Korovkin's Linear operators and Approximation Theory and Sequence of Functions

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Declaration

I declare that this written submission represents my project work, and where ideas or words of others have been included, I have adequately referenced the original sources. I own the mistake, if any, crept into this report and do not hold anybody or any reference responsible for such mistakes.

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Abstract

In my project I am doing my work on the Korovkin's theorem and some standard version of Korovkin's theorem and thier applications. The small description of my topic is

(1) The section Linear Positive Functional defined on an existence domain of functions has some definitions of linear posisitive functionals and examples. If a sequence of linear positive functionals define on $C_b(\mathbb{R})$ for which conditions it will be convergent and where converge.

(2) The section Positive Linear Operator I state the Korovkin's first theorem and its corollary. In this theorem it says that if a sequence of positive linear operators defined on $C[a, b]$ satisfies some conditions then it is uniformly convergent on $C[a, b]$.

(3) The section Approximation theory I state small thing of orthogonality of a set of functions and Fejer's operator.

Contents

Chapter 1

Introduction

1.1 Introduction

The approximation theory has a close relationship with other branches of mathematics(like Fourier analysis). Existence of such a relationship is explained by the fact that many important problems of the approximation theory are formulated.

The relationship between approximation theory and functional analysis is basicaly very close. In fact, all well known methods of approximation of functions by means of algebric or trigonometric polynomials (which are partial sum of Taylor's series, interpolating polynomials, Bernstein and Landau polynomials, partial sums Of Fourier series, Fejer sums, etc.) are linear oeparators.

1.2 Some definitions and formulas

We want to define what is linear operator on an existence domains of functions. Also, we want to look at the nature of linear functionals and its some examples.

Definition 1. We say that a functional $\psi(f)$ is defined on the set F of functions $f(x)$ if a real numbers $\psi(f)$ is associated to every function $f(x)$ belonging the set F, $f(x) \in F$. The set F is called the domain of exixtence of the functional.

Definition 2. The functional $\psi(f)$ is said to be linear if the domain of its exixtence together with functions $f(x)$ and $g(x)$ satisfies the equation such that

$$\psi(af + bg) = a\psi(f) + b\psi(g),$$

Where a and b are any real numbers.Let $\psi(f) = Af(\alpha)$ where A is a real number. It is easy to check that it is a linear functional.

Definition 3. An operator $L(f; x)$ is said to be linear if the domain of its existence together with the functions $f(t)$ and $g(t)$ contains the functions $af(t)+bg(t)$ and if their holds the equality

$$L(af+bg;x)=aL(f;x)+bL(g;x),$$

Where a and b are real numbers.

Let $K(x,t)$ be a function continuous with respect to t in the interval $a\leq t\leq b$ for every value of x of the set B. We take the integral such that

$$L(f;x)=\int_a^b f(t)K(x,t)dt.$$

It is easy to check that it is a linear operator of the function $f\in F$ which is integrable in the interval $[a,b]$.

Lemma: Now the foloowing relations are true,

$$D_N(\alpha)=\frac{1}{2}+\sum_{n=1}^{N}\cos n\alpha=\frac{\sin(N+\frac{1}{2})\alpha}{2\sin\frac{\alpha}{2}}$$

and

$$F_N(\alpha)=\sum_{n=1}^{N}\sin\frac{(2n-1)\alpha}{2}=\frac{\sin^2(\frac{N\alpha}{2})}{\sin\frac{\alpha}{2}}$$

Now in this topic by using this lemma we have defined application the standard version of Korovkin's theorem on $C_{2\pi}[-\pi,\pi]$.

Chapter 2

Korovkin's theorem

2.1 Proofs of Korovkin's theorem via inequalities

Let $C[a, b]$ be the vector space of all real valued continuous functions on $[a, b]$ and let T be a linear transformation on $C[a, b]$. Throughout our discussion, we assume that any linear transformation on $C[a, b]$ is bounded. We say that T is positive if $T(f) \geq 0$ whenever $f \geq 0$. Now $\| f \| = max\{|f(x)| : a \leq x \leq b\}$ and $\| T \|$ is called the operator norm is defined by $max\{\| T(f) \| \quad : \quad \| f \| \leq 1\}$.

2.1.1 Theorem

Let (T_n) be a sequence of positive linear operators on $C[a, b]$. If

$$\| T_n(h) - h \| \to 0 \quad as \quad (n \to \infty).$$

holds for $h = 1, x$, and x^2, then it holds for every h in $C[a, b]$. Here the norm is sup norm.

2.1.2 Lemma

A subspace V of $C[a, b]$ is called a subalgebra if fg belongs to V whenever f and g are members of V. Let V be a normed closed subalgebra which contains the function $1(x)$. If T is a positive linear functional on V with $T(1)(x) \leq 1$ for all $x \in [a, b]$, then

$$M(h) = T(h^2) - (T(h))^2 \geq 0$$

for every h in V.

$$|T(fg) - T(f)T(g)|^2 \leq M(f)M(g)...........(1)$$

$$\| T(fg) - T(f)T(g) \| \leq \sqrt{\| M(f) \|} \sqrt{\| M(g) \|} ....(2)$$

$$\| T(fg) - T(f)T(g) \| \leq \sqrt{\|}M(f) \| \sqrt{\|}M(g) + M(k) \| \text{..........(3)}$$

Proof of lemma. Since $T((h+t))^2 \geq 0$ for every h in V and every real constant function t, we get

$$T(h^2) + 2tT(h) + t^2T(1) \geq 0$$

for all t in V, which implies that $(T(h))^2 - T(h^2)T(1) \leq 0$. Because this quadratic equation has no distinct positive real roots. Hence, as $T(1)(x) \leq 1$ for all $x \in [a, b]$, we get

$$T(h^2) - (T(h))^2 \geq 0.$$

The substitution of $f + tg$ for h in this inequality gives that

$$t^2(T^2(g^2) - T(g)^2) + 2t(T(fg) - T(f)T(g) + T(f^2) - T(f)^2 \geq 0$$

for all t. This gives the inequality (1), from which (2) follows immeadiately and the inequality

$$0 \leq M(g) \leq M(g) + M(k)$$

it gives the ineuality (3).

Proof of theorem: Since $\| T_n(1) - 1 \| \to 0$ for $n \to \infty$ and the sequence $\| T_n \|$ converges to 1. By considering $\frac{T_n}{\|T_n(1)\|}$ in place of T_n, we may assume that $T_n(1) \leq 1$ for all $n \in \mathbb{N}$. This gives that $\| T_n \| \leq 1$ for all $n \in \mathbb{N}$. By (2),

$$\| T_n(xf) - T_n(x)T_n(f) \|^2 \leq \| T_n(x^2) - T_n(x)^2 \| \| \| T_n(f^2) - T_n(f)^2 \|$$

holds for arbitrary f in $C[a, b]$. since

$$\| T_n(f^2) - T_n(f)^2 \| \leq 2 \| f \|^2 \text{......}(i)$$

and

$$\lim_{n\to\infty} T_n(x)^2$$

$$= x^2$$

$$= \lim_{n\to\infty} T_n(x^2)$$

the right-hand side of (i) tends to zero for $n \to \infty$. If $f(x) = x^n$ and $n \in \mathbb{N}$ then we get

$$\| T_n(xf) - xf \| \to 0$$

and

$$\| T_n(f) - f \| \to 0.$$

Thus the sequence of positive linear operators (T_n) holds for every polynomials $P(x)$. Since $\| T_n \| \leq 1$ for every n, as the polynomials are dense in $C[a, b]$. Hence this completes proof.

Chapter 3

Linear Positive Functionals

3.1 Some definitions and examples

Now we shall consider operators on functions space, which helps us to understand about linear functionals.

1. We put $T(f) = sup\{|f(x)| : 0 \leq x \leq 1\}$. What is the difference between the values of $T(f)$ and the functions f? and what is the common between them?

The value of $T(f)$ depends on a variables quantity. So, if $f_1(x) = x$ and $f_2(x) = x^2 + 1$, then the values of $T(f_1)$ and $T(f_2)$ are 1,2. The dependent variable quantity is a function. The differences between the values of $T(f)$ and the function are not very essential. If the function $f(x)$ is bounded on $[0, 1]$ is an argument of the quantity $T(f)$. A set F of the functions $f(x)$ bounded on $[0, 1]$ is a dommain of existence of the quantity $T(f)$.

2. Let $\psi(x)$ be a function continuous in the interval $[a, b]]$. We put

$$I(f) = \int_a^b f(x)\psi(x)dx.$$

The quantity $I(f)$, whose value depends on $f(x)$ and $\psi(x)$ is a fixed continuos function on $[a, b]$.

Definition 3.1.1. *We say that a functional $T(f)$ is defined on the set F of functions $f(x)$ if a real numbers T is associated to every function $f(x)$ belonging to the set F and $f(x) \in F$. Here the set F is called the domain of existence of the functional.*

Definition 3.1.2. *The functional $T(f)$ is said to be linear if the domain of its existence together with the functions $f(x)$ and $\psi(x)$ contains the $af(x) + b\psi(x)$ and there holds the equality*

$$T(af + b\psi) = aT(f) + b\psi$$

,

Where a and b are any two real numbers.

example 3.1.3. $T(f) = Af(c)$.
The functional $T(f)$ exists on the set F of functions $f(x)$ defined at point $x = c$. It is a linear functional on F.

Definition 3.1.4. *A linear functional $T(f)$ is said to be positive if $T(f) \geq 0$ for every positive function $f(x)$ in F.*

It is easy to check that if $f_1(x) \geq f_2(x)$ then $f_1(x) - f_2(x) \geq 0$. As T is linear, hence we get $T(f_1) \geq T(f_2)$. Therefore T is monotonicaly increasing. The linear functional $T(f) = Af(c)$ is positive if $A \geq 0$.

Theorem: Let $(T_n(f))$ be a sequence of positive linear functionals on a domain of exixtence of all functionals of this sequence. If two conditions are satisfied such that

$$T_n(1) \to 1,$$

$$T_n(\psi) \to 0$$

for $n \to \infty$. Where $\psi(x) = (x - c)^2$, then

$$\lim_{n\to\infty} T_n(f) \to f(c)$$

for any function $f(x)$ continuos at the point $x = c$ and bounded on the real axis.

Proof: At first we shall construct the two inequalities of $f(x)$ by its conditions that it is bounded on real axis and continuos at $x = c$. As the function $f(x)$ is bounded on real axis, there exists a positive real number M such that

$$|f(x)| \leq M$$

for all $x \in \mathbb{R}$. Now

$$-M \leq f(x) \leq M.......(1)$$

and

$$-M \leq f(c) \leq M.......(2).$$

From the inequalities (1) and (2) we get that,

$$-2M \leq f(x) - f(c) \leq 2M......(3)$$

This is true for all $x \in \mathbb{R}$. Again $f(x)$ is continuous at $x = c$, then for every $\epsilon > 0$, there exists $\delta > 0$ with $|x - c| < \delta$ such that,

$$|f(x) - f(c)| < \epsilon.......(4)$$

Now for the two inequalities $|x-c|<\delta$ and $|x-c|\geq\delta$ combinig the inequalities (3) and (4) we get that,

$$-\epsilon-\frac{2M}{\delta^2}\psi(x)<f(x)-f(c)<\epsilon+\frac{2M}{\delta^2}\psi(x)......(5)$$

for all $x\in\mathbb{R}$. As (T_n) be a sequence of positive linear functionals. Hence we get from (5) such that,

$$-\epsilon T_n(1)-\frac{2M}{\delta^2}T_n(\psi)\leq T_n(f)-f(c)T_n(1)\leq\epsilon T_n(1)+\frac{2M}{\delta^2}T_n(\psi).........(6)$$

As $\epsilon>0$ is arbitrary and the sequence (T_n) of positive linear functionals converges to 1 and 0 respectively at the two functions 1 and ψ for $n\to\infty$. Hence there exists a natural number k which depends on ϵ and for all $n\geq k$, from (6) we get,

$$-2\epsilon<T_n(f)-f(c)<2\epsilon$$

Finally we get the sequence (T_n) of linear functionals converges to $f(c)$ for the function f.

Corollary 1. If the three conditions,

$$\lim_{n\to\infty}T_n(1)\to 1.....(1),$$
$$\lim_{n\to\infty}T_n(x)\to c.....(2),$$
$$\lim_{n\to\infty}T_n(x^2)\to c^2.....(3)$$

are satisfied for the sequence of positive linear functionals (T_n), then the sequence of real numbers $(T_n(f))$ converges to $f(c)$ for any function f bounded on the real axis and continuous at the point $x=c$.

Proof: We put

$$\psi(x)=(x-c)^2=x^2-2cx+c^2.$$

Since the (T_n) be a sequence of linear functionals, therefore we get

$$T_n(\psi)=T_n(x^2)-2cT_n(x)+c^2T_n(1)$$

Hence the sequence of real numbers $(T_n)(\psi)$ converges to 0 for $n\to\infty$. Hence we can see that every condition of our previous theorem is fulfilled. Therefore the sequence of linear positive functionals (T_n) converges to $f(c)$ for the function f.

Theorem 2: If two conditions

$$\lim_{n\to\infty}T_n(1)\to 1......(1)$$

,

$$\lim_{n\to\infty}T_n(\psi)\to 0......(2)$$

Where $\psi(x) = \sin^2(\frac{x-c}{2})$, are satisfied for the sequence of linear positive functionals (T_n), then

$$\lim_{n\to\infty} T_n(f) = f(c)$$

in case the function $f(x)$ has period 2π, is continuous at the point $x = c$ and bounded on real axis.

Corollary 2. If three conditions

$$\lim_{n\to\infty} T_n(1) \to 1.....(1),$$

$$\lim_{n\to\infty} T_n \cos(x) \to \cos(c).....(2),$$

$$\lim_{n\to\infty} T_n \sin(x) \to \sin(c).....(3)$$

are satisfied for the sequence of linear positive functionals (T_n), then

$$\lim_{n\to\infty} T_n(f) \to f(c)$$

in case the function $f(x)$ has 2π period, continuous at the point $x = c$ and bounded on real axis.

Proof: We get

$$\psi(x) = \sin^2(\frac{x-c}{2})$$

$$= \frac{1-\cos(x-c)}{2}$$

$$= \frac{1-\cos(c)\cos(x)-\sin(c)\sin(x)}{2}$$

As the property of linearty we get,

$$T_n(\psi) = \frac{T_n(1) - \cos(c)T_n(\cos(x)) - \sin(c)T_n(\sin(x))}{2}$$

Hence by the conditions of this corollary we get that,

$$\lim_{n\to\infty} T_n(\psi) \to 0$$

Therefore this corollary satisfied the all conditions of our previous result, hence

$$\lim_{n\to\infty} T_n(f) \to f(c).$$

(A) Linear Positive Operators:

we shall at first consider an example which helps us to concept on an opertor, close to the concept of a function.

Let $u_1(x), u_2(x), ..., u_n(x)$ be function given on a set E and $t_1, t_2, ..., t_n$ be real numbers. We put

$$H(f;x) = H(f(t);x)$$
$$= \sum_{k=1}^{n} f(t_k)u_k(x) = \psi(x)$$

According to this equality a function $\psi(x) = H(f;x)$ is associated to every function $f(t)$ given on the set of points $t_1, t_2, ..., t_n$.

Definition 3.1.5. *We say that an operator $H(f;x) = H(f(t);x)$ is given on the set F of functions $f(t)$ if the function $\psi(x)$ is associated to every function $f(t)$ of the set F. $\psi(x) = H(f;x)$.*

Definition 3.1.6. *An operator $L(f;x)$ is said to be linear if the domain of its existence together with the functions $f(t)$ and $\psi(t)$ contains the function $af(t) + b\psi(t)$ and if there holds the equality*

$$L(af + b\psi; x) = aL(f;x) + bL(\psi;x)$$

where a and b are real numbers.

Examples: Let $u_1(x), u_2(x), ...u_n(x)$ be functions given on a set E. we put

$$L(f;x) = \sum_{k=1}^{n} f(t_k)u_k(x)$$

we have

$$L(af + b\psi; x) = \sum_{k=1}^{n}(af(t_k) + b\psi(t_k))u_k(x)$$
$$= a\sum_{k=1}^{n} f(t_k)u_k(x) + b\sum_{k=1}^{n} \psi(t_k)u_k(x)$$
$$= aL(f;x) + bL(\psi;x)$$

Thus we have proved the liniearity $L(f;x)$.

Theorem: If the three conditions

$$L_n(1;x) = 1 + \alpha_n(x).....(1)$$

$$L_n(t;x) = x + \beta_n(x).....(2)$$

$$L_n(t^2;x) = x^2 + \gamma_n(x).....(3)$$

are satisfied for the sequence of linear positive operators $L_n(f;x)$, where $\alpha_n(x)$, $\beta_n(x)$, $\gamma_n(x)$ converge uniformly to zero in the interval $a \leq x \leq b$, then the sequence $L_n(f;x)$ converges uniformly to the function $f(x)$ in this interval, if $f(t)$ is bounded, continuous in the interval $[a, b]$.

Proof: We want to prove this theorem by contradiction. Assuming that the conclusion of the theorem does hold, then we get a function $f(t)$ which shall satisfie the conditions of this theorem, for which the sequence $L_n(f;x)$ would not converge uniformly to the function $f(x)$ in the interval $[a,b]$. It means that tehre exists an $\epsilon > 0$, a sequence of points (x_k) for all $k \in \mathbb{N}$, $a \leq x \leq b$ and a sequence of numbers n_k, for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} n_k \to \infty$, such that there holds the inequality

$$|L_{n_k}(f;x_k) - f(x_k)| \geq \epsilon$$

.

Since the sequence (x_k) is bounded, so by Bolzano-weierstrass theorem we get a subsequence (x_{n_s}) from the sequence (x_k) which convergent and converges a point, say c in the interval $[a,b]$. We shall show that the sequence of functionals $L_{n_{k_s}}(f;x_{k_s})$ satisfies the conditions of this theorem

[If the three conditions,

$$\lim_{n\to\infty} T_n(1) \to 1.....(1),$$
$$\lim_{n\to\infty} T_n(x) \to c.....(2),$$
$$\lim_{n\to\infty} T_n(x^2) \to c^2.....(3)$$

are satisfied for the sequence of positive linear functionals (T_n), then the sequence of real numbers $(T_n(f))$ converges to $f(c)$ for any function f bounded on the real axis and continuous at the point $x = c$.]

In fact, since $x_{k_s} \to c$, and $\alpha_n(x)$, $\beta_n(x)$, $\gamma_n(x)$ converges uniformly to zero in the interval $a \leq x \leq b$, so

$$L_{n_{k_s}}(1;x_{k_s}) = 1 + \alpha_{n_{k_s}}(x_{k_s}) \to 1,$$
$$L_{nk_s}(t;x_{k_s}) = x_{k_s} + \beta_{nk_s}(x_{k_s}) \to c,$$
$$L_{nk_s}(t^2;x_{k_s}) = x_{k_s}^2 + \gamma_{n_{k_s}}(x_{k_s}) \to c^2,$$

for $s \to \infty$. As the function $f(x)$ is continuous at the point $x = c$. By using the theorem in our bracket we have

$$\lim_{s\to\infty} L_{n_{k_s}}(f;x_{k_s}) = f(c).$$

Since $f(x)$ is continuous at the point $x = c$, then property continuity we get

$$f(x_{k_s}) \to f(c)$$

So,

$$L_{n_{k_s}}(f;x_{k_s}) - f(x_{k_s}) \to 0$$

and therefore there exists a natural number P, for all $s \geq P$ such that,

$$|L_{n_{k_s}}(f; x_{k_s}) - f(x_{k_s})| < \epsilon$$

Therefore our assumption is contradicted. Hence the theorem is proved.

Theorem: If the three conditions

$$L_n(1; x) = 1 + \alpha_n(x),$$

$$L_n(\cos(t); x) = \cos(x) + \beta_n(x),$$

$$L_n(\sin(t); x) = \sin(x) + \gamma_n(x)$$

are satisfied for the sequence of linear positive operators $L_n(f; x)$, where

$$\lim_{n\to\infty} \alpha_n(x) \to 0$$

$$\lim_{n\to\infty} \beta_n(x) \to 0$$

$$\lim_{n\to\infty} \gamma_n(x) \to 0$$

and they are uniformly convergent in the interval $[a, b]$, then the sequence $L_n(f; x)$ converges uniformly to the function $f(x)$ in this interval $[a, b]$ and the function $f(t)$ is bounded, has 2π period, is continuous in the interval $[a, b]$, continuous on the right at the point b and on the left at the point a.

Lemma If a function $\psi(x)$ satisfies the conditions :

(1) $\psi(x)$ is continuous in the interval $-c \leq x \leq c$, $c > 0$.
(2) $\psi(0) = 1$, $0 \leq \psi(x) < 1$, if $x \neq 0$, $-c \leq x \leq c$ and if we put

$$I_n = \int_{-c}^{c} \psi^n(x)dx;$$

and

$$I_n(\delta) = \int_{-\delta}^{\delta} \psi^n(x)dx,$$

$0 < \delta \leq c$, then

$$\lim_{n\to\infty} \frac{I_n(\delta)}{I_n} = 1.$$

Proof: We have

$$I_n = \int_{-c}^{c} \psi^n(x)dx = \int_{-c}^{-\delta} \psi^n(x)dx + \int_{-\delta}^{\delta} \psi^n(x)dx + \int_{\delta}^{c} \psi^n(x)dx$$

$$= \int_{-c}^{-\delta} \psi^n(x)dx + \int_{\delta}^{c} \psi^n(x)dx + I_n(\delta).$$

The function $\psi(x)$ is continuous in the interval $[-c, -\delta]$. Let p and q are the maximum value of the function $\psi(x)$ in this interval $[-c, -\delta]$ and $[\delta, c]$ respectively.

$Q = Q(\delta)$ be the greater of the numbers p and q, as the function $\psi(x)$ attains the maximum value at $x = 1$, then the function $\psi(x)$ satisfies the inequality

$$0 \leq \psi(x) \leq Q < 1$$

on the set $[-c, -\delta] \cup [\delta, c]$. Therefore we have

$$0 \leq \int_{-c}^{-\delta} \psi^n(x)dx + \int_{\delta}^{c} \psi^n(x)dx < Q^n(c-\delta) + Q^n(c-\delta) < 2cQ^n.....(1)$$

Now we shall calculate the value of the integral $I_n(\delta)$. Since the function $\psi(x)$ is continuous at $x = 0$ and $\psi(0) = 1$, then there exists a $\delta_1 > 0$, $\delta_1 < \delta$ for $\epsilon = \frac{1-Q}{2} > 0$ such that the inequality

$$\psi(x) \geq 1 - \epsilon = \frac{1+Q}{2} = Q_1 > Q$$

will hold in case $|x| < \delta_1$. Thus since the function $\psi(x)$ is positive, we have

$$I_n(\delta) = \int_{-\delta}^{\delta} \psi^n(x)dx \geq \int_{-\delta_1}^{\delta_1} > 2Q_1^n\delta_1.....(2)$$

The inequality from (1) and (2) we get the inequality

$$I_n(\delta) \leq I_n \leq I_n(\delta) + 2cQ^n.$$

Now dividing the all parts by $I_n(\delta)$ and the inequality

$$1 \leq \frac{I_n}{I_n(\delta)} < 1 + \frac{2cQ^n}{I_n(\delta)} < 1 + \frac{2cQ^n}{2\delta_1 Q_1^n}.$$

Since $Q_1 > Q$, if we take the limit as $n \to \infty$ we get the right hand side limit is 1 and apply the Sandwitch theorem on above inequality. Hence we get oue desire result.

This completes proof.

Theorem If a function $\psi(x)$ satisfies the conditions of previous lemma and

$$I_n = \int_{-c}^{c} \psi^n(x)dx$$

then the sequence of operators

$$L_n(f;x) = \frac{\int_a^b f(t)\psi^n(t-x)dx}{I_n},$$

where $0 \leq b-a \leq c$ converges uniformly to the function $f(x)$ in the interval $[a+\delta, b-\delta]$, $\delta > 0$, if the function $f(x)$ is continuous in the interval $[a,b]$.

Proof Before proving this theorem we want to recall a theorem which is that " If a sequence of linear functionals (T_n) converges to 1 for the function $1(t)$ and converges to 0 for the function $(t-\alpha)^2$. Then $(T_n(f))$ converges to $f(\alpha)$ if $f(t)$ is continuous at $t = \alpha$ and bounded on real axis".

Now we want to check the values of the linear operators at the functions $f(t) = 1$ and $f(t) = (t-x)^2$ respectively where $t, x \in [a,b]$. Hence we get

$$L_n(1;x) = \frac{\int_a^b \psi^n(t-x)dt}{I_n}.$$

Putting $z = t - x$ we obtain

$$L_n(1;x) = \frac{\int_{a-x}^{b-x} \psi^n(z)dz}{I_n}.$$

By observing that $x \in [a+\delta, b-\delta]$ we have

$$a - x \geq a - (b-\delta) = \delta - (b-a) \geq \delta - c > -c,$$

$$a - x \leq a - (a+\delta) = -\delta,$$

$$b - x \geq b - (b-\delta) = \delta,$$

$$b - x \leq b - (a+\delta) = (b-a) - \delta \leq c - \delta < c.$$

Therefore, since the $\psi(x)$ is positive, so

$$I_n(\delta) = \int_{-\delta}^{\delta} \psi^n(z)dz \leq \int_{a-x}^{b-x} \psi^n(z)dz \leq \int_{-c}^{c} \psi^n(z)dz = I_n,$$

$$\frac{I_n(\delta)}{I_n} \leq \frac{\int_{a-x}^{b-x} \psi^n(z)dz}{I_n} = L_n(1;x) \leq 1.$$

The left-hand side of this inequality converges to 1 for $n \to \infty$ (according our previous lemma). Thus for every $\epsilon > 0$, there exists a natural number k, for all $n > k$ and $x \in [a+\delta, b-\delta]$ such that

$$-\epsilon \leq L_n(1;x) - 1 \leq 0 < \epsilon$$

Hence the sequence of linear operators $L_n(1;x)$ converges uniformly to function $1(x)$.

It remains to prove uniform convergence of the sequence of linear operators $L_n(\psi;x)$, where

$\psi(t) = (t-x)^2$, to the zero function. We have

$$0 < L_n(\psi; x) = \frac{\int_a^b (t-x)^2 \psi^n(t-x)dt}{I_n} = \frac{\int_{a-x}^{b-x} z^2 \psi^n(z)dz}{I_n}.$$

Since $a - x \geq -c$, $b - x \leq c$ and the function $\psi(x)$ is positive in this interval $[-c, c]$, hence we get

$$0 \leq L_n(\psi; x) \leq \frac{\int_{a-x}^{b-x} z^2 \psi^n(z)dz}{I_n}$$

$$= \frac{\int_{-c}^{-\alpha} z^2 \psi^n(z) + \int_{\alpha}^{c} z^2 \psi^n(z)dz}{I_n} + \frac{\int_{-\alpha}^{\alpha} z^2 \psi^n(z)dz}{I_n}.$$

As $z^2 \leq c^2$ in the first and second integrands, and $z^2 \leq \alpha^2$ in the third integrands. Thus we get

$$0 \leq L_n(\psi; x) \leq \frac{c^2 \int_{-c}^{-\alpha} \psi^n(z)dz + c^2 \int_{\alpha}^{c} \psi^n(z)dz}{I_n} + \frac{\alpha^2 \int_{-\alpha}^{\alpha} \psi^n(z)dz}{I_n}$$

. By using our previous lemma we get

$$0 < L_n(\psi; x) < \frac{c^2.2cQ^n}{I_n} + \frac{\alpha^2 I_n(\alpha)}{I_n} ....(1)$$

Now for every $\epsilon > 0$ and $\alpha^2 = \frac{\epsilon}{2}$. Then there exists a natural no. k, $\forall n > k$ and using our lemma we get

$$0 < L_n(\psi; x) < \epsilon$$

Hence it follows that the sequence $L_n(\psi; x)$ converges uniformly to zero in the interval $[a, b]$.

This completes proof.

3.2 Application of this theorem

Now we want to see some application of this theorem.

Weierstrass first theorem: If a function $f(x)$ is continuous in the interval $[a, b]$ and $\epsilon > 0$, then we can find a polynomial $P(x)$ such that the inequality

$$|f(x) - P(x)| < \epsilon$$

would hold for all $x \in [a, b]$.

Proof by Weierstrass: Let the function $f(x)$ be continuous in the interval $[a, b]$. Without loss of generality we can regard this function continuous on the whole real axis by putting $f(x) = f(a)$ if $x \leq a$, and $f(x) = b$ if $x \geq$. The function so obtained and denoted by $f(x)$ is continuous on the real axis.

Now we take $a_1 = a - \delta$, $b_1 = b + \delta$, $\delta > 0$, and

$$W_n(f;x) = \frac{\int_{a_1}^{b_1} f(t)e^{-n(t-x)^2}dt}{I_n}$$

where

$$I_n = \int_{-c}^{c} e^{-nt^2}dt,$$

where $c = b_1 - a_1$.

Proof by Landau(1908). As in this case of proof given Weierstrass, we shall regard the function $f(x)$ continuous on the whole real axis, $a_1 = a - \delta$, $b_1 = b + \delta$, $\delta > 0$. We get

$$L_n(f;x) = \frac{\int_{a_1}^{b_1} f(t)\{\frac{c^2-(t-x)^2}{c^2}\}^n dt}{I_n}$$

where

$$I_n = \int_{-c}^{c} (\frac{c^2 - x^2}{c^2})^n dt.$$

Here $c = b_1 - a_1$.

3.3 Approximation of functions by trigonometric polynomials

Definition: Two functions $f(x)$ and $g(x)$ are said to be orthogonal in the interval $[a, b]$, if

$$\int_a^b f(x)g(x)dx = 0.$$

Definition: A finite or infinite system of functions $f_1(x), f_2(x), ...$, is said to be orthogonal in the interval $[a, b]$, if any two functions of this system are orthogonal in this interval that is

$$\int_a^b f_i(x)f_k(x)dx = 0, i \neq k.$$

.

Now the trigonometric system of functions 1, $\cos(x)$, $\sin(x)$, $\cos(2x)$, $sin(2x)$......is orthogonal in the interval $[-\pi, \pi]$.

Definition: The function

$$T_n(x) = \sum_{i=0}^{n}(a_i \cos ix + b_i \sin ix)$$

is called a trigonometric polynomial of order n if $a_i^2+b_i^2 \neq 0$, and the series $\sum_{i=0}^{\infty}(a_i \cos ix + b_i \sin ix)$ is called a trigonometric series.

Lemma: Now the foloowing relations are true,

$$D_N(\alpha) = \frac{1}{2} + \sum_{n=1}^{N} \cos n\alpha = \frac{\sin(N+\frac{1}{2})\alpha}{2\sin\frac{\alpha}{2}}$$

and

$$F_N(\alpha) = \sum_{n=1}^{N} \sin\frac{(2n-1)\alpha}{2} = \frac{\sin^2(\frac{N\alpha}{2})}{\sin\frac{\alpha}{2}}$$

Now we shall find an integral representation of the partial sum of Fourier series of the function $f(x)$, which we shall use in the sequel to establish uniform convergence of this series for a sufficiently wide class of continuous and periodic functions. We put

$$S_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k sinkx)$$

On account of we get

$$S_n(f;x) = \frac{\int_{-\pi}^{\pi} f(t)\frac{\sin(n+\frac{1}{2})(t-x)}{2\sin\frac{t-x}{2}}dt}{\pi} ....(1)$$

Now we put

$$F_n(f;x) = \frac{S_0(f;x) + S_1(f;x) + S_2(f;x) + ... + S_{n-1}(f;x)}{n}$$

The operators $F_n(f;x)$ are called Fejer operators. In view of the equalities and from (1) we get

$$F_n(f;x) = \frac{\int_{-\pi}^{\pi} f(t)\frac{\sin^2\frac{n(t-x)}{2}}{2\sin^2\frac{t-x}{2}}dt}{n\pi}$$

In particular, putting $f(t) = 1$, $\cos t$, $\sin t$ we obtain that

$$F_n(1;x) = \frac{\sum_{k=0}^{n-1} S_k(1;x)}{n}$$

$$= \frac{1+1+......+1}{n} = 1,$$

$$F_n(\cos t;x) = \frac{0+\cos x+\cos x+.....+\cos x}{n} = \frac{(n-1)\cos x}{n}$$

$$F_n(\sin t;x) = \frac{(n-1)\sin x}{n}$$

Using the Fejers operators on $C_2\pi[-\pi,\pi]$. I want to prove a important theorem.

Theorem If a function $f(x)$ has period 2π and is continuous on the real axis, then we can find a trigonometric polynomial $T(x)$ for $\epsilon > 0$ such that there holds the inequality

$$|T(x) - f(x)|\epsilon, \quad -\pi \leq x \leq \pi.$$

Proof The sequence of linear positive operators $F_n(f;x)$ is uniformly convergent in the interval $[-\pi, \pi]$ for each of the functions 1, $\cos t$, $\sin t$, as it follows from our previous thing. Hence by Korovkin's second theorem for every $\epsilon > 0$, there exists a natural no. k, for all $n \geq k$ and $-\pi \leq x \leq \pi$ such that

$$|F_n(f;x) - f(x)| < \epsilon$$

As $F_n(f;x)$ being arithmetic mean of trigonometric polynomails is trigonometric polynomial and it is true for $\forall n \in \mathbb{N}$. This completes the proof.

SEQUENCE OF FUNCTIONS

SUBINOY HATAI

Definition 1. *(Sequence of functions) Let us consider a set F such that F contains all real valued functions. Then a map $j : \mathbb{N} \to F$ is called the sequence of functions. But we are interested to know about such collections function on same domain. Let D be a domain and a set $A = \{f | f : D \subset \mathbb{R} \to \mathbb{R}\}$. Then the map $j : \mathbb{N} \to \mathbb{R}$ is called the sequence of functions on the domain D.*

Example 0.1. *Let us consider the function $f_n(x) = x^n, n \in \mathbb{N}, x \in [0, 1]$. Then (f_n) is a sequence of functions on $[0, 1]$*

Now we want to draw some concepts about convergence ,pointwise convergence of sequence of functions.

Definition 2. *(Pointwise convergence) Let (f_n) be sequence of functions on the domain $D \subset \mathbb{R}$. Then the the sequence (f_n) is said to be pointwise convergent on D, if for each $x \in D$, the sequence $(f_n(x))$ converges.*

Let the sequence (f_n) be pointwise convergent on D and let $c \in D$ .Then the sequence $(f_n(c))$ is convergent .

Let $\lim f_n(c) = f(c)$.Since for all $x \in D$,$(f_n(x))$ converges to a limit , then $f(x)$ eixts for all $x \in D$. Then $f : D \to \mathbb{R}$ is called the limit function.And $\lim f_n = f$ on D.

Let (f_n) be a sequence of functions on a domain D .Then (f_n) is said to be pointwise convergent to a function f on D, if $\forall x \in D$, $\forall \epsilon > 0$,$\exists k \in \mathbb{N}$,$\forall n \geqslant k$ such that

$$|f_n(x) - f(x)| < \epsilon$$

.

Example 0.2. *For each $n \in \mathbb{N}$,let (f_n) be a sequence of functions defined by $f_n(x) = \frac{x}{n}$, $x \in \mathbb{R}$. For all $x \in \mathbb{R}$ the sequence $f_n(x)$ converges to 0.Because $\lim_{n\to\infty} f_n(x) = 0$.Therefore the sequence (f_n) is pointwise convergent on $\mathbb{R}$ and the limit function f is defined by $f(x) = 0$, $x \in \mathbb{R}$.*

Example 0.3. *Let $f_n(x) = x^n, x \in (0, 1)$. Now for all $n \in \mathbb{N}$,(f_n) is a sequence of functions.Then $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0, 1)$.*

Now the sequence of functions (f_n) is pointwise convergent on $(0, 1)$.Let f be the limit function and $f(x) = 0$ for all $x \in (0, 1)$

Definition 3. *(Uniform convergence) Let (f_n) be sequence of functions on a domain D.The sequence (f_n) is said to be uniformly convergent on D to a function f if $\forall \epsilon > 0$,$\exists K \in \mathbb{N}$,$\forall n \geq K, \forall x \in D$ such that*

$$|f_n(x) - f(x)| < \epsilon$$

.

In this case we write $\lim_{f_n} = f$ uniformly on D, or $f_n \to f$ on D then f is said to be the uniform limit of the sequence (f_n) on D.

Example 0.4. *Let* (f_n) *be a sequence of functions defined by* $f_n(x) = x^n$*, for all* $x \in [0, a], a < 1$ *for all* $n \in \mathbb{N}$*. Let* $f(x)$ *be the pointwise limit function for the sequence* (f_n) *on* $[0, a]$*. Then* $f(x) = 0$ *for all* $x \in [0, a]$*. We want to prove the uniform convergence of this sequence. Now,* $|f_n(x) - f(x)| = |x^n - 0| = x^n$*. As* $0 \leq x < a \Rightarrow x^n < a^n$*, suppose* $\epsilon > 0$*. As this sequence* (a^n) *is a convergent sequence and* $a^n \to 0$ *for* $n \to \infty$*. Then there exists a natural number* k_1*, for all* $n \geq k_1$ *such that*

$$|a^n| < \epsilon \Rightarrow \log a^n < \log \epsilon \Rightarrow n \log a < \log \epsilon$$

. As $\log a < 0$*, therefore* $n > \frac{\log \epsilon}{\log a}$*. Let* $k_2 = [\frac{\log \epsilon}{\log a}] + 1$ *and* $k = \max\{k_1, k_2\}$*, therefore* $|f_n(x) - f(x)| < \epsilon$ *for all* $n > k$ *and for all* $x \in [0, a]$*. Hence by definition this sequence of functions* (f_n) *is uniformly convergent to the function* $f(x)$ *on* $[0, a]$*.*

Example 0.5. *Let* (f_n) *be a sequence of functions defined by* $f_n(x) = x^n$*,* $n \in \mathbb{N}, x \in [0, 1]$*. Now we shall prove that* (f_n) *is not uniformly convergent on* $[0, 1]$ *as well as* $(0, 1)$*.*

Let f *be the limit function for the sequence of functions* (f_n) *on the domain* $[0, 1]$*. Then*

$$f(x) = 0 \ x \in [0, 1)$$
$$= 1$$

if $x = 1$*.*

Let $c \in (0, 1)$ *and* $|f_n(c) - f(c)| = |c^n - 0| = c^n$*. As this sequence* (c^n) *is a convergent sequence and* $c^n \to 0$ *for* $n \to \infty$*. Then there exists a natural number* k_1*, for all* $n \geq k_1$ *such that*

$$|c^n| < \epsilon$$
$$\Rightarrow \log c^n < \log \epsilon$$
$$\Rightarrow n \log c < \log \epsilon$$

. As $\log c < 0$*, therefore* $n > \frac{\log \epsilon}{\log c}$*. Let* $k_2 = [\frac{\log \epsilon}{\log c}] + 1$ *and* $k = \max\{k_1, k_2\}$*, therefore* $|f_n(c) - f(c)| < \epsilon$ *for all* $n \geq k$*.*

As $c \to 1$ *then* $k \to \infty$*. So we are not able to find a finite* k *such that which satisfies the uniform convergence condition. Hence the sequence of functions* (f_n) *is not uniform convergent on* $[0, 1]$ *as well as* $(0, 1)$*.*

The Negation statement of uniform convergence:

Let (f_n) *is a sequence of functions on a domain* D *and pointwise covergent to a function* f *on* D *. Then the sequence of functions* (f_n) *is not uniformly convergent on* D *, if*

$\exists \epsilon > 0$ *,* $\forall k \in \mathbb{N}$*,* $\exists n \geq k$*,* $\exists x \in D$ *such that*

$$|f_n(x) - f(x)| \geq \epsilon.$$

Now we want to apply this condition for the previous examples . let $\epsilon = \frac{1}{2}$ *and a natural number* $k \in \mathbb{N}$*. Now*

$$|f_k(x) - f(x)| \geq \frac{1}{2}.$$

$$\Rightarrow |x^k - 0| \geq \frac{1}{2}.$$

$$\Rightarrow x \geq \left(\frac{1}{2}\right)^{\frac{1}{k}}.$$

As $\forall\ k \in \mathbb{N}$ *the point* $\left(\frac{1}{2}\right)^{\frac{1}{k}}$ *belongs to the set* $(0,1)$.*It is enough to take this point and this same k to satisfie our condition.Hence the sequence of function* (f_n) *is not uniform convergent on* $[0,1]$ *as well as* $(0,1)$.*If we change the interval such that* $[0,a]$ *where* $a < 1$.*Then* $\forall k \in \mathbb{N}$ *there does not exist points in* $[0,a]$.*Therefore the sequence of functions* (f_n) *is uniform convergent on* $[0,a]$.

Theorem 1: Let $D \subset \mathbb{R}$ and let (f_n) be a sequence of functions pointwise convergent on D to a function f. Let $M_n = sup_{x\in D}|f_n(x) - f(x)|$. Then (f_n) is uniformly convergent on D to f if and only if $limM_n = 0$.

Proof: Let the sequence (f_n) be uniformly convergent on D to f. Let $\epsilon > 0$. Then there exists a natural number k(depending only on ϵ) such that for all $x \in D$, $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $n \geq k$.

This implies $sup_{x\in D}|f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon$ for all $n \geq k$

or, $|M_n| < \epsilon$ for all $n \geq k$. This proves that $limM_n = 0$.

Conversely, let $limM_n = 0$.

Let $\epsilon > 0$. Then there exists a natural number k such that $|M_n| < \epsilon$ for all $n \geq k$. or, $sup_{x\in D}|f_n(x) - f(x)| < \epsilon$ for all $n \geq k$.

Therefore for all $x \in D$,

$$|f_n(x) - f(x)| \leq sup_{x\in D}|f_n(x) - f(x)| < \epsilon$$

for all $n \geq k$.

This proves that the sequence (f_n) is uniformly convergent to f on D.

Example 0.6. *For each natural number n, let* $f_n(x) = 1 - \frac{x^n}{n}, x \in [0,1]$. *Show that the sequence* (f_n) *is uniformly convergent on* $[0,1]$.

For $0 \leq x \leq 1$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}(1 - \frac{x^n}{n}) = 1$.

Hence the sequence (f_n) *converge pointwise on* $[0,1]$ *to the function f where* $f(x) = 1, x \in [0,1]$. *Then* $M_n = sup_{x\in D}|f_n(x) - f(x)|$.

Then $M_n = sup_{x\in[0,1]}\frac{|x|^n}{n} = \frac{1}{n}$ *and* $\lim\limits_{n\to\infty} M_n = 0$.

Hence the sequence (f_n) *converges uniformly on* $[0,1]$.

Theorem2: Let (f_n) be a sequence of functions on $D \subset \mathbb{R}$. Then (f_n) is said to be uniformly convergent to a function f on D iff $\forall\ \epsilon > 0, \exists K \in \mathbb{N}, \forall\ m,n \geq K, \forall\ x \in D$. such that

$$|f_m(x) - f_n(x)| < \epsilon.$$

Proof: Let the sequence (f_n) be uniformly convergent on D and let the limit function be f. Then $\forall\ \epsilon > 0, \exists\ k \in \mathbb{N}, \forall\ n \geq k, \forall\ x \in D$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}.$$

Now $m \geq n \geq k$ such that $|f_m(x) - f(x)| < \frac{\epsilon}{2}$.

Now $|f_m(x) - f_n(x)| = |f_m(x) - f(x) + f(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Therefore $\forall\ \epsilon > 0, \exists\ k \in \mathbb{N}, \forall\ m, n \geq k, \forall\ x \in D$ such that

$$|f_m(x) - f_n(x)| < \epsilon.$$

Conversely, Let the condition be satisfied. Then $\forall \epsilon > 0, \exists k \in \mathbb{N}, \forall m, n \geq k, \in D$. such that

$$|f_m(x) - f_n(x)| < \epsilon \text{ ....(1)}$$

Let $x_0 \in D$. Then $\forall \epsilon > 0, \forall m, n \geq k$ such that

$$|f_m(x_0) - f_n(x_0)| < \epsilon.$$

It follows that the sequence $(f_n(x_0))$ is Cauchy Sequence in $\mathbb{R}$ and therefore it is convergent. Consequently the sequence (f_n) is pointwise convergent on D. Let the limit function be f .

Now $\lim_{n\to\infty} f_n(x) = f(x)$, for all $x \in D$.

From the equation (1) we can write that

$$|f_n(x) - f_k(x)| < \epsilon.$$

$$\Rightarrow f_n(x) - \epsilon < f_k(x) < f_n(x) + \epsilon \text{.....(2)}$$

If we take the limit $n \to \infty$ in equation (2), then we get $f(x) - \epsilon < f_k(x) < f(x) + \epsilon$. Which is also true for $k + 1, k + 2, ...$ Therefore $\forall \epsilon > 0, \exists k \in \mathbb{N}, \forall n \geq k, \forall x \in D$ such that

$$|f_n(x) - f(x)| < \epsilon.$$

Therefore the sequence of functions (f_n) is uniformly convergent to the function f on D. Hence complete the proof.

Example 0.7. *For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{n}, x \in \mathbb{R}$. Then $(f_n(x))$ is a sequence of functions on $\mathbb{R}$. For each $x \in \mathbb{R}$, the sequence $(f_n(x))$ coverges to 0.*

Now

$$|f_m(x) - f_n(x)| = |\frac{x}{m} - \frac{x}{n}| \leq \frac{x}{m} + \frac{x}{n} \leq \frac{2x}{n}$$

if $m \geq n$. As the sequence $\left(\frac{x}{n}\right)$ converge to 0 for all $x \in [0, 1]$. Then $\forall \epsilon > 0, \exists k \in \mathbb{N}, \forall m, n \geq k, \forall x \in [0, 1]$ such that

$$|f_m(x) - f_n(x)| < \epsilon.$$

Therefore the sequence of functions (f_n) is uniformly convergent on $[0, 1]$.

Negation statement of Cauchy criterion: Let (f_n) be a sequence of functions on D and (f_n) pintwise convergent to f on D.Then (f_n) is not uniformly convergent to f on D if, $\exists \epsilon > 0$, $\forall K \in \mathbb{N}$,$\exists m, n \geq K$,$\exists x \in D$ such that

$$|f_m(x) - f_n(x)| \geq \epsilon$$

Example 0.8. *Let $r_1, r_2, r_3, ...$ be an enumeration of the set of all rationals points in $[0,1]$ and a sequence of functions (f_n) is defined by ,*

$$f_n(x) = \begin{cases} 0, & x = r_1, r_2, \ldots, r_n. \\ 1, & x \in [0,1] - \{r_1, r_2, ...r_n\} \end{cases}$$

Let us take $\epsilon = \frac{1}{2}$, $\forall K \in \mathbb{N}$,

$$f_k(x) = \begin{cases} 0, & x = r_1, r_2, \ldots, r_k. \\ 1, & x \in [0,1] - \{r_1, r_2, ...r_k\} \end{cases}$$

$\forall K \in \mathbb{N}$, $\exists r_k + 1 \in [0,1]$ such that

$$|f_k(r_{k+1}) - f_{k+1}(r_{k+1})| = |1 - 0| = 1$$

Hence the sequence of functions (f_n) is uniform convergent on $[0,1]$

Theorm3: Let D be a subset of $\mathbb{R}$ and a sequence of functions (f_n) be uniformly convergent on D to a function f. Let $x_0 \in A$ $(thederivedsetofD)$ and $\lim_{x \to x_0} f_n(x) = a_n$.Then the sequence (a_n) is convergent and $\lim_{x \to x_0} f(x) exists$ and equals $\lim_{n \to \infty} a_n$

proof: Let us choose $\epsilon > 0$.Since the sequence (f_n) is uniformly convergent, $\exists k \in \mathbb{N}$,$\forall m, n \geq k$,$\forall x \in D$ such tht

$$|f_m(x) - f_n(x)| < \epsilon..........(i)$$

As $\lim_{x \to x_0} f_n(x) = a_n$ and $\lim_{x \to x_0} f_m(x) = a_m$, it follows that

$$\lim_{x \to x_0} (f_m(x) - f_n(x)) = a_m - a_n$$

and therefore $\lim_{x \to x_0} |f_m(x) - f_n(x)| = |a_m - a_n|$

It follows from (i) that the sequence (a_n) is a Cauchy sequence in $\mathbb{R}$ and it is a convergent sequence.

Let $\lim a_n = l$.Let us choose $\epsilon > 0$.

Since the the sequence of functions (f_n) converges uniformly on D, $\exists p \in \mathbb{N}$, $\forall n \geq p$, $\forall x \in D$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

Since $lima_n = l$, there exists a natural number q and for all $n \geq q$ such that

$$|a_n - l| < \frac{\epsilon}{3}$$

Let $P = max(p,q)$.Then $|f_P(x) - f(x)| < \frac{\epsilon}{3}$ and $|a_P - l| < \frac{\epsilon}{3}$.It is true for all x in D.

Since $\lim_{x \to x_0} f_P(x) = a_P$, there exists a positive δ and for all $x\ inD$ with $|x - x_0| < \delta$ such that

$$|f_P(x) - a_P| < \frac{\epsilon}{3}$$

,

By triangle inequality,

$$|f(x) - l| \leq |f(x) - f_P(x)| + |f_P(x) - a_P| + |a_P - l| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} (= \epsilon)$$

it is true for all x beolngs to D with $|x - x_0| < \delta$,This proves that $\lim_{x \to x_0} f(x) = l$.Therefore $\lim_{x \to x_0} f(x) = \lim_{n \to \infty} a_n$:

UNIFORM CONVERGENCE OF A SEQUENCE OF BOUNDED FUNCTIONS.

Let $D \subset \mathbb{R}$ and the sequence of bounded functions (f_n) on D such that the sequence of functions (f_n) be pointwise convergent to a function f on D.Then the limit function may not be bounded on D.

Example 0.9. *Let* $f_n(x) = 1 + x + x^2 + ... + x^{n-1}, for\ x \in (0,1)$

Then $\lim_{n \to \infty} f_n(x) = \frac{1}{1-x}$.

The sequence of functions (f_n) *converges on* $[0,1]$ *to the function* f *defined by,*

$$f(x) = \frac{1}{1-x}, for\ x \in (0,1)$$

Now for all $n \in \mathbb{N}$ *the the sequence of functions are bounded functions on* $(0,1)$.*But the limit function* f *is not bounded on* $(0,1)$

Thm4: Let $D \subset \mathbb{R}$ and let (f_n) be sequence of bounded functions on D.If the sequence (f_n) be uniformly convergent to a function f on D, then f is also bounded on D.

textitProof: Let us chose $\epsilon = 1$.Since the sequence of functions (f_n) is uniformly convergent the function f on D , $\exists K \in \mathbb{N}, \forall n \geq K, \forall x \in D$ such that

$$|f_n(x) - f(x)| < 1$$

Now $|f(x)| \leq |f(x)-f_K(x)|+|f_K(x)$.As the function $f_K(x)$ is bounde on D ,then there exists a positvie real number B such that

$$|f_K(x)| \leq B$$

and it is true for all $x \in D$.

therefore for all $x \in D$,$|f(x)| \leq B+1$ and this proves that f is bounded on D.

UNIFORM CONVERGENCE OF A SEQUENCE OF CONTINUOUS FUNCTIONS

Let $D \subset \mathbb{R}$ and the sequence of continuous functions (f_n) be pointwise convergent to a function f on D ,then the limit function may not be continuos functin on D.

Example 0.10. *Let $f_n(x) = x^{n-1}, x \in [0,1]$.Then for each $n \in \mathbb{N}$ the function f_n is continuos on $[0,1]$.The sequence (f_n) is pointwise convergent to the function f on $[0,1]$ defined by*

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

The limit function is not continuos on $[0,1]$.

Theorem(5): Let $D \subset \mathbb{R}$ and the sequence of continuos functions (f_n) be uniformly convergent to a function f on D ,then the function f is continuos on D.

Proof: Let $c \in D$.Let us choose $\epsilon > 0$.
Since the sequence of functions (f_n) is uniformly convergent to the function f on D ,$\exists K \in \mathbb{N}$, $\forall n \geq K$, $\forall x \in D$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

Therefore $\forall x \in D$, $|f_K(x) - f(x)| < \frac{\epsilon}{3}$.Now $|f_K(x) - f(x)| < \frac{\epsilon}{3}$,since the function f_K is continuos on at c ,there exists a positive δ and for all $x \in D$ with $0 \leq |x-c| < \delta$ such that

$$|f_K(x) - f_K(c)| < \frac{\epsilon}{3}$$

By triangle inequality, $|f(x)-f(c)| \leq |f(x)-f_K(x)|+|f_K(x)-f_K(c)|+|f_K(c)-f(c) < \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}$. That is ,$|f(x)-f(c)| < \epsilon$ for all $x \in N(c,\delta) \cap D$.This proves that f is continuos on D.Since c is arbitrary , f is continuos on D.

Note 0.11. *If each f_n be continuos on D, the uniform convergence of the sequence of functions (f_n) on D is a sufficient but not a necessary condition for continuty of the limit function f on D.*

Example 0.12. *Let* $f_n(x) = \frac{nx}{1+n^2.x^2}$, *for* $x \in [0,1]$. *Each* f_n *is continuos on* $[0,1]$.*The sequence of functions* (f_n) *is pointwise convergent to the function* f *on* D *defined by*

$$f(x) = 0, x \in [0,1]$$

the limit function f *is continuos on* $[0,1]$.*But the convergence of the sequence of functions* (f_n) *is not uniform on* $[0,1]$ *by the supremum test.*

Note 0.13. *If each* f_n *be continuos on* D *and the sequence of functions* (f_n) *converges pointwise on* D *to a function* f *which is not continuos on* D *,then it follows from the theorem that the convergence of the sequence is not uniform on* D.

Theorem(Dini): Let D be a compact subset of $\mathbb{R}$ and the sequence of continuos functions (f_n) converges pointwise to a continuos function f on D.If the sequence of functions (f_n) be monotone sequence of functions on D,the convergence of the sequence of functions (f_n) is uniform to the function f on D.

Proof: If the sequence of functions (f_n) be monotone increasing ,let us consider a sequence of functions (g_n) on D such that $g_n = f - f_n$.If the sequence of functions (f_n) be monotone decreasing ,then $g_n = f_n - f$. Then $\forall n \in \mathbb{N}$,$\forall x \in D$ such that

$$g_{n+1} - g_n \leq 0$$

So the sequence of functions (g_n) is a monotone decreasing sequence of continuos functions on D.Let g be the limit function on D defined by , $\forall x \in D$,$g(x) = 0$.Therefore the limit functions g is continuos the compact set D.

Let $M_n = sup\{g_n(x) : x \in D\}$.Then $M_{n+1} \leq M_n$, for all $n \in \mathbb{N}$.

since $n \in \mathbb{N}$, (g_n) be a sequence of continuos functions on D,the g_n attains the supremum M_n at a point ,say x_n in D, that is $g_n(x_n) = M_n$ for all $n \in \mathbb{N}$. The sequence (x_n) is a sequence in a compact set D.Therefore there exists a subsequence (x_{r_n}) of (x_n) such that (x_{r_n}) converges to a point c in D.

Since $\lim_{n\to\infty} g_n(c) = 0$, $\forall \epsilon > 0$, $\exists k \in \mathbb{N}$, $\forall n \in \mathbb{N}$ such that

$$g_n(c) < \frac{\epsilon}{2}$$

Since the function g_k is continuos at c ,there exists a neighbourhood U of c and $\forall x \in U \cap D$ such that

$$|g_k(x) - g(c)| < \frac{\epsilon}{2}$$

It follows that $g_k(x) < \epsilon$ for all $x \in U \cap D$.

Since the subsequence (x_{r_n}) converges to the point c in D ,there exists a natural number k_1, $\forall n \geq k_1$ such that $x_{r_n} \in U \cap D$.Also there exists a natural number k_2 , $\forall n \geq k_2$ such that $r_n > k$.

Let $m = max(k_1, k_2)$.Then $g_{r_n}(x_{r_n}) < g_m(x_{r_n}) < \epsilon$ for all $n \geq m$.Therefore $\forall n \geq m$, $M_{r_n} < \epsilon$ .Hence the

sequence (M_{r_n}) converges to 0.

Since (M_n) is a monotone decreasing sequence having a convergent subsequence (M_{r_n}) with limit 0,the sequence M_n converges to 0.

Therefore the sequence of functions (g_n) converges uniformly on D and therefore the sequence of functions (f_n) converges uniformly to the function f on D.
This completes the proof.

Another proof(A): To prove this theorem wee need some theorems , lemmas.They are,
(I) Let (X, d) and (Y, d_1) be two metric spaces and let $f : X \to Y$ be a continuos map iff for all open sets V in Y, $f^{-}(V)$ open in X.

(III) Any closed subset of a compact set is compact.

By previous work we define a sequence of functions (g_n) on D which is monotone decreasing sequence of functions converges pointwise to a function g define by,

$$g(x) = 0, x \in D$$

Now we have to prove (g_n) converges uniformly on D.Let us choose $\epsilon > 0$.Let us consider a sequence of subsets of D say, (K_n) such that $K_n = \{x \in D : g_n(x) \geq \epsilon\}$ where $n \in \mathbb{N}$.Since the function g_n is continuos on D ,therefore the set K_n is a closed subset of D.Therefore by (I) and by (III) the set K_n is compact subset of D,it is also true for all $n \in \mathbb{N}$.There the sequence (K_n) is a sequence of compact subsets in D, also we get $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}$.Fix $x \in D$, since the sequence of functions (g_n) converges pointwise on D,there exists a natural number k, $\forall n \geq k$ such that

$$|g_n(x)| < \epsilon$$

AS the (K_n) is monotone decreasing ,then x does not belong the set K_{k-1} .Therefore the point x does not belong $\cap_{n=1}^{\infty} K_n$.As x is an arbitrary point in D,proceeding in this way we get $\cap_{n=1}^{\infty} K_n$ is empty .By (II) there exists a finite subcollection from the sequence (K_n) such that their intersection is empty.As these subsets are monotone ,there exists a natural number m, $\forall n \geq m$, $\forall x \in D$ such that

$$g_n(x) < \epsilon$$

.

AS $\epsilon > 0$ is arbitrary,hence the sequence of functions (g_n) converges uniformly the limit function g on D.Consequently the sequence of functions (f_n) converges uniformly to the limit function f on D.

This completes the proof.

In Dini's theorem compactness property are necessary ,if we drop the condition the sequence of functions (f_n) is not unifromly convergent on D.

Example 0.14. *Let us consider a sequence of functions (f_n) defined by ,*

$$f_n(x) = \frac{1}{1+nx}, for x \in (0,1)$$

Now the sequence of functions (f_n) converges pointwisely to a limit function f on $(0,1)$ defined by, $f(x) = 0, for\ x \in (0,1)$.Now the limit function f is continuos on $(0,1)$ and the sequence of functions (f_n) is monotone and continuos.But it does not converge uniformly on $(0,1)$.

UNIFORM CONVERGENCE OF A SEQUENCE OF INTEGRABLE FUNCTIONS:

Let $I = [a,b]$ be closed and bounded interval in $\mathbb{R}$ and for each $n \in \mathbb{N}$,let $f_n :\to \mathbb{R}$ be Riemann integrable on I.If the sequence of functions (f_n) be pointwise convergent on I to a function f then f may not be Riemann integrable on I.

Example 0.15. *Let $I = [0,1]$.Let $r_1, r_2, r_3, ...$ be an enumeration of the set of all rational points in I.Let us consider a sequence of function (f_n) defined by,*

$$f_n(x) = \begin{cases} 0 & x = r_1, r_2, r_3, ...r_n \\ 1 & x \in \ I - \{r_1, r_2, r_3, ...r_n\} \end{cases}$$

For all $n \in \mathbb{N}$,the function f_n is continuos on $[0,1]$ except only at m points.Therefore each f_n is Riemann integrable on $[0,1]$.Now the sequence of functions (f_n) converges pointwise to the function f on I defined by,

$$f(x) = \begin{cases} 0 & x \in [0,1] \cap \mathbb{Q} \\ 1 & x \in [0,1] - \mathbb{Q} \end{cases}$$

f is discontinuos at everypoint of [0,1].So f is not Riemann integrable on $[0,1]$

Let $I = [a,b]$ be a closed and bounded interval and for each $n \in \mathbb{N}$, let $f_n : I \to \mathbb{R}$ be integrable on I and the sequence of functions (f_n) converges pointwise to a function f which may not be integrable on I.

Example 0.16. *Let $f_n(x) = nxe^{-nx^2}, x \in [0,1]$. when $x = 0$ the sequence is $\{0, 0, 0......\}$.this converges to 0. When $0 < x \leq 1, e^{nx^2} > \frac{n^2.x^4}{2}$.For all $x \in [0,1]$, we have $0 < nxe^{-nx^2} < \frac{2}{nx^3}$.*

By Sandwich theorem, $\lim_{n\to\infty} nxe^{-nx^2} = 0, for\ x \in (0,1]$. therefore the sequence of functions (f_n) converges on $[0,1]$ to the function f defined by

$$f(x) = 0, x \in [0,1]$$

Each f_n is integrable on $[0,1]$ and f is also integrable on $[0,1]$. $\int_0^1 f_n(x)dx = [-\frac{1}{2}e^{-nx^2}]_0^1 = \frac{1}{2}(1 - e^{-n})$.
Now $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \frac{1}{2}(1 - e^{-n}) = \frac{1}{2}$.
Hence the sequence $(\int_0^1 f_n)$ converges to $\frac{1}{2}$ but $\int_0^1 f(x)dx = 0$.

Therefore $\lim_{n\to\infty}(\int_0^1 f_n)$ is not equal with $\int_0^1(\lim_{n\to\infty} f_n)$.

Theorem6: Let $I = [a,b]$ be a closed and bounded interval and for each $n \in \mathbb{N}$, $f_n : I \to \mathbb{R}$ be Riemann integrable function on I.If the sequence of functions (f_n) converges uniformly to a function f on I then f is Riemann integrable on I and moreover, the sequence $\{\int_a^b f_n\}$ converges to $\int_a^b f$.

Proof: Let us choose $\epsilon > 0$.Since the sequence of functions (f_n) is uniformly convergent on $[a,b]$ to the function f.Therefore $\exists k \in \mathbb{N}$, $\forall n \geq k$, $\forall x \in [a,b]$ such that,

$$|f_n(x) - f(x)| < \frac{\epsilon}{4(b-a)}$$

Therefore for all $x \in [a,b]$ and for $n = k$ we get $|f_k(x) - f(x)| < \frac{\epsilon}{4(b-a)}$.
Hence $f_k(x) - \frac{\epsilon}{2} < f(x) < f_k(x) - \frac{\epsilon}{2}$.......$(i)$ and this inequality holds for all $x \in [a,b]$.

Since the function f_k is integrable on $[a,b]$, there exists a partition $P = (x_0, x_1, x_2, ...x_n)$ of $[a,b]$ such that $U(P, f_k) - L(P, f_k) < \frac{\epsilon}{2}$.......$(ii)$.
Let

$$M_r = \sup_{x\in[x_{r-1},x_r]} f(x)$$

,

$$m_r = \inf_{x\in[x_{r-1},x_r]} f(x)$$

.

$$N_r = \sup_{x\in[x_{r-1},x_r]}$$

,

$$n_r = \inf_{x\in[x_{r-1},x_r]}$$

, where $r = 1, 2, 3......n$.

From (i) it follows that $m_r \geq n_r - \frac{\epsilon}{4(b-a)}$ and $M_r \leq N_r + \frac{\epsilon}{4(b-a)}$.
Now $U(P, f) = M_1(x_1 - x_2) + M_2(x_2 - x_1) + M_3(x_3 - x_2) + ... + M_n(x_n - x_{n-1}) \leq N_1(x_1 - x_0) + N_2(x_2 - x_1) + ... + N_n(x_n - x_{n-1}) + \frac{\epsilon}{4}$.

$L(P, f_k) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + ... + m_n(x_n - x_{n-1} \geq n_1(x_1 - x_0) + n_2(x_2 - x_1) + ... + n_n(x_n - x_{n-1}) - \frac{\epsilon}{4}$

Therefore $U(P, f) - L(P, f_k) \leq U(P, f_k) - L(P, f_k) + \frac{\epsilon}{2} < \epsilon$,....by using (ii)
This proves that f is Riemann integrable on $[a, b]$

Second part.
Let us choose $\epsilon > 0$.Since the sequence of functions (f_n) converges uniformly to the function f on $[a, b]$.Therefore $\exists k \in \mathbb{N}$, $\forall n \geq k$, $x \in [a, b]$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2(b-a)}$$

We have for all $n \geq k$,

$$|\int_a^b [f_n(x) - f(x)]dx| \leq \int_a^b |f_n(x) - f(x)|dx \leq \frac{\epsilon}{2(b-a)}.(b-a)$$

Therefore for all $n \geq k$, $|\int_a^b f_n(x)dx - \int_a^b f(x)dx| < \epsilon$.This implies that $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.
In the otherwords, the sequence $(\int_a^b f_n)$ converges to $\int_a^b f$.
This completes the proof.

Remarks: Symbolic notation will be $\lim_{n\to\infty}(\int_a^b f_n) = \int_a^b(\lim_{n\to\infty} f_n)$.
This says that if the convergence of the sequence of functions (f_n) be uniform on the interval $[a, b]$,then we can interchange the limit and the integration.

Corollary: For all $x \in [a, b]$, the sequence $(\int_a^x f_n)$ converges to $\int_a^x f$.

Note 0.17. *If each n the function f_n be integrable on $[a, b]$ and the sequence of functions (f_n) converges pointwise to a function f on $[a, b]$ which is also integrable on $[a, b]$,the uniform convergence of the sequence of functions (f_n) is a sufficient but not a necessary condition for the convergence of the sequence $(\int_a^b f_n)$ to $\int_a^b f$.*

Example 0.18. *Let us consider a sequence of functions* (f_n) *on a closed and bounded interval* $[0, 1]$ *defined by,*

$$f_n(x) = \frac{nx}{1 + n^2x^2}, x \in [0, 1]$$

Now the sequence of functions (f_n) *converges on* $[0, 1]$ *to the function* f *where* $f(x) = 0, x \in [0, 1]$.

Each n the function f_n *is integrable on* $[0, 1]$ *and also the function* f *is integrable on* $[0, 1]$.

Now $\int_0^1 f_n(x)dx = [\frac{1}{2n} \log(1 + n^2x^2)]_0^1 = \frac{1}{2n} \log(1 + n^2)$.

As the limit of function $\frac{\log(1+x^2)}{2x}$ *tends to* 0 *when* $x \to \infty$.*By sequential criterian for limits we get ,* $\lim\limits_{n\to\infty} \frac{\log(1+n^2)}{2n} = 0$.*Hence* $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = 0$ *and* $\int_0^1 f(x)dx = 0$

Thus the sequence $(\int_0^1 f_n)$ *converges to* $\int_0^1 f$ *but the convergence of the sequence of functions* (f_n) *is not uniform on* $[0, 1]$.

UNIFORM CONVERGENCE OF A SEQUENCE OF DIFFERENTIABLE FUNCTIONS.

Let us consider a sequence of functions (f_n) on $[a, b]$ such that for all $n \in \mathbb{N}$, the function f_n is differentiable on $[a, b]$.Let the sequence of function (f_n) be pointwise convergent to a function f on $[a, b]$.Then $\frac{d}{dx}f(x)$ may not exist for all $x \in [a, b]$.

Example 0.19. *Let* (f_n) *be a sequence of functions defined by* $f_n(x) = x^n, x \in [0, 1]$.*Then the limit function* f *is given by,*

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$$

Now for all $n \in \mathbb{N}$, $\frac{d}{dx}f_n(x)$ *exists for all* $x \in [0, 1]$.*But for the limit function* f *such that* $\frac{d}{dx}f(1)$ *does not exist.*

Let us consider a sequence of functions (f_n) on a closed and bounded interval $[a, b]$.Let the sequence of functions (f_n) converges pointwise to a function f on $[a, b]$ and $\frac{d}{dx}f(x)$ exists for all $x \in [a, b]$.Then the sequence of functions $(\frac{d}{dx}f_n)$ does not converge to the function $\frac{d}{dx}f$ on $[a, b]$.

Example 0.20. *Let us consider a sequence of functions* (f_n) *on* $[0, 1]$ *where*

$$f_n(x) = x - \frac{x^n}{n}, x \in [0, 1]$$

.

Therefore the sequence of functions (f_n) *converges pointwise to the function* f *is given by,*

$$f(x) = x, x \in [0, 1]$$

.

Now $\frac{d}{dx}f_n(x) = 1 - x^-n - 1$ *and* $\frac{d}{dx}f(x) = 1$ *for* $x \in [0,1]$.
For each $n \in \mathbb{N}$, $\frac{d}{dx}f_n$ *exits for all* $x \in [0,1]$.*Also* $\frac{d}{dx}f(x)$ *for all* $x \in [0,1]$.

Now

$$f_n(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & x = 1 \end{cases}$$

This shows that the sequence of functions $(\frac{d}{dx}f_n)$ *does not converge to the function* $\frac{d}{dx}f$ *on* $[0,1]$.

Theorem7: *Let* (f_n) *be a sequence of functions on a closed and bounded interval* $[a,b]$ *such that for all* $n \in \mathbb{N}$ *, the function* f_n *is differentiable on* $[a,b]$.*If the sequence of functions* $(\frac{d}{dx}f_n)$ *converges uniformly on* $[a,b]$ *and the sequence of functions* (f_n) *converges at least at one point* $x_0 \in [a,b]$*, then the sequence of functions* (f_n) *is uniformly convergent on* $[a,b]$ *and let limit function be* f*, then the sequence of functions* $(\frac{d}{dx}f_n)$ *converges to the function* $\frac{d}{dx}f$ *on* $[a,b]$.

Proof; Let us choose $\epsilon > 0$.*As the sequence of functions* (f_n) *converges at a point* $x_0 \in [a,b]$.*Therefore* $\exists k_1 \in \mathbb{N}$, $\forall m,n \geq k_1$ *such that,*

$$|f_m(x_0) - f_n(x_0)| < \frac{\epsilon}{2}.......(i)$$

and

The sequence of functions $(\frac{d}{dx}f_n)$ *converges uniformly on* $[a,b]$,*then by Cauchy criterion we get* $\exists k_2 \in \mathbb{N}$, $\forall m,n \geq k_2$, $\forall x \in [a,b]$ *such that,*

$$|\frac{d}{dx}f_m(x) - \frac{d}{dx}f_n(x)| < \frac{\epsilon}{2(b-a)}$$

Let $k = \max(k_1,k_2)$.*Now we apply the Mean value theorem (Lagrange) for the function* $f_m - f_n$ *on the interval* $[x,x_0]$ *or* $[x_0,x]$ *where* $x \in [a,b]$.*Hence we get,*

$$|f_m(x) - f_n(x) - f_m(x_0) + f_n(x_0)| \leq \frac{|x-x_0|\epsilon}{2(b-a)} \leq \frac{\epsilon}{2}.......(ii)$$

as $|x - x_0| < (b-a) for x \in [a,b]$.

Therefore $\forall x \in [a,b]$, $\forall m,n \geq k$, *the inequality becomes*

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_n(x) - f_m(x_0) + f_n(x_0)| + |f_m(x_0) - f_n(x_0)|$$

Therefore from (i) and (ii) and for arbitrary $\epsilon > 0$ *we get* $\exists k \in \mathbb{N}$, $\forall m, n \geq k$, $\forall x \in [a, b]$ *such that,*

$$|f_m(x) - f_n(x)| < \epsilon$$

Hence by Cauchy principle the sequence of functions (f_n) *is uniformly convergent on* $[a, b]$*.Let* f *be the limit function,then* $\lim_{n\to\infty} f_n(x) = f(x), for\ x \in [a, b]$.

Second part: Let us now fix a point $x \in [a, b]$ *and define*

$$g_n(t) = \frac{f_n(t) - f_n(x)}{t - x}$$

,

$$g(t) = \frac{f(t) - f(x)}{t - x}$$

For $a \leq t \leq b$, $t \neq x$*.Then*

$$\lim_{t\to x} g_n(t) = \frac{d}{dx} f_n(x), for\ n = 1, 2, 3, ...$$

Now by the inequality (ii) we get that $\exists k \in \mathbb{N}$, $\forall m, n \geq k$, $\forall x \in [a, b]$ *such that*

$$|g_m(t) - g_n(t)| \leq \frac{\epsilon}{2} < \epsilon$$

Therefore by Cauchy criterion the sequence of functions (g_n) *is uniformly convergent* $[a, b]$ *for* $t \neq x$*.Since the sequence of functions* (f_n) *converges to* f*, we conclude that*

$$\lim_{n\to\infty} g_n(t) = g(t)$$

uniformly for $t \in [a, b]$ *with* $t \neq x$.

Now by a consequence of a sequence of functions such that (f_n) *be a sequence of functions on* $D \subset \mathbb{R}$ *converges uniformly to a function* f *on* D*.Let* x_0 *be a limit point on* D *and* $\lim_{x\to x_0} f_n(x) = a_n$*.Then the sequence* (a_n) *converges and* $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} a_n$*.Hence from this result we get*

$$\lim_{t\to x} g(t) = \lim_{n\to\infty} \frac{d}{dx} f_n(x)$$

.This completes the proof.

Note 0.21. *For a sequence of functions* (f_n) *where each function* f_n *is differentiable on a closed and bounded interval* $[a, b]$ *and the sequence of functions* (f_n) *is pointwise convergent on* $[a, b]$*, the uniform convergence of the sequence of functions* $\frac{d}{dx} f_n$ *is only a sufficient but not a necessary condition for the uniform convergence of the sequence of functions on* $[a, b]$*.*

Example 0.22. *let* (f_n) *be a sequence of functions on* $[0, 1]$ *define by*

$$f_n(x) = x - \frac{x^n}{n}, x \in [0, 1]$$

Here the limit function f *is* $f(x) = x, x \in [0, 1]$*.Now the sequence of functions converges uniformly to the function* f *on* $[0, 1]$ *and* f *is differentiable on* $[a, b]$*.*

Now $\frac{d}{dx} f_n(x) = 1 - x^-n - 1, x \in [0, 1]$ *and* g *be the limit function for the sequence of functions* $(\frac{d}{dx} f_n)$ *define by,*

$$g(x) = \begin{cases} 1 & x \in [0, 1) \\ 0 & x = 1 \end{cases}$$

As the limit function g *is not continuous on* $[0, 1]$*.Therefore the sequence of continuous functions* $(\frac{d}{dx} f_n)$ *is not uniformly convergent on* $[0, 1]$*.*

Thus the sequence of functions (f_n) *is uniformly convergent on* $[0, 1]$ *but the sequence of functions* $(\frac{d}{dx} f_n)$ *is not uniformly convergent on* $[0, 1]$ *and our assertion is established.*

SOME IMPORTANT THEOREM ON UNIFORM CONVERGENCE AND APPLICATION.

A function $P : C[a, b] \to C[a, b]$ *is said to be positive linear if* $P(f) \geq 0$ *whenever* $f \geq 0$*.Now for* $f \leq g$ *we get* $P(f) \leq P(g)$*.*

Theorem (Korovkin, 1953) *Consider the three function on closed and bounded interval* $[a, b]$ *say* f_1, f_2, f_3 *such that*

$$f_1(t) = 1$$
$$f_2(t) = t$$
$$f_3(t) = t^2$$

for $t \in [a, b]$*.For each* $n \in \mathbb{N}$ *,let* $P_n : C[a, b] \to C[a, b]$ *be a positvie linear map.If the sequence of functions* $(P_n(f_1))$*,* $(P_n(f_2))$*,* $(P_n(f_3))$ *converge uniformly to the function*

f_1, f_2, f_3 respectively on $[a,b]$, then then sequence of functions $(P_n(f))$ converges uniformly to the function f on $[a,b]$ for every function $f \in C[a,b]$.

Proof: *Let $f \in C[a,b]$ be real valued function.Since f is bounde, there exists a positive real number α such that*

$$|f(t)| \leq \alpha$$

for all $t \in [a,b]$.For any two points $t, s \in [a,b]$,we have

$$-\alpha \leq f(t) \leq \alpha$$
$$-\alpha \leq f(s) \leq \alpha$$

This implies that

$$-2\alpha \leq f(t) - f(s) \leq 2\alpha........(i)$$

.

Let $\epsilon > 0$.Since f is uniformly continuous on $[a,b]$, there exists some $\delta > 0$ such that for $t, s \in [a,b]$ with $|t-s| < \delta$, we have

$$-\epsilon < f(t) - f(s) < \epsilon.......(ii)$$

.

Now fix $s \in [a,b]$ and consider the function $f_s(t) = (t-s)^2$, $t \in [a,b]$.Then for $|t-s| \geq \delta$, we have $f_s(t) \geq \delta^2$.Combinig the inequalities $|t-s| < \delta$ and $|t-s| \geq \delta$ we see for all $t \in [a,b]$ from $(i), (ii), (iii)$ such that,

$$-\epsilon - \frac{2\alpha f_s(t)}{\delta^2} \leq f(t) - f(s) \leq \epsilon + \frac{2\alpha f_s(t)}{\delta^2}$$

.

Since each P_n is positive and linear,we have

$$-\epsilon P_n(f_1) - \frac{2\alpha P_n(f_s)}{\delta^2} \leq P_n(f) - f(s)P_n(f_1) \leq \epsilon P_n(f_1) + \frac{2\alpha P_n(f_s)}{\delta^2}$$

By assumption, the sequence of functions $(P_n(f_1)(s)$ converges to 1 uniformly for $s \in [a,b]$.Also,since $f_s = f_3 - 2sf_2 + s^2 f_1$ we have

$$P_n(f_s)(s) = P_n(f_3)(s) - 2sP_n(f_2)(s) + s^2 P_n(f_1)(s).$$

Hence by assumption, $(P_n(f_s(s))$ converges $s^2 - 2s.s + s^2.1 = 0$ uniformly for $s \in [a,b]$.Thus the sequence $(P_n(f)(s))$ converges to $f(s)$ uniformly for $s \in [a,b]$.As the function f is arbitrary on $C[a,b]$.For all $f \in C[a,b]$ the sequence of functions $(P_n(f))$ converges uniformly to the function f on $[a,b]$.

This completes the proof.

Corollary (Weierstrass, 1885):
The set of all polynomil functions on $[a,b]$ is dense in $C[a,b]$ with respect to sup metric.

Proof:
Without loss of generality ,assume $a=0$ and $b=1$.For $n=1,2,......,$ let

$$B_n(f)(t)=\sum_{k=0}^{n} n_{c_k} t^k (1-t)^{n-k} f(\frac{k}{n})$$

for $t \in [0,1]$.Then each B_n is positive linear map on $C[0,1]$.Also we get that for all $n \in \mathbb{N}$,

$$B_n(f_1)=f_1$$
$$B_n(f_2)=f_2$$
$$B_n(f_3)=(1-\frac{1}{n})f_3+\frac{f_2}{3}$$

.

As the sequence of functions $(B_n(f_1))$,$(B_n(f_2))$,$(B_n(f_3))$ converges uniformly to the functions f_1,f_2,f_3 respectevly on $[0,1]$.Hence by Korovin's theorem we get that for all $f \in C[0,1]$ the sequence of functions $(B_n(f))$ converges uniformly to the function f on $[0,1]$.As B_n stands for Bernstein polynomial.This completes proof.

Theorem(Korovkin (ii)) *Let $X=\{f \in C[-\pi,\pi] : f(\pi)=f(-\pi)\}$.Consider $f_1(t)=1, f_2(t)=\cos(t), f_3(t)=\sin(t)$ for $t \in [-\pi,\pi]$.*
Let $P_n : X \to X$ be a positive linear map for $n=1,2,3$.......If the sequence of functions $(P_n(f_i))$ converges uniformly to the function f_i for $i \in \{1,2,3\}$ on $[-\pi,\pi]$,then the sequence of functions $(P_n(f))$ converges uniformly to the function f on $[-\pi,\pi]$ for all $f \in X$.

Proof: Now proof is same as before one.Here we consider the function $f_s(t)=\frac{\sin^2(t-s)}{2}$ instead of the function $f_s(t)=(t-s)^2$.

EQUICONTINUOS FAMILIES OF FUNCTIONS

We know that every bounded sequence of real numbers has a convergent subsequence.The question is that if a sequence of functions (f_n) *is bounded on a domain* $D \subset \mathbb{R}$.*Then what will be the subsequence of functions ,they converges or does not converge.We shall know from our next topic.*

Definition 4. *Let* (f_n) *be a sequence of functions defined on a set* D.

We say that the sequence of functions (f_n) *is pointwise bounded on* D *if the sequence* $(f_n(x))$ *is bounded for every* $x \in D$*, that is,if there exists a finite valued function* g *defined on* D *such that,*

$$|f_n(x)| < g(x)$$

for $x \in D$ *and* $n = 1, 2, ...$

we say the sequence of functions (f_n) *is uniformly bounded on* D *if there exists a number* $M > 0$ *such that*

$$|f_n(x)| < M$$

for $x \in D$ *and* $n = 1, 2, 3.....$

Now if the sequence of functions (f_n) *is pointwise bounded on* D *and* D_1 *is a countable subset of* D*,it is always possible to find a subsequence of functions* (f_{n_k}) *such that the subsequence* (f_{n_k}) *converges for every point* $x \in D_1$.*This can be done by diagonal process.*

However,even the sequence of continuous functions (f_n) *is a uniformly bounded on a compact set* D.*Then it need not exist a subsequence of functions on* D *which converges pointwise on* D.

Example 0.23. *Let* (f_n) *be sequence of continuous functions on a compact subset* $[0, \pi]$ *defined by,*

$$f_n(x) = \sin(nx)$$

Now the sequence of functions (f_n) *is uniformly bounded on* $[0, \pi]$ *but there does not exists a subsequence of functions* (f_{n_k}) *such that the subsequence* $(f_{n_k}(x))$ *converges for every* $x \in [0, \pi]$.

Definition 5. *A family* F *of real functions* f *defined on a set* D *in a metric space* X *is said to be equicontinuous on* D *if for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *with* $d(x, y) < \delta$,$x, y \in D$ *and* $\forall f \in F$ *such that*

$$|f(x) - f(y)| < \epsilon$$

.Here d *denotes the metric of* X*.It is clear that every member of an equicontinuous family is uniformly continuous.*

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