Numerical modelling of fluid structure interaction using fictitious domain method

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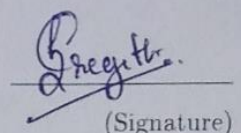
Declaration

I declare that this written submission represents my ideas in my own words, and where ideas or words of others have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.



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Approval Sheet

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Dedication

I dedicate this work to all my friends in IITH, who offered me unconditional love and support.

Abstract

The interaction of rigid or deformable solids with an internal or surrounding fluid are complex non linear multi-physics problems and are considered of great importance in the design of many engineering systems. The fictitious domain method is a widely used numerical method for solving fluid structure interaction problems and has been sucessful in obtaining solutions for FSI problems dealing with incompressible flow. The objective of this work is to implement the fictitious domain method in solving fluid structure interaction problems involving compressible flow. This work uses Eulerian and Lagrangian finite element formulation for describing the solid domain and fluid domain respectively. The coupling between the two is provided using a Lagrangian multiplier. This multiplier lets the solid and fluid mesh to sweep across each other or in other words the mesh will be non conforming. This avoids the requirement of mesh update as in the case of other numerical methods for FSI problems.

Contents

Declaration ii
Approval Sheet iii
Acknowledgements iv
Abstract vi

Nomenclature **vii**

1 Introduction **1**
1.1 Fluid-Structure Interaction 1
1.1.1 Classification 1
1.2 Objective and Scope 3
1.3 Outline 3

2 Formulation **4**
2.1 Solid dynamics model 4
2.2 Fluid dynamics model 5
2.3 Fluid-Structure Interaction 15

3 Numerical Examples **17**
3.1 Heat transfer by conduction and convection caused by a rigid oscillating beam . . . 18

4 Conclusion **24**

References **25**

Appendix A **26**
A.1 Kinematics 26
A.1.1 Lagrangian and Eulerian description 26
A.2 Governing equations of fluid. 28
A.2.1 Conservation of mass 30
A.2.2 Balance of linear momentum 30
A.2.3 Energy equation 34

Chapter 1

Introduction

1.1 Fluid-Structure Interaction

Fluid-structure interaction problem involves a coupling between a fluid dynamics problem and a structural dynamics problem. In fluid mechanics problems, we have a fixed computational grid across the boundary of which flow occurs. That is, we use the Eulerian description (refer Appendix A.1) of motion. However, in solid mechanics problems, the solid particles and the computational grid experiences the same amount of deformation. That is, we follow the Lagrangian description of motion (refer Appendix A.1). When a structure interacts with a fluid, it imposes a velocity on the fluid, and as a result of the reaction force, it experiences the fluid drag force, which may be either inertial or viscous. The structural deformation occurs due to this drag force which in turn alters the velocity field. This cycle repeats until the system reaches equilibrium.
Thus fluid-structure interaction refers to the interaction of rigid or deformable structure with an internal or surrounding flow. It is a coupled nonlinear multi-physics problem in which forces and velocity have to be determined simultaneously. This interaction has been a crucial consideration for many engineering applications which includes aircraft, engines, bridges, parachute dynamics, arterial blood flow, etc.
Obtaining an analytical solution for FSI problem is too complicated. Hence we have to rely on experimental and numerical solutions. Numerical methods are classified according to the solution approach used and also based on the treatment of meshes.

1.1.1 Classification

According to the solution approach followed

Based on the solution approach followed we can classify the FSI problems as:

- **Monolithic approach**
  Monolithic approach is the one in which the equations governing the flow (Fluid dynamics equations) and the displacement of structure (solid dynamics equations) are solved simultaneously for each time step with a single solver. These methods are unconditionally stable.
- **Partitioned approach**
  Partitioned approach is the one in which the equations governing the flow (Fluid dynamics

equations) and the displacement of structure (solid dynamics equations) are solved alternatively. Also, the kinematic and dynamic conditions are enforced asynchronously.

Based on the treatment of meshes

- **Conformal mesh method**

  In conforming mesh method, the finite element meshes located at the interface of solid and fluid meshes are assumed to be conforming. i.e., there is an interconnection between each node of solid and fluid mesh at the interface. Hence the interface will remain intact. Due to this property, any movement of solid boundary results in displacement of the fluid boundary which in turn is transmitted through the fluid domain. This displacement will lead to the requirement of simultaneous mesh refinement of fluid mesh. Also if the solid undergoes large deformation, the fluid mesh will experience excessive distortion and remeshing is the only remedy. FSI methods with conformal mesh include three fields describing the fluid dynamics, structural dynamics and mesh movement respectively. In this method, the focus is on the coordination of data transfer and the consistency between the fluid and structural nodes. Here we solve the fluid equations at a given time instance using the assumed initial interface location. The resulting drag forces are then imposed on the structural interface as external forces. Solid dynamics equations are solved to update the position of the structural surface. Finally, fluid remeshing is performed to accommodate the new interface location which is used for the next time instance. The process is continued iteratively until the convergence criteria are attained. One popular example is the Arbitrary Lagrangian-Eulerian method where the normal Eulerian fluid mesh is transformed into a moving Eulerian fluid mesh by incorporating an extra convective term into the fluid dynamics equation. Since the moving Eulerian fluid mesh is connected to a Lagrangian solid mesh at the interface, the nodes at the fluid-solid interface will move in a Lagrangian manner. This lets the interface remain intact and accurate solutions are obtained near the interface.

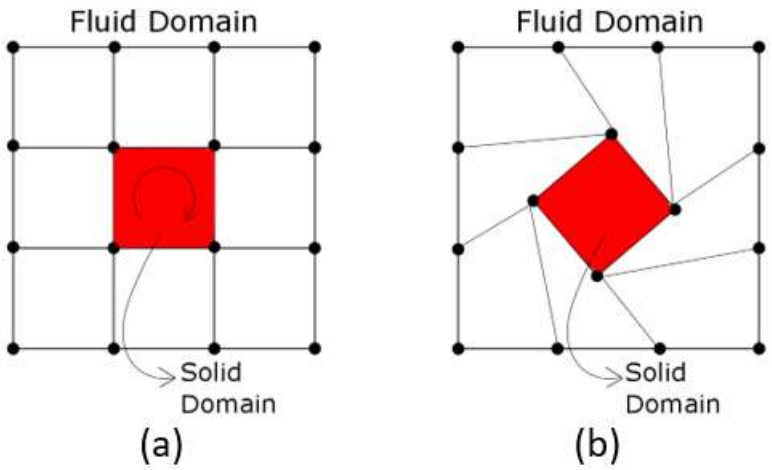

Figure 1.1: shows schematic representation of conformal mesh method where (a) the solid at rest is given an (b) angular displacement. Reproduced from [1]

- **Non-conformal mesh method**

  In the non-conforming mesh method, the fluid mesh is fixed in space, and the solid mesh sweeps across the fluid domain. In this case, there is no matching between the fluid and solid nodes at the interface, and hence the fluid and solid mesh are allowed to overlap. Since the nodes at the interface are not conforming, there is no distortion of the fluid element. Hence mesh refinement not required. In FSI methods with non-conformal meshing, the coupling is done with the help of Lagrange multiplier. This multiplier will impose a kinematic condition at the fluid-structure interface in an approximate manner. So the fluid mesh near the interface requires interpolation for coupling with the solid mesh. This leads to inaccurate solutions near the interface. But at regions away from the interface the solution appears to have good accuracy.

  Fictitious domain method is an example which uses non-conforming mesh. Here the computation time will be lesser compared to ALE method since mesh refinement is not needed.

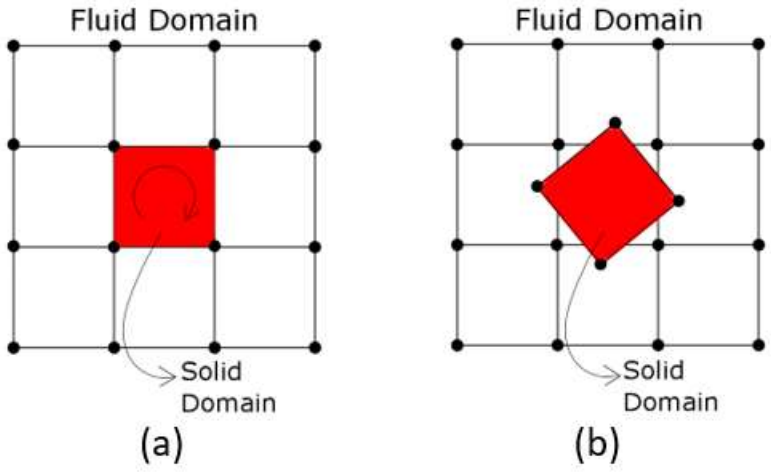

Figure 1.2: shows schematic representation of Non conformal mesh method where (a) the solid at rest is given an (b) angular displacement. Reproduced from [1]

1.2 Objective and Scope

The objective of this work is to develop a numerical model to solve fluid structure interaction problems where deformable solid bodies come in contact with a fluid domain. In this work we first concentrate on the interaction between the fluid domain and rigid solids.

1.3 Outline

The outline of the thesis is as discussed below. Chapter 2 focus on the finite element formulation where the variational formulation,its linearisation, finite element formulation and coupling of solid and fluid equations is discussed. Chapter 3 discuss on the bench mark problems solved for validating fluid formulation and also the coupled numerical problem solved using the FSI solver. Chapter 4 presents the conclusions.

Chapter 2

Formulation

2.1 Solid dynamics model

We consider the structure member as a thin Euler-Bernoulli beam with plane strain conditions in the z-directions. Also we assume that the solid material is isotropic and linearly elastic. Using the principle of virtual work we can write,

$$\int_V (\sigma\delta\epsilon + \rho(\ddot{u}\delta u + \ddot{v}\delta v))dV = \int (f_x\delta u + f_y\delta v)dV + \int (t_x\delta u + t_y\delta v)dA \tag{2.1}$$

where, u and v denotes the axial and transverse displacements, and ρ represents the mass density of beam. σ and ϵ represents the axial stress and the corresponding non linear strain. Axial and transverse body forces are denoted as f_x and f_y respectively; and t_x and t_y represents the surface tractions. For discretization , we use a beam element with two node at the ends each having three degrees of freedom which are u (axial displacement), v (transverse displacement) and $\phi = \frac{\partial v}{\partial x}$ (rotation) where x denotes the axial cordinates along the beam. Axial displacement is given linear interpolation and cubical interpolation is provided for transverse displacement.The finite element formulation for solids used in this work closely follows Reference [2] and in the following we review this in detail.

Let u= N_u p and ; v= N_v p

where, N_u and N_v are the interpolation matrices and $p = \{\ u_1\ v_1\ \phi_1\ l_0\ u_2\ v_2\ \phi_2\ l_0\ \}^T$. The subscripts 1 and 2 are the node numbers and l_0 is the reference length.

We have the formula for non linear axial strain in the beam as:

$$\begin{aligned}\epsilon &= \frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial v}{\partial x}^2 - y(\frac{\partial^2 v}{\partial x^2}) \\ &= \bar{\epsilon} - y\chi\end{aligned}$$

A body is said to be in equilibrium if the virtual work done by the internal forces are equal to that done by external forces. Hence we can write as follows.

$$\delta W_{int}^{t+\Delta t} = \delta W_{ext}^{t+\Delta t}$$

We have

$$\delta W_{int}^{t+\Delta t} = \int_V (\sigma \delta\epsilon + \rho(\ddot{u}\delta u + \ddot{v}\delta v))dV$$

We define $\int \sigma dA = P$ and $-\int \sigma y dA = M$ and substitute this along with the strain into the above equation. We get,

$$\begin{aligned}
\delta W_{int}^{t+\Delta t} &= \int_{x_0} (P^{t+\Delta t}\delta\bar{\epsilon}^{t+\Delta t} + M^{t+\Delta t}\delta\chi + \rho A(\ddot{u}^{t+\Delta t}\delta u + \ddot{v}^{t+\Delta t}\delta v))dx \\
\delta W_{int}^{t+\Delta t} &= \int_{x_0} [(P^t\delta\bar{\epsilon}^t + M^t\delta\chi) + (\Delta P\delta\bar{\epsilon}^t + \Delta M\delta\chi) + P^t\Delta\delta\bar{\epsilon} + \rho A(\ddot{u}^{t+\Delta t}\delta u + \ddot{v}^{t+\Delta t}\delta v)]dx \\
\delta W_{int}^{t+\Delta t} &= \delta p^T f_{int}^t + \delta p^T K\Delta p + \delta p^T M\ddot{p}^{t+\Delta t}
\end{aligned}$$

where

$$\begin{aligned}
f_{int}^t &= \int [P^t B_u^T + M^t C_v^T]dx \\
K &= \int EAB_u^T B_u dx + \int EIC_v^T C_v dx + \int P^t B_v^t B_v dx \\
M &= \int \rho A(N_u^T N_u + N_v^T N_v)dx \\
\delta W_{ext}^{t+\Delta t} &= \int [f_x^{t+\Delta t}\delta u + f_y^{t+\Delta t}\delta v]Adx + \int [t_x^{t+\Delta t}\delta u + t_y^{t+\Delta t}\delta v]bdx \\
&= \delta p \int [(f_x^{t+\Delta t}N_u^T + f_y^{t+\Delta t}N_v^T)A + b(t_x^{t+\Delta t}N_u^T + t_y^{t+\Delta t}N_v^T)]dx \\
&= \delta p^T f_{ext}^{t+\Delta t}
\end{aligned}$$

Since δp is arbitrary, we can write

$$K\Delta P + M\ddot{p}^{t+\Delta t} = f_{ext}^{t+\Delta t} - f_{int}^t \tag{2.2}$$

Newmark's Algorithm can be used to solve the above equations after applying appropriate initial and boundary conditions.

2.2 Fluid dynamics model

Numerous finite element formulations are available for the analysis of compressible fluid flow. Most among these are stabilised formulations where variational formulation is provided with additional stabilising terms such as lest square operators and shock capturing terms. Addition of these terms lets the formulation attain improved stabilising properties and also field variables cn be given equal order interpolation. Least square finite element method [3, 4] follows a distinct approach where in governing equations are converted to equalent 1st order equations which allow C^0 continuity to be maintained across the element boundaries. Reference [5, 6] uses a characteristics based split method in which a numerical technique is developed with the help of theory of characteristics. This formulation also requires the addition of stabilising terms.

In Bristeau's [7] work a different strategy was proposed which makes use of the primitive variables namely velocity, density and temperature. Inf-sup conditions were satisfied by using lower order in-

terpolations for density term compared to velocity. Capon [8] also used a similar scheme and found that even though low Reynolds number problems can be dealt with sucessfully using this scheme but as the Reynolds number is increased the oscillations starts to reappear. This problem was solved by the implementation of state equation in a weak form and also pressure interpolation is chosen such that it obeys inf-sup conditions.

The Finite element formulation used for fluids model used in this work is closely related to the one used in Reference [9]. Let Ω denote the domain and Γ denote its boundary. Let Γ_u, Γ_θ and Γ_q denote respectively the portions of Γ where velocity, temperature and heat flux are prescribed. The fluid flow should satisfy the three governing equations : Mass balance , Momentum balance and Energy balance.These governing equations (refer Appendix A.2) are :

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\nabla \mathbf{u})\,\mathbf{u}\right] = \nabla\cdot\tau+\rho\mathbf{b}, \tag{2.3}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{u}) = 0, \tag{2.4}$$

$$\rho C_v\left[\frac{\partial \theta}{\partial t}+\mathbf{u}\cdot(\nabla\theta)\right] = -p\nabla\cdot\mathbf{u}+\sigma:D-\nabla\cdot\mathbf{q}+\rho Q_h\cdot \tag{2.5}$$

where $\rho, \mathbf{u}, \theta$ and p represents density, velocity, temperature and pressure respectively. $\sigma = \lambda(\rho,\theta)(trD)I + 2\mu(\rho,\theta)D$, is the Cauchy stress and σ is the deviatoric or viscous stress. $\mathbf{q}$=-k$\nabla\theta$ is the heat flux and k is the thermal conductivity. Heat input per unit mass is denoted by Q_h and C_v is the specific heat at constant volume.D is the rate of deformation tensor.

So we have five unknowns $(\mathbf{u}, \rho, \theta, p)$ but only 4 governing equations. In order to complete the set of equations, we use the following equation of state for the pressure

$$p = \tilde{p}\,(\rho,\theta)\,. \tag{2.6}$$

where $\tilde{p}$ represents the function form for the pressure. This set of non-linear equations are to be solved after applying appropriate initial and boundary conditions on the field variables. The weak form of these governing equations is obtained (after performing integration by parts) as

$$\int_\Omega \rho\mathbf{u}_\delta^T\left[\frac{\partial \mathbf{u}}{\partial t}+(\boldsymbol{\nabla}\mathbf{u})\,\mathbf{u}\right]d\Omega-\int_\Omega \boldsymbol{\nabla}\cdot\mathbf{u}_\delta\; p\; d\Omega+\int_\Omega \left[\boldsymbol{D}_c\left(\mathbf{u}_\delta\right)\right]^T\boldsymbol{C}_c\boldsymbol{D}_c = \int_\Omega \rho\mathbf{u}_\delta^T\mathbf{b}\; d\Omega+\int_{\Gamma_t}\mathbf{u}_\delta^T\bar{\mathbf{t}}\; d\Gamma \quad \forall\mathbf{u}_\delta,$$

$$\int_\Omega \rho_\delta\left[\frac{\partial \rho}{\partial t}+\rho\boldsymbol{\nabla}\cdot\mathbf{u}+\mathbf{u}\cdot(\nabla\rho)\right]d\Omega = 0,$$

$$\int_\Omega \rho C_v\theta_\delta\left[\frac{\partial \theta}{\partial t}+\mathbf{u}\cdot(\boldsymbol{\nabla}\theta)+p\boldsymbol{\nabla}\cdot\mathbf{u}-\boldsymbol{\sigma}:\boldsymbol{D}\right]d\Omega+\int_\Omega k\boldsymbol{\nabla}\theta_\delta\cdot\boldsymbol{\nabla}\theta\; d\Omega = \int_\Omega \rho\theta_\delta Q_h-\int_{\Gamma_q}\theta_\delta\bar{q_n}d\Gamma \quad \forall\theta_\delta,$$

$$\int_\Omega p_\delta \left[p - \tilde{p}\left(\rho, \theta\right)\right] d\Omega = 0 \quad \forall p_\delta.$$

where $\boldsymbol{D}_c$ and $\boldsymbol{C}_c$ represents the rate of deformation and the material constitutive tensor and is expressed as

$$\boldsymbol{D}_c = \begin{bmatrix} D_{xx} \\ D_{yy} \\ 2D_{xy} \end{bmatrix} \quad \boldsymbol{C}_c = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

$\bar{\mathbf{t}} = \boldsymbol{\tau}\mathbf{n}$ are the prescribed tractions, and $\bar{q}_n = \mathbf{q} \cdot \mathbf{n}$ is the prescribed normal heat flux on Γ_q. Generalized trapezoidal rule is used for carrying out the time discretization on the time interval $\left[t^n, t^{n+1}\right]$, we have $\alpha t_\Delta \dot{\mathbf{u}}^{t^{n+1}} = \mathbf{u}^{t^{n+1}} - \mathbf{u}^{t^n} - (1-\alpha)\, t_\Delta \dot{\mathbf{u}}^{t^n}$. Similar descritisation for time is used for density and temperature fields also .

Let $(\mathbf{u}^n, \rho^n, \theta^n, p^n)$ denote the values of the velocity, density, temperature and pressure field variables at the n iterative step and $\left(\mathbf{u}^{n+1}, \rho^{n+1}, \theta^{n+1}, p^{n+1}\right)$ denotes the corresponding values at $n+1^{th}$ iterative steps at time step t_{n+1}, and let $(\Delta\mathbf{u}, \Delta\rho, \Delta\theta, \Delta p)$ denote the change in the velocity, density, temperature and pressure fields at t_{n+1} and t_n time step .

The linearization and variational formulation of Continuity, Momentum, Energy and State equations are described below.

Variational formulation

Continuity equation

$$\int_\Omega \left[\frac{\partial \rho}{\partial t} + \nabla.(\rho u)\right] d\Omega = 0$$
$$\int_\Omega \delta\rho \left[\frac{\partial \rho}{\partial t} + \rho(\nabla.u) + u.(\nabla\rho)\right] d\Omega = 0$$

$$\begin{aligned}
\int_\Omega \delta\rho[\rho(\nabla.u)+u.(\nabla\rho)]d\Omega &= \int_\Omega \left[\delta\rho\rho u_{j,j} + \delta\rho\rho_{,j}u_j\right] d\Omega \\
&= \int_\Omega \left[[\delta\rho\rho^n u^n_{j,j} + \delta\rho\rho^n_{,j}u^n_j] + \delta\rho[\Delta\rho]u^n_{j,j} + \delta\rho\rho^n[\Delta u_{j,j}] + \delta\rho[\Delta\rho_{,j}]u^n_j + \delta\rho\rho^n_{,j}[\Delta u_j]\right] d\Omega \\
&= \int_\Omega [[\delta\rho\rho^n u^n_{j,j} + \delta\rho\rho^n_{,j}u^n_j] + \delta\rho[\rho^{n+1} - \rho^n]u^n_{j,j} + \delta\rho\rho^n[u^{n+1}_{j,j} - u^n_{j,j}] \\
&\qquad + \delta\rho[\rho^{n+1}_{,j} - \rho^n_{,j}]u^n_j + \delta\rho\rho^n_{,j}[u^{n+1}_j - u^n_j]]d\Omega \\
&= \int_\Omega [\delta\rho[\rho^n u^n_{j,j} + \rho^n_{,j}u^n_j] + \delta\rho[\rho^{n+1}u^n_{j,j} - \rho^n u^n_{j,j}] + \delta\rho[\rho^n u^{n+1}_{j,j} - \rho^n u^n_{j,j}] \\
&\qquad + \delta\rho[\rho^{n+1}_{,j}u^n_j - \rho^n_{,j}u^n_j] + \delta\rho[\rho^n_{,j}u^{n+1}_j - \rho^n_{,j}u^n_j]]d\Omega \\
&= \int_\Omega \left[\delta\rho[\rho^{n+1}u^n_{j,j} + \rho^n u^{n+1}_{j,j} - \rho^n u^n_{j,j} + \rho^{n+1}_{,j}u^n_j + \rho^n_{,j}u^{n+1}_j - \rho^n_{,j}u^n_j]\right] d\Omega
\end{aligned}$$

$$\int_\Omega \left[\frac{\partial \rho}{\partial t}\right] d\Omega = \int_\Omega \delta\rho \left[\frac{\partial \rho}{\partial t}\right] d\Omega$$
$$= \int_\Omega \delta\rho \left[\rho^{t_n+1} - \rho^{t_n} - (1-\alpha)\Delta t \dot{\rho}^{t_n}\right] d\Omega$$

The variational formulation is obtained as:

$$\begin{aligned}&\int_\Omega \delta\rho[\rho^{t_n+1} - \rho^{t_n} - (1-\alpha)\Delta t\dot{\rho}^{t_n}]d\Omega \\ &\quad + \int_\Omega [\delta\rho[\rho^{n+1}u^n_{j,j} + \rho^n u^{n+1}_{j,j} - \rho^n u^n_{j,j} + \rho^{n+1}_{,j}u^n_j + \rho^n_{,j}u^{n+1}_j - \rho^n_{,j}u^n_j]]d\Omega = 0\end{aligned} \tag{2.7}$$

Navier stokes equation

$$\int_\Omega \rho\delta u^T \left[\frac{\partial u}{\partial t} + (\nabla u)\, u\right] d\Omega - \int_\Omega \nabla.\delta u p d\Omega + \int_\Omega \left[D_c(\delta u)\right]^T C_c D_c d\Omega$$
$$= \int_\Omega \rho\delta u^T b d\Omega + \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma$$

$$\begin{aligned}\int_\Omega [\rho\delta u^T (\nabla u)\, u]d\Omega &= \int_\Omega \left[\rho\delta u_i^T u_{i,j} u_j\right] d\Omega \\ &= \rho^n\delta u_i u^n_{i,j}u^n_j + \Delta\rho\delta u_i u^n_{i,j}u^n_j + \rho^n\delta u_i \Delta u_{i,j}u^n_j + \rho^n\delta u_i u^n_{i,j}\Delta u_j \\ &= \rho^n\delta u_i u^n_{i,j}u^n_j + [\rho^{n+1} - \rho^n]\delta u_i u^n_{i,j}u^n_j + \rho^n\delta u_i[u^{n+1}_{i,j} - u^n_{i,j}]u^n_j + \rho^n\delta u_i u^n_{i,j}[u^{n+1}_j - u^n_j] \\ &= \rho\delta u_i u^n_{i,j}u^n_j + \rho^n\delta u_i[u_{i,j}u^n_j - 2u^n_{i,j}u^n_j + u^n_{i,j}u_j] \\ &= \delta u_i \rho u^n_{i,j}u^n_j + \delta u_i\rho^n u_{i,j}u^n_j - 2\delta u_i\rho^n u^n_{i,j}u^n_j + \delta u_i\rho^n u^n_{i,j}u_j\end{aligned}$$

$$-\int_\Omega \nabla.\delta u p d\Omega + \int_\Omega \left[D_c(\delta u)\right]^T C_c D_c d\Omega + \int_\Omega \rho\delta u^T b d\Omega + \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma$$
$$= -\delta u_{i,i}P + \delta D_{ij}C_{ijkl}D_{kl} + \int_\Omega \rho\delta u_i^T b_i d\Omega + \int_{\Gamma_t} \delta u_i^T \bar{t}_i d\Gamma$$

$$\begin{aligned}&\int_\Omega \rho\delta u^T[u^{tn+1} - u^{tn} - (1-\alpha)\Delta t\dot{u}^{tn}]d\Omega \\ &= \int_\Omega \left[\rho^{n+1}\delta u^T u^{tn+1} - \rho^{n+1}\delta u^T u^{tn} - \rho^{n+1}\delta u^T(1-\alpha)\Delta t\dot{u}^{tn}\right] d\Omega \\ &= \int_\Omega \left[\rho^n\delta u^T u^{tn} + \Delta\rho\delta u^T u^{tn} + \rho^n\delta u^T\Delta u - \rho^{n+1}\delta u^T u^{tn} - \rho^{n+1}\delta u^T(1-\alpha)\Delta t\dot{u}^{tn}\right] d\Omega \\ &= \int_\Omega [\rho^n\delta u^T u^{tn} + \left[\rho^{n+1} - \rho^n\right]\delta u^T u^{tn} + \rho^n\delta u^T\left[u^{n+1} - u^n\right] \\ &\qquad - \rho^{n+1}\delta u^T u^{tn} - \rho^{n+1}\delta u^T(1-\alpha)\Delta t\dot{u}^{tn}]d\Omega \\ &= \int_\Omega \left[\rho^{n+1}\delta u^T u^{tn} + \rho^n\delta u^T u^{n+1} - \rho^n\delta u^T u^{tn} - \rho^{n+1}\delta u^T u^{tn} - \rho^{n+1}\delta u^T(1-\alpha)\Delta t\dot{u}^{tn}\right] d\Omega\end{aligned}$$

The variational formulation is obtained as:

$$\begin{aligned}\int_\Omega [\rho^{n+1}\delta u^T u^{tn} + \rho^n \delta u^T u^{n+1} - \rho^n \delta u^T u^{tn} - \rho^{n+1}\delta u^T u^{tn} - \rho^{n+1}\delta u^T (1-\alpha)\Delta t \dot{u}^{tn}] d\Omega + \\ \delta u_i \rho u^n_{i,j} u^n_j + \delta u_i \rho^n u_{i,j} u^n_j - 2\delta u_i \rho^n u^n_{i,j} u^n_j + \delta u_i \rho^n u^n_{i,j} u_j - \delta u_{i,i} P + \\ \delta D_{ij} C_{ijkl} D_{kl} = \int_\Omega \rho \delta u^T_i b_i d\Omega + \int_{\Gamma_t} \delta u^T_i \bar{t}_i d\Gamma\end{aligned} \tag{2.8}$$

Energy equation

$$\begin{aligned}\int_\Omega \rho C_v \left[\frac{\partial \theta}{\partial t} + u.\nabla\theta\right] d\Omega = \int_\Omega \left[-p\nabla.u + \sigma : D - \nabla.q + \rho Q_h\right] d\Omega \\ \int_\Omega [\rho C_v \delta\theta \left(\frac{\partial \theta}{\partial t} + u.\nabla\theta\right) + \delta\theta \left(p\nabla.u - \sigma : D\right)] d\Omega + \int_\Omega k\nabla\delta\theta.\nabla\theta d\Omega \\ = \int_\Omega \rho\delta\theta Q_h d\Omega - \int_{\Gamma_q} \delta\theta \bar{q_n} d\Gamma\end{aligned}$$

$$\begin{aligned}\int_\Omega \rho C_v u.\nabla\theta d\Omega &= \int_\Omega \rho C_v \delta\theta u.\nabla\theta d\Omega \\ &= \int_\Omega \left[\delta\theta^T [C_v(\rho_n u^n_i \theta^n_{,i} + \rho_n u^n_i \Delta\theta_{,i} + \rho_n \Delta u_i \theta^n_{,i} + \Delta\rho u^n_i \theta^n_{,i})\right] d\Omega \\ &= \int_\Omega \delta\theta^T C_v \left[\rho_n u^n_i \theta^n_{,i} + \rho_n u^n_i (\theta^{n+1}_{,i} - \theta^n_{,i}) + \rho_n (u^{n+1}_i - u^n_i)\theta^n_{,i} + (\rho^{n+1} - \rho^n) u^n_i \theta^n_{,i}\right] d\Omega \\ &= \int_\Omega \delta\theta^T C_v \left[\rho_n u^n_i \theta_{,i^{n+1}} + \rho_n \theta^n_{,i} u^{n+1}_i + \rho^{n+1} u^n_i \theta^n_{,i} - 2\rho^n u^n_i \theta^n_{,i}\right] d\Omega\end{aligned}$$

$$\begin{aligned}\int_\Omega [p\nabla.u - \sigma : D] d\Omega &= \int_\Omega \delta\theta [p\nabla.u - \sigma : D] d\Omega \\ &= \int_\Omega \delta\theta [p u_{i,i} - \sigma ij : D_{ij}] d\Omega \\ &= \int_\Omega [\delta\theta P^n u^n_{i,i} + \delta\theta \Delta P u^n_{i,i} + \delta\theta P^n \Delta u_{i,i} - \delta\theta (D^T_{ij}(u^n) C_{ijkl} D_{kl}(u^n) \\ &\qquad + \Delta D^T_{ij} C_{ijkl} D_{kl}(u^n) + D^T_{ij}(u^n) C_{ijkl} \Delta D_{kl})] d\Omega \\ &= \int_\Omega [\delta\theta P^n u^n_{i,i} + \delta\theta (P^{n+1} - P^n) u^n_{i,i} + \delta\theta P^n (u^{n+1}_{i,i} - u^n_{i,i}) - \delta\theta (D^T_{ij}(u^n) C_{ijkl} D_{kl}(u^n) \\ &\qquad + (D^T_{ij}(u^{n+1}) - D^T_{ij}(u^n)) C_{ijkl} D_{kl}(u^n) + D^T_{ij}(u^n) C_{ijkl} (D_{kl}(u^{n+1}) - D_{kl}(u^n)))] d\Omega \\ &= \int_\Omega [\delta\theta P^{n+1} u^n_{i,i} + \delta\theta P^n u^{n+1}_{i,i} - \delta\theta P^n u^n_{i,i} - \delta\theta (D^T_{ij}(u^{n+1}) C_{ijkl} D_{kl}(u^n) \\ &\qquad + D^T_{ij}(u^n) C_{ijkl} D_{kl}(u^{n+1}) - D^T_{ij}(u^n) C_{ijkl} D_{kl}(u^n))] d\Omega \\ &= \int_\Omega [\delta\theta P^{n+1} u^n_{i,i} + \delta\theta P^n u^{n+1}_{i,i} - \delta\theta P^n u^n_{i,i} - \delta\theta (2D^T_{ij}(u^{n+1}) C_{ijkl} D_{kl}(u^n) \\ &\qquad - D^T_{ij}(u^n) C_{ijkl} D_{kl}(u^n))] d\Omega\end{aligned}$$

$$\int_\Omega k\nabla\delta\theta.\nabla\theta d\Omega + \int_\Omega \rho\delta\theta Q_h d\Omega - \int_{\Gamma_q} \delta\theta\bar{q_n} d\Gamma = \int_\Omega k\delta\theta_{,i}.\theta^{n+1}_{,i} d\Omega + \int_\Omega \rho^{n+1}\delta\theta Q_h d\Omega - \int_{\Gamma_q} \delta\theta\bar{q_n} d\Gamma$$

$$\begin{aligned}
&\int_\Omega \rho C_v \delta\theta^T[\theta^{tn+1} - \theta^{tn} - (1-\alpha)\Delta t\dot{\theta}^{tn}]d\Omega \\
&\qquad = \int_\Omega C_v \left[\rho^{n+1}\delta\theta^T\theta^{tn+1} - \rho^{n+1}\delta\theta^T\theta^{tn} - \rho^{n+1}\delta\theta^T(1-\alpha)\Delta t\dot{\theta}^{tn}\right] d\Omega \\
&\qquad = \int_\Omega C_v \left[\rho^{n}\delta\theta^T\theta^{tn} + \Delta\rho\delta\theta^T\theta^{tn} + \rho^{n}\delta\theta^T\Delta\theta - \rho^{n+1}\delta\theta^T\theta^{tn} - \rho^{n+1}\delta\theta^T(1-\alpha)\Delta t\dot{\theta}^{tn}\right] d\Omega \\
&\qquad = \int_\Omega C_v [\rho^{n}\delta\theta^T\theta^{tn} + \left[\rho^{n+1} - \rho^{n}\right]\delta\theta^T\theta^{tn} + \rho^{n}\delta\theta^T\left[\theta^{n+1} - \theta^{n}\right] \\
&\qquad\qquad - \rho^{n+1}\delta\theta^T\theta^{tn} - \rho^{n+1}\delta\theta^T(1-\alpha)\Delta t\dot{\theta}^{tn}]d\Omega \\
&\qquad = \int_\Omega C_v \left[\rho^{n+1}\delta\theta^T\theta^{tn} + \rho^{n}\delta\theta^T\theta^{n+1} - \rho^{n}\delta\theta^T\theta^{tn} - \rho^{n+1}\delta\theta^T\theta^{tn} - \rho^{n+1}\delta\theta^T(1-\alpha)\Delta t\dot{\theta}^{tn}\right] d\Omega
\end{aligned}$$

The variational formulation is obtained as:

$$\begin{aligned}
&\int_\Omega C_v[\rho^{n+1}\delta\theta^T\theta^{tn} + \rho^{n}\delta\theta^T\theta^{n+1} - \rho^{n}\delta\theta^T\theta^{tn} - \rho^{n+1}\delta\theta^T\theta^{tn} - \rho^{n+1}\delta\theta^T(1-\alpha)\Delta t\dot{\theta}^{tn}]d\Omega + \\
&\int_\Omega \delta\theta^T C_v \left[\rho_n u^n_i \theta_{,i^{n+1}} + \rho_n\theta^n_{,i}u^{n+1}_i + \rho^{n+1}u^n_i\theta^n_{,i} - 2\rho^n u^n_i\theta^n_{,i}\right] d\Omega + \int_\Omega k\delta\theta_{,i}.\theta^{n+1}_{,i} d\Omega + \\
&\int_\Omega [\delta\theta P^{n+1}u^n_{i,i} + \delta\theta P^n u^{n+1}_{i,i} - \delta\theta P^n u^n_{i,i} - \delta\theta\left(2D^T_{ij}(u^{n+1})C_{ijkl}D_{kl}(u^n) - D^T_{ij}(u^n)C_{ijkl}D_{kl}(u^n)\right)]d\Omega \\
&\qquad\qquad = \int_\Omega \rho^{n+1}\delta\theta Q_h d\Omega - \int_{\Gamma_q} \delta\theta\bar{q_n} d\Gamma
\end{aligned} \tag{2.9}$$

State equation

$$\int_\Omega \delta P\left[P - \tilde{P}\right] d\Omega = 0$$

The variational formulation is obtained as:

$$\begin{aligned}
\int_\Omega \delta P\left[P - \tilde{P}\right] d\Omega &= \int_\Omega \delta P\left[P - \tilde{P}_n - \tilde{P}_\rho\Delta\rho - \tilde{P}_\theta\Delta\theta\right] d\Omega \\
&= \int_\Omega \delta P\left[P - \tilde{P}_n - \tilde{P}_\rho(\rho^{n+1} - \rho^n) - \tilde{P}_\theta(\theta^{n+1} - \theta^n)\right] d\Omega
\end{aligned} \tag{2.10}$$

The above linearized variational statements are subject to finite element formulation. The fluid domain is discretized with quadrilateral, isoparametric finite elements as shown in Fig. 2.1 where in the velocity is interpolated quadratically within each element, while all other field variables i.e, density, temperature and pressure are interpolated linearly.

Finite element formulation

Let the velocity, density, temperature and pressure fields, and their variations be interpolated as

$$\mathbf{u} = N\hat{\mathbf{u}}, \quad \rho = N_\rho \hat{\rho}, \quad \theta = N_\theta \hat{\theta}, \quad p = N_p \hat{p}, \tag{2.11}$$

N are the standard shape functions of a 9-noded quadratic element, while N_ρ, N_θ and N_p are the standard shape function for a 4-noded linear element. Using the above interpolations, we have

$$\boldsymbol{D}_c\left(\mathbf{u}_c\right) = \mathbf{B}\hat{\mathbf{u}}, \tag{2.12}$$

$$\left(\boldsymbol{\nabla}\mathbf{u}\right)\mathbf{u}^k = \mathbf{R}\mathbf{B}_{NL}\hat{\mathbf{u}}, \tag{2.13}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \mathbf{B}_p\hat{\mathbf{u}}, \tag{2.14}$$

$$\rho^k\left(\boldsymbol{\nabla}\cdot\mathbf{u}\right) + \left(\boldsymbol{\nabla}\rho^k\right)\cdot\mathbf{u} = \mathbf{B}_{\rho 1}\hat{\mathbf{u}}, \tag{2.15}$$

$$\left(\boldsymbol{\nabla}\cdot\mathbf{u}^k\right)\rho + \mathbf{u}^k\cdot\left(\boldsymbol{\nabla}\rho\right) = \mathbf{B}_{\rho 2}\hat{\rho}, \tag{2.16}$$

$$\boldsymbol{\nabla}\theta = \mathbf{B}_\theta\hat{\theta}, \tag{2.17}$$

where,

$$\mathbf{B} = \begin{bmatrix} N_{1,x} & 0 & N_{2,x} & 0 & . & . & . & N_{9,x} & 0 \\ 0 & N_{1,y} & 0 & N_{2,y} & . & . & . & 0 & N_{9,y} \\ N_{1,y} & N_{1,x} & N_{2,y} & N_{2,x} & . & . & . & N_{9,y} & N_{9,x} \end{bmatrix},$$

$$\mathbf{B}_p = \begin{bmatrix} N_{1,x} & N_{1,y} & N_{2,x} & N_{2,y} & . & . & . & N_{9,x} & N_{9,y} \end{bmatrix},$$

$$\mathbf{R} = \begin{bmatrix} u_x^k & u_y^k & 0 & 0 \\ 0 & 0 & u_x^k & u_y^k \end{bmatrix},$$

$$\mathbf{B}_{NL} = \begin{bmatrix} N_{1,x} & 0 & N_{2,x} & 0 & . & . & . & N_{9,x} & 0 \\ N_{1,y} & 0 & N_{2,y} & 0 & . & . & . & N_{9,y} & 0 \\ 0 & N_{1,x} & 0 & N_{2,x} & . & . & . & 0 & N_{9,x} \\ 0 & N_{1,y} & 0 & N_{2,y} & . & . & . & 0 & N_{9,y} \end{bmatrix},$$

$$\mathbf{B}_{\rho 1} = \begin{bmatrix} \rho^k N_{1,x} + \frac{\partial \rho^k}{\partial x} N_1 & \rho^k N_{1,y} + \frac{\partial \rho^k}{\partial y} N_1 & . & . & . & \rho^k N_{9,x} + \frac{\partial \rho^k}{\partial x} N_9 & \rho^k N_{9,y} + \frac{\partial \rho^k}{\partial y} N_9 \end{bmatrix},$$

$$\mathbf{B}_{\rho 2} = \begin{bmatrix} \left(\boldsymbol{\nabla}\cdot\mathbf{u}^k\right) N_1 + \left(\mathbf{u}^k\cdot\boldsymbol{\nabla}\right) N_1 & \left(\boldsymbol{\nabla}\cdot\mathbf{u}^k\right) N_2 + \left(\mathbf{u}^k\cdot\boldsymbol{\nabla}\right) N_2 & . & . & . & \left(\boldsymbol{\nabla}\cdot\mathbf{u}^k\right) N_9 + \left(\mathbf{u}^k\cdot\boldsymbol{\nabla}\right) N_9 \end{bmatrix},$$

$$\mathbf{B}_\theta = \begin{bmatrix} N_{1,x} & N_{2,x} & . & . & . & N_{9,x} \\ N_{1,y} & N_{2,y} & . & . & . & N_{9,y} \end{bmatrix},$$

Finally we obtain the finite element formulation for the fluid element using the above mentioned shape functions.

Continuity Equation

$$\int_\Omega \delta\rho^T \left[\rho^{t_{n}+1} - \rho^{t_n} - (1-\alpha)\Delta t\dot{\rho}^{t_n}\right] d\Omega + \alpha\Delta t \int_\Omega [\delta\rho^T[\rho^{n+1}u^n_{j,j} + \rho^n u^{n+1}_{j,j}$$
$$- \rho^n u^n_{j,j} + \rho^{n+1}_{,j} u^n_j + \rho^n_{,j} u^{n+1}_j - \rho^n_{,j} u^n_j]]d\Omega$$
$$= \int_\Omega \delta\rho^T[N^T_\rho N_\rho \rho - N^T_\rho \rho^{tn}]d\Omega + \alpha\Delta t \int_\Omega [(N^T_\rho \rho^n_{,j} N_u + N^T_\rho \rho^n B_{\rho u})u$$
$$+ (N^T_p u^n_{j,j} N_\rho + N^T_\rho u^n_j B_\rho)\rho - N^T_\rho(\rho^n u^n_{j,j} + u^n_j \rho^n_{,j})]d\Omega$$
$$= \delta\rho^T \int_\Omega [N^T_\rho N_\rho \rho - N^T_\rho \rho^{tn}]d\Omega + \alpha\Delta t \int_\Omega [N^T_\rho B_{\rho 1} u + N^T_\rho B_{\rho 2}\rho - N^T_\rho(\rho^n u^n_{j,j} + u^n_j \rho^n_{,j})]d\Omega$$

$$\delta\rho^T[M_{\rho\rho}\rho - g_\rho + \alpha\Delta t(K_{\rho u}u + K_{\rho\rho}\rho - f_\rho)] = 0$$

Navier Stokes Equation

$$\int_\Omega \left[\rho^{n+1}\delta u^T u^{tn} + \rho^n \delta u^T u^{n+1} - \rho^n \delta u^T u^{tn} - \rho^{n+1}\delta u^T u^{tn} - \rho^{n+1}\delta u^T(1-\alpha)\Delta t\dot{u}^{tn}\right] d\Omega$$
$$+ \alpha\Delta t \int_\Omega [\delta u_i \rho u^n_{i,j} u^n_j + \delta u_i \rho^n u_{i,j} u^n_j - 2\delta u_i \rho^n u^n_{i,j} u^n_j + \delta u_i \rho^n u^n_{i,j} u_j - \delta u_{i,i} P + \delta D_{ij} C_{ijkl} D_{kl}]d\Omega$$
$$= \alpha\Delta t[\int_\Omega \rho\delta u^T_i b_i d\Omega + \int_{\Gamma_t} \delta u^T_i \bar{t}_i d\Gamma]$$
$$\int_\Omega [\rho^n N^T N u + N^T(u^n - u^{tn} - (1-\alpha)\Delta t\dot{u}^{tn})N_p - \rho^n u^{tn} N^T]d\Omega + \alpha\Delta t \int_\Omega \delta u^T [(B^T_u C_c B_u + N^T \rho^n R B_{NL}$$
$$+ N^T_u \rho^n u^n_{i,j} N)u + N^T u^n_{i,j} u^n_j N_\rho \rho - B^T_p N_p P - 2N^T \rho^n u^n_{i,j} u^n_j]d\Omega + \alpha\Delta t \int_{\Gamma_t} N^T \bar{t} d\Gamma = 0$$

$$\delta u^T[M_{uu}u + M_{u\rho} - g_u + \alpha\Delta t(K_{uu}u + K_{u\rho}\rho + K_{up}P - f_u)] = 0$$

Energy Equation

$$\int_\Omega C_v[\rho^{n+1}\delta\theta^T\theta^{tn} + \rho^n\delta\theta^T\theta^{n+1} - \rho^n\delta\theta^T\theta^{tn} - \rho^{n+1}\delta\theta^T\theta^{tn} - \rho^{n+1}\delta\theta^T(1-\alpha)\Delta t\dot{\theta}^{tn}]d\Omega +$$
$$\int_\Omega \delta\theta^T C_v \left[\rho_n u^n_i \theta_{,i^{n+1}} + \rho_n \theta^n_{,i} u^{n+1}_i + \rho^{n+1} u^n_i \theta^n_{,i} - 2\rho^n u^n_i \theta^n_{,i}\right] d\Omega + \int_\Omega k\delta\theta_{,i}.\theta^{n+1}_{,i} d\Omega +$$
$$\int_\Omega [\delta\theta P^{n+1} u^n_{i,i} + \delta\theta P^n u^{n+1}_{i,i} - \delta\theta P^n u^n_{i,i} - \delta\theta\left(2D^T_{ij}(u^{n+1})C_{ijkl}D_{kl}(u^n) - D^T_{ij}(u^n)C_{ijkl}D_{kl}(u^n)\right)]d\Omega$$
$$= \int_\Omega \rho^{n+1}\delta\theta Q_h d\Omega - \int_{\Gamma_q} \delta\theta \bar{q_n} d\Gamma$$

$$\int_\Omega [C_v N_\theta^T(\theta^n - \theta^{tn} - (1-\alpha)\Delta t\dot{\theta}^{tn})N_\rho + C_v\rho^n N_\theta^T N_\theta - C_v\rho^n\theta^n N_\theta^T]d\Omega + \int_\Omega \delta\theta^T[(C_v N_\theta^T\rho^n\theta_{,i}^n N$$
$$+ N_\theta^T P^n B_p - 2N_\theta^T D(u)^T C_c B_c)u + (C_v N_\theta^T\theta_{,i}^n u_i^n N_\rho)\rho + N_\theta^T Q_n N_\rho rho$$
$$+ (C_v\rho^n N_{theta}^T u_i^n B_{theta} + B_\theta^T kB_\theta)\theta + (N_\theta^T u_{i,i}^n N_p)P + 2N_\theta^T\rho^n u_i^n\theta_{,i}^n + N_\theta^T P^n u_{i,i}^n$$
$$- D(u)^T C_c D(u)N_\theta^T]d\Omega - \int_{\Gamma_q} \bar{q_n} N_\theta^T d\Gamma = 0$$

$$\delta\theta^T[M_{\theta\rho}u + M_{\theta\theta} - g_\theta + \alpha\Delta t(K_{\theta u}u + K_{\theta\rho}\rho + K_{\theta\theta}\theta + K_{\theta P} - f_\theta)] = 0$$

State Equation

$$\int_\Omega \delta P[P - \tilde{P}_n - \tilde{P}_\rho(\rho^{n+1} - \rho^n) - \tilde{P}_\theta(\theta^{n+1} - \theta^n)]d\Omega$$
$$= \delta P^T[-\tilde{P}_\rho N_p^T N_\rho\rho - \tilde{P}_\theta N_p^T N_\theta\theta + N_p^T N_p P - (\tilde{P}_n - \tilde{P}_\rho\rho^n - \tilde{P}_\theta\theta^n)]$$

$$\delta P^T[K_{p\rho}P + K_{p\theta} + K_{pp} - f_p] = 0$$

From the above FE formulation , we obtain the general matrix form as written below

$$(\mathbf{M} + \alpha\Delta t\mathbf{K})\,\hat{\mathbf{x}} = \alpha\Delta t\mathbf{f} + \mathbf{g}, \tag{2.18}$$

where,

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{\mathbf{u}} \\ \hat{\rho} \\ \hat{\theta} \\ \hat{p} \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{uu} & \mathbf{M}_{u\rho} & 0 & 0 \\ 0 & \mathbf{M}_{\rho\rho} & 0 & 0 \\ 0 & \mathbf{M}_{\theta\rho} & \mathbf{M}_{\theta\theta} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\rho} & 0 & \mathbf{K}_{up} \\ \mathbf{K}_{\rho u} & \mathbf{K}_{\rho\rho} & 0 & 0 \\ \mathbf{K}_{\theta u} & \mathbf{K}_{\theta\rho} & \mathbf{K}_{\theta\theta} & \mathbf{K}_{\theta p} \\ \mathbf{K}_{pu} & \mathbf{K}_{p\rho} & \mathbf{K}_{p\theta} & 0 \end{bmatrix},$$

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_u \\ f_\rho \\ f_\theta \\ f_p \end{bmatrix},$$

$$\mathbf{g} = \begin{bmatrix} \mathbf{g}_u \\ g_\rho \\ g_\theta \\ 0 \end{bmatrix},$$

with

$$\mathbf{M}_{uu} = \int_\Omega \rho^k N^T N d\Omega,$$

$$\mathbf{M}_{u\rho} = \int_\Omega N^T \left[\mathbf{u}^n - \mathbf{u}^{t^n} - (1-\alpha)\,\Delta t \dot{\mathbf{u}}^{t^n} \right] N_\rho d\Omega,$$

$$\mathbf{M}_{\rho\rho} = \int_\Omega N_\rho^T N_p d\Omega,$$

$$\mathbf{M}_{\theta\rho} = \int_\Omega C_v \left[\theta^n - \theta^{t^n} - (1-\alpha)\,\Delta t \dot{\theta}^{t^n} \right] N_\theta N_\rho d\Omega,$$

$$\mathbf{M}_{\theta\theta} = \int_\Omega \rho^n C_v N_\theta^T N_\rho d\Omega,$$

$$\mathbf{K}_{uu} = \int_\Omega \rho^n N^T \left[(\boldsymbol{\nabla} \mathbf{u}^n)\, N + \mathbf{R}\mathbf{B}_{NL} \right] d\Omega + \int_\Omega \mathbf{B}^T \boldsymbol{C}_c \mathbf{B} d\Omega,$$

$$\mathbf{K}_{u\rho} = \int_\Omega N^T \left[(\boldsymbol{\nabla} \mathbf{u}^n)\, \mathbf{u}^n - \mathbf{b} \right] N_\rho d\Omega,$$

$$\mathbf{K}_{up} = -\int_\Omega \mathbf{B}_p^T N_p d\Omega,$$

$$\mathbf{K}_{\rho u} = \int_\Omega N_\rho^T \mathbf{B}_{\rho 1} d\Omega,$$

$$\mathbf{K}_{\rho\rho} = \int_\Omega N_\rho^T \mathbf{B}_{\rho 2} d\Omega,$$

$$\mathbf{K}_{\theta u} = \int_\Omega N_\theta^T \left[\rho^n C_v \left(\boldsymbol{\nabla} \theta^n \right)^T N + p^n \mathbf{B}_p - 2 \boldsymbol{D}_c^T \left(\mathbf{u}^n \right) \boldsymbol{C}_c \mathbf{B} \right] d\Omega,$$

$$\mathbf{K}_{\theta\rho} = \int_\Omega \left[C_v \mathbf{u}^n \cdot (\boldsymbol{\nabla} \theta^n) - Q_h \right] N_\theta^T N_\rho d\Omega,$$

$$\mathbf{K}_{\theta\theta} = \int_\Omega \left[\rho^n C_v N_\theta^T \left(\mathbf{u}^n \right)^T \mathbf{B}_\theta + k \mathbf{B}_\theta^T \mathbf{B}_\theta \right] d\Omega,$$

$$\mathbf{K}_{\theta p} = \int_\Omega \left(\boldsymbol{\nabla} \cdot \mathbf{u}^n \right) N_\theta^T N_p d\Omega,$$

$$\mathbf{K}_{p\rho} = \int_\Omega \frac{\partial \tilde{p}}{\partial \rho}\Big|_k N_p^T N_\rho d\Omega,$$

$$\mathbf{K}_{p\theta} = \int_\Omega \frac{\partial \tilde{p}}{\partial \theta}\Big|_k N_p^T N_\theta d\Omega,$$

$$\mathbf{K}_{pp} = \int_\Omega N_p^T N_p d\Omega,$$

$$\mathbf{f}_u = \int_{\Gamma_t} N^T \bar{t} d\Gamma - \int_{\Omega} \left[2\rho^n N^T \left(\boldsymbol{\nabla} \mathbf{u}^n \right) \mathbf{u}^n \right] d\Omega,$$

$$f_\rho = - \int_{\Omega} N_\rho^T \left[\rho^n \left(\boldsymbol{\nabla} \cdot \mathbf{u}^n \right) + \mathbf{u}^n \cdot \left(\boldsymbol{\nabla} \rho^n \right) \right] d\Omega,$$

$$f_\theta = - \int_{\Gamma_q} N_\theta^T \bar{q_n} d\Gamma + \int_{\Omega} \left[2\rho^n \mathbf{u}^n \cdot \left(\boldsymbol{\nabla} \theta^n \right) - \boldsymbol{D}_c^T \left(\mathbf{u}^n \right) \boldsymbol{C}_c \boldsymbol{D}_c \left(\mathbf{u}^n \right) \right] N_\theta^T d\Omega + \int_{\Omega} \mathbf{p}^n \boldsymbol{\nabla} \theta^n N_\theta^T d\Omega,$$

$$f_p = \int_{\Omega} \left[p^n - \tilde{p} \left(\rho^n, \theta^n \right) \right] N_p^T d\Omega,$$

$$\mathbf{g}_u = \int_{\Omega} \rho^n N^T \left[\mathbf{u}^{t^n} \right] d\Omega,$$

$$g_\rho = \int_{\Omega} \left[\rho^{t^n} \right] N_\rho^T d\Omega,$$

$$g_\theta = \int_{\Omega} \rho^n C_v \left[\theta^{t^n} \right] N_\theta^T d\Omega.$$

For obtaining steady state solutions, the transient terms are omitted and final set of equations takes the form

$$\mathbf{K}\hat{\mathbf{x}} = \mathbf{f}.$$

2.3 Fluid-Structure Interaction

The Eulerian finite element fluid model is coupled with the Lagrangian beam element formulation. The interaction between solid and fluid element is modelled using the fictitious domain method [2]. Coupling is achieved by using the constraint that fluid and solid nodes have equal velocity at the interface. So no slip boundary condition has to be imposed at the interface.This coupling is provided using the method of Lagrange multipliers [10].

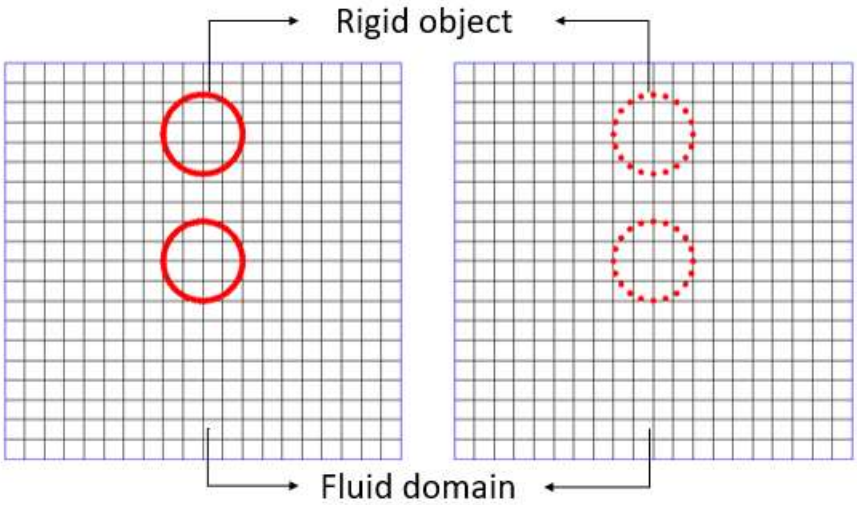

Figure 2.1: shows (Left) a non moving rigid solid body present within a fluid domain and (Right) its descritisation into a finite number of points.

$$u_i = u_i^s,$$

$$\sum_{I=1}^{9} N_I U_{Ii} = u_i^s.$$

where N_I is the I^{th} shape function, U_{Ii} is the fluid velocity at the node I in the i^{th} direction and u_i^s is the velocity of a point representing a solid in the i^{th} direction. Figure shows an example in which a rigid body is present within a fluid domain. This rigid object is descritized into a finite number of points and a condition is applied such that at each node on the interface the velocity of the fluid node and solid node are equal.

Combining the fluid and solid equations of motion we obtain the final set of equations [2] in matrix form as:

$$\begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\rho} & \mathbf{K}_{u\theta} & \mathbf{K}_{up} & 0 & \phi \\ \mathbf{K}_{\rho u} & \mathbf{K}_{\rho\rho} & 0 & 0 & 0 & 0 \\ \mathbf{K}_{\theta u} & \mathbf{K}_{\theta\rho} & \mathbf{K}_{\theta\theta} & \mathbf{K}_{\theta p} & 0 & 0 \\ \mathbf{K}_{pu} & \mathbf{K}_{p\rho} & \mathbf{K}_{p\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{K}_s & \mathbf{-A}^T \\ \phi^T & 0 & 0 & 0 & \mathbf{-A} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \rho \\ \theta \\ \mathbf{P} \\ \mathbf{u^s} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f_u} \\ \mathbf{f}_\rho \\ \mathbf{f}_\theta \\ \mathbf{f_p} \\ \mathbf{F_s} \\ 0 \end{bmatrix}$$

Chapter 3

Numerical Examples

Several benchmark tests were done to verify the performance of compressible fluid flow solver [11]. The equation of state used for all these benchmark tests was taken from Reference [9] depending upon the fluid in consideration. We can classify the different terms in the Euler's equations as steady, transient, linear and non linear. The accuracy of the Finite element implementation of these terms are tested using different benchmark problems.
In this approach we validate each term either using the previous results or the analytical solutions and then validate the complete coupled equations. Stokes lid driven cavity(A) problem was solved to validate the steady part and Navier stokes lid driven cavity(B) problem for non linear part of momentum equation. Convection diffusion equation(C) was solved to validate the energy equation. Plane couette flow(D) and Oscillatory couette flow(E) was solved which includes unsteady terms. Sod's shock tube(F) problem which involves unsteady as well as non linear terms were solved and compared with analytical results. The fully coupled equations were validated by solving the Compressible couette flow (G) problem. FSI analysis was done for Lid driven cavity with two rigid objects(H) and Longitudinal flow past a rigid circular object(I) where we verified that no slip boundary condition is perfectly imposed between the rigid object and fluid domain.

The coupled FSI solver was used to solve the problem involving heat transfer due to conduction and convection caused by a rigid vertical oscillating beam.

TEST CASE	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**
STEADY ANALYSIS	✓	✓	✓	-	-	-	✓	✓	-
TRANSCIENT ANALYSIS	-	-	-	✓	✓	✓	-	-	✓
NON LINEAR ANALYSIS	-	✓	-	-	-	✓	-	✓	✓
FSI ANALYSIS	-	-	-	-	-	-	-	✓	✓

Table 3.1: Test cases solved to verify compressible fluid flow solver

3.1 Heat transfer by conduction and convection caused by a rigid oscillating beam

The problem involves a rigid vertical beam oscillating at a certain frequency (or Time period) inside square cavity filled with static compressible fluid.The cavity is a square of unit size ($1x1mm^2$) and the vertical beam is of length 5mm with its bottom point positioned at cordinate (5,0). The fluid domain is descritised into 1600 elements and the structure member is descritised into 20 elements. The geometry and boundary conditions are as shown in the figure. All four boundaries are assumed to be of zero velocity. Top and bottom boundary of the fluid domain were also provided with temperature boundary conditions as 1^0C and 0^0C respectively.

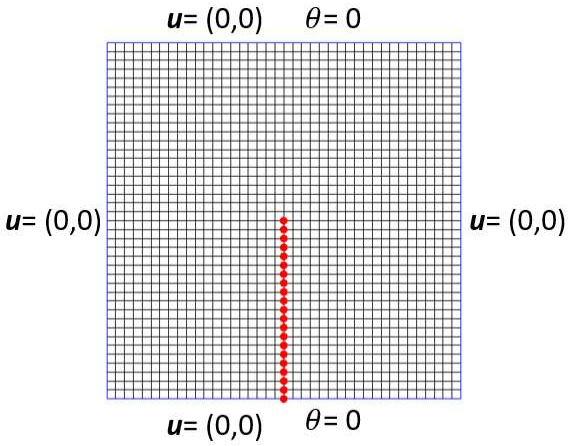

Figure 3.1: Schematic representation of the problem of flow caused by a rigid oscillating beam with fluid boundary conditions.

The velocity boundary conditions for the vertical beam is as follows:

$$V_x = -2 * n * \frac{\pi}{TP} * (\frac{i-1}{nnod-1}) * length * sin(\frac{\pi}{8}) * sin(4 * (w - \frac{\pi}{8}))$$
$$V_y = 2 * n * \frac{\pi}{TP} * (\frac{i-1}{nnod-1}) * length * (1 - cos(\frac{\pi}{8})) * sin(8 * (w - \frac{\pi}{8}))$$

where,

t (Time)	w	n
0 - $\frac{TP}{4}$	$\frac{\pi t}{2TP}$	-1
$\frac{TP}{4} - \frac{3TP}{4}$	$-\frac{\pi(TP-4t)}{8TP}$	1
$\frac{3TP}{4} - TP$	$\frac{\pi(TP-t)}{2TP}$	-1

The results obtained from the simulation are discussed below. The simulation was run for different values of Peclet numbers and Reynolds numbers. We investigated the variation of heat flux with change in Peclet number and Reynolds number.

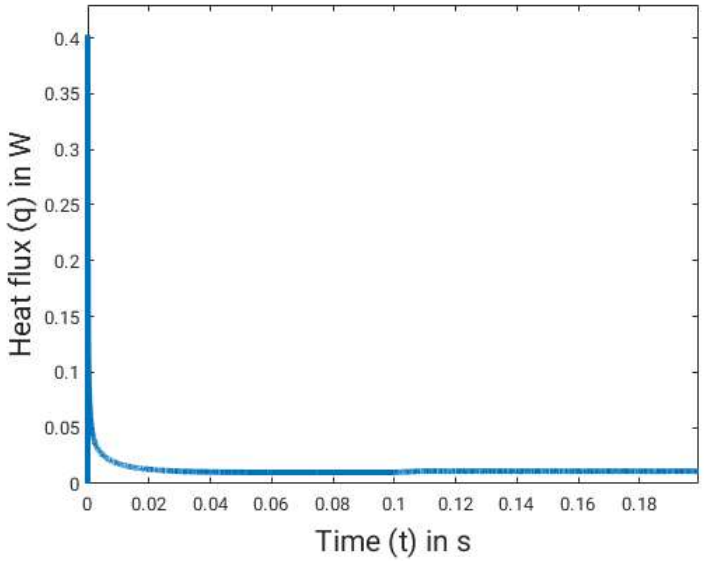

Figure 3.2: shows the variation of heat flux (q) with time (t)

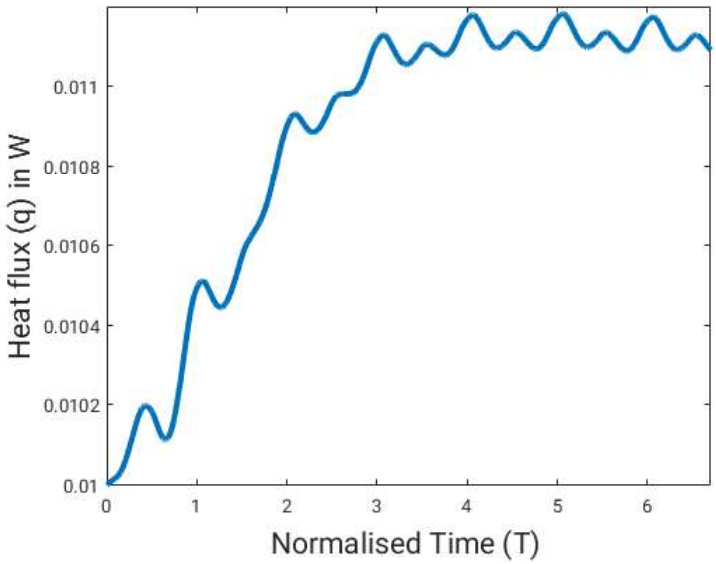

Figure 3.3: shows the variation of heat flux (q) with normalised time (T) (enlarged view when steady state is acheived)

Figure 3.1 shows the variation of heat flux with normalised time, where normalised time is the ratio between time step and time period of oscillation. Fig:3.2 shows a enlarged view of the time period when the heat flux achieves steady state. The contour plots presented below shows the change in temperature at different time steps. Fig: 3.3 corresponds to the case when heat has not reached steady state and fig: 3.4 corresponds to the case when heat flux has achieved steady state.

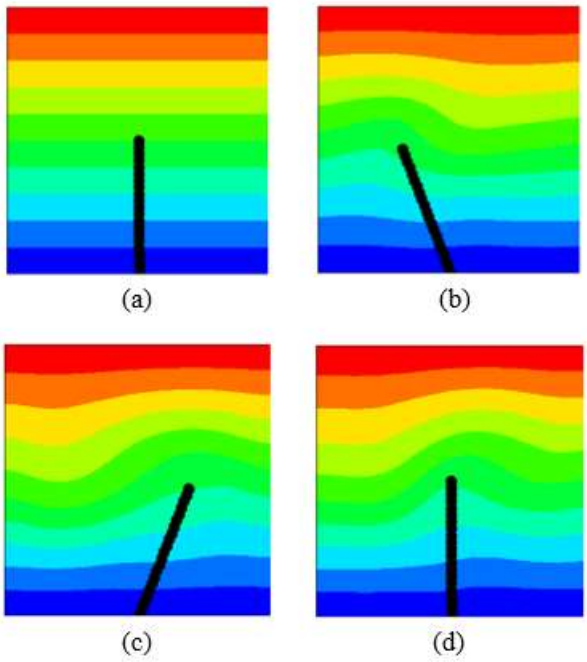

Figure 3.4: shows temperature contour behaviour Re= 8.33 and Pe= 8.33 at (a)T = 0 ,(b)T = 0.3 ,(c)T = 0.7 and (d)T = 1 (when heat flux is unsteady)

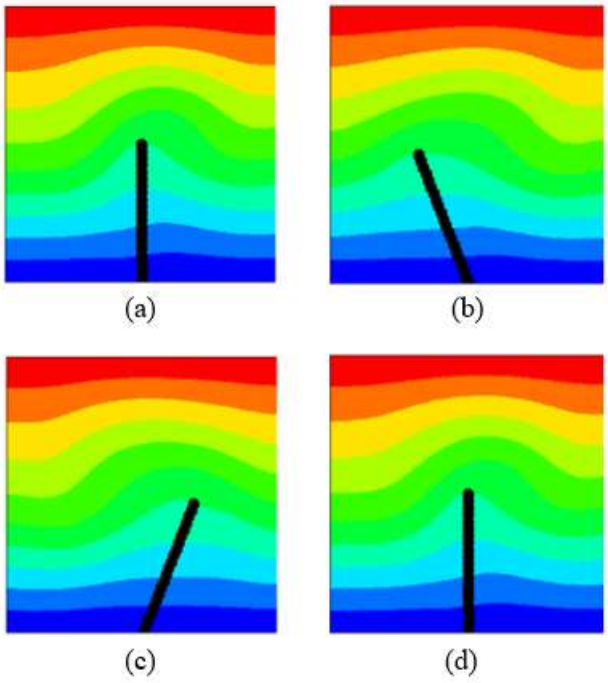

Figure 3.5: shows temperature contour behaviour Re= 8.33 and Pe= 8.33 at (a)T = 0 ,(b)T = 0.3 ,(c)T = 0.7 and (d)T = 1 and (d)time step = (after heat flux reached steady state)

Figure 3.5 shows the variation of heat flux ratio $\bar{q}$ with normalised time T, where $\bar{q}$ is the ratio of heat flux when beam is oscillating to the heat flux when beam is at rest; and T is the ratio between time step and time period of oscillation.

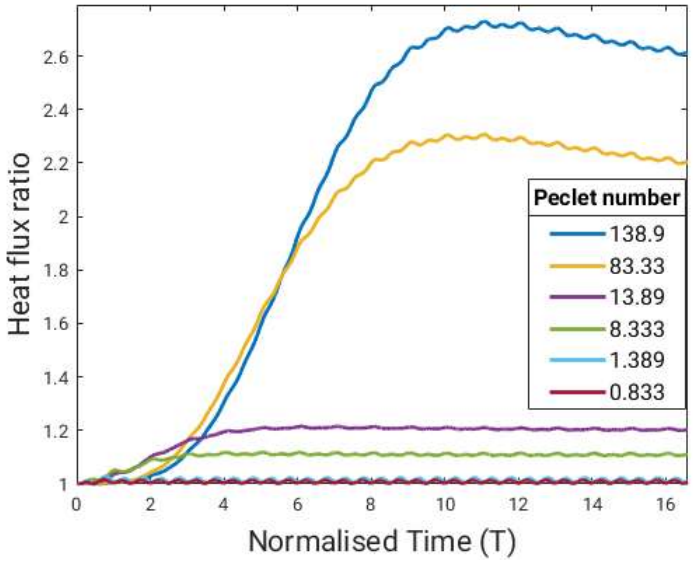

Figure 3.6: shows the variation of heat flux ratio with normalised time T at different values of Peclet numbers (Re =8.333).

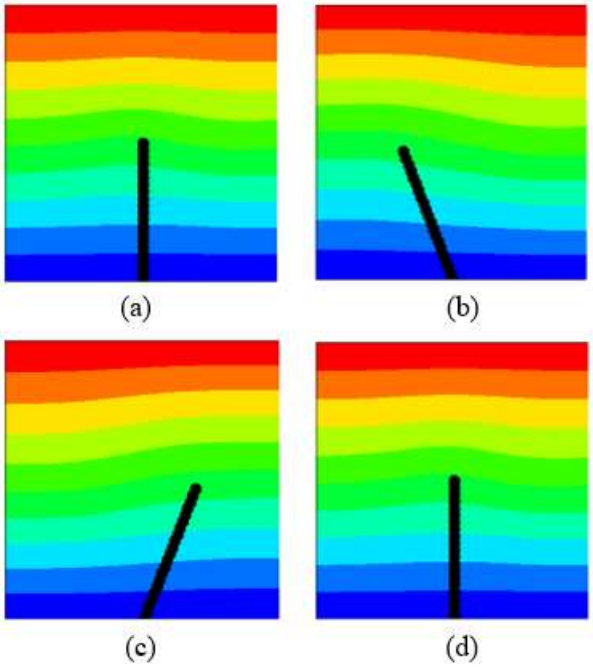

Figure 3.7: shows temperature contour behaviour for Re = 8.33 and Pe = 0.833 at (a) T = 5, (b) T = 5.3, (c) T = 5.7 and (d) T = 6.

We observe that with the increase in Peclet number the ratio of heat flux is also increasing. This behaviour can be explained with the help of the temperature contour plots provided below. We compared the temperature contour plots of the cases where Peclet number is very high (Pe= 138.9) and very low (Pe=0.833) and observed that the temperature diffuses more in the case of higher Peclet number leading to higher heat flux.

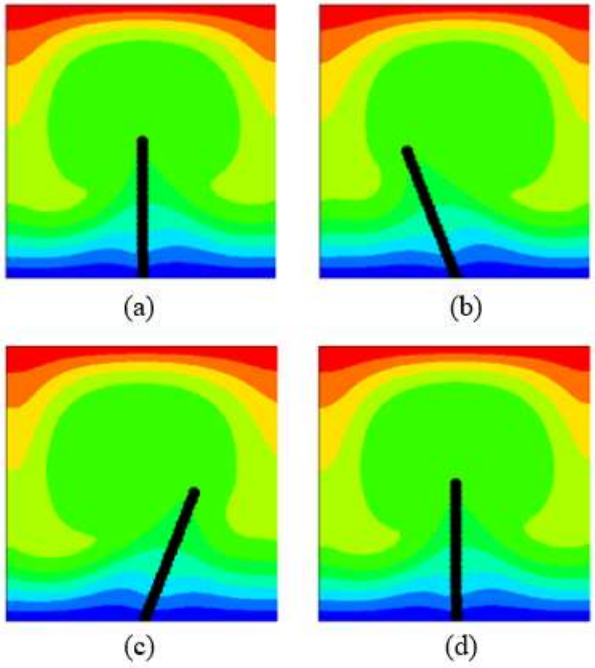

Figure 3.8: shows temperature contour behaviour for Re = 8.33 and Pe = 138.9 at (a) T = 67, (b) T = 67.3, (c) T = 67.7 and (d) T = 68.

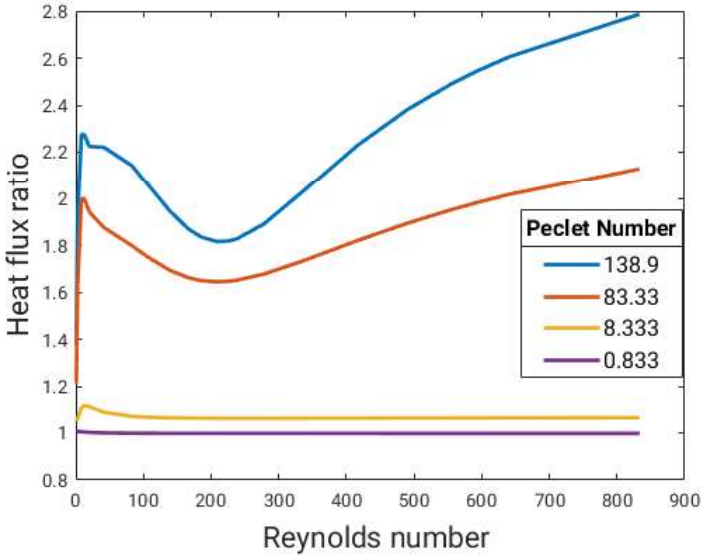

Figure 3.9: shows the variation of heat flux ratio with Reynolds number at different values of Peclet numbers.

We observe that in the beginning the normalised heat flux increases with increase in Reynold's number. The reason for this behaviour is that the heat transfer due to convection is increased due to the decrease in viscosity. Further the curve tends to decrease and take a 'U' turn and increase with further increase in Reynold's number. The reason for this behaviour is under investigation which need to be found out.

Chapter 4

Conclusion

We first solved some benchmark problems to validate the performance of the Finite element model for fluids. Then we solved problems involving compressible fluid flow coupled with embedded rigid bodies using fictitious domain method. Next the fluid dynamics equations were coupled with structural dynamics equations and the Fluid-Structure interaction problem was solved using monolithic method. A problem involving fluid flow caused by a vertical oscillating beam was solved using the present finite element model. Next step is to extent this problem where a number of vertical oscillating beams are arranged in parallel.

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Appendix A

A.1 Kinematics

A.1.1 Lagrangian and Eulerian description

Consider a arbitrary shaped fluid element occupying a region in space V_0, which is capable of undergoing motion on application of some force so as to occupy a different region in space $V(t)$, as shown in Fig. A.1. The intial configuration V_0 serves as the reference configuration with respect to which the deformations in the current configuration $V(t)$ are measured. The reference configuration can be taken to be any random configuration, and as a matter of convenience, the time is set at t=0 at this configuartion. As shown in Fig. A.1, a particle with a position vector $\boldsymbol{X}$ in the reference configuration V_0 takes the position $\boldsymbol{x}$ in the current configuration $V(t)$. The points $\boldsymbol{X} \in V_0$ will often be called as material points.
The relation between $\boldsymbol{X}$ and $\boldsymbol{x}$ is expressed as

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t) \tag{A.1}$$

The coordinates $\boldsymbol{X}$ and $\boldsymbol{x}$ are known as the material and spatial coordinates, respectively. The mapping $\boldsymbol{\chi}$ taking $\boldsymbol{X}$ to $\boldsymbol{x}$ is assumed to be one-to-one and orientation-preserving. Due to the one-to-one nature of the mapping $\boldsymbol{\chi}$, we can invert it to obtain $\boldsymbol{X}$ as a function of $\boldsymbol{x}$ and t, i.e.,

$$\boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t)$$

The formulation based on the material coordinates is called as the Lagrangian formulation, while the formulation based on spatial coordinates is called Eulerian formulation. Physically, in Lagrangian formulation we tag each individual fluid particle and observe the change in flow properties of those particle with respect to time whereas in case of Eulerian formulation we emphasise on fixed region in space where we focus on change in flow properties at each point in space of fixed region with repect to time.

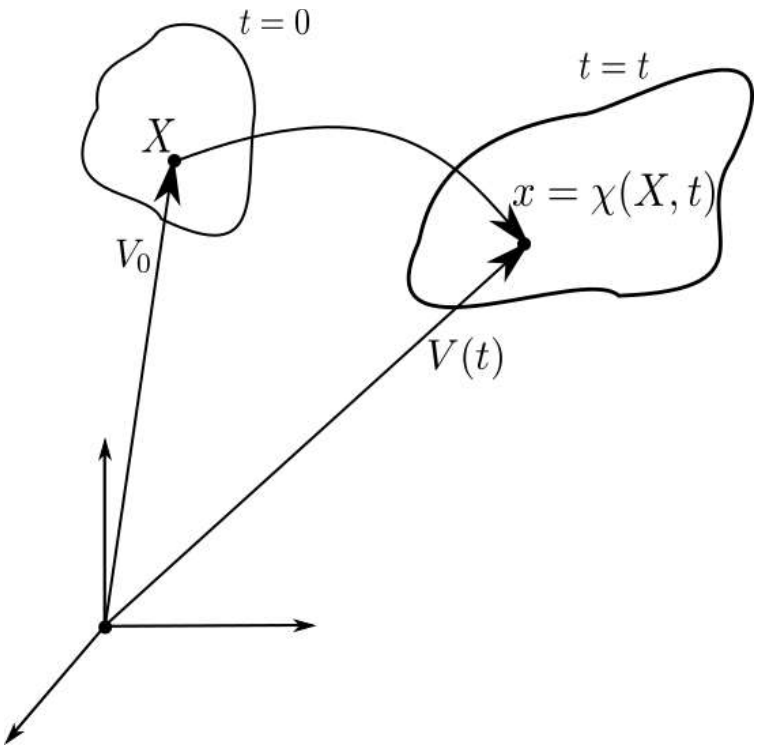

Figure A.1: Reference and current configuration.

At each point in the domain, we define the deformation gradient $\boldsymbol{F}$ by

$$F_{ij} = \frac{\partial \chi_i}{\partial X_j} = \begin{bmatrix} \frac{\partial \chi_1}{\partial X_1} & \frac{\partial \chi_1}{\partial X_2} & \frac{\partial \chi_1}{\partial X_3} \\ \frac{\partial \chi_2}{\partial X_1} & \frac{\partial \chi_2}{\partial X_2} & \frac{\partial \chi_2}{\partial X_3} \\ \frac{\partial \chi_3}{\partial X_1} & \frac{\partial \chi_3}{\partial X_2} & \frac{\partial \chi_3}{\partial X_3} \end{bmatrix}$$

The relationship between the length element in the deformed configuration $\boldsymbol{dx}$ and the length element in the reference configuration $\boldsymbol{dX}$ at given instant of time is given by

$$\boldsymbol{dx} = \boldsymbol{F}\boldsymbol{dX}; \quad dx_i = F_{ij} dX_j = \frac{\partial \chi_i}{\partial X_j} dX_j; \quad J = det\boldsymbol{F} \tag{A.2}$$

J represents, locally, the volume after deformation per unit original volume. By any deformation process, we can not make the material body to vanish so we have $J \neq 0$. In order to avoid the change in orientation of reference configuration, we need to have

$$J > 0 \quad \forall \boldsymbol{X} \in V_0 \quad t \in [0, \infty) \tag{A.3}$$

Given a material field $\phi(\boldsymbol{X}, t)$, the particle or material derivative of that field, denoted by $D\phi/Dt$, is defined as the partial derivative of ϕ with respect to time, i.e.,

$$\frac{D\phi}{Dt} = \left(\frac{\partial \phi}{\partial t}\right)_{\boldsymbol{X}}$$

In the Lagrangian approach, the velocity and acceleration are defined as the first and second material

derivatives, respectively, of the mapping $\boldsymbol{\chi}$. i.e.,

$$\tilde{\boldsymbol{u}}\left(\boldsymbol{X},t\right)=\frac{D\boldsymbol{\chi}}{Dt}=\left(\frac{\partial\chi}{\partial t}\right)_{\boldsymbol{X}},$$
$$\tilde{\boldsymbol{a}}\left(\boldsymbol{X},t\right)=\frac{D\tilde{\boldsymbol{u}}}{Dt}=\left(\frac{\partial\tilde{u}}{\partial t}\right)_{\boldsymbol{X}}=\left(\frac{\partial^2\chi}{\partial t^2}\right)_{\boldsymbol{X}}$$

However, in fluid mechanics the problem with finding the velocities and accelerations from the above formulae is that, in general, at any given time t, we do not know the reference position $\boldsymbol{X}$ occupied by a particle now at $\boldsymbol{x}$. In such a case, the computation can be carried out using Eulerian approach, in which the field quantities are now expressed as functions of the spatial position $\boldsymbol{x}$ and time. Since $\boldsymbol{X}=\boldsymbol{\chi}^{-1}\left(\boldsymbol{x},t\right)$, the Eulerian description of the velocity is given by

$$\boldsymbol{u}\left(\boldsymbol{x},t\right)=\tilde{\boldsymbol{u}}\left(\boldsymbol{X},t\right)=\tilde{\boldsymbol{u}}\left(\boldsymbol{\chi}^{-1}\left(\boldsymbol{x},t\right),t\right)$$

The expression for the acceleration using spatial description (i.e., Eulerian approach) of the velocity $\boldsymbol{u}\left(\boldsymbol{x},t\right)$, is given by

$$\boldsymbol{a}\left(\boldsymbol{x},t\right)=\left(\frac{\partial\boldsymbol{u}}{\partial t}\right)_{\boldsymbol{x}}+\left(\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}\right)\boldsymbol{u}$$

The above expression for the acceleration shows that the acceleration at a fixed point depends on the material velocity that changes with time i.e., $\left(\partial\boldsymbol{u}n\partial t\right)_{\boldsymbol{x}}$ and the material point (with a specific velocity) carried past the fixed point in space i.e., $\left(\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{u}\right)\boldsymbol{u}$. The argument used to determine the acceleration is rather general and can be used to compute the material derivative of any spatial field, be it scalar, vector or tensor-valued. For example, if $\phi\left(\boldsymbol{x},t\right)$ is a scalar-valued field, then its material derivative is given by

$$\frac{D\phi}{Dt}=\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x_i}\frac{\partial\chi_i}{\partial t}=\frac{\partial\phi}{\partial t}+\boldsymbol{u}\cdot\left(\boldsymbol{\nabla}\phi\right)\tag{A.4}$$

A.2 Governing equations of fluid.

The following transport theorem is used for deriving the governing equation of fluid.

Transport theorem 1

Let $f(\boldsymbol{x},t)$ be a continuous and differentiable scalar-valued function, and let $V(t)$ be the material volume (i.e., a volume comprising of definite set of particles and moving with the medium) and $S(t)$ be the boundary of the material volume $V(t)$. Then $\int_{V(t)}$ is a function of time alone and its derivative is given by

$$\frac{d}{dt}\int_{V(t)}f(\boldsymbol{x},t)dV=\int_{V(t)}\frac{\partial f}{\partial t}dV+\int_{S(t)}f(\boldsymbol{u}\cdot n)dS\tag{A.5}$$

$$\text{Let}\quad f(\boldsymbol{x},t)=f(\chi(\boldsymbol{x},t),t)=\tilde{f}(\boldsymbol{X},t)$$

then the volume integral in the current configuration $V(t)$ and the reference configuration at $t=0$, V_0 are related by

$$\int_{V(t)}f(\boldsymbol{x},t)dV=\int_{V_0}\tilde{f}JdV_0\qquad(\because J=dV\backslash dV_0)$$

taking the time derivative on both sides, we have

$$\frac{d}{dt}\int_{V(t)} f(\boldsymbol{x},t)dV = \frac{d}{dt}\int_{V_0} \tilde{f}JdV_0$$

using Leibniz integral rule, we have

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} f(\boldsymbol{x},t)dV &= \int_{V_0}\left(\frac{\partial(\tilde{f}J)}{\partial t}\right)_{\boldsymbol{X}} dV_0 \\ &= \int_{V_0}\left[J\left(\frac{\partial\tilde{f}}{\partial t}\right)_{\boldsymbol{X}} + \tilde{f}\left(\frac{\partial J}{\partial t}\right)_{\boldsymbol{X}}\right] \\ &= \int_{V_0}\left[J\frac{D\tilde{f}}{Dt} + \frac{\tilde{f}}{J}\frac{DJ}{Dt}\right]JdV_0\end{aligned}$$

changing the volume integral V_0 back to $V(t)$, we get

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} f(\boldsymbol{x},t)dV &= \int_{V_t}\left[\frac{Df}{Dt} + f\left(\boldsymbol{\nabla}\cdot\boldsymbol{u}\right)\right]dV && \text{(by Eqn. A.4 and ??)} \\ &= \int_{V_t}\left[\frac{\partial f}{\partial t} + \boldsymbol{u}\cdot(\boldsymbol{\nabla}f) + f(\boldsymbol{\nabla}\cdot\boldsymbol{u})\right]dV \\ &= \int_{V_t}\left[\frac{\partial f}{\partial t} + \boldsymbol{\nabla}\cdot(f\boldsymbol{u})\right]dV && (\boldsymbol{\nabla}\cdot(f\boldsymbol{u}) = \boldsymbol{u}\cdot(\boldsymbol{\nabla}f) + f(\boldsymbol{\nabla}\cdot\boldsymbol{u})) \\ &= \int_{V_t}\frac{\partial f}{\partial t}dV + \int_{V_t}\boldsymbol{\nabla}\cdot(f\boldsymbol{u})\,dV \\ &= \int_{V_t}\frac{\partial f}{\partial t}dV + \int_{S(t)} f\left(\boldsymbol{u}\cdot n\right)dS && \left(\int_{V_t}\boldsymbol{\nabla}\cdot(f\boldsymbol{u})dV = \int_{S(t)} f(\boldsymbol{u}\cdot n)dS\right)\end{aligned}$$

If $\boldsymbol{g}\left(\boldsymbol{x},t\right)$ is a vector-valued function, then the above equation is applies to each component of $\boldsymbol{g}\left(x,t\right)$ to get

$$\frac{d}{dt}\int_{V(t)}\boldsymbol{g}\left(\boldsymbol{x},t\right)dV = \int_{V(t)}\frac{\partial\boldsymbol{g}}{\partial t}dV + \int_{S(t)} f(\boldsymbol{g}\cdot n)dS$$

Transport theorem 2

Let $f(\boldsymbol{x},t)$ and $\boldsymbol{g}(\boldsymbol{x},t)$ be scalar and vector fields defined on $V(t)$. Then,

$$\frac{d}{dt}\int_{V(t)}\rho f\left(\boldsymbol{x},t\right)dV = \int_{V(t)}\rho\frac{Df}{Dt}dV = \int_{V(t)}\rho\left[\frac{\partial f}{\partial t} + \boldsymbol{u}\cdot(\boldsymbol{\nabla}f)\right]dV, \tag{A.6}$$

$$\frac{d}{dt}\int_{V(t)}\rho\boldsymbol{g}\left(\boldsymbol{x},t\right)dV = \int_{V(t)}\rho\frac{D\boldsymbol{g}}{Dt}dV = \int_{V(t)}\rho\left[\frac{\partial\boldsymbol{g}}{\partial t} + \left(\boldsymbol{\nabla}\boldsymbol{g}\right)\boldsymbol{u}\right]dV \tag{A.7}$$

By letting ρf play the role of f in transport theorem 1, we get

$$\begin{aligned}\frac{d}{dt}\int_{V(t)}\rho f\left(\boldsymbol{x},t\right)dV &= \int_{V(t)}\left[\frac{\partial\left(\rho f\right)}{\partial t}+\boldsymbol{\nabla}\cdot\left(\rho f\boldsymbol{u}\right)\right]dV\\ &= \int_{V(t)}\left[\rho\frac{\partial f}{\partial t}+f\left(\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{u}\right)\right)+\left(\rho\boldsymbol{u}\right)\cdot\left(\boldsymbol{\nabla}f\right)\right]dV\end{aligned}$$

we use conservation of mass i.e., $\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{u}\right)=0$ which will be discussed later (see section A.3.1) to get

$$\begin{aligned}&= \int_{V(t)}\rho\left[\frac{\partial f}{\partial t}+\boldsymbol{u}\cdot\left(\boldsymbol{\nabla}f\right)\right]dV\\ &= \int_{V(t)}\rho\frac{Df}{Dt}dV \qquad \text{(by Eqn. A.4)}\end{aligned}$$

A.2.1 Conservation of mass

Consider a specific mass of medium enclosed in a material volume $V(t)$, which is arbitrarily chosen. By the statement of conservation of mass, we know that as time progresses the $V(t)$ changes in shape and size, but the mass contained in $V(t)$ remains constant i.e., the mass in V_0 and $V(t)$ is the same except in the case of $V(t)$ with mass sources inside it. Thus, the mass m contained within $V(t)$ using conservation of mass is given by

$$\begin{aligned}m=\int_{V_0}\rho_0\left(\boldsymbol{X}\right)dV_0 &= \int_{V(t)}\rho\left(\boldsymbol{x},t\right)dV\\ \int_{V(t)}\frac{dm}{dt} &= \frac{d}{dt}\int_{V(t)}\rho dV=0\end{aligned} \tag{A.8}$$

Applying the transport theorem 1 with ρ as the scalar valued function, we have

$$\int_{V(t)}\frac{\partial\rho}{\partial t}dV+\int_{S(t)}\rho(\boldsymbol{u}\cdot n)dS=0 \qquad \text{(By Eqn. A.5)}$$

$$\int_{V_t}\left[\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{u}\right)\right]dV=0 \qquad \left(\int_{V(t)}\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{u}\right)dV=\int_{S(t)}\rho(\boldsymbol{u}\cdot n)dS\right)$$

we know that the $V(t)$ is arbitrary, so we obtain

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{u}\right)=0 \tag{A.9}$$

A.2.2 Balance of linear momentum

Balance of linear momentum states that the rate of change of momentum of material volume is equal to the net external force acting on the material volume. The external forces acting on material volume can be classified as body force such as gravitational force etc. and surface force such as pressure or viscous forces that arises due to the surface contact between the bulk of continua. Let $\boldsymbol{b}(\boldsymbol{x},t)$ be the body force per unit mass of the material volume, then the body force acting on the material volume is given by $\int_{V(t)}\rho\boldsymbol{b}dV$.

If $\boldsymbol{n}$ be the outward normal vector acting on the bounding surface $S(t)$ of the arbitrary portion of

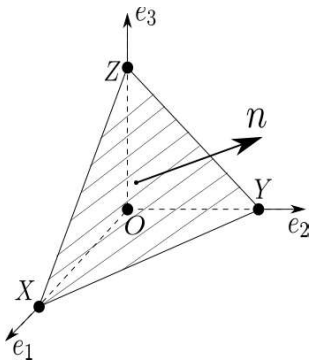

Figure A.2: Schematic representation of tetrahedron considered.

material volume $V(t)$, then by Cauchy hypothesis there exist a traction vector $\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$ due to the action of material outside the $S(t)$. The net surface force acting on the material volume is given by $\int_{S(t)} \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) dS$. Let m be the mass contained within the material volume $V(t)$, then by the statement of balance of linear momentum we get

$$\begin{aligned} \int_{V(t)} \frac{d\,(m\boldsymbol{u})}{dt} &= \int_{S(t)} \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) dS + \int_{V(t)} \rho \boldsymbol{b} dV \\ \frac{d}{dt} \int_{V(t)} \rho \boldsymbol{u} dV &= \int_{S(t)} \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) dS + \int_{V(t)} \rho \boldsymbol{b} dV \qquad \left(\because m = \int_{V(t)} \rho dV \right) \end{aligned} \tag{A.10}$$

*Using transport theorem 2 to left hand side of Eqn. A.10, we have Thus, D_{11} is

$$\begin{aligned} \int_{V(t)} \rho \frac{D\boldsymbol{u}}{Dt} dV &= \int_{S(t)} \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) dS + \int_{V(t)} \rho \boldsymbol{b} dV \\ \int_{V(t)} \rho \left[\frac{D\boldsymbol{u}}{Dt} - \boldsymbol{b} \right] dV &= \int_{S(t)} \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) dS \end{aligned} \tag{A.11}$$

Applying the above equation to the tetrahedron considered, we have

$$\rho \left[\frac{Df}{Dt} - \boldsymbol{b} \right] \frac{hdA}{3} = \left[\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) + \boldsymbol{t}(\boldsymbol{x}, t, -e_1) n_1 + \boldsymbol{t}(\boldsymbol{x}, t, -e_2) n_2 + \boldsymbol{t}(\boldsymbol{x}, t, -e_3) n_3 \right] dA$$

We now shrink the tetrahedron to the point O as h tends to zero by keeping normal to the face $\triangle XYZ$ to be same as $\boldsymbol{n}$. This turns the left hand side of above equation to zero giving

$$\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) + \boldsymbol{t}(\boldsymbol{x}, t, -e_1) n_1 + \boldsymbol{t}(\boldsymbol{x}, t, -e_2) n_2 + \boldsymbol{t}(\boldsymbol{x}, t, -e_3) n_3 = 0 \tag{A.12}$$

Let $\boldsymbol{n} = e_1 = (1, 0, 0)$ then the above equation reduces to

$$\boldsymbol{t}(\boldsymbol{x}, t, e_1) = -\boldsymbol{t}(\boldsymbol{x}, t, -e_1) \tag{A.13}$$

Similarly by taking $\boldsymbol{n} = e_2$ and $\boldsymbol{n} = e_3$, we get

$$\boldsymbol{t}(\boldsymbol{x}, t, e_2) = -\boldsymbol{t}(\boldsymbol{x}, t, -e_2) \tag{A.14}$$

$$\boldsymbol{t}(\boldsymbol{x}, t, e_3) = -\boldsymbol{t}(\boldsymbol{x}, t, -e_3) \tag{A.15}$$

By substituting Eqn. (A.13, A.14, A.15) in Eqn. (A.12), we have

$$\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n}) = \boldsymbol{t}(\boldsymbol{x},t,e_1)n_1 + \boldsymbol{t}(\boldsymbol{x},t,e_2)n_2 + \boldsymbol{t}(\boldsymbol{x},t,e_3)n_3$$

Let us know define the cauchy stress tensor $\boldsymbol{\tau}$ such that

$$\boldsymbol{\tau}(\boldsymbol{x},t)\boldsymbol{n} = \boldsymbol{t}(\boldsymbol{x},t,e_1)n_1 + \boldsymbol{t}(\boldsymbol{x},t,e_2)n_2 + \boldsymbol{t}(\boldsymbol{x},t,e_3)n_3 \tag{A.16}$$

Using Eqn. (A.4, A.16) in Eqn. (A.11) and rewriting Eqn. (A.11), we have

$$\int_{V(t)} \rho \left[\frac{d\boldsymbol{u}}{dt} + \boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{u})\right] dV = \int_{S(t)} \boldsymbol{\tau}\boldsymbol{n} dS + \int_{V(t)} \rho\boldsymbol{b} dV$$

$$\int_{V(t)} \rho \left[\frac{d\boldsymbol{u}}{dt} + \boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{u}) - \boldsymbol{\nabla}\cdot\boldsymbol{\tau} - \rho\boldsymbol{b}\right] dV = 0 \quad \left(\int_{S(t)} \boldsymbol{\tau}\boldsymbol{n} dS = \int_{V(t)} \boldsymbol{\nabla}\cdot\boldsymbol{\tau} dV\right)$$

Since the volume $V(t)$ is arbitrary, the integrand of above equation must be equal to zero, therefore

$$\begin{aligned}
\rho\left[\frac{d\boldsymbol{u}}{dt} + \boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{u})\right] - \boldsymbol{\nabla}\cdot\boldsymbol{\tau} - \rho\boldsymbol{b} &= 0 \\
\rho\left[\frac{d\boldsymbol{u}}{dt} + \boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{u})\right] &= \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \rho\boldsymbol{b} \\
\rho\frac{D\boldsymbol{u}}{Dt} &= \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \rho\boldsymbol{b}
\end{aligned} \tag{A.17}$$

Constitutive relation for a Newtonian fluid

In this section, we establish relation between Cauchy stress tensor $\boldsymbol{\tau}$ and the fluid properties of Newtonian fluid (such as water, air etc.). The Cauchy stress tensor can be decomposed as hydrostatic stress and deviatoric stress as shown below

$$\boldsymbol{\tau} = \sigma_m \boldsymbol{I} + \boldsymbol{\sigma}$$

In indicial notation, we have

$$\tau_{ij} = \sigma_m \delta_{ij} + \sigma_{ij} \tag{A.18}$$

The Cauchy stress tensor $\boldsymbol{\tau}$ for a Newtonian fluid needs to satisfy the following condition

1. When the fluid is at rest, only hydrostatic stress acts on it which is given by the thermodynamic pressure p.

$$\begin{aligned}
\sigma_m &= -p && \text{(-ve sign} \rightarrow \text{compressive stress)} \\
\tau_{ij} &= -p\delta_{ij} + \sigma_{ij} && \text{(by Eqn. A.18)}
\end{aligned} \tag{A.19}$$

2. The stress tensor $\boldsymbol{\tau}$ is directly proportional to velocity gradient $\boldsymbol{L}$.
   Since the $\sigma_m = -p$, it is independent of deformation tensor so the only part of τ_{ij} that is directly proportional to velocity is σ_{ij}.

σ_{ij} has 9 components each of which must be directly proportional to the 9 components of L_{kl}, so we need 81 proportionality constants which is given by 4th order tensor β_{ijkl}

$$\begin{aligned}\sigma_{ij} &= \beta_{ijkl} L_{kl} \\ \sigma_{ij} &= \beta_{ijkl} \left(D_{kl} + W_{kl}\right) \qquad \text{(by Eqn. ??)}\end{aligned} \tag{A.20}$$

3. The shear stress is zero for the solid body rotation of fluid.
   The deviatoric stress of τ_{ij} i.e., σ_{ij} denotes the shear stress of fluid. For solid body rotation, there will be no contribution to the rate of deformation i.e. $D_{kl} = 0$. So the only part of velocity gradient L_{kl} contributing to solid body rotation is vorticity tensor W_{kl}. By using the above statement, we get

$$\begin{aligned}\beta_{ijkl} \left(D_{kl} + W_{kl}\right) &= 0 \quad \text{(by Eqn. A.20)} \\ \beta_{ijkl} W_{kl} &= 0 \\ W_{kl} &= 0 \qquad (\because \beta_{ijkl} \neq 0)\end{aligned} \tag{A.21}$$

   Using Eqn.(A.21) in Eqn.(A.20), we have

$$\begin{aligned}\sigma_{ij} &= \beta_{ijkl} D_{kl} \\ \sigma_{ij} &= \beta_{ijkl} \left[\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)\right] \qquad \left(\because D_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)\right)\end{aligned} \tag{A.22}$$

4. Condition of isotropy for fluid properties.
   This statement describes that the fluid properties such as viscosity should be independent of the orientation of coordinate system chosen. In order to do so, we choose a general 4th order tensor shown below

$$\beta_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \gamma \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)$$

   Due to the symmetry nature of τ_{ij}, the contribution from γ term will be zero, therefore we have

$$\beta_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{A.23}$$

Using Eqn.(A.23) in Eqn.(A.22), we have

$$\begin{aligned}\sigma_{ij} &= \left[\lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right] \times \left[\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)\right] \\ &= \lambda \delta_{ij} \left[\frac{1}{2}\left(\frac{\partial u_k}{\partial x_k} + \frac{\partial u_k}{\partial x_k}\right)\right] + \frac{\mu}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \\ \sigma_{ij} &= \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\end{aligned} \tag{A.24}$$

Using Eqn.(A.24) in Eqn.(A.19), we have

$$\tau_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$
$$\tau_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + 2\mu D_{ij} \qquad \left(\because D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)$$

In matrix form, we have

$$\boldsymbol{\tau} = -p\boldsymbol{I} + \lambda(tr\boldsymbol{D})\boldsymbol{I} + 2\mu\boldsymbol{D} \tag{A.25}$$

where λ and μ are scalar constants found by experiments

A.2.3 Energy equation

In order to derive the total energy equation, we must first derive the mechanical energy balance equation which is solely based on the momentum equation (Eqn. A.17), so mechanical energy balance equation is independent of thermodynamic properties.
Taking dot product on both sides of Eqn.(A.17) with $\boldsymbol{u}$, we have

$$\rho\boldsymbol{u}\cdot\frac{D\boldsymbol{u}}{Dt} = \boldsymbol{u}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \rho\boldsymbol{u}\cdot\boldsymbol{b}$$

which can be rewritten as

$$\begin{aligned}
\rho\frac{D}{Dt}\left(\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right) &= \boldsymbol{u}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\tau}) + \rho\boldsymbol{u}\cdot\boldsymbol{b} \\
&= \boldsymbol{\nabla}\cdot\left(\boldsymbol{\tau}^T\boldsymbol{u}\right) - \boldsymbol{\tau} : \boldsymbol{\nabla}\boldsymbol{u} + \rho\boldsymbol{u}\cdot\boldsymbol{b} \qquad \left(\boldsymbol{\nabla}\cdot\left(\boldsymbol{T}^T\boldsymbol{u}\right) = \boldsymbol{T} : \boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{u}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{T})\right) \\
&= \boldsymbol{\nabla}\cdot\left(\boldsymbol{\tau}^T\boldsymbol{u}\right) - \boldsymbol{\tau} : \boldsymbol{L} + \rho\boldsymbol{u}\cdot\boldsymbol{b} \\
&= \boldsymbol{\nabla}\cdot\left(\boldsymbol{\tau}^T\boldsymbol{u}\right) - \boldsymbol{\tau} : (\boldsymbol{D} + \boldsymbol{W}) + \rho\boldsymbol{u}\cdot\boldsymbol{b}
\end{aligned} \tag{A.26}$$

If $\boldsymbol{A}$ and $\boldsymbol{B}$ is symmetric and anti-symmetric tensor, then it follows that

$$\boldsymbol{A} : \boldsymbol{B} = 0$$

Since $\boldsymbol{\tau}$ is symmetry tensor whereas $\boldsymbol{W}$ is anti-symmetry tensor, so using the above statement we have $\boldsymbol{\tau} : \boldsymbol{W} = 0$. This reduces the Eqn.(A.26) to the below form

$$\rho\frac{D}{Dt}\left(\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right) = \boldsymbol{\nabla}\cdot\left(\boldsymbol{\tau}^T\boldsymbol{u}\right) - \boldsymbol{\tau} : \boldsymbol{D} + \rho\boldsymbol{u}\cdot\boldsymbol{b} \tag{A.27}$$

Integrating the above equation with respect to material volume $V(t)$, we have

$$\int_{V(t)}\rho\frac{D}{Dt}\left(\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right)dV = \int_{V(t)}\boldsymbol{\nabla}\cdot\left(\boldsymbol{\tau}^T\boldsymbol{u}\right)dV + \int_{V(t)}\left[-\boldsymbol{\tau} : \boldsymbol{D} + \rho\boldsymbol{u}\cdot\boldsymbol{b}\right]dV \tag{A.28}$$

Applying transport theorem 2 to left-hand side and divergence theorem to the first term of the right-hand side, we have

$$\begin{aligned}\int_{V(t)} \rho \frac{D}{Dt}\left(\frac{\boldsymbol{u}\cdot\boldsymbol{u}}{2}\right) dV &= \int_{S(t)} \boldsymbol{u}\cdot(\boldsymbol{\tau}\boldsymbol{n})\, dS + \int_{V(t)} \left[-\boldsymbol{\tau} : \boldsymbol{D} + \rho \boldsymbol{u}\cdot\boldsymbol{b}\right] dV \\ &= \int_{S(t)} \boldsymbol{t}\cdot\boldsymbol{u} dS + \int_{V(t)} \left[-\boldsymbol{\tau} : \boldsymbol{D} + \rho \boldsymbol{u}\cdot\boldsymbol{b}\right] dV \quad \text{(Cauchy stress relation} \rightarrow \boldsymbol{t} = \boldsymbol{\tau}\boldsymbol{n})\end{aligned}$$

Using transport theorem 2 to left-hand side of the above expression, we can write

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} \frac{1}{2}\rho\,(\boldsymbol{u}\cdot\boldsymbol{u})\, dV &= \int_{S(t)} \boldsymbol{t}\cdot\boldsymbol{u} dS + \int_{V(t)} \left[-\boldsymbol{\tau} : \boldsymbol{D} + \rho \boldsymbol{u}\cdot\boldsymbol{b}\right] dV \\ \frac{d\boldsymbol{k}}{dt} &= \int_{S(t)} \boldsymbol{t}\cdot\boldsymbol{u} dS + \int_{V(t)} \left[-\boldsymbol{\tau} : \boldsymbol{D} + \rho \boldsymbol{u}\cdot\boldsymbol{b}\right] dV\end{aligned}$$

where the kinetic energy $\boldsymbol{k}$ is defined by $\boldsymbol{k} = \int_{V(t)} \frac{1}{2}\rho\,(\boldsymbol{u}\cdot\boldsymbol{u})\, dV$
We know define the first law of thermodynamics which forms the basis for the total energy equation. Mathematically, we can define the first law of thermodynamics as follows

$$\frac{dU}{dt} = \frac{dW}{dt} + \frac{dQ}{dt} \tag{A.29}$$

where U, W and Q denotes total energy of the system, work done on the system and heat supplied to the system respectively. The time derivative represents the rate of change of the respective thermodynamic quantities.
Internal energy of the system is the summation of kinetic energy ($\boldsymbol{k}$) and internal energy per unit mass (e) of the system. So we have

$$\begin{aligned}U &= \int_{V(t)} \rho\left[\frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{u}) + e\right] dV \\ \frac{dU}{dt} &= \frac{d}{dt}\int_{V(t)} \rho\left[\frac{1}{2}\,(\boldsymbol{u}\cdot\boldsymbol{u}) + e\right] dV\end{aligned} \tag{A.30}$$

The external forces acting on the system are surface traction $\boldsymbol{t}$ and body force $\boldsymbol{b}$. So the rate of change of work done on the system is given by rate of change of work done by external forces acting on it. Let us consider a small element of surface area dS on surface S, then the surface traction force acting on the small element is given by $\boldsymbol{t}dS$ and the rate of change of work done by this force is given by $\int_{S(t)} \boldsymbol{t}\cdot\boldsymbol{u} dS$. Similarly, the rate of change of work done by body force per unit mass is given by $\int_{V(t)} \rho\boldsymbol{b}\cdot\boldsymbol{u} dV$. The total rate of change of work done on the system is given by

$$\frac{dW}{dt} = \int_{S(t)} \boldsymbol{t}\cdot\boldsymbol{u} dS + \int_{V(t)} \rho\boldsymbol{b}\cdot\boldsymbol{u} dV \tag{A.31}$$

If Q_h is the heat generation per unit mass per unit time by any heat source and $\boldsymbol{q}$ is the heat flux vector, then the rate of heat generated within system is given by $\int_{V(t)} \rho Q_h dV$ and the rate of heat dissipated from the system is given by $-\int_{S(t)} \boldsymbol{q}\cdot\boldsymbol{n} dS$. The net heat flow rate of the system is given

by

$$\frac{dQ}{dt} = \int_{V(t)} \rho Q_h dV - \int_{S(t)} \boldsymbol{q}.\boldsymbol{n} dS \tag{A.32}$$

Using Eqn. (A.30, A.31, A.32) in Eqn. (A.29), we get

$$\frac{d}{dt}\int_{V(t)} \rho \left[\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}) + e\right] dV = \int_{S(t)} \boldsymbol{t}\cdot\boldsymbol{u} dS + \int_{V(t)} \rho \boldsymbol{b}\cdot\boldsymbol{u} dV + \int_{V(t)} \rho Q_h dV - \int_{S(t)} \boldsymbol{q}\cdot\boldsymbol{n} dS \tag{A.33}$$

Applying divergence theorem to the first and last term of the right-hand side of above equation yields

$$\int_{S(t)} \boldsymbol{q}\cdot\boldsymbol{n} dS = \int_{V(t)} \boldsymbol{\nabla}\cdot\boldsymbol{q} dV$$
$$\int_{S(t)} \boldsymbol{t}\cdot\boldsymbol{u} dS = \int_{S(t)} (\boldsymbol{\tau}\boldsymbol{n})\cdot\boldsymbol{u} dS = \int_{V(t)} \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}^T\boldsymbol{u}) dV$$

Using these above expressions in Eqn. (A.33), we have

$$\frac{d}{dt}\int_{V(t)} \rho \left[\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}) + e\right] dV = \int_{V(t)} \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}^T\boldsymbol{u}) dV + \int_{V(t)} \rho\boldsymbol{b}\cdot\boldsymbol{u} dV + \int_{V(t)} \rho Q_h dV - \boldsymbol{\nabla}\cdot\boldsymbol{q} dV$$

Applying transport theorem 2 to the left-hand side of above equation gives

$$\int_{V(t)} \rho \frac{D}{Dt}\left[\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}) + e\right] dV = \int_{V(t)} \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}^T\boldsymbol{u}) dV + \int_{V(t)} \rho\boldsymbol{b}\cdot\boldsymbol{u} dV + \int_{V(t)} \rho Q_h dV - \boldsymbol{\nabla}\cdot\boldsymbol{q} dV$$
$$\int_{V(t)} \left\{\rho \frac{D}{Dt}\left[\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}) + e\right] - \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}^T\boldsymbol{u}) - \rho\boldsymbol{b}\cdot\boldsymbol{u} - \rho Q_h + \boldsymbol{\nabla}\cdot\boldsymbol{q}\right\} dV = 0$$

Since $V(t)$ can be chosen arbitrarily, we can make the integrand to be zero to obtain differential form of energy equation.

$$\rho\frac{D}{Dt}\left[\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}) + e\right] = \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}^T\boldsymbol{u}) + \rho\boldsymbol{b}\cdot\boldsymbol{u} + \rho Q_h - \boldsymbol{\nabla}\cdot\boldsymbol{q} \tag{A.34}$$

From Eqn.(A.27) and Eqn.(A.34), we have

$$\rho\frac{De}{Dt} = \boldsymbol{\tau} : \boldsymbol{D} - \boldsymbol{\nabla}\cdot\boldsymbol{q} + \rho Q_h$$

From thermodynamic relation for perfect gas, we have $e = C_v\theta$ where C_v is specific heat at constant volume. This reduces the above equation to

$$\rho C_v \frac{D\theta}{Dt} = \boldsymbol{\tau} : \boldsymbol{D} - \boldsymbol{\nabla}\cdot\boldsymbol{q} + \rho Q_h$$
$$\rho C_v \left[\frac{d\theta}{dt} + \boldsymbol{u}\cdot(\boldsymbol{\nabla}\theta)\right] = \boldsymbol{\tau} : \boldsymbol{D} - \boldsymbol{\nabla}\cdot\boldsymbol{q} + \rho Q_h \quad \text{(by Eqn.(1.4))} \tag{A.35}$$

Substituting Eqn.(A.19) in Eqn.(A.35), we have

$$\rho C_v \left[\frac{d\theta}{dt} + \boldsymbol{u} \cdot (\boldsymbol{\nabla}\theta) \right] = -p \cdot \boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\sigma} : \boldsymbol{D} - \boldsymbol{\nabla} \cdot \boldsymbol{q} + \rho Q_h$$