

Holographic computational complexity in non-relativistic field theories

Ankita Guptaroy

A Thesis Submitted to
Indian Institute of Technology Hyderabad
In Partial Fulfilment of the Requirements for
The Degree of Master of Science, Physics

Department of Physics

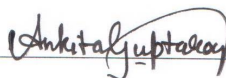


भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

May 8, 2018

Declaration

I declare that this written submission represents my ideas in my own words, and where ideas or words of others have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.



(Signature)

ANKITA GUPTARROY

(Ankita Guptaroy)

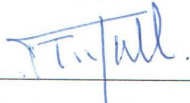
PH16MSEST11003

(Roll number)

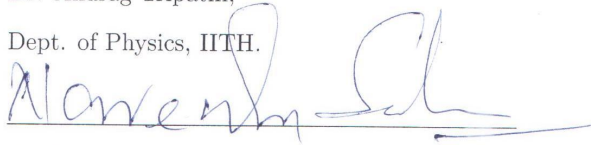
Approval Certificate

This is to certify that the thesis entitled **Holographic computational complexity in non-relativistic field theories** by *Ankita Guptaroy*, Roll no.: *ph16mscst11003*, is approved for the partial fulfillments for the degree of Master of Science, Physics, from IIT Hyderabad.

Examiners :



Dr. Anurag Tripathi,
Dept. of Physics, IITH.



Dr. Narendra Sahu,
Dept. of Physics, IITH.

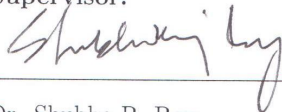


Dr. Srikanth Hundi,
Dept. of Physics, IITH.



Dr. Shantanu Desai,
Dept. of Physics, IITH.

Supervisor:



Dr. Shubho R. Roy,
Dept. of Physics, IITH

Acknowledgements

I sincerely express my gratitude for my advisor, Dr. Shubho R. Roy for introducing me to all the concepts in this project and helping me throughout the course of it. I've learned a lot in this process.

I would also like to thank Mr. Gaurav Katoch, PhD scholar, who helped me in doing and understanding a number of calculations of this project

I'm also thankful to all the faculty members of the department of Physics for helping me to learn the various concepts of the subject.

Finally I will thank my family and friends for always supporting me and encouraging me to carry out my work.

Abstract

The aim of the project is to study the holographic complexity of black holes and black branes. We discuss basic concept of complexity. We see how quantum computational complexity is related to black hole physics. In the process we look at the difference between classical and quantum complexity. We solve for the holographic complexity of pure AdS and pure Lifshitz spacetime. We then study the AdS black hole in detail. We first try to find the nature of the maximal volume slice at a finite time, and then see the small black hole limit of it using the perturbation method. We perform similar late time analysis on the Lifshitz black hole as well. Lastly we try to understand the notion of complexity from first principle calculations. We retrace the procedure of calculating the complexity of a system and try to compare with the holographic quantity. However, although the two approaches are similar in a lot of aspects, they don't match perfectly.

Contents

Declaration	i
Approval Certificate	i
Acknowledgements	i
Abstract	ii
1 Introduction	1
1.1 Introduction	1
2 Complexity	3
2.1 Computational Complexity	3
2.1.1 Comparison of Classical and Quantum Complexity	3
3 CV and CA Conjectures	6
3.1 CV and CA Conjectures	6
3.1.1 CV conjecture	6
3.1.2 CA Conjecture	7
4 AdS Spacetime	8
4.1 Warm up Example for CV conjecture, pure AdS in Poincarè coordinates	8
4.2 Pure AdS Black Brane in Poincare coordinates	10
4.3 AdS Black hole	12
4.3.1 Small AdS black hole as perturbation in mass over pure AdS	12

4.3.2	Late time slice	15
5	Lifshitz spacetime	17
5.1	Lifshitz geometry	17
5.2	Lifshitz Black Brane	20
5.3	Lifshitz Black Hole	21
5.3.1	Late time slice	21
5.3.2	Arbitrary d	22
6	QFT method of computing complexity	23
6.1	QFT method of computing complexity	23
7	Conclusion and Outlook	29
7.1	Conclusion and Outlook	29
	Appendices	33
A	Appendix : Mathematica code snippet	34

List of Figures

- 2.1 Complexity vs Time plot 5

- 3.1 Maximal Volume slice and WDW patch shown in case of a two sided AdS black hole. 7

- 4.1 The final maximal slice at a late time 16

Chapter 1

Introduction

1.1 Introduction

The aim of this project comprises multiple ideas. We try to understand this subject in general, and try to see some of the unsolve problems in this field. The basic idea of the project is based on the gauge-gravity duality. This Principle of Duality implies [?] that a theory, typically a gravity theory defined on a D dimensional space or spacetime(bulk) is equivalent to a theory, typically a gauge theory, defined on a $D - 1$ dimensional space or spacetime that forms the bulks boundary. This duality is said to be reflection of holography which means that the physics of the bulk is encoded adequately in the boundary. Both theories describe the same physics; however the advantage of using duality principle is that one theory's strong coupling regime where problems are hard to solve by usual methods can be translated into the other theory's weak coupling regime, where problems are easier to solve and vice-versa.

The most common example of this is the Anti de Sitter-Conformal Field Theory(AdS-CFT) conjecture. AdS spacetime is a maximally symmetric spacetime having negative curvature. Conformal Field Theories refer to theories that are invariant under a larger group of spacetime transformations called conformal transformations, which include the Poincarè transformations, dilatations and special conformal transformations. This conjecture thus states that gravity theory(String theory) on an Anti de Sitter spacetime of D dimensions is thus dual to a gauge theory(Conformal Field Theory) on its boundary. More specifically, the anti-de Sitter/conformal field theory correspondence relates $N = 4$ supersymmetric $SU(N)$ gauge theory to superstrings

in 10 dimensions [?].

The principle of duality has been generalized to non relativistic field theories. It has several applications in various branches of physics, such as hydrodynamics, physics of the quark/gluon plasma, variety of condensed matter systems(for example, to find the shear velocity of plasma), ranging from superconductors to (non) Fermi liquids. These studies in condensed matter physics can be related to non relativistic Lifshitz theories both of which have the same scaling symmetry.

Here we mainly focus on one particular application which is the relation between growth of Einstein-Rosen Bridges(ERBs) and Quantum Complexity. ERBs connect two entangled regions in space, as stated in the $ER = EPR$ paradox. They have the property that they eternally grow with time. However we know that all obvious properties of a chaotic system become static by the time the system thermalises. This property of ERBs which keeps growing is conjectured to be dual to quantum complexity.

Thus in our project we use the well founded conjecture of Susskind that relates the bulk geometry and complexity of the dual boundary.

In the first part of the thesis, we discuss the concepts of complexity in both classical and quantum regime. Then we take the case of AdS spacetime and perform the calculations of holographic complexity in the pure case as well as for black hole. Next we consider the Lifshitz spacetime and calculate the holographic complexity for both the pure case as well as for the black hole. Lastly we try and calculate the complexity from a first principle approach and see how it compares with the holographic value.

Chapter 2

Complexity

2.1 Computational Complexity

Computational Complexity is a notion from Computer Science and it deals with quantifying the difficulty of solving problems. The basic task is to generally start with a simple state and transform the system to some other state.

Classically, complexity of a state is by definition the minimum number of simple operations that are required to carry out the task. The basic ingredients are system, a space of states, simple state, simple operation, task. This task transforms the simple state to some target state, and the minimum number of steps give us the complexity.

2.1.1 Comparison of Classical and Quantum Complexity

To compare the classical and quantum point of view we take the very simple example of N spins. For the classical system, we have N classical spins which can exist in either UP state or DOWN state.

$$(\uparrow \downarrow \uparrow \uparrow \downarrow \dots)$$

We assume the reference state to have all spin DOWN : $(\downarrow \downarrow \downarrow \downarrow \dots)$

We define a simple operator to be the flipping of a spin. This operator will take the reference state to some generic state. Thus the complexity can be defined as the minimum number of simple operators that are required to carry out the task to convert to some generic state. It is thus very easy to see that the maximum complexity will be $\frac{N}{2}$. Since each spin can exist in two

possible states, the total number of states of the system is 2^N . Thus, the Boltzmann's Entropy is given by

$$S_{max} = N \ln 2.$$

We define the thermalisation time as the time taken by the system to achieve the maximum entropy, and complexity time as the time taken by the system to attain maximum complexity. Both these quantities are of the same order,

$$t_{therm} = t_{comp} < N^P$$

where P is some polynomial. The recurrence time is defined as the time after which the system comes back to its initial state after traversing the entire phase space and is given by,

$$t_{rec} = e^N.$$

Coming to the quantum case, the system will be N quantum spins. The difference from classical system is that here the system may have superposition of UP and DOWN states. The reference state remains the same

System : $(\uparrow \downarrow \uparrow \uparrow \downarrow \dots)$

Reference state : $(\downarrow \downarrow \downarrow \downarrow \dots)$

Here, the operator cannot be a simple flipping of spin. This is because, a single flip cannot create entanglement between two spins, which we refer to as quantum bits or qubits. So basically the operator has to be a 2-qubit unitary operator. This operator is referred to as a gate. We will see later how a sequence of gates form quantum circuits. The total number of possible states in this case is of the same order, that is 2^N and thus

$$S_{max} = N \ln 2.$$

The thermalisation time remains of the same order, but the complexity time is e^N . Therefore, the recurrence time increases as well and we have,

$$t_{rec} = e^{e^N}$$

which is practically unreachable.

The following figure shows the growth rate of complexity.

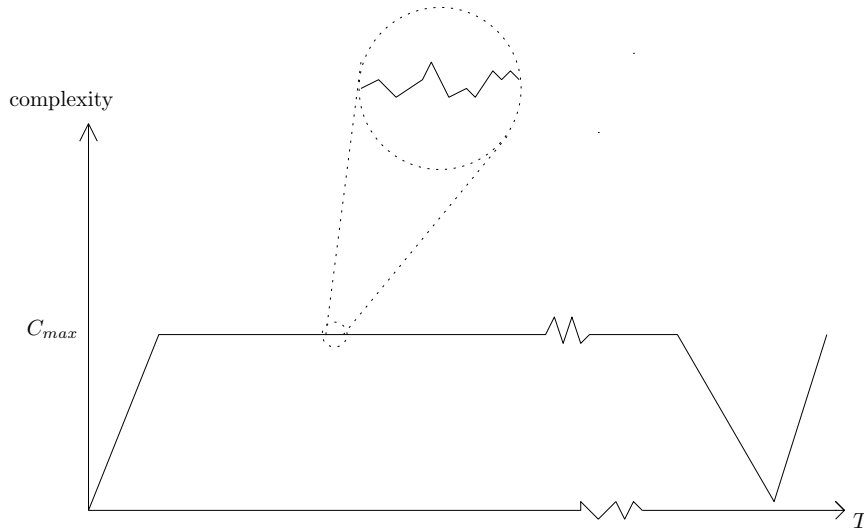


Figure 2.1: Complexity vs Time plot

The following table compares the values obtained for the same system in the classical and quantum contexts:

	CLASSICAL	QUANTUM
No. of States	2^N	$2^N \times \mathbb{C}$
Computational Complexity	$\mathcal{CC}_{max} = \frac{N}{2}$	$\mathcal{CC}_{max} = e^N$
Entropy	$S_{max} = N \log 2$	$S_{max} = N \log 2$
Thermalization time	$t_{therm} < N^P$	$t_{therm} < N^P$
Complexity time	$t_{compl} < N^P$	$t_{comp} = e^N$
Recurrence time	$t_{rec} = e^N$	$t_{rec} = e^{e^N}$

We observe that for the classical case, the thermalisation time and the complexity time are of same order. However for the quantum case, the two quantities are different in scale. Unlike the classical case, the system still continues to evolve even after reaching maximum entropy which suggests that some changes on the quantum scale are still taking place within the system. The recurrence time for the quantum case is a huge number, and practically the system never reaches this state.

Chapter 3

CV and CA Conjectures

3.1 CV and CA Conjectures

The growth of ERBs have been linked to the rate of change in the complexity. There are two different approaches to evaluate this complexity. The complexity-volume(CV) conjecture and the complexity-action(CA) conjecture.

3.1.1 CV conjecture

The Complexity-Volume(CV) Conjecture states [?] that the complexity of the boundary state is dual to the volume of the extremal hyper-surface bulk hyper-surface which meets the asymptotic boundary on the desired time slice. More precisely the CV duality states that the complexity of a state on a time slice Σ is given by

$$C_V(\Sigma) = \max_{\Sigma=\partial B} \left[\frac{V(B)}{G_N l} \right] \quad (3.1)$$

where B is the corresponding bulk surface and l is some length scale associated with the bulk geometry.

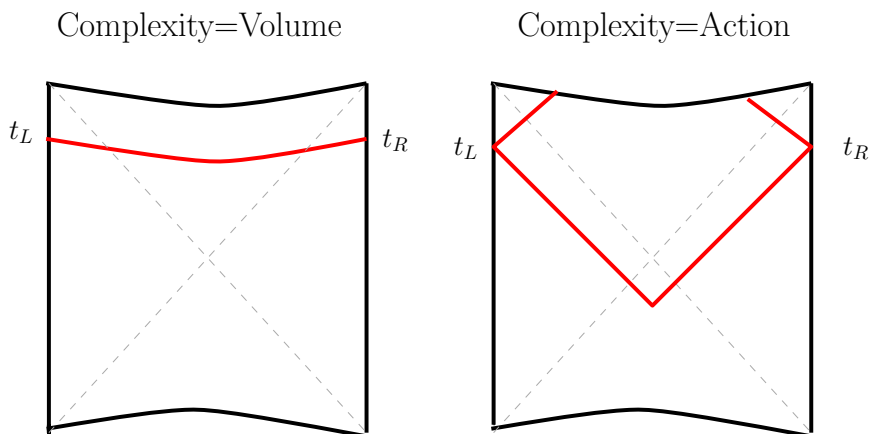


Figure 3.1: Maximal Volume slice and WDW patch shown in case of a two sided AdS black hole.

3.1.2 CA Conjecture

However, there is an ambiguity in choosing the length scale appearing in the CV conjecture and thus this led to the motivation for developing the CA conjecture which does not involve any free parameter like l . The complexity-action(CA) conjecture [?] equates the complexity with the gravitational action evaluated on a particular bulk region, now known as the Wheeler-DeWitt (WDW) patch:

$$C_A(\Sigma) = \frac{I_{WDW}}{\pi h}. \quad (3.2)$$

The WDW patch can be defined as the domain of dependence of any Cauchy surface in the bulk which asymptotically approaches the time slice Σ on the boundary.

Chapter 4

AdS Spacetime

4.1 Warm up Example for CV conjecture, pure AdS in Poincaré coordinates

A d -dimensional anti-de Sitter space (AdS) is a maximally symmetric Lorentzian manifold with constant negative scalar curvature. As a warm up example we consider the Pure AdS in Poincaré coordinates, the metric of which in $d + 1$ dimensions is given as

$$ds^2 = \frac{l^2}{z^2} dz^2 + \frac{l^2}{z^2} dx^2 - \frac{l^2}{z^2} dt^2. \quad (4.1)$$

We assume the following function that we need to determine

$$t = f(z, x) \implies t = f(z)(x - \text{translation invariant}) \quad (4.2)$$

and then we construct the induced metric as follows.

$$ds^2 = \left[\left(1 - \left(\frac{df}{dz} \right)^2 \right) \frac{l^2}{z^2} \right] dz^2 + \frac{l^2}{z^2} dx^2. \quad (4.3)$$

From this metric we take the coefficients and compute the volume element as follows

$$V = \int d^{d-1}x \int dz \left(\frac{l}{z} \right)^d \sqrt{1 - \left(\frac{df}{dz} \right)^2}. \quad (4.4)$$

We extremise this quantity using calculus of variation and infer that

$$\begin{aligned} f' &= 0, \\ \Rightarrow f &= \text{constant}. \end{aligned}$$

With $f' = 0$,

$$V = (2\Lambda)^{d-1} \int_{\epsilon}^{\infty} dz \left(\frac{l}{z}\right)^d \quad (4.5)$$

or

$$V = \frac{2\Lambda^{d-1} l^d}{(d-1)\epsilon^{d-1}} \quad (4.6)$$

where 2Λ is the volume of the $d-1$ dimensional hypersphere Therefore the Complexity turns out to be

$$C_V = \frac{1}{d-1} \left(\frac{2\Lambda}{\epsilon}\right)^{d-1} \frac{l^{d-1}}{G_N}. \quad (4.7)$$

Here we have considered ϵ as the UV cutoff parameter. The quantity 2Λ gives the extensive size of the system and acts as the IR cutoff. Therefore $\frac{IRcutoff}{UVcutoff}$ gives the value of total number of lattice sites. Furthermore, $\frac{l^{D-2}}{G_N} \propto N^2$, which gives the number of degrees of freedom for each lattice site. Thus complexity is an extensive quantity and it gives us a measure of the total number of degrees of freedom of the system as we intuitively expect [?].

4.2 Pure AdS Black Brane in Poincare coordinates

Next we consider the example of a black brane in the AdS spacetime. In general relativity, a black brane is a solution of the equations that generalises a black hole solution but it is also extended, and translationally symmetric. The metric in $d + 1$ dimensions is given by

$$ds^2 = \frac{L^2}{z^2} \left(\frac{dz^2}{f(z)} - f(z) dt^2 + dx_i^2 \right) \quad (4.8)$$

where

$$f(z) = 1 - \tilde{M} \frac{z^d}{z_0^d}, \quad (4.9)$$

$$\tilde{M} = \frac{M}{L^d}. \quad (4.10)$$

Like before we assume some function g as follows and get the induced metric

$$t = g(z, x_i) = g(z), \quad (4.11)$$

$$ds^2 = \frac{L^2}{z^2} \left(\frac{1}{f(z)} - f(z) \left(\frac{dg}{dz} \right)^2 \right) dz^2 + \frac{L^2}{z^2} dx_i^2. \quad (4.12)$$

The volume is therefore given by,

$$V = \int d^{d-1} x_i \int dz \left(\frac{L}{z} \right)^d \left(\frac{1}{f(z)} - f(z) \left(\frac{dg}{dz} \right)^2 \right)^{\frac{1}{2}}. \quad (4.13)$$

Using calculus of variation, we obtain the following condition

$$2fzg'' + g'[3f'z - 2df] + g'^3[2df^3 - f^2f'z] = 0. \quad (4.14)$$

Putting $f = 1, f' = 0$ we get the relation for pure AdS approximation

$$zg'' + dg'(g'^2 - 1) = 0. \quad (4.15)$$

We next try to solve this equation by the power series method which gives us the solution

$$g = T + \alpha z \tag{4.16}$$

where T is the initial time at which the slice is anchored. From our power series solution α can only take values 0, +1,-1. If $\alpha = 0$ then our slice is space-like and the volume turns out positive and thus the only possible solution. Otherwise for $\alpha = +1$ or -1 , the solution is null and volume becomes zero. Therefore we get a constant time slice as the maximal slice.

4.3 AdS Black hole

An anti-de Sitter (AdS) black hole is a black hole solution of general relativity or its extensions which represents an isolated massive object, but with a negative cosmological constant. Such a solution asymptotically approaches anti-de Sitter space at spatial infinity, and is a generalisation of the Kerr vacuum solution, which asymptotically approaches Minkowski spacetime at spatial infinity. We first take the small black hole limit, and solve for the maximal slice using the perturbation method and then try to probe the slice at a late time.

4.3.1 Small AdS black hole as perturbation in mass over pure AdS

The metric for AdS Black Hole is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2 \quad (4.17)$$

where,

$$f(r) = 1 + \frac{r^2}{l^2} - mr^{3-D}. \quad (4.18)$$

Since there are logarithmic branch cuts across the horizon, this metric is not defined across the horizon. To avoid this difficulty, we introduce the ingoing Eddington-Finkelstein coordinates and replace with the new coordinate v as follows

$$v = t + r^* \quad (4.19)$$

where

$$r^* = \int_{\infty}^r \frac{dr'}{f(r')}, \quad (4.20)$$

where plugging in the zeroth solution of $f(r)$ we get

$$r^* = \int_{\infty}^r \frac{dr'}{1+r^2} = \tan^{-1}(r) - \frac{\pi}{2}. \quad (4.21)$$

$$v - \int_{\infty}^{\infty} \frac{dr'}{f(r')} = a. (\text{say}) \quad (4.22)$$

Therefore,

$$v = a = v^*. \quad (4.23)$$

Thus the complete solution is given by

$$v(r) = v^* + \tan^{-1}(r) - \frac{\pi}{2}. \quad (4.24)$$

Using this new coordinates, the metric is written as

$$ds^2 = - \left(f(r) + 2 \frac{dr}{dv} \right) dv^2 + r^2 d\Omega_{D-2}^2. \quad (4.25)$$

The Volume is given by

$$V = \int d^{D-2}\Omega \int r^{D-2} \sqrt{-f(r) + 2 \frac{dr}{dv}} dv. \quad (4.26)$$

To find the zeroth order solution first we assume $v = g(r)$. In this case the metric becomes,

$$ds^2 = (2g(r)' - f(r)g(r)'^2) dr^2 + r^2 d\Omega_{D-2}^2. \quad (4.27)$$

Therefore the volume is given by

$$V = \omega_{D-2} \int dr r^{D-2} \sqrt{2g' - fg'^2}. \quad (4.28)$$

For simplicity we put $D = 3$. Solving for the extremal volume and using calculus of variation, we get

$$\begin{aligned} & -rg''(r) + 2r^4g'(r)^3 + \epsilon(-3r^2g'(r)^3 - 2g'(r)^3 + 3g'(r)^2) \\ & + 3r^2g'(r)^3 - 6r^2g'(r)^2 + \epsilon^2g'(r)^3 + g'(r)^3 - 3g'(r)^2 + 2g'(r) = 0. \end{aligned} \quad (4.29)$$

This equation is naturally satisfied by plugging in the solution $v(r) = v^* + \tan^{-1}(r) - \frac{\pi}{2}$ and putting $\epsilon = 0$.

We introduce the dimensionless quantity $\epsilon = l^{D-3}m$. For simplicity we put $D = 3$ and $l = 1$ without loss of generality. To solve for the first order in $g(r)$, we take a perturbation in ϵ as follows

$$g(r) = g_0(r) + \epsilon g_1(r). \quad (4.30)$$

We know that the zeroth order solution is given by $g_0(r) = v^* + \tan^{-1}(r) - \frac{\pi}{2}$, (giving the pure AdS case). Since ϵ is infinitesimal, we ignore higher order terms and get on simplification

$$-r^7 g_1'' - 3r^5 g_1'' - 3r^3 g_1'' - r g_1'' - 4r^6 g_1' - 9r^4 g_1' - 6r^2 g_1' - g_1' + 1 = 0. \quad (4.31)$$

This equation has the following solution

$$g_1(r) = \frac{r}{2(1+r^2)} + \frac{1}{2}\tan^{-1}(r) + \frac{c_1}{\sqrt{1+r^2}} + c_2 + c_1 \log \left[\frac{r}{1+\sqrt{1+r^2}} \right]. \quad (4.32)$$

We put $v^* = T$, as the initial time at which the slice is anchored at the boundary ($r = \infty$). Therefore the complete solution is given by

$$v(r) = T + \tan^{-1}(r) - \frac{\pi}{2} + \epsilon \left(\frac{r}{2(1+r^2)} + \frac{1}{2}\tan^{-1}(r) + \frac{c_1}{\sqrt{1+r^2}} + c_2 + c_1 \log \left[\frac{r}{1+\sqrt{1+r^2}} \right] \right). \quad (4.33)$$

We find the constants c_1 and c_2 by applying the boundary condition $v(r) = T$ when $r \rightarrow \infty$. This gives

$$\lim_{r \rightarrow \infty} v(r) = T + \epsilon \left(\frac{1}{2r} + \frac{\pi}{4} + c_1 \left[\frac{1}{r} + \log(1) \right] + c_2 \right) = T = T + \epsilon \left(\frac{1}{r} \left[c_1 + \frac{1}{2} \right] + c_2 + \frac{\pi}{4} \right), \quad (4.34)$$

$$c_1 = -\frac{1}{2},$$

$$c_2 = -\frac{\pi}{4}.$$

Therefore the complete solution is given by

$$v(r) = T + \tan^{-1}(r) - \frac{\pi}{2} + \epsilon \left(\frac{r}{2(1+r^2)} + \frac{1}{2}\tan^{-1}(r) - \frac{1}{2\sqrt{1+r^2}} - \frac{1}{2} \log \left[\frac{r}{1+\sqrt{1+r^2}} \right] - \frac{\pi}{4} \right), \quad (4.35)$$

Thus we see that the maximal slice is not effectively a flat one, and there are some undulations.

4.3.2 Late time slice

We then try to study the constant t slice at a late time. The metric for AdS black hole is given by [?]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\omega_{D-2}^2 \quad (4.36)$$

where for a three dimensional BTZ black hole ($D=3$)

$$f(r) = \frac{1}{l^2}(r^2 - \mu^2), \quad (4.37)$$

and

$$\mu^2 = 8GMl^2. \quad (4.38)$$

where M is the mass of the black hole. The volume of a constant t slice is given by (we substitute $dr = 0$, because, r turns time-like inside the horizon)

$$V = 2\omega_{D-2}r_f^{D-2}\sqrt{|f(r_f)|}t \quad (4.39)$$

where ω_n is the volume of a n dimensional hypersphere and

$$r_f = \frac{\mu}{\sqrt{2}} \quad (4.40)$$

where r_f being the final slice radius is obtained from maximising the volume, that is, by solving $dg(r_f)/dr = 0$, $g(r) = f(r)r^{2(D-2)}$.

Therefore, the volume ($D = 3$ case) is

$$V = \frac{2\pi\mu^2t}{l} \quad (4.41)$$

and the complexity is given by

$$C = \frac{V}{Gl} = 16\pi Mt. \quad (4.42)$$

The following figure represents the Penrose diagram for BTZ foliated by maximal surfaces. The green curve is the final slice, which the other purple curves approach with time.

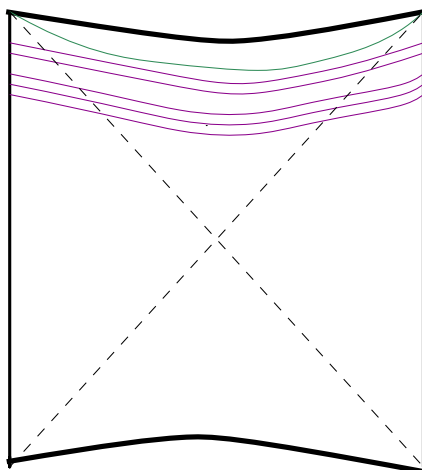


Figure 4.1: The final maximal slice at a late time

Therefore,

$$\frac{dC}{dt} = 16\pi M. \tag{4.43}$$

We see that the complexity is linearly proportional to time as expected from our intuitions.[?]

Chapter 5

Lifshitz spacetime

5.1 Lifshitz geometry

We start with the metric of pure Lifshitz spacetime [?], in $d + 1$ dimensions,

$$ds^2 = \frac{dr^2}{r^2} + \frac{dx^i dx_i}{r^2} - \frac{dt^2}{r^{2z}}. \quad (5.1)$$

The Lifshitz metric is characterised by a scaling symmetry given by,

$$\begin{aligned} x &\longrightarrow \lambda x \\ t &\longrightarrow \lambda^z t \end{aligned}$$

where z is called the dynamical scaling exponent.

Like in the previous case, we assume the following function that we need to determine.

$$t = f(r, x_i) = f(r), \quad (5.2)$$

and get the induced metric as

$$ds^2 = \frac{dx^i dx_i}{r^2} + \left(\frac{1}{r^2} - \frac{f'^2}{r^{2z}} \right) dr^2. \quad (5.3)$$

The volume is therefore given by,

$$V = \int \int \frac{d^{d-1}x}{r^{d-1}} dr \sqrt{\frac{1}{r^2} - \frac{f'^2}{r^{2z}}}. \quad (5.4)$$

To find the maximal volume, we use calculus of variation, and get the following condition for our function f

$$f'^2 = \frac{r^{2(z-1)}a^2}{a^2 + r^{-2z-2d}}. \quad (5.5)$$

Therefore we have

$$\frac{1}{r^2} - \frac{f'^2}{r^{2z}} = \frac{a^2 + r^{-2z-2d} - r^{-2z-2d}a^2}{a^2 + r^{-2z-2d}} \quad (5.6)$$

or

$$\frac{1}{r^2} - \frac{f'^2}{r^{2z}} = \frac{1}{r^2} \frac{r^{-2z-2d}}{(a^2 + r^{-2z-2d})}. \quad (5.7)$$

For extremum value, the denominator has to be minimum, which implies $\rightarrow a^2 = 0$ or $a^2 < 0$ (complex quantity).

We have the following two cases: Case 1 : $a^2 = 0$

$$\Rightarrow f'^2 = 0 \Rightarrow f = \text{const} = t.$$

$$V = \int d^{d-1}x \int_{\epsilon}^{\infty} \frac{dr}{r^d} = \frac{L^d}{d-1} \left(\frac{2\Lambda}{\epsilon} \right)^{d-1}. \quad (5.8)$$

Case 2 : $a^2 < 0 \Rightarrow$ imaginary $a = i\alpha$

$$f'^2 = \frac{-r^{2(\Delta-1)}\alpha^2}{-\alpha^2 + r^{-2\Delta-2d}} \quad (5.9)$$

or

$$\frac{1}{r^2} - \frac{f'^2}{r^{2\Delta}} = \left[\frac{1}{1 - \frac{\alpha^2}{r^{-2\Delta-2d}}} \right]. \quad (5.10)$$

Again, the volume will be maximum when the denominator is minimum, that is,

$$\begin{aligned} & 1 - \frac{\alpha^2}{r^{-2\Delta-2d}} \text{ is minimum,} \\ \Rightarrow \alpha^2 r^{2\Delta+2d} \leq 1 & \Rightarrow \alpha^2 \leq 1/r^{2\Delta+2d} \Rightarrow \alpha = 0 \end{aligned}$$

The final volume thus match with the pure AdS case

$$V = \frac{2\Lambda^{d-1}l^d}{(d-1)\epsilon^{d-1}}, \quad (5.11)$$

where 2Λ is the volume of the $d-1$ dimensional hypersphere and as before ϵ is the UV cutoff.

Therefore the Complexity turns out to be

$$C_V = \frac{1}{d-1} \left(\frac{2\Lambda}{\epsilon} \right)^{d-1} \frac{l^{d-1}}{G_N}. \quad (5.12)$$

We thus note that this result matches exactly with the complexity value for the pure AdS case. Moreover, we see that the holographic complexity is independent of the scaling factor z . This may be because of the fact that our maximal volume slices are constant t slices. Since the scaling factor z is responsible for the scaling of time only and not the space, it doesn't contribute to the dynamics of complexity.

5.2 Lifshitz Black Brane

We next consider the Lifshitz Black Brane The metric is given by :

$$ds^2 = \frac{l^2}{r^2} \frac{dr^2}{b_0} - \frac{r^{2z}}{l^{2z}} b_0 dt^2 + \frac{r^2}{l^2} dx_{d-1}^2, \quad (5.13)$$

where

$$b_0 = 1 - mr^{-(d+z+1)}. \quad (5.14)$$

z being the dynamic critical exponent Like in the previous cases, we assume an undetermined function

$$t = f(r, x_i) = f(r). \quad (5.15)$$

Then the induced metric is given by

$$ds^2 = \left[\frac{l^2}{r^2} \frac{1}{b_0} - \frac{r^{2z}}{l^{2z}} b_0 \left(\frac{df}{dr} \right)^2 \right] dr^2 + \frac{r^2}{l^2} dx_{d-1}^2. \quad (5.16)$$

The expression for volume thus comes out to be

$$V = \int d^{d-1}x \int dr \left(\frac{r}{l} \right)^{d-2} \left(\frac{1}{b_0} - \left(\frac{r}{l} \right)^{2z+2} b_0 \left(\frac{df}{dr} \right)^2 \right)^{\frac{1}{2}}. \quad (5.17)$$

We extremise the volume and using the calculus of variation, we obtain the following expression

$$f'' + f'^3 \left(\frac{1}{2} b_0 l^{-2(1+z)} r \left(2b_0 r^{2z} (1+z) - l^{2z} \left(\frac{r}{l} \right)^{2z} (b_0' r + 2b_0 (d+2z)) \right) \right) + f' \left(\frac{3}{2} \frac{b_0'}{b_0} + \frac{d+2z}{r} \right), \quad (5.18)$$

$b_0 = 1$ gives back the pure Lifshitz case. In addition, if we put $z = 1$ we get the pure AdS space

$$uf'' - df'(1 - f'^2) = 0 \quad (5.19)$$

with $u = \frac{1}{r}$ and for which we already have the solution as given in section 4.2

5.3 Lifshitz Black Hole

5.3.1 Late time slice

The metric is given by [?]

$$ds^2 = \frac{l^2}{b_k} \frac{dr^2}{r^2} - \frac{r^{2z}}{l^{2z}} b_k dt^2 + r^2 d\Omega_{k,d-1}^2, \quad (5.20)$$

where

$$b_k = k \left(\frac{d-2}{d+z-3} \right)^2 \frac{l^2}{r^2} + 1 - mr^{-(d+z-1)}. \quad (5.21)$$

The volume of a constant t slice is given by

$$V = 2\omega_{D-2} r_f^{D-2} \sqrt{|f(r_f)|} t \quad (5.22)$$

as before, and in this case

$$g(r) = r^{2(d-1)} \frac{r^{2z}}{l^{2z}} b_k. \quad (5.23)$$

Thus $dg(r_f)/dr = 0$ gives

$$r_f = \left(\frac{m}{2} \right)^{\frac{1}{z+1}} \quad (5.24)$$

as the final slice radius. Therefore after substituting the value, the volume is

$$V = \frac{4\pi}{l^z} \left(\frac{m}{2} \right) t. \quad (5.25)$$

The relation between m and the mass M of the black hole is given by [?]

$$M = \frac{V_{d-1}}{16\pi G_{d+1}} m l^{-1-z} (d-1) \quad (5.26)$$

which has been calculated from the fundamental relation $dM = TdS$. For $d = 2$, we get

$$m = \frac{16\pi G M l^{z+1}}{2\pi}. \quad (5.27)$$

Substituting this, we get the complexity

$$C = 16\pi Mt, \quad (5.28)$$

and thus

$$\frac{dC}{dt} = 16\pi M \quad (5.29)$$

Thus we see that just like in the pure Lifshitz case, the rate of change of complexity is independent of the scaling factor z . This confirms the same interpretation that since we are considering a constant t slice, the factor z doesn't contribute to the dynamics of the complexity. Hence the final expressions are independent of z .

5.3.2 Arbitrary d

We try and compute this same complexity, now for arbitrary d . In the large black hole limit, the horizon radius is greater than the AdS radius and we have

$$b = 1 - mr^{-(d+z-1)}. \quad (5.30)$$

Therefore, on solving for the final slice radius, we have

$$r_f = \exp\left(\frac{\log(-dml^{-2z} + ml^{-2z} - mzl^{-2z}) - \log(2dl^{-2z} + 2zl^{-2z} - 2l^{-2z}) - i\pi}{d+z-1}\right) \quad (5.31)$$

on solving which r_f simplifies to

$$r_f = \left(\frac{m}{2}\right)^{\frac{1}{-1+d+z}}. \quad (5.32)$$

Thus the volume becomes

$$V = \frac{\pi^{\frac{d-1}{2}} t 2^{\frac{d-2(d+1)+z+3}{d+z-1}} l^{-z} m^{\frac{2(d-1)}{d+z-1}} \left(2^{-\frac{1}{d+z-1}} m^{\frac{1}{d+z-1}}\right)^z \sqrt{1 - m \left(2^{-\frac{1}{d+z-1}} m^{\frac{1}{d+z-1}}\right)^{-d-z+1}}}{\Gamma\left(\frac{d+1}{2}\right)}. \quad (5.33)$$

Chapter 6

QFT method of computing complexity

6.1 QFT method of computing complexity

In this section, we review the concept of complexity from a different perspective. We try to see how the calculation of complexity from first principle calculations compare with those obtained from holographic considerations. The method suggested by Robert Myers,[?] is retraced although this process doesn't give us the expected results.

We take the case of simple harmonic oscillator, a bosonic oscillator in particular and proceed with a field theory approach of calculating the complexity. The idea is to consider two coupled oscillators, find the complexity and extend the derivation to an infinite number of them.

The hamiltonian of a free scalar field is given by

$$H = \frac{1}{2} \int d^{d-1}x [\pi(x)^2 + \nabla\phi(x)^2 + m^2\phi(x)^2]. \quad (6.1)$$

We can then apply the concept of lattice QFT to regulate the hamiltonian on a lattice and thus get an approximation of an ordinary continuum field theory

$$H = \frac{1}{2} \sum_n \left[\frac{P(n)^2}{\delta^{d-1}} + \delta^{d-1} \left(\frac{1}{\delta^2} \sum_i (\phi(n) - \phi(n - x_i))^2 + m^2\phi(n)^2 \right) \right]. \quad (6.2)$$

Now the problem is converted to an infinite number of coupled harmonic oscillators. For simplification, we consider two coupled harmonic oscillators and write the hamiltonian in terms of the normal modes

$$H = \frac{1}{2} [p_+^2 + \omega_+^2 x_+^2 + p_-^2 + \omega_-^2 x_-^2]. \quad (6.3)$$

where $x_{+-} = \frac{1}{\sqrt{2}}(x_+ \pm x_-)$ and

$$\omega_+^2 = \omega^2, \quad \omega_-^2 = \omega^2 + 2\Omega^2$$

Next we construct the wave function by taking the product of the two normal mode wave functions as follows,

$$\Psi_0(x_+, x_-) = \frac{(\omega_+ \omega_-)^{\frac{1}{4}}}{\sqrt{\pi}} \exp \left[-\frac{1}{2} (\omega_+ x_+^2 + \omega_- x_-^2) \right], \quad (6.4)$$

or

$$\Psi_0(x_1, x_2) = \frac{(\omega_+ \omega_-)^{\frac{1}{4}}}{\sqrt{\pi}} \exp \left[-\frac{1}{4} (\omega_+ (x_1 + x_2)^2 + \omega_- (x_1 - x_2)^2) \right]. \quad (6.5)$$

We fix our target wave function to be

$$\Psi_T = \frac{(\omega_1^2 - \beta^2)^{\frac{1}{4}}}{\sqrt{\pi}} \exp \left[-\frac{1}{2} \omega_1 x_1^2 - \frac{1}{2} \omega_1 x_2^2 - \beta x_1 x_2 \right], \quad (6.6)$$

We can basically choose any form of the function for both the reference and target states. However for easier calculations, we choose a similar factorised Gaussian for the reference state

$$\Psi_R = \exp \left[-\frac{1}{2} \omega_0 x_1^2 - \frac{1}{2} \omega_0 x_2^2 \right] \quad (6.7)$$

where ω_0 has to be determined.

In this state the two masses are not entangled. Our next task is to construct gates that will take the reference state to the target state. The number of such gates required for an optimal design will roughly give us the complexity. Robert Myers defines the following five simple gates in his work, that is made from the natural quantum mechanical operators x and p .

$Q_{00} = \exp[i\epsilon x_0 p_0]$ - adds a small phase, ϵ being an infinitesimal parameter.

$Q_{0i} = \exp[i\epsilon x_0 p_i]$ - shifts x_i by ϵx_0 .

$Q_{i0} = \exp[i\epsilon x_i p_0]$ - shifts p_i by ϵp_0 .

$Q_{ij} = \exp[i\epsilon x_i p_j]$ - shifts x_j by ϵx_i (Entangling gate).

$$\begin{aligned} Q_{ii} &= \exp \left[i \frac{\epsilon}{2} (x_i p_i + p_i x_i) \right] \\ &= e^{\frac{\epsilon}{2}} \exp[i\epsilon x_i p_j]. \text{ - rescales } x_i \text{ to } e^\epsilon x_i \text{ (Scaling gate)}. \end{aligned}$$

The process of taking the reference state to the target state can be represented as

$$\Psi_T(x_1, x_2) = Q_{22}^{\alpha_3} Q_{21}^{\alpha_2} Q_{11}^{\alpha_1} \Psi_R(x_1, x_2) \quad (6.8)$$

where $\alpha_1, \alpha_2, \alpha_3$ represent the number of each type of gates required. Thus the complexity is given by

$$C = |\alpha_1| + |\alpha_2| + |\alpha_3| = \frac{1}{2\epsilon} \log \left[\frac{\omega_1^2 - \beta^2}{\omega_0^2} \right] + \frac{|\beta|}{\epsilon} \sqrt{\frac{\omega_0}{\omega_1}} (\omega_1^2 - \beta^2)^{-\frac{1}{2}}. \quad (6.9)$$

Now, there is no way to infer from this expression if this represents the most optimal circuit or not Myers then shifts to the approach adopted by Nielsen [?][?] where he states that it is easier to work with smooth functions in smooth space rather than discrete ones. He then defines the function as

$$\Psi_T = U \Psi_R \quad (6.10)$$

such that

$$U = P \exp \left[\int_0^1 ds Y^I(s) O_I \right], \quad (6.11)$$

where

$$O_{11} = \frac{i}{2} (x_1 p_1 + p_1 x_1), O_{12} = i x_1 p_2 \quad (6.12)$$

$$O_{21} = i x_2 p_1, O_{22} = \frac{i}{2} (x_2 p_2 + p_2 x_2) \quad (6.13)$$

where P is the path ordering operator, s is like some position label of the gates, Y^I is some function determining which gate in the sequence is on or off. We can define $Y^I(s)$ as

$$Y^I(s) = Tr[\delta_s U(s) U^{-1}(s) M_I]. \quad (6.14)$$

Y^I thus represents some sort of velocity along the trajectory. Therefore, the situation is like

moving through a space of circuits along the trajectory in order to evolve to the final circuit. So the problem of minimising the action to find the geodesic comes down to the task of minimising the cost function D

$$D = \int_0^1 ds \Sigma |Y^I(s)|. \quad (6.15)$$

Thus minimising this cost function will be equivalent to finding the extremal path $U(s)$ of a geodesic in Finsler geometry. We convert it to the more convenient Riemannian geometry by writing the following familiar cost function.

$$D = \int_0^1 ds \sqrt{\Sigma_{IJ} \delta_{IJ} Y^I(s) Y^J(s)}. \quad (6.16)$$

Minimising the action thus implies minimising

$$D = \int_0^1 ds [(Y^{11}(s))^2 + (Y^{22}(s))^2 + (Y^{21}(s))^2 + (Y^{12}(s))^2]. \quad (6.17)$$

For easy computation, we shift the problem to matrix method. The wave-function can now be represented as

$$\Psi = \exp \left[\frac{1}{2} x_i A_{ij} x_j \right] \quad (6.18)$$

where the reference state is given by

$$A_R = \begin{bmatrix} \omega_0 & 0 \\ 0 & \omega_0 \end{bmatrix}$$

and the target state can be given by

$$A_R = \begin{bmatrix} \omega_1 & \beta \\ \beta & \omega_1 \end{bmatrix}$$

The unitary gates that we had defined can now be written as

$$Q_{ij} = \exp[\epsilon M_{ij}] \quad (6.19)$$

where

$$[M_{ij}]_{ab} = \delta_{ia}\delta_{jb}. \quad (6.20)$$

We see that M_{ij} are the generators of the General Linear group ($GL(2, \mathbb{R})$). To construct the required geodesics, we parametrise U as follows :

$$U = \begin{bmatrix} x_0 - x_3 & x_2 - x_1 \\ x_2 + x_1 & x_0 + x_3 \end{bmatrix}$$

where

$$x_0 = e^y \cos(\tau) \cosh(\rho), \quad (6.21)$$

$$x_1 = e^y \sin(\tau) \cosh(\rho) \quad x_2 = e^y \cos(\theta) \sinh(\rho), \quad (6.22)$$

$$x_3 = e^y \sin(\theta) \sinh(\rho) \quad (6.23)$$

where τ , ρ , θ are the time, radius and angle of global coordinates on AdS_3 . Thus, after using the appropriate boundary conditions, we find the shortest geodesic from the family of geodesics, which is given by

$$\tau(s) = 0, \theta(s) = \pi, \quad (6.24)$$

$$y(s) = y_1 s, \rho(s) = \rho_1 s. \quad (6.25)$$

Therefore we can write

$$U(s) = P \exp \left[\begin{bmatrix} y_1 & -\rho_1 \\ -\rho_1 & y_1 \end{bmatrix} s \right]$$

The complexity is then given by

$$C = \sqrt{2\rho_1^2 + 2y_1^2} \quad (6.26)$$

$$= \frac{1}{2} \sqrt{\log^2 \left(\frac{\omega_+}{2\omega_0} \right) + \log^2 \left(\frac{\omega_-}{2\omega_0} \right)} \quad (6.27)$$

where $\omega_+^2 = \omega^2$ and $\omega_-^2 = \omega^2 + 2\Omega^2$.

Extending this idea to a lattice of N oscillators,

$$A_T = mI, A_R = \omega_0 I \tag{6.28}$$

and thus,

$$C = \frac{1}{2} \sqrt{\Sigma \log^2 \left(\frac{m}{\omega_0} \right)} = \frac{N^{\frac{1}{2}}}{2} \log \left(\frac{\omega_0}{m} \right). \tag{6.29}$$

Thus we see that in this case $C \propto N^{\frac{1}{2}}$ or thus $C \propto V^{\frac{1}{2}}$ where V is the volume of the system. However, from our holographic considerations we know that $C \propto N$ or $C \propto V$. Therefore the results don't match. This difference in order by a factor of $\frac{1}{2}$ can be attributed to a bad choice of cost function, since the action maybe the origin of the square root. So maybe, if we had taken a square of this cost function, we might have got a result where $C \propto V$.

In principle, there is no particular rule regarding defining the cost function. It can be anything. Moreover, ω_0 introduces a kind of scale factor too. So our choice of ω_0 affects the final value of the shortest geodesic. Myers thus answers the question that the optimal quantum circuit indeed depends on our choices. This idea of matching the holographic results is yet to be concretised properly, and there is a lot of scope still for developing new theories.

Chapter 7

Conclusion and Outlook

7.1 Conclusion and Outlook

We thus get an idea of how holographic complexity looks like for different spacetimes. The values match for pure AdS and pure Lifshitz. The growth rate of complexity too matched for both AdS and Lifshitz black holes. Moreover, we noted that the complexity results for the Lifshitz case is independent of the scaling exponent z , as we expect for the dynamics of a constant t slice being independent of scaling factor. Similar calculations to find the complexity, using the complexity-action(CA) conjecture can be done and then we would be able to compare the results with those obtained from CV considerations.

We had actually intended to calculate the CA results for the Lifshitz spacetime until some paper was already published. Moreover, we also have to try and get the complete solution for the maximal slice in case of AdS and Lifshitz black branes.

A better understanding of quantum complexity and quantum information will help us tackle problems of quantum gravity.

. As we have seen in the last section, the choice of cost function was not an appropriate one. Therefore finding an appropriate cost function remains an unsolved problem. Maybe we can inspect different cost functions and devise some way to find the optimal one, then somewhere the holographic results will match. One alternative problem which can be tried out is to retrace all the steps for the case of coupled fermionic oscillators. A fermionic oscillator has two states only, and we should be easily be able to solve for the cost function in this case.

If the geometric approach to compute complexity doesn't work, we can study tensor networks alternatively. In this case, the complexity is proportional to the size of tensor networks.

Bibliography

- [1] Sebastian De Haro et. al., "Conceptual Aspects of Gauge/Gravity Duality", *Found Phys* (2016) 46:1381?1425
- [2] M. H. Dehghani and R. B. Mann, "Thermodynamics of Lovelock-Lifshitz black branes", *Physical Review D* 82, 064019 (2010)
- [3] Dean Carmi, Robert C. Myers and Pratik Rath, "Comments on Holographic Complexity", arXiv: 1612.00433v3 [hep-th]
- [4] Leonard Susskind, "Entanglement is not Enough", arXiv:1411.0690v1 [hep-th]
- [5] Adam R. Brown and Leonard Susskind, "The Second Law of Quantum Complexity", arXiv:1701.01107v1
- [6] Run-Qiu Yang, "A Complexity for Quantum Field Theory States and Application in Thermofield Double States", arXiv:1709.00921v3 [hep-th]
- [7] Marika Taylor, "Lifshitz holography", arXiv:1512.03554 [hep-th]
- [8] Javier Tarrio, Stefan Vandoren, "Black holes and black branes in Lifshitz spacetimes", arXiv:1105.6335 [hep-th]
- [9] Robert A. Jefferson and Robert C. Myers, "Circuit complexity in quantum field theory", arXiv:1707.08570v1
- [10] Michael A. Nielsen, "A geometric approach to quantum circuit lower bounds", arXiv:quant-ph/0502070v1

- [11] Michael A. Nielsen?, Mark R. Dowling, Mile Gu, and Andrew C. Doherty, "Quantum Computation as Geometry", arXiv:quant-ph/0603161v2

Appendices

Appendix A

Appendix : Mathematica code snippet

Complexity of late time slice in Lifshitz spacetime, for d dimensions:

$$\frac{1}{G G_d \Gamma\left[\frac{1+d}{2}\right]} 2^{\frac{1-d+z}{-1+d+z}} \mathfrak{l}^{-2-z} \pi^{\frac{1}{2}(-1+d)} \mathfrak{t} \left(\frac{2^{4+\frac{-1+d}{-1+d+z}} \mathfrak{l}^{1+z} \mathfrak{m}^{\frac{1-d}{-1+d+z}} \mathfrak{M} \pi^{\frac{3-d}{2}-\frac{d}{2}} G_{1+d} \Gamma\left[\frac{1+d}{2}\right]}{-1+d} \right)^{\frac{2(-1+d)}{-1+d+z}}$$

$$\left(\left(\frac{2^{3+\frac{-1+d}{-1+d+z}} \mathfrak{l}^{1+z} \mathfrak{m}^{\frac{1-d}{-1+d+z}} \mathfrak{M} \pi^{\frac{3-d}{2}-\frac{d}{2}} G_{1+d} \Gamma\left[\frac{1+d}{2}\right]}{-1+d} \right)^{\frac{1}{-1+d+z}} \right)^z$$

$$\sqrt{\left(\frac{1}{-1+d} \left(-1+d - 2^{4+\frac{-1+d}{-1+d+z}} \mathfrak{l}^{1+z} \mathfrak{m}^{\frac{1-d}{-1+d+z}} \mathfrak{M} \pi^{\frac{3-d}{2}-\frac{d}{2}} G_{1+d} \right) \right)}$$

$$\Gamma\left[\frac{1+d}{2}\right] \left(\left(\frac{2^{3+\frac{-1+d}{-1+d+z}} \mathfrak{l}^{1+z} \mathfrak{m}^{\frac{1-d}{-1+d+z}} \mathfrak{M} \pi^{\frac{3-d}{2}-\frac{d}{2}} G_{1+d} \Gamma\left[\frac{1+d}{2}\right]}{-1+d} \right)^{\frac{1}{-1+d+z}} \right)^{1-d-z} \right)$$