Finite deformations and incremental axisymmetric motions of a magnetoelastic tube

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Abstract

A thick-walled circular cylindrical tube made of an incompressible magnetoelastic material is subjected to a finite static deformation in the presence of an internal pressure, an axial stretch, and an azimuthal or an axial magnetic field. The dependence of the static magnetoelastic deformation on the intensity of the applied magnetic field is analysed for two different magnetoelastic energy density functions. Then, superimposed on this static configuration, incremental axisymmetric motions of the tube and their dependence on the applied magnetic field and deformation parameters are studied. In particular, we show that magnetoelastic coupled waves exist only for particle motions in the azimuthal direction. For particle motion in radial and axial directions, only purely mechanical waves are able to propagate when magnetic field is absent. The wave speeds as well as the stability of the tube can be controlled by changing the internal pressure, axial stretch, and applied magnetic field that demonstrates the applicability of magneto-elastomers as wave guides and vibration absorbers.

Keywords: Nonlinear magnetoelasticity; magnetoacoustics; wave propagation; waveguide; finite deformation; instability analysis

Note: The definitive version of this article is published in the journal "Mathematics and Mechanics of Solids" and can be downloaded from their website. DOI: 10.1177/1081286517697502

Contents

1 Introduction

Magnetoelastic solids are smart materials in which the magnetic and the mechanical response have a nonlinear coupling with each other. This paper deals with the problem of large axisymmetric deformations and propagation of magnetoelastic waves in such materials.

Most modern magnetoelastic materials are magnetorheological elastomers (MREs) that are composites of an elastomeric matrix filled with ferromagnetic particles (usually micron sized iron particles) embedded in their bulk. Their manufacturing process and experimental methods to measure changes in overall parameters like stiffness are detailed in papers, such as those by Jolly et al. [1], Lokander and Stenberg [2], Varga et al. [3], Boczkowska and Awietjan [4], and Schubert and Harrison [5]. Their applicability in development of sensors, actuators, and vibration control systems have been demonstrated in some recent papers by Mayer et al. [6], Xin et al. [7], and Ying and Ni [8], among others.

The magnetoelastic behaviour of such materials is analysed using the general theory of electromagnetic interactions in deformable continua that can be found in texts by Brown [9], Pao [10], Maugin [11], and Eringen and Maugin [12, 13]. These classical works along with the recent advances in development of MREs have inspired a great deal of mathematical and computational work in this area in recent years. From a pheomenological perspective, Brigadnov and Dorfmann [14] and Dorfmann and Ogden [15] proposed a constitutive formulation of nonlinear magnetoelasticity based on the definition of a total energy density function. Kankanala and Triantafyllidis [16] approached this problem through a variational formulation and arrived at similar equations. Later, based on the micromechanical constitution of the MREs, formulations for magnetoelastic coupled equations using homogenization techniques were proposed by Castañeda and Galipeau [17, 18] and Chatzigeorgiou et al. [19]. We note here that a parallel track of research exists in the field of nonlinear electroelasticity motivated by the development of elctro-active polymers (EAPs) [20]. The mathematical formulation is quite similar and can be seen in the papers on phenomenological models by Dorfmann and Ogden [21], McMeeking and Landis [22], and micromechanical models using homogenization technique by Castañeda and Siboni [23], among others.

Since both EAPs and MREs are polymer based composites, the deformation process is usually dissipative. Effect of magnetic field on the viscoelasticity has been studied by Bellan and Bossis [24], effect on the Mullin's effect by Coquelle and Bossis [25], magneto-viscoelasticity of isotropic and anisotropic MREs by Saxena et al. [26, 27] and Haldar et al. [28], and a micromechanical approach to study magneto-viscoelasticity has been taken by Ethiraj and Miehe [29]. Similar case of dissipation in the elctro-viscoelastic case has been studied by Ask et al. [30], Saxena et al. [31], and Denzer and Menzel [32]. For a more exhaustive survey of the literature on electro- and magnetoelasticity, we refer to the recent books on the subject by Hutter et al. [33], Ogden and Steigmann [34], and Dorfmann and Ogden [35].

Based on the modelling techniques of electro- and magnetoelasticity mentioned above, Otténio et al. [36] studied the instabilities on the surface of a finitely deformed magnetoelastic half-space using an incremental formulation, Kankanala and Triantafyllidis [37] studied the geometric instabilties of a finitely deformed magnetoelastic block, and Rudykh and Bertoldi [38] studied the stability of anisotropic MREs using a micromechanical approach. Similar studies in the case of electroelasticity were done by Dorfmann and Ogden [39] who analysed the instabilities of an electroelastic plate, Rudykh et al. [40] who studied snap-through instabilities of a thick-walled electroactive balloons, and Miehe et al. [41] who did a computational study of structural and material instabilities in EAPs.

The incremental formulation (small deformations superposed on large deformations) used by Otténio et al. [36] was further generalised by Ogden [42] to derive the general governing equations for time-dependent incremental elastic deformation and electromagnetic fields. Using this Dorfmann and Ogden [43] studied waves in a finitely deformed electroelastic solid, while Destrade and Ogden [44] studied waves in a finitely deformed magnetoelastic bulk under a quasimagnetostatic approximation. Later Saxena and Ogden [45–47] used the above-mentioned theory to study Rayleigh type wave propagation on the surface of a homogeneously finitely deformed magnetoelastic half-space, and to consider Bleustein–Gulyaev type and Love type waves in a homogeneously finitely deformed magnetoelastic layered half-space. We use the same formulation in the present work to study a boundary value problem in cylindrical geometry concerning a thick-walled tube made of a magnetoelastic material.

Study of finite deformation and incremental motions of a cylindrically shaped elastic solid has been done by many researchers, some of which are listed as follows (and references therein). Stability and vibrations of a thick-walled tube under finite torsion and external pressure have been studied by Ertepinar [48], and Wang and Ertepinar [49]. Static bifurcation for the axisymmetric case in the presence of an internal pressure and axial stretch has been analysed by Haughton and Ogden [50, 51], while waves in rotating thick-walled elastic tubes have been studied by Haughton [52] using the equations derived in [53]. More recently, Akbarov and Guz [54] studied axisymmetric waves in pre-stressed linear elastic cylinders, El-Raheb [55] studied transient waves in infinite inhomogeneous linearly elastic tubes, Chen and Lin [56] analysed thick-walled linearly elastic cylinders made of functionally graded materials with an internal pressure, and Shams [57] studied axisymmetric wave propagation in a residually stressed tube. For magnetoelastic materials, the problem of static nonlinear deformations in the presence of an underlying magnetic field has been studied briefly by Dorfmann and Ogden [58], numerical solutions for static deformations of a thick hollow cylinder in the presence of an axial magnetic field have been obtained by Bustamante et al. [59], and the same for a solid cylinder have been obtained by Salas and Bustamante [60]. Role of internal pressure in the finite deformations of an electroelastic tube has been studied by Melnikov and Ogden [61]. In a recent paper, Shmuel and DeBotton [62] have studied waves in a hollow electroelastic tube with an electric field in the radial direction.

In the first part of the present paper, we study static finite deformations of an incompressible magnetoelastic tube of finite length in the presence of an azimuthal magnetic field, mechanical internal pressure, and an axial stretch. In the second part we consider the effects of these quantities on infinitesimal axisymmetrical motions of the finitely deformed cylinder. It is shown that for a coupled magneto-mechanical problem, longitudinal motions (in the(r, z) plane) are not possible and we get waves with particle motion in the azimuthal direction only.

The governing equations for nonlinear magnetoelastic deformations and for the increments in the deformation and magnetic fields are summarised in Section 2, along with appropriate boundary conditions. Constitutive relations based on a total energy density function are summarised in Section 3 where we also define the magnetoelastic moduli tensors.

In Section 4, the governing equations are expressed in cylindrical coordinates and finite magnetoelastic deformations of a thick-walled cylindrical tube are examined. Dependence of the total internal pressure and the axial force in the tube on the axial stretch and the underlying magnetic field is studied. For the purpose of numerical calculations we use two forms of the energy density function, one being similar to a Mooney–Rivlin magnetoelastic energy function introduced by Otténio et al. [36] and the other is an extension of the Ogden energy function to the magnetoelastic context. The computations show that the presence of an azimuthal magnetic field requires additional internal pressure and compressive axial force if a given state of deformation has to be maintained. The axial magnetic field is, however, decoupled from the internal pressure and requires only a change in the axial force in order to maintain a given configuration.

From Section 5 onwards, we consider time-dependent axisymmetric incremental motions superimposed on the finitely deformed cylinder. For both axial and azimuthal magnetic fields, the equations governing incremental motion in the azimuthal direction are decoupled from the equations governing incremental motion in the axial and radial directions. We therefore consider the four cases separately. Considering wave type solutions for displacement and incremental magnetic field, we obtain high-order ordinary differential equations which are converted to systems of first-order ODEs for the purpose of obtaining a numerical solution. Wave speeds are computed numerically for the Mooney–Rivlin type magnetoelastic energy density function.

It is observed that the differential equations governing the displacement in axial and radial directions yield a unique solution only when the underlying magnetic field is zero, thereby reducing the problem to a pure elastic case. Solutions do exist for magnetic and mechanical waves in the case of mechanical displacement in azimuthal direction and we obtain multiple modes of wave propagation. Existence of such waves and dependence of their propagation velocity on various magnetomechanical loading parameters (axial stretch, internal pressure, applied magnetic field, tube thickness) is demonstrated graphically. An underlying azimuthal magnetic field always reduces the wave speed while an axial magnetic field can increase or decrease the wave speed for different conditions. The same procedure also helps to study geometric instabilities and we observe that increasing of both the axial and azimuthal magnetic field can cause the tube to become unstable in the long wavelength region.

Finally we report our conclusions in Section 8.

2 Governing equations of finite magnetoelasticity

Consider a continuous solid body made of a magnetoelastic material. We denote its static, stress-free reference (Lagrangian) configuration by $\mathcal{B}_\mathrm{r}$ and the boundary by $\partial\mathcal{B}_\mathrm{r}$. Under the effects of a time-dependent magnetic field and surface tractions, the current (Eulerian) configuration of the body at time t is denoted by $\mathcal{B}_t$ and its boundary by $\partial\mathcal{B}_t$. Material points in $\mathcal{B}_\mathrm{r}$ are denoted by $\mathbf{X}$, which becomes $\mathbf{x}$ in $\mathcal{B}_t$.

The motion of the body is described by an invertible function $\boldsymbol{\chi}$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ and it is assumed that $\boldsymbol{\chi}$ and its inverse at each instant t are sufficiently differentiable. The deformation gradient tensor is defined as $\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}$ and its determinant by $J = \det \mathbf{F} > 0$. The standard differential operators are denoted by grad, div, and curl with respect to $\mathbf{x}$ and by Grad, Div, and Curl with respect to $\mathbf{X}$. For an incompressible material we have the constraint

$$J = \det \mathbf{F} \equiv 1. \tag{1}$$

We first consider a purely magneto-elastostatic finitely deformed configuration, denoted $\mathcal{B}$ with boundary $\partial\mathcal{B}$. Subsequently, when considering incremental motions, we shall adopt the *quasimagnetostatic approximation* (see, for example, [42]), in which the effects of electric fields can be neglected, so that there are no surface or volume electric charges and no electric currents. We also assume that there are no mechanical body forces. The balance equations to be satisfied in $\mathcal{B}$ are

$$\mathrm{div}\,\boldsymbol{\tau} = \mathbf{0}, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^\mathrm{T}, \quad \mathrm{curl}\,\mathbf{H} = \mathbf{0}, \quad \mathrm{div}\,\mathbf{B} = 0, \tag{2}$$

where $\boldsymbol{\tau}$ is the total Cauchy stress tensor, which incorporates the magnetic body forces, $\boldsymbol{\tau}^\mathrm{T}$ is its transpose, $\mathbf{H}$ is the magnetic field vector, and $\mathbf{B}$ is the magnetic induction vector. Outside the material in vacuum (or a non-conducting, non-magnetic material), the relevant equations to be satisfied are

$$\mathrm{curl}\,\mathbf{H}^* = \mathbf{0}, \quad \mathrm{div}\,\mathbf{B}^* = 0, \tag{3}$$

with the constitutive relation $\mathbf{B}^* = \mu_0 \mathbf{H}^*$. Here μ_0 is the magnetic permeability of vacuum with the numerical value of $4\pi \times 10^{-7}$ N/A^2, and here and henceforth we denote the quantities in vacuum by a superscript $*$. The boundary conditions are given as

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_\mathrm{m} + \mathbf{t}_\mathrm{a}, \quad \mathbf{n} \times [\![\mathbf{H}]\!] = \mathbf{0}, \quad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0, \tag{4}$$

on $\partial\mathcal{B}$, where $\mathbf{n}$ is the unit outward normal to the boundary, $\mathbf{t}_\mathrm{m}$ and $\mathbf{t}_\mathrm{a}$ are the tractions due to magnetic and (applied) mechanical forces, respectively, and $[\![\bullet]\!]$ represents jump in a field variable across a boundary.

The total nominal stress tensor $\mathbf{T}$ and the Lagrangian forms of $\mathbf{H}$ and $\mathbf{B}$ are defined by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}, \quad \mathbf{H}_l = \mathbf{F}^{\mathrm{T}}\mathbf{H}, \quad \mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B}, \tag{5}$$

using which we can write the balance equations and boundary conditions in Lagrangian form as

$$\operatorname{Div}\mathbf{T} = \mathbf{0}, \quad \mathbf{F}\mathbf{T} = (\mathbf{F}\mathbf{T})^{\mathrm{T}}, \quad \operatorname{Curl}\mathbf{H}_l = \mathbf{0}, \quad \operatorname{Div}\mathbf{B}_l = 0 \quad \text{in } \mathcal{B}_{\mathrm{r}}, \tag{6}$$

$$\mathbf{T}^{\mathrm{T}}\mathbf{N} = \mathbf{t}_{\mathrm{M}} + \mathbf{t}_{\mathrm{A}}, \quad \mathbf{N}\times[\![\mathbf{H}_l]\!] = \mathbf{0}, \quad \mathbf{N}\cdot[\![\mathbf{B}_l]\!] = 0 \quad \text{on } \partial\mathcal{B}_{\mathrm{r}}, \tag{7}$$

where $\mathbf{N}$ is the unit outward normal to $\partial\mathcal{B}_{\mathrm{r}}$, and $\mathbf{t}_{\mathrm{M}}$ and $\mathbf{t}_{\mathrm{A}}$ are magnetic and mechanical traction per unit reference area. In particular, $\mathbf{t}_{\mathrm{M}}$ is given by $\mathbf{t}_{\mathrm{M}} = \boldsymbol{\tau}^*\mathbf{F}^{-\mathrm{T}}\mathbf{N}$ on $\partial\mathcal{B}_{\mathrm{r}}$, where $\boldsymbol{\tau}^*$ is the Maxwell stress given by

$$\boldsymbol{\tau}^* = \mathbf{B}^*\otimes\mathbf{H}^* - \frac{1}{2}(\mathbf{B}^*\cdot\mathbf{H}^*)\,\mathbf{I}, \tag{8}$$

$\mathbf{I}$ being the identity tensor.

2.1 Incremental equations

Superimposed on the underlying deformation and the initial magnetic field, we consider an infinitesimal incremental motion denoted by $\dot{\boldsymbol{\chi}}(\mathbf{X},t) = \mathbf{u}(\mathbf{x},t)$ and an increment in the magnetic field denoted by $\dot{\mathbf{H}}_l(\mathbf{X},t)$ [1]. Incremented quantities referred to $\mathcal{B}$ are denoted by the subscript 0 using the relations

$$\dot{\mathbf{T}}_0 = J^{-1}\mathbf{F}\dot{\mathbf{T}}, \quad \dot{\mathbf{B}}_{l0} = J^{-1}\mathbf{F}\dot{\mathbf{B}}_l, \quad \dot{\mathbf{H}}_{l0} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{H}}_l. \tag{9}$$

Next, we form the increments of the governing equations (6) and then update them to $\mathcal{B}$ to obtain

$$\operatorname{div}\dot{\mathbf{T}}_0 = \rho\mathbf{u}_{,tt}, \quad \mathbf{L}\boldsymbol{\tau} + \dot{\mathbf{T}}_0 = \boldsymbol{\tau}\mathbf{L}^{\mathrm{T}} + \dot{\mathbf{T}}_0^{\mathrm{T}}, \quad \operatorname{curl}\dot{\mathbf{H}}_{l0} = \mathbf{0}, \quad \operatorname{div}\dot{\mathbf{B}}_{l0} = 0, \tag{10}$$

in the first of which the inertia term is now included, where ρ is the mass density and $\mathbf{L} = \operatorname{grad}\mathbf{u}$ is the displacement gradient. The updated incremented boundary conditions are

$$\dot{\mathbf{T}}_0^{\mathrm{T}}\mathbf{n} = \dot{\boldsymbol{\tau}}^*\mathbf{n} - \boldsymbol{\tau}^*\mathbf{L}^{\mathrm{T}}\mathbf{n}, \quad \mathbf{n}\times(\dot{\mathbf{H}}_{l0} - \mathbf{L}^{\mathrm{T}}\mathbf{H}^* - \dot{\mathbf{H}}^*) = \mathbf{0}, \quad \mathbf{n}\cdot(\dot{\mathbf{B}}_{l0} - \dot{\mathbf{B}}^* + \mathbf{L}\mathbf{B}^*) = 0, \tag{11}$$

on $\partial\mathcal{B}$, where the increment in the mechanical traction has been set to zero.

In vacuum, the incremented governing equations are

$$\operatorname{curl}\dot{\mathbf{H}}^* = \mathbf{0}, \quad \operatorname{div}\dot{\mathbf{B}}^* = 0, \tag{12}$$

with the relation $\dot{\mathbf{B}}^* = \mu_0\dot{\mathbf{H}}^*$. Thus, the increment in the Maxwell stress is given by

$$\dot{\boldsymbol{\tau}}^* = \mu_0\left[\dot{\mathbf{H}}^*\otimes\mathbf{H}^* + \mathbf{H}^*\otimes\dot{\mathbf{H}}^* - (\dot{\mathbf{H}}^*\cdot\mathbf{H}^*)\mathbf{I}\right]. \tag{13}$$

3 Constitutive relations

The material is assumed henceforth to be incompressible ($J \equiv 1$) and is defined by a total energy density function $\Omega(\mathbf{F},\mathbf{H}_l)$ as given in [15] which results in the constitutive equations

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}} - p\mathbf{F}^{-1}, \quad \mathbf{B}_l = -\frac{\partial\Omega}{\partial\mathbf{H}_l}, \tag{14}$$

where p is a Lagrange multiplier associated with the constraint of incompressibility.

For an incompressible isotropic magnetoelastic material, the energy density function can be expressed in terms of five scalar invariants which we choose to be

$$\begin{gathered} I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \frac{1}{2}[I_1^2 - \operatorname{tr}(\mathbf{C}^2)], \\ K_4 = \mathbf{H}_l\cdot\mathbf{H}_l, \quad K_5 = (\mathbf{C}\mathbf{H}_l)\cdot\mathbf{H}_l, \quad K_6 = (\mathbf{C}^2\mathbf{H}_l)\cdot\mathbf{H}_l, \end{gathered} \tag{15}$$

where $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is the right Cauchy–Green tensor. We use K_4, K_5, K_6 above instead of I_4, I_5, I_6 to maintain consistency as the latter are used in literature to define invariants in terms of $\mathbf{B}_l$; see, for example, [15]. Also, $I_3 = \det\mathbf{C}$ is unity because of incompressibility. Hence, the constitutive relations (14) can be expanded as

$$\begin{aligned} \mathbf{T} = & -p\mathbf{F}^{-1} + 2\Omega_1\mathbf{F}^{\mathrm{T}} + 2\Omega_2(I_1\mathbf{F}^{\mathrm{T}} - \mathbf{C}\mathbf{F}^{\mathrm{T}}) + 2\Omega_5\mathbf{H}_l\otimes\mathbf{F}\mathbf{H}_l \\ & + 2\Omega_6(\mathbf{H}_l\otimes\mathbf{F}\mathbf{C}\mathbf{H}_l + \mathbf{C}\mathbf{H}_l\otimes\mathbf{F}\mathbf{H}_l), \end{aligned} \tag{16}$$

[1] Note that for the sake of brevity, we depart from the usual convention and use a superposed bold dot to denote increment in a quantity and not a time derivative that is the usual case. Derivatives with respect to time are denoted using a subscript ‘,t’ as can be seen in equation (10), for example.

and

$$\mathbf{B}_l = -2(\Omega_4 \mathbf{H}_l + \Omega_5 \mathbf{C}\mathbf{H}_l + \Omega_6 \mathbf{C}^2\mathbf{H}_l), \tag{17}$$

where $\Omega_k = \partial\Omega/\partial I_k$ for $k = 1, 2$ and $\Omega_k = \partial\Omega/\partial K_k$ for $k = 4, 5, 6$. In Eulerian form the equations are

$$\boldsymbol{\tau} = -p\mathbf{I} + 2\Omega_1 \mathbf{b} + 2\Omega_2(I_1\mathbf{b} - \mathbf{b}^2) + 2\Omega_5 \mathbf{bH} \otimes \mathbf{bH} + 2\Omega_6(\mathbf{bH} \otimes \mathbf{b}^2\mathbf{H} + \mathbf{b}^2\mathbf{H} \otimes \mathbf{bH}), \tag{18}$$

and

$$\mathbf{B} = -2(\Omega_4 \mathbf{bH} + \Omega_5 \mathbf{b}^2\mathbf{H} + \Omega_6 \mathbf{b}^3\mathbf{H}), \tag{19}$$

where $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is the left Cauchy–Green tensor.

On incrementing the equations (14), we obtain

$$\dot{\mathbf{T}} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{F}}\dot{\mathbf{F}} + \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{H}_l}\dot{\mathbf{H}}_l - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{20}$$

$$\dot{\mathbf{B}}_l = -\frac{\partial^2\Omega}{\partial\mathbf{H}_l\partial\mathbf{F}}\dot{\mathbf{F}} - \frac{\partial^2\Omega}{\partial\mathbf{H}_l\partial\mathbf{H}_l}\dot{\mathbf{H}}_l. \tag{21}$$

Using the relations in equation (9) with $J = 1$, we update the above equations to obtain

$$\dot{\mathbf{T}}_0 = \mathbf{F}\boldsymbol{\mathcal{A}}\dot{\mathbf{F}} + \mathbf{F}\boldsymbol{\mathcal{C}}\dot{\mathbf{H}}_l - \dot{p}\mathbf{I} + p\mathbf{L}, \quad \dot{\mathbf{B}}_{l0} = -\mathbf{F}\boldsymbol{\mathcal{C}}^{\mathrm{T}}\dot{\mathbf{F}} - \mathbf{F}\mathsf{K}\dot{\mathbf{H}}_l, \tag{22}$$

where we have used the notations

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{F}}, \quad \boldsymbol{\mathcal{C}} = \frac{\partial^2\Omega}{\partial\mathbf{F}\partial\mathbf{H}_l}, \quad \boldsymbol{\mathcal{C}}^{\mathrm{T}} = \frac{\partial^2\Omega}{\partial\mathbf{H}_l\partial\mathbf{F}}, \quad \mathsf{K} = \frac{\partial^2\Omega}{\partial\mathbf{H}_l\partial\mathbf{H}_l}, \tag{23}$$

which define the *magnetoelastic moduli tensors*. In component form the updated magnetoelastic tensors, $\boldsymbol{\mathcal{A}}_0, \boldsymbol{\mathcal{C}}_0$ and K_0, respectively, are given by

$$\mathcal{A}_{0ipjq} = \mathcal{A}_{0jqip} = F_{i\alpha}F_{j\beta}\mathcal{A}_{\alpha p\beta q}, \tag{24}$$

$$\mathcal{C}_{0ijk} = F_{i\alpha}F^{-1}_{\beta k}\mathcal{C}_{\alpha j\beta}, \tag{25}$$

$$\mathsf{K}_{0ij} = \mathsf{K}_{0ji} = F^{-1}_{\alpha i}F^{-1}_{\beta j}\mathsf{K}_{\alpha\beta}. \tag{26}$$

It is to be noted here that the first two indices of the tensors $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}_0$ correspond to the derivative with respect to $\boldsymbol{F}$ and the third index corresponds to the derivative with respect to $\boldsymbol{H}_l$.

Explicit formulas for these components for an isotropic magnetoelastic material referred to the principal axes of the left Cauchy–Green tensor $\mathbf{b}$ are listed in [45]. These are substituted into the updated incremented constitutive equations above to give

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{H}}_{l0} - \dot{p}\mathbf{I} + p\mathbf{L}, \quad \dot{\mathbf{B}}_{l0} = -\boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} - \mathsf{K}_\mathbf{0}\dot{\mathbf{H}}_{l0}. \tag{27}$$

On substituting the above forms of constitutive equations in $(10)_{1,4}$ (assuming no mechanical body forces) we obtain

$$\operatorname{div}(\boldsymbol{\mathcal{A}}_0\mathbf{L} + \boldsymbol{\mathcal{C}}_0\dot{\mathbf{H}}_{l0} + p\mathbf{L}) - \operatorname{grad}\dot{p} = \rho\mathbf{u}_{,tt}, \tag{28}$$

$$\operatorname{div}(\boldsymbol{\mathcal{C}}_0^{\mathrm{T}}\mathbf{L} + \mathsf{K}_\mathbf{0}\dot{\mathbf{H}}_{l0}) = 0. \tag{29}$$

Using $(27)_1$, and the symmetry of the total stress tensor in the incremental form $(10)_2$, we obtain the identities

$$\mathcal{A}_{0ipjq} + \delta_{iq}(\tau_{jp} + p\delta_{jp}) = \mathcal{A}_{0pijq} + \delta_{pq}(\tau_{ij} + p\delta_{ij}), \quad \mathcal{C}_{0ijk} = \mathcal{C}_{0jik}, \tag{30}$$

the first of which can be used to obtain the useful relation

$$p = \mathcal{A}_{01313} - \mathcal{A}_{01331} - \tau_{11} = \mathcal{A}_{01212} - \mathcal{A}_{01221} - \tau_{11}. \tag{31}$$

4 Specialization to a cylindrical geometry

We consider an infinite circular cylindrical tube made of an incompressible non-conducting magnetoelastic material. We work in terms of cylindrical polar coordinates, which in the reference configuration $\mathcal{B}_\mathrm{r}$ are denoted by (R, Θ, Z) and in the deformed configuration $\mathcal{B}$ by (r, θ, z). In the reference configuration, let the internal and external radii of the tube be given by A and B, respectively.

The tube is deformed by an internal pressure, stretching in the axial direction, and the application of a magnetic field in the azimuthal and the axial directions to maintain axisymmetry. One way of applying an azimuthal magnetic field is by placing a current carrying wire through the axis of the hollow cylinder while a uniform axial magnetic field can be applied by placing permanent magnets at the two ends of the tube. After the deformation, the new inner and outer radii are a and b such that $a \leq r \leq b$. The deformation assumes the form

$$r = \left[a^2 + \frac{1}{\lambda_z}\left(R^2 - A^2\right)\right]^{\frac{1}{2}}, \quad z = \lambda_z Z, \quad \theta = \Theta, \tag{32}$$

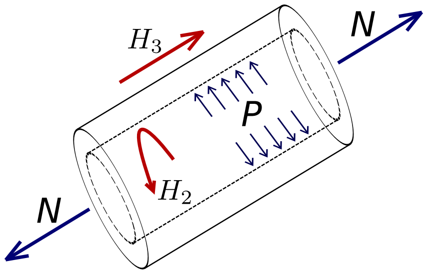


Figure 1: Sketch of the problem statement. A combination of internal pressure (P), axial force (N), axial magnetic field (H_3), and azimuthal magnetic field (H_2) is applied on a thick-walled magnetoelastic cylinder.

where the first relation is due to incompressibility and λ_z is the (uniform) axial stretch.

From here onwards, we take $(1, 2, 3)$ to correspond to (r, θ, z). Thus $\lambda_1, \lambda_2, \lambda_3$ correspond to principal stretches in r, θ, z coordinates and the underlying magnetic field is given as $\mathbf{H} = (0, H_2, H_3)$.

Using the constraint of incompressibility ($\lambda_1\lambda_2\lambda_3 = 1$), the principal stretches in the azimuthal, axial and radial directions are given by

$$\lambda_2 = \lambda = \frac{r}{R}, \quad \lambda_3 = \lambda_z, \quad \lambda_1 = \lambda^{-1}\lambda_z^{-1}, \tag{33}$$

respectively, wherein the notation λ is introduced.

From Equation (18), we obtain

$$\begin{aligned} \tau_{11} &= -p + 2\Omega_1\lambda_1^2 + 2\Omega_2\lambda_1^2\left(\lambda_2^2 + \lambda_3^2\right), \\ \tau_{22} &= -p + 2\Omega_1\lambda_2^2 + 2\Omega_2\lambda_2^2\left(\lambda_1^2 + \lambda_3^2\right) + 2\Omega_5\lambda_2^4H_2^2 + 4\Omega_6\lambda_2^6H_2^2, \\ \tau_{33} &= -p + 2\Omega_1\lambda_3^2 + 2\Omega_2\lambda_3^2\left(\lambda_1^2 + \lambda_2^2\right) + 2\Omega_5\lambda_3^4H_3^2 + 4\Omega_6\lambda_3^6H_3^2. \end{aligned} \tag{34}$$

The equilibrium equation $\operatorname{div}\boldsymbol{\tau} = \mathbf{0}$ gives

$$\frac{d\tau_{11}}{dr} = \frac{1}{r}(\tau_{22} - \tau_{11}), \tag{35}$$

which on substituting the values for τ_{11} and τ_{22} becomes

$$\frac{d\tau_{11}}{dr} = \frac{1}{r}\left[2\Omega_1\left(\lambda_2^2 - \lambda_1^2\right) + 2\Omega_2\lambda_3^2\left(\lambda_2^2 - \lambda_1^2\right) + 2\Omega_5\lambda_2^4H_2^2 + 4\Omega_6\lambda_2^6H_2^2\right]. \tag{36}$$

Boundary conditions on the lateral surfaces of the cylinder are given by the balance of traction $(4)_1$ as

$$\tau_{11} = \tau_{11}^* - P_{\text{in}} \quad \text{on } r = a, \quad \text{and} \quad \tau_{11} = \tau_{11}^* - P_{\text{out}} \quad \text{on } r = b, \tag{37}$$

where, P_{in} and P_{out} are the mechanically applied internal and external pressures, respectively, while τ_{11}^* obtains the value $-\mu_0(H_2^2 + H_3^2)/2$ from Equation (8).

We note here that in the case of a tube of finite length, the magnetic boundary conditions at the two ends of the tube are easily satisfied if the magnetic field is in azimuthal direction. For an axial magnetic field, a numerical analysis for a tube of finite length has been done by Bustamante et al. [59].

Since the independent parameters of the deformation process are $\lambda, \lambda_z, H_{l2}$, and H_{l3}, we can write the energy function as

$$\Omega(\mathbf{F}, \mathbf{H}_l) = \hat{\Omega}(\lambda, \lambda_z, H_{l2}, H_{l3}). \tag{38}$$

Equation (36) can then be rewritten as

$$\frac{d\tau_{11}}{dr} = \frac{\lambda}{r}\frac{\partial\hat{\Omega}}{\partial\lambda}. \tag{39}$$

We also mention the easily derived relations (see, for example, [50])

$$r\frac{d\lambda}{dr} = \lambda(1 - \lambda^2\lambda_z), \tag{40}$$

$$A^{-2}B^2\left(\lambda_b^2\lambda_z - 1\right) = \lambda_a^2\lambda_z - 1 = R^2A^{-2}(\lambda^2\lambda_z - 1), \qquad \frac{\partial\lambda_b}{\partial\lambda_a} = \frac{\lambda_a A^2}{\lambda_b B^2}, \tag{41}$$

for a finitely deformed tube, where we have defined $\lambda_a = \lambda|_{r=a}$ and $\lambda_b = \lambda|_{r=b}$.

On integrating equation (36) using the boundary conditions (37), we obtain

$$\begin{aligned} \int_a^b \frac{1}{r}\left[2\Omega_1\left(\lambda_2^2 - \lambda_1^2\right) + 2\Omega_2\lambda_3^2\left(\lambda_2^2 - \lambda_1^2\right) + 2\Omega_5\lambda_2^4H_2^2 + 4\Omega_6\lambda_2^6H_2^2\right] dr \\ = P_{\text{in}} - P_{\text{out}} + \frac{\mu_0}{2}\left(H_2^2|_a - H_2^2|_b\right), \end{aligned} \tag{42}$$

using which, along with $(32)_1$, we can determine the inner and outer radii (a, b) of the tube after deformation for a given pressure difference and magnetic field. (The contributions due to H_3 in the above formula cancel out.) Note that as we are working with an incompressible material, the terms P_{in} and P_{out} always appear together in the form of pressure difference. For the sake of brevity, we only consider P_{in} in the equations from here onwards. A negative value of P_{in} will correspond to a higher value of external pressure compared to the internal pressure. We now use the above calculated value of the inner radius a to obtain an expression for τ_{11} as a function of r by integrating equation (36) as

$$\begin{aligned}\tau_{11} = &-\frac{\mu_0}{2}H_{2a}^2 - P_{\text{in}}\\ &+\int_a^r \frac{1}{r}\left[2\Omega_1\left(\lambda_2^2-\lambda_1^2\right)+2\Omega_2\lambda_3^2\left(\lambda_2^2-\lambda_1^2\right)+2\Omega_5\lambda_2^4H_2^2+4\Omega_6\lambda_2^6H_2^2\right]dr.\end{aligned} \tag{43}$$

The above process can be repeated equivalently by using equation (39) instead of (36) depending on the requirements of the energy density function used.

In the following subsections we study the pressure and the axial force generated in the tube due to static nonlinear axisymmetric deformations in the presence of an underlying magnetic field.

4.1 Total internal pressure in the tube

We define the net total internal pressure P_T as the difference between the surface traction per unit area on the inside and on the outside of the tube.

$$P_T = (P_{\text{in}} - \tau_{11}^*|_a) + \tau_{11}^*|_b \tag{44}$$

$$= \left(P_{\text{in}} + \frac{\mu_0}{2}(H_2^2+H_3^2)|_a\right) - \frac{\mu_0}{2}(H_2^2+H_3^2)|_b. \tag{45}$$

Consider the case when there is either only a uniform axial magnetic field $(H_2 = 0)$. Then $H_3|_a = H_3|_b$ and hence $P_T = P_{\text{in}}$. For the case when there is only an axisymmetric magnetic field in the azimuthal direction $(H_3 = 0)$, the increment in pressure in comparison to a purely mechanical case is

$$P_T - P_{\text{in}} = \frac{\mu_0}{2}\left[H_2^2|_a - H_2^2|_b\right]. \tag{46}$$

On integrating Equation (39) using the boundary conditions (37), we obtain

$$P_T = \int_a^b \frac{\lambda}{r}\frac{\partial\hat{\Omega}}{\partial\lambda}dr. \tag{47}$$

which is slightly more general than the formula (127) given in [58]. We use Equation (40) to change the variable of integration from r to λ and hence obtain

$$P_T = \int_{\lambda_b}^{\lambda_a} \frac{1}{(\lambda^2\lambda_z - 1)}\frac{\partial\hat{\Omega}}{\partial\lambda}d\lambda. \tag{48}$$

On differentiating this with respect to λ_a and using Equation $(41)_1$, we obtain

$$\frac{(\lambda_a^2\lambda_z-1)}{\lambda_a}\frac{\partial P_T}{\partial\lambda_a} = \frac{1}{\lambda_a}\frac{\partial}{\partial\lambda}\hat{\Omega}(\lambda,\lambda_z,H_{l2},H_{l3})|_{\lambda=\lambda_a} - \frac{1}{\lambda_b}\frac{\partial}{\partial\lambda}\hat{\Omega}(\lambda,\lambda_z,H_{l2},H_{l3})|_{\lambda=\lambda_b}. \tag{49}$$

This is similar to the formula (15) obtained by Haughton and Ogden [50] in the context of pure elasticity. It is evident from the above equation that a necessary condition for an extremum to exist is

$$\frac{\partial}{\partial\lambda}\left(\frac{1}{\lambda}\frac{\partial}{\partial\lambda}\hat{\Omega}(\lambda,\lambda_z,H_{l2},H_{l3})\right) = 0, \qquad \text{for at least one} \quad \lambda \in (\lambda_b, \lambda_a). \tag{50}$$

For rubber-like solids it is observed experimentally (at least for thin-walled tubes) that as λ increases, the internal pressure increases up to a maximum, then decreases until it attains a minimum and then again increases monotonically until rupture [63]. We can predict a similar behaviour for the total pressure P_T if the above condition is satisfied. To show this we use a generalisation of the Ogden elastic energy density function to the magnetoelastic context. This is given by

$$\Omega = \hat{\Omega}(\lambda,\lambda_z,H_{l2}) = \sum_{r=1}^{3}\frac{\mu_r}{\alpha_r}\left(\lambda^{\alpha_r}+\lambda_z^{\alpha_r}+\lambda^{-\alpha_r}\lambda_z^{-\alpha_r}-3\right)+qK_5, \tag{51}$$

where the last term is $K_5 = (\lambda^2H_{l2}^2+\lambda_z^2H_{l3}^2)$. Here, μ_r are material constants with the dimension of stress, while the α_r are dimensionless constants and q is a magnetoelastic coupling parameter with q/μ_0 being dimensionless.

We also analyse the same problem for a generalised Mooney–Rivlin magnetoelastic material defined by

$$\Omega = \frac{\mu}{4}[(1+\gamma)(I_1-3)+(1-\gamma)(I_2-3)]+\tilde{q}K_5. \tag{52}$$

Table 1: List of material parameters for the generalised Ogden magnetoelastic energy density function (51). The mechanical parameters are similar to the ones used by Haughton and Ogden [51] for rubber elasticity.

α_1	α_2	α_3	μ	μ_1	μ_2	μ_3	q
1.3	5	-2	2.6×10^5 N/m^2	1.491μ	0.003μ	-0.023μ	$4\pi \times 10^{-7}$ N/A^2

Table 2: Numerical values of material parameters for the generalised Mooney–Rivlin magnetoelastic energy density function (52) as used by Otténio et al. [36].

μ	γ	$\tilde{q}$
2.6×10^5 N/m^2	0.3	$4\pi \times 10^{-7}$ N/A^2

Here μ is the shear modulus of the material in the absence of a magnetic field, γ is a dimensionless parameter in the range $-1 \leq \gamma \leq 1$, and $\tilde{q}$ is a magnetoelastic coupling constant such that $\tilde{q}/\mu_0$ is dimensionless. We note in passing that the energy density functions used here are prototype functions commonly used while solving computational problems in nonlinear magnetoelasticity [17, 36, 58]. Although they are able to capture the influence of magnetic loading in nonlinear magnetoelastic materials, they are not suitable to model some effects like magnetic saturation or damage due to particle motion in MREs. One can also perform the ensuing analysis by using material models that are more grounded in experimental data, such as those given by Bustamante [64], Danas et al. [65], and Itskov and Khiem [66].

The non-dimensionalised total internal pressure $\hat{P}_T = P_T/\mu$ is plotted against the inflation (λ_a = final internal radius/initial internal radius) in Figures 2 and 3 for the numerical values of material parameters listed in Tables 1 and 2. For the purpose of comparison we show the plots for the above two different energy density functions side-by-side.

In order to understand the combined influence of tube thickness B/A, radial stretch λ_a, and axial stretch λ_z, we solve a purely mechanical problem and show the results in Figure 2. Our computations show that for higher values of inflation, a lower value of axial stretch λ_z tends to increase the internal pressure but the situation reverses at lower values of λ_a where a higher axial stretch requires a higher pressure for the given deformation to be maintained. In a state of no inflation $\lambda_a = 1$ and axial compression $\lambda_z < 1$, we observe negative values of internal pressure – this corresponds to nothing but the requirement of an external pressure to obtain this configuration. Thicker the tuber, higher is the pressure required for the same amount of inflation. In general, the internal pressure required to maintain the geometrical configuration rises with an increase in the inflation λ_a. For the Mooney–Rivlin type function, the increase is monotonic while for the Ogden type function the pressure increases up to a maximum and then falls as discussed previously in accordance with Equation (50).

The axial magnetic field H_3 has no effect on the internal pressure on account of Equation (46). A reference value H_0 is taken for the azimuthal magnetic field so that at radius r, H_2 is given by $H_2 = H_0 B/r$. We plot the dependence of the internal pressure on the magnetic field in Figure 3 for the case of $B/A = 1.3$ and $\lambda_z = 0.9$. An azimuthal magnetic field tends to reduce the inflation of the cylindrical tube given constant pressure and constant axial stretch. In other words, if a given deformation is to be maintained, application of an azimuthal magnetic field leads to a reduction of the required internal pressure.

4.2 Total axial load on the cylinder

The principal stress in the axial direction is given as

$$\tau_{33} = \lambda_3 \frac{\partial \Omega}{\partial \lambda_3} - p, \tag{53}$$

which on using equations (35) and (38) can be rewritten in terms of principal stretches

$$\tau_{33} = \frac{1}{2}\left(2\lambda_z \frac{\partial \hat{\Omega}}{\partial \lambda_z} - \lambda \frac{\partial \hat{\Omega}}{\partial \lambda}\right) + \frac{1}{2r}\frac{d}{dr}\left(r^2 \tau_{11}\right), \tag{54}$$

or the invariants as

$$\begin{aligned}\tau_{33} =& \Omega_1(3\lambda_3^2 - I_1) + \Omega_2(I_2 - 3\lambda_1^2\lambda_2^2) \\ &+ \Omega_5\left(2\lambda_3^4 H_3^2 - \lambda_2^4 H_2^2\right) + 2\Omega_6\left(2\lambda_3^6 H_3^2 - \lambda_2^6 H_2^2\right) + \frac{1}{2r}\frac{d}{dr}\left(r^2\tau_{11}\right).\end{aligned} \tag{55}$$

For detailed derivation of above expression, we refer to Appendix C of [47].

The total axial force on the cylinder is given as

$$N = \int_0^{2\pi}\int_a^b \tau_{33}\, r\, dr\, d\theta, \tag{56}$$

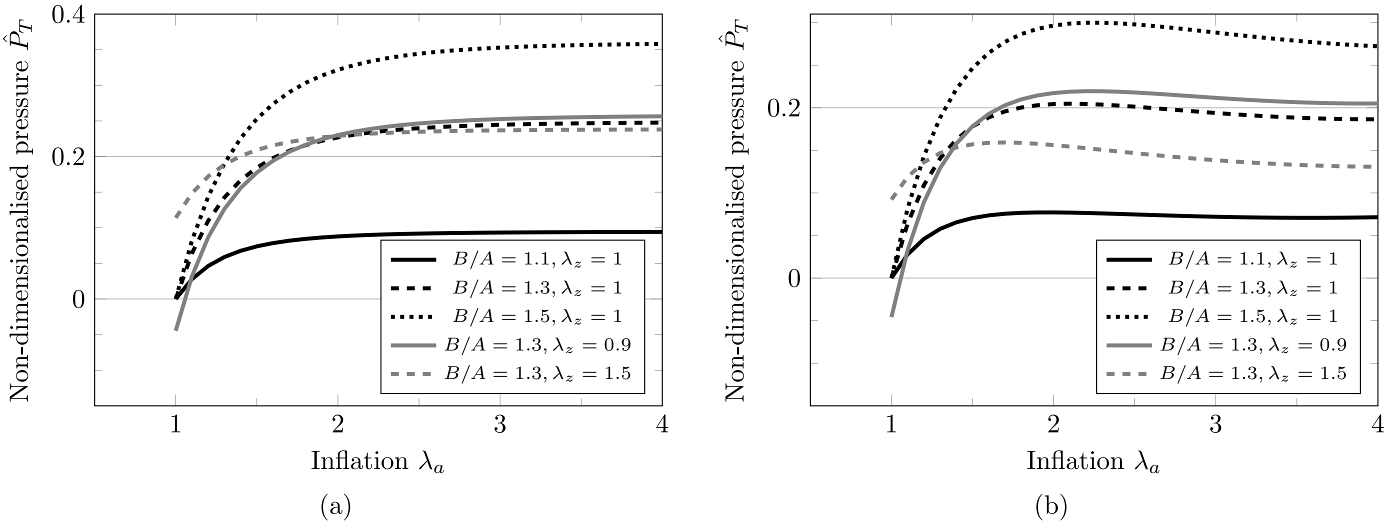


Figure 2: Plot of non-dimensionalised internal pressure vs inflation for different values of axial stretch and tube thickness for purely mechanical deformation. (a) Mooney-Rivlin type magnetoelastic material (b) Ogden type magnetoelastic material.

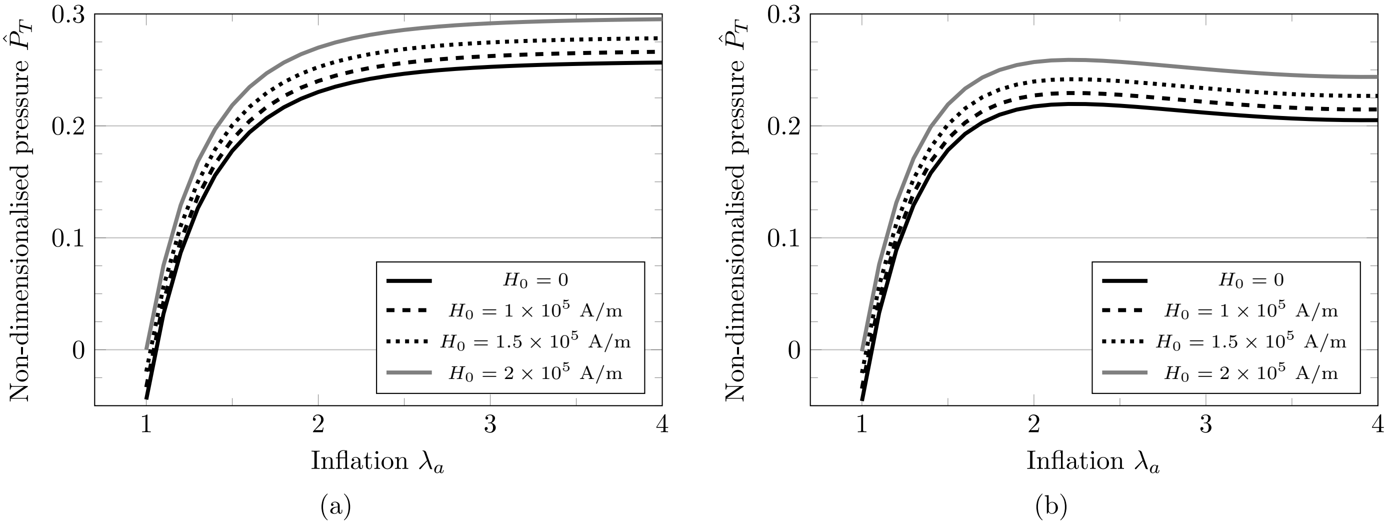


Figure 3: Plot of non-dimensionalised internal pressure vs inflation for different values of underlying azimuthal magnetic field $H_2 = H_0 B/A$. $B/A = 1.3, \lambda_z = 0.9$. (a) Mooney-Rivlin type magnetoelastic material (b) Ogden type magnetoelastic material.

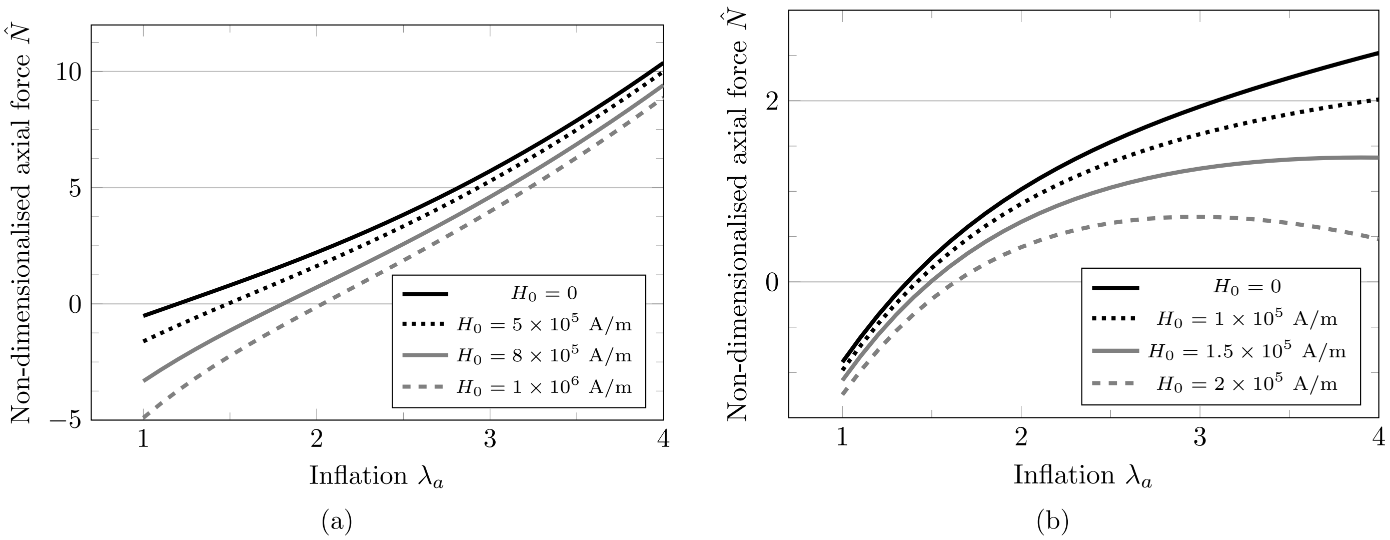


Figure 4: Plot of non-dimensionalised axial force vs inflation for different values of the underlying azimuthal magnetic field $H_2 = H_0 B/A$. $B/A = 1.3, \lambda_z = 0.9$. (a) Mooney–Rivlin type magnetoelastic material (b) Ogden type magnetoelastic material.

which on using the value of τ_{33} from Equation (54) can be rewritten as

$$N = \pi \int_a^b \left(2\lambda_z \frac{\partial \hat{\Omega}}{\partial \lambda_z} - \lambda \frac{\partial \hat{\Omega}}{\partial \lambda} \right) r dr + \pi a^2 P_{\text{in}} - \frac{\pi \mu_0}{2} H_3^2 \left(b^2 - a^2 \right), \tag{57}$$

which is similar to the formula (128) obtained by Dorfmann and Ogden [58]. Using Equation (40), we can change the variable of integration in the first term from r to λ to get

$$N = \pi A^2 \left(\lambda_a^2 \lambda_z - 1 \right) \int_{\lambda_b}^{\lambda_a} \frac{\lambda}{(\lambda^2 \lambda_z - 1)^2} \left(2\lambda_z \frac{\partial \hat{\Omega}}{\partial \lambda_z} - \lambda \frac{\partial \hat{\Omega}}{\partial \lambda} \right) d\lambda$$
$$+ \pi a^2 P_{\text{in}} - \frac{\pi \mu_0}{2} H_3^2 \left(b^2 - a^2 \right). \tag{58}$$

Similar rearrangements to the invariant based expressions in equations (55) and (56) gives

$$N = \pi \int_a^b [\Omega_1 (3\lambda_3^2 - I_1) + \Omega_2 (I_2 - 3\lambda_1^2 \lambda_2^2) + \Omega_5 \left(2\lambda_3^4 H_3^2 - \lambda_2^4 H_2^2 \right)$$
$$+ 2\Omega_6 \left(2\lambda_3^6 H_3^2 - \lambda_2^6 H_2^2 \right)] r \, dr + \pi a^2 P_{\text{in}} - \frac{\pi \mu_0}{2} H_3^2 \left(b^2 - a^2 \right). \tag{59}$$

Upon changing the variable of integration from r to λ, we obtain

$$N = \pi A^2 \left(\lambda_a^2 \lambda_z - 1 \right) \int_{\lambda_b}^{\lambda_a} \frac{\lambda}{(\lambda^2 \lambda_z - 1)^2} [\Omega_1 (3\lambda_3^2 - I_1) + \Omega_2 (I_2 - 3\lambda_1^2 \lambda_2^2)$$
$$+ \Omega_5 \left(2\lambda_3^4 H_3^2 - \lambda_2^4 H_2^2 \right) + 2\Omega_6 \left(2\lambda_3^6 H_3^2 - \lambda_2^6 H_2^2 \right)] \, d\lambda$$
$$+ \pi a^2 P_{\text{in}} - \frac{\pi \mu_0}{2} H_3^2 \left(b^2 - a^2 \right). \tag{60}$$

The non-dimensionalised axial force $\hat{N} = N/\mu$ is plotted against the inflation (λ_a = final internal radius/initial internal radius) in Figure 4 for the numerical values of material parameters listed in Tables 1 and 2. For the purpose of comparison we show the plots for the Mooney–Rivlin type and Ogden type magnetoelastic energy density functions side-by-side. For the Mooney–Rivlin material, axial force increases with a positive upwards curvature while the opposite is true for an Ogden elastic material. Upon application of an azimuthal magnetic field ($\mathbf{H} = (0, H_2, 0)$), axial force is reduced suggesting that the magnetic field tends to cause an extensional loading in the axial direction. The effect of magnetic field is more pronounced for the Mooney–Rivlin type magnetoelastic material in the case of smaller values of inflation λ_a. Reverse happens in the case of Ogden type magnetoelastic material in which the effect of magnetic field is more noticeable in the case of higher λ_a.

5 Incremental motions

We now consider time-dependent increments in the displacement and the magnetic field superimposed on the underlying finite static deformation. Consider a small increment $\mathbf{u}$ in the deformation such that $\mathbf{u} = \{u_1, u_2, u_3\}$. The constraint of incompressibility requires $\mathbf{u}$ to satisfy the condition $\operatorname{div}\mathbf{u} = 0$. We consider only axisymmetric motions so that there is no dependence on θ and the components of the displacement gradient and the increment in the deformation gradient are given in matrix form by

$$[\mathbf{L}] = [\operatorname{grad}\mathbf{u}] = \begin{bmatrix} u_{1,1} & -u_2/r & u_{1,3} \\ u_{2,1} & u_1/r & u_{2,3} \\ u_{3,1} & 0 & u_{3,3} \end{bmatrix}, \tag{61}$$

$$[\dot{\mathbf{F}}] = [\operatorname{Grad}\,\mathbf{u}] = [\mathbf{LF}] = \begin{bmatrix} \lambda^{-1}\lambda_z^{-1}u_{1,1} & -\lambda u_2/r & \lambda_z u_{1,3} \\ \lambda^{-1}\lambda_z^{-1}u_{2,1} & \lambda u_1/r & \lambda_z u_{2,3} \\ \lambda^{-1}\lambda_z^{-1}u_{3,1} & 0 & \lambda_z u_{3,3} \end{bmatrix}, \tag{62}$$

where here and henceforth we use the subscript i followed by a comma to denote a derivative with respect to the ith coordinate, $i \in \{1, 3\}$.

The incremental incompressibility constraint $\operatorname{tr}\mathbf{L} = 0$ is then given as

$$u_{1,1} + \frac{u_1}{r} + u_{3,3} = 0. \tag{63}$$

Turning to the boundary conditions, we consider first the curved faces ($\mathbf{n} = \mathbf{e}_1$ for the outer surface and $\mathbf{n} = -\mathbf{e}_1$ for the inner surface) of the tube. Continuity of the underlying magnetic field requires that $H_2 = H_2^*$ and $H_3 = H_3^*$ on each of these boundaries. For the incremental traction boundary condition (11) we obtain, after making use of the above expressions for the Maxwell stress and its increment,

$$\dot{T}_{011} = \dot{\tau}_{11}^* - \tau_{11}^* L_{11}, \quad \dot{T}_{012} = \dot{\tau}_{12}^* - \tau_{22}^* L_{12}, \quad \dot{T}_{013} = \dot{\tau}_{13}^* - \tau_{33}^* L_{13}, \tag{64}$$

on each face, and for the incremental magnetic boundary conditions

$$\dot{B}_{l01} - \dot{B}_1^* + B_2^* L_{12} + B_3^* L_{13} = 0, \tag{65}$$

$$\dot{H}_{l02} - \dot{H}_2^* - H_2 L_{22} = 0, \tag{66}$$

$$\dot{H}_{l03} - \dot{H}_3^* - H_2 L_{23} - H_3 L_{33} = 0. \tag{67}$$

If one were to study finite geometry, the components of the incremental mechanical traction required on the ends $z = 0, \lambda_z L$ ($\mathbf{n} = \pm\mathbf{e}_3$) of the cylinder to maintain any imposed displacement condition are, from $(11)_1$,

$$\dot{T}_{031} + \tau_{11}^* L_{31}, \quad \dot{T}_{032} - \dot{\tau}_{23}^* + \tau_{22}^* L_{32}, \quad \dot{T}_{033} - \dot{\tau}_{33}^* + \tau_{33}^* L_{33}, \tag{68}$$

and the incremental magnetic boundary conditions are obtained from equations $(11)_{2,3}$ as

$$\dot{H}_{l01} - \dot{H}_1^* - H_2 L_{21} = 0, \quad \dot{H}_{l02} - \dot{H}_2^* - H_2 L_{22} = 0, \quad \dot{B}_{l03} - \dot{B}_3^* = 0. \tag{69}$$

However, these equations will not be used in this paper since we restrict the analysis to an infinite cylinder.

We now consider the two cases of the underlying magnetic field being in the axial and in the azimuthal directions separately.

6 Axial magnetic field $\mathbf{H} = (0, 0, H_3)$

In this first case, we consider an infinite tube with a uniform initial magnetic field in the axial direction. The Maxwell stress and its increment are given in component form by

$$[\boldsymbol{\tau}^*] = \frac{B_3^{*2}}{2\mu_0} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\dot{\boldsymbol{\tau}}^*] = \frac{B_3^*}{\mu_0} \begin{bmatrix} -\dot{B}_3^* & 0 & \dot{B}_1^* \\ 0 & -\dot{B}_3^* & \dot{B}_2^* \\ \dot{B}_1^* & \dot{B}_2^* & \dot{B}_3^* \end{bmatrix}. \tag{70}$$

In the presence of an axial magnetic field, the non-zero components of the magnetoelastic tensors are $\mathcal{A}_{0iiii}$, $\mathcal{A}_{0iijj}$, $\mathcal{A}_{0ijij}$, $\mathcal{A}_{0ijji}$, $\mathcal{C}_{0ii3}$, $\mathcal{C}_{0i3i}$, K_{0ii} for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Explicit formulas for these components for the generalized Mooney–Rivlin magnetoelastic material are given in the Appendix. Expanding the incremental governing equations $(10)_3$, (28), and (29) in component form, we obtain

$$\dot{H}_{l02,3} = 0, \quad \dot{H}_{l02,1} + \frac{\dot{H}_{l02}}{r} = 0, \tag{71}$$

$$\dot{H}_{l01,3} - \dot{H}_{l03,1} = 0, \tag{72}$$

$$\frac{1}{r}\left\{r\left(\mathcal{C}_{0131}(u_{1,3}+u_{3,1})+\mathsf{K}_{011}\dot{H}_{l01}\right)\right\}_{,1}$$
$$+\left(\mathcal{C}_{0113}u_{1,1}+\mathcal{C}_{0223}u_1/r+\mathcal{C}_{0333}u_{3,3}+\mathsf{K}_{033}\dot{H}_{l03}\right)_{,3}=0, \tag{73}$$

$$\frac{1}{r}\left\{r\left((\mathcal{A}_{01111}+p)u_{1,1}+\mathcal{A}_{01122}u_1/r+\mathcal{A}_{01133}u_{3,3}+\mathcal{C}_{0113}\dot{H}_{l03}\right)\right\}_{,1}$$
$$-\frac{1}{r}\left(\mathcal{A}_{01122}u_{1,1}+(\mathcal{A}_{02222}+p)u_1/r+\mathcal{A}_{02233}u_{3,3}+\mathcal{C}_{0223}\dot{H}_{l03}\right)$$
$$+\left(\mathcal{A}_{03131}u_{1,3}+(\mathcal{A}_{03113}+p)u_{3,1}+\mathcal{C}_{0131}\dot{H}_{l01}\right)_{,3}-\dot{p}_{,1}=\rho u_{1,tt}, \tag{74}$$

$$\frac{1}{r}\left[\left\{r\left(\mathcal{A}_{01212}u_{2,1}-(\mathcal{A}_{01221}+p)\frac{u_2}{r}\right)\right\}_{,1}-\mathcal{A}_{02121}u_2/r+(\mathcal{A}_{01221}+p)u_{2,1}\right]$$
$$+\left\{\mathcal{A}_{03232}u_{2,3}+\mathcal{C}_{0322}\dot{H}_{l02}\right\}_{,3}=\rho u_{2,tt}, \tag{75}$$

$$\frac{1}{r}\left\{r\left(\mathcal{A}_{01313}u_{3,1}+(\mathcal{A}_{01331}+p)u_{1,3}+\mathcal{C}_{0131}\dot{H}_{l01}\right)\right\}_{,1}$$
$$+\left\{\mathcal{A}_{01133}u_{1,1}+\mathcal{A}_{02233}u_1/r+(\mathcal{A}_{03333}+p)u_{3,3}+\mathcal{C}_{0333}\dot{H}_{l03}\right\}_{,3}-\dot{p}_{,3}=\rho u_{3,tt}. \tag{76}$$

In vacuum, equations (12) give

$$\dot{H}^*_{1,1}+\frac{\dot{H}^*_1}{r}+\dot{H}^*_{3,3}=0,\quad \dot{H}^*_{1,3}-\dot{H}^*_{3,1}=0,\quad \dot{H}^*_{2,3}=0,\quad \dot{H}^*_{2,1}+\frac{\dot{H}^*_2}{r}=0. \tag{77}$$

Note that if we consider the purely elastic case (neglecting $\dot{\mathbf{H}}_{l0}$ and $\boldsymbol{\mathcal{C}}$) and only quasi-static bifurcations (no dependence on time), then the equations (74) and (76) reduce to equations (47) and (48) of Haughton and Ogden [50] after taking into account the differences in notation. We can eliminate $\dot{p}$ from equations (74) and (76) to get

$$-\frac{1}{r}\left\{r\left(\mathcal{A}_{01313}u_{3,1}+(\mathcal{A}_{01331}+p)u_{1,3}+\mathcal{C}_{0131}\dot{H}_{l01}\right)\right\}_{,11}+\frac{1}{r^2}\left\{r\left(\mathcal{A}_{01313}u_{3,1}\right.\right.$$
$$\left.\left.+(\mathcal{A}_{01331}+p)u_{1,3}+\mathcal{C}_{0131}\dot{H}_{l01}\right)\right\}_{,1}+\frac{1}{r}\left\{r\left((\mathcal{A}_{01111}+p)u_{1,1}+\mathcal{A}_{01122}u_1/r\right.\right.$$
$$\left.\left.+\mathcal{A}_{01133}u_{3,3}+\mathcal{C}_{0113}\dot{H}_{l03}\right)\right\}_{,13}-\left(\mathcal{A}_{01133}u_{1,1}+\mathcal{A}_{02233}u_1/r+(\mathcal{A}_{03333}+p)u_{3,3}\right.$$
$$\left.+\mathcal{C}_{0333}\dot{H}_{l03}\right)_{,13}+\left\{\mathcal{A}_{03131}u_{1,3}+(\mathcal{A}_{01331}+p)\,u_{3,1}+\mathcal{C}_{0311}\dot{H}_{l01}\right\}_{,33}$$
$$-\frac{1}{r}\left\{\mathcal{A}_{01122}u_{1,1}+(\mathcal{A}_{02222}+p)\,\frac{u_1}{r}+\mathcal{A}_{02233}u_{3,3}+\mathcal{C}_{0223}\dot{H}_{l03}\right\}_{,3}=\rho(u_{1,3}-u_{3,1})_{,tt}. \tag{78}$$

It can be seen from the equations above that u_2 and $\dot{H}_{l02}$ are coupled with each other and are independent of $u_1, u_3, \dot{H}_{l01}$, and $\dot{H}_{l03}$ which are related to each other. We now consider both these cases separately.

6.1 Displacement in the (r, z) plane

In this section, we work only with the equations that have incremental motion in the radial and the axial directions. Considering that the magnetoelastic moduli tensors are uniform along the z direction, we can rewrite Equation (78) as

$$A_1u_{1,3}+A_2u_{1,13}+A_3u_{1,113}+A_4u_{3,1}+A_5u_{3,11}+A_6u_{3,111}+A_7u_{3,133}+A_8u_{3,33}$$
$$A_9u_{1,333}+A_{10}\dot{H}_{l01}+A_{11}\dot{H}_{l01,1}+A_{12}\dot{H}_{l01,11}+A_{13}\dot{H}_{l03,3}+A_{14}\dot{H}_{l03,13}$$
$$-A_{12}\dot{H}_{l01,33}=\rho\left(u_{1,3}-u_{3,1}\right)_{,tt}, \tag{79}$$

where we have used the fact that the magnetoelastic moduli tensors are independent of z, and the coefficients $A_1, ..., A_{14}$ are defined by

$$A_1 = -(\mathcal{A}_{01331} + p)_{,11} + \frac{1}{r}(\mathcal{A}_{01122} - \mathcal{A}_{02233} - \mathcal{A}_{01331} - p)_{,1} + \frac{1}{r^2}(\mathcal{A}_{01331} + \mathcal{A}_{02233} - \mathcal{A}_{02222}), \quad A_2 = (\mathcal{A}_{01111} - 2\mathcal{A}_{01331} - \mathcal{A}_{01133} - p)_{,1} + \frac{1}{r}(\mathcal{A}_{01111} - \mathcal{A}_{01331} - \mathcal{A}_{02233}),$$
$$A_3 = \mathcal{A}_{01111} - \mathcal{A}_{01313} - \mathcal{A}_{01133}, \quad A_4 = -\mathcal{A}_{01313,11} - \frac{\mathcal{A}_{01313,1}}{r} + \frac{\mathcal{A}_{01313}}{r^2},$$
$$A_5 = -\frac{\mathcal{A}_{01313}}{r} - 2\mathcal{A}_{01313,1}, \quad A_6 = -\mathcal{A}_{01313}, \quad A_7 = \mathcal{A}_{01133} - \mathcal{A}_{03333} + \mathcal{A}_{01331},$$
$$A_8 = (\mathcal{A}_{01133} - \mathcal{A}_{03333} - p)_{,1} + \frac{\mathcal{A}_{01133} - \mathcal{A}_{02233}}{r}, \quad A_9 = \mathcal{A}_{03131},$$
$$A_{10} = \frac{\mathcal{C}_{0131}}{r^2} - \frac{\mathcal{C}_{0131,1}}{r} - \mathcal{C}_{0131,11}, \quad A_{11} = -\frac{\mathcal{C}_{0131}}{r} - 2\mathcal{C}_{0131,1}, \quad A_{12} = -\mathcal{C}_{0131},$$
$$A_{13} = (\mathcal{C}_{0113} - \mathcal{C}_{0333})_{,1} + \frac{\mathcal{C}_{0113} - \mathcal{C}_{0223}}{r}, \quad A_{14} = \mathcal{C}_{0113} - \mathcal{C}_{0333}. \tag{80}$$

Equation (73) gives

$$\left(\mathcal{C}_{0131,1} + \frac{\mathcal{C}_{0131}}{r}\right) u_{3,1} + \left(\mathcal{C}_{0131,1} + \frac{\mathcal{C}_{0131} + \mathcal{C}_{0223}}{r}\right) u_{1,3} + (\mathcal{C}_{0131} + \mathcal{C}_{0113})\, u_{1,13} + \mathcal{C}_{0131} u_{3,11} + \mathcal{C}_{0333} u_{3,33} + \left(\mathsf{K}_{011,1} + \frac{\mathsf{K}_{011}}{r}\right) \dot{H}_{l01} + \mathsf{K}_{011} \dot{H}_{l01,1} + \mathsf{K}_{033} \dot{H}_{l03,3} = 0, \tag{81}$$

while from the boundary conditions $(64)_{1,3}$, (66), and (67), we get

$$\mathcal{A}_{01111} u_{1,1} + \mathcal{A}_{01122} \frac{u_1}{r} + \mathcal{A}_{01133} u_{3,3} + \mathcal{C}_{0113} \dot{H}_{l03} - \dot{p} + p u_{1,1} = -\mu_0 H_3 \dot{H}_3^* + \frac{\mu_0 H_3^2}{2} u_{1,1}, \tag{82}$$

$$\mathcal{A}_{01313} u_{3,1} + \mathcal{A}_{01331} u_{1,3} + \mathcal{C}_{0131} \dot{H}_{l01} + p u_{1,3} = \mu_0 H_3 \dot{H}_1^* - \frac{\mu_0 H_3^2}{2} u_{1,3}, \tag{83}$$

$$-\mathcal{C}_{0131}\,(u_{1,3} + u_{3,1}) - \mathsf{K}_{011} \dot{H}_{l01} - \mu_0 \dot{H}_1^* + \mu_0 H_3 u_{1,3} = 0, \tag{84}$$

$$\dot{H}_{l03} - H_3 u_{3,3} - \dot{H}_3^* = 0, \tag{85}$$

at $r = a$ and $r = b$.

We differentiate Equation (82) with respect to z and replace $\dot{p}_{,3}$ using Equation (76) to get

$$\xi_1 u_{1,3} + \xi_2 u_{1,13} + \xi_3 u_{3,1} + \xi_4 u_{3,33} + A_6 u_{3,11} + \xi_5 \dot{H}_{l01} + A_{12} \dot{H}_{l01,1} + A_{14} \dot{H}_{l03,3} + \mu_0 H_3 \dot{H}_{3,3}^* + \rho u_{3,tt} = 0, \tag{86}$$

where the coefficients $\xi_1, ..., \xi_5$ are defined by

$$\xi_1 = \frac{1}{r}\left(\mathcal{A}_{01122} - \mathcal{A}_{01331} - \mathcal{A}_{02233} - p\right) - \left(\mathcal{A}_{01331} + p\right)_{,1},$$
$$\xi_2 = \mathcal{A}_{01111} - \mathcal{A}_{01331} - \mathcal{A}_{01133} - \frac{\mu_0 H_3^2}{2}, \quad \xi_3 = -\left(\mathcal{A}_{01313,1} + \frac{\mathcal{A}_{01313}}{r}\right),$$
$$\xi_4 = \mathcal{A}_{01133} - \mathcal{A}_{03333} - p, \quad \xi_5 = -\left(\mathcal{C}_{0131,1} + \frac{\mathcal{C}_{0131}}{r}\right). \tag{87}$$

Since u_1 and u_3 satisfy Equation (63), $\dot{H}_{l01}$ and $\dot{H}_{l03}$ satisfy Equation (72), and $\dot{H}_1^*$ and $\dot{H}_3^*$ satisfy Equation $(77)_2$, we can define the potentials $\phi(r,z,t)$, $\psi(r,z,t)$, and $\psi^*(r,z,t)$ such that

$$u_1 = \frac{\phi_{,3}}{r}, \quad u_3 = \frac{-\phi_{,1}}{r}, \quad \dot{H}_{l01} = \psi_{,1}, \quad \dot{H}_{l03} = \psi_{,3}, \quad \dot{H}_1^* = \psi_{,1}^*, \quad \dot{H}_3^* = \psi_{,3}^*. \tag{88}$$

Substituting the potentials and their derivatives in the governing equations $(77)_1$, (79), and (81), we get

$$\phi_{,1}\left(\frac{A_4}{r^2} - 2\frac{A_5}{r^3} + 6\frac{A_6}{r^4}\right) + \phi_{,11}\left(-\frac{A_4}{r} + 2\frac{A_5}{r^2} - 6\frac{A_6}{r^3}\right) + \phi_{,111}\left(-\frac{A_5}{r} + 3\frac{A_6}{r^2}\right) - \frac{A_6}{r}\phi_{,1111} + \phi_{,33}\left(\frac{A_1}{r} - \frac{A_2}{r^2} + 2\frac{A_3}{r^3}\right) + \phi_{,133}\left(\frac{A_2}{r} - 2\frac{A_3}{r^2} - \frac{A_8}{r} + \frac{A_7}{r^2}\right) + \phi_{,1133}\left(\frac{A_3}{r} - \frac{A_7}{r}\right) + \frac{A_9}{r}\phi_{,3333} + A_{10}\psi_{,1} + A_{11}\psi_{,11} + A_{12}\psi_{,111} + A_{13}\psi_{,33} + (A_{14} - A_{12})\,\psi_{,133} = \rho\left(\frac{\phi_{,33}}{r} + \frac{\phi_{,11}}{r} - \frac{\phi_{,1}}{r^2}\right)_{,tt}, \tag{89}$$

$$\phi_{,1}\left(\frac{\mathcal{C}_{0131,1}}{r^2}-\frac{\mathcal{C}_{0131}}{r^3}\right)+\phi_{,11}\left(-\frac{\mathcal{C}_{0131,1}}{r}+\frac{\mathcal{C}_{0131}}{r^2}\right)-\frac{\mathcal{C}_{0131}}{r}\phi_{,111}$$
$$+\left(\frac{\mathcal{C}_{0131,1}}{r}+\frac{\mathcal{C}_{0223}-\mathcal{C}_{0113}}{r^2}\right)\phi_{,33}+\frac{\phi_{,133}}{r}\left(\mathcal{C}_{0131}+\mathcal{C}_{0113}-\mathcal{C}_{0333}\right)$$
$$+\left(\mathsf{K}_{011,1}+\frac{\mathsf{K}_{011}}{r}\right)\psi_{,1}+\mathsf{K}_{011}\psi_{,11}+\mathsf{K}_{033}\psi_{,33}=0, \tag{90}$$

for $a < r < b$ and

$$\psi^*_{,11}+\frac{1}{r}\psi^*_{,1}+\psi^*_{,33}=0, \tag{91}$$

for $r < a$ and $r > b$.

The boundary conditions become

$$\frac{\mathcal{A}_{01313}}{r^2}\phi_{,1}-\frac{\mathcal{A}_{01313}}{r}\phi_{,11}+\phi_{,33}\left(\frac{\mathcal{A}_{01331}+p}{r}+\frac{\mu_0H_3^2}{2r}\right)+\mathcal{C}_{0131}\psi_{,1}-\mu_0H_3\psi^*_{,1}=0, \tag{92}$$

$$\frac{\mathcal{C}_{0131}}{r^2}\phi_{,1}-\frac{\mathcal{C}_{0131}}{r}\phi_{,11}+\left(\frac{\mathcal{C}_{0131}-\mu_0H_3}{r}\right)\phi_{,33}+\mathsf{K}_{011}\psi_{,1}+\mu_0\psi^*_{,1}=0, \tag{93}$$

$$\psi_{,3}+\frac{H_3^*}{r}\phi_{,13}-\psi^*_{,3}=0, \tag{94}$$

$$\phi_{,1}\left(\frac{\xi_3}{r^2}-2\frac{A_6}{r^3}\right)+\phi_{,11}\left(\frac{-\xi_3}{r}+2\frac{A_6}{r^2}\right)+\left(-\frac{\xi_2}{r^2}+\frac{\xi_1}{r}\right)\phi_{,33}+\phi_{,133}\left(\frac{\xi_2-\xi_4}{r}\right)$$
$$-\frac{A_6}{r}\phi_{,111}+\xi_5\psi_{,1}+A_{12}\psi_{,11}+A_{14}\psi_{,33}+\mu_0H_3\psi^*_{,33}-\frac{\rho}{r}\phi_{,1tt}=0, \tag{95}$$

at $r = a$ and $r = b$.

6.1.1 Wave propagation solutions

For the above partial differential equations, by separation of variables we consider wave type solutions of the form

$$\phi = F(r)\exp\left(ikz-i\omega t\right), \quad a<r<b, \tag{96}$$
$$\psi = G(r)\exp\left(ikz-i\omega t\right), \quad a<r<b, \tag{97}$$
$$\psi^* = M_1(r)\exp\left(ikz-i\omega t\right), \quad r<a, \tag{98}$$
$$\psi^* = M_2(r)\exp\left(ikz-i\omega t\right), \quad r>b, \tag{99}$$

which convert the equations to a system of coupled ODEs as follows

$$\left\{\frac{k^4}{r}A_9-k^2\left(\frac{A_1}{r}-\frac{A_2}{r^2}+2\frac{A_3}{r^3}+\frac{\rho\omega^2}{r}\right)\right\}F$$
$$+\left\{\frac{A_4}{r^2}-2\frac{A_5}{r^3}+6\frac{A_6}{r^4}-k^2\left(\frac{A_2-A_8}{r}+\frac{A_7-2A_3}{r^2}\right)-\frac{\rho\omega^2}{r}\right\}F'$$
$$+\left\{-\frac{A_4}{r}-2\frac{A_5}{r^2}-6\frac{A_6}{r^3}-k^2\frac{A_3-A_7}{r}+\frac{\rho\omega^2}{r}\right\}F''+\left(3\frac{A_6}{r^2}-\frac{A_5}{r}\right)F'''$$
$$-\frac{A_6}{r}F''''-k^2A_{13}G+\left\{A_{10}+k^2\left(A_{12}-A_{14}\right)\right\}G'+A_{11}G''+A_{12}G'''=0, \tag{100}$$

$$-k^2\left(\frac{\mathcal{C}_{0131,1}}{r}+\frac{\mathcal{C}_{0223}-\mathcal{C}_{0113}}{r^2}\right)F+\left(-\frac{\mathcal{C}_{0131,1}}{r}+\frac{\mathcal{C}_{0131}}{r^2}\right)F''$$
$$+\left\{-\frac{\mathcal{C}_{0131}}{r^3}+\frac{\mathcal{C}_{0131,1}}{r^2}-\frac{k^2}{r}\left(\mathcal{C}_{0131}+\mathcal{C}_{0113}-\mathcal{C}_{0333}\right)\right\}F'-\frac{\mathcal{C}_{0131}}{r}F'''-k^2\mathsf{K}_{033}G$$
$$+\left(\mathsf{K}_{011,1}+\frac{\mathsf{K}_{011}}{r}\right)G'+\mathsf{K}_{011}G''=0, \tag{101}$$

for $a < r < b$, and

$$M_1''+\frac{1}{r}M_1'-k^2M_1=0, \quad r<a, \qquad M_2''+\frac{1}{r}M_2'-k^2M_2=0, \quad r>b. \tag{102}$$

Here and henceforth, a prime denotes a derivative with respect to r. The boundary conditions reduce to

$$-k^2\left(\frac{\mathcal{A}_{01331}+p}{r}+\frac{\mu_0H_3^2}{2r}\right)F+\frac{\mathcal{A}_{01313}}{r^2}F'-\frac{\mathcal{A}_{01313}}{r}F''$$
$$+\mathcal{C}_{0131}G'-\mu_0H_3M_1'=0, \tag{103}$$

$$-\frac{k^2}{r}\left(\mathcal{C}_{0131}-\mu_0 H_3\right)F+\frac{\mathcal{C}_{0131}}{r^2}F'-\frac{\mathcal{C}_{0131}}{r}F''+\mathsf{K}_{011}G'+\mu_0 M_1'=0, \tag{104}$$

$$G+\frac{H_3}{r}F'-M_1=0, \tag{105}$$

$$-k^2\left(-\frac{\xi_2}{r^2}+\frac{\xi_1}{r}\right)F+\left\{\frac{\xi_3}{r^2}-2\frac{A_6}{r^3}-\frac{k^2}{r}\left(\xi_2-\xi_4\right)+\frac{\rho c^2 k^2}{r}\right\}F'$$
$$+\left(-\frac{\xi_3}{r}+2\frac{A_6}{r^2}\right)F''-\frac{A_6}{r}F'''-k^2A_{14}G+\xi_5 G'+A_{12}G''-k^2\mu_0 H_3 M_1=0, \tag{106}$$

at $r=a$ and

$$-k^2\left(\frac{\mathcal{A}_{01331}+p}{r}+\frac{\mu_0 H_3^2}{2r}\right)F+\frac{\mathcal{A}_{01313}}{r^2}F'-\frac{\mathcal{A}_{01313}}{r}F''$$
$$+\mathcal{C}_{0131}G'-\mu_0 H_3 M_2'=0, \tag{107}$$

$$-\frac{k^2}{r}\left(\mathcal{C}_{0131}-\mu_0 H_3\right)F+\frac{\mathcal{C}_{0131}}{r^2}F'-\frac{\mathcal{C}_{0131}}{r}F''+\mathsf{K}_{011}G'+\mu_0 M_2'=0, \tag{108}$$

$$G+\frac{H_3}{r}F'-M_2=0, \tag{109}$$

$$-k^2\left(-\frac{\xi_2}{r^2}+\frac{\xi_1}{r}\right)F+\left\{\frac{\xi_3}{r^2}-2\frac{A_6}{r^3}-\frac{k^2}{r}\left(\xi_2-\xi_4\right)+\frac{\rho c^2 k^2}{r}\right\}F'$$
$$+\left(-\frac{\xi_3}{r}+2\frac{A_6}{r^2}\right)F''-\frac{A_6}{r}F'''-k^2A_{14}G+\xi_5 G'+A_{12}G''-k^2\mu_0 H_3 M_2=0, \tag{110}$$

at $r=b$.

Let the governing equations be written in the form

$$p_1F+p_2F'+p_3F''+p_4F'''+p_5F''''+p_6G+p_7G'+p_8G''+p_9G'''=0, \tag{111}$$
$$q_1F+q_2F'+q_3F''+q_4F'''+q_5G+q_6G'+q_7G''=0, \tag{112}$$

p's and q's being the coefficients in (100) and (101), and let

$$y_1=F,\quad y_2=F',\quad y_3=F'',\quad y_4=F''',\quad y_5=G,\quad y_6=G',\quad y_7=G'', \tag{113}$$

then the above equations can be written as a system of first order ODEs of the form

$$\Pi\mathbf{y}'=\mathbf{g}. \tag{114}$$

Here $\Pi,\mathbf{y}'$, and $\mathbf{g}$ are matrices of size 7×7, 7×1 and 7×1, respectively, and are given by

$$\Pi=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&p_5&0&0&p_9\\0&0&q_4&0&0&q_7&0\end{bmatrix},\quad \mathbf{y}'=\begin{Bmatrix}y_1'\\y_2'\\y_3'\\y_4'\\y_5'\\y_6'\\y_7'\end{Bmatrix}, \tag{115}$$

$$\mathbf{g}=\left\{\begin{array}{c}y_2\\y_3\\y_4\\y_6\\y_7\\-p_1y_1-p_2y_2-p_3y_3-p_4y_4-p_6y_5-p_7y_6-p_8y_7\\-q_1y_1-q_2y_2-q_3y_3-q_5y_5-q_6y_6\end{array}\right]. \tag{116}$$

Here we have eight boundary conditions but have to solve for nine variables, viz. $y_1,...,y_7,M_1$, and M_2. Hence we have infinitely many solutions to this problem and a unique solution is only possible when $H_3=0$. Vanishing of the underlying magnetic field would cause the increments in magnetic field to be identically zero ($G=M_1=M_2\equiv0$) and only the increments in mechanical displacement F remain. Such purely elastic waves have already been studied in papers, such as those by, Haughton [52, 53], and Wang and Ertepinar [49].

6.2 Displacement in the azimuthal direction

Now considering the set of equations that contain only u_2 and $\dot{H}_{l02}$, the governing equations (71) and (75) are written in component form as

$$-\left(\frac{\mathcal{A}_{01212,1}-\tau_{11,1}}{r}+\frac{\mathcal{A}_{02121}}{r^2}\right)u_2+\left(\mathcal{A}_{01212,1}+\frac{\mathcal{A}_{01212}}{r}\right)u_{2,1}+\mathcal{A}_{01212}u_{2,11} +\mathcal{A}_{03232}u_{2,33}=\rho u_{2,tt}, \tag{117}$$

$$\dot{H}_{l02,3}=0,\quad \dot{H}_{l02,1}+\frac{\dot{H}_{l02}}{r}=0, \tag{118}$$

in $a<r<b$, along with $(77)_{3,4}$ in vacuum. The boundary conditions $(64)_2$ and (66) give

$$\mathcal{A}_{01212}u_{2,1}-\left(\mathcal{A}_{01221}+p-\frac{\mu_0H_3^2}{2}\right)\frac{u_2}{r}=0, \tag{119}$$

$$\dot{H}_{l02}-\dot{H}_2^*=0, \tag{120}$$

at $r=a$ and $r=b$.

Due to $(71)_1$, the governing equations for u_2 and $\dot{H}_{l02}$ are decoupled. So, Equation (117) is of the form what one would normally obtain for a purely mechanical problem except that the coefficients still depend on H_3. The governing equations for $\dot{H}_{l02}$ and $\dot{H}_2^*$ can be integrated analytically to give $\dot{H}_{l02}=c_1/r$ in $a<r<b$, $\dot{H}_2^*=c_2/r$ in $r<a$, and $\dot{H}_2^*=c_3/r$ in $r>b$. The boundary conditions (120) at $r=a,b$ require that $c_1=c_2=c_3$.

For the mechanical displacement, if we consider propagating wave type solution of the form

$$u_2=F(r)\exp\left(\mathrm{i}kz-\mathrm{i}\omega t\right), \tag{121}$$

the governing equations and boundary conditions are transformed to

$$\left(\frac{-\mathcal{A}_{01212,1}+\tau_{11,1}}{r}-\frac{\mathcal{A}_{02121}}{r^2}-k^2\mathcal{A}_{03232}+\rho\omega^2\right)F+\left(\mathcal{A}_{01212,1}+\frac{\mathcal{A}_{01212}}{r}\right)F' +\mathcal{A}_{01212}F''=0, \tag{122}$$

for $a<r<b$, and

$$\mathcal{A}_{01212}F'-\left(\mathcal{A}_{01212}-\tau_{11}-\frac{\mu_0H_3^2}{2}\right)\frac{F}{r}=0, \tag{123}$$

at $r=a,b$.

The above set of equations can be non-dimensionalised by defining

$$\zeta=\frac{\rho\omega^2}{k^2\mu},\quad \hat{r}=\frac{r}{A},\quad \hat{k}=Ak,\quad \hat{F}(\hat{r})=\frac{F(r)}{A}\quad \hat{\boldsymbol{\mathcal{A}}}=\frac{\boldsymbol{\mathcal{A}}}{\mu},\quad \hat{\boldsymbol{\tau}}=\frac{\boldsymbol{\tau}}{\mu}, \tag{124}$$

and are rewritten as

$$\left\{\frac{1}{\hat{r}\hat{k}^2}\left(\hat{\tau}_{11}'-\hat{\mathcal{A}}_{01212}'\right)-\frac{\hat{\mathcal{A}}_{02121}}{\hat{r}^2\hat{k}^2}-\hat{\mathcal{A}}_{03232}+\zeta\right\}\hat{F}+\left(\hat{\mathcal{A}}_{01212}'+\frac{\hat{\mathcal{A}}_{01212}}{\hat{r}}\right)\frac{\hat{F}'}{\hat{k}^2} +\frac{\hat{\mathcal{A}}_{01212}}{\hat{k}^2}\hat{F}''=0, \tag{125}$$

for $\hat{a}<\hat{r}<\hat{b}$, and

$$\hat{\mathcal{A}}_{01212}\hat{F}'-\left(\hat{\mathcal{A}}_{01212}-\hat{\tau}_{11}-\frac{\mu_0H_3^2}{2\mu}\right)\frac{\hat{F}}{\hat{r}}=0, \tag{126}$$

at $\hat{r}=\hat{a},\hat{b}$.

6.2.1 Numerical results

The above equations are converted to a system of two first order ODEs and solved numerically using the algorithm described in Section 7.1.2. Variation of non-dimensionalised squared wave speed ($\zeta=\frac{\rho\omega^2}{k^2\mu}$ from Equation (124)) with various deformation parameters is illustrated in the following plots. The tube becomes unstable (onset of non-homogeneity of magnetoelastic field variables in the axial direction) when $\zeta\to0$ as discussed for the purely mechanical case by Haughton and Ogden [50]. We observe existence of more than one mode of wave propagation due to presence of a finite length scale $(B-A)$ in the problem. These are illustrated in Figure 5 for the Mooney–Rivlin type magnetoelastic material of Equation (52) and the material parameters listed in Table 2. The first mode for small magnetic fields (Figure 5(a), $H_3=800$ A/m) has qualitatively very different dispersion curve in comparison to all other cases. The cylindrical tube becomes unstable for very small wave numbers (for the given loading conditions, $\hat{k}\approx0.5$ in Figure 5(a)) and upon

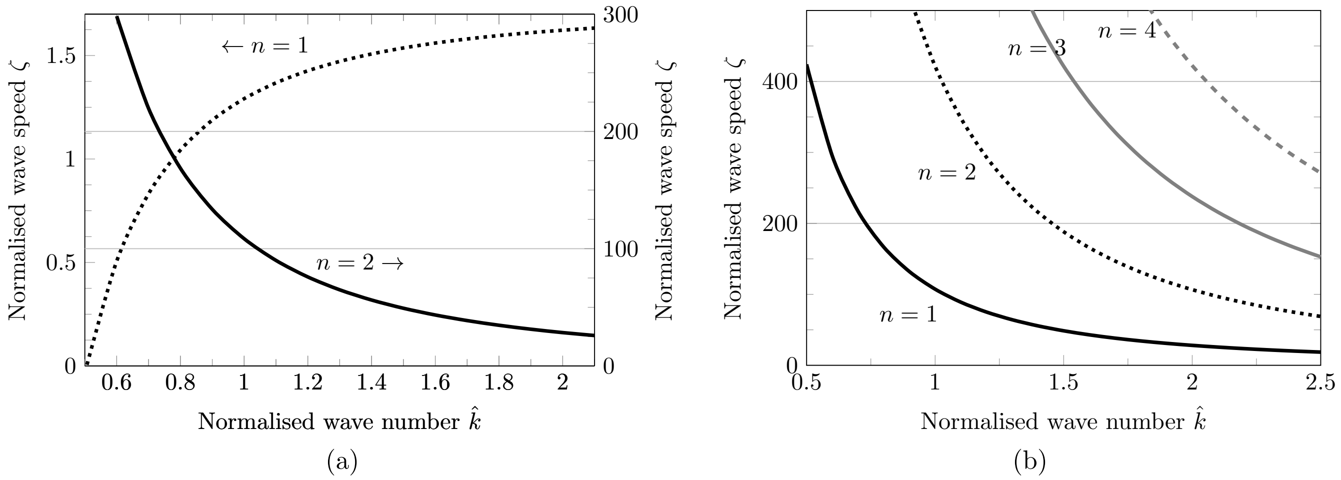


Figure 5: First few modes of wave propagation for $\lambda_z = 1.5, P_{\text{in}} = 0.1\mu, B/A = 1.3$. (a) $H_3 = 800$ A/m, (b) $H_3 = 1 \times 10^4$ A/m. Note that here n corresponds to the mode of wave propagation and is different from the unit outward normal $\mathbf{n}$ as discussed in Equation (4).

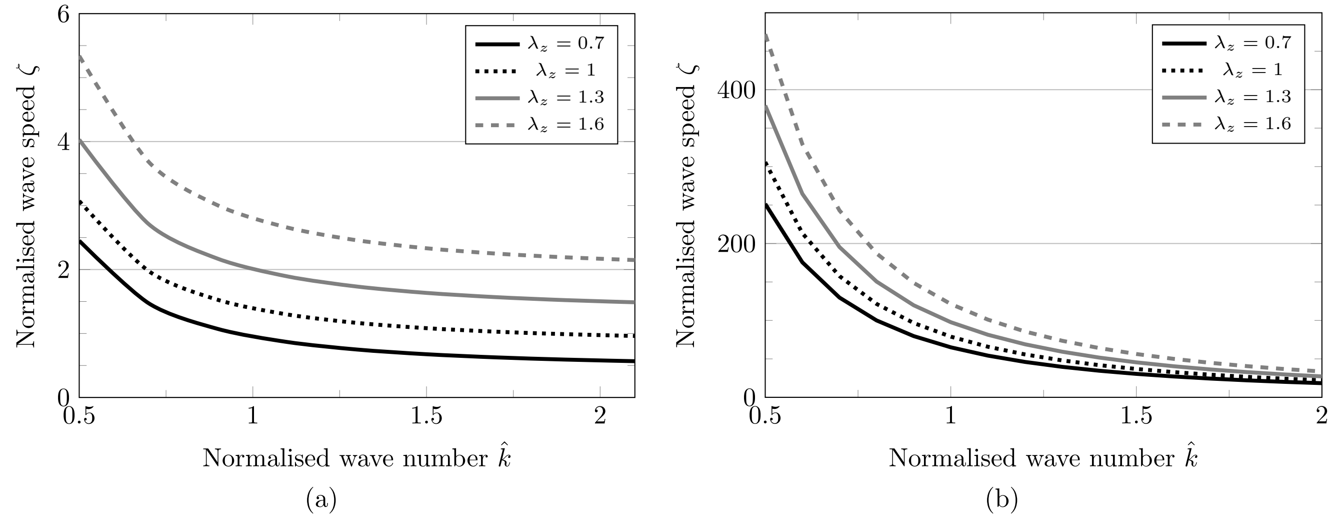


Figure 6: Variation of wave speed with wave number for different values of the axial stretch λ_z. $P_{\text{in}} = 0.1\mu, B/A = 1.3$. (a) $H_3 = 0$, (b) $H_3 = 5 \times 10^5$ A/m.

increasing the value of magnetic field (Figure 8). The wave speed increases from zero to reach an asymptotic value with an increasing wave number. In general, for all other modes and conditions, wave speeds decrease with an increase in the wave number. Increasing the mode number results in a significantly higher wave speed in comparison to the first mode. For higher magnetic field (Figure 5(b)), modes corresponding to very low speeds do not exist as is also shown later in Figure 9.

Dispersion relations for different values of the underlying axial stretch are plotted in Figures 6(a,b). It is seen that stretching the tube in the axial direction, results in an increase in the speed of wave propagation. For the case of large magnetic field as shown in Figure 6(b), lower modes cease to exist and we obtain higher modes with larger wave velocities, although with the similar shape of the dispersion curve. This is in accordance with the observations in Figure 9.

Effect of tube thickness on the wave speed and the dispersion relations is demonstrated in Figure 7. An increase in the tube thickness results in a significant decrease in the corresponding wave speed. The changes in wave speed due to wall thickness are very significant in the range of low wave numbers. For higher wave numbers, the effect of wall thickness is diminished.

Next, we consider the effect of an axial magnetic field on the wave speed. As observed from Figure 8, for very small magnetic fields ($H_3 = 100$ A/m), the dispersion curve follows a decreasing wave speed with an increasing wave number. As the underlying magnetic field is increased further ($H_3 > 600$ A/m), the shape of the dispersion curve is inverted and the wave speeds tend to increase with an increasing wave number eventually reaching an asymptotic value. For very small wave numbers, the given configuration tends to become unstable for the first mode in the case of moderate to high axial magnetic fields ($H_3 \geq 800$ A/m). Essentially this means that upon an increased magnetic loading, waves with longer wavelength cannot propagate in the tube.

It is seen that for low and moderate magnetic fields ($H_3 < 1 \times 10^4$ A/m), the wave speed tends to decrease with an increase in the magnetic field. As shown in Figure 9(a), wave speed eventually reaches a value of zero that corresponds to

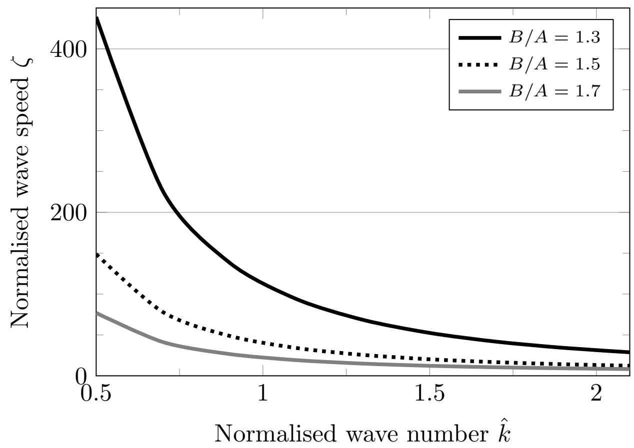


Figure 7: Variation of wave speed with wave number for different values of the tube thickness B/A. $P_{\text{in}} = 0.1\mu, \lambda_z = 1.5, H_3 = 5 \times 10^5$ A/m.

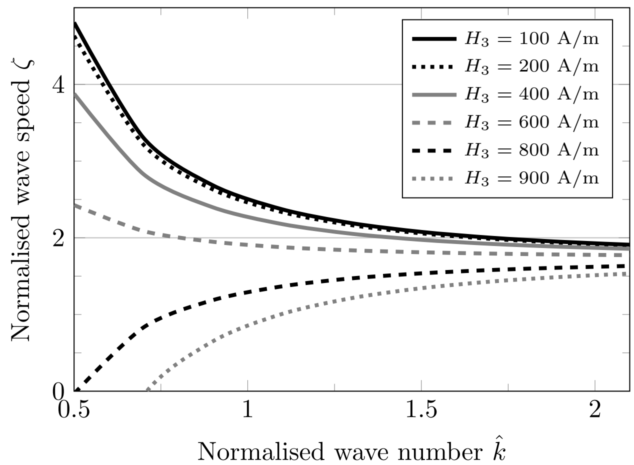
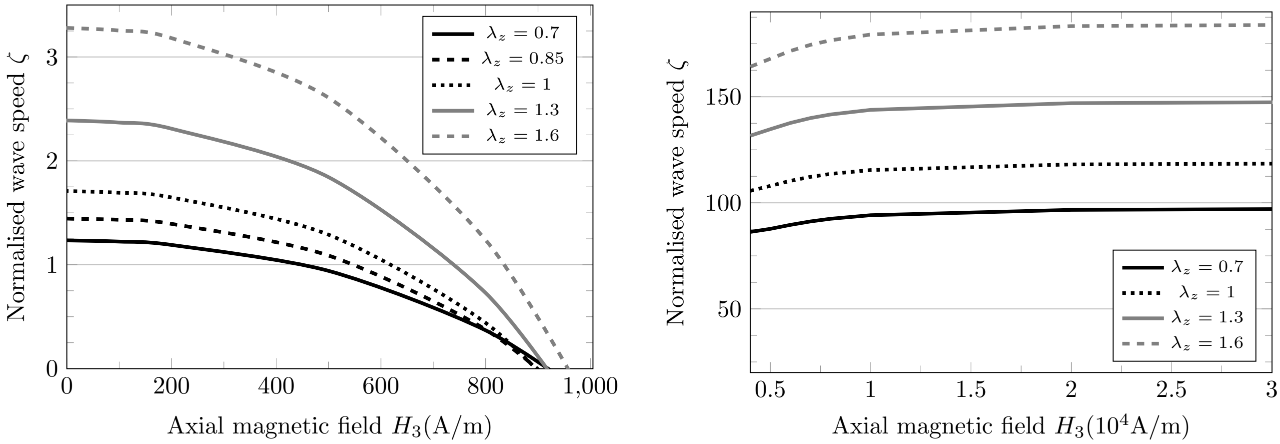


Figure 8: Variation of wave speed with wave number for different relatively small values of the underlying magnetic field H_3. $P_{\text{in}} = 0.1\mu, \lambda_z = 1.5, B/A = 1.3$.



(a) (b)

Figure 9: Variation of wave speed with the axial magnetic field for different values of the axial stretch λ_z. $P_{\text{in}} = 0.1\mu$, $\hat{k} = 0.8$, $B/A = 1.3$.

onset of instability. Note the nonlinear coupling between the critical magnetic field and the axial stretch λ_z at the point of instability ($\zeta \to 0$). Wave speed consistently decrease with the decrease in λ_z except for the case of large magnetic fields close to the point of instability, at which point the behaviour is uncertain. When the magnetic field is increased even further (Figure 9(b)), no waves are observed from the first mode and we have waves only for higher modes. The wave speeds are almost two orders of magnitude higher in this case and the speed increases with a rise in magnetic field eventually reaching an asymptotic value.

7 Azimuthal magnetic field $\mathbf{H} = (0, H_2, 0)$

We now consider an initial magnetic field in the azimuthal direction. Such a field can be generated by a long current carrying wire placed on the axis of the hollow tube so that H_2 has dependence only on r. For this specialisation, the Maxwell stress and its increment are given in the component form by

$$[\boldsymbol{\tau}^*] = \frac{B_2^{*2}}{2\mu_0}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad [\dot{\boldsymbol{\tau}}^*] = \frac{B_2^*}{\mu_0}\begin{bmatrix} -\dot{B}_2^* & \dot{B}_1^* & 0 \\ \dot{B}_1^* & \dot{B}_2^* & \dot{B}_3^* \\ 0 & \dot{B}_3^* & -\dot{B}_2^* \end{bmatrix}. \tag{127}$$

To work with the governing equations (29) and (28) in the presence of an azimuthal magnetic field, the non-zero components of the magnetoelastic tensors are $\mathcal{A}_{0iiii}$, $\mathcal{A}_{0iijj}$, $\mathcal{A}_{0ijij}$, $\mathcal{A}_{0ijji}$, $\mathcal{C}_{0ii2}$, $\mathcal{C}_{0i2i}$, K_{0ii} for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Explicit formulas for these components for the Mooney–Rivlin magnetoelastic material are given in the Appendix.

Expanding the incremental governing equations $(10)_3$, (28), and (29) in component form, we obtain (71), (72), and

$$\frac{1}{r}\left[r\left\{\mathcal{C}_{0121}\left(u_{2,1} - \frac{u_2}{r}\right) + \mathsf{K}_{011}\dot{H}_{l01}\right\}\right]_{,1} + \left(\mathcal{C}_{0323}u_{2,3} + \mathsf{K}_{033}\dot{H}_{l03}\right)_{,3} = 0, \tag{128}$$

$$\begin{aligned}\frac{1}{r}\Bigg[\Big\{r\Big((\mathcal{A}_{01111} + p)u_{1,1} + \mathcal{A}_{01122}\frac{u_1}{r} + \mathcal{A}_{01133}u_{3,3} + \mathcal{C}_{0112}\dot{H}_{l02}\Big)\Big\}_{,1} \\ - \Big\{(\mathcal{A}_{02222} + p)\frac{u_1}{r} + \mathcal{A}_{01122}u_{1,1} + \mathcal{A}_{02233}u_{3,3} + \mathcal{C}_{0222}\dot{H}_{l02}\Big\}\Bigg] \\ + \{\mathcal{A}_{03131}u_{1,3} + (\mathcal{A}_{01331} + p)u_{3,1}\}_{,3} - \dot{p}_{,1} = \rho u_{1,tt},\end{aligned} \tag{129}$$

$$\begin{aligned}\frac{1}{r}\Bigg[\Big\{r\Big(\mathcal{A}_{01212}u_{2,1} - (\mathcal{A}_{01221} + p)\frac{u_2}{r} + \mathcal{C}_{0121}\dot{H}_{l01}\Big)\Big\}_{,1} + (\mathcal{A}_{01221} + p)\,u_{2,1} \\ -\mathcal{A}_{02121}\frac{u_2}{r} + \mathcal{C}_{0121}\dot{H}_{l01}\Bigg] + \Big(\mathcal{A}_{03232}u_{2,3} + \mathcal{C}_{0323}\dot{H}_{l03}\Big)_{,3} = \rho u_{2,tt},\end{aligned} \tag{130}$$

$$\begin{aligned}\Big(\mathcal{A}_{01133}u_{1,1} + \mathcal{A}_{02233}\frac{u_1}{r} + (\mathcal{A}_{03333} + p)u_{3,3} + \mathcal{C}_{0332}\dot{H}_{l02}\Big)_{,3} \\ \frac{1}{r}\left[r\left\{\mathcal{A}_{01313}u_{3,1} + (\mathcal{A}_{01331} + p)\,u_{1,3}\right\}\right]_{,1} - \dot{p}_{,3} = \rho u_{3,tt},\end{aligned} \tag{131}$$

in the material along with the equations (77) in vacuum and the constraint of incompressibility (63). We can eliminate $\dot{p}$ from equations (129) and (131) and use $(71)_1$ to get

$$
\begin{aligned}
A_1 u_{1,3} + A_2 u_{1,13} + A_3 u_{1,113} + A_4 u_{3,1} + A_5 u_{3,11} + A_6 u_{3,111} \\
+ A_7 u_{3,133} + A_8 u_{3,33} + A_9 u_{1,333} = \rho (u_{1,3} - u_{3,1})_{,tt},
\end{aligned} \tag{132}
$$

where we have assumed that the magnetoelastic moduli tensors are uniform along the axial z direction and $A_1, ..., A_9$ are defined in (80).

From the above governing equations, we observe that $u_2, \dot{H}_{l01}$, and $\dot{H}_{l03}$ are related to each other and independent of u_1, u_3, and $\dot{H}_{l02}$. Hence we analyse these two cases separately.

7.1 Displacement in the azimuthal direction

We now consider the set of equations with $u_2, \dot{H}_{l01}, \dot{H}_{l03}, \dot{H}_1^*$, and $\dot{H}_3^*$. Since $\dot{H}_{l01}$ and $\dot{H}_{l03}$ satisfy Equation (72) while $\dot{H}_1^*$ and $\dot{H}_3^*$ satisfy equation $(77)_2$, we can define the potential functions ψ and ψ^* that satisfy equations $(88)_{3,4,5,6}$. On substituting them in to the governing equations (128) and (130), we obtain

$$
\begin{aligned}
\mathcal{C}_{0121,1}\left(u_{2,1} - \frac{u_2}{r}\right) + \mathcal{C}_{0121} u_{2,11} + \mathcal{C}_{0323} u_{2,33} + \left(\frac{\mathsf{K}_{011}}{r} + \mathsf{K}_{011,1}\right)\psi_{,1} \\
+ \mathsf{K}_{011}\psi_{,11} + \mathsf{K}_{033}\psi_{,33} = 0,
\end{aligned} \tag{133}
$$

$$
\begin{aligned}
-\left(\frac{\mathcal{A}_{02121}}{r^2} + \frac{(\mathcal{A}_{01221} + p)_{,1}}{r}\right) u_2 + \left(\frac{\mathcal{A}_{01212}}{r} + \mathcal{A}_{01212,1}\right) u_{2,1} + \mathcal{A}_{01212} u_{2,11} \\
+ \mathcal{A}_{03232} u_{2,33} + \left(2\frac{\mathcal{C}_{0121}}{r} + \mathcal{C}_{0121,1}\right)\psi_{,1} + \mathcal{C}_{0121}\psi_{,11} + \mathcal{C}_{0323}\psi_{,33} = \rho u_{2,tt},
\end{aligned} \tag{134}
$$

for $a < r < b$ along with equations (91) in vacuum. Boundary conditions are given by the equations $(64)_2$, (65), and (67) as

$$
-\left(\mathcal{A}_{01221} + p + \frac{\mu_0 H_2^2}{2}\right)\frac{u_2}{r} + \mathcal{A}_{01212} u_{2,1} + \mathcal{C}_{0121}\psi_{,1} - \mu_0 H_2^* \psi^*_{,1} = 0, \tag{135}
$$

$$
(\mathcal{C}_{0121} - \mu_0 H_2)\frac{u_2}{r} - \mathcal{C}_{0121} u_{2,1} - \mathsf{K}_{011}\psi_{,1} - \mu_0 \psi^*_{,1} = 0, \tag{136}
$$

$$
\psi_{,3} - \psi^*_{,3} - H_2 u_{2,3} = 0, \tag{137}
$$

at $r = a$ and $r = b$.

7.1.1 Wave propagation solutions

Using separation of variables we assume solutions of the form

$$
u_2 = F(r)\exp(\mathrm{i}kz - \mathrm{i}\omega t) \quad \text{for} \quad a < r < b, \tag{138}
$$
$$
\psi = G(r)\exp(\mathrm{i}kz - \mathrm{i}\omega t) \quad \text{for} \quad a < r < b, \tag{139}
$$
$$
\psi^* = M_1(r)\exp(\mathrm{i}kz - \mathrm{i}\omega t) \quad \text{for} \quad r < a, \tag{140}
$$
$$
\psi^* = M_2(r)\exp(\mathrm{i}kz - \mathrm{i}\omega t) \quad \text{for} \quad r > b, \tag{141}
$$

with $\mathrm{i} = \sqrt{-1}$, k being the wave number, and ω being the frequency.

On substituting these solutions in the governing equations we obtain

$$
\begin{aligned}
-\left(\frac{\mathcal{C}_{0121,1}}{r} + k^2\mathcal{C}_{0323}\right) F + \mathcal{C}_{0121,1} F' + \mathcal{C}_{0121} F'' - k^2 \mathsf{K}_{033} G \\
+ \left(\mathsf{K}_{011,1} + \frac{\mathsf{K}_{011}}{r}\right) G' + \mathsf{K}_{011} G'' = 0,
\end{aligned} \tag{142}
$$

$$
\begin{aligned}
\left(-\frac{\mathcal{A}_{02121}}{k^2 r^2} - \frac{(\mathcal{A}_{01221} + p)_{,1}}{k^2 r} - \mathcal{A}_{03232} + \rho v^2\right) F + \frac{1}{k^2}\left(\frac{\mathcal{A}_{01212}}{r} + \mathcal{A}_{01212,1}\right) F' \\
+ \frac{1}{k^2}\mathcal{A}_{01212} F'' - \mathcal{C}_{0323} G + \frac{1}{k^2}\left(2\frac{\mathcal{C}_{0121}}{r} + \frac{\mathcal{C}_{0121,1}}{k^2}\right) G' + \mathcal{C}_{0121} G'' = 0,
\end{aligned} \tag{143}
$$

for $a < r < b$, and

$$
M_1'' + \frac{M_1'}{r} - k^2 M_1 = 0 \quad \text{for} \;\; r < a, \qquad M_2'' + \frac{M_2'}{r} - k^2 M_2 = 0 \quad \text{for} \;\; r > b, \tag{144}
$$

where we have taken a prime to denote a derivative with respect to r and $v = \omega/k$ is the wave speed. The boundary conditions are

$$-\left(\mathcal{A}_{01221} + p + \frac{\mu_0 H_2^2}{2}\right)\frac{F}{r} + \mathcal{A}_{01212}F' + \mathcal{C}_{0121}G' - \mu_0 H_2^* M' = 0, \tag{145}$$

$$(\mathcal{C}_{0121} - \mu_0 H_2)\frac{F}{r} - \mathcal{C}_{0121}F' - \mathsf{K}_{011}G' - \mu_0 M' = 0, \tag{146}$$

$$G - H_2 F - M_1 = 0, \tag{147}$$

at $r = a$, and

$$-\left(\mathcal{A}_{01221} + p + \frac{\mu_0 H_2^2}{2}\right)\frac{F}{r} + \mathcal{A}_{01212}F' + \mathcal{C}_{0121}G' - \mu_0 H_2^* M' = 0, \tag{148}$$

$$(\mathcal{C}_{0121} - \mu_0 H_2)\frac{F}{r} - \mathcal{C}_{0121}F' - \mathsf{K}_{011}G' - \mu_0 M' = 0, \tag{149}$$

$$G - H_2 F - M_2 = 0, \tag{150}$$

at $r = b$.

To obtain numerical solutions, we non-dimensionalise the above governing equations and boundary conditions. For this purpose we define $H_{2a} = H_2|_{r=a}$ and define the following non-dimensional quantities (with a superposed hat) in addition to those in Equation (124)

$$\hat{\boldsymbol{\mathcal{C}}} = \frac{\boldsymbol{\mathcal{C}}}{H_{2a}\mu_0}, \quad \hat{\mathsf{K}} = \frac{\mathsf{K}}{\mu_0}, \quad \hat{G}(\hat{r}) = \frac{G(r)}{H_{2a}A}, \quad \hat{M}(\hat{r}) = \frac{M(r)}{H_{2a}A},$$
$$\hat{M}_1(\hat{r}) = \frac{M_1(r)}{H_{2a}A}, \quad \hat{M}_2(\hat{r}) = \frac{M_2(r)}{H_{2a}A}. \tag{151}$$

On non-dimensionalisation, the governing equations become

$$-\left(\frac{\hat{\mathcal{C}}'_{0121}}{\hat{r}} + \hat{k}^2\hat{\mathcal{C}}_{0323}\right)\hat{F} + \hat{\mathcal{C}}'_{0121}\hat{F}' + \hat{\mathcal{C}}_{0121}\hat{F}'' - \hat{k}^2\hat{\mathsf{K}}_{033}\hat{G}$$
$$+\left(\hat{\mathsf{K}}'_{011} + \frac{\hat{\mathsf{K}}_{011}}{\hat{r}}\right)\hat{G}' + \hat{\mathsf{K}}_{011}\hat{G}'' = 0, \tag{152}$$

$$-\left(\frac{\hat{\mathcal{A}}_{02121}}{\hat{k}^2\hat{r}^2} + \frac{\hat{\mathcal{A}}'_{01221} + \hat{p}'}{\hat{k}^2\hat{r}} + \hat{\mathcal{A}}_{03232} - \zeta\right)\hat{F} + \frac{1}{\hat{k}^2}\left(\frac{\hat{\mathcal{A}}_{01212}}{\hat{r}} + \hat{\mathcal{A}}'_{01212}\right)\hat{F}'$$
$$+\frac{\hat{\mathcal{A}}_{01212}}{\hat{k}^2}\hat{F}'' + \frac{\mu_0 H_{2b}^2}{\hat{k}^2\mu}\left\{-\hat{k}^2\hat{\mathcal{C}}_{0323}\hat{G} + \left(2\frac{\hat{\mathcal{C}}_{0121}}{\hat{r}} + \hat{\mathcal{C}}'_{0121}\right)\hat{G}' + \hat{\mathcal{C}}_{0121}\hat{G}''\right\} = 0, \tag{153}$$

for $\hat{a} \leq \hat{r} \leq \hat{b}$ and

$$\hat{M}_1'' + \frac{\hat{M}_1'}{\hat{r}} - \hat{k}^2\hat{M}_1 = 0 \quad \text{for}\, \hat{r} < \hat{a}, \qquad \hat{M}_2'' + \frac{\hat{M}_2'}{\hat{r}} - \hat{k}^2\hat{M}_2 = 0 \quad \text{for}\, \hat{r} > \hat{b}, \tag{154}$$

where a prime now denotes a derivative with respect to $\hat{r}$. The boundary conditions are

$$-\left(\hat{\mathcal{A}}_{01221} + \hat{p} + \frac{\mu_0 H_2^2}{2\mu}\right)\frac{\hat{F}}{\hat{r}} + \hat{\mathcal{A}}_{01212}\hat{F}' + \frac{\mu_0 H_{2a}^2}{\mu}\left(\hat{\mathcal{C}}_{0121}\hat{G}' - \hat{M}_1'\right) = 0, \tag{155}$$

$$\left(\hat{\mathcal{C}}_{0121} - 1\right)\frac{\hat{F}}{\hat{r}} - \hat{\mathcal{C}}_{0121}\hat{F}' - \hat{\mathsf{K}}_{011}\hat{G}' - \hat{M}_1' = 0, \tag{156}$$

$$\hat{G} - \hat{F} - \hat{M}_1 = 0, \tag{157}$$

at $\hat{r} = \hat{a}$, and

$$-\left(\hat{\mathcal{A}}_{01221} + \hat{p} + \frac{\mu_0 H_2^2}{2\mu}\right)\frac{\hat{F}}{\hat{r}} + \hat{\mathcal{A}}_{01212}\hat{F}' + \frac{\mu_0 H_{2a}^2}{\mu}\left(\hat{\mathcal{C}}_{0121}\hat{G}' - \frac{H_2}{H_{2a}}\hat{M}_2'\right) = 0, \tag{158}$$

$$\left(\hat{\mathcal{C}}_{0121} - \frac{H_2}{H_{2a}}\right)\frac{\hat{F}}{\hat{r}} - \hat{\mathcal{C}}_{0121}\hat{F}' - \hat{\mathsf{K}}_{011}\hat{G}' - \hat{M}_2' = 0, \tag{159}$$

$$\hat{G} - \frac{H_2}{H_{2a}}\hat{F} - \hat{M}_2 = 0, \tag{160}$$

at $\hat{r} = \hat{b}$.

7.1.2 Numerical solution procedure

Equations (154) are modified Bessel's equations and the solution not diverging at $\hat{r}=0$ and $\hat{r}=\infty$ are $\hat{M}_1 = e_5 J_0(\mathrm{i}\hat{r}/\hat{k})$, $\hat{M}_2 = e_6 J_0(\mathrm{i}\hat{r}/\hat{k})$, where J_0 is the Bessel's function of first kind and order zero, and e_5 and e_6 are scaling parameters. To obtain a numerical solution of the system of coupled ODEs, we convert them into a system of first order ODEs by defining

$$y_1 = \hat{F},\;\; y_2 = \hat{F}',\;\; y_3 = \hat{G},\;\; y_4 = \hat{G}'. \tag{161}$$

Let the ODEs be then given by

$$p_1 y_1 + p_2 y_2 + p_3 y_2' + p_4 y_3 + p_5 y_4 + p_6 y_4' = 0, \tag{162}$$
$$q_1 y_1 + q_2 y_2 + q_3 y_2' + q_4 y_3 + q_5 y_4 + q_6 y_4' = 0, \tag{163}$$

where p_is and q_is $(i = 1, ..., 6)$ correspond to the coefficients in the equations (152) and (153) respectively.

Hence, we obtain the following system of first order ODEs

$$\Pi \mathbf{y}' = \mathbf{g}, \tag{164}$$

to be solved for $\hat{a} < \hat{r} < \hat{b}$ where the matrices $\Pi, \mathbf{y}'$, and $\mathbf{g}$ are given by

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & p_3 & 0 & p_6 \\ 0 & q_3 & 0 & q_6 \end{pmatrix}, \quad \mathbf{y}' = \begin{Bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{Bmatrix}, \tag{165}$$

$$\mathbf{g} = \begin{Bmatrix} y_2 \\ y_4 \\ -p_1 y_1 - p_2 y_2 - p_4 y_3 - p_5 y_4 \\ -q_1 y_1 - q_2 y_2 - q_4 y_3 - q_5 y_4 \end{Bmatrix}. \tag{166}$$

Given the internal pressure, the underlying magnetic field, and the axial stretch, we first evaluate a using Equation (42) and then consider the initial value problem defined by

$$y_i(a) = \delta_{ik}, \quad i = 1, ..., 4 \tag{167}$$

for each of $k = 1, ..., 4$, δ_{ik} being the Kronecker delta. Subject to these initial conditions, we solve the differential equation described by Equation (164) using the '`ode15s`' solver in Matlab R2015b. The four solutions thus generated are denoted by $\mathbf{y}^k$ $(k = 1, ..., 4)$ and a general solution to the problem is expressed as

$$\mathbf{y} = \sum_{k=1}^{4} e_k \mathbf{y}^k, \quad \hat{M}_1 = e_5 J_0(\mathrm{i}\hat{r}/\hat{k}), \quad \hat{M}_2 = e_6 J_0(\mathrm{i}\hat{r}/\hat{k}), \tag{168}$$

where e_k $(k = 1, ..., 6)$ are constants. For the solutions to exist, there should be a set of non-trivial constants $\{e_k\}_{k=1}^{6}$ such that the general solution (168) satisfies the following boundary conditions.

$$-\left(\hat{\mathcal{A}}_{01221} + \hat{p} + \frac{\mu_0 H_2^2}{2\mu}\right)\frac{y_1}{\hat{r}} + \hat{\mathcal{A}}_{01212} y_2 + \frac{\mu_0 H_{2a}^2}{\mu}\left(\hat{\mathcal{C}}_{0121} y_4 - \hat{M}_1'\right) = 0, \tag{169}$$

$$\left(\hat{\mathcal{C}}_{0121} - 1\right)\frac{y_1}{\hat{r}} - \hat{\mathcal{C}}_{0121} y_2 - \hat{\mathsf{K}}_{011} y_4 - \hat{M}_1' = 0, \tag{170}$$

$$y_3 - y_1 - \hat{M}_1 = 0, \tag{171}$$

at $\hat{r} = \hat{a}$, and

$$-\left(\hat{\mathcal{A}}_{01221} + \hat{p} + \frac{\mu_0 H_2^2}{2\mu}\right)\frac{y_1}{\hat{r}} + \hat{\mathcal{A}}_{01212} y_2 + \frac{\mu_0 H_{2a}^2}{\mu}\left(\hat{\mathcal{C}}_{0121} y_4 - \frac{H_2}{H_{2a}}\hat{M}_2'\right) = 0, \tag{172}$$

$$\left(\hat{\mathcal{C}}_{0121} - \frac{H_2}{H_{2a}}\right)\frac{y_1}{\hat{r}} - \hat{\mathcal{C}}_{0121} y_2 - \hat{\mathsf{K}}_{011} y_4 - \hat{M}_2' = 0, \tag{173}$$

$$y_3 - \frac{H_2}{H_{2a}} y_1 - \hat{M}_2 = 0, \tag{174}$$

at $\hat{r} = \hat{b}$.

This results in six linear equations in e_k that can be written in the form $Z_{jk} e_k = 0$, Z_{jk} being a 6×6 matrix. For a non-trivial solution to exist, we require the condition $\det(Z_{jk}) = 0$, which gives the relationship between ζ and other parameters. This solution process is similar to the numerical routine described by Haughton and Ogden [50].

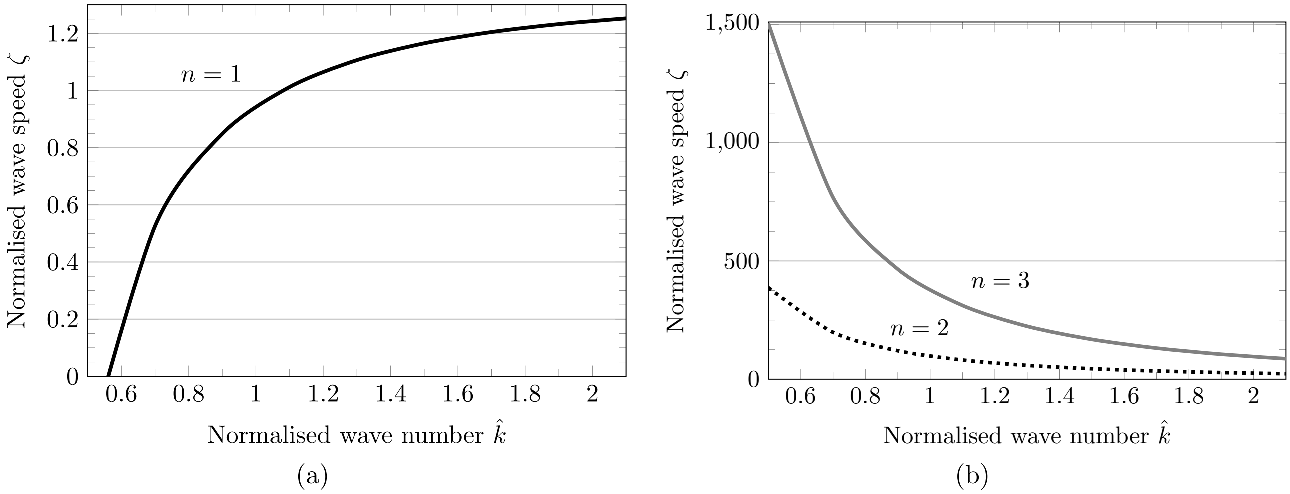


Figure 10: First three modes of wave propagation for $\lambda_z = 1.3, P_{\text{in}} = 0.1\mu$, $B/A = 1.3$, $H_0 = 800$ A/m

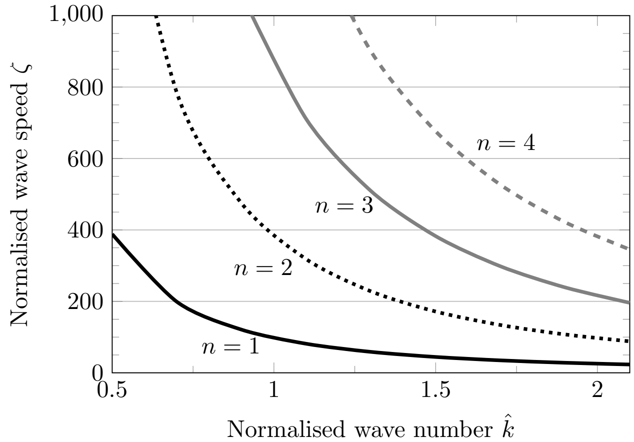


Figure 11: First four modes of wave propagation for $\lambda_z = 1.3, P_{\text{in}} = 0.1\mu$, $B/A = 1.3, H_0 = 1 \times 10^4$ A/m.

7.1.3 Numerical results

We solve the above equations numerically for the Mooney–Rivlin type magnetoelastic material defined in Equation (52) and plot the non-dimensionalised squared wave speed ($\zeta = \frac{\rho\omega^2}{k^2\mu}$ as discussed in Equation (124)). Numerical values of the ratio of radii (B/A), the internal pressure P_{in}, the axial stretch λ_z and the magnetic field H_0 are specified along with the calculations in each graph. Numerical values of the material parameters are taken from the Table 2.

Multiple modes of wave propagation are obtained as shown in Figures 10 and 11, a case similar to the previous section because of the existence of a finite length scale. The first mode for small magnetic fields (Figure 10(a), $H_0 = 800$ A/m) has qualitatively very different dispersion curve in comparison to all other cases. The configuration seems to be unstable ($\zeta < 0$) for very small wave numbers and the wave speed increases from zero to reach an asymptotic value with an increasing wave number. For all other modes as well as for higher values of magnetic field ($H_0 = 1 \times 10^4$ A/m), the wave speed decreases with an increase in the wave number. For a given set of conditions, wave of a higher mode propagate with a higher speed. In the figures to follow, we plot only the first modes. When the first or other lower modes cease to exist, we proceed to plot the next lowest mode.

Effect of the underlying magnetic field H_0 and the underlying axial stretch on the speed of wave propagation is demonstrated through the Figures 12, 13, and 14. For the smaller values of magnetic field ($H_0 < 1000$ A/m), we observe a similar behaviour of dispersion curves as has been shown in the previous section of axial magnetic field. For very small magnetic fields ($H_0 = 150$ A/m), the dispersion curve follows a decreasing wave speed with an increasing wave number. As the underlying magnetic field is increased further ($H_0 > 600$ A/m), the shape of the dispersion curve is inverted and the wave speeds tend to increase with an increasing wave number eventually reaching an asymptotic value. For very small wave numbers, the given material configuration tends to become unstable for the first mode hinting that large wavelengths cannot be accommodated in the tube when the magnetic loading crosses a certain threshold.

It can also be observed from the graphs in all the three figures that the wave speed tends to decrease as the azimuthal magnetic field is increased. For the case of smaller magnetic fields ($H_0 < 1200$ A/m), the wave speed decreases rapidly with an increase in the magnetic field and approaches the point of the onset of instability ($\zeta \to 0$) for the first mode. Upon a further increase in the underlying magnetic field, only the second and higher modes are observed. For the second mode of wave propagation, the wave speed again decreases with an increase in the magnetic field, but with a decreasing

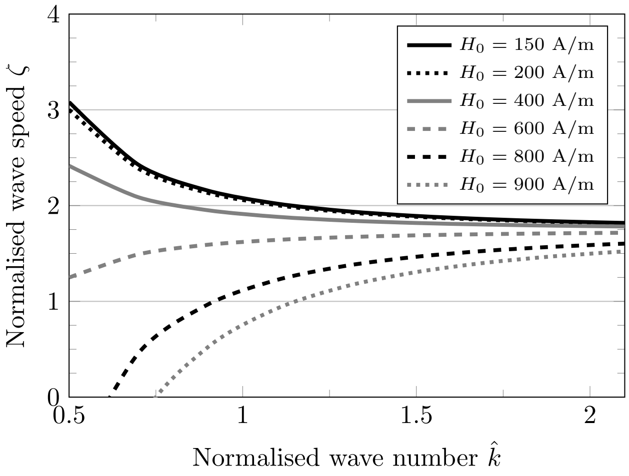
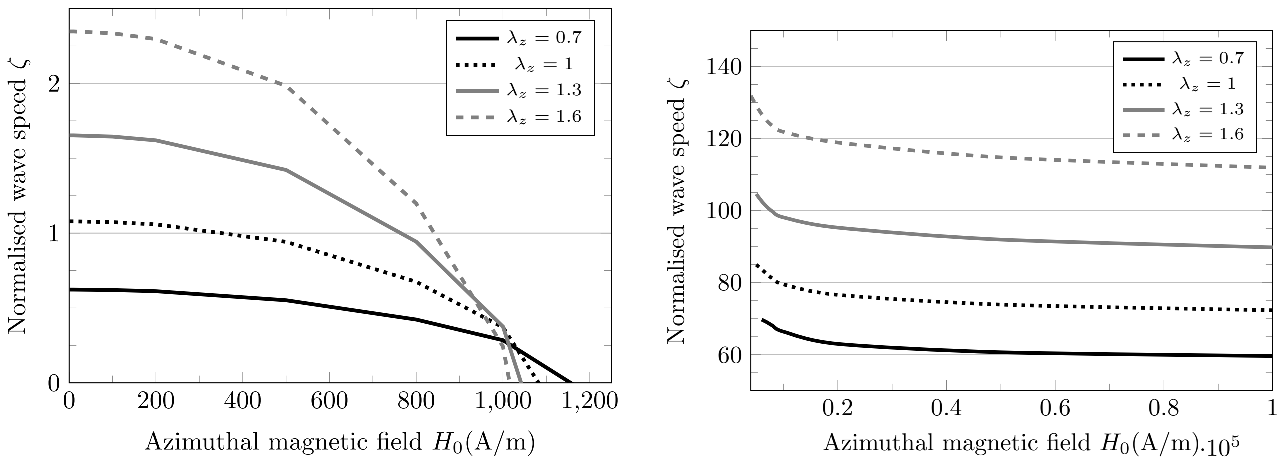


Figure 12: Dispersion curves for several magnetic fields. $\lambda_z = 1.5, P_{\text{in}} = 0.1\mu,\ B/A = 1.3$.



(a) (b)

Figure 13: Wave speeds for different values of the underlying azimuthal magnetic field $H_2 = H_0 B/r$ for various values of the axial stretch $\lambda_z . P_{\text{in}} = 0.1\mu, B/A = 1.3$.

slope in an asymptotic manner.

As can be seen from Figures 12–14, barring a few exceptions, an increase in the axial stretch λ_z results in an increase in the wave speed. However, an increased axial stretch also results in an early onset of instability for lower wave numbers (Figure 14(a)). Similarly, it is observed from Figure 13(a) that a higher axial stretch results in an early onset of instability at even lower values of the underlying magnetic field. In the case of high magnetic fields (Figures 13(b) and 14(b), $H_0 > 1 \times 10^4$ A/m), an increased axial stretch uniformly increases the value of wave speed. However, the influence of axial stretch is considerable only for lower wave numbers and it becomes minimal at higher wave numbers.

Effect of tube thickness on the wave speed and dispersion relations is shown in Figure 15. For both cases of higher and lower magnetic fields, a thicker tube tends to have smaller wave speed in comparison to a thinner tube. The difference in wave speeds is more prominent for lower wave numbers and tends to be minimal for higher wave numbers.

7.2 Displacement in the (r, z) plane

We now consider the incremental displacements in the radial and axial directions and hence deal with the equations involving u_1, u_3, and $\dot{H}_{l02}$. Since u_1 and u_3 satisfy Equation (63), we can define a potential ϕ that satisfy equations $(88)_{1,2}$ and substitute in the governing equation (132) to get

$$\begin{aligned}\phi_{,1}\left(\frac{A_4}{r^2} - 2\frac{A_5}{r^3} + 6\frac{A_6}{r^4}\right) + \phi_{,11}\left(-\frac{A_4}{r} + 2\frac{A_5}{r^2} - 6\frac{A_6}{r^3}\right) + \phi_{,111}\left(-\frac{A_5}{r} + 3\frac{A_6}{r^2}\right)\\ -\frac{A_6}{r}\phi_{,1111} + \phi_{,33}\left(\frac{A_1}{r} - \frac{A_2}{r^2} + 2\frac{A_3}{r^3}\right) + \phi_{,133}\left(\frac{A_2 - A_8}{r} + \frac{A_7 - 2A_3}{r^2}\right)\\ +\phi_{,1133}\frac{A_3 - A_7}{r} + \frac{A_9}{r}\phi_{,3333} = \rho\left(\frac{\phi_{,11} + \phi_{,33}}{r} - \frac{\phi_{,1}}{r^2}\right)_{,tt}.\end{aligned} \tag{175}$$

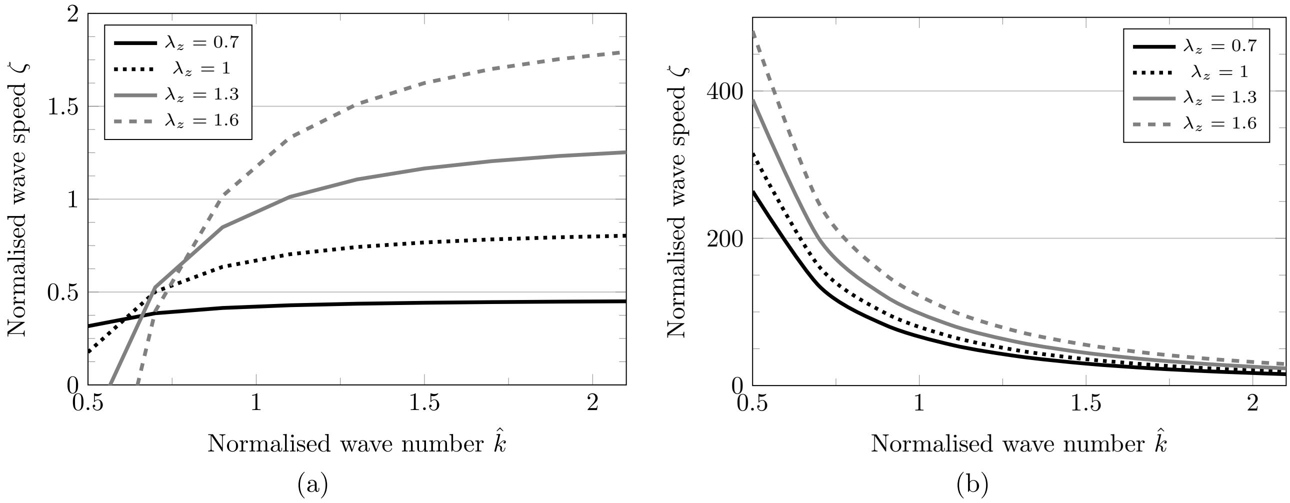


Figure 14: Wave speeds for different values of the underlying axial stretch λ_z for $P_{\text{in}} = 0.1\mu$, $B/A = 1.3$. (a) $H_0 = 800$ A/m, (b) $H_0 = 1 \times 10^4$ A/m.

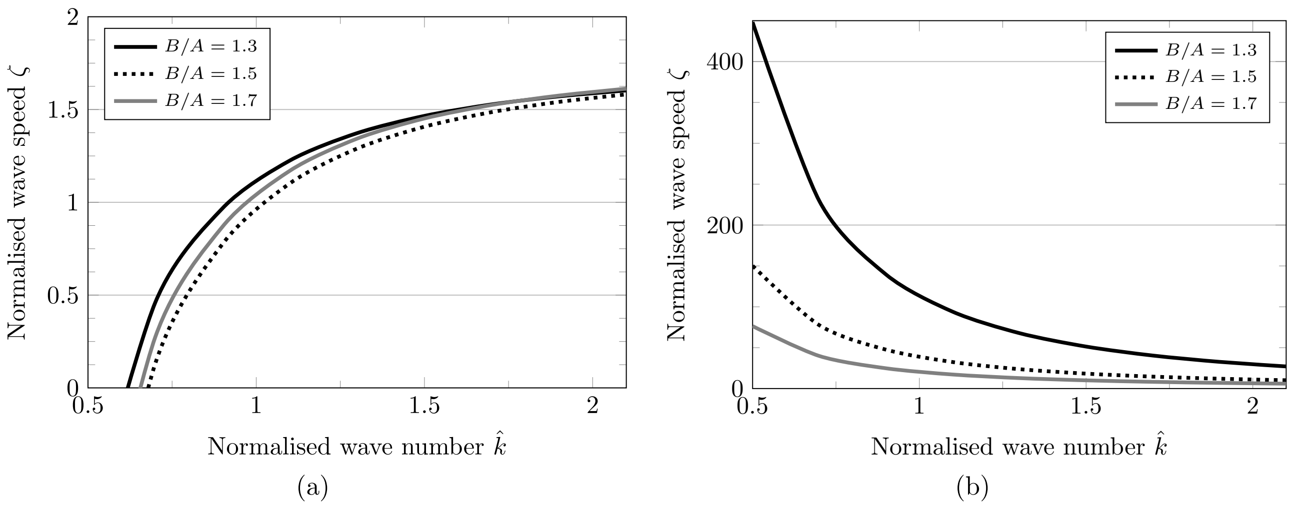


Figure 15: Wave speeds for different values of the tube wall thickness B/A for $\lambda_z = 1.5, P_{\text{in}} = 0.1\mu$. (a) $H_0 = 800$ A/m, (b) $H_0 = 1 \times 10^4$ A/m.

Equations (71) govern $\dot{H}_{l02}$ while $\dot{H}_2^*$ satisfies $(77)_{3,4}$. The boundary conditions $(64)_{1,3}$ and (66) give

$$\left(\mathcal{A}_{01111}+p-\frac{\mu_0 H_2^{*2}}{2}\right)u_{1,1}+\mathcal{A}_{01122}\frac{u_1}{r}+\mathcal{A}_{01133}u_{3,3}$$
$$+\mathcal{C}_{0112}\dot{H}_{l02}+\mu_0 H_2\dot{H}_2^*-\dot{p}=0, \tag{176}$$
$$\left(\mathcal{A}_{01331}+p-\frac{\mu_0 H_2^{2}}{2}\right)u_{1,3}+\mathcal{A}_{01331}u_{3,1}=0, \tag{177}$$
$$\dot{H}_{l02}-H_2\frac{u_1}{r}-\dot{H}_2^*=0, \tag{178}$$

at $r=a$ and $r=b$. We differentiate Equation (176) with respect to z and substitute $\dot{p}_{,3}$ from Equation (131) to get

$$u_{1,3}\left(\frac{1}{r}\left(\mathcal{A}_{01122}-\mathcal{A}_{02233}-\mathcal{A}_{01331}-p\right)-\left(\mathcal{A}_{01331}+p\right)_{,1}\right)-\mathcal{A}_{01313}u_{3,11}$$
$$-u_{3,1}\left(\frac{\mathcal{A}_{01313}}{r}+\mathcal{A}_{01313,1}\right)+u_{1,13}\left(\mathcal{A}_{01111}-\mathcal{A}_{01133}-\mathcal{A}_{01331}-\frac{\mu_0 H_2^{*2}}{2}\right)$$
$$+\left(\mathcal{A}_{01133}-\mathcal{A}_{03333}-p\right)u_{3,33}+\rho u_{3,tt}=0. \tag{179}$$

Using the definition of ϕ from Equation $(88)_{1,2}$ in the above boundary conditions, we get

$$\phi_{,1}\left(\frac{\mathcal{A}_{01313}}{r^3}-\frac{\mathcal{A}_{01313,1}}{r^2}\right)+\phi_{,11}\left(\frac{\mathcal{A}_{01313,1}}{r}-\frac{\mathcal{A}_{01313}}{r^2}\right)+\frac{\mathcal{A}_{01313}}{r}\phi_{,111}$$
$$+\phi_{,33}\left\{-\frac{\left(\mathcal{A}_{01331}+p\right)_{,1}}{r}+\frac{1}{r^2}\left(\frac{\mu_0 H_2^2}{2}+\mathcal{A}_{01122}-\mathcal{A}_{02233}-\mathcal{A}_{01111}+\mathcal{A}_{01133}\right.\right.$$
$$\left.\left.-p\right)\right\}+\frac{\phi_{,133}}{r}\left(\mathcal{A}_{01111}+\mathcal{A}_{03333}-2\mathcal{A}_{01133}-\mathcal{A}_{01331}+p-\frac{\mu_0 H_2^2}{2}\right)$$
$$-\frac{\rho}{r}\phi_{,1tt}=0, \tag{180}$$
$$\mathcal{A}_{01313}\left(-\frac{\phi_{,11}}{r}+\frac{\phi_{,1}}{r^2}\right)+\left(\mathcal{A}_{01331}+p-\frac{\mu_0 H_2^2}{2}\right)\frac{\phi_{,33}}{r}=0, \tag{181}$$
$$\dot{H}_{l02}-H_2\frac{\phi_{,3}}{r^2}-\dot{H}_2^*=0, \tag{182}$$

at $r=a$ and $r=b$.

7.2.1 Wave propagation solutions

We consider the solutions of the above mentioned differential equations of the form

$$\phi=F(r)\exp\left(\mathrm{i}kz-\mathrm{i}\omega t\right),\quad a<r<b, \tag{183}$$
$$\dot{H}_{l02}=G(r)\,\mathrm{e}^{-\mathrm{i}\omega t},\quad a<r<b, \tag{184}$$
$$\dot{H}_2^*=M_1(r)\mathrm{e}^{-\mathrm{i}\omega t}\quad\text{for } r<a,\qquad \dot{H}_2^*=M_2(r)\mathrm{e}^{-\mathrm{i}\omega t}\quad\text{for } r>b. \tag{185}$$

Substituting these solutions in the governing equations (71), $(77)_4$, and (175) gives

$$\left(-k^2b_4+b_7k^4-\rho\omega^2\frac{k^2}{r}\right)F+\left(b_1-k^2b_5-\frac{\rho\omega^2}{r^2}\right)F'$$
$$-\left(rb_1+k^2b_6-\frac{\rho\omega^2}{r}\right)F''+b_2F'''+b_3F''''=0,\quad a<r<b, \tag{186}$$
$$G'+\frac{G}{r}=0,\quad a<r<b, \tag{187}$$
$$M_1'+\frac{M_1}{r}=0,\quad r<a,\qquad M_2'+\frac{M_2}{r}=0,\quad r>b, \tag{188}$$

where prime denotes a derivative with respect to r and we have defined

$$b_1=\frac{A_4}{r^2}-2\frac{A_5}{r^3}+6\frac{A_6}{r^4},\quad b_2=\frac{-A_5}{r}+3\frac{A_6}{r^2},\quad b_3=-\frac{A_6}{r},\quad b_7=\frac{A_9}{r},$$
$$b_4=\frac{A_1}{r}-\frac{A_2}{r^2}+2\frac{A_3}{r^3},\quad b_5=\frac{A_2-A_8}{r}+\frac{A_7-2A_3}{r^2},\quad b_6=\frac{A_3-A_7}{r},$$
$$C_1=\left\{\frac{1}{r^2}\left(-\mathcal{A}_{01111}+\mathcal{A}_{01122}+\mathcal{A}_{01133}-\mathcal{A}_{02233}-p+\frac{\mu_0 H_2^2}{2}\right)\right.$$
$$\left.-\frac{1}{r}\left(\mathcal{A}_{01331,1}+p_{,1}\right)\right\},\quad C_2=\left(\frac{\mathcal{A}_{01313}}{r^3}-\frac{\mathcal{A}_{01313,1}}{r^2}\right),\quad C_4=\frac{\mathcal{A}_{01313}}{r},$$
$$C_3=\frac{1}{r}\left(\mathcal{A}_{01111}-\mathcal{A}_{01331}-2\mathcal{A}_{01133}+\mathcal{A}_{03333}+p-\frac{\mu_0 H_2^2}{2}\right). \tag{189}$$

The boundary conditions (180)–(182) become

$$-k^2C_1F + \left(C_2 - k^2C_3 + \frac{\rho\omega^2}{r}\right)F' - rC_2F'' + C_4F''' = 0 \quad \text{at } r = a, r = b, \tag{190}$$

$$-k^2\left(C_4 + \frac{p}{r} - \frac{\mu_0 H_2^2}{2r}\right)F + \frac{C_4}{r}F' - C_4F'' = 0, \quad \text{at } r = a, r = b, \tag{191}$$

$$G - \frac{1}{r^2}\mathrm{i}kH_2F\mathrm{e}^{\mathrm{i}kz} - M_1 = 0 \quad \text{at } r = a, \tag{192}$$

$$G - \frac{1}{r^2}\mathrm{i}kH_2F\mathrm{e}^{\mathrm{i}kz} - M_2 = 0 \quad \text{at } r = b. \tag{193}$$

Since the last two boundary conditions apply for all z and considering the fact that G, M_1, and M_2 do not depend on z, they can be split into

$$G = M_1 \quad \text{at } r = a, \quad G = M_2 \quad \text{at } r = b, \tag{194}$$

$$H_2F = 0 \quad \text{at } r = a, b. \tag{195}$$

The governing equations for G, M_1 and M_2 can be integrated analytically to get $G = c_1/r$ in $a < r < b$, $M_1 = c_2/r$ in $r < a$, and $M_2 = c_3/r$ in $r > b$. The boundary conditions (194), however, require that $c_1 = c_2 = c_3$.

To solve a fourth order ODE (186), we have six boundary conditions (190), (191), and (195) which makes the system overdetermined and therefore a solution is only possible when $H_2 = 0$. This would lead to a purely elastic wave propagation problem as discussed at the end of Section 6.1.1. A non-trivial solution (for incremental magnetic field) can be obtained only in a very special case when the parameters $C_1, ..., C_4$ obtain values such that two of the boundary conditions become linearly dependent.

8 Concluding remarks

In this paper, we have studied finite deformation and propagation of axisymmetric waves in a long, thick magnetoelastic tube governed by Ogden type and Mooney–Rivlin type incompressible magnetoelastic energy density functions. The mathematical procedure for computing internal pressure and axial force on the tube are demonstrated and the influence of magnetomechanical loading parameters is studied. An underlying azimuthal magnetic field results in a deflation (reduction of radius) and reduction in the axial force for a given geometry.

We analyse axisymmetric motions superimposed on a finitely deformed configuration and show that for a magnetoelastic coupled problem, waves with particle motion in (r, z) direction are not possible. They exist only for a purely mechanical problem in the absence of magnetic field. We solve the governing equations numerically to study wave propagation with particle motion in the azimuthal direction. Due to the presence of a length scale (tube thickness) in the problem, several modes of wave propagation are obtained and we analyse the influence of various parameters by computing how they effect the relationship between wave speed and wavenumber.

We observe that the first mode for small magnetic fields usually has a different qualitative behaviour in comparison to all other modes and conditions. An underlying azimuthal magnetic field tends to reduce the speed of wave propagation. In the case of an axial magnetic field, the wave speed is reduced by the magnetic field if the magnetic field is of low magnitude; but for high magnetic fields ($H_3 > 1 \times 10^4$ A/m), the wave speed is enhanced by the magnetic field. Waves in a thicker tube travel slower in comparison to those in a thinner tube and in most cases, application of an axial stretch results in a higher wave speed. It is demonstrated that in many cases, increasing the underlying magnetic field can result in vanishing of the wave velocity for large wavelengths suggesting that those waves cannot propagate in the magnetomechanically loaded tube.

Vanishing wave speeds $\zeta \to 0$ correspond to onset of geometric instability and it is shown that the first mode becomes quickly unstable upon increasing the magnetic field (Figures 9(a) and 13). A nonlinear relation exists between the critical magnetic field that leads to instability and the underlying axial stretch λ_z.

Thus, we have demonstrated the ability to control the internal pressure, axial force, stability conditions, and wave speeds in a magnetomechanically loaded tube. The results and analysis presented in this paper should be applicable while designing waveguides and vibration controllers from magnetoelastic polymers.

Appendix. Magnetoelastic moduli tensors

Here we list some components of the magnetoelastic moduli tensors for a generalised Mooney–Rivlin material given by Equation (52) which are useful in this paper. For this material the only non-zero derivatives of the energy function are

$$\Omega_1 = (1 + \gamma)\frac{\mu}{4}, \quad \Omega_2 = (1 - \gamma)\frac{\mu}{4}, \quad \Omega_5 = \tilde{q}.$$

In the presence of an axial or azimuthal magnetic field $\{\mathbf{H} = (0, H_2, 0)$ or $\mathbf{H} = (0, 0, H_3)\}$, the non-zero components of the magnetoelastic moduli tensors in terms of derivatives defined above are

$$\begin{aligned}\mathcal{A}_{01111} &= 2\lambda_1^2 \left\{\Omega_1 + (\lambda_2^2 + \lambda_3^2)\Omega_2\right\}, \\ \mathcal{A}_{02222} &= 2\lambda_2^2 \left\{\Omega_1 + (\lambda_1^2 + \lambda_3^2)\Omega_2 + \lambda_2^2 H_2^2 \Omega_5\right\}, \\ \mathcal{A}_{03333} &= 2\lambda_3^2 \left\{\Omega_1 + (\lambda_1^2 + \lambda_2^2)\Omega_2 + \lambda_3^2 H_3^2 \Omega_5\right\},\end{aligned}$$

$$\mathcal{A}_{01122} = 4\lambda_1^2\lambda_2^2\Omega_2, \quad \mathcal{A}_{01133} = 4\lambda_1^2\lambda_3^2\Omega_2, \quad \mathcal{A}_{02233} = 4\lambda_2^2\lambda_3^2\Omega_2,$$

$$\begin{aligned}\mathcal{A}_{01212} &= 2\lambda_1^2\left(\Omega_1 + \lambda_3^2\Omega_2\right), \quad \mathcal{A}_{02121} = 2\lambda_2^2\left(\Omega_1 + \lambda_3^2\Omega_2\right) + 2\lambda_2^4 H_2^2\Omega_5, \\ \mathcal{A}_{01313} &= 2\lambda_1^2\left(\Omega_1 + \lambda_2^2\Omega_2\right), \quad \mathcal{A}_{03131} = 2\lambda_3^2\left(\Omega_1 + \lambda_1^2\Omega_2\right) + 2\lambda_3^4 H_3^2\Omega_5, \\ \mathcal{A}_{02323} &= 2\lambda_2^2\left(\Omega_1 + \lambda_1^2\Omega_2\right) + 2\lambda_2^4 H_2^2\Omega_5, \quad \mathcal{A}_{03232} = 2\lambda_3^2\left(\Omega_1 + \lambda_1^2\Omega_2\right) + 2\lambda_3^4 H_3^2\Omega_5,\end{aligned}$$

$$\mathcal{A}_{01221} = -2\lambda_1^2\lambda_2^2\Omega_2, \quad \mathcal{A}_{01331} = -2\lambda_1^2\lambda_3^2\Omega_2, \quad \mathcal{A}_{02332} = -2\lambda_2^2\lambda_3^2\Omega_2,$$

$$\mathcal{C}_{0112} = 0 = \mathcal{C}_{0332}, \quad \mathcal{C}_{0222} = 4\lambda_2^2 H_2\Omega_5 = 2\mathcal{C}_{0121} = 2\mathcal{C}_{0323},$$

$$\mathcal{C}_{0113} = 0 = \mathcal{C}_{0223}, \quad \mathcal{C}_{0333} = 4\lambda_3^2 H_3\Omega_5 = 2\mathcal{C}_{0131} = 2\mathcal{C}_{0232},$$

$$\mathsf{K}_{011} = 2\lambda_1^{-2}\Omega_4 + 2\Omega_5, \quad \mathsf{K}_{022} = 2\lambda_2^{-2}\Omega_4 + 2\Omega_5, \quad \mathsf{K}_{033} = 2\lambda_3^{-2}\Omega_4 + 2\Omega_5.$$

Acknowledgements:
This work was started while author was pursuing PhD under the supervision of Prof. Ray W. Ogden at the University of Glasgow. His advice during this work has been very helpful. Consequently, some analysis and results in this paper can also be found in author's PhD thesis [47].
The author would also like to thank the anonymous reviewers for their constructive comments that led to several improvements in this paper.
This project is currently supported by a research grant through the Indian Science and Engineering Research Board's Ramanujan fellowship.

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