Cubic Spline and Their Application

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Declaration

I declare that this written submission represents the ideas which i have learned from certain books that are mentioned in the references. I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission.

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CERTIFICATE

This is to certify that the work contained in this report entitled **"CUBIC SPLINE AND THEIR APPLICATION"** submitted by **MAHENDRA KUMAR** (**Roll No: MA15MSCST11004.**) to Department of Mathematics, Indian Institute of Technology Hyderabad towards the requirement of the course **MA5980 Project** has been carried out by him under my supervision.

Hyderabad
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Dr.P.A.L.Narayana
(Project supervisor)

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Mahendra Kumar

ABSTRACT

In this dissertation, we describe Cubic Splines and their applications.In particular Cubic Splines are discussed in detail with their applications in the Interpolation,Solutions of Initial Value Problems and Solutions of Boundary Value Problems.These are easy to implement on a computer. Several numerical examples have been solved to demonstrate the applicability of Cubic Splines.In the case of Boundary Value Problems, computational results are presented for constant coefficient,variable coefficient of homogeneous,non-homogeneous linear two point Boundary Value Problems.Computational results have been taken for different step sixes and these results are compared with exact solutions.It is observed from the computational results that the Cubic Spline approximate the exact solution very well.

1 Introduction

Spline function constitute an important subject in analysis. During the past decade both the theory of Splines and experience with their use in numerical analysis have undergone a considerable degree of development. Discoveries of new and significant results are of frequent occurrence.

It is useful at this juncture,nevertheless, to make some serious effort to organize and present material already developed up to this time.Much of this has become standardized.

1.1 What is a Spline

It seems appropriate to begin a dissertation on Splines by defining Spline in its simplest and most widely used from, and also to indicate the motivation leading to this definition. For many years,long,thin strips of wood or some other material have been used much like French curves by draftsman to fair in a smooth curve between specified points. These strips or Splines are anchored in place by attaching lead weights called "ducks" at the points along the Spline. By varying the points where the ducks are attached to the Spline itself and the position of both the Spline and the duck relative to the drafting surface, the Spline can be made to pass through the specified points provided a sufficient number of ducks are used.

If we regard the draftsman's Spline as a thin beam,then the Bernoulli-Euler law

$$M(x) = EI[1/r(x)]$$

is satisfied. Here $M(x)$ is the bending moment,E is the Young's modulus, I is the geometric moment of inertia, and $R(x)$ is the radius of the curvature of the elastica,i.e.,the curve assumed by the deformed the beam.For small deflections,$R(x)$ is replaced by $1/y^{"}(x)$ where $y^{"}(x)$ denotes the elastica.Thus we have

$$y^{"}(x) = (1/EI)M(x)$$

Since the ducks act effectively as simple supports, the variation of M(x) between duck position is linear. The mathematical Spline is the result of replacing the draftsman's Spline by its elastica and then approximating the latter by a piecewise cubic(normally a different cubic between each pair of adjacent ducks) woth certain discontinuities of derivatives permitted at the junction points(the ducks)where two cubics join.In it's simple form, the mathematical Spline is continuous and has both a continuous first derivative and a continuous second derivative.Normally,however, there is a jump discontinuity in its third derivative at the junction points. This corresponds to the draftsman's Spline having continuous curvature with jumps occurring in the rate of change of curvature at the ducks. For many important applications, this mathematical model of the draftsman's Spline is highly realistic.In practice, the draftsman does not place the ducks at the specified points through which his Spline must pass.Moreover,there is not usually a one-to-one correspondence between the specified points are the junction points (including the endpoints) the same.

2 CUBIC SPLINE INTERPOLATION

Instead of trying to approximate a function f(x) over the entire interval by one polynomial,divide the interval into sub-intervals and approximate the function f(x) by Spline functions(polynomials) in each sub-interval of the original interval.

The name is derived from the draftsman device flexible strip i.e., to draw a curve through the points in such a way that not only the curve but also its slope and curvature are continuous functions. Let [a,b] be a finite interval $a = x_0 < x_1 < x_2 < ...... < x_n = b$ be a partition of $f_0, f_1.....f_n$ are the value of function defined at $x_0, x_1...x_n$.

Spline function of degree 'm' is a function S(x) with following properties-
1.S(x)is (n-1) time continuously differentiable on [a,b]
2.On each sub-interval $[x_{i-1}, x_i]$,S_i is a polynomial of degree m
3.$S(x_i) = f_i$,for i=0,1,2,3....n

If m=3 this polynomial is called cubic spline. A cubic spline function is a function S(x) such that
1. S(x),,S(x'),S(x") are continuous on [a,b]
2. S(x) is a cubic polynomial $S_i(x)$ on each sub interval $[x_i, x_{i-1}]$
3.$S(x_i) = f_i$ for i=1,2,3,...n

Method-I Construction of $S_i(x)$-
Let us define $h_i = x_i - x_{i-1}$ and $d_i = \frac{f_i - f_{i-1}}{h_i}$ and $t_i = \frac{x - x_{i-1}}{h_i}$ since S(x) is a cubic in $[x_{i-1}, x - i]$ satisfy the condition $S_i(x_{i-1}) = f_{i-1}$, $S_i(x_i) = f_i$
Now we can think of-

$$S_i(x) = (1-t)f_{i-1} + tf_i + h_i t(1-t)[(1-t)A_i + tB_i] \tag{1}$$

where A,B are determined Differentiate equation (1) w.r.to t we get

$$S_i^{'} = \frac{f_i - f_{i-1}}{h_i} + t(1-t)[B_i - A_i] + (1-2t)[(1-t)A_i + tB_i] = 0 \tag{2}$$

$$S_i^{'} = d_i + t(1-t)[B_i - A_i] + (1-2t)[(1-t)A_i + tB_i] \tag{3}$$

let $S_i^{'}(x_i) = k_i$, $S_i^{'}(x_{i-1}) = k_{i-1}$ putting t=o and t=1 in equation (3) then we find-
$A_i = k_{i-1} - d_i$ and $B_i = d_i - k_i$
put these value in equation (1)

$$S_i(x) = (1-t)f_{i-1} + tf_i + h_i t(1-t)[(1-t)(k_{i-1} - d_i) + t(d_i - k_i)] \tag{4}$$

diff equation (4)

$$S_i^{'} = d_i + t(1-t)[d_i - k_i - k_{i-1} + d_i] + (1-2t)[(1-t)(k_{i-1} - d_i) + t(d_i - k_i)] \tag{5}$$

Now,we can observe that- $S_{i+1}(x_i) = S_i^{'}(x_i) = k_i$
$S_i^{'}$ is continuous Equation (5) differentiate twice wrt t we get -

$$S_i^{''}(x) = \frac{2}{h_i}[3d_i - 2k_{i-1} - k_i - 3t(2d_i - k_{i-1} - k_i)] \tag{6}$$

$$S_i^{''}(x) = \frac{2}{h_i}[k_{i-1} - 3d_i + 2k_i] \quad on \quad [x_{i-1}, x_i] \tag{7}$$

since S(x) is continuous
now we have-
$S_{i+1}^{''}(x_i) = S_i^{''}(x_i)$,

$$\frac{k_{i-1}}{h_i} + 2(\frac{1}{h_i} + \frac{1}{h_{i+1}})k_i + \frac{k_{i+1}}{h_{i+1}} = 3(\frac{d_i}{h_i} + \frac{d_{i+1}}{h_{i+1}}) \tag{8}$$

this equation represents (n-1) equation in (n+1) unknowns $k_0, k_1, ...k_n$
Now we find two more equation to solve the system we assume-
$S_i^{''}(x_0) = S_n^{''}(x_n) = 0$

$$2k_0 + k_1 = 3d_i$$

$$k_{n-1} + 2k_n = 3d_n$$

In Matrix form-

$$\begin{bmatrix} 2 & 1 & 0 & 0\cdots & 0 & \\ \frac{1}{h_1} & 2(\frac{1}{h_1}+\frac{1}{h_2}) & \frac{1}{h_2} & 0 & \cdots & 0 \\ 0 & \frac{1}{h_2} & 2(\frac{1}{h_2}+\frac{1}{h_3}) & \frac{1}{h_3}\cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_n \end{bmatrix} = 3 \begin{bmatrix} d_1 \\ d_1 + d_2 \\ \vdots \\ d_n \end{bmatrix}$$

Solving this for $k_0, k_1, k_2, ..., k_n$ and put we get $S_i(x)$ if $h_1 = h_2 = \cdots = h_n$,then

$$\begin{bmatrix} 2 & 1 & 0 & 0\cdots & 0 & \\ 1 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_n \end{bmatrix} = 3 \begin{bmatrix} d_1 \\ d_1 + d_2 \\ \vdots \\ d_n \end{bmatrix}$$

Method II -Construction of $S_i(x)$- $S_{i+1}(x)$ is cubic $S_{i+1}^{'}$is quardic and $S_{i+1}^{''}$ is linear equation on $[x_i, x_{i-1}]$
We want $S(x)$ over $[x_i, x_{i-1}]$
Now, $h_i = x_{i+1} - x_i$

$$S_{i+1}^{"} = \frac{1}{h_i}[(x_{i+1} - x)S^{"}(x_i) + (x - x_i)S^{"}(x_{i+1})]$$

Since $S_{i+1}^{"}(x_i) = S_i^{"}(x_i)$, $S_{i+1}^{"}(x_{i+1}) = S_i^{"}(x_{i+1})$ Integrate the equation Twice
Then we get-

$$S_{i+1}^{'}(x) = \frac{1}{h_i}[\frac{-(x_{i+1} - x)^2}{2}S^{"}(x_i) + \frac{(x - x_i)^2}{2}S^{"}(x_{i+1})] + C_1(x_{i+1} - x) \tag{9}$$

$$S_{i+1}(x) = \frac{1}{h_i}[\frac{(x_{i+1} - x)^3}{3!}S^{"}(x_i) + \frac{(x - x_i)^3}{3!}S^{"}(x_{i+1})] + C_1(x_{i+1} - x) + C_2(x - x_i) \tag{10}$$

Where C_1 and C_2 are constant to be determined-
Since $S_{i+1}(x_i) = y_i$ and $S_{i+1}(x_{i+1}) = y_{i+1}$
Through initial Value condition $C_1 = \frac{1}{h_i}[f_i - \frac{h_i^2}{3!}S^{"}(x_i)]$
and $C_2 = \frac{1}{h_i}[f_i - \frac{h_i^2}{3!}S^{"}(x_{i+1})]$

lets denote $S^{"}(x_i) = M_i$ and $S^{"}(x_{i+1}) = M_{i+1}$
Put them together we get

$$S(x) = \frac{1}{h_i}[x_{i+1}M_i + (x - x_i)^3 M_{i+1}] + \frac{1}{h_i}(x_i - x)(f_i - \frac{h_i^2}{6}M_{i+1}) + \frac{1}{h_i}(x - x_i)(f_{i+1} - \frac{h_i^2}{6}M_i) \quad (11)$$

differentiate equation (11)

$$S'_{i+1}(x) = \frac{-h_i}{6}(2M_i + M_{i+1}) + \frac{1}{h_i}[f_{i+1} - f_i] \quad (12)$$

For interval $[x_{i-1}, x_i]$

$$S'_i(x_i) = \frac{h_{i-1}}{6}(2M_i + M_{i-1}) + \frac{1}{h_i}[f_i - f_{i-i}]$$

by continuity both are same-
$S'_i(x_i) = S'_{i+1}(x_i)$

$$h_{i-1}M_{i-1} + 2(h_{i-1} + h-_i)M_i + h_iM_{i+1} = 6\frac{f_{i+1} - f_i}{h_i} - \frac{f_i - f_{i-i}}{h_{i-1}} \quad (13)$$

this gives (n-1)equations in (n+1) unknowns

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 & 0\cdots & 0 & \\ \cdots & 0 & & & & \\ \cdots & 0 & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & h_{n-1} & 2(h_{n-2}+h_{n-1}) \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{n-1} \end{bmatrix} = 6 \begin{bmatrix} f_2 - f_1 h_1 - \frac{f_1 - f_0}{h_0} \\ \vdots \\ \frac{f_n - f_{n-1}}{h_{n-1}} - f - n \end{bmatrix}$$

In special case-$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[f_{i+1} - 2f_i + f_{i-1}](14)$
$then system of equation yield the matrix-$

$$\begin{bmatrix} 4 & 1 & 0 & 0\cdots & 0 & \\ 1 & 4 & 1 & 0\cdots & 0 & \\ 0 & 1 & 4 & 1\cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 4 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \vdots \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} {}_2 - 2f_1 + f_0 \\ f_3 - 2f_2 + f_1 \\ \vdots \\ f_n - 2f_{n-1} + f_{n-2} \end{bmatrix}$$

Problem 1:- Fit Cubic Spline for the data and find f(1.5) and f(1)
x 1 2 3
f -8 -1 -8
Solution:
Here $h_1 = h_2 = h_3 = 1$.S(x) needs M_0 and M_1 over$[2, 3]$.Put $M_0 = M_2 = 0$
we know $M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[f_{i+1} - 2f_i + f_{i-1}]$.Put i=1
$M_0 + 4M_1 + M_2 = \frac{6}{h^2}[f_2 - 2f_1 + f_0]$

substituting the M_0, M_2, f_0, f_1, f_2 values we get,$M_1 = 18$
$S_1(x)$ over [1,2]: Using equation(4)
$S_1(x) = \frac{1}{6h}[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1] + \frac{1}{6}(x_1 - x)(f_0 - \frac{1}{6}M_0) + \frac{1}{6}(x - x_0)(f_1 - \frac{1}{6}M_1)$
$S_1(x) = \frac{1}{6}[(x-1)^3 18] - 8(2-x) - 4(x-1)$
Similarly $S_2(x)$ over [1,2]:
$S_2(x) = 3(3-x)^3 - 4(x-2) + 18(x-2)$
to find f(1.5),we have to consider S(x) in the interval [1,2], i.e.,f(1.5)=s(1.5)=-45/8
similarly $f^{'}(1) = S^{'}$=4.

Problem 2:- Fit the Cubic Spline for the unequally spaced data

x	0	1	2
f	1	0	2

Solution: $h_0 = 1$ and $h_1 = 2$ $h_0 M_0 + 2(h_0 + h_1)M_1 + h_1 M_1 = 6(\frac{f_2 - f_1}{h_1} - \frac{f_1 - f_0}{h_0})$ set$M_0 = M_2 = 0$,We get$M_1 = 2$

$S_1(x)$ **over [0,1]:**

$S_1(x) = \frac{x^3}{3} - \frac{4x}{3} + 1$

$S_2(x)$ **over [1,3]:**

$S_2(x) = \frac{-(x-3)^3}{6} + \frac{4}{6}(x-3) + (x-1).$

3 CUBIC SPLINE SOLUTION OF INITIAL VALUE PROBLEM

Spline function can be used to solve initial value problem using Hermite interpolation formula we get the following for cubic spline interpolation $S(x)$ in $x_{i-1} \preceq x \preceq x_i$ in the terms of first derivatives $S^{'}(x) = M_i$,

$S(x) = M_{i-1}\frac{(x_i-x)^2(x-x_{i-1})}{h^2} - M_i\frac{(x-x_{i-1})^2(x_i-x)}{h^2} + y_{i-1}\frac{(x_i-x)^2[2(x-x_{i-1})+h]}{h^3} + y_i\frac{(x-x_{i-1})^2[2(xi-x)+h]}{h^3}$
$h_i = x_i - x_{i-1}$ Equation (13) differentiate twice with respect to x

$$S_x^{'} = \frac{M_{i-1}}{h^2}(x_i-x)(2x_(i-1)+x_i-3x) - \frac{M_i}{h^2}(x-x_{i-1})(x_{i-1}+2x_i-3x) + \frac{6}{h^3}(y_i-y_{i-1})(x-x_{i-1})(x_i-x) \tag{15}$$

$$S_x^{"} = -2\frac{M_{i-1}}{h^2}(x_{i-1}+2x_i-3x) - 2\frac{M_i}{h^2}(2x_{i-1}+x_i-3x) + 6\frac{(y_i-y_{i-1})}{h^3}(x_{i-1}+x_i-2x) \tag{16}$$

We get from equation (15)

$$S_x^{"} = 2\frac{M_{i-1}}{h} + 4\frac{M_i}{h} - 6\frac{S_i - S_{i-1}}{h^2} \tag{17}$$

Consider the Initial Value Problem

$$\frac{dy}{dx} = f(x,y), y(x_0) = y_0,$$

$$\frac{d^2y}{dx^2} = \frac{df}{dx} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = f_x^{'}(x_i, S_i) + f_y^{'}(x_i, S_i)f(x_i, y_i) \tag{18}$$

from (17) and (18) S_i can be calculated.Substituting in equation we get required solution.

Problem 1: Solve the Initial Value Problem $y^{'} = 3x + 1/2y, y(0) = 1$
Take h=0.5,n=2 compute y(0.5)
Solution:
Exact solution fo the above problem is given by $y = 13e^{x/2} - 6x - 12$
Take h=0.5,n=2 f(x,y)=3x+1/2y, $f_x^{'} = 3$, $f_y^{'} = 1/2$
$f(x_i, S_i) = 3x_i + \frac{1}{2}S_i$
$4m_0 + 8m_1 - 24(S_1 - S_0) = 3 + S_1$
and $4m_1 + 8m_2 - 24(S_2 - S_0) = 3 + \frac{7}{4}S_2$
since $m_0 = .5, m_1 = \frac{3}{2} + \frac{1}{2}S_1$ and $m_2 = 3 + \frac{1}{2}S_2$
$S_1 = 1.691358, S_2 = 3.430879.$

4 Cubic Spline For Boundary Value Problem

When a practical problem in Science or Technology permits Mathematical formulation the chances or rather good that it leads to one or more differential equations.
Differential Equations play an important roll in Science,Engineering and Social Sicneces. Many phenomena in these branches of knowledge or interpreted in terms of differential equations and their solutions.Every one knows that Ordinary Differential Equations seve as Mathematical Models for many exciting 'Real World' problems,not only in Science and Technology, but also in such diverse fields as Ecomomics,Psychology,Defense and Demography. As a resultthe stydy of differential equations and their solutions is attaining importance.
It is a well-known fact that the majority of differential equation in science and engineering cannot be integrated analytically. In these cases it is necessary to apply some method of approximation.There exists a large number of different approximation methods for solving differential equations the most important of which are the methods of Spline function-

Conditions 1.-

Two point boundary value problem given by $y^{"} + f(x)y^{'} + g(x)y = \sigma(x)$
(I)Dirichlet boundary value problem- $y(a) = \alpha, y(b) = \beta$
(II)Neumann boundary condition- $y^{'}(a) = \alpha, y^{'}(b) = \beta$
(III)Mixed boundary condition - $a_0y(a) + a_1y^{'}(a) = \alpha$,$b_0y(b) + b_1y^{'}(b) = \beta$
if r(x)=0 then differential equation called homogenous equation.

Cubic Spline Method:

The Cubic spline method is one-step method and at the same time a global one.The step-size cab be changed during computations and,under certain conditions, The cubic Spline method gives $o(h^4)$ convergence.The method can also be extended to systems of ordinary Differential Equations.
Consider a two-point boundary value problem of the form:

$$y^{"}(x) + p(x)y^{'}(x) + q(x)y(x) = r(x), 0 < x < 1 \tag{19}$$

with the boundary conditions $y(0) = \alpha, \quad y(1) = \beta$
We consider mesh with a grid points $0 = x_0 < x_1 < ..... < x_n = 1$ with $h = x_i - x_{i-1} > 0$. we have shown that the Cubic Spline S(x) interpolating the function y(x) at the mesh $x_i = x_0 + ih$,which is continuous together with the first and second derivatives on the interval [0,1],corresponds to a Cubic Polynomial in each sub-interval $x_{i-1} \preceq x \preceq x_i$ and satisfies $S(x_i) = y_i$.The Splines function S(x) also approximates y(x) to fourth order in 'h' at all points in [0,1].
If S(x) is a cubic polynomial on$[x_{i-1}, x_i]$,then in general,S(x)can be written as

$$S(x) = M_{i-1}\frac{(x_i - x)^2}{6h} + M_i\frac{(x - x_{i-1})^3}{6h} + (y_{i-1} - \frac{h^2}{6}M_{i-1})(\frac{x_i - x}{h}) + (y_i - \frac{h^2}{6}M_{i-1})(\frac{x - x_{i-1}}{h})$$

Where $M_i = S^{"(x_i)}$ and $y_i = y(x_i)$.The unknown derivatives M_i are related by the continuity condition on $S^{'}(x)$ in such a way that-
$S^{'}(x_i^+) = S^{'}(x_i^-)$ where $S^{'}(x_i^+)$ and $S^{'}(x_i^-)$ are right hand and left hand derivatives over the interval $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ respectively.
The we can find

$$S^{'}(x) = M_{i-1}[\frac{h}{6} - \frac{(x_i - x)^2}{2h}] + M_i[\frac{(x - x_{i-1})^2}{2h} - \frac{h}{6}] + \frac{y_i - y_{i-1}}{h}$$

Now we have a one sided limits of the derivative as

$$S^{'}(x_i^+) = -\frac{h}{3}M_i - \frac{h}{6}M_{i+1} + \frac{y_{i+1} - y_i}{h}$$

$$S^{'}(x_i^-) = -\frac{h}{3}M_i + \frac{h}{6}M_{i-1} + \frac{y_i - y_{i-1}}{h}$$

The continuity condition of first order derivatives implies

$$\frac{h}{6}M_{i-1} + \frac{2h}{3}M_i + \frac{h}{6}M_{i+1} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h} \tag{20}$$

consider the equation

$$M_i + p_i[-\frac{h_i}{3}M_i - \frac{h}{6}M_{i-1} + \frac{y_{i+1} - y_i}{h}] + q_i y_i = r_i \tag{21}$$

$$M_i + p_i[\frac{h_i}{3}M_i + \frac{h}{6}M_{i-1} + \frac{y_i - y_{i-1}}{h}] + q_i y_i = r_i \tag{22}$$

from equation (19) and (20)

$$(1 - \frac{hp_i}{3})M_i + \frac{hp_i}{6}M_{i-1} = r_i - q_i y_i - p_i\frac{y_{i+1} - y_i}{h} \tag{23}$$

$$(1 + \frac{hp_i}{3})M_i - \frac{hp_i}{6}M_{i+1} = r_i - q_i y_i - p_i\frac{y_i - y_{i-1}}{h} \tag{24}$$

adding both equation

$$\frac{hp_i}{6}M_{i-1} + 2M_i - \frac{hp_i}{6}M_{i+1} = 2(r_i - q_i y_i) - p_i\frac{y_{i+1} - y_{i-1}}{h} \tag{25}$$

Verify equation (19) and (20) for M_{i-1}, M_{i+1}

$$M_{i-1} = \frac{1}{e_i}[(r_{i-1}+\frac{hp_i}{3}r_{i-1}+\frac{hp_{i-1}}{6}r_i)-(q_{i-1}+\frac{hp_i}{3}q_{i-1}-\frac{p_{i-1}}{h}-\frac{p_ip_{i-1}}{2})y_{i-1}-(\frac{p_{i-1}}{h}+\frac{p_ip_{i-1}}{2}+\frac{hp_{i-1}}{6}q_i)y_i] \tag{26}$$

Where $[e_i = 1 - \frac{hp_{i-1}}{3} + \frac{hp_{i+1}}{3} - \frac{h^2p_ip_{i-1}}{12}]$

$$M_{i+1} = \frac{1}{f_i}[(r_{i+1}-\frac{hp_i}{3}r_{i+1}-\frac{hp_{i+1}}{6}r_i)-(q_{i+1}+\frac{hp_i}{3}q_{i+1}-\frac{p_{i+1}}{h}-\frac{p_ip_{i+1}}{2})y_{i+1}-(\frac{p_{i+1}}{h}+\frac{p_ip_{i+1}}{2}+\frac{hp_{i+1}}{6}q_i)y_i] \tag{27}$$

Where $[f_i = 1 - \frac{hp_i}{3} + \frac{hp_{i+1}}{3} - \frac{h^2p_ip_{i+1}}{12}]$

Now Spline approximation is given b three term reccurence relation-

$A_iy_{i+1} + B_iy_i + C_iy_{i-1} = D_i, \quad i = 1, 2, 3...n-1$

Where

$A_i = e_i[1 + \frac{hp_{i+1}}{2} + \frac{h^2}{6}q_{i+1}]$

$B_i = [\frac{2h^2}{3}q_ig_i - e_i(1 + \frac{hp_{i+1}}{2}) - f_i(1 - \frac{hp_{i-1}}{2})]$

$C_i = f_i[1 - \frac{hp_{i-1}}{2} + \frac{h^2}{6}q_{i-1}]$

$D_i = \frac{h^2}{6}(f_ir_{i-1} + 4g_ir_i + e_ir_{i+1})$

and $g_i = [1 - \frac{h^2p_{i-1}p_{i+1}}{12} + \frac{7h}{24}(p_{i+1} - p_{i-1})$ From equation (19)and(20) lead to a system of 2n equations with (2n+2) unknowns $M_0, M_1, ..., M_n$ and $y_o, y_1, ..., y_n$.Eliminating M_i's lead to a system of (n-1) equations with (n+1) unknowns.The two given boundary conditions together with (n-1) are then sufficient to solve for the unknowns. The three-term recurrence relationship for the Spline approximation can ve solved by "Thomas Algorithm" .

5 Thomas Algorithm:

For finding a three term recurrence relation we will use Thomas algorithm that is quick and simple way to find a three term recurrence relation- In order to solve tridiagonal system $A_i y_{i+1} + B_i y_i + C_i y_{i-1} = D_i$ Here we know A_i, B_i, C_i, D_i respectively
$y_o = \alpha$
$y_n = \beta$
we set-up difference relation of the form
$y_i = w_i y_{i+1} + t_i$
where w_i and t_i corresponds to $w(x_i)$ and $t(x_i)$ that are to be determined from equation-
$y_{i-1} = w_{i-1} y_i + t_{i-1}$ by putting in equation then we get
$A_i y_{i+1} + B_i y_i + C_i(w_{i-1} y_i + t_{i-1}) = D_i$

$$y_i = \frac{-A_i}{B_i + C_i w_{i-1}} y_{i+1} + \frac{D_i - C_i t_{i-1}}{B_i + C_i w_{i-1}} \tag{28}$$

by comparing the equation we get the recurrence relations
$w_i = \frac{-A_i}{B_i + C_i w_{i-1}}$
$t_i = \frac{D_i - C_i t_{i-1}}{B_i + C_i w_{i-1}}$
To solve the recurrence relations for i=1,2,3,..n-1, we need to know the initial conditions for w_0 and t_0. these can be considered in the equation-
$y_o = \alpha = w_0 y_1 + t_0$
choose $w_0 = 0, t_0 = \alpha$.Using these initial values we compute $w_i$and t_i for i=1,2,3,...,n-1 and then obtain the solution y_i in the backward process from the equations.

5.1 Numerical Results

Example 1: Consider the Boundary Value Problem

$$y^{"} - y = 0$$

with the boundary conditions y(0)=0 and y(2)=3.6268.
$h = \frac{1}{20}$ and $h = \frac{1}{40}$ The Exact solution is y(x)=$\sinh x$.The Numerical results are given in Table 1.
Example 2: Consider the Boundary Value Problem

$$y^{"} - y = -x$$

with the boundary conditions y(0)=0 and y(1)=0.
$h = \frac{1}{40}$ and $h = \frac{1}{80}$ The exact solution is $y(x) = \frac{\sin(x)}{\sin(1)} - x$
The numerical result are given in Table 2.

x	Y(x) Spline	Y(x) Spline	Exact
0.0	0.000000	0.000000	0.000000
0.1	0.100150	0.100151	0.100167
0.2	0.201303	0.201305	0.201336
0.3	0.304471	0.304474	0.304520
0.4	0.410687	0.410691	0.410752
0.5	0.521014	0.521019	0.521095
0.6	0.636557	0.636563	0.636654
0.7	0.758474	0.758480	0.758584
0.8	0.887983	0.887990	0.888106
0.9	1.026382	1.026390	1.026517
1.0	1.175056	1.175065	1.175201
1.1	1.335494	1.335503	1.335647
1.2	1.509302	1.509311	1.509461
1.3	1.698219	1.698229	1.698282
1.4	1.904138	1.904148	1.904301
1.5	2.129120	2.129128	2.129279
1.6	2.375416	2.375424	2.375567
1.7	2.645493	2.645499	2.645630
1.8	2.942054	2.942058	2.942172
1.9	3.268068	3.268071	3.268160
2.0	3.626800	3.626800	3.626857

x	Y(x) Spline	Y(x) Spline	Exact
0.0	0.000000	0.000000	0.000000
0.1	0.018641	0.018640	0.018642
0.2	0.036096	0.036095	0.036098
0.3	0.051192	0.051190	0.051195
0.4	0.062779	0.062778	0.062783
0.5	0.069743	0.069741	0.069747
0.6	0.071015	0.071013	0.071018
0.7	0.065582	0.065580	0.065585
0.8	0.052499	0.052499	0.052503
0.9	0.030900	0.030900	0.030902
1.0	0.000000	0.000000	0.000000

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