbf{f})\}$ are all real, then there are infinitely many sign changes on $\{C(\mathfrak{m}, \mathbf{f})\}$.

Furthermore, an equidistribution of signs for the Fourier coefficients of primitive Hilbert forms of integral weight without CM can be obtained as, similar to the case of classical forms, a consequence of the Sato-Tate equidistribution theorem due to Barnet-Lamb, Gee, and Geraghty [9] (cf. Theorem 3.5.4 in the text). To the best of authors knowledge, similar results are not available in the literature for Hilbert modular forms of half-integral weight.

In this chapter, we prove sign change results for the Fourier coefficients of half-integral weight by using Shimura correspondence between the half-integral weight modular forms and the integral weight modular forms. The chapter is organized as follows.

In §3.3.1, we recall the half-integral modular forms followed by integral weight Hilbert modular forms. In §3.4, we state the Shimura correspondence between them. In §3.5, we recall the Sato-Tate equidistribution Theorem for automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$. In §3.6, the main result of this chapter is stated as Theorem 3.6.1 with its proof immediately after.

## 3.3 Hilbert modular forms

### 3.3.1 Half-integral weight:

Let $k = (k_1, \ldots, k_n)$ be an integral or a half-integral weight, i.e., $k_j \in \mathbb{Z}_{>0}$ for all $j$ or $k_j = \frac{1}{2} + m_j$ with $m_j \in \mathbb{Z}_{>0}$ for all $j$, respectively. Both cases together, we denote $k = u/2 + m$ while it is understood that $u \in \{0, 1\}$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>0}$. Given a holomorphic function $g$ on $\mathfrak{h}^n$ and an element $\gamma = (\gamma_1, \ldots, \gamma_n)$ of $\mathrm{SL}_2(\mathbb{R})^n$ with $\gamma_j = \left[\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right]$, define

$$g\|_k\gamma(z) = h(\gamma, z)^{-u} \prod_j (c_j z_j + d_j)^{-m_j} g(\gamma z) \tag{3.1}$$

where $z \in \mathfrak{h}^n$ and $h(\gamma, z)$ is some non-vanishing holomorphic function on $\mathfrak{h}^n$. See [43, Proposition 2.3] for the precise definition for $h$. We note that the function $h$ is only defined when $\gamma$ is in a "nice" subgroup of $\mathrm{SL}_2(\mathbb{R})$, but we shall not worry about the details as we only consider such a congruence subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{R})$ as in (3.2). We refer the reader to Shimura [43, Section 2] for the details.

For the rest of this section, we assume $k$ is half-integral and $\mathfrak{D}_F$ denote the absolute different of $F$. Let $\mathfrak{c}$ be an integral ideal of $F$ that is divisible by $4$ and define a congruence subgroup $\Gamma = \Gamma(\mathfrak{c})$ of $\mathrm{SL}_2(\mathbb{R})$ by

$$\Gamma(\mathfrak{c}) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{R}) \ : \ \begin{matrix} a \in \mathcal{O}_F, & b \in 2\mathfrak{D}_F^{-1} \\ c \in 2^{-1}\mathfrak{c}\mathfrak{D}_F, & d \in \mathcal{O}_F \end{matrix} \right\}. \tag{3.2}$$

Let us take a Hecke character $\psi$ on the idele group $\mathbb{A}_F^\times$ of $F$ whose conductor divides $\mathfrak{c}$ and infinite part $\psi_\infty = \prod_j \psi_{\eta_j}$ satisfies the following condition;

$$\psi_\infty(-1) = (-1)^{\sum_j m_j}. \tag{3.3}$$

Such a character $\psi$ can be extended to a character of $\Gamma(\mathfrak{c})$, which is again denoted by $\psi$, as $\psi(\gamma) = \psi(a)$ where $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We denote by $M_k(\Gamma(\mathfrak{c}), \psi)$ the set of all holomorphic functions $g$ on $\mathfrak{h}^n$ satisfying

$$g\|_k\gamma = \psi_{\mathfrak{c}}(\gamma) g$$

for all $\gamma \in \Gamma(\mathfrak{c})$, where $\psi_{\mathfrak{c}}$ is the "$\mathfrak{c}$-part" of $\psi$, i.e., $\psi_{\mathfrak{c}} = \prod_{\mathfrak{p}|\mathfrak{c}} \psi_{\mathfrak{p}}$. In other words $M_k(\Gamma(\mathfrak{c}), \psi)$ is the space of Hilbert modular forms of half-integral weight $k$ with respect to $\Gamma(\mathfrak{c})$ with character $\psi$. It should be noted that our choice of Hecke character $\psi$ only depends on finitely many places, namely at archimedean places and the $\mathfrak{c}$-part. However, we will keep the notation $M_k(\Gamma(\mathfrak{c}), \psi)$ without replacing $\psi$ with $\psi_{\mathfrak{c}}$, as the choice of characters become more crucial in §3.6.

Such a form $g \in M_k(\Gamma(\mathfrak{c}), \psi)$ is known to have the Fourier expansion corresponding to any given fractional ideal $\mathfrak{a}$. Its coefficients are denoted by $\{\lambda_g(\xi, \mathfrak{a})\}_{\xi,\mathfrak{a}}$ where $\xi$ varies over totally positive elements in $F$. One can treat $\{\lambda_g(\xi, \mathfrak{a})\}$ as a two parameter family of Fourier coefficients for $g$, as varies over $\xi$ in $F$ and fractional ideals $\mathfrak{a}$. It should be noted that $\lambda_g(\xi, \mathfrak{a}) = 0$ unless $\xi \in (\mathfrak{a}^{-2})^+$ or $\xi = 0$. A modular form $g$ is said to be a cusp form if $\lambda_{g|_\gamma}(0, \mathfrak{a}) = 0$ for every fractional ideal $\mathfrak{a}$ and every $\gamma \in \mathrm{SL}_2(F)$. The space of such $g$ is denoted by $S_k(\Gamma(\mathfrak{c}), \psi)$. For more details, we refer the reader to [43, Proposition 3.1].

Our aim in this chapter is to study a signs change result for a family of Fourier coefficients $\{\lambda_g(\xi, \mathfrak{a})\}$ with $\mathfrak{a}$ varying over a certain family of fractional ideals. In the next section, we shall recall the definition and basic properties of Hilbert modular forms of integral weight. This section is essentially a summary of Raghuram and Tanabe [38, Section 4.1] with some modifications following Shimura [43, Section 6].

### 3.3.2 Integral weight:

We assume that $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{>0}$ throughout this section. For a non-archimedean place $\mathfrak{p}$ of $F$, let $F_{\mathfrak{p}}$ be a completion of $F$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be integral ideals of $F$, and define a subgroup $K_{\mathfrak{p}}(\mathfrak{a}, \mathfrak{b})$ of $\mathrm{GL}_2(F_{\mathfrak{p}})$ as

$$K_{\mathfrak{p}}(\mathfrak{a}, \mathfrak{b}) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(F_{\mathfrak{p}}) \ : \ \begin{array}{lll} a \in \mathcal{O}_{\mathfrak{p}}, & b \in \mathfrak{a}_{\mathfrak{p}}^{-1}\mathfrak{D}_{\mathfrak{p}}^{-1}, & \\ c \in \mathfrak{b}_{\mathfrak{p}}\mathfrak{D}_{\mathfrak{p}}, & d \in \mathcal{O}_{\mathfrak{p}}, & |ad - bc|_{\mathfrak{p}} = 1 \end{array} \right\}$$

where the subscript $\mathfrak{p}$ means the $\mathfrak{p}$-parts of given ideals. Furthermore, we put

$$K_0(\mathfrak{a}, \mathfrak{b}) = \mathrm{SO}(2)^n \cdot \prod_{\mathfrak{p}<\infty} K_{\mathfrak{p}}(\mathfrak{a}, \mathfrak{b}) \quad \text{and} \quad W(\mathfrak{a}, \mathfrak{b}) = \mathrm{GL}_2^+(\mathbb{R})^n K_0(\mathfrak{a}, \mathfrak{b}).$$

In particular, if $\mathfrak{a} = \mathcal{O}_F$, we simply write $K_{\mathfrak{p}}(\mathfrak{b}) = K_{\mathfrak{p}}(\mathcal{O}_F, \mathfrak{b})$ and $W(\mathfrak{b}) = W(\mathcal{O}_F, \mathfrak{b})$. Then, we have the following disjoint decomposition of $\mathrm{GL}_2(\mathbb{A}_F)$,

$$\mathrm{GL}_2(\mathbb{A}_F) = \cup_{\nu=1}^{h} \mathrm{GL}_2(F) x_\nu^{-\iota} W(\mathfrak{b}), \tag{3.4}$$

where $x_\nu^{-\iota} = \left[\begin{smallmatrix} t_\nu^{-1} & \\ & 1 \end{smallmatrix}\right]$ with $\{t_\nu\}_{\nu=1}^h$ taken to be a complete set of representatives of the narrow class group of $F$. We note that such $t_\nu$ can be chosen so that the infinity part $t_{\nu,\infty}$ is $1$ for all $\nu$. For each $\nu$, we also put

$$\begin{aligned}\Gamma_\nu(\mathfrak{a},\mathfrak{b}) &= \mathrm{GL}_2(F) \cap x_\nu W(\mathfrak{a},\mathfrak{b}) x_\nu^{-1} \\ &= \left\{ \left[\begin{smallmatrix} a & t_\nu^{-1}b \\ t_\nu c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(F) : \begin{matrix} a \in \mathcal{O}_F, & b \in \mathfrak{a}^{-1}\mathfrak{D}_F^{-1}, & \\ c \in \mathfrak{b}\mathfrak{D}_F, & d \in \mathcal{O}_F, & ad - bc \in \mathcal{O}_F \end{matrix} \right\}.\end{aligned}$$

It is understood that $\Gamma_\nu(\mathfrak{b}) = \Gamma_\nu(\mathcal{O}_F, \mathfrak{b})$ as before.

Let $\psi$ be a Hecke character of $\mathbb{A}_F^\times$ such that its conductor divides $\mathfrak{b}$ and its infinite part $\psi_\infty$ is of the form

$$\psi_\infty(x) = \mathrm{sgn}(x_\infty)^k |x_\infty|^{i\mu}$$

where $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ with $\sum_{j=1}^n \mu_j = 0$. We let $M_k(\Gamma_\nu(\mathfrak{b}), \psi_\mathfrak{b}, \mu)$ denote the space of all functions $f_\nu$ that are holomorphic on $\mathfrak{h}^n$ and at cusps, satisfying

$$f_\nu \|_k \gamma = \psi_\mathfrak{b}(\gamma) \det \gamma^{i\mu/2} f_\nu$$

for all $\gamma$ in $\Gamma_\nu(\mathfrak{b})$. We note that such a function $f_\nu$ affords a Fourier expansion, and its coefficients are denoted as $\{a_\nu(\xi)\}_\xi$ where $\xi$ runs over all the totally positive elements in $t_\nu^{-1}\mathcal{O}_F$ and $\xi = 0$. Similar to the case of half-integral weight forms, a Hilbert modular form is called a cusp form if, for all $\gamma \in \mathrm{GL}_2^+(F)$, the constant term of $f\|_k\gamma$ in its Fourier expansion is $0$, and the space of cusp forms with respect to $\Gamma_\nu(\mathfrak{b})$ is denoted by $S_k(\Gamma_\nu(\mathfrak{b}), \psi_\mathfrak{b}, \mu)$.

Now, put $\mathbf{f} = (f_1, \ldots, f_h)$ where $f_\nu$ belongs to $M_k(\Gamma_\nu(\mathfrak{b}), \psi_\mathfrak{b}, \mu)$ for each $\nu$, and define $\mathbf{f}$ to be a function on $\mathrm{GL}_2(\mathbb{A}_F)$ as

$$\mathbf{f}(g) = \mathbf{f}(\gamma x_\nu^{-\iota} w) := \psi_\mathfrak{b}(w^\iota) \det w_\infty^{i\mu/2} (f_\nu ||_k w_\infty)(i) \tag{3.5}$$

where $\gamma x_\nu^{-\iota} w \in \mathrm{GL}_2(F) x_\nu^{-\iota} W(\mathfrak{b})$ as in (3.4), $w^\iota = \omega_0 ({}^t w) \omega_0^{-1}$ with $\omega_0 = \left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]$, and $i = (i, \ldots, i)$. The space of such $\mathbf{f}$ is denoted as

$$\mathfrak{M}_k(\psi_\mathfrak{b}, \mu) = \prod_\nu M_k(\Gamma_\nu(\mathfrak{b}), \psi_\mathfrak{b}, \mu).$$

Furthermore, the space consisting of all $\mathbf{f} = (f_1, \ldots, f_h) \in \mathfrak{M}_k(\psi_{\mathfrak{b}}, \mu)$ satisfying

$$\mathbf{f}(xg) = \psi(x)\mathbf{f}(g) \quad \text{for any } x \in \mathbb{A}_F^\times \quad \text{and} \quad g \in \mathrm{GL}_2(\mathbb{A}_F)$$

is denoted as $\mathfrak{M}_k(\mathfrak{b}, \psi)$, which denotes the space of Hilbert modular forms of integral weight $k$ of level $\mathfrak{b}$ and character $\psi$.

In particular, if each $f_\nu$ belongs to $S_k(\Gamma_\nu(\mathfrak{b}), \psi_{\mathfrak{b}}, \mu)$, then the space of such $\mathbf{f}$ is denoted by $\mathfrak{S}_k(\mathfrak{b}, \psi)$. A cusp form $\mathbf{f}$ is called primitive if it is a normalized new form and a common eigenfunction of all Hecke operators.

Let $\mathfrak{m}$ be an ideal of $F$ and write $\mathfrak{m} = \xi t_\nu^{-1}\mathcal{O}_F$ with a totally positive element $\xi$ in $F$. Then the Fourier coefficient of $\mathbf{f}$ at $\mathfrak{m}$ is defined as

$$c(\mathfrak{m}, \mathbf{f}) = \begin{cases} a_\nu(\xi)\xi^{-(k+i\mu)/2} & \text{if} \quad \mathfrak{m} = \xi t_\nu^{-1}\mathcal{O}_F \subset \mathcal{O}_F \\ 0 & \text{if} \quad \mathfrak{m} \text{ is not integral.} \end{cases} \tag{3.6}$$

To make the calculation simpler at a later point, we re-normalize $c(\mathfrak{m}, \mathbf{f})$ as follows: define

$$C(\mathfrak{m}, \mathbf{f}) = \mathrm{N}(\mathfrak{m})^{k_0/2} c(\mathfrak{m}, \mathbf{f}), \tag{3.7}$$

where $k_0 = \max_j\{k_j\}$ with $k = (k_1, \ldots, k_n)$ being the weight of $\mathbf{f}$.

## 3.4 Shimura correspondence

In this section, we shall recall the Shimura correspondence, which states that, given a non-zero half-integral weight cusp form, there is an integral automorphic form associated with it.

For any integral ideal $\mathfrak{a}$ in $\mathcal{O}_F$, we introduce a formal symbol $\mathrm{M}(\mathfrak{a})$ satisfying that $\mathrm{M}(\mathcal{O}_F) = 1$ and $\mathrm{M}(\mathfrak{a}\mathfrak{b}) = \mathrm{M}(\mathfrak{a})\mathrm{M}(\mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_F$. Then one can consider the ring of formal series in these symbols, indexed by integral ideals.

The following result is the Shimura correspondence for Hilbert modular forms [43, Theorems 6.1 and 6.2]. We assume for simplicity that $\psi$ is a quadratic character, as it will be the case in our setting.

**Theorem 3.4.1.** *Let $0 \neq g \in S_k(\Gamma(\mathfrak{c}), \psi)$ with a half-integral weight $k = \frac{1}{2} + m$ with $m \geq 1$, an integral ideal $\mathfrak{c}$ of $F$ divisible by 4, and $\psi$ being a Hecke character of $F$ such that it satisfies* (3.3) *and its conductor divides $\mathfrak{c}$. Further, we assume that $\psi_\infty(x) = |x|^{i\mu}$ for any totally positive element $x$ in $\mathbb{A}_{F,\infty}^\times$ with some $\mu \in \mathbb{R}^n$ such that $\sum_j \mu_j = 0$.*

*Let $\tau$ be an arbitrary element in $\mathcal{O}_F^+$, and write $\tau\mathcal{O}_F = \mathfrak{a}^2\mathfrak{r}$ for some integral ideal $\mathfrak{a}$ and a square free integral ideal $\mathfrak{r}$. Then the following assertions hold.*

1. *Let $\mathfrak{b}$ be a fractional ideal of $F$ and define $\Gamma = \mathrm{GL}_2(F) \cap W(\mathfrak{b}, 2^{-1}\mathfrak{c}\mathfrak{b})$. Then there exists $f \in M_{2m}(\Gamma, \psi^2_{2^{-1}\mathfrak{c}}, \mu)$ so that*

$$\sum_{\xi \in \mathfrak{b}/\mathcal{O}_F^{\times,+}} a(\xi)\xi^{-m-i\mu}\mathrm{M}(\xi\mathfrak{b}^{-1}) = \sum_{\mathfrak{m}\subseteq\mathcal{O}_F} \lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{m})\mathrm{M}(\mathfrak{m}) \sum_{\mathfrak{n}\mathfrak{m}\mathfrak{b}\sim 1} (\psi\epsilon_\tau)^*(\mathfrak{n})\mathrm{N}(\mathfrak{n})^{-1}\mathrm{M}(\mathfrak{n}), \tag{3.8}$$

*where $\lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{m})$ is the Fourier coefficient of $g$ at a cusp corresponding to $\mathfrak{a}^{-1}\mathfrak{m}$, $\psi_{\{2^{-1}\mathfrak{c}\}} = \prod_{\mathfrak{p}|2^{-1}\mathfrak{c}} \psi_\mathfrak{p}$, $\epsilon_\tau$ is the Hecke character of $F$ corresponding to $F(\sqrt{\tau})/F$, and $(\psi\epsilon_\tau)^*$ is the character induced from $\psi\epsilon_\tau$. In the second sum of the right hand side in (3.8), $\mathfrak{n}$ runs through all the integral ideal of $F$ that are prime to $\mathfrak{c}\mathfrak{r}$ and equivalent to $(\mathfrak{m}\mathfrak{b})^{-1}$.*

2. *Let $\mathbf{f}_\tau = (f_1, \ldots, f_h)$ where $f_\nu$ is of the form given in* (1) *with $\mathfrak{b} = t_\nu\mathcal{O}_F$ for all $\nu = 1, \ldots, h$. Then $\mathbf{f}_\tau \in \mathfrak{M}_{2m}(2^{-1}\mathfrak{c}, \psi^2)$ and it satisfies*

$$\sum_{\mathfrak{m}} c(\mathfrak{m}, \mathbf{f}_\tau)\mathrm{M}(\mathfrak{m}) = \sum_{\mathfrak{m}} \lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{m})\mathrm{M}(\mathfrak{m}) \prod_{\mathfrak{p}\nmid\mathfrak{c}\mathfrak{r}} \left(1 - \frac{(\psi\epsilon_\tau)^*(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})}\mathrm{M}(\mathfrak{p})\right)^{-1} \tag{3.9}$$

*where $\mathfrak{m}$ runs over all the integral ideals of $F$ and $\mathfrak{p}$ over all the prime ideals which do not divide $\mathfrak{c}\mathfrak{r}$.*

3. *The function $f$ given in* (1) *is a cusp form if $m_j > 1$ for some $j$.*

## 3.5 Sato-Tate equidistribution theorem for Hilbert modular forms

In this section, we shall recall the Sato-Tate equidistribution Theorem due to Barnet-Lamb, Gee, and Geraghty in [9].

**Theorem 3.5.1.** ([9, Corollary 7.1.7]) Let $F$ be a totally real number field of degree $n$ and $\Pi$ a non-CM regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Write $\mu = (\mu_1, \ldots, \mu_n)$ for an integral weight for the diagonal torus of $\mathrm{GL}_2(\mathbb{R})^n$ with $\mu_j = (a_j, b_j)$ and $a_j \geq b_j$ for all $j$. We note that the values $a_j + b_j$ are the same for all $j$, and therefore we may put $\omega_\Pi = a_j + b_j$. Let $\chi$ be the product of the central character of $\Pi$ with $|\cdot|^{\omega_\pi}$, so that $\chi$ is a finite order character. Let $\zeta$ be a root of unity such that $\zeta^2$ is in the image of $\chi$. For any place $\mathfrak{p}$ of $F$ such that $\Pi_{\mathfrak{p}}$ is unramified, let $\lambda_{\mathfrak{p}}$ denote the eigenvalue of the Hecke operator

$$\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})\begin{pmatrix} \varpi_{\mathfrak{p}} & \\ & 1 \end{pmatrix}\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}) \tag{3.10}$$

on $\Pi_{\mathfrak{p}}^{\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})}$, where $\varpi_{\mathfrak{p}}$ is a uniformizer of $\mathcal{O}_{\mathfrak{p}}$.

Then as $\mathfrak{p}$ ranges over the unramified places of $F$ such that $\chi_{\mathfrak{p}}(\varpi_{\mathfrak{p}}) = \zeta^2$, the number given by

$$\frac{\lambda_{\mathfrak{p}}}{2\mathrm{N}(\mathfrak{p})^{(1+\omega_\Pi)/2}\zeta}$$

belongs to $[-1, 1]$, and furthermore they are equidistributed in $[-1, 1]$ with respect to the measure $(2/\pi)\sqrt{1-t^2}dt$.

Now, we rewrite the above theorem in terms of the Fourier coefficients of primitive Hilbert modular eigenforms. The following lemma provides the relation between the Fourier coefficients and the eigenvalues. More precisely,

**Lemma 3.5.2.** *Let $\mathbf{f} \in \mathfrak{S}_k(\mathfrak{c}, \mathbb{1})$ be a primitive form in the new space, and $\Pi = \Pi_{\mathbf{f}}$ an irreducible cuspidal automorphic representation corresponding to $\mathbf{f}$. Let $\mathfrak{p}$ be a prime ideal of $F$ such that $\mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F$. Let $c(\mathfrak{p}, \mathbf{f})$ be the Fourier coefficients at $\mathfrak{p}$ defined as in* (3.6)*, and $\lambda_{\mathfrak{p}}$ be the eigenvalue of the Hecke operator defined as in* (3.10)*, then*

$$\lambda_{\mathfrak{p}} = c(\mathfrak{p}, \mathbf{f})\mathrm{N}(\mathfrak{p}).$$

*Proof.* For a proof, please refer to [38, Page 305-306]. □

Let $\mathbb{P}$ denote the set of all prime ideals of $\mathcal{O}_F$. Now, we recall the notion of natural density for a subset of $\mathbb{P}$.

**Definition 3.5.3.** *Let $F$ be a number field and $S$ be a subset of $\mathbb{P}$. We define the natural density of $S$ to be*

$$d(S) = \lim_{x\to\infty} \frac{\#\{\mathfrak{p} : \mathrm{N}(\mathfrak{p}) \le x, \mathfrak{p} \in S\}}{\#\{\mathfrak{p} : \mathrm{N}(\mathfrak{p}) \le x, \mathfrak{p} \in \mathbb{P}\}}, \tag{3.11}$$

*provided the limit exists.*

The following theorem is a consequence of Theorem 3.5.1 above:

**Theorem 3.5.4.** *Let $\mathbf{f} \in \mathfrak{S}_k(\mathfrak{c}, \mathbb{1})$ be a primitive form of weight $k = (k_1, \ldots, k_n)$ such that $k_1 \equiv \cdots \equiv k_n \equiv 0 \pmod 2$ and each $k_j \ge 2$. Suppose that $\mathbf{f}$ does not have complex* *multiplication. Then, for any prime ideal* $\mathfrak{p}$ *of* $F$ *such that* $\mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F$*, we have*

$$B(\mathfrak{p}) := \frac{C(\mathfrak{p}, \mathbf{f})}{2\mathrm{N}(\mathfrak{p})^{\frac{k_0-1}{2}}} \in [-1, 1].$$

*Furthermore,* $\{B(\mathfrak{p})\}_{\mathfrak{p}}$ *are equidistributed in* $[-1, 1]$ *with respect to the measure* $\mu = (2/\pi)\sqrt{1-t^2}dt$*. In other words, for any subinterval* $I$ *of* $[-1, 1]$*, we have*

$$\lim_{x \to \infty} \frac{\#\{\mathfrak{p} \in \mathbb{P} : \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F, \mathrm{N}(\mathfrak{p}) \le x, B(\mathfrak{p}) \in I\}}{\#\{\mathfrak{p} \in \mathbb{P} : \mathrm{N}(\mathfrak{p}) \le x\}} = \mu(I) = \frac{2}{\pi} \int_I \sqrt{1-t^2}dt,$$

*i.e., the natural density of the set* $\{\mathfrak{p} : B(\mathfrak{p}) \in I\}$ *is* $\mu(I)$*.*

*Proof.* Let $\Pi = \Pi_{\mathbf{f}}$ be the non-CM irreducible cuspidal automorphic representation corresponding to $\mathbf{f}$. Since $k_1 \equiv \cdots \equiv k_n \equiv 0 \pmod 2$ and each $k_j \ge 2$, it follows that $\Pi$ is algebraic and regular (cf. [38, Theorem 1.4]). By Lemma 3.5.2, we have

$$\frac{\lambda_{\mathfrak{p}}}{2\mathrm{N}(\mathfrak{p})^{(1+\omega_\Pi)/2}} = \frac{\mathrm{N}(\mathfrak{p})c(\mathfrak{p}, \mathbf{f})}{2\mathrm{N}(\mathfrak{p})^{(1+\omega_\Pi)/2}} = \frac{c(\mathfrak{p}, \mathbf{f})}{2\mathrm{N}(\mathfrak{p})^{(\omega_\Pi-1)/2}}.$$

Given that the highest weight vector $\mu = (\mu_1, \ldots, \mu_n)$ of $\Pi$ being $\mu_j = ((k_j - 2)/2, -(k_j - 2)/2)$, we have $\omega_\Pi = \frac{k_j-2}{2} - \frac{k_j-2}{2} = 0$. Hence, we get

$$\frac{\lambda_{\mathfrak{p}}}{2\mathrm{N}(\mathfrak{p})^{1/2}} = \frac{c(\mathfrak{p}, \mathbf{f})}{2\mathrm{N}(\mathfrak{p})^{\frac{-1}{2}}} = \frac{C(\mathfrak{p}, \mathbf{f})}{2\mathrm{N}(\mathfrak{p})^{\frac{k_0-1}{2}}}.$$

Now, the theorem follows from Theorem 3.5.1 by taking $\zeta = 1$. □

## 3.6 Main theorem

Let $\tau$ be an element of $\mathcal{O}_F^+$ and $\mathfrak{a}$ be an integral ideal of $F$. We are interested in studying sign change results over a certain family of Fourier coefficients of a half-integral cusp form $g$, namely $\{\lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{p})\}_{\mathfrak{p}}$ where $\mathfrak{p}$ varies over prime ideals. For a fixed $g$, we put

$$\mathbb{P}_{>0}(\tau, \mathfrak{a}) = \{\mathfrak{p} \in \mathbb{P} : \mathfrak{p} \nmid \mathfrak{c}\mathfrak{D}_F,\ \lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{p}) > 0\},$$

and similarly $\mathbb{P}_{<0}(\tau, \mathfrak{a})$, $\mathbb{P}_{\ge 0}(\tau, \mathfrak{a})$, $\mathbb{P}_{\le 0}(\tau, \mathfrak{a})$, and $\mathbb{P}_{=0}(\tau, \mathfrak{a})$. We also write $\mathbb{P}_{\mathfrak{c}}$ for the set of all prime ideals not dividing $\mathfrak{c}$. We are now ready to state the main result of this chapter.

**Theorem 3.6.1.** *Let $0 \neq g \in S_k(\Gamma(\mathfrak{c}), \psi)$ with a half-integral weight $k = \frac{1}{2} + m$ with $m_j > 1$ for some $j$, an integral ideal $\mathfrak{c}$ of $F$ divisible by $4$, and $\psi$ a Hecke character of $\mathbb{A}_F$ satisfying the following conditions:*

*a. the conductor of $\psi$ divides $\mathfrak{c}$,*

*b. $\psi_\infty(-1) = (-1)^{\sum_j m_j}$, and*

*c. for any totally positive element $x$ in $\mathbb{A}_{F,\infty}^\times$, $\psi_\infty(x) = |x|^{i\mu}$ with some $\mu \in \mathbb{R}^n$ such that $\sum_j \mu_j = 0$.*

*Furthermore, we suppose that the Fourier coefficients of $g$ are real and the character $\psi$ of $g$ is quadratic.*

*Let $\tau$ be an arbitrary element in $\mathcal{O}_F^+$, and write $\tau\mathcal{O}_F = \mathfrak{a}^2\mathfrak{r}$ for some integral ideal $\mathfrak{a}$ and a square free integral ideal $\mathfrak{r}$. Then, there is a lift $\mathbf{f}_\tau$ of $g$ under the Shimura correspondence (as in Theorem 3.4.1). Assume that $\mathbf{f}_\tau$ is a non-CM primitive Hilbert modular form.*

*Then, the natural density of $\mathbb{P}_{>0}(\tau, \mathfrak{a})$ (resp., of $\mathbb{P}_{<0}(\tau, \mathfrak{a})$ ) is $1/2$, i.e., $d(\mathbb{P}_{>0}(\tau, \mathfrak{a})) = 1/2$ (resp., $d(\mathbb{P}_{<0}(\tau, \mathfrak{a})) = 1/2$ ), and $d(\mathbb{P}_{=0}(\tau, \mathfrak{a})) = 0$.*

The rest of this chapter is devoted to proving the theorem. From now on, we simply write $\mathbb{P}_{>0}$ for $\mathbb{P}_{>0}(\tau, \mathfrak{a})$, etc. Let us also define

$$\pi(x) = \#\{\mathfrak{p} \in \mathbb{P} : \mathrm{N}(\mathfrak{p}) \leq x\} \quad \text{and} \quad \pi_{>0}(x) = \#\{\mathfrak{p} \in \mathbb{P}_{>0} : \mathrm{N}(\mathfrak{p}) \leq x\}.$$

Then we have the following proposition.

**Proposition 3.6.2.** *Assume that all the hypotheses in Theorem 3.6.1 hold. Then, we have*

$$\liminf_{x\to\infty} \frac{\pi_{>0}(x)}{\pi(x)} \geq \mu([0,1]) = \frac{1}{2} \quad \textit{and} \quad \liminf_{x\to\infty} \frac{\pi_{\leq 0}(x)}{\pi(x)} \geq \mu([0,1]) = \frac{1}{2}.$$

*Proof.* The equality of the formal sums in (3.9) can also be re-interpreted as

$$\prod_{\mathfrak{p}\nmid\mathfrak{c}\mathfrak{r}} \left(1 - \frac{(\psi\epsilon_\tau)^*(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})}\mathrm{M}(\mathfrak{p})\right) \sum_{\mathfrak{m}} c(\mathfrak{m}, \mathbf{f}_\tau)\mathrm{M}(\mathfrak{m}) = \sum_{\mathfrak{m}} \lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{m})\mathrm{M}(\mathfrak{m}).$$

For any non-zero prime ideal $\mathfrak{p} \nmid \mathfrak{c}\mathfrak{r}$, comparing the coefficients of $\mathrm{M}(\mathfrak{p})$ on both sides gives

$$c(\mathfrak{p}, \mathbf{f}_\tau) - \frac{(\psi\epsilon_\tau)^*(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})} = \lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{p}) \tag{3.12}$$

since $\mathbf{f}_\tau$ is primitive (i.e., $c(\mathcal{O}_F, \mathbf{f}_\tau) = 1$). There are exactly two terms on the left side of (3.12) because the unique factorization of ideals holds in $\mathcal{O}_F$, and hence the only integral ideals which divides $\mathfrak{p}$ are $\mathfrak{p}$ and $\mathcal{O}_F$ itself.

Since $\psi$ is a quadratic character, the primitive Hilbert modular form $\mathbf{f}_\tau$ has trivial nebentypus. Hence, the Fourier coefficients $c(\mathfrak{p}, \mathbf{f}_\tau)$ are real numbers (cf. [42, Proposition 2.5]). This implies that $(\psi\epsilon_\tau)^*(\mathfrak{p}) \in \{\pm 1\}$ since, by our assumption, $(\psi\epsilon_\tau)^*(\mathfrak{p})$ is a root of unity and $\lambda_g(\tau, \mathfrak{m})$ is real for all fractional ideals $\mathfrak{m}$.

By (3.12), we have

$$\lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{p}) > 0 \Leftrightarrow c(\mathfrak{p}, \mathbf{f}_\tau) > \frac{(\psi\epsilon_\tau)^*(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})},$$

which gives us

$$\lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{p}) > 0 \Leftrightarrow B(\mathfrak{p}) > \frac{(\psi\epsilon_\tau)^*(\mathfrak{p})}{2\mathrm{N}(\mathfrak{p})^{\frac{1}{2}}}$$

since $B(\mathfrak{p}) = \frac{C(\mathfrak{p}, \mathbf{f}_\tau)}{2\mathrm{N}(\mathfrak{p})^{\frac{k_0-1}{2}}}$.

For any $\epsilon > 0$, we have the following inequality:

$$\pi_{>0}(x) + \pi\left(\frac{1}{4\epsilon^2}\right) \geq \#\{\mathfrak{p} \in \mathbb{P}_{\mathfrak{c}\mathfrak{r}\mathfrak{D}_F} : \mathrm{N}(\mathfrak{p}) \leq x \text{ and } B(\mathfrak{p}) > \epsilon\}$$

since $|\frac{(\psi\epsilon_\tau)^*(\mathfrak{p})}{2N(\mathfrak{p})^{1/2}}| = \frac{1}{2N(\mathfrak{p})^{1/2}} < \epsilon$ if $\mathrm{N}(\mathfrak{p}) > 1/4\epsilon^2$. Now divide the above inequality by $\pi(x)$ to obtain

$$\frac{\pi_{>0}(x)}{\pi(x)} + \frac{\pi\left(\frac{1}{4\epsilon^2}\right)}{\pi(x)} \geq \frac{\#\{\mathfrak{p} \in \mathbb{P}_{\mathfrak{c}\mathfrak{r}\mathfrak{D}_F} : \mathrm{N}(\mathfrak{p}) \leq x \text{ and } B(\mathfrak{p}) > \epsilon\}}{\pi(x)}.$$

The term $\pi\left(\frac{1}{4\epsilon^2}\right)/\pi(x)$ tends to $0$, as $x \to \infty$, since $\pi\left(\frac{1}{4\epsilon^2}\right)$ is finite. On the other hand, Theorem 3.5.4 gives

$$\frac{\#\{\mathfrak{p} \in \mathbb{P} : \mathrm{N}(\mathfrak{p}) \leq x \text{ and } B(\mathfrak{p}) > \epsilon\}}{\pi(x)} \to \mu([\epsilon, 1])$$

as $x \to \infty$, and therefore we have

$$\liminf_{x\to\infty} \frac{\pi_{>0}(x)}{\pi(x)} \geq \mu([\epsilon, 1]) \text{ for all } \epsilon > 0.$$

Hence, we can conclude that

$$\liminf_{x\to\infty} \frac{\pi_{>0}(x)}{\pi(x)} \geq \mu([0,1]) = \frac{1}{2}.$$

A similar proof shows that

$$\liminf_{x\to\infty} \frac{\pi_{\leq 0}(x)}{\pi(x)} \geq \mu([0,1]) = \frac{1}{2}.$$

$\square$

***Proof of Theorem of 3.6.1.*** By Proposition 3.6.2, we have

$$\frac{1}{2} \leq \liminf_{x\to\infty} \frac{\pi_{>0}(x)}{\pi(x)}.$$

Since $\pi_{>0}(x) = \pi(x) - \pi_{\leq 0}(x)$, we have

$$\limsup_{x\to\infty} \frac{\pi_{>0}(x)}{\pi(x)} \leq \mu([0,1]) = \frac{1}{2}.$$

Hence,

$$\frac{1}{2} \leq \liminf_{x\to\infty} \frac{\pi_{>0}(x)}{\pi(x)} \leq \limsup_{x\to\infty} \frac{\pi_{>0}(x)}{\pi(x)} \leq \mu([0,1]) = \frac{1}{2},$$

and therefore, $\lim_{x\to\infty} \frac{\pi_{>0}(x)}{\pi(x)}$ exists and equals $\frac{1}{2}$. Thus, the set $\mathbb{P}_{>0}$ has natural density $\frac{1}{2}$. The same argument yields that $\mathbb{P}_{<0}$ has natural density $\frac{1}{2}$ as well. This proves that $\mathbb{P}_{=0}$ has natural density $0$. $\square$

**Corollary 3.6.3.** *Assume that all the hypotheses of Theorem 3.6.1 hold. Then, the set* $\{\lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{p})\}_{\mathfrak{p}\in\mathbb{P}}$ *changes signs infinitely often. In particular, there exist infinitely many primes* $\mathfrak{p} \in \mathbb{P}$ *for which* $\lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{p}) > 0$ *(resp.,* $\lambda_g(\tau, \mathfrak{a}^{-1}\mathfrak{p}) < 0$*).*

# Chapter 4

# Simultaneous behaviour of the Fourier coefficients of two Hilbert modular cusp forms

## 4.1 Introduction

The sign changes and non-vanishing of the Fourier coefficients of modular forms over number fields has been an interesting area of research in the recent years. In this chapter, we study the simultaneous sign changes and simultaneous non-vanishing of the Fourier coefficients of distinct Hilbert cusp forms.

In [28], the authors show that, if $f$ and $g$ are two normalized cusp forms of the same level and different weights with totally real algebraic Fourier coefficients, then there exist a Galois automorphism $\sigma$ such that $f^\sigma$ and $g^\sigma$ have infinitely many Fourier coefficients of the opposite signs. Their proof uses Landau's theorem on Dirichlet series with non-negative coefficients, the properties of the Rankin-Selberg zeta function attached to cusp forms, and the bounded denominators argument.

In [17], the authors, by using an observation about real zeros of Dirichlet series instead of bounded denominators argument, strengthen the results of [28] by doing away with the Galois conjugacy condition. In fact, they extended the result to cusp forms with arbitrary real Fourier coefficients.

In [16], the authors investigated simultaneous non-vanishing of the Fourier coefficients at the powers of a prime ideal of two different Hecke eigenforms of integral weight over $\mathbb{Q}$. They proved that if $f$ and $g$ are two Hecke eigenforms of integral weights and $a_f(n)$ and $a_g(n)$ are Fourier coefficients of $f$ and $g$, respectively, then for all primes $p$, the set $\{m \in \mathbb{N} | a_f(p^m)a_g(p^m) \neq 0\}$ has positive density.

In [24], we extended some results of Gun, Kohnen and Rath [17] and Gun, Kumar and Paul [16] to the Hilbert modular forms case. Firstly, we show that two Hilbert cuspidal forms of different integral weights and same level have infinitely many Fourier coefficients of same sign (resp., of opposite sign). Secondly, we show that the simultaneous non-vanishing of the Fourier coefficients, of two non-zero distinct primitive Hilbert cuspidal non-CM eigenforms, at the powers of a prime ideal has positive density.

## 4.2 Statements of the main results

In this section, we shall state the main results of this chapter. Firstly, we prove a result on the simultaneous sign changes of the Fourier coefficients of two Hilbert cuspidal forms of different integral weights. More precisely, we prove:

**Theorem 4.2.1.** *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be non-zero Hilbert cusp forms over* $F$ *of level* $\mathfrak{c}$ *and different integral weights* $k = (k_1, \dots, k_n)$, $l = (l_1, \dots, l_n)$, *respectively. For each ideal* $\mathfrak{m} \subseteq \mathcal{O}_F$, *let* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$ *denote the Fourier coefficients (as defined in* (3.7)*) of* $\mathbf{f}$ *and* $\mathbf{g}$, *respectively. Further, assume that* $C(\mathfrak{m}, \mathbf{f}), C(\mathfrak{m}, \mathbf{g})$ *are real numbers. If* $C(\mathcal{O}_F, \mathbf{f})C(\mathcal{O}_F, \mathbf{g}) \neq 0$, *then there exist infinitely many ideals* $\mathfrak{m} \subseteq \mathcal{O}_F$ *such that* $C(\mathfrak{m}, \mathbf{f})C(\mathfrak{m}, \mathbf{g}) > 0$ *and infinitely many ideals* $\mathfrak{m} \subseteq \mathcal{O}_F$ *such that* $C(\mathfrak{m}, \mathbf{f})C(\mathfrak{m}, \mathbf{g}) < 0$.

The second result is about the simultaneous non-vanishing of the Fourier coefficients, of two non-zero distinct primitive Hilbert cuspidal eigenforms, at powers of a prime ideal has positive density. More precisely, we prove:

**Theorem 4.2.2.** *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be distinct primitive Hilbert cuspidal non-CM eigenforms over* $F$ *with trivial nebentypus and of levels* $\mathfrak{c}_1, \mathfrak{c}_2$ *and with integral weights* $k = (k_1, \dots, k_n)$, $l = (l_1, \dots, l_n)$, *respectively. We further assume that* $k_1 \equiv \cdots \equiv k_n \equiv l_1 \equiv \cdots \equiv l_n \equiv 0 \pmod 2$ *and each* $k_j, l_j \geq 2$.

*For each ideal* $\mathfrak{m} \subseteq \mathcal{O}_F$, *let* $C(\mathfrak{m}, \mathbf{f})$ *and* $C(\mathfrak{m}, \mathbf{g})$ *denote the Fourier coefficients of* $\mathbf{f}$ *and* $\mathbf{g}$, *respectively. Then, for any prime ideal* $\mathfrak{p} \subseteq \mathcal{O}_F$ *such that* $\mathfrak{p} \nmid \mathfrak{c}_1\mathfrak{c}_2\mathfrak{D}_F$, *the set*

$$\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}$$

*has positive density.*

**Corollary 4.2.3.** *Assume that the hypothesis of the above theorem holds. Then, for any prime ideal* $\mathfrak{p} \nmid \mathfrak{c}_1\mathfrak{c}_2\mathfrak{D}_F$, *there exists infinitely* $m \in \mathbb{N}$ *such that* $C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0$.

## 4.3 Proof of Theorem 4.2.1

For the proof of Theorem 4.2.1, we need the following basic results.

**Lemma 4.3.1.** *Let $s \in \mathbb{C}$ and*

$$R(s) = \sum_{n \geq 1} \frac{a(n)}{n^s}$$

*be a Dirichlet series with real coefficients $a(n)(n \in \mathbb{N})$. Assume that $a(n) \geq 0$ or $a(n) \leq 0$ for all $n \geq 1$. If $R(s)$ has a real zero $\alpha$ in the region of convergence, then $R(s)$ is identical zero.*

*Proof.* Without loss of generality, we can assume that $a(n) \geq 0$ for all $n \geq 1$. Denote the sequence of partial sums of $R(\alpha)$ by $s_i = \sum_{n=1}^{i} \frac{a(n)}{n^\alpha}$, for $i \geq 1$. Since $a(n) \geq 0$, the sequence $\{s_i\}$ is a monotonically increasing sequence. Hence, the sequence $\{s_i\}$ converges to its least upper bound. Since, $R(\alpha) = 0$, we get that, for $i \geq 1$, $s_i$ is zero. We can deduce that each $a(i) = 0$ for each $i$. Hence, $R(s)$ is identical zero.

Now, if $a(n) \leq 0$ for all $n \geq 1$, then we get the required result by applying above argument with $-R(s)$. $\square$

**Lemma 4.3.2.** ([17, Lemma 6]) Let $s \in \mathbb{C}$ and $a(n) \in \mathbb{R}$. For $m \geq 1$, consider the Dirichlet polynomial

$$R_m(s) := \sum_{1 \leq n \leq m} \frac{a(n)}{n^s}.$$

If $R_m(s)$ has infinitely many real zeros, then $R_m(s)$ is identically zero.

**Proposition 4.3.3.** ([42, Proposition 2.3]) For any integral ideal $\mathfrak{q} \subseteq \mathcal{O}_F$ and every $\mathbf{f} \in S_k(\mathfrak{c}, \psi)$, there exists an unique element of $S_k(\mathfrak{q}\mathfrak{c}, \psi)$, written as $\mathbf{f}|\mathfrak{q}$, such that

$$C(\mathfrak{m}, \mathbf{f}|\mathfrak{q}) = C(\mathfrak{q}^{-1}\mathfrak{m}, \mathbf{f}) \tag{4.1}$$

**Proposition 4.3.4.** ([37, Page 124]) For any integral ideal $\mathfrak{q} \subseteq \mathcal{O}_F$ and every $\mathbf{f} \in S_k(\mathfrak{c}, \psi)$, there exists an unique element of $S_k(\mathfrak{q}\mathfrak{c}, \psi)$, written as $\mathbf{f}|U(\mathfrak{q})$, such that

$$C(\mathfrak{m}, \mathbf{f}|U(\mathfrak{q})) = C(\mathfrak{q}\mathfrak{m}, \mathbf{f}) \tag{4.2}$$

We need the following proposition in the proof Theorem 4.2.1.

**Proposition 4.3.5.** *Let $\mathbf{f} \in S_k(\mathfrak{c}, \psi)$ and $\mathfrak{q}$ be an integral ideal of $\mathcal{O}_F$. Then $\mathbf{g} = \mathbf{f} - (\mathbf{f}|U(\mathfrak{q}))|\mathfrak{q}$ is a Hilbert cusp form of weight $k$ and level $\mathfrak{q}^2\mathfrak{c}$. Further, it has the property that $C(\mathfrak{m}\mathfrak{q}, \mathbf{g}) = 0$ and $C(\mathfrak{m}, \mathbf{g}) = C(\mathfrak{m}, \mathbf{f})$, if $(\mathfrak{m}, \mathfrak{q}) = 1$.*

*Proof.* Observe that $C(\mathfrak{m}\mathfrak{q},\mathbf{g}) = C(\mathfrak{m}\mathfrak{q},\mathbf{f}-(\mathbf{f}|U(\mathfrak{q}))|\mathfrak{q}) = C(\mathfrak{m}\mathfrak{q},\mathbf{f})-C(\mathfrak{m}\mathfrak{q},(\mathbf{f}|U(\mathfrak{q}))|\mathfrak{q})$. Now, let us compute $C(\mathfrak{m}\mathfrak{q},\mathbf{f}|U(\mathfrak{q})|\mathfrak{q}) = C(\mathfrak{m},\mathbf{f}|U(\mathfrak{q})) = C(\mathfrak{m}\mathfrak{q},\mathbf{f})$. Hence, $C(\mathfrak{m}\mathfrak{q},\mathbf{g}) = 0$.

Now, let us look at the expression when $(\mathfrak{m},\mathfrak{q}) = 1$.

$$C(\mathfrak{m},\mathbf{g}) = C(\mathfrak{m},\mathbf{f}-(\mathbf{f}|U(\mathfrak{q}))|\mathfrak{q}) = C(\mathfrak{m},\mathbf{f}) - C(\mathfrak{m},(\mathbf{f}|U(\mathfrak{q}))|\mathfrak{q}).$$

However, $C(\mathfrak{m},\mathbf{f}|U(\mathfrak{q})|\mathfrak{q}) = C(\mathfrak{q}^{-1}\mathfrak{m},\mathbf{f}|U(\mathfrak{q})) = 0$, since $\mathfrak{q}^{-1}\mathfrak{m}$ is not an integral ideal. Hence, $C(\mathfrak{m},\mathbf{g}) = C(\mathfrak{m},\mathbf{f})$, if $(\mathfrak{m},\mathfrak{q}) = 1$.

□

Now, we are in a position to prove Theorem 4.2.1.

*Proof.* By hypothesis, we have $C(\mathcal{O}_F,\mathbf{f})C(\mathcal{O}_F,\mathbf{g}) \neq 0$. First, we will show that there exist infinitely many $\mathfrak{m} \subseteq \mathcal{O}_F$ such that

$$\frac{C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g})}{C(\mathcal{O}_F,\mathbf{f})C(\mathcal{O}_F,\mathbf{g})} < 0. \tag{4.3}$$

Without loss of generality, we can assume that $C(\mathcal{O}_F,\mathbf{f})C(\mathcal{O}_F,\mathbf{g}) > 0$ as otherwise we replace g by $-$g.

If (4.3) is not true, then there exist an ideal $\mathfrak{m}' \subseteq \mathcal{O}_F$ such that

$$C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g}) \geq 0 \tag{4.4}$$

for all $\mathfrak{m} \subseteq \mathcal{O}_F$ with $N(\mathfrak{m}) \geq N(\mathfrak{m}')$. Set $\mathfrak{n} := \prod_{N(\mathfrak{p})\leq N(\mathfrak{m}')} \mathfrak{p}$, where $\mathfrak{p}$ are prime ideals of $\mathcal{O}_F$.

Suppose $\mathbf{f}_1$ and $\mathbf{g}_1$ are Hilbert modular cusp forms obtained from $\mathbf{f}$ and $\mathbf{g}$ respectively, by applying the Proposition 4.3.5 to $\mathbf{f}$ and $\mathbf{g}$ with the ideal $\mathfrak{n}$. Clearly, $\mathbf{f}_1$ and $\mathbf{g}_1$ are also Hilbert cusp forms of level $k$ and $l$ respectively, and of level $\mathfrak{c}_1$. We just say that the level is $\mathfrak{c}_1$, because as such we do not need the explicit level in the further calculations.

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 1$, the Rankin-Selberg $L$-function of $\mathbf{f}_1$ and $\mathbf{g}_1$ is defined by

$$R_{\mathbf{f}_1,\mathbf{g}_1}(s) := \sum_{\mathfrak{m}\subseteq\mathcal{O}_F,(\mathfrak{m},\mathfrak{n})=1} \frac{C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g})}{N(\mathfrak{m})^s}. \tag{4.5}$$

In above summation $C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g}) \geq 0$, since, if $N(\mathfrak{m}) \leq N(\mathfrak{m}')$ then $\mathfrak{m} = \prod_{\mathfrak{p}_i|\mathfrak{n}} \mathfrak{p}_i^{e_i}$

implies $(\mathfrak{m},\mathfrak{n}) \neq 1$. For $\mathrm{Re}(s) \gg 1$, we set

$$L_{\mathbf{f}_1,\mathbf{g}_1}(s) := \zeta_F^{\mathfrak{c}_1}(2s-(k_0+l_0)+2)R_{\mathbf{f}_1,\mathbf{g}_1}(s),$$

where $\zeta_F^{\mathfrak{c}_1}(s) = \prod_{\mathfrak{p}|\mathfrak{c}_1,\mathfrak{p}:\text{prime}}(1-N(\mathfrak{p})^{-s})\zeta_F(s)$, where $\zeta_F(s) = \sum_{\mathfrak{m}\subseteq\mathcal{O}_F} N(\mathfrak{m})^{-s}$ is Dedekind zeta function of $F$. By the Euler expansion of Dedekind zeta function of $F$, we get that

$$\begin{aligned}
\zeta_F^{\mathfrak{c}_1}(s) &= \prod_{\mathfrak{p}|\mathfrak{c}_1,\mathfrak{p}:\text{prime}}(1-N(\mathfrak{p})^{-s}) \prod_{\mathfrak{p}:\text{prime}}(1-N(\mathfrak{p})^{-s})^{-1} \\
&= \sum_{\mathfrak{m}\subseteq\mathcal{O}_F,(\mathfrak{m},\mathfrak{c}_1)=1} \frac{1}{N(\mathfrak{m})^s} = \sum_{n=1}^{\infty}\frac{a_n(\mathfrak{c}_1)}{n^s},
\end{aligned}$$

where $a_n(\mathfrak{c}_1)$ is the number of integral ideals of norm $n$ that are co-prime to $\mathfrak{c}_1$. Hence, we can write

$$L_{\mathbf{f}_1,\mathbf{g}_1}(s) = \sum_{n=1}^{\infty}\frac{a_n(\mathfrak{c}_1)n^{k_0+l_0-2}}{n^{2s}} \times \sum_{\mathfrak{m}\subseteq\mathcal{O}_F,(\mathfrak{m},\mathfrak{n})=1}\frac{C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g})}{N(\mathfrak{m})^s}.$$

Now, we can re-write

$$L_{\mathbf{f}_1,\mathbf{g}_1}(s) = \sum_{m=1}^{\infty}\frac{\mathfrak{b}_m^{\mathfrak{c}_1}(\mathbf{f}_1,\mathbf{g}_1)}{m^s},$$

where

$$\mathfrak{b}_m^{\mathfrak{c}_1}(\mathbf{f}_1,\mathbf{g}_1) = \sum_{n|m}\left(a_n(\mathfrak{c}_1)n^{k_0+l_0-2}\sum_{(\mathfrak{m},\mathfrak{n})=1,N(\mathfrak{m})=m/n^2}C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g})\right).$$

In the above summation $\mathfrak{b}_m^{\mathfrak{c}_1}(\mathbf{f}_1,\mathbf{g}_1) \geq 0$ for all $m$ because $C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g}) \geq 0$, for all $(\mathfrak{m},\mathfrak{n}) = 1$, by (4.4). Observe that $\mathfrak{b}_1^{\mathfrak{c}_1}(\mathbf{f}_1,\mathbf{g}_1) = C(\mathcal{O}_F,\mathbf{f})C(\mathcal{O}_F,\mathbf{g})$.

Denote $k_0 := \max\{k_1,k_2,\ldots,k_n\}$ and $l_0 := \max\{l_1,l_2,\ldots,l_n\}$. Define, for any $j$, $k_j' := k_0 - k_j$, and similarly, define $l_j'$.

Now, look at the complete $L$-function, defined by the product

$$\Lambda_{\mathbf{f}_1,\mathbf{g}_1}(s) = \prod_{j=1}^{n}\Gamma\left(s+1+\frac{k_j-l_j-k_0-l_0}{2}\right)\Gamma\left(s-\frac{k_j'+l_j'}{2}\right)L_{\mathbf{f}_1,\mathbf{g}_1}(s)$$

can be continued to a holomorphic function on the whole plane, since the weights are different (cf. [42, Proposition 4.13]). As the $\Gamma$-function is extended by analytic continuation to all complex numbers except the non-positive integers, where the function has simple poles, we get that the function $L_{\mathbf{f}_1,\mathbf{g}_1}(s)$ is also entire.

By Landau's Theorem it follows that the Dirichlet series $L_{\mathbf{f}_1,\mathbf{g}_1}(s)$ converges everywhere. Observe that the function $L_{\mathbf{f}_1,\mathbf{g}_1}(s)$ has real zeros because the $\Gamma$-factors have poles at non-positive integers. By Lemma 4.3.1, we have that $\mathfrak{b}_m^{\mathfrak{c}_1}(\mathbf{f}_1,\mathbf{g}_1) = 0$ for all $m$. This contradicts the assumption that $C(\mathcal{O}_F,\mathbf{f})C(\mathcal{O}_F,\mathbf{g}) \neq 0$ This completes the proof of (4.3).

In order to complete the proof of the Theorem 4.2.1, we need to show that there exist infinitely many $\mathfrak{m} \subseteq \mathcal{O}_F$ such that

$$\frac{C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g})}{C(\mathcal{O}_F,\mathbf{f})C(\mathcal{O}_F,\mathbf{g})} > 0.$$

It is sufficient to assume that $C(\mathcal{O}_F,\mathbf{f})C(\mathcal{O}_F,\mathbf{g}) > 0$. We then have to show that there exist infinitely many integral ideals $\mathfrak{m}$ such that $C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g}) > 0$. If not, then $C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g}) \leq 0$ for all ideals $\mathfrak{m} \subseteq \mathcal{O}_F$ with $N(\mathfrak{m}) \gg 0$. Note that, $C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g})$ cannot be equal to zero for almost all ideals $\mathfrak{m} \subseteq \mathcal{O}_F$. For in this case $\sum_{\mathfrak{m}\subseteq\mathcal{O}_F} \frac{C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g})}{N(\mathfrak{m})^s}$ is a Dirichlet polynomial and

$$\Lambda_{\mathbf{f},\mathbf{g}}(s) = \prod_{j=1}^{n} \Gamma(s+1+\frac{k_j-l_j-k_0-l_0}{2})\Gamma(s-\frac{k_j'+l_j'}{2})L_{\mathbf{f},\mathbf{g}}(s)$$

is entire. The presence of the multiple $\Gamma$-factors ensures that $\sum_{\mathfrak{m}\subseteq\mathcal{O}_F} \frac{C(\mathfrak{m},\mathbf{f})C(\mathfrak{m},\mathbf{g})}{N(\mathfrak{m})^s}$ has infinitely many zeros. Hence, by Lemma 4.3.2, we get that $C(\mathcal{O}_F,\mathbf{f})C(\mathcal{O}_F,\mathbf{g}) = 0$, which is a contradiction. Hence, there exists an integral ideal $\mathfrak{d} \subseteq \mathcal{O}_F$ such that $C(\mathfrak{d},f)C(\mathfrak{d},g) < 0$.

Now, by Proposition 4.3.4, $\mathbf{f}|U(\mathfrak{d})$ and $\mathbf{g}|U(\mathfrak{d})$ are Hilbert cusp forms of weights $k_1, k_2$, respectively and weight $\mathfrak{d}\mathfrak{c}$. Observe that

$$C(\mathcal{O}_F,\mathbf{f}|U(\mathfrak{d}))C(\mathcal{O}_F,\mathbf{g}|U(\mathfrak{d})) = C(\mathfrak{d},\mathbf{f})C(\mathfrak{d},\mathbf{g}) < 0.$$

Now, by (4.3), we have $C(\mathfrak{m},\mathbf{f}|U(\mathfrak{d}))C(\mathfrak{m},\mathbf{g}|U(\mathfrak{d})) > 0$ for infinitely many $\mathfrak{m} \subseteq \mathcal{O}_F$. This proves our claim.

□

## 4.4 Proof of Theorem 4.2.2

By [42, (2.23)], the Fourier coefficients $C(\mathfrak{m}, \mathbf{f})$ of $\mathbf{f}$ satisfy the following Hecke relations

$$C(\mathfrak{m}, \mathbf{f})C(\mathfrak{n}, \mathbf{f}) = \sum_{\mathfrak{m}+\mathfrak{n}\subset\mathfrak{a}} N(\mathfrak{a})^{k_0-1}C(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, \mathbf{f}),$$

where $k_0 = \max\{k_1, \ldots, k_n\}$. In particular, for any $m \geq 1$, the following relation holds:

$$C(\mathfrak{p}^{m+1}, \mathbf{f}) = C(\mathfrak{p}, \mathbf{f})C(\mathfrak{p}^m, \mathbf{f}) - N(\mathfrak{p})^{k_0-1}C(\mathfrak{p}^{m-1}, \mathbf{f}). \tag{4.6}$$

For any integral ideal $\mathfrak{a} \subseteq \mathcal{O}_F$, define

$$\beta(\mathfrak{a}, f) := \frac{C(\mathfrak{a}, \mathbf{f})}{N(\mathfrak{a})^{\frac{k_0-1}{2}}}.$$

For any prime ideal $\mathfrak{p} \subseteq \mathcal{O}_F$, by (4.6), we have the following

$$\beta(\mathfrak{p}^{m+1}, \mathbf{f}) = \beta(\mathfrak{p}, \mathbf{f})\beta(\mathfrak{p}^m, \mathbf{f}) - \beta(\mathfrak{p}^{m-1}, \mathbf{f}). \tag{4.7}$$

It is well-known that for a primitive Hilbert cuspidal eigenform $\mathbf{f}$ over $F$, there is an irreducible cuspidal automorphic representation $\Pi = \Pi_{\mathbf{f}}$ of $GL_2(\mathbb{A}_F)$ corresponding to it. For any place $\mathfrak{p}$ of $F$ such that $\Pi_{\mathfrak{p}}$ is unramified, let $\lambda_{\mathfrak{p}}$ denote the eigenvalue of the Hecke operator

$$\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}) \begin{pmatrix} \varpi_{\mathfrak{p}} & \\ & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}) \tag{4.8}$$

on $\Pi_{\mathfrak{p}}^{\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})}$, where $\varpi_{\mathfrak{p}}$ is a uniformizer of $\mathcal{O}_{\mathfrak{p}}$. For such a prime $\mathfrak{p}$, by [23, Lemma 3.2], we see that the eigenvalues $\lambda_{\mathfrak{p}}$ and the Fourier coefficient $C(\mathfrak{p}, \mathbf{f})$ are related by

$$\lambda_{\mathfrak{p}} = \frac{C(\mathfrak{p}, \mathbf{f})}{N(\mathfrak{p})^{\frac{k_0-2}{2}}}.$$

For any fixed prime ideal $\mathfrak{p} \nmid \mathfrak{c}_1\mathfrak{c}_2\mathfrak{D}_F$, by [23, Theorem 3.3], we have

$$\beta(\mathfrak{p}, \mathbf{f}) := \frac{C(\mathfrak{p}, \mathbf{f})}{N(\mathfrak{p})^{\frac{k_0-1}{2}}} = \frac{\lambda_{\mathfrak{p}}}{N(\mathfrak{p})^{\frac{1}{2}}} \in [-2, 2]. \tag{4.9}$$

Since $\beta(\mathfrak{p}, \mathbf{f}) \in [-2, 2]$, we can write $\beta(\mathfrak{p}, \mathbf{f}) = 2\cos\alpha_{\mathfrak{p}}$, for some $0 \leq \alpha_{\mathfrak{p}} \leq \pi$. Before getting into the proof of Theorem 4.2.2, we need the following proposition.

**Proposition 4.4.1.** *For any fixed prime ideal $\mathfrak{p} \nmid \mathfrak{c}_1\mathfrak{c}_2\mathfrak{D}_F$ and for any $m \geq 1$, we have*

$$\beta(\mathfrak{p}^{\mathfrak{m}}, \mathbf{f}) = \begin{cases} (-1)^m(m+1) & \text{if } \alpha_{\mathfrak{p}} = \pi; \\ m+1 & \text{if } \alpha_{\mathfrak{p}} = 0; \\ \frac{\sin(m+1)\alpha_{\mathfrak{p}}}{\sin \alpha_{\mathfrak{p}}} & \text{if } 0 < \alpha_{\mathfrak{p}} < \pi. \end{cases} \tag{4.10}$$

*Proof.* The first two cases are easy to prove by induction. So WLOG assume that $0 < \alpha_{\mathfrak{p}} < \pi$. When $m = 1$, we have $\beta(\mathfrak{p}, \mathbf{f}) = \frac{\sin 2\alpha_{\mathfrak{p}}}{\sin \alpha_{\mathfrak{p}}} = \frac{2 \sin \alpha_{\mathfrak{p}} \cos \alpha_{\mathfrak{p}}}{\sin \alpha_{\mathfrak{p}}} = 2\cos \alpha_{\mathfrak{p}}$. Assume that $\beta(\mathfrak{p}^{\mathfrak{m}}, \mathbf{f}) = \frac{\sin(m+1)\alpha_{\mathfrak{p}}}{\sin \alpha_{\mathfrak{p}}}$ for some $m \geq 1$. By (4.7), we have

$$\begin{aligned} \beta(\mathfrak{p}^{m+1}, \mathbf{f}) &= \beta(\mathfrak{p}, \mathbf{f})\beta(\mathfrak{p}^m, \mathbf{f}) - \beta(\mathfrak{p}^{m-1}, \mathbf{f}) \\ &= 2\cos \alpha_{\mathfrak{p}} \frac{\sin(m+1)\alpha_{\mathfrak{p}}}{\sin \alpha_{\mathfrak{p}}} - \frac{\sin m\alpha_{\mathfrak{p}}}{\sin \alpha_{\mathfrak{p}}} \\ &= \frac{2\sin(m+1)\alpha_{\mathfrak{p}} \cos \alpha_{\mathfrak{p}} - \sin m\alpha_{\mathfrak{p}}}{\sin \alpha_{\mathfrak{p}}} \\ &= \frac{\sin(m+2)\alpha_{\mathfrak{p}} + \sin m\alpha_{\mathfrak{p}} - \sin m\alpha_{\mathfrak{p}}}{\sin \alpha_{\mathfrak{p}}} \\ &= \frac{\sin(m+2)\alpha_{\mathfrak{p}}}{\sin \alpha_{\mathfrak{p}}}. \end{aligned}$$

□

Now, we are in a position to prove Theorem 4.2.2. Let $\mathfrak{p} \nmid \mathfrak{c}_1\mathfrak{c}_2\mathfrak{D}_F$ be a prime ideal. By (4.9), one can write

$$\beta(\mathfrak{p}, \mathbf{f}) = 2\cos \alpha_{\mathfrak{p}} \text{ and } \beta(\mathfrak{p}, \mathbf{g}) = 2\cos \beta_{\mathfrak{p}}$$

with $0 \leq \alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}} \leq \pi$. Now, the proof of Theorem 4.2.2 follows from following cases.

**Case (1):** When $\alpha_{\mathfrak{p}} = 0$ or $\pi$ and $\beta_{\mathfrak{p}} = 0$ or $\pi$, then by Proposition 4.4.1, we see that

$$\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\} = \mathbb{N}.$$

In this case all elements of the sequence $\{C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g})\}_{m\in\mathbb{N}}$ are non-zero.

**Case (2):** Suppose that exactly one of $\alpha_{\mathfrak{p}}$, $\beta_{\mathfrak{p}}$ is $0$ or $\pi$, say $\alpha_{\mathfrak{p}} = 0$ or $\pi$ and $\beta_{\mathfrak{p}} \in (0, \pi)$. If $\beta_{\mathfrak{p}}/\pi \notin \mathbb{Q}$, there is nothing to prove. If $\beta_{\mathfrak{p}}/\pi \in \mathbb{Q}$, say $\beta_{\mathfrak{p}} = \frac{r}{s}$, where $r, s \in \mathbb{N}$ and $(r, s) = 1$, then we have $\sin m\alpha_{\mathfrak{p}} = 0$ if and only if $m$ is an integer multiple of $s$, then we have

$$\#\{m \leq x | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\} = \#\{m \leq x | C(\mathfrak{p}^m, \mathbf{g}) \neq 0\} = [x] - \left[\frac{x}{s}\right].$$

Hence the set $\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}$ has positive density.

**Case (3):** Suppose that $\alpha_{\mathfrak{p}} = \beta_{\mathfrak{p}} \in (0, \pi)$, i.e., $\alpha_{\mathfrak{p}}/\pi = \beta_{\mathfrak{p}}/\pi \in (0, 1)$. If $\alpha_{\mathfrak{p}}/\pi \notin \mathbb{Q}$, then $C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0$ for all $m \in \mathbb{N}$ as $\sin m\alpha_{\mathfrak{p}} \neq 0$ for all $m \in \mathbb{N}$. If $\alpha_{\mathfrak{p}}/\pi \in \mathbb{Q}$, say $\alpha_{\mathfrak{p}} = \frac{r}{s}$, where $r, s \in \mathbb{N}$ and $(r, s) = 1$, then we have $\sin m\alpha_{\mathfrak{p}} = 0$ if and only if $m$ is an integer multiple of $s$ and hence

$$\#\{m \leq x | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\} = [x] - \left[\frac{x}{s}\right].$$

Hence the set $\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}$ has positive density.

**Case (4):** Suppose that $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}} \in (0, \pi)$ with $\alpha_{\mathfrak{p}} \neq \beta_{\mathfrak{p}}$. If both $\alpha_{\mathfrak{p}}/\pi, \beta_{\mathfrak{p}}/\pi \notin \mathbb{Q}$, then there is nothing to prove. Next suppose that one of them, say $\alpha_{\mathfrak{p}}/\pi = \frac{r}{s}$ with $(r, s) = 1$ and $\beta_{\mathfrak{p}}/\pi \notin \mathbb{Q}$. Then we have

$$\#\{m \leq x | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\} = \#\{m \leq x | C(\mathfrak{p}^m, \mathbf{f}) \neq 0\} = [x] - \left[\frac{x}{s}\right].$$

Hence the set $\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}$ has positive density.

Now let both $\alpha_{\mathfrak{p}}/\pi, \beta_{\mathfrak{p}}/\pi \in \mathbb{Q}$. If $\alpha_{\mathfrak{p}}/\pi = \frac{r_1}{s_1}$ and $\beta_{\mathfrak{p}}/\pi = \frac{r_2}{s_2}$ with $(r_i, s_i) = 1$, for $1 \leq i \leq 2$, then

$$\#\{m \leq x | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\} = \#[\{m \leq x | C(\mathfrak{p}^m, \mathbf{f}) \neq 0\} \cap \{m \leq x | C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}].$$

Since

$$\begin{aligned}\#\{m \leq x | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) = 0\} &= \#[\{m \leq x | C(\mathfrak{p}^m, \mathbf{f}) = 0\} \cup \{m \leq x | C(\mathfrak{p}^m, \mathbf{g}) = 0\}] \\ &\leq \left[\frac{x}{s_1}\right] + \left[\frac{x}{s_2}\right].\end{aligned}$$

Hence the set $\{m \in \mathbb{N} | C(\mathfrak{p}^m, \mathbf{f})C(\mathfrak{p}^m, \mathbf{g}) \neq 0\}$ has positive density. This completes the proof of Theorem 4.2.2.

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