Non-Linear Analysis and Application to Partial Differential Equations

Saransh Bali

A Thesis Submitted to

Indian Institute of Technology Hyderabad

In Partial Fulfillment of the Requirements for

The Degree of Master of Science

Department of Mathematics

May 2018

Declaration

I declare that this written submission represents my ideas in my own words, and where ideas or words of others have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

Saransh

(Signature)

(Saransh Bali)

MA16MSCST11013.

(Roll No.)

Approval Sheet

This Thesis entitled Non-Linear Analysis and Application to Partial Differential Equations by Saransh Bali is approved for the degree of Master of Science from IIT Hyderabad

08/05/18

(Dr.Bhakti Bhusan Manna) Adviser
Dept. of Mathematics
IITH

8/5/18

(Dr. C. S Sastry) Faculty-Adviser
Dept. of Mathematics
IITH

(Dr. J Bala Subramaniam) H.O.D
Dept. of Mathematics
IITH

Acknowledgements

First and foremost, I have to thank my supervisor Dr.Bhakti Bhusan Manna. Without his assistance and dedicated involvement in every step through out the process, this project would have never been accomplished. His willingness to give his time so generously is beyond appreciation. At last I thank him for all the support and trust he has put on me.
Most importantly, none of this could have happened without my family. I would like to express my gratitude towards my parents for their love and encouragement. My thanks and appreciation also goes to all my classmates. Atlast, I would like to thank Phd Scholar Alok Kumar Sahoo for working with me and helping me clearing my doubts.

Dedication

Dedicated to all my Well-wishers.

Abstract

In the past couple of decades, when Linear Functional Analysis was quite widely and completely established, the interest of mathematicians towards Non-linear Analysis has increased a lot. At one hand the treatment of various classical problems has been unified, on the other, theories specifically non-linear one of great significance and applicability have come out.

In Non-linear Functional Analysis we study the properties of (continuous) mappings between the normed linear spaces and we describe the methods for solving non-linear equaions involving such mappings. For finding the solution of non-linear equations there are primarily two major approaches which are known as topological methods and variational methods. Topological methods are derived from fixed point theorems and one of the important tools used in this direction are the Topological Degree and Morse Theory. Variational methods describe the solutions as critical points of a suitable functional and study ways of locating them. Moreover, there is an important fact to be noted. The fact is that the problems that are often considered to be dificult, once they are framed in an appropriate functional setting, may be faced and solved quite easily.

Here in this project, we provide an introduction to the basic aspect of Non-Linear Analysis mainly those which are based on differential calculus in Banach spaces. We have expressed the results here in geometric style in such a way that they are often a transposition of infinite dimensions of events, which are intutive in $\mathbb{R}^2$ or $\mathbb{R}^3$. A particular nature of Non-Linear Analysis is that its theory has direct applications, especially to those related to differntial equations, where the power of non-linear methods is expressed in a more striking way like Degree Theory, Bifurcation Theory and Morse Theory.

Contents

Chapter 1

Preliminaries and Notations

1.1 Some notations:

- $\mathbb{R}^n$ will donate the n dimensional Euclidean space with scalar product $x.y$ and the norm given by $|x|^2 = x.x$
- $X, Y, Z, ...,$ denote (real) Banach spaces with norm $\|.\|_X, \|.\|_Y$,etc respectively (the subscript will be omitted if no possible confusion arises)
- $B(x^*, r)$ denotes the ball $\{x \in X : \|x - x^*\| < r\}$ and $B(r)$ stands for $B(0, r)$.
- If X^* is the topological dual of X the symbol $(.,.)$ will indicate the duality pairing between X and X^*
- Let $\{x_n\}$ be a sequence in X. We say that x_n converges (strongly) to $x \in X$ written as $x_n \to x$, if $\|x_n - x\| \to 0$ as $n \to \infty$, we say that x_n converges weakly to x, if $(\psi, x_n - x) \to 0$ as $n \to \infty$ for all $\psi \in X^*$
- Let X be a Banach space and let V be a closed subspace of X. A topological complement of V in X is a closed subspace W of X such that $W \bigcap V = \{0\}$ and $X = V \bigoplus W$, $V \bigoplus W$ is called a splitting of X.

1.2 Continuous mappings

We will deal with the continuous maps $f : U \to Y$, where U is an open subset of X. Continuity means $f(x_n) \to f(x)$ (strongly) for any sequence x_n strongly convergent to $x \in X$. The set of all continuous $f : U \to Y$ will be denoted by $C(U, Y)$.

Remark 1.2.1. *In a Normed Linear space a linear map is continuous if and only if it is bounded.*

example 1.2.2 (Identity map)**.** *The identity map $I : X \to X$ on a normed space $X \neq \{0\}$ is bounded and thus continuous.*

example 1.2.3 (Zero map)**.** *The zero map $0 : X \to Y$ on a normed space X is bounded and hence continuous.*

example 1.2.4 (Norm). *The norm* $\|.\| : X \to \mathbb{R}$ *on a normed space* $(X, \|.\|)$ *is continuous but not linear.*

1.3 Linear Continuous maps

The space of linear continuous maps $A : X \to Y$ will be denoted by $L(X, Y)$. The Range of A, $R(A)$, is the linear space $\{A(x) : x \in X\}$. Sometimes, when $Y = X$ we use the notation $L(X)$ instead of $L(X, X)$

Theorem 1.3.1 (Finite dimension). *If a normed space* X *is finite dimensional, then every linear map on* X *is bounded and hence continuous.*

Proof : Let X and Y be normed spaces and f be a linear map from $X \to Y$. If X is finite dimensional, choose a basis $(e_1, e_2, ..., e_n)$ in X. Then,
$x = \sum_{i=1}^{n} x_i e_i$ that implies $f(x) = \sum_{i=1}^{n} x_i f(e_i)$
So by the triangle inequality

$$\|f(x)\| = \|\sum_{i=1}^{n} x_i f(e_i)\|$$

Letting $M = \sup_i\{\|f(e_i)\|\}$, and using the fact that

$$\sum_{i=1}^{n} |x_i| \leqslant c\|x\|$$

for some $c > 0$ which follows from the fact that any two norms on a finite dimensional space are equivalent, thus one finds

$$\begin{aligned}\|f(x)\| &\leqslant \left\{\sum_{i=1}^{n} |x_i|\right\} M \\ &\leqslant cM\|x\|\end{aligned}$$

Thus, f is a bounded linear map and so is continuous.
If X is infinite-dimensional, this proof will fail as there is no guarantee that supremum M exists.

1.4 Integration

Definition 1.4.1 (Cauchy Integral). *A Cauchy Integral is a definite integral of a continuous function of one real variable. Let* $f(x)$ *be a continuous function on an interval* (a, b) *and let* $a = x_0 < ... < x_{i-1} < x_i < ... < x_n = b$, $\triangle x_i = x_i - x_{i-1}, i = 1, 2, ..., n$. *The limit*

$$\lim_{max \triangle x_i \to 0} \sum_{i=1}^{n} f(x_{i-1}) \triangle x_i$$

is called the definite integral in Cauchy sense of $f(x)$ over a, b and is denoted by

$$\int_a^b f(x)dx$$

For the continuous map $f : [a,b] \to Y$, the definition of the cauchy integral is given as in the elementary case, as the (strong) limit of the finite sums $\sum f(\xi_i)(t_i - t_{i-1})$ (with obvious meaning). From

$$\begin{aligned} \|\sum_i f(\xi_i)(t_i - t_{i-1})\| &\leqslant \sum_i \|f(\xi_i)(t_i - t_{i-1})\| \\ &\leqslant \sum_i \|f(\xi_i)\|(t_i - t_{i-1}) \end{aligned}$$

thus follows immediately the inequality

$$\|\int_a^b f(t)dt\| \leqslant \int_a^b \|f(t)\|dt$$

1.5 Some useful Function Spaces

In any discussion of functions of n variables the term multi-index denotes an ordered n-tuple.

$$\alpha = (\alpha_1, ..., \alpha_n), \quad \alpha_i > 0 \quad are \quad non-ve \quad integers.$$

with each multi-index α we associate the differential operator

$$D^\alpha = (\frac{\delta}{\delta x_i})^{\alpha_1} ....(\frac{\delta}{\delta x_n})^{\alpha_n}$$

whoose order is $|\alpha| = \sum_{i=1}^n \alpha_i$. If $\alpha = 0$ then $D^\alpha f = f$

Definition 1.5.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. For any non-ve integer m , let $C^m(\Omega)$ denote the vector space consisting of all functions ϕ which together with all their partial derivatives $D^\alpha \phi$ of order $|\alpha| \leqslant m$ are continuous on Ω.*

We abbreviate $C^0(\Omega) \equiv C(\Omega)$, $\quad C^\infty(\Omega) = \bigcap_{m=0}^\infty C^m(\Omega)$
The subspaces $C_0(\Omega)$ and $C_0^\infty(\Omega)$ consists of all those functions $C(\Omega)$ and $C^\infty(\Omega)$ respectively that have compact supports in Ω.

1.5.2 The space of bounded continuous functions:

Since Ω is open, functions in $C^m(\Omega)$ need not be bounded in Ω. We define $C_B^m(\Omega)$ to consists of those functions for which $D^\alpha u$ is bounded and continuous on Ω for $0 \leqslant |\alpha| \leqslant m$.
$C_B^m(\Omega)$ is a Banach space with the norm given by:

$$\|\phi\|_{C_B^m(\Omega)} = \max_{0 \leqslant \alpha \leqslant m} \quad \sup_{x \in \Omega} |D^\alpha \phi(x)|$$

1.5.3 The space of uniformly continuous functions:

We define the vector space $C^m(\overline{\Omega})$ to consist of all those functions $\phi \in C^m(\Omega)$ for which $D^\alpha\phi$ is bounded and uniformly continuous on Ω for $0 \leqslant |\alpha| \leqslant m$.
$C^m(\overline{\Omega})$ with the norm $\|.\|_{C^m(\overline{\Omega})}$ is a Banach space with respect to the norm given by:

$$\|\phi\|_{C^m(\overline{\Omega})} = \max_{0\leqslant\alpha\leqslant m} \quad \sup_{x\in\Omega} |D^\alpha\phi(x)|$$

1.5.4 The space of Holder continuous functions:

Let $0 < \lambda \leqslant 1$. We define $C^{m,\lambda}(\overline{\Omega})$ to be the subspace of $C^m(\overline{\Omega})$ consisting of those functions ϕ for which $D^\alpha\phi$ satisfies the Holder condition of exponent λ in Ω, for all $0 \leqslant |\alpha| \leqslant m$, in other words there exists a constant k such that

$$|D^\alpha\phi(x) - D^\alpha\phi(y)| \leqslant \quad k|x-y|^\lambda \quad \forall \quad x, y \in \Omega$$

$C^{m,\lambda}(\overline{\Omega})$ is a Banach space with the norm given by:

$$\|\phi\|_{C^{m,\lambda}(\overline{\Omega})} := \|\phi\|_{C^m(\overline{\Omega})} + \max_{0\leqslant|\alpha|\leqslant m} \quad \sup_{x,y\in\Omega, x\neq y} \frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\lambda}$$

1.5.5 The Lebesgue spaces L^p, $\quad 1 \leqslant p < \infty$:

If $1 \leqslant p < \infty$, the space $L^p = L^p(X, \Sigma, \mu)$ consists of all μ-equivalence classes of Σ-measurable real-valued functions f for which $|f|^p$ has finite integral with respect to μ over X. Two functions are μ-equivalent if they are equal μ-almost every where.
L^p space is a Banach space with the norm given by:

$$\|f\|_p = \left\{ \int |f|^p d\mu \right\}^{1/p}$$

.

Remark 1.5.6. *For only p=2, L^p is a Hilbert space.*

1.5.7 The Lebesgue space L^∞ :

The space $L^\infty = L^\infty(X, \Sigma, \mu)$ consists of all the equivalence classes of Σ-measurable real valued functions which are almost everywhere bounded, two functions equivalent when they are equal μ-almost everywhere.
The space L^∞ is a Banach space with norm given by:

$$\|f\|_\infty = \inf\{S(N) : N \in \Sigma, \mu(N) = 0\}$$

where

$$S(N) = \sup\{|f(x)| : x \notin N\}$$

An element of L^∞ is called an essentially bounded function.

1.5.8 The Sobolev spaces $W^{k,p}(\Omega)$:

let $1 \le p \le \infty$ and $k \ge 0$ is an integer. The Sobolev space $W^{k,p}$ consists of all $u : \Omega \to \mathbb{R}$ with $u \in L^1_{loc}(\Omega)$ such that, for each multi-index α with $|\alpha| \le k$, $D^\alpha(u)$ exists in weak sense and belongs to $L^p(\Omega)$
For each $k = 1, 2, ...$ and $1 \le p \le \infty$, the space $W^{k,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left\{ \sum_{|\alpha| \le k} \int_\Omega |D^\alpha u|^p \right\}^{1/p} \quad : 1 \le p < \infty$$
$$= \left\{ \sum_{|\alpha| \le k} ess \sup_\Omega |D^\alpha u| \right\} \quad : p = \infty$$

Remark 1.5.9. *For case $p = 2$ we usually write $H^k(\Omega) := W^{k,2}(\Omega)$ and $H^k(\Omega)$ is a Hilbert space for $k = 0, 1, 2, ...$*

Definition 1.5.10 (Segment)**.** *For u, $v \in U$, let $[u, v]$ denote the segment $\{tu + (1 - t)v : t \in [0, 1]\}$*

1.6 Some important theorems

Theorem 1.6.1 (Hahn-Banach Theorem)**.** *Let X be a Normed Linear space over $\mathbb{R}$ and Y be any linear subspace. Let $f : Y \to \mathbb{R}$ be a continuous linear functional. Then there exists a $g \in X^*$ such that $\|g\| = \|f\|$ and $g(y) = f(y)$ for all $y \in Y$.*

An important application of Hahn-Banach theorem can be stated as:
If $0 \ne x \in X$, then there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

Theorem 1.6.2 (Riesz Representation Theorem)**.** *Every bounded linear functional f on a Hilbert space H can be represented in terms of the inner product, namely*

$$f(x) = (x|z)$$

where z depends on f, is uniquely determined by f and has norm $\|z\| = \|g\|$.

Theorem 1.6.3 (Lebesgue Dominated Convergence Theorem)**.** *Let (f_n) be a sequence of integrable functions which converges almost everywhere to a real-valued measurable function f. If there exists an integrable function g such that $|f_n| \le g$ for all n, then f is integrable and*

$$\int f d\mu = \lim_{n \to \infty} \int f_n d\mu$$

Theorem 1.6.4 (The Inverse Function Theorem)**.** *The inverse function theorem states, roughly speaking, that a continuously differentiable mapping f is invertible in a neighborhood of any point x at which the linear transformation $f'(x)$ is invertible.*
Statement of the Theorem*: Suppose f is a C^1-mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $f'(a)$*

is invertible for some $a \in E$*, and* $b = f(a)$*. Then*

(a) there exist open sets U *and* V *in* $\mathbb{R}^n$ *such that* $a \in U$, $b \in V$ f *is one-to-one on* U*, and* $f(U) = V$

(b) if g *is the inverse of* f *[which exists, by (a)], defined in* V *by*

$$g(f(x)) = x \qquad (x \in U),$$

then $g \in C^1(V)$

Chapter 2

Differential Calculus

2.1 Differentiation

Definition 2.1.1 (Differentiation in $\mathbb{R}^n$)**.** *Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$ and $x \in E$, if there exists a linear transformation A of $\mathbb{R}^n$ into $\mathbb{R}^m$ such that:*

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|_{\mathbb{R}^m}}{\|h\|_{\mathbb{R}^n}} = 0$$

then we say that f is differentiable at x and we write $f^{'}(x) = A$.
If f is differentiable at every $x \in E$, we say that f is differentiable in E

2.1.2 Frechet and Gateaux Derivative

The Frechet-differential is nothing else than the natural extension to Banach Spaces of the usual definition of a differential map in Euclidean spaces.
Let U be an open subset of X and consider a map $F : U \to Y$

Definition 2.1.3. *Let $u \in U$. We say that F is (Frechet) differentiable at u if there exists $A \in L(X, Y)$ such that, if we set*

$$R(h) = F(u + h) - F(u) - A(h),$$

there results

$$R(h) = o(\|h\|_X) \tag{2.1}$$

that is

$$\frac{\|R(h)\|_Y}{\|h\|_X} \to 0 \quad as \quad \|h\|_X \to 0$$

Such an A is uniquely determined and will be called the (Frechet) differerntial of F at u and is denoted by $A = DF(u)$. If F is differentiable at all $u \in U$, we say that F is differentiable in U.

Now there should be no misunderstanding between differentiablity and Frechet differentiablity, as they are refered in same context. A few comments on the preceding definition are in order.

Lemma 2.1.4. *If a function f is differentiable at a pioint $u \in U$ then there exists $A \in L(X,Y)$ satisfying* (2.1) *and this A is unique.*

Proof. Let us prove the above by assuming contrary ie there is a $B \in L(X,Y)$ which is also satisfying (2.1), then

$$\begin{aligned}\frac{\|Ah - Bh\|_Y}{\|h\|_X} &= \frac{\|Ah - Bh - F(u+h) - F(u) + F(u+h) + F(u)\|_Y}{\|h\|_X}\\ &= \frac{\|F(u+h) - F(u) - Bh - (F(u+h) - F(u) - Ah)\|_Y}{\|h\|_X}\\ &\leq \frac{\|F(u+h) - F(u) - Ah\|_Y + \|F(u+h) - F(u) - Bh\|_Y}{\|h\|_X}\end{aligned}$$

Since F is differentiable at u, thus
$\|F(u+h) - F(u) - Ah\|_Y \to 0$ as $\|h\|_X \to 0$ and $\|F(u+h) - F(u) - Bh\|_Y \to 0$ as $\|h\|_X \to 0$ and hence

$$\frac{\|Ah - Bh\|_Y}{\|h\|_X} \to 0 \quad as \quad \|h\|_X \to 0. \tag{2.2}$$

if $A \neq B$ there exists $h^* \in X$ such that $a := \|Ah^* - Bh^*\|_Y \neq 0$, taking $h = th^*$, $t \in \mathbb{R} - \{0\}$, one has

$$\frac{\|A(th^*) - B(th^*)\|_Y}{\|th^*\|_X} = \frac{|t|\|Ah^* - Bh^*\|_Y}{|t|\|h^*\|_X} = \frac{a}{\|h^*\|_X}$$

a constant in contradiction to (2.2). □

Lemma 2.1.5. ***Continuity of*** F : *If F is differentiable at u, then,*
$F(u+h) = F(u) + DF(u)h + o(\|h\|)$ and F is continuous at same point.
To prove this, we see that

$$F(u+h) - F(u) = A(h) + o(\|h\|_X)$$

$$\|F(u+h) - F(u)\|_Y = \|A(h) + \frac{o(\|h\|_X)}{\|h\|_X}\|h\|_X\|_Y$$

$$\|F(u+h) - F(u)\|_Y \leq \|A\|\|h\|_X + \|\frac{o(\|h\|_X)}{\|h\|_X}\|h\|_X\|_Y$$

Thus when $\|h\|_X \to 0$, $\|F(u+h) - F(u)\|_Y \to 0$ proving the continuity of F at u.
Conversely if $F \in C(U,Y)$, then it is not necessary to have the continuity of A in definition(2.1.3). In fact (2.1)*,yields*

$$A(h) = F(u+h) - F(u) - o(\|h\|)$$

and the continuity of F implies the continuity of A.

example 2.1.6. *$F : U \to Y$.The constant map $F(u) = c$ is differentiable at any u and $DF(u) = 0$ for all $u \in X$.*

Since $F(u+h) = c$ and $F(u) = c$. Also if F is differentiable then there exists $A \in L(X,Y)$ such that (**??**) holds. Also $0 \in L(X,Y)$ and when $A = 0$ then (2.1) always holds for any $u \in U$. Thus, $DF(u) = 0$ for all $u \in U$.

example 2.1.7. *If $A \in L(X,Y)$.Then $DA(u) = A$.*

Since $A(u+h) = A(u) + A(h)$. Then $A(u+h) - A(u) = A(h)$. Thus, if A is differentiable then there exists $G \in L(X,Y)$ such that (2.1) holds. Thus if $G = A$, then (2.1) always holds. Thus $G = DA(u) = A$.

Definition 2.1.8 (Bilinear map)**.** *Let A,B and C are the vector spaces over the same base field F. A bilinear map is a function.*
$T : A \times B \to C$, such that for any $b \in B$ the map $a \to T(a,b)$ is a linear map from A to C and for any $a \in A$ the map $b \to T(a,b)$ is a linear map from $B \to C$

example 2.1.9. *Let $B : X \times Y \to Z$ be a bilinear continuous map. Then*

$$\begin{aligned} B(u+h, v+k) &= B(u, v+k) + B(h, v+k) \\ &= B(u,v) + B(u,k) + B(h,v) + B(h,k) \end{aligned}$$

From the continuity of B at the origin, we have

$$\|B(h,k)\|_Z \le c\|h\|_X \|k\|_Y$$

. Then B is differentiable at any $(u,v) \in X \times Y$ and $DB(u,v)$ is the map $(h,k) \to B(h,v) + B(u,k)$.

example 2.1.10. *Let H be a Hilbert space with scalar product $(.|.)$ and consider the map*

$$F : u \to \|u\|^2 = (u|u).$$

Now,

$$\begin{aligned} \|u+h\|^2 - \|u\|^2 &= (u+h, u+h) - (u,u) \\ &= (u,u) + (h,u) + (u,h) + (h,h) - (u,u) \end{aligned}$$

since here we are considering only real vector spaces. Thus

$$\|u+h\|^2 - \|u\|^2 = \|h\|^2 + 2(u,h)$$

It follows that F is differentiable at any u and $DF(u)h = 2(u,h)$. Note that $\|.\|$ is not differentiable at $u = 0$. For otherwise, $\|h\| = Ah + o(\|h\|)$ for some $A \in L(H, \mathbb{R})$. Replacing h with $-h$ we get that $\|h\| = -Ah + o(\|h\|)$ and hence $\|h\| = o(\|h\|)$, a contradiction.

example 2.1.11. *If $X = \mathbb{R}$, $U = (a,b)$ and $F : U \to Y$ is differentiable at $t \in U$, the differential $DF(t)$ can be identified with $DF(t)[1] \in Y$ through the canonical isomorphism $i : L(\mathbb{R}, Y) \to Y$, i(A)= A(1). For example, if $Y = \mathbb{R}^n$ and $F(t) = (f_i(t))_{i=1,\dots,n}$, $DF(t)$ is the vector with components $\dfrac{Df_i}{dt}$.*

Proposition 2.1.12. *The main differentiation rules are collected in the following proposition.*
Let F,G : $U \to Y$. If F and G are differentiable at $u \in U$, then $aF + bG$ is differentiable at u for any a, $b \in \mathbb{R}$ and $D(aF + bG)(u)h = aDF(u)h + bDG(u)h$

Definition 2.1.13. *Let $F : U \to Y$ be differentiable in U. The map $F^{'} : u \to DF(u)$ is called the*

(Frechet) derivative of F. If $F^{'}$ is continuous as a map from $U \to L(X,Y)$ we say that F is C^1 and write $F^1 \in C^1(U,Y)$

Now we shall introduce the concept of variational (or potential) operator. If $Y = \mathbb{R}$, maps $J : U \to \mathbb{R}$ are usually called functionals and $J^{'}$ turns out to be a map from U to $L(X,\mathbb{R}) = X^*$ (the dual of X). In particular, if $X = H$ is a Hilbert space, $J^{'}(u) \in H^*$ for all u and the Riesz-Representation Theorem allows us to identify $J^{'}(u)$ with an element of H.

Definition 2.1.14. *Given a differentiable functional $J : U \to \mathbb{R}$, where $U \subset H$, the gradient of J at u denoted by $\nabla J(u)$, is the element of H, denoted by*

$$(\nabla J(u)|h) = DJ(u)h, \quad for \quad all \quad h \in H.$$

A map $F : U \to H$ with the property that there exists a differentiable functional $J : U \to \mathbb{R}$ such that $F = \nabla J$ is called a variational (or potential) operator.

As for maps in $\mathbb{R}^n$, we can also define here a directional derivative, usually called the Gateaux differential (for short G-differential).

Definition 2.1.15. *Let $F : U \to Y$ be given and let $x \in U$. We say that F is G-differentiable at u if there exists $A \in L(X,Y)$ such that for all $h \in X$ there results*

$$\frac{F(u+\epsilon h) - F(u)}{\epsilon} \to Ah \quad as \quad \epsilon \to 0$$

The map A is uniquely determined , called the G-differential of F at u and is denoted by $D_G F(u)$.

Theorem 2.1.16. *If F is Frechet-differentiable at $u \in U$, then F is also G-differentiable there and the two differentials coincide.*

Proof : Given $F : U \to Y$, where $U \subset X$ is open, is Frechet differentiable at a point $u \in U$, then

$$F(u+h) - F(u) = Ah + o(\|h\|), \text{ where } \frac{o(\|h\|)}{\|h\|} \to 0 \text{ as } \|h\| \to 0$$

Also $\epsilon h \to 0$ as $\epsilon \to 0$, replacing h by ϵh in above ,we have

$$F(u+\epsilon h) - F(u) = A\epsilon h + o(\|\epsilon h\|)$$

Since A is linear and multipling and dividing $o(\|\epsilon h\|)$ by $\|\epsilon h\|$, we have

$$\frac{F(u+\epsilon h) - F(u)}{\epsilon} = Ah + \|h\| \frac{o(\|\epsilon h\|)}{\|\epsilon h\|}$$

Taking $\epsilon \to 0$ in above , we have

$$\frac{F(u+\epsilon h) - F(u)}{\epsilon} \to Ah \quad as \quad \epsilon \to 0.$$

Thus, $A = D_G F(u)$ and hence

$$D_G F(u) : h \to Ah$$

Conversely, the G-differentiablity of F doesnot even imply the continuity of F. Recall the elementary example. let $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here in this example, we find that $f^{'}(0;y)$ (directional derivative of f wrt y at $a = (0,0)$ exists for all directions y without the function even being continuous at point $a = (0,0)$.

Theorem 2.1.17. *Let $F : U \to Y$ be $G-$differentiable at any point of U. Given $u,\, v \in U$ such that $[u,v] \subset U$, there results*

$$\|F(u) - F(v)\| \leqslant \sup\{\|D_G F(w)\| : w \in [u,v]\}\|u - v\| \tag{2.3}$$

Proof : If $F(u) = F(v)$ then (2.3) is trivial, thus we can assume here that $F(u) \neq F(v)$ i.e $F(u) - F(v) \neq 0$. Thus by an application of the Hahn Banach theorem there exists $\psi \in Y^*$ such that $\|\psi\| = 1$ and

$$(\psi, (F(u) - F(v))) = \|F(u) - F(v)\| \tag{2.4}$$

Let $\gamma(t) = tu + (1-t)v, t \in [0,1]$ and consider the map $h : [0,1] \to \mathbb{R}$ defined by setting

$$\begin{aligned} h(t) &= (\psi, F[\gamma(t)]) \\ &= (\psi, F(tu + (1-t)v)) \end{aligned}$$

From, $\gamma(t+\tau) = \gamma(t) + \tau(u - v)$, it follows that

$$\frac{h(t+\tau) - h(t)}{\tau} = \left(\psi, \frac{F[\gamma(t+\tau)] - F[\gamma(t)]}{\tau}\right)$$

$$\frac{h(t+\tau) - h(t)}{\tau} = \left(\psi, \frac{F[\gamma(t) + \tau(u-v)] - F[\gamma(t)]}{\tau}\right) \tag{2.5}$$

Since F is G-differentiable in U, and ψ is linear thus by passing the limit in (2.5) as $\tau \to 0$, we find

$$h^{'}(t) = (\psi, D_G F(tu + (1-t)v)(u - v)) \tag{2.6}$$

Applying the Mean-Value Theorem to h one has

$$h(1) - h(0) = h^{'}(\theta) \quad for \quad some \quad \theta \in (0,1) \tag{2.7}$$

Now substituting (2.4) and (2.6) into (2.7), we have

$$\begin{aligned} \|F(u) - F(v)\| &= h(1) - h(0) = h^{'}(\theta) \\ &= (\psi, D_G F(\theta u + (1-\theta)v)(u - v)) \\ &\leqslant \|\psi\|\|D_G F(\theta u + (1-\theta)v)(u - v)\| \\ &\leqslant \|\psi\|\|D_G F(\theta u + (1-\theta)v)\|\|u - v\| \end{aligned}$$

Since $\|\psi\| = 1$ and $\theta u + (1-\theta)v \in [u, v]$, thus

$$\|F(u) - F(v)\| \leqslant \sup\{\|D_G F(w)\| : w \in [u, v]\}\|u - v\|$$

and the theorem is proved.
As a consequence of the above theorem we find a classical criteria of Frechet differentiablity as the same we used to have in $\mathbb{R}^n$ case.

Theorem 2.1.18. *Suppose $F : U \to Y$ is G-differentiable in U and let*

$$F_G^{'} : U \to L(X, Y), \quad F_G^{'}(U) = D_G F(u),$$

be continuous at $v \in U$. Then F is Frechet-differentiable at v and $DF(v) = D_G F(v)$

$Proof$: We start by setting

$$R(h) = F(v + h) - F(v) - D_G F(v)h$$

Since F is G-differentiable for every u in U and $D_G F(u) \in L(X, Y)$, so is F-differentiable and hence G-differentiable also. Thus R being sum of three G-differentiable functions is also G-differentiable in B_ϵ, for every $\epsilon > 0$ small enough and

$$R(h + \epsilon h) - R(h) = F(v + h + \epsilon h) - F(v + h) - \epsilon D_G F(v)h$$

Thus,

$$\frac{R(h + \epsilon h) - R(h)}{\epsilon} = \frac{F(v + h + \epsilon h) - F(v + h)}{\epsilon} - D_G F(v)h$$

Now letting $\epsilon \to 0$, we get

$$D_G R(h) : k \to D_G F(v + h)k - D_G F(v)k \tag{2.8}$$

Now applying Mean Value Theorem with $[u, v] = [0, h]$,we find

$$\|R(h)\| \leqslant \sup_{0 \leqslant t \leqslant 1} \|D_G R(th)\|\|h\| \tag{2.9}$$

From (2.8), using th instead of h, we have

$$\|D_G R(th)\| = \|D_G F(v + th) - D_G F(v)\|$$

Substituting above into (2.9) we find

$$\|R(h)\| \leqslant \sup_{0 \leqslant t \leqslant 1} \|D_G F(v + th) - D_G F(v)\|\|h\|$$

Since $F_G^{'}$ is continuous at v, we thus have

$$\sup_{0 \leqslant t \leqslant 1} \|D_G F(v + th) - F(v)\| \to 0 \quad as \quad \|h\| \to 0$$

and therefore $\dfrac{\|R(h)\|}{\|h\|} \to 0$ as $\|h\| \to 0$ proving the differentiablity of F at v.
Thus by the use of above theorem one can find the Frechet differential of F by determing $D_G F$ and showing the continuity of $F_G^{'}$

Chapter 3

Nemitski operators

Here we are going to study the continuity and differentiablity of an important class of operators arising in nonlinear analysis, the so called Nemitski operators.

3.1 Nemitski operators

Let Ω be any open bounded subset of $\mathbb{R}^n$ and let $M(\Omega)$ denote the class of real-valued functions $u : \Omega \to \mathbb{R}$ that are measurable on Ω. Here, and always hereafter, the measure is the Lebesgue one and will be denoted by μ; all the functions we deal here are taken in $M(\Omega)$.
Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be given

Definition 3.1.1. *The Nemitski operator associated to f is the map defined on $M(\Omega)$ by setting*

$$u(x) \to f(x, u(x))$$

The same symbol f will be used to denote both function f and its Nemitski operator.
We shall assume that f is a Caratheodory function. More precisely, we will say that f satifies (C) if

1. $s \to f(x, s)$ is continuous for almost every $x \in \Omega$
2. $x \to f(x, s)$ is measurable for all $s \in \mathbb{R}$

For the purpose of analysis it is particularly interesting when the Nemitski operators act on Lebesgue spaces $L^p = L^p(\Omega)$. (Now onwards L^p will be used for $L^p(\Omega)$)
We start by noticing that

$$f(u) \in M(\Omega) \quad \forall u \in M(\Omega)$$

The above assertion is true, if $u \in M(\Omega)$ there is a sequence $\{\phi_n\}$ of simple functions such that $\phi_n \to u$ a.e in Ω. From (C), it follows that

$$f(\phi_n) \text{ is measurable and } \quad f(\phi_n) \to f(u) \text{ a.e in } \Omega$$

and since the limit of a sequence of mesaurable functions is measurable, thus $f(u) \in M(\Omega)$.

3.1.2 Continuity of Nemitski operators:

Let p, $q \geq 1$ and suppose

$$|f(x,s)| \leq a + b|s|^\alpha, \quad \alpha = \frac{p}{q} \tag{3.1}$$

for some constants a, $b > 0$

Theorem 3.1.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded and suppose f satisfies (C) and (3.1). Then the Nemitski operator f is a continuoius map from L^p to L^q.*

$Proof$: For the proof we need the following mesure-theoretic result
If $\mu(\Omega) < \infty$ and also $u_n \to u$ in L^p. Then there exist a sub-sequence $\{u_{n_k}\}$ and $h \in L^p$ such that

$$u_{n_k} \to u \text{ a.e. in } \Omega, \tag{3.2}$$

$$|u_{n_k}| \leq h \text{ a.e in } \Omega. \tag{3.3}$$

Since as $u \in L^p$ and from (3.1), we have

$$|f(x,u)| \leq a + b|u|^\alpha, \quad \alpha = \frac{p}{q}$$

$$\begin{aligned} \int_\Omega |f(u)|^q d\mu &\leq \int_\Omega (a + b|u|^\alpha)^q d\mu \\ &\leq 2^{q-1}\left\{|a|^q \mu(\Omega) + b^q \int_\Omega |u|^p d\mu\right\} \end{aligned}$$

Since a, b, q are finite, Ω is bounded and $u \in L^p$, thus

$$\int_\Omega |f(u)|^q d\mu \quad < \infty$$

Hence, $f(u) \in L^q$ whenever $u \in L^p$
To show that f is continuous from L^p to L^q, let u_n ,$u \in L^p$ be such that

$$\|u_n - u\|_{L^p} \to 0$$

By measure theoretic result, we can find a sub-sequence $\{u_{n_k}\}$ of $\{u_n\}$ and $h \in L^p$ satisfying (3.2) and (3.3). Since u_{n_k} converges almost everywhere to u, it readily follows from (C) that

$$f(u_{n_k}) \to f(u) \text{ a.e in } \Omega \tag{3.4}$$

Moreover, from (3.1) and (3.2) we infer

$$|f(u_{n_k})| \leq a + b|u_{n_k}|^\alpha \leq a + b|h|^\alpha \in L^q \tag{3.5}$$

As an immediate consequence of the Lebesgue Dominated-Convergence Theorem, (3.4) and (3.5), we get

$$\|f(u_{n_k}) - f(u)\|^p_{L^q} = \int_\Omega |f(u_{n_k}) - f(u)|^q \to o$$

Since any sequence u_n converging to u in L^p has a subsequence u_{n_k} such that $f(u_{n_k}) \to f(u)$ in L^q, we can conclude that f is continuous at u, as a map from L^p to L^q.

Remark 3.1.4. *The above theorem can also be proved assuming that f satisfies* (3.1) *with a replaced by $a(x) \in L^q$.*

Remark 3.1.5. *It is possible to show that, if (C) holds and $f(u) \in L^q$ for all $u \in L^p$, then $f \in C(L^p, L^q)$.*

3.1.6 Differentiablity of Nemitski operators :

Here in this section we deal with the differentiablity of Nemitski operators. Now some remarks are in order.
Let $p > 2$ and suppose f has partial derivative $f_s = \dfrac{\delta f}{\delta s}$ satisfying (C) and such that

$$|f_s(x,s)| \leq a + b|s|^{p-2} \tag{3.6}$$

for some constants $a,\ b > 0$. Since f_s satisfies (3.6), Theorem (3.1.3) implies that the Nemitski operator f_s is continuous from L^p to L^r, with $r = p/(p-2)$. As a cosequence, for the function $f_s(u)v$ defined by

$$f_s(u)v : x \to f_s(x, u(x))v(x)$$

Proposition 3.1.7. *$f_s(u)v \in L^{p'}$ for all $u,\ v \in L^p$ where $p' = p/(p-1)$ is the conjugate exponent of p.*

Proof : When $u \in L^p$, $f_s(u) \in L^r$ where $r = p/(p-2)$. Thus, by the generalised Holder inequality $f_s(u)v \in L^{p/(p-1)}$

Theorem 3.1.8. *Let $\Omega \subset \mathbb{R}^n$ be bounded and suppose that $p > 2$ and f satisfies (C). Moreover, we suppose that $f(x,0)$ is bounded and that f has partial derivative f_s satisfying (C) and* (3.6)*.*
Then $f : L^p \to L^{p'}$ is Frechet-differentiable on L^p with differential

$$Df(u) : v \to f_s(u)v.$$

Proof. Intergrating (3.6) w.r.to s form 0 to s, we get

$$\begin{aligned}
\int_0^s |f_s(x,s)|ds &\leq \int_0^s (a + b|s|^{p-2})ds \\
|f(x,s)|_0^s &\leq as + \left[\frac{b|s|^{p-1}}{p-1}\right]_0^s \\
|f(x,s)| - |f(x,0)| &\leq as + \frac{b|s|^{p-1}}{p-1}
\end{aligned}$$

Since, $f(x,0)$ is bounded, we find constants c, d such that

$$|f(x,s) \leq c + d|s|^{p-1}$$

Thus, by previous theorem we find that f is continuous as a map from L^p to $L^{p'}$, where $p' = \dfrac{p}{p-1}$

For $u, v \in L^p$, we evaluate

$$w(u,v) = \|f(u+v) - f(u) - f_s(u)v\|_{L^{p'}}$$

$$= \left[\int |f(x,u(x)) + v(x)| - f(x,u(x)) - f_s(x,u(x))v(x)|^{p'}\right]^{\dfrac{1}{p'}}$$

Now, consider

$$|f(u+v) - f(u) - f_s(u)v| = |\int_u^{u+v} f_s(s)ds - f_s(u)v|$$

using change of variables,

$$s - u = vt, \qquad t \in [0,1]$$

$$\begin{aligned}
|f(u+v) - f(u) - f_s(u)v| &= |\int_0^1 f_s(u+tv)dt - f_s(u)v| \\
&= |v\int_0^1 [f_s(u+tv) - f_s(u)]\, dt| \\
&= |v(x)\int_0^1 [f_s(x,u(x)+tv(x)) - f_s(x,u(x))]dt| \\
&= |v(x)w(x)|
\end{aligned}$$

where

$$w(x) = \int_0^1 [f_s(x,u(x)+tv(x)) - f_s(x,u(x))]dt$$

Thus,

$$w(u,v) = \left[\int_\Omega |v(x)w(x)|^{p'} dx\right]^{1/p'}$$

Now, using Holder's inequality we have

$$w(u,v) \le \|v\|_{L^p}\|w\|_{L^r}, \text{ where r} = \frac{p}{p-2} \tag{3.7}$$

Now, the norm $\|w\|_{L^r}$ can be estimated as follows,

$$\begin{aligned}
\|w\|_{L^r}^r &= \int_\Omega |w|^r dx \\
&= \int_\Omega dx |\int_0^1 [f_s(x,u(x)+tv(x)) - f_s(x,u(x))]dt|^r \\
&\le \int_\Omega dx \int_0^1 |[f_s(x,u(x)+tv(x)) - f_s(x,u(x))]|^r dt \\
&= \int_0^1 dt \int_\Omega |[f_s(x,u(x)+tv(x)) - f_s(x,u(x))]|^r dt
\end{aligned}$$

Hence,

$$\|w\|_{L^r}^r \le \int_0^1 \|f_s(u+tv) - f_s(u)\|_r^r dt \tag{3.8}$$

Also, we know that f_s is continuous from L^p to L^r, hence, $\|f_s(u+tv) - f_s(u)\|_{L^r}^r \to 0$ as $\|v\|_{L^p} \to 0$, $t \in [0,1]$.
Thus, by (3.7) and (3.8), it follows that $w(u,v) = o(\|v\|_{L^p})$ and also $\dfrac{o(\|v\|_{L^p})}{\|v\|_{L^p}} \to 0$ as $\|v\|_{L^p} \to 0$.
Hence, we find that $f : L^p \to L^{p'}$ is Frechet-differentiable on L^p. $\square$

For case $p = 2$, above result doesnot hold, in general. Indeed, under the preceeding assumptions, the Nemitski operator f is G-differentiable, but possibly not Frechet differentiable. To be precise, let us assume that (C) holds for both f, f_s and

$$\|f_s(x,s)\| \leq const. \tag{3.9}$$

As before, it follows plainly that f is continuous from L^2 to L^2 and the map $v \to f_s(u)v$ from L^2 to L^2 is linear and bounded. In addition one has the following

Theorem 3.1.9. *Let $\Omega \subset \mathbb{R}^n$ be bounded and let f and f_s satisfy (C) and (3.9). Then $f : L^2 \to L^2$ is G-differentiable and $D_G f(u)[v] = f_s(u)v$.*

Proof. By the definition of G-differentiablity, we have to show that for all u, $v \in L^2$, there results

$$\|\frac{f(u+tv) - f(u)}{t} - f_s(u)v\|_{L^2} \to 0 \text{ as } t \to 0 \tag{3.10}$$

Now we know that,

$$\frac{f(u+tv) - f(u)}{t} - f_s(u) = \int_u^{u+tv} \frac{f_s(s)}{t} ds - f_s(u)v$$

Now by using change of variables.

$$s - u = \lambda tv$$

thus,

$$\begin{aligned}\frac{f(u+tv) - f(u)}{t} - f_s(u) &= \int_0^1 \frac{f_s(u+\lambda tv)}{t} tv d\lambda - f_s(u)v \\ &= v \int_0^1 \left[f_s(u+\lambda tv) - f_s(u)\right] d\lambda\end{aligned}$$

letting $w_t = w_t(u,v) = \int_0^1 \left[f_s(u+\lambda tv) - f_s(u)\right] d\lambda$
One has

$$\begin{aligned}\|\frac{f(u+tv) - f(u)}{t} - f_s(u)v\|_{L^2}^2 &= \int_\Omega v^2 |\int_0^1 \left[f_s(u+\lambda tv) - f_s(u)\right] d\lambda|^2 \\ &\leq \int_\Omega v^2 \int_0^1 |f_s(u+\lambda tv) - f_s(u)|^2 d\lambda\end{aligned}$$

when $t \to 0$, $t\lambda v \to 0$ a.e in Ω and hence

$$f_s(u+\lambda tv) - f_s(u) \to 0 \text{ a.e in } \Omega$$

Since,

$$|f_s(x, u(x) + \lambda t v(x)) - f_s(x, u(x))|^2 \leq const.$$

Then, the Lebesgue Dominated Convergence theorem implies

$$\int_0^1 |f_s(u + \lambda t v) - f_s(u)|^2 d\lambda \to 0 \text{ as } t \to 0, \tag{3.11}$$

and hence (3.10) follows $\square$

Chapter 4

Higher Derivatives

Let $F \in C(U,Y)$ be differentiable in the open set $U \subset X$ and consider $F^{'} : U \to L(X,Y)$

Definition 4.0.1. *Let $u^* \in U$: F is twice (Frechet) differentiable at u^* if $F^{'}$ is differentiable at u^*. The second (Frechet) differential of F at u^* is defined as*

$$D^2F(u^*) = DF^{'}(u^*)$$

If F is twice differentiable at all points of U we say that F is twice differentiable in U.

According to the above definition $D^2F(u^*)$ is a linear continuous map from X to $L(X,Y)$:

$$D^2F(u^*) \in L(X, L(X,Y))$$

It is convenient to see $D^2F(u^*)$ as a bilinear map on X. For this,

$$D^2F(u^*) \in L(X, L(X,Y))$$

$$D^2F(u^*)[h_1] \in L(X,Y)$$

$$D^2F(u^*)[h_1] : X \to Y$$

$$D^2F(u^*)[h_1,h_2] = F^{''}(u^*)[h_1,h_2]$$

In the following we will use the same symbol $D^2F(u^*)$ to denote the continuous bilinear map. The value of $D^2F(u^*)$ at a pair (h,k) will be denoted by

$$D^2F(u^*)[h,k]$$

If F is twice differentiable in U, the second (Frechet) derivative of F is the map $F^{''} : U \to L_2(X,Y)$,

$$F^{''} : u \to D^2F(u)$$

If $F^{''}$ is continuous from U to $L_2(X,Y)$ we say that $F \in C^2(U,Y)$.

example 4.0.2. *If $A \in L(X,Y)$ then $A \in C^2(X,Y)$ and $D^2A[h,k] = 0$ for all $(h,k) \in X \times X$.*

example 4.0.3. *Let* $X = C([0,1])$ *and* $F : X \to X$, $F : u(t) \to u^2(t)$. $F \in C^2(X, X)$ *and*

$$D^2F(u) : (h(t), k(t)) \to 2h(t)k(t)$$

The following proposition can be useful for evaluating $D^2F(u)$.

Proposition 4.0.4. *Let* $F : U \to Y$ *be twice differentiable at* $u^* \in U$. *Then for all fixed* $h \in X$ *the map* $F_h : X \to Y$ *defined by setting*

$$F_h(u) = DF(u)h$$

is differentiable at u^* *and* $DF_h(u^*)k = F^{''}(u^*)[h, k]$.

Proof. F_h is obtained by composition

$$DF : U \to L(X, Y) \quad \text{and} \quad \mathcal{E}_h : L(X, Y) \to Y$$

$$u \to DF(u) \to DF(u)h$$

between the derivative $u \to DF(u)$ and the "evaluation map" $\mathcal{E}_h$ which associates to each $A \in L(X, Y)$ the value $A(h) \in Y$. Since $\mathcal{E}_h$ is linear, the result follows from the composition mapping formula. □

We have seen that $F^{''}(u)$ can be regarded as a bilinear map. So, we have the following theorem

Theorem 4.0.5. *If* $F : U \to Y$ *is twice differentiable at* $u \in U$, *then* $F^{''}(u) \in L_2(X, Y)$ *is symmetric.*

Proof. For h , $k \in X$ with h , $k \in B(\epsilon)$(ϵ -small enough), we set

$$\psi(h, k) = F(u + h + k) - F(u + h) - F(u + k) + F(u),$$

$$\gamma_h(\xi) = F(u + h + \xi) - F(u + \xi),$$

and consider, for h fixed, the map $g_h : B(\epsilon) \to Y$,

$$g_h : k \to \psi(h, k) - F^{''}(u)[h, k] = \gamma_h(k) - \gamma_h(0) - F^{''}(u)[h, k].$$

Since F is differentiable in U and $F^{''}(u)(h) : k \to F^{''}(u)[h, k]$ is linear (as a map from X to $L(X, Y)$), Mean Value Theorem yields

$$\begin{aligned} \|\psi(h, k) - F^{''}(u)[h, k]\| &\leqslant \sup\left\{\|D\gamma_h(tk) - F^{''}(u)(h)\| : 0 \leqslant t \leqslant 1\right\}\|k\| \\ &= \sup\left\{\|DF(u + h + tk) - DF(u + tk) - F^{''}(u)(h)\| : 0 \leqslant t \leqslant 1\right\}\|k\| \end{aligned} \tag{4.1}$$

Since F is twice differentiable at $u \in U$, one has

$$F^{'}(u + h + tk) = F^{'}(u) + F^{''}(u)(h + tk) + \omega(h + tk)$$

$$F^{'}(u+tk) = F^{'}(u) + F^{''}(u)(tk) + \omega(tk) \text{ with } \omega(v) = o(\|v\|)$$

Hence,

$$F^{'}(u+h+tk) - F^{'}(u+tk) = F^{''}(u)h + \omega(h+tk) - \omega(tk) \tag{4.2}$$

using 4.1 and 4.2 and taking into account $\omega(v) = o(\|v\|)$. We get that

$$\begin{aligned}
\|\psi(h,k) - F^{''}(u)[h,k]\| &\leqslant \sup\{\|\omega(h+tk) - \omega(tk)\| : 0 \leqslant t \leqslant 1\} \|k\| \\
&\leqslant \sup\{\|\omega(h+tk)\| + \omega\|(tk)\| : 0 \leqslant t \leqslant 1\} \|k\| \\
&\leqslant \| \{\omega(h+k)\| + \omega\|(tk)\|\} \|k\| \\
&\leqslant \{\epsilon(\|h+k\|) + \epsilon(\|k\|)\} \|k\| \\
&\leqslant \epsilon(\|h\| + 2\|k\|)\|k\| \qquad (4.3)
\end{aligned}$$

Exchanging the roles of h,k, we get (for $\|h\|$, $\|k\|$ small)

$$\begin{aligned}
\|\psi(h,k) - F^{''}(u)[h,k]\| &\leqslant \sup\{\|\omega(k+th) - \omega(th)\| : 0 \leqslant t \leqslant 1\} \|h\| \\
&\leqslant \epsilon(\|k\| + 2\|h\|)\|h\| \qquad (4.4)
\end{aligned}$$

Using $\|A - B\| \leqslant \|A\| + \|B\|$ for (4.3) and (4.4)

$$\begin{aligned}
\|\psi(h,k) - F^{''}(u)[h,k]\| &\leqslant \epsilon(\|h\| + 2\|k\|)\|k\| + \epsilon(\|k\| + 2\|h\|)\|h\| \\
&\leqslant \epsilon(\|h\|\|k\| + 2\|k\|^2 + \|h\|\|k\| + 2\|h\|^2) \\
&\leqslant 2\epsilon(\|h\|\|k\| + \|k\|^2 + \|h\|^2) \\
&\leqslant 3\epsilon(\|h\|^2 + \|k\|^2) \qquad (4.5)
\end{aligned}$$

(4.5) has been proved for $\|h\|$,$\|k\|$ small enough but holds true for all $\|h\|$, $\|k\|$ because $F^{''}(u)[h,k]$ is homogeneous of degree 2. Since ϵ is arbitrary, (4.5) implies that $F^{''}(u)[h,k] = F^{''}(u)[k,h]$ for all h,k. □

To define $(n+1)^{th}$ derivatives ($n \geqslant 2$) we can proceed by induction. Given $F : U \to Y$, let F be n times differentiable in U. The n^{th} differential at a point $x \in U$ will be identified with a continuous n-linear map from $X \times X \times ... \times X$ (n times) to Y.
Let $F^n : U \to L_n(X,Y)$ denote the map
$F^n : u \to DF(u)$
The $(n+1)^{th}$ differential at u^* will be defined as the differential of F^n namely

$$DF^n : U \to L(X, L_n(X,Y))$$

$D^{n+1}F(u^*) = D^nF(u^*) \in L(X, L_n(X,Y)) \approx L_{n+1}(X,Y)$.
We will say that $F \in C^n(U,Y)$, if F is n-times (Frechet) differentiable in U. and the n^{th} derivative F^n is continuous from $U \to L_n(X,Y)$. The value of $D^nF(U^*)$ at $(h_1, h_2, ..., h_n)$ will be denoted by $D^nF(u^*)[h_1, h_2, ..., h_n]$.
If $h_1 = h_2 = ... = h_n$, we will write for short $D^nF(u^*)[h]^n$. In order to extend symmetric definition

for higher derivatives some preliminaries are in order. Given a map $G : U \to L_m(X,Y)$ and the point $h = (h_1, h_2, ..., h_m) \in X \times X \times ... \times X$, we can associate

$$G[h] : U \to Y \text{ defined by setting}$$

$$G[h](u) = G(u)[h_1, h_2, ..., h_m]$$

We can immediately see that if G is differentiable at u then $G[h]$ is differentiable at u and there results.

$$D(G[h])(u) : v \to DG(u)[v, h_1, h_2, ..., h_m] \tag{4.6}$$

Let F be n-times differentiable on U and set $h = (h_2, h_3, ..., h_m)$.
Applying (4.6) to $G = D^{n-1}F$, we find that

$$D(D^{n-1}F[h])(u^*) : h_1 \to D^nF(u^*)[h_1, h_2, ..., h_m]$$

$$D(D^{n-1}F[h])(u^*)[h_1] = D^nF(u^*)[h_1, h_2, ..., h_m] \tag{4.7}$$

Theorem 4.0.6. *If $F : U \to Y$ is n-times differentiable in U, then the map $(h_1, h_2, ..., h_n) \to D^nF(u^*)[h_1, h_2, ..., h_n]$ is symmetric.*

Proof. The result is true for $n = 2$. By induction on n, let the claim hold for $n - 1 \geqslant 2$. Then

$$D^{n-1}F(u)[h_2, ..., h_i, ..., h_j, ..., h_n] = D^{n-1}F(u)[h_2, ..., h_j, ..., h_i, ..., h_n]$$

Applying (4.7) to $h(u) = D^{n-1}F(u)[h_2, ..., h_i, ..., h_j, ..., h_n]$ we get that

$$D^nF(u^*)[h_1, ..., h_i, ..., h_j, ..., h_n] = D^nF(u^*)[h_1, ..., h_j, ..., h_i, ..., h_n] \tag{4.8}$$

Similarly, letting $G(u) = D^{n-2}F(u)[h_3, ..., h_n]$, one has $D^2G(u^*)[h_1, h_2] = D^nF(u)[h_1, h_2, ..., h_n]$ and from the symmetry of $L_2(X,Y)$, we have

$$D^2G(u^*)[h_1, h_2] = D^nF(u^*)[h_1, h_2, ..., h_n] = D^2G(u^*)[h_2, h_1] = D^nF(u^*)[h_2, h_1, ..., h_n] \tag{4.9}$$

From (4.8) and (4.9), we conclude the symmetry of $D^nF(u^*)$. $\square$

4.1 Partial Derivatives, Taylor's Formula

4.1.1 Partial Derivatives

Let us consider two Banach Spaces X , Y and let $(u^*, v^*) \in X \times Y$. Define mappings $\sigma_{v^*} : X \to X \times Y$ and $\tau_{u^*} : Y \to X \times Y$ as follows.

$$\sigma_{v^*}(u) = (u, v^*).$$

$$\tau_{u^*}(v) = (u^*, v).$$

Notice that the derivatives of σ_{v^*} and τ_{u^*} are respectively, the linear maps

$$\sigma := D\sigma_{v^*} : h \to (h, 0),$$

$$\tau := D\tau_{u^*} : k \to (0, k).$$

Let Q be an open subset of $X \times Y$, $(u^*, v^*) \in Q$ and $F : Q \to Z$.

Definition 4.1.2. *If the map $F \circ \sigma_{v^*}$ is differentiable at u^* we say that F is differentiable with respect to u at (u^*, v^*). The linear map $D[F \circ \sigma_{v^*}](u^*) \in L(X, Z)$ is called the partial derivative of F at (u^*, v^*) with respect to u and denoted by $D_u F(u^*, v^*)$.*
Similarly, if $F \circ \tau_{u^*}$ is differentiable at v^* we say that F is differentiable with respect to v at (u^*, v^*) and the linear map $D[F \circ \tau_{u^*}](v^*) \in L(Y, Z)$ is called the v-partial derivative of F at (u^*, v^*) and denoted by $D_v F(u^*, v^*)$.

The preceeding definition is equivalent to requiring that there exist a linear map $A_u \in L(X, Z)$ (respectively $A_v \in L(Y, Z)$), such that

$$F(u^* + h, v^*) - F(u^*, v^*) = A_u(h) + o(\|h\|)$$

$$F(u^*, v^* + k) - F(u^*, v^*) = A_v(k) + o(\|k\|)$$

Proposition 4.1.3. *If F is differentiable at (u^*, v^*) then F has partial derivatives with respect to u and v at (u^*, v^*) and we have*

$$D_u F(u^*, v^*)(h) = DF(u^*, v^*)\sigma(h) = DF(u^*, v^*)(h, 0).$$

$$D_v F(u^*, v^*)(k) = DF(u^*, v^*)\tau(k) = DF(u^*, v^*)(o, k).$$

Proof. Applying the derivative of composition map formula to the definition of Partial Derivatives we obtain the above proposition. $\square$

In quite similar way one can define higher partial derivatives. For example, if F has u partial derivative at all $(u, v) \in Q$, we can define the map $F_u : Q \to L(X, Z)$ by setting

$$F_u(u, v) = D_u F(u, v)$$

Then the partial derivative $D_{u,v} F(u^*, v^*)$ is the v^{th} derivative at (u^*, v^*) of F_u namely

$$D_{u,v} F(u^*, v^*) = D_v[F_u](u^*, v^*)$$

The map $F_{u,v} : Q \to L(Y, L(X, Z))$ will be defined by setting

$$F_{u,v}(u, v) = D_{u,v} F(u, v) = D_v[F_u](u, v) = D_v D_u F(u, v)$$

Moreover, if F is twice differentiable at (u^*, v^*) then $D_{u,v} F(u^*, v^*)$ is the bilinear map from $X \times Y \to Z$ given by

$$(h, k) \to F^{''}(u^*, v^*)[\sigma h, \tau k] \tag{4.10}$$

The notation $D^m_{u^l,v^{m-l}}$ will be employed to indicate

$$D^m_{u^l,v^{m-l}} = D^l_{u^l}(D^{m-l}_{v^{m-l}})$$

The definition of partial derivative given above permits us to obtain in a rather straight forward way all the classical results of calculus.
For example one can prove the following

Theorem 4.1.4. *Suppose that*
(i) F has $u-$ and $v-$ derivatives in a neighbourhood N of $(u^*, v^*) \in Q$,
(ii) F_u and F_v are continuous in N.
Then F is differentiable at (u^*, v^*)

As another example, we can use (4.10) and theorem (symmetry of $L_2(X, Y)$) to show

$$\begin{aligned} D_{u,v}F(u^*, v^*)[h, k] &= F^{''}(u^*, v^*)[\sigma h, \tau k] \\ &= F^{''}(u^*, v^*)[\tau k, \sigma h] \\ &= D_{v,u}F(u^*, v^*)[k, h] \end{aligned}$$

which is nothing other than the classical Schwarz Theorem.

4.1.5 Taylor's Formula

Let $F \in C^n(Q, Y)$ and let u, $u + v \in Q$ be such that the interval $[u, u + v] \subset Q$.
Set $\gamma(t) = u + tv$, $t \in [0, 1]$ and let $\phi : [0, 1] \to Y$ be defined by

$$\phi(t) = F(\gamma(t)).$$

Using differentiatioin of composition map theorem and (4.7) it follows readily that the function ϕ is C^n and there result

$$\phi^{'}(t) = DF(u + tv)[v],$$

$$\phi^{''}(t) = D^2F(u + tv)[v]^2,$$

$$\dots\dots\dots\dots\dots\dots$$

$$\phi^n(t) = D^nF(u + tv)[v]^n$$

By elementary calculations one has
$\phi(1) = \phi(0) + \phi^{'}(0) + \dfrac{1}{2!}\phi^{''}(0) + ... + \dfrac{1}{(n-1)!}\phi^{n-1}(0) + \dfrac{1}{(n-1)!}\int_0^1 (1-t)^{n-1}\phi^n(t)dt,$
and hence
$F(u + v) = F(u) + DF(u)[v] + ... + \dfrac{1}{(n-1)!}\int_0^1 (1-t)^{n-1}D^nF(u + tv)[v]^n dt$
The last integral can be written as

$$\frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1}D^{(n)}F(u+tv)dt[v]^n = \frac{1}{n!}D^nF(u)[v]^n + \epsilon(u, v)[v]^n, \tag{4.11}$$

where
$\epsilon(u,v) = \dfrac{1}{(n-1)!}\int_0^1 (1-t)^{n-1}[D^{(n)}F(u+tv) - D^{(n)}F(u)]dt \to 0 \quad as \quad v \to 0.$

Lastly, let us write explicitly the form of (4.11) when $F = F(u,v)$ is defined on $Q \subset X \times Y$ with values in Z and is C^n, that is, F has continuous partial derivaters up to order n. We write (u,v) instead of u and write $\omega = (h,k) = \sigma h + \tau k$. If we use proposition 4.1.3, the mth term in (4.11) becomes

$$\frac{1}{m!}D^m F(u,v)[w]^m = \frac{1}{m!}D^m F(u,v)[\sigma h, \tau k]$$

$$= \frac{1}{m!}D^{(m)}F(u,v)\sum \binom{m}{l}[\sigma h]^l[\tau k]^{m-l}$$

$$= \frac{1}{m!}\sum \binom{m}{l} D^{(m)}F(u,v)[\sigma h]^l[\tau k]^{m-l}$$

$$= \frac{1}{m!}\sum \binom{m}{l} D^m_{u^l,v^{m-l}}F(u,v)[h]^l[k]^{m-l}.$$

Remark 4.1.6 (on notation)**.** *Hereafter we will often deal with maps $F : \mathbb{R} \times X \to Y$ depending on a real parameter λ. In such a case the mixed derivative $F_{u,\lambda}(\lambda_0, u_0)$ is a linear map from $\mathbb{R} \to L(X,Y) : F_{u,\lambda}(\lambda_0, u_0) \in L(\mathbb{R}, L(X,Y))$. Then, in accordance with what we remarked in example 2.1.11., we can and will identify $F_{u,\lambda}(\lambda_0, u_0)$ with the linear map $h \to F_{u,\lambda}(\lambda_0, u_0)[h, 1]$.*

Chapter 5

Topological Degree

The topological degree (in short, degree) of a map is a classical tool which is very useful for solving functional equations. It was introduced by L. Brouwer for finite dimension and extended by J. Leray and J. Schauder to infinite dimension. Applications include The Brouwer fixed theorem, Leray-Schauder fixed point theorem, the Krasnoselski bifurcation theorem. There are many examples where the degree is used to solve elliptic differential equations.

5.1 Preliminaries for Degree

Showing that two spaces are homeomorphic is a matter of constructing a continuous mapping from one to another having a continuous inverse.
For showing that two spaces are not homeomorphic is a different matter. For that one must show that a continuous function with continuous inverse does not exist. If one can find some topological property that holds for one space but not for the other, then the problem is solved, then the spaces are not homeomorphic. Let us consider

example 5.1.1. $(0,1)$ *is not homeomorphic to* $[0,1]$, *as* $[0,1]$ *is compact whereas* $(0,1)$ *is non-compact.*

example 5.1.2. $\mathbb{R}$ *is not homeomorphic to* $\mathbb{R} \cup (-\infty, +\infty)$ *because* $\mathbb{R}$ *has a countable basis where as later doesnot have a countable basis.*

example 5.1.3. $\mathbb{R}$ *is not homeomorphic to* $\mathbb{R}^2$ *as deleting a point from* $\mathbb{R}^2$ *leaves* $\mathbb{R}^2$ *into a connected space, and deleting a point from* $\mathbb{R}$ *does not.*

But the topological properties we have studied upto now donot lead us to very far in solving the problem. For instance, how does one show that plane $\mathbb{R}^2$ is not homeomorphic to three-dimensional space $\mathbb{R}^3$? Topological properties like compactness, connectedness, local connectedness, metrizablity and soon, cannot distinguish between them.
For this we must introduce new properties and techniques. One of the most natural such properties is that of simple connectedness. We say a space X is simply connected if every closed curve in X is shrunk to a point in X. The property of simple connectivity, it turns out, will distinguish between

$\mathbb{R}^2$ and $\mathbb{R}^3$, deleting a point from $\mathbb{R}^3$ leaves a simply connected space remaining, but deleting a point from $\mathbb{R}^2$ doesnot.
There is an idea more general than idea of simple connectedness, an idea that includes simple connectedness as a special case. It involves a certain group that is called fundamental group of the space. Two spaces that are homeomorphic have fundamental groups that are isomorphic, and the condtion of the simple connectedness is just the condition that the fundamental group of X is the trivial group.
Applications include theorems about fixed points and antipode preserving maps of the sphere, as well as the well known fundamental theorem of algebra, Jordan Curve theorem.

5.1.4 Homotopy of Paths

Before defining the fundamental group of a space X, we shall consider paths on X and an equivallence relation called path homotopy between them.

Definition 5.1.5. *If f, $f^{'}$ are continuous maps of the space X into the space Y, we say that f is homotopic to $f^{'}$ if there is a continuous map $F : X \times I \to Y$ such that*

$$F(x,0) = f(x) \text{ and } F(x,1) = f^{'}(x) \text{ for each } x\ .$$

(Here $I = [0,1]$). The map F is called a homotopy between f and $f^{'}$. If f is homotopic to $f^{'}$, we write $f \approx f^{'}$. If $f \approx f^{'}$ and $f^{'}$ is a constant map, we say that f is nulhomotopic. We think of a homotopy as a continuous one parameter family of maps from $X \to Y$. If we imagine the parameter t as representing time, then the homotopy F represents a continuous deforming of the map f to the map $f^{'}$, as t goes from 0 to 1.
Now we consider the special case in which f is a path in X. Recall that if $f : [0,1] \to X$ is a continuous map such that $f(0) = x_0$ and $f(1) = x_1$ we say that f is a path in X from x_0 to x_1 where x_0 is an initial point and x_1 is a final point of the path f.
If f and $f^{'}$ are two paths in X, there is a stronger relation between them than mere homotopy. It is defined as follows

Definition 5.1.6. *Two paths f and $f^{'}$, mapping the interval $I = [0,1]$ into X are said to be path homotopic if they have the same initial point x_0 and same final point x_1, and if there is a continuous map $F : I \times I \to X$ such that*

$$F(s,0) = f(s) \text{ and } F(s,1) = f^{'}(s)$$

$$F(0,t) = x_0 \text{ and } F(1,t) = x_1$$

for each $s \in I$ and each $t \in I$. We call F a path homotopy between f and $f^{'}$. If f is path homotopic to $f^{'}$, we write $f \approx_p f^{'}$.
The first condition says simply that F is homotopy between f and $f^{'}$ and the second condition says that for each t, the path f_t defined by the equation $f_t(s) = F(s,t)$ is a path from x_0 to x_1. Said differently, the first condition says that F represents a continuous way of deforming the path f to the path $f^{'}$, and the second condition says that the end points of the path remains fixed during the

deformation.
If f is a path, we shall denote its path-homotopy equivalence class by $[f]$.

Lemma 5.1.7. *The relation $\approx$ and $\approx_p$ are equivalence relations.*

Proof. Let us verify the properties of an equivalence relation.
Given f it is trivial that $f \approx f$, the map $F(x,t) = f(x)$ is the required homotopy. If f is a path, F is a path homotopy.
Given $f \approx f^{'}$, we show that $f^{'} \approx f$. Let F be a homotopy between f and $f^{'}$. Then $G(x,t) = F(x, 1-t)$ is a homotopy between $f^{'}$ and f. If F is a path homotopy, so is G.
Suppose that $f \approx f^{'}$ and $f^{'} \approx f^{''}$. We show that $f \approx f^{''}$. Let F be a homotopy between f and $f^{'}$, let $F^{'}$ be a homotopy between $f^{'} \approx f^{''}$. Define $G : X \times I \to Y$ by the equation

$$G(x,t) = \begin{cases} F(x,2t) & t \in [0,1/2] \\ F(x,2t-1) & t \in [1/2,1] \end{cases}$$

The map G is well defined, since if $t = \frac{1}{2}$,
$F(X,2t) = F(x,1) = f^{'}(x) = F^{'}(x,0) = F^{'}(x,2t-1)$.
Because G is continuous on the two closed subsets $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ of $X \times I$, by the pasting lemma. Thus G is the required homotopy between f and $f^{''}$
We can also see that if F and $F^{'}$ are path homotopies, so is G. □

example 5.1.8. *Let f and g be any two continuous maps of a space X into $\mathbb{R}^2$. it is easy to see that f and g are homotopic, the map*

$$F(x,t) = (1-t)f(x) + tg(x) \qquad \textit{for} \quad 0 \leqslant t \leqslant 1$$

. is a homotopy between them. It is called a stright-line homotopy, because it moves the point $f(x)$ to the point $g(x)$ along the straight-line joining them.
If f and g are paths from x_0 to x_1, then F will be a path homotopy.

example 5.1.9. *Let X denote the punctured plane, $\mathbb{R}^2 - 0$, the following paths in X,*

$$f(s) = (cos\pi s, sin\pi s),$$

$$g(s) = (cos\pi s, 2sin\pi s)$$

are path homotopic, the straight-line homotopy between them is an acceptable path homotopy. But the straight-line homotopy between f and the path

$$h(s) = (cos\pi s, -sin\pi s)$$

is not acceptable, for its image does not lie in the space X.
Indeed, there exists no path homotopy in X between paths f and h. This result is hardly surprising; it is intuitively clear that one cannot deform f past the hole at 0 without introducing a discontinuity.

This example illustrates the fact that we must know what range space is before we can tell whether two paths are homotopic or not. The paths f and h would be path homotopic if they were paths in $\mathbb{R}^2$.
Now we introduce some algebra into this geometric situation. We define a certain operation on path-homotopy classes as follows:

Definition 5.1.10. *If f is a path in X from x_0 to x_1 and if g is a path in X from x_1 to x_2, we define the composition $f * g$ of f and g to be path h given by the equations*

$$h(s) = \begin{cases} f(2s) & s \in [0, 1/2] \\ g(2s-1) & s \in [1/2, 1] \end{cases}$$

The function h is well defined and continuous, by the pasting lemma; and it is a path in X from x_0 to x_2. We will think of h as a path whoose first half is the path f and whoose second half is the path g.
Furthermore, the operation $*$ on path-homotopy classes turns out to satisfy properties that look like very much like the axioms for a group. They are called the groupoid properties of $*$. The only difference from the properties of a group is that $[f] * [g]$ is not defined for every pair of classes, but only for those pairs $[f], [g]$ for which $f(1) = g(0)$.

Theorem 5.1.11. *The operation $*$ is well-defined on path-homotopy classes. It has the following properties:*
(1) *(Associativity): If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$ and they are equal.*
(2) *(Right and Left identities): Given $x \in X$, let e_x denote the constant path $e_x :\to X$ carrying all of I to point x. If f is a path in X from x_0 to x_1, then*

$$[f] * [e_{x_1}] = [f] \quad \text{and} \quad [e_{x_0}] * [f] = [f].$$

(3) *(Inverse): Given the path f in X from x_0 to x_1, let $\tilde{f}$ be the path defined by $\tilde{f} = f(1-s)$, is called the reverse of f. Then*

$$[f] * [\tilde{f}] = [e_{x_0}] \quad \text{and} \quad [\tilde{f}] * [f] = [e_{x_1}]$$

5.1.12 Fundamental Group

The set of path-homotopy classes of paths in a space X doesnot form a group under the operation $*$, only a groupoid. But suppose we pick out a point x_0 of X to serve as a base point and restrict ourselves to those paths that begin and end at x_0. The set of these path-homotopy classes does form a group under $*$. It will be called the **fundamental group** of X.

Definition 5.1.13. *Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a loop based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the* ***fundamental group*** *of X relative to the base point x_0. It is denoted by $\pi_1(X, x_0)$.*

It follows from the preceding theorem that the operation $*$, when restricted to this set, satisfies the axioms for a group. Given two loops f and g based at x_0, the composition $[f] * [g]$ is always defined and it a loop based at x_0. Associativity, the existence of an identity element $[e_{x_0}]$ and the existence of an inverse $[\tilde{f}]$ for $[f]$ are immediate.

Definition 5.1.14. *A space X is said to be* ***simply connected*** *if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$, and hence for every $x_0 \in X$. We often express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.*

Lemma 5.1.15. *In a simply connected space X, any two paths having the same initial and final points are path homotopic.*

Definition 5.1.16. *A homotopy is admissible (with respect to Ω and p) if $h(\lambda, x) \neq p$ for all $(\lambda, x) \in [0, 1] \times \partial\Omega$.*

5.2 Brouwer degree and its properties

Let us assume that:
(a) Ω is an open bounded set in $\mathbb{R}^n$, with boundary $\partial\Omega$;
(b) f is continuous map from $\overline{\Omega}$ to $\mathbb{R}^n$; the comonents of f will be denoted by f_i;
(c) p is a point in $\mathbb{R}^n$ such that $p \notin f(\partial\Omega)$.
To each triple (f, Ω, p) satisying $(a) - (c)$, one can associate an integer $deg(f, \Omega, p)$ called the degree of f (with respect to Ω and p), with the following properties.
(P.1) Normalization: If $I_{\mathbb{R}^n}$ denotes the identity map in $\mathbb{R}^n$, then

$$deg(I_{\mathbb{R}^n}, \Omega, p) = \begin{cases} 1 & \text{if} \quad p \in \Omega \\ 0 & \text{if} \quad p \notin \Omega \end{cases}$$

(P.2) Solution Property: If $deg(f, \Omega, p) \neq 0$, then there exists $z \in \Omega$ such that $f(z) = p$.
(P.3) $deg(f, \Omega, p) = deg(f - p, \Omega, 0)$.
(P.4) Decomposition: If $\Omega_1 \cap \Omega_2 = \phi$ then we have

$$deg(f, \Omega_1 \cup \Omega_2, p) = deg(f, \Omega_1, p) + deg(f, \Omega_2, p).$$

(P.5) Homotopy invariance: If h is an admissible homotopy, then $deg(h(\lambda, .), \Omega, p)$ is constant with respect to $\lambda \in [0, 1]$. In particular, if $f(x) = h(0, x)$ and $g(x) = h(1, x)$ then

$$deg(f, \Omega, p) = deg(g, \Omega, p).$$

(P.6) Continuity: If $f_k \to f$ uniformly in $\overline{\Omega}$, then $deg(f_k, \Omega, p) \to deg(f, \Omega, p)$. Moreover, $deg(f, \Omega, p)$ is continuous with respect to p.
(P.7) Excision property: Let $\Omega_0 \subset \Omega$ be an open set such that $f(x) \neq p$, for all $x \in \Omega \setminus \Omega_0$. Then $deg(f, \Omega, p) = deg(f, \Omega_0, p)$
We fill now prove each of the above property for a general case at the end of this section
First we consider a C^1 map f and a regular value p. We say p is a regular value for f, if the jacobian $J_f(x)$ is different from zero for every $x \in f^{-1}(p)$. The jacobian is basiaclly the determinant of the

matrix $f^{'}(x)$ with entries

$$a_{ij} = \frac{\partial f_i}{\partial x_j}$$

Lemma 5.2.1. *If p is a regular value for $f \in C^1$ then the set $f^{-1}(p)$ is finite.*

Proof. Since $f \in C^1$ and $f(x) = p$ is a closed set then the set $f^{-1}(p)$ is also closed and being subset of a bounded set in $\mathbb{R}^n$ is also bounded and hence is also compact. Also p being a regular value implies the set $J_f(x_0)$ is different from zero from all $x \in f^{-1}(p)$ and thus by Inverse Function Theorem, there exists a diffeomorphism between a ball $B_\epsilon(x_0)\epsilon > 0$ small enough and its image $U_\epsilon = f(B_\epsilon(x_0))$. Also, $B_\epsilon(x_i), x_i \in f^{-1}(p)$ is an open cover for $f^{-1}(p)$ and also $f^{-1}(p)$ is also compact then it has finite subcovers, and by diffeomorphism between $B_\epsilon(x_i) = f(B_\epsilon(x_i))$ no member of $f^{-1}(p)$ can come in any two balls and thus $f^{-1}(p)$ is finite. □

By previous lemma, we conclude that when p is a regular value for $f \in C^1$ the set $f^{-1}(p)$ is finite and hence one can define the degree by setting

$$deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} sgn[J_f(x)] \tag{5.1}$$

where, for $b \in \mathbb{R} \setminus 0$, we set

$$sgn[b] = \begin{cases} 1 & \text{if} \quad b > 0 \\ -1 & \text{if} \quad b < 0. \end{cases}$$

In order to extend the preceding definition to any continuous function f and any point p, one uses an approximation procedure. First, in order to approximate p with regular values p_k we have to apply Sard Theorem.

Theorem 5.2.2 (Sard Theorem)**.** *Let $f \in C^1(\Omega, \mathbb{R}^n)$ and set $S_f = \{x \in \Omega : J_f(x) = 0\}$. Then $f(S_f)$ is a set of zero measure.*
The set S_f is called the set of singular points of f. Any u such that $f(u) = p$ is called a nonsingular solution of the equation $f = p$, provided $u \notin S_f$.

According to Sard theorem, there exists a sequence $p_k \notin S_f$, such that $p_k \to p$. When p_k is sufficiently close to p, p_k verifies (c) and hence it makes sense to consider the $deg(f, \Omega, p_k)$, given by (5.1). Moreover, one can show that, for k >> 1, $deg(f, \Omega, p_k)$ is a constant which is independent of the approximating sequence p_k. Hence one can define the degree of $f \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ at any point p by setting $deg(f, \Omega, p) = \lim_k deg(f, \Omega, p_k)$.
Similarly, given $f \in C(\overline{\Omega}, \mathbb{R}^n)$, let $f_k \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ be such that $f_k \to f$ uniformly on $\overline{\Omega}$. If $k >> 1$, then any f_k, Ω, p_k satisfies $(a) - (c)$ and one can show that $\lim deg(f, \Omega, p_k)$. Similarly again, we can show that $\lim deg(f_k, \Omega, p)$ does not depend upon the choice of the sequence f_k and thus one can define the degree of f by setting $deg(f, \Omega, p) = deg(f_k, \Omega, p)$.
Now we have the following theorem which is an application of $(P.4)$ (Homotopy Invariance)

Theorem 5.2.3 (Dependence on the boundary values)**.** *Let f, $g \in C(\Omega, \mathbb{R}^n)$ be such that $f(x) = g(x)$ for all $x \in \partial\Omega$ and let $p \notin f(\partial\Omega) = g(\partial\Omega)$. Then $deg(f, \Omega, p) = deg(g, \Omega, p)$.*

Proof. Consider the homotopy

$$h(\lambda, x) = \lambda g(x) + (1 - \lambda) f(x).$$

For all $x \in \partial\Omega$ one has that $f(x) = g(x)$ and hence $h(\lambda, x) = f(x) \neq p$. Hence, h is an admissible homotopy and by $(P.4)$ (Homotopy invariance), we have:

$$\begin{aligned} deg(f, \Omega, p) &= deg(h(., 0), \Omega, p) \\ &= deg(h(., 1), \Omega, p) \\ &= deg(g, \Omega, p) \end{aligned}$$

Hence, the required result is established. □

Now, we begin our discussion by seeing that the excision property of the degree allows us to define the index of an isolated solution of $f(x) = p$. Let $x_0 \in \Omega$ be such that $f(x_0) = p$ and also we suppose that there exists $r > 0$ such that $f(x) \neq p$ for all $x \in \overline{B_r(x_0)} \setminus x_0$. Now using the excision property, with $\Omega = B_r(x_0)$ and $\Omega_0 = B_\rho(x_0)$, $\rho \in (0, r)$, we deduce that

$$deg(f, B_\rho(x_0), p) = deg(f, B_r(x_0), p), \qquad \forall \rho \in (0, r)$$

This common value is, defined as the index of f with respect to x_0 and is calculated as

$$i(f, x_0) = \lim_{\rho \to 0} deg(f, B_r(x_0), p), \qquad p = f(x_0).$$

Lemma 5.2.4. *If $f^{-1}(p) = x_1, ..., x_k$, $x_j \in \Omega$, then we have*

$$deg(f, \Omega, p) = \sum_1^k i(f, x_j)$$

Proof. To see this, it suffices to take $\rho > 0$ such that $B_\rho(x_i) \cap B_\rho(x_j) = \phi$ for all $i \neq j$. Letting $\Omega_0 = B_\rho(x_1) \cup B_\rho(x_2) \cup ... \cup B_\rho(x_k)$, using the excision property $(P.7)$ and the decomposition property $(P.4)$, we find

$$\begin{aligned} deg(f, \Omega, p) &= deg(f, \Omega_0, p) \\ &= \sum_1^k deg(f, B_\rho(x_j), p) \\ &= \sum_1^k i(f, x_j) \end{aligned}$$

proving the lemma. □

Now, let us consider $f \in C^1$ and let p be a regular value of f (i.e $J_f(x_0) \neq 0$ for all $x_0 \in f^{-1}(p)$). We also know that, if p is a regular value of f then the set $f^{-1}(p)$ is finite. In particular, any solution x_0 of $f(x) = p$ is isolated and it makes sense to consider the index $i(f, x_0)$.

Theorem 5.2.5. *Suppose that $f \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ and let $x_0 \in \Omega$ be such that $p = f(x_0)$ is a regular value of f. Then*

$$i(f, x_0) = (-1)^{\beta}$$

where β is the sum of the algebraic multiplicities of all the negative eigen values of $f^{'}(x_0)$.

Proof. Let $r > 0$ be such that the only solution of $f(x) = p$ in $B_r = B_r(x_0)$ is x_0.
Then $i(f, x_0) = deg(f, B_r, p)$ and (5.1) yields $i(f, x_0) = sgn[J_f(x_0)]$. Using the Jordan normal form, we know that the Jacobian determinant $J_f(x_0)$ is given by

$$J_f(x_0) = \lambda_1 \times ... \times \lambda_n,$$

where λ_j are the eigen values of $f^{'}(x_0)$ repeated according to their algebraic multiplicity. Now, we will use the following facts
(i): Since x_0 is a regular value for f and hence Jacobian determinant is different from zero and also product of the eigen values cannot be zero and hence each λ_j is non-zero.
(ii): If an eigen value is complex, let it be equal to $a + \iota b$, as complex roots always comes in pair, thus its complex conjugate $a - \iota b$ is also an eigen value of $f^{'}(x_0)$, and their product is $a^2 + b^2 > 0$.
Hence, from above two facts it follows that $sgn[J_f(x_0)] = (-1)^{\beta}$. □

5.3 Application: The Brouwer fixed point theorem

Here in this section we assume that a degree $deg(f, \Omega, p)$ satisfying the properties $(P.1)$ to $(P.8)$ definead earlier has been defined and we will show that it can be used to obtain classical Brouwer fixed point theorem.
Let $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ denote the unit ball in $\mathbb{R}^n$.
We start with a preliminary result.

Theorem 5.3.1. *The unit sphere ∂B_1 is not a retract of the unit ball B_1. Namely, there is no continuous map $f : \overline{B_1} \to \partial B_1$ such that $f(x) = x$, for all $x \in \partial B_1$.*

Proof. We prove this theorem by contradiction, thus assuming the contrary i.e there exists a continuous map $f : \overline{B_1} \to \partial B_1$ such that $f(x) = x$, for all $x \in \partial B_1$. Consider $g(x) = x$ and thus by dependence on boundary value, we have

$$deg(f, \Omega, 0) = deg(g, \Omega, 1) = 1$$

Now, by using the solution property of degree we infer that there exists $x \in \overline{B_1}$ such that $f(x) = 0$ and this is a contradiction with the assumption that $f(B_1) \subseteq \partial B_1$ □

Remark 5.3.2. *More in general, it is easy to see that, if Ω is a bounded open convex set, or else if Ω is a bounded domain homeomorphic to a convex set, then it is not possible to retract Ω onto its boundary $\partial\Omega$.*

We are now ready to prove the Brouwer fixed point theorem.

Theorem 5.3.3. *If f is a continuous map from a bounded closed convex set $C \subset \mathbb{R}^n$ into itself, then there exists $z \in C$ such that $f(z) = z$.*

Proof. First, let C be the closure of the unit ball B_1 in $\mathbb{R}^n$. If $f(x) \neq x$ for all $x \in \overline{B_1}$, we then define a map $\tilde{f} : \overline{B_1} \to \partial B_1$ by letting $\tilde{f}(x)$ be the intersection of ∂B_1 with the half-line starting from $f(x)$ and crossing x,
It is easy to see that $\tilde{f}$ is continuous. Moreover, $\tilde{f}(x) = x$ for all $x \in \partial B_1$, and hence ∂B_1 is a deformation retract of B_1, a contradiction to previous theorem. The case in which C is a bounded closed convex set, follows similarly, by using Remark(6.2.2). □

Remark 5.3.4. *According to Remark 6.2.2, one can extend the Brouwer fixed point theorem proving that if Ω is a bounded domain homeomorphic to a convex set then any continuous map from $\overline{\Omega}$ into itself has atleast a fixed point $z \in \overline{\Omega}$*

An alternate proof of Brouwer fixed point theorem can be carried out using the homotopy invariance of the degree. One can show that the homotoy

$$(\lambda, x) \to x - \lambda f(x) \quad \text{is admissible and thus}$$

$$deg(I - f, B_1, 0) = deg(I, B_1, 0) = 1$$

Now using the solution property, we deduces that there exists zero of $x - f(x)$, which is a fixed point of f.

Theorem 5.3.5 (Borsuk-Ulam theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded, open, symmetric (namely $x \in \Omega \iff -x \in \Omega$) and such that $0 \notin \Omega$. Suppose that $f \in C(\overline{\Omega}, \mathbb{R}^n)$ is an odd map such that $0 \notin f(\partial\Omega)$. Then $deg(f, \Omega, 0) = 1 \quad (mod\ 2)$*

5.4 An analytic definition of the degree

Here in this section we give a complete account of the topological degree and of its properties.

5.4.1 Degree for C^2 maps

We start the case of C^2 maps. We always assume that the preceding conditions $(a) - (c)$ are verified. Let $f \in C^2(\Omega, \mathbb{R}^n)$ and let $J_f(x)$ denote the Jacobian of f. From (c) one has that $min\,\{|f(x) - p| : x \in \partial\Omega\} > 0$ and we can choose $\alpha > 0$ such that

$$\alpha < \min_{x \in \partial\Omega} |f(x) - p|.$$

Consider a real valued continuous function ϕ defined on $[0, \infty)$ and such that
(i) $\quad supp[\phi] \subset (0, \alpha)$
(ii) $\quad \int_{\mathbb{R}^n} \phi(|x|)dx = 1$

Definition 5.4.2. *For $f \in C^1(\Omega, \mathbb{R}^n)$ we set*

$$deg(f, \Omega, p) = \int_\Omega \phi(|f(x) - p|) J_f(x) dx$$

Remark 5.4.3. *Trivially, one can see that $deg(f, \Omega, p) = deg(f - p, \Omega, 0)$, thus property (P.3) holds.*

We shall justify the definition by showing that it is independent of the choice of α and ϕ satisfying (i) and (ii). Precisely, let us show that if α_1, ϕ_1, α_2, ϕ_2 satisfy (i) and (ii), then

$$\int_\Omega \phi_1(|f(x) - p|) J_f(x) dx = \int_\Omega \phi_2(|f(x) - p|) J_f(x) dx \tag{5.2}$$

We can take $p = 0$ and also letting $\tilde{\phi} = \phi_1 - \phi_2$, (5.2) becomes

$$\int_\Omega \tilde{\phi}(|f(x)|) J_f(x) dx = 0$$

We know that ϕ_1,α_1 and ϕ_2,α_2 satisfy (i) and (ii). Thus, by (ii) we infer that

$$\int_{\mathbb{R}^n} \phi_1(|x|) dx = 1 \qquad \text{and} \qquad \int_{\mathbb{R}^n} \phi_2(|x|) dx = 1$$

Thus,

$$\int_{\mathbb{R}^n} \tilde{\phi}(|x|) dx = 0$$

Now, changing integral ovcr $\mathbb{R}^n$ to integral over boundary of ball, we get

$$\int_0^\infty r^{n-1} \tilde{\phi}(r) dr = 0 \tag{5.3}$$

Moreover, if $supp[\phi_i] \subset (0, \alpha_i), (i = 1, 2)$, then

$$supp[\tilde{\phi}] \subset (0, \alpha), \tag{5.4}$$

where $\alpha = \max[\alpha_1, \alpha_2]$. We set

$$\psi(r) = \begin{cases} r^{-n} \int_0^r s^{n-1} \tilde{\phi}(s) ds & \text{if} \quad r > 0 \\ 0 & \text{if} \quad r = 0. \end{cases}$$

From (5.3) and (5.4) it follows that for $r > \alpha$ one has that

$$\psi(r) = r^{-n} \int_0^r s^{n-1} \tilde{\phi}(s) ds = r^{-n} \int_0^\infty s^{n-1} \tilde{\phi}(s) ds = 0$$

Thus $supp[\psi] \subset (0, \alpha)$. Moreover, ψ is of class C^1 and

$$\tilde{\phi}(r) = r\psi^{'}(r) + n\psi(r).$$

Let A_{ij} denote the cofactor of $a_{ij} = \dfrac{\partial f_i}{\partial x_j}$ in the Jacobian J_f and consider the vector field $V \in$

$C^1(\Omega, \mathbb{R}^n)$ with components

$$V_i(x) = \sum_{j=1}^{n} A_{ji}(x)\psi(|f(x)|)f_j(x).$$

Taking into account the following property of the cofactors A_{ij}

$$\sum_{i=1}^{n} \frac{\partial A_{ji}}{\partial x_i} = 0 \qquad \forall j = 1, ..., n$$

we calculate $div(V(x))$ and it is

$$\begin{aligned} div(V(x)) &= \sum_{i=1}^{n} \frac{\partial V_i}{\partial x_i} \\ &= J_f(x)(|f(x)|\psi^{'}(|f(x)|) + n\psi(|f(x)|)) = \tilde{\phi}(|f(x)|)J_f(x) \end{aligned}$$

Integrating on Ω and using Guass Divergence theorem, we find that

$$\int_{\Omega} \tilde{\phi}(|f(x)|)J_f(x)dx = \int_{\partial\Omega} V(x).v\partial\sigma, \tag{5.5}$$

where v denotes the unit outer normal at $\partial\Omega$. Now, let us remark that for all $x \in \partial\Omega$ one has that $|f(x)| \geqslant min\{|f(x)| : x \in \partial\Omega\} > \alpha$ and thus

$$V(x) = 0 \qquad \forall x \in \partial\Omega \tag{5.6}$$

Using (5.6), we infer that the last integral in (5.5) is zero and hence

$$\int_{\Omega} \tilde{\phi}(|f(x)|)J_f(x)dx = 0$$

This proves that (5.2) holds true.

example 5.4.4. *Let $f = A \in L(\mathbb{R}^n)$ with A nonsingular. Then*

$$deg(A, \Omega, p) = \int_{\Omega} \phi(|Ax - p|)J_A(x)dx$$

Since, derivative of a linear map evaluated at a point is the linear map itself. Thus,

$$deg(A, \Omega, p) = \int_{\Omega} \phi(|Ax - p|)det(A)dx$$

Now, using the change of variables $y = Ax$, we have

$$deg(A, \Omega, p) = \int_{A(\Omega)} \phi(|y - p|)sgn[det(A)]dy$$

Now, we can take $\alpha < \min_{x\in\partial\Omega} |Ax - p|$ such that $B_\alpha(p) \subset A(\Omega)$ if $p \in A(\Omega)$, $B_\alpha(p) \subset (A(\Omega))^{\complement}$ if $p \notin A(\Omega)$.

We also know that

$$\int_{\mathbb{R}^n} \phi(|y|)dy = 1$$

Since the support of $\phi(. - p)$ is contained in the ball $B_\alpha(p)$, then one gets

$$deg(A, \Omega, p) = \begin{cases} sgn[det(A)] & \text{if} \quad p \in A(\Omega) \\ 0 & \text{if} \quad p \notin A(\Omega). \end{cases}$$

In particular, we find

$$deg(I_{\mathbb{R}^n}, \Omega, p) = \begin{cases} 1 & \text{if} \quad p \in \Omega \\ 0 & \text{if} \quad p \notin \Omega. \end{cases}$$

Thus, by above we have the property $(P.1)$ of degree.
As a second example, we consider the case in which p is a regular value of f.

example 5.4.5. *Let D be an open subset of $\mathbb{R}^n$, consider $f \in C^2(D, \mathbb{R}^n)$ and let $x_o \in D$ be such that $p = f(x_0)$ is a regular value of f. By inverse function Theoem, we know that f induces a diffeomorphism between a ball $B_\epsilon(x_0)$, $\epsilon > 0$ small enough, and its image $U_\epsilon = f(B_\epsilon(x_0))$. We also know that J_f has a constant sign in $B_\epsilon(x_0)$. With the same arguments used in previous example, we get*

$$\begin{aligned} deg(A, \Omega, p) &= \int_{B_\epsilon(x_0)} \phi(|f(x) - p|)J_f(x)dx \\ &= sgn[J_f(x_0)] \int_{B_\epsilon(x_0)} \phi(|f(x) - p|)|J_f(x)|dx \\ &= sgn[J_f(x_0)] \int_{U_\epsilon} \phi(|y - p|)dy. \end{aligned}$$

In the definition of the degree we can take α such that $B_\alpha(p) \subset U_\epsilon$. Since $\phi(|y - p|) = 0$ for $|y - p| > \alpha$, then

$$\int_{U_\epsilon} \phi(|y - p|)dy = \int_{\mathbb{R}^n} \phi(|y - p|)dy = 1,$$

and hence

$$deg(f, B_\epsilon(x_0), p) = sgn[J_f(x_0)].$$

5.4.6 Degree for continuous maps

Now we are going to define the degree for any continuous f verifying the conditions $(a) - (c)$ stated at the begining. We need the following lemma for defining degree for continuous maps

Lemma 5.4.7. *For $i = 1, 2$, let $f_i \in C^2(\Omega, \mathbb{R}^n)$ be such that*

$$|f_i(x) - p| > \alpha > 0, \qquad \forall x \in \partial\Omega$$

Given $\epsilon \in (o, \frac{\alpha}{6})$, we suppose that

$$|f_2(x) - f_1(x)| < \quad \epsilon \qquad \forall x \in \overline{\Omega}$$

Then $deg(f_1, \Omega, p) = deg(f_2, \Omega, p)$.

Proof. We know that $deg(f, \Omega, p) = deg(f - p, \Omega, 0)$. Thus, we can take $p = 0$. Let $\chi \in C^1(0, \infty)$ be a function such that

$$\chi(r) = \begin{cases} 1 & \text{if} \quad 0 \leqslant r \leqslant 2\epsilon \\ 0 & \text{if} \quad r \leqslant 2\epsilon, \end{cases}$$

and define $f_3 \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ by setting

$$f_3(x) = (1 - \chi(|f_1(x)|))f_1(x) + \chi(|f_1(x)|)f_2(x)$$

From the definition of X, we get:

$$f_3(x) = f_1(x) \quad if \quad \chi(|f_1(x)|) = 0 \quad if \quad \forall x \in \overline{\Omega} : |f_1(x)| > 3\epsilon \tag{5.7}$$

$$f_3(x) = f_2(x) \quad if \quad \chi(|f_1(x)|) = 1 \quad if \quad \forall x \in \overline{\Omega} : |f_1(x)| < 2\epsilon \tag{5.8}$$

In particular, since $|f_1(x)| > \alpha$, for all $x \in \partial\Omega$, and $\alpha > 6\epsilon$ ($since, \epsilon < \dfrac{\alpha}{6} \quad implies \quad \alpha > 6\epsilon$), then $|f_3(x)| = |f_1(x)| > \alpha$ for all $x \in \partial\Omega$. Furthermore, for all $x \in \overline{\Omega}$ one has that

$$f_3(x) - f_1(x) = \chi(|f_1(x)|).(f_2(x) - f_1(x)),$$

$$f_3(x) - f_2(x) = (1 - \chi(|f_1(x)|)).(f_1(x) - f_2(x)),$$

and therefore, for $i = 1, 2$

$$|f_3(x) - f_i(x)| < \epsilon, \qquad \forall x \in \overline{\Omega}. \tag{5.9}$$

Choose two functions $\phi_i \in C(0, \infty)$ with the following properties:

$$\int_{\mathbb{R}^n} \phi_i(|x|)dx = 1, \quad i = 1, 2,$$

$$supp[\phi_1] \subset (4\epsilon, 5\epsilon),$$

$$supp[\phi_2] \subset (0, \epsilon).$$

According to definition of degree, the functions ϕ_1 and ϕ_2 can be used to evaluate the degree of $f_i, i = 1, 2, 3$. In particular, one has

$$deg(f_3, \Omega, 0) = \int_{\Omega} \phi_1(|f_3(x)|)J_{f_3}(x)dx,$$

$$deg(f_1, \Omega, 0) = \int_{\Omega} \phi_1(|f_1(x)|)J_{f_1}(x)dx,$$

Now, since $supp[\phi_1] \subset (4\epsilon, 5\epsilon)$ we get:

$$\phi_1(|f_3(x)|)J_{f_3}(x) \neq 0 \qquad \text{if and only if} \quad 4\epsilon < |f_3(x)| < 5\epsilon.$$

Using (5.9), we deduce that $3\epsilon < |f_1(x)| < 6\epsilon$ provided $4\epsilon < |f_3(x)| < 5\epsilon$ and then (5.7) yields $f_3(x) = f_1(x)$ for all $x \in \overline{\Omega}$ such that $4\epsilon < |f_3(x)| < 5\epsilon$. In other words,

$$\phi_1(|f_3(x)|)J_{f_3}(x) = \phi_1(|f_1(x)|)J_{f_1}(x), \qquad \forall x \in \overline{\Omega}$$

and this, jointly witt the degree defined above, implies that $deg(f_3, \Omega, 0) = deg(f_1, \Omega, 0)$.
Similarly, we see that

$$\phi_2(|f_3(x)|)J_{f_3}(x) = \phi_2(|f_2(x)|)J_{f_2}(x), \qquad \forall x \in \overline{\Omega}$$

and we deduce that $deg(f_3, \Omega, 0) = deg(f_2, \Omega, 0)$.
In conclusion, one has $deg(f_1, \Omega, 0) = deg(f_3, \Omega, 0) = deg(f_2, \Omega, 0)$ proving the lemma. □

Now, we are in a position to define the degree for any continuous map $f : \overline{\Omega} \to \mathbb{R}^n$ st $f(x) \neq p \quad \forall x \in \partial\Omega$. By density there exists a sequence of functions $f_k \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ converging to f uniformly on $\overline{\Omega}$. Clearly, for $k >> 1$, we have $f_k(x) \neq p$ on $\partial\Omega$ and hence the degree $deg(f_k, \Omega, p)$ is well defined. Moreover, previous lemma implies that $deg(f_k, \Omega, p)$ is constant for k sufficiently large. This allows us to define the degree of f wrt Ω and p, by setting

$$deg(f, \Omega, p) = \lim_{k\to\infty} deg(f_k, \Omega, p)$$

5.5 Properties of the degree

Here, in this section we will prove that properties $(P.1) - (P.7)$ hold. We have already proved the property $(P.1)$ with the help of example(6.3.4) and by the definition of degree and remark (6.3.3) property $(P.3)$ also hold. According to the definition and to lemma (6.3.7), it suffices to carry out the proofs under the assumption that $f \in C^2$. Let us point out that $p \notin f(\partial\Omega)$ implies that $p \notin f_k(\partial\Omega)$ for $k >> 1$, as well as $q \notin f(\partial\Omega)$ for all q near p, so that it makes sense to consider $deg(f_k, \Omega, p) = deg(f, \Omega, q)$.

Proof of $(P.2)$. If $f(x) \neq p$ for all $x \in \Omega$, then $f(x) \neq p$ on all the compact set $\overline{\Omega}$ and hence $\exists\delta > 0, \delta \leq \alpha$, such that $|f(x) - p| > \delta$, for all $x \in \overline{\Omega}$.
Choose ψ such that $\int_{\mathbb{R}^n} \psi(|x|)dx = 1$ and $\text{supp}[\psi] \subset (0, \delta)$. From this and the fact that $|f(x) - p| > \delta$ in Ω, we infer that $\psi(|f(x) - p|) \equiv 0$ in Ω and one finds

$$\deg(f, \Omega, p) = \int_\Omega \psi(|f(x) - p)J_f dx = 0.$$

This is in contradiction with our assumptions.

□

Proof of $(P.4)$. Since $\Omega_1 \cup \Omega_2 = \psi$ one has

$$\int_{\Omega_1\cap\Omega_2} \psi|(f(x) - p|)J_f(x)dx = \int_{\Omega_1} \psi|(f(x) - p|)J_f(x)dx + \int_{\Omega_2} \psi|(f(x) - p|)J_f(x)dx.$$

Then $(P.4)$ immediately follows from Definition of the degree. □

Proof of $(P.5)$. For any $\epsilon > 0$ small, we can find $\delta(\epsilon) > 0$ such that $|h(x, \lambda_1) - h(x, \lambda_2)| < \epsilon$ for all $x \in \overline{\Omega}$ provided $|\lambda_1 - \lambda_2| < \delta$. Using Lemma 5.4.7 we have infer

$$\deg(h(., \lambda_1), \Omega, p) = \deg(h(, .\lambda_2), \Omega, p).$$

Covering the interval $[0, 1]$ with a finite number of subintervals with length smaller than δ, and applying the preceding equation, the result follows. □

Proof of $(P.6)$. This follows immediately from Lemma 5.4.7. Since $\deg(f, g, p)$=$\deg(f - p, \Omega, 0)$ we also deduce the continuity of the degree with respect to p. □

Proof of $(P.7)$. Since $f(x) \neq p, \forall x \in \Omega \backslash \Omega_0$, then $f(x) \neq p$, on the compact set $\overline{\Omega} \backslash \Omega_0$. In the definition of the degree, let us choose ψ in such a way that $\text{supp}[\psi] \subset (0, \alpha)$. Then $\psi(|f(x) - p|) \equiv 0$ on $\Omega \backslash \Omega_0$ and this yields

$$\int_{\Omega} \psi(|f(x) - p|) J_f(x) dx = \int_{\Omega_0} \psi(|f(x) - p|) J_f dx.$$

Since, by defintion, the former integral equals $\deg(f, g, p)$ while the latter equals $\deg(f, \Omega_0, p)$, we conclude that $\deg(f, \Omega, p) = \deg(f, \Omega_0, p)$. □

Let us point out that the definition and properties of the index depend on $(P.1) - (P.7)$ only. In particular Lemma 5.2.1 holds. Moreover, arguing as example 5.4.5, we immedialtely deduce the following corollary, which is nothing but the definition of the degree for regular values and C^1 functions, given in (5.1).

Corollary 5.5.1. *If p is a regular value of $f \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$, then*

$$deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} sgn[J_f(x)].$$

Proof. In accordance with example 5.4.5, when x_0 is an inverse image of $f^{-1}(p)$, we have the $\deg(f, B_\epsilon(x_0), p) = sgn[J_f(x_0)]$, where $B_\epsilon(x_0)$ is ball at x_0 for small ϵ. Also, when p is a regular value then the set $f^{-1}(p)$ is finite and by the excision property and property(P.4) we have
$deg(f, \Omega, p) = deg(f, B_\epsilon(x_0), p) + deg(f, B_\epsilon(x_1), p) + ... + deg(f, B_\epsilon(x_n), p)$, where $f^{-1}(p) = x_0, x_1, ..., x_n$
Thus, we obtain

$$deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} sgn[J_f(x)]$$

proving the corollary. □

We end this section by showing that $\deg(f, \Omega, p)$ defined, is always an integer. As usual, it suffices to consider C^1 maps. If p is a regular value, the claim follows from the corollary above. Otherwise, let S_f denote the set of points $x \in \Omega$ such that $J_f(x) = 0$. The Sard theorem ensure that $f(S_f)$ has zero lebesgue measure. Hence, there exists a sequence of regular values p_k with $p_k \to p$. Since $\deg(f, \Omega, p_k)$ is an integer then, by continuity, we conclude that $\deg(f, \Omega, p)$ is an integer too.

Bibliography

[1] Antonio Ambrosetti and Giovanni Prodi, *A Primer of Nonlinear Analysis*

[2] Erwin Kreyszig, *Introductory Functional Analysis With Applications*

[3] H.L.Royden and P.M.Fitzpatrick, *Real Analysis*

[4] Antonio Ambrosetti and Andrea Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*

[5] James R. Munkres, *Topology*