Modulation & Coding for the Gaussian Channel

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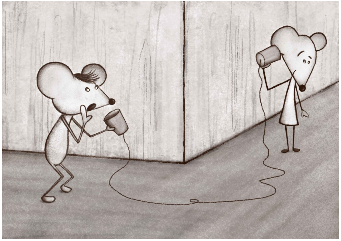
Lakshmi Prasad Natarajan
Dept. of Electrical Engineering
Indian Institute of Technology Hyderabad
lakshminatarajan@iith.ac.in

Digital Communication

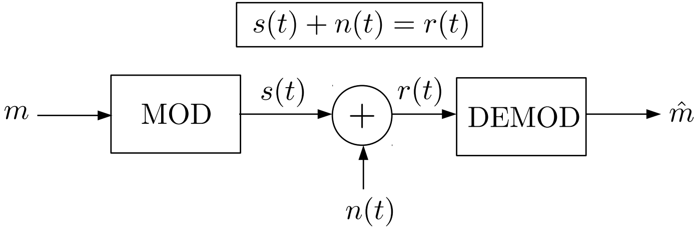
Convey a message from *transmitter* to *receiver* in a finite amount of time, where the message can assume only finitely many values.



- *'time'* can be replaced with any resource:
  space available in a compact disc, number of cells in flash memory

Picture courtesy `brigetteheffernan.wordpress.com`

The Additive Noise Channel



- Message m
  - takes finitely many, say M, distinct values
  - Usually, not always, $M = 2^k$, for some integer k
  - assume m is uniformly distributed over $\{1, \ldots, M\}$
- Time duration T
  - transmit signal $s(t)$ is restricted to $0 \le t \le T$
- Number of message bits $k = \log_2 M$ (not always an integer)

Modulation Scheme

- The transmitter & receiver agree upon a set of waveforms $\{s_1(t), \ldots, s_M(t)\}$ of duration T.
- The transmitter uses the waveform $s_i(t)$ for the message $m = i$.
- The receiver must guess the value of m given $r(t)$.
- We say that a decoding error occurs if the guess $\hat{m} \neq m$.

Definition

An M-ary modulation scheme is simply a set of M waveforms $\{s_1(t), \ldots, s_M(t)\}$ each of duration T.

Terminology

- Binary: $M = 2$, modulation scheme $\{s_1(t), s_2(t)\}$
- Antipodal: $M = 2$ and $s_2(t) = -s_1(t)$
- Ternary: $M = 3$, Quaternary: $M = 4$

Parameters of Interest

- Bit rate $R = \dfrac{\log_2 M}{T}$ bits/sec

  Energy of the i^{th} waveform $E_i = \|s_i(t)\|^2 = \int_{t=0}^{T} s_i^2(t)\mathrm{d}t$

- Average Energy

$$E = \sum_{i=1}^{M} P(m = i)E_i = \sum_{i=1}^{M} \frac{1}{M} \int_{t=0}^{T} \|s_i(t)\|^2$$

- Energy per message bit $E_b = \dfrac{E}{\log_2 M}$

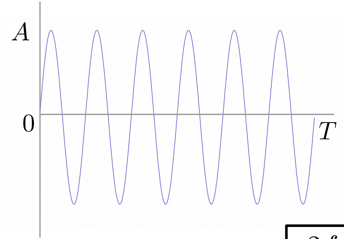
- Probability of error $P_e = P(m \neq \hat{m})$

Note

P_e depends on the modulation scheme, noise statistics and the demodulator.

Example: On-Off Keying, $M = 2$

$s_1(t) = A\sin(2\pi f_c t)$



$s_2(t) = 0$ for $0 \le t \le T$

$$\boxed{2 f_c T = \text{integer}}$$

$$\text{Rate } R = \frac{\log_2 M}{T} = \frac{1}{T}$$

$$E_1 = \int_{t=0}^{T} A^2 \sin^2(2\pi f_c t)\mathrm{d}t = \frac{A^2 T}{2} \text{ and } E_2 = 0$$

$$\text{Average Energy } E = \frac{E_1 + E_2}{2} = \frac{A^2 T}{4}$$

Objectives

1. Characterize and analyze a modulation scheme in terms of energy, rate and error probability.
   - What is the best/optimal performance that one can expect?

2. Design a good modulation scheme that performs close to the theoretical optimum.

Key tool: Signal Space Representation

- Represent waveforms as vectors: 'geometry' of the problem
- Simplifies performance analysis and modulation design
- Leads to efficient modulation/demodulation implementations

References

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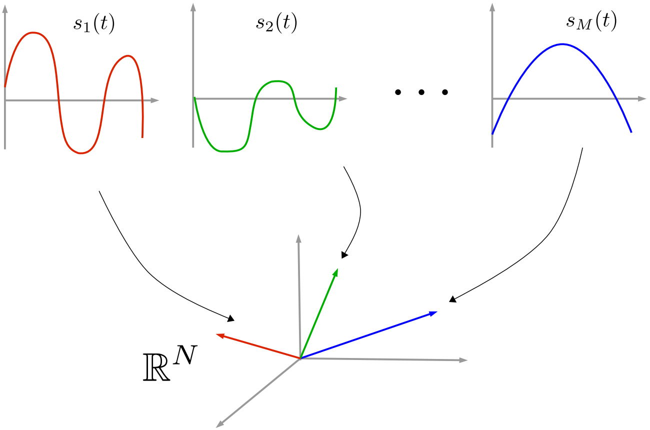
- D. Slepian and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty I," in *The Bell System Technical Journal*, vol. 40, no. 1, pp. 43-63, Jan. 1961.
- H. J. Landau and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty III: The dimension of the space of essentially time- and band-limited signals," in *The Bell System Technical Journal*, vol. 41, no. 4, pp. 1295-1336, July 1962.

Goal

Map waveforms $s_1(t), \ldots, s_M(t)$ to M vectors in a Euclidean space $\mathbb{R}^N$, so that the map preserves the mathematical structure of the waveforms.



Quick Review of $\mathbb{R}^N$: N-Dimensional Euclidean Space

$$\mathbb{R}^N = \left\{(x_1, x_2, \ldots, x_N) \,|\, x_1, \ldots, x_N \in \mathbb{R}\right\}$$

Notation: $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$ and $\mathbf{0} = (0, 0, \ldots, 0)$

Addition Properties:

- $\boldsymbol{x} + \boldsymbol{y} = (x_1, \ldots, x_N) + (y_1, \ldots, y_N) = (x_1 + y_1, \ldots, x_N + y_N)$
- $\boldsymbol{x} - \boldsymbol{y} = (x_1, \ldots, x_N) - (y_1, \ldots, y_N) = (x_1 - y_1, \ldots, x_N - y_N)$
- $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ for every $\boldsymbol{x} \in \mathbb{R}^N$

Multiplication Properties:

- $a\,\boldsymbol{x} = a\,(x_1, \ldots, x_N) = (ax_1, \ldots, ax_N)$, where $a \in \mathbb{R}$
- $a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$
- $(a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$
- $a\boldsymbol{x} = \mathbf{0}$ if and only if $a = 0$ or $\boldsymbol{x} = \mathbf{0}$

Quick Review of $\mathbb{R}^N$: Inner Product and Norm

Inner Product

- $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$
- $\langle \boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle$ (distributive law)
- $\langle a\boldsymbol{x}, \boldsymbol{y} \rangle = a \langle \boldsymbol{x}, \boldsymbol{y} \rangle$
- If $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ we say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal

Norm

- $\|\boldsymbol{x}\| = \sqrt{x_1^2 + \cdots + x_N^2} = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$ denotes the length of $\boldsymbol{x}$
- $\|\boldsymbol{x}\|^2 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$ denotes the energy of the vector $\boldsymbol{x}$
- $\|\boldsymbol{x}\|^2 = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$
- If $\|\boldsymbol{x}\| = 1$ we say that $\boldsymbol{x}$ is of unit norm
- $\|\boldsymbol{x} - \boldsymbol{y}\|$ is the distance between two vectors.

Cauchy-Schwarz Inequality

- $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$
- Or equivalently, $-1 \leq \dfrac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{x}\| \, \|\boldsymbol{y}\|} \leq 1$

Waveforms as Vectors

The set of all finite-energy waveforms of duration T and the Euclidean space $\mathbb{R}^N$ share *many* structural properties.

Addition Properties

- We can add and subtract two waveforms $x(t) + y(t)$, $x(t) - y(t)$
- The all-zero waveform $0(t) = 0$ for $0 \leq t \leq T$ is the additive identity

$$x(t) + 0(t) = x(t) \text{ for any waveform } x(t)$$

Multiplication Properties

- We can scale $x(t)$ using a real number a and obtain $a\,x(t)$
- $a(x(t) + y(t)) = ax(t) + ay(t)$
- $(a + b)x(t) = ax(t) + bx(t)$
- $ax(t) = 0(t)$ if and only if $a = 0$ or $x(t) = 0(t)$

Inner Product and Norm of Waveforms

Inner Product

- $\langle x(t), y(t) \rangle = \langle y(t), x(t) \rangle = \int_{t=0}^{T} x(t)y(t)\mathrm{d}t$
- $\langle x(t), y(t) + z(t) \rangle = \langle x(t), y(t) \rangle + \langle x(t), z(t) \rangle$ (distributive law)
- $\langle ax(t), y(t) \rangle = a\langle x(t), y(t) \rangle$
- If $\langle x(t), y(t) \rangle = 0$ we say that $x(t)$ and $y(t)$ are orthogonal

Norm

- $\|x(t)\| = \sqrt{\langle x(t), x(t) \rangle} = \sqrt{\int_{t=0}^{T} x^2(t)\mathrm{d}t}$ is the norm of $x(t)$
- $\|x(t)\|^2 = \int_{t=0}^{T} x^2(t)\mathrm{d}t$ denotes the energy of $x(t)$
- If $\|x(t)\| = 1$ we say that $x(t)$ is of unit norm
- $\|x(t) - y(t)\|$ is the distance between two waveforms

Cauchy-Schwarz Inequality

- $|\langle x(t), y(t) \rangle| \leq \|x(t)\| \, \|y(t)\|$ for any two waveforms $x(t)$, $y(t)$

We want to map $s_1(t), \ldots, s_M(t)$ to vectors $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_M \in \mathbb{R}^N$ so that the addition, multiplication, inner product and norm properties are preserved.

Orthonormal Waveforms

Definition

A set of N waveforms $\{\phi_1(t), \ldots, \phi_N(t)\}$ is said to be orthonormal if

1. $\|\phi_1(t)\| = \|\phi_2(t)\| = \cdots = \|\phi_N(t)\| = 1$ (unit norm)
2. $\langle \phi_i(t), \phi_j(t) \rangle = 0$ for all $i \neq j$ (orthogonality)

The role of orthonormal waveforms is similar to that of the standard basis

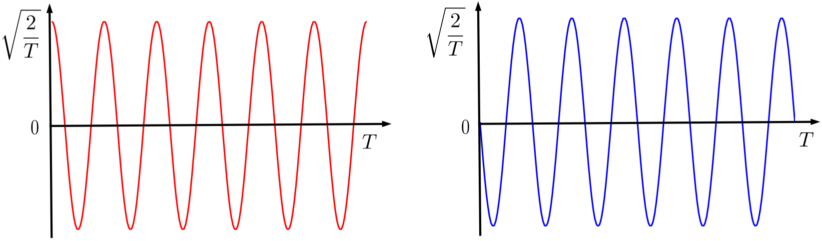
$$\boldsymbol{e}_1 = (1, 0, 0, \ldots, 0), \boldsymbol{e}_2 = (0, 1, 0, \ldots, 0), \cdots, \boldsymbol{e}_N = (0, 0, \ldots, 0, 1)$$

Remark

Say $x(t) = x_1\phi_1(t) + \cdots x_N\phi_N(t)$, $y(t) = y_1\phi_1(t) + \cdots + y_N\phi_N(t)$

$$\begin{aligned}
\langle x(t), y(t) \rangle &= \left\langle \sum_{i=1}^{N} x_i\phi_i(t), \sum_{j=1}^{N} y_j\phi_j(t) \right\rangle = \sum_i \sum_j x_i y_j \langle \phi_i(t), \phi_j(t) \rangle \\
&= \sum_i \sum_{j=i} x_i y_j = \sum_i x_i y_i \\
&= \langle \boldsymbol{x}, \boldsymbol{y} \rangle
\end{aligned}$$

Example



$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \qquad \phi_2(t) = -\sqrt{\frac{2}{T}}\sin(2\pi f_c t)$$

$$\boxed{2 f_c T = \text{integer}}$$

$$\|\phi_1(t)\|^2 = \|\phi_2(t)\|^2 = 1$$

$$\langle \phi_1(t), \phi_2(t)\rangle = -\int_{t=0}^{T} \frac{2}{T}\cos(2\pi f_c t)\sin(2\pi f_c t)\mathrm{d}t = 0$$

Orthonormal Basis

Definition

An orthonormal basis for $\{s_1(t), \ldots, s_M(t)\}$ is an orthonormal set $\{\phi_1(t), \ldots, \phi_N(t)\}$ such that

$$s_i(t) = s_{i,1}\phi_i(t) + s_{i,2}\phi_2(t) + \cdots + s_{i,M}\phi_N(t)$$

for some choice of $s_{i,1}, s_{i,2}, \ldots, s_{i,N} \in \mathbb{R}$

- We associate $s_i(t) \to \boldsymbol{s}_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,N})$
- A given modulation scheme can have many orthonormal bases.
- The map $s_1(t) \to \boldsymbol{s}_1, s_2(t) \to \boldsymbol{s}_2, \ldots, s_M(t) \to \boldsymbol{s}_M$ depends on the choice of orthonormal basis.

Example: M-ary Phase Shift Keying

Modulation Scheme

- $s_i(t) = A\cos(2\pi f_c t + \frac{2\pi i}{M})$, $i = 1, \ldots, M$
- Expanding $s_i(t)$ using $\cos(C + D) = \cos C \cos D - \sin C \sin D$

$$s_i(t) = A\cos\left(\frac{2\pi i}{M}\right)\cos(2\pi f_c t) - A\sin\left(\frac{2\pi i}{M}\right)\sin(2\pi f_c t)$$

Orthonormal Basis

- Use $\phi_1(t) = \sqrt{2/T}\cos(2\pi f_c t)$ and $\phi_2(t) = \sqrt{2/T}\sin(2\pi f_c t)$

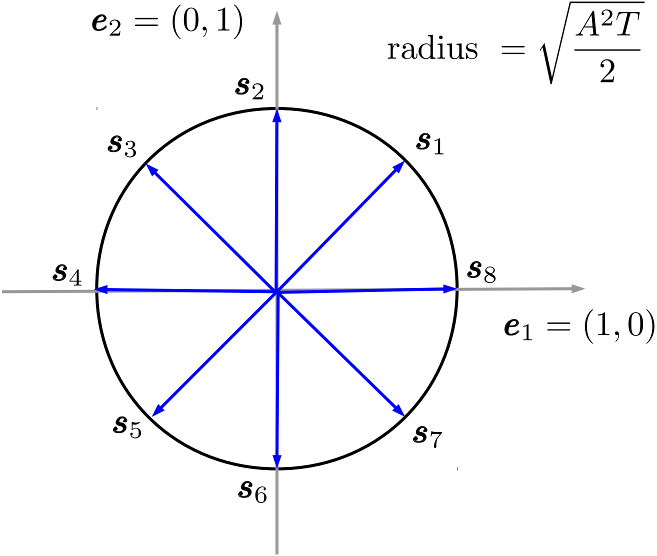
$$s_i(t) = A\sqrt{\frac{T}{2}}\cos\left(\frac{2\pi i}{M}\right)\phi_1(t) + A\sqrt{\frac{T}{2}}\sin\left(\frac{2\pi i}{M}\right)\phi_2(t)$$

- Dimension $N = 2$

Waveform to Vector

$$s_i(t) \to \left(\sqrt{\frac{A^2 T}{2}}\cos\left(\frac{2\pi i}{M}\right), \sqrt{\frac{A^2 T}{2}}\sin\left(\frac{2\pi i}{M}\right)\right)$$

8-ary Phase Shift Keying



How to find an orthonormal basis

Gram-Schmidt Procedure

Given a modulation scheme $\{s_1(t), \ldots, s_M(t)\}$, constructs an orthonormal basis $\phi_1(t), \ldots, \phi_N(t)$ for the scheme.

Similar to QR factorization of matrices

$$\boldsymbol{A} = [\boldsymbol{a}_1 \ \boldsymbol{a}_2 \ \cdots \ \boldsymbol{a}_M] = [\boldsymbol{q}_1 \ \boldsymbol{q}_2 \ \cdots \ \boldsymbol{q}_N] \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,M} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,M} \\ \vdots & \vdots & \ldots & \vdots \\ r_{N,1} & r_{N,2} & \cdots & r_{N,M} \end{bmatrix} = \boldsymbol{QR}$$

$$[s_1(t) \ \cdots \ s_M(t)] = [\phi_1(t) \ \cdots \ \phi_N(t)] \begin{bmatrix} s_{1,1} & s_{2,1} & \cdots & s_{M,1} \\ s_{1,2} & s_{2,2} & \cdots & s_{M,2} \\ \vdots & \vdots & \ldots & \vdots \\ s_{1,N} & s_{2,N} & \cdots & s_{M,N} \end{bmatrix}$$

Waveforms to Vectors, and Back

Say $\{\phi_1(t), \ldots, \phi_N(t)\}$ is an orthonormal basis for $\{s_1(t), \ldots, s_M(t)\}$.

$$\text{Then, } s_i(t) = \sum_{j=1}^{N} s_{i,j}\phi_j(t) \text{ for some choice of } \{s_{i,j}\}$$

Waveform to Vector

$$\langle s_i(t), \phi_j(t) \rangle = \langle \sum_k s_{i,k}\phi_k(t), \phi_j(t) \rangle = \sum_k s_{i,k} \langle \phi_k(t), \phi_j(t) \rangle = s_{i,j}$$

$$s_i(t) \to (s_{i,1}, s_{i,2}, \ldots, s_{i,N}) = \boldsymbol{s}_i$$

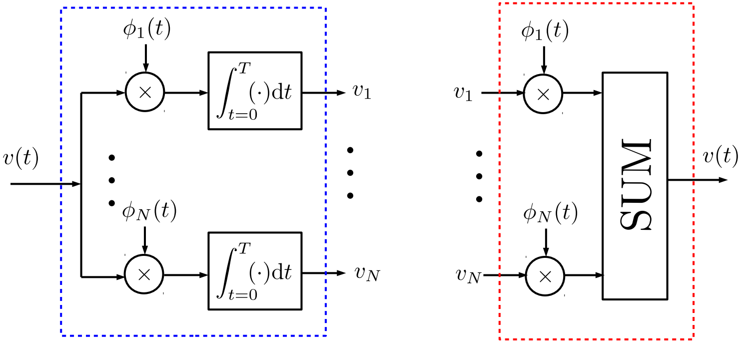
where $s_{i,1} = \langle s_i(t), \phi_1(t) \rangle$, $s_{i,2} = \langle s_i(t), \phi_2(t) \rangle$,..., $s_{i,N} = \langle s_i(t), \phi_N(t) \rangle$

Vector to Waveform

$$\boldsymbol{s}_i = (s_{i,1}, \ldots, s_{i,N}) \to s_{i,1}\phi_1(t) + s_{i,2}\phi_2(t) + \cdots + s_{i,N}\phi_N(t)$$

- Every point in $\mathbb{R}^N$ corresponds to a unique waveform.
- Going back and forth between vectors and waveforms is easy.

Waveforms to Vectors, and Back



Waveform to Vector   Vector to Waveform

Caveat

$v(t) \to$ Waveform to vector $\to \boldsymbol{v}$ $\boldsymbol{v} \to$ Vector to waveform $\to \hat{v}(t)$

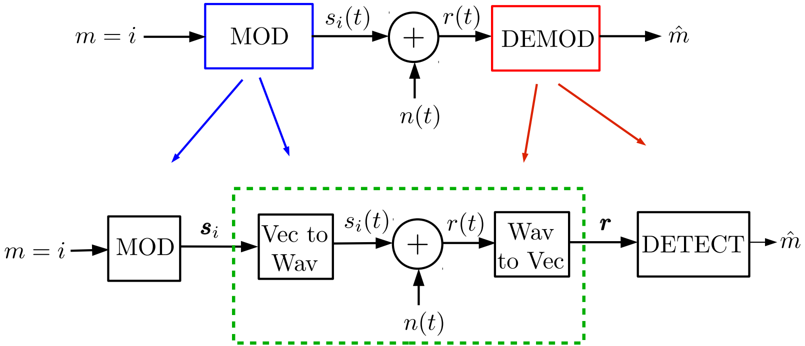
$\hat{v}(t) = v(t)$ *iff* $v(t)$ is some linear combination of $\phi_1(t), \ldots, \phi_N(t)$, or equivalently, $v(t)$ is some linear combination of $s_1(t), \ldots, s_M(t)$

Equivalence Between Waveform and Vector Representations

Say $v(t) = v_1\phi_1(t) + \cdots + v_N\phi_N(t)$ and $u(t) = u_1\phi_1(t) + \cdots + u_N\phi_N(t)$

Addition	$v(t) + u(t)$	$\boldsymbol{v} + \boldsymbol{u}$
Scalar Multiplication	$a\,v(t)$	$a\,\boldsymbol{v}$
Energy	$\lVert v(t)\rVert^2$	$\lVert\boldsymbol{v}\rVert^2$
Inner product	$\langle v(t), u(t)\rangle$	$\langle\boldsymbol{v}, \boldsymbol{u}\rangle$
Distance	$\lVert v(t) - u(t)\rVert$	$\lVert\boldsymbol{v} - \boldsymbol{u}\rVert$
Basis	$\phi_i(t)$	$\boldsymbol{e}_i$ (Std. basis)

Vector Gaussian Channel



Definition

An M-ary modulation scheme of dimension N is a set of M vectors $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_M\}$ in $\mathbb{R}^N$

- Average energy $E = \frac{1}{M}\left(\|\boldsymbol{s}_1\|^2 + \cdots + \|\boldsymbol{s}_M\|^2\right)$

Vector Gaussian Channel

Relation between received vector $\boldsymbol{r}$ and transmit vector $\boldsymbol{s}_i$

The j^{th} component of received vector $\boldsymbol{r} = (r_1, \ldots, r_N)$

$$\begin{aligned} r_j &= \langle r(t), \phi_i(t) \rangle = \langle s_i(t) + n(t), \phi_j(t) \rangle \\ &= \langle s_i(t), \phi_j(t) \rangle + \langle n(t), \phi_j(t) \rangle \\ &= s_{i,j} + n_j \end{aligned}$$

Denoting $\boldsymbol{n} = (n_1, \ldots, n_N)$ we obtain

$$\boldsymbol{r} = \boldsymbol{s}_i + \boldsymbol{n}$$

If $n(t)$ is a Gaussian random process, noise vector $\boldsymbol{n}$ follows Gaussian distribution.

Note

Effective noise at the receiver $\hat{n}(t) = n_1\phi_1(t) + \cdots + n_N\phi_N(t)$

In general, $n(t)$ not a linear combination of basis, and $\hat{n}(t) \neq n(t)$,

Designing a Modulation Scheme

1. Choose an orthonormal basis $\phi_1(t), \ldots, \phi_N(t)$
   - Determines bandwidth of transmit signals, signalling duration T
2. Construct a (vector) modulation scheme $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_M \in \mathbb{R}^N$
   - Determines the signal energy, probability of error

An N-dimensional modulation scheme exploits 'N uses' of a scalar Gaussian channel

$$r_j = s_{i,j} + n_j \text{ where } j = 1, \ldots, N$$

With limits on bandwidth and signal duration, how large can N be?

Dimension of Time/Band-limited Signals

Say transmit signals $s(t)$ must be time/band limited

1. $s(t) = 0$ if $t < 0$ or $t \geq T$, and (time-limited)
2. $S(f) = 0$ if $f < f_c - \frac{W}{2}$ or $f > f_c + \frac{W}{2}$ (band-limited)

Uncertainty Principle: No non-zero signal is both time- and band-limited.

$\Rightarrow$ No signal transmission is possible!

We relax the constraint to approximately band-limited

1. $s(t) = 0$ if $t < 0$ or $t > T$, and (time-limited)
2. $\displaystyle\int_{f=f_c-W/2}^{f_c+W/2} |S(f)|^2 \mathrm{d}f \geq (1-\delta) \int_0^{+\infty} |S(f)|^2 \mathrm{d}f$ (approx. band-lim.)

Here $\delta > 0$ is the fraction of out-of-band signal energy.

> What is the largest dimension N of time-limited/approximately band-limited signals?

Dimension of Time/band-limited Signals

Let $T > 0$ and $W > 0$ be given, and consider any $\delta, \epsilon > 0$.

Theorem (Landau, Pollak & Slepian 1961-62)

If TW is sufficiently large, there exists $N = 2TW(1-\epsilon)$ orthonormal waveforms $\phi_1(t), \ldots, \phi_N(t)$ such that

1. $\phi_i(t) = 0$ if $t < 0$ or $t > T$, and (time-limited)
2. $\int_{f=f_c-W/2}^{f_c+W/2} |\Phi_i(f)|^2 \mathrm{d}f \geq (1-\delta) \int_0^{+\infty} |\Phi_i(f)|^2 \mathrm{d}f$ (approx. band-lim.)

In summary

- We can 'pack' $N \approx 2TW$ dimensions if the time-bandwidth product TW is large enough.
- Number of dimensions/channel uses normalized to 1 sec of transmit duration and 1 Hz of bandwidth

$$\frac{N}{TW} \approx 2 \text{ dim/sec/Hz}$$

Relation between Waveform & Vector Channels

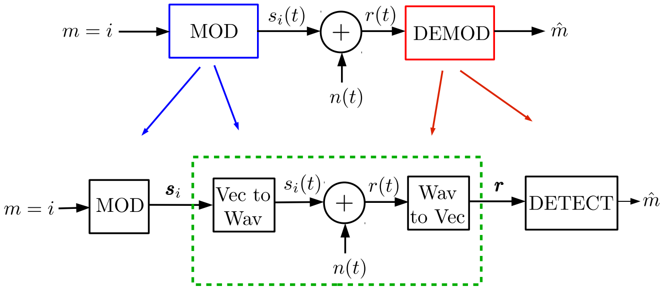
Assume $N = 2TW$

Signal energy	E_i	$\|s_i(t)\|^2$	$\|\boldsymbol{s}_i\|^2$
Avg. energy	E	$\frac{1}{M}\sum_i \|s_i(t)\|^2$	$\frac{1}{M}\sum_i \|\boldsymbol{s}_i\|^2$
Transmit Power	S	$\dfrac{E}{T}$	$\dfrac{E}{N}2W$
Rate	R	$\dfrac{\log_2 M}{T}$	$\dfrac{\log_2 M}{N}2W$

Parameters for Vector Gaussian Channel

- Spectral Efficiency $\eta = 2\log_2 M/N$ (unit: bits/sec/Hz)
  - Allows comparison between schemes with different bandwidths.
  - Related to rate as $\eta = R/W$
- Power $P = E/N$ (unit: Watt/Hz)
  - Related to actual transmit power as $S = 2WP$

Detection in the Gaussian Channel



Definition

Detection/Decoding/Demodulation is the process of estimating the message m given the received waveform $r(t)$ and the modulation scheme $\{s_1(t), \ldots, s_M(t)\}$.

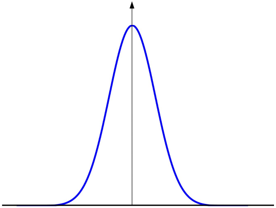
Objective: Design the decoder to minimize $P_e = P(\hat{m} \neq m)$.

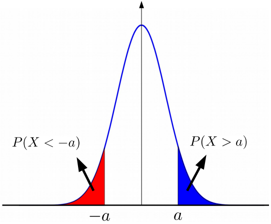
The Gaussian Random Variable

$$\underline{X \sim \mathcal{N}(0,1)}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$Q(a) = P(X > a) = \int_a^{\infty} f(x)\mathrm{d}x$$





- $P(X < -a) = P(X > a) = Q(a)$
- $Q(\cdot)$ is a decreasing function
- $Y = \sigma X$ is Gaussian with mean 0 and var σ^2, i.e., $\mathcal{N}(0, \sigma^2)$
- $P(Y > b) = P(\sigma X > b) = P(X > \frac{b}{\sigma}) = Q\left(\frac{b}{\sigma}\right)$

White Gaussian Noise Process $n(t)$

Noise waveform $n(t)$ modelled as a white Gaussian random process, i.e., as a a collection of random variables $\{n(\tau) \,|\, -\infty < \tau < +\infty\}$ such that

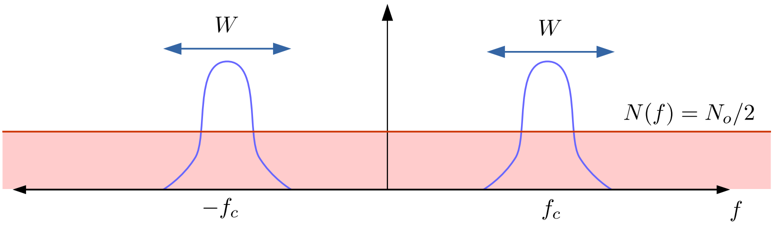
- Stationary random process
  Statistics of the processes $n(t)$ and $n(t - \text{constant})$ are identical
- Gaussian random process
  Any linear combination of finitely many samples of $n(t)$ is Gaussian

$$a_1 n(t_1) + a_2 n(t_2) + \cdots + a_\ell n(t_\ell) \sim \text{Gaussian distributed}$$

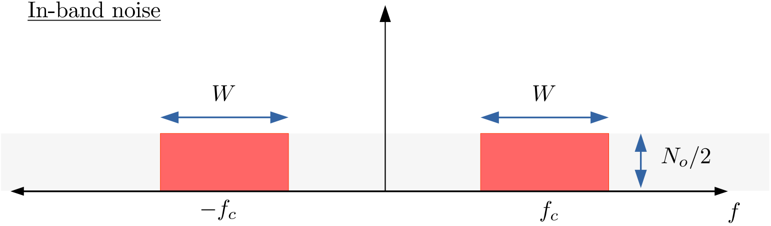
- White random process
  The power spectrum $N(f)$ of the noise process is 'flat'

$$N(f) = \frac{N_o}{2} \text{ W/Hz, for } -\infty < f < +\infty$$

Power spectrum of received waveform (Signal + Noise)

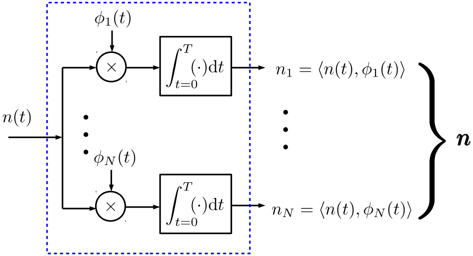


In-band noise



In-band noise power $= 2 \times W \times N_o/2 = N_o W$

Noise Process Through Waveform-to-Vector Converter



Properties of the noise vector $\boldsymbol{n} = (n_1, \ldots, n_N)$

- $n_1, n_2, \ldots, n_N$ are independent $\mathcal{N}(0, N_o/2)$ random variables

$$f(n_i) = \frac{1}{\sqrt{\pi N_o}} \exp\left(-\frac{n_i^2}{N_o}\right)$$

- Noise vector $\boldsymbol{n}$ describes only a part of $n(t)$

$$\hat{n}(t) = n_1\phi_1(t) + \cdots + n_N\phi_N(t) \neq n(t)$$

The noise component not captured by waveform-to-vector converter:

$$\Delta n(t) = n(t) - \hat{n}(t) \neq 0$$

White Gaussian Noise Vector $\boldsymbol{n}$

$$\boxed{\boldsymbol{n} = (n_1, \ldots, n_N)}$$

- Probability density of $\boldsymbol{n} = (n_1, \ldots, n_N)$ in $\mathbb{R}^N$

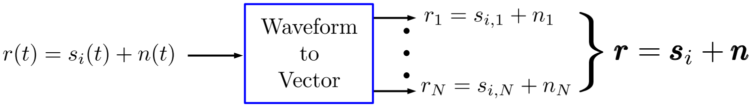
$$f_{\text{noise}}(\boldsymbol{n}) = f(n_1, \ldots, n_N) = \prod_{i=1}^{N} f(n_i) = \frac{1}{(\sqrt{\pi N_o})^N} \exp\left(-\frac{\|\boldsymbol{n}\|^2}{N_o}\right)$$

  - Probability density depends only on $\|\boldsymbol{n}\|^2 \Rightarrow$
    Spherically symmetric: Isotropic distribution
  - Density highest near $\mathbf{0}$ and decreasing in $\|\boldsymbol{n}\|^2 \Rightarrow$
    noise vector of larger norm less likely than a vector with smaller norm

- For any $\boldsymbol{a} \in \mathbb{R}^N$, $\langle \boldsymbol{n}, \boldsymbol{a} \rangle \sim \mathcal{N}\left(0, \|\boldsymbol{a}\|^2 \frac{N_o}{2}\right)$

- $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_K$ are orthonormal $\Rightarrow \langle \boldsymbol{n}, \boldsymbol{a}_1 \rangle, \ldots, \langle \boldsymbol{n}, \boldsymbol{a}_K \rangle$ are independent $\mathcal{N}(0, N_o/2)$

$\Delta n(t)$ Carries Irrelevant Information



- $\boldsymbol{r} = \boldsymbol{s}_i + \boldsymbol{n}$ does not carry all the information in $r(t)$

$$\hat{r}(t) = r_1\phi_1(t) + \cdots + r_N\phi_N(t) \neq r(t)$$

- The information about $r(t)$ not contained in $\boldsymbol{r}$

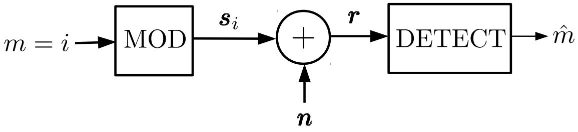
$$r(t) - \sum_j r_j\phi_j(t) = s_i(t) + n(t) - \sum_j s_{i,j}\phi_j(t) - \sum_j n_j\phi_j(t) = \Delta n(t)$$

Theorem

The vector $\boldsymbol{r}$ contains all the information in $r(t)$ that is relevant to the transmitted message.

- $\Delta n(t)$ is irrelevant for the optimum detection of transmit message.

The (Effective) Vector Gaussian Channel



- Modulation Scheme/Code is a set $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_M\}$ of M vectors in $\mathbb{R}^N$
- Power $P = \frac{1}{N} \cdot \frac{\|\boldsymbol{s}_1\|^2 + \cdots + \|\boldsymbol{s}_M\|^2}{M}$
- Noise variance $\sigma^2 = \frac{N_o}{2}$ (per dimension)
- Signal to noise ratio $\mathrm{SNR} = \frac{P}{\sigma^2} = \frac{2P}{N_o}$
- Spectral Efficiency $\eta = \frac{2 \log_2 M}{N}$ bits/s/Hz (assuming $N = 2TW$)

Optimum Detection Rule

Objective

Given $\{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_M\}$ & $\boldsymbol{r}$, provide an estimate $\hat{m}$ of the transmit message m, so that $P_e = P(\hat{m} \neq m)$ is as small as possible.

Optimal Detection: Maximum a posteriori (MAP) detector

Given received vector $\boldsymbol{r}$, choose the vector $\boldsymbol{s}_j$ that has the highest probability of being transmitted

$$\hat{m} = \arg \max_{k \in \{1,\ldots,M\}} P(\boldsymbol{s}_k \text{ transmitted} \,|\, \boldsymbol{r} \text{ received})$$

In other words, choose $\hat{m} = k$ if

$$P(\boldsymbol{s}_k \text{ transmitted} \,|\, \boldsymbol{r} \text{ received}) > P(\boldsymbol{s}_j \text{ transmitted} \,|\, \boldsymbol{r} \text{ received}) \text{ for every } j \neq k$$

- In case of a tie, can choose one of the indices arbitrarily. This does not increase P_e.

Optimum Detection Rule

Use Bayes' rule $P(A|B) = \dfrac{P(A)P(B|A)}{P(B)}$

$$\hat{m} = \arg\max_k P(\boldsymbol{s}_j|\boldsymbol{r}) = \arg\max_k \frac{P(\boldsymbol{s}_k)f(\boldsymbol{r}|\boldsymbol{s}_k)}{f(\boldsymbol{r})}$$

$P(\boldsymbol{s}_j) =$ Probability of transmitting $\boldsymbol{s}_j = 1/M$ (equally likely messages)
$f(\boldsymbol{r}|\boldsymbol{s}_k) =$ Probability density of $\boldsymbol{r}$ when $\boldsymbol{s}_k$ is transmitted
$f(\boldsymbol{r}) =$ Probability density of $\boldsymbol{r}$ averaged over all possible transmissions

$$\hat{m} = \arg\max_k \frac{1/M \cdot f(\boldsymbol{r}|\boldsymbol{s}_k)}{f(\boldsymbol{r})} = \arg\max_k f(\boldsymbol{r}|\boldsymbol{s}_k)$$

Likelihood function $f(\boldsymbol{r}|\boldsymbol{s}_k)$, Max. likelihood rule $\hat{m} = \arg\max_k f(\boldsymbol{r}|\boldsymbol{s}_k)$

If all the M messages are equally likely

Max. a posteriori detection = Max. likelihood (ML) detection

Maximum Likelihood Detection in Vector Gaussian Channel

Use the model $\boldsymbol{r} = \boldsymbol{s}_i + \boldsymbol{n}$ and the assumption $\boldsymbol{n}$ is independent of $\boldsymbol{s}_i$

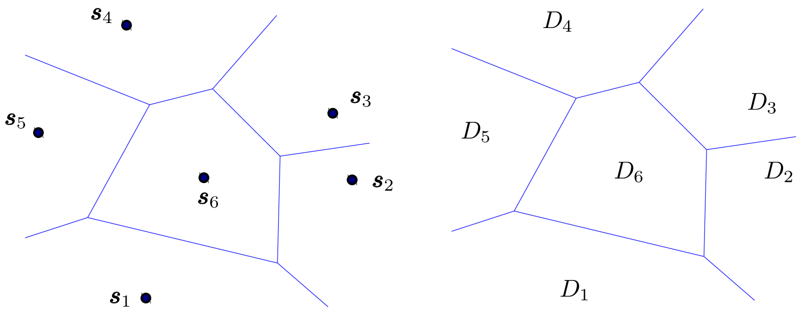
$$\begin{aligned}
\hat{m} = \arg\max_k f(\boldsymbol{r}|\boldsymbol{s}_k) &= \arg\max_k f_{\text{noise}}(\boldsymbol{r} - \boldsymbol{s}_k|\boldsymbol{s}_k) \\
&= \arg\max_k f_{\text{noise}}(\boldsymbol{r} - \boldsymbol{s}_k) \\
&= \arg\max_k \frac{1}{(\sqrt{\pi N_o})^N} \exp\left(-\frac{\|\boldsymbol{r} - \boldsymbol{s}_k\|^2}{N_o}\right) \\
&= \arg\min_k \|\boldsymbol{r} - \boldsymbol{s}_k\|^2
\end{aligned}$$

ML Detection Rule for Vector Gaussian Channel

Choose $\hat{m} = k$ if $\|\boldsymbol{r} - \boldsymbol{s}_k\| < \|\boldsymbol{r} - \boldsymbol{s}_j\|$ for every $j \neq k$

- Also called minimum distance/nearest neighbor decoding
- In case of a tie, choose one of the contenders arbitrarily.

Example: $M = 6$ vectors in $\mathbb{R}^2$

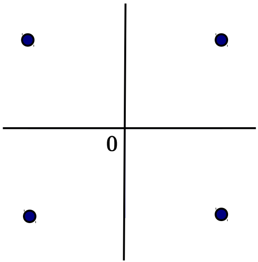


The k^{th} Decision region D_k

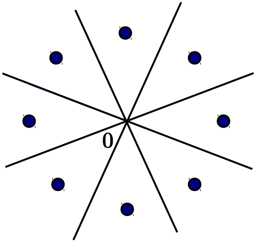
$$
\begin{aligned}
D_k &= \text{set of all points closer to } \boldsymbol{s}_k \text{ than any other } \boldsymbol{s}_j \\
&= \left\{ \boldsymbol{r} \in \mathbb{R}^N \mid \|\boldsymbol{r} - \boldsymbol{s}_k\| < \|\boldsymbol{r} - \boldsymbol{s}_j\| \text{ for all } j \neq k \right\}
\end{aligned}
$$

The ML detector outputs $\hat{m} = k$ if $\boldsymbol{r} \in D_k$.

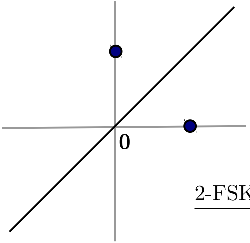
Examples in $\mathbb{R}^2$



4-QAM (Quadrature Amplitude Modulation)



8-PSK (Phase Shift Keying)



2-FSK (Frequency Shift Keying)

Error Probability when $M = 2$

Scenario

Let $\{\boldsymbol{s}_1, \boldsymbol{s}_2\} \subset \mathbb{R}^N$ be a binary modulation scheme with

- $P(\boldsymbol{s}_1) = P(\boldsymbol{s}_2) = 1/2$, and
- detected using the nearest neighbor decoder

- Error $\mathcal{E}$ occurs if $(\boldsymbol{s}_1 \text{ tx}, \hat{m} = 2)$ or $(\boldsymbol{s}_2 \text{ tx}, \hat{m} = 1)$
- Conditional error probability

$$P(\mathcal{E}|\boldsymbol{s}_1) = P\left(\hat{m} = 2|\boldsymbol{s}_1\right) = P\left(\|\boldsymbol{r} - \boldsymbol{s}_2\| < \|\boldsymbol{r} - \boldsymbol{s}_1\| \,|\, \boldsymbol{s}_1\right)$$

- Note that
$$P(\mathcal{E}) = P(\boldsymbol{s}_1)P(\mathcal{E}|\boldsymbol{s}_1) + P(\boldsymbol{s}_2)P(\mathcal{E}|\boldsymbol{s}_2) = \frac{P(\mathcal{E}|\boldsymbol{s}_1) + P(\mathcal{E}|\boldsymbol{s}_2)}{2}$$
- $P(\mathcal{E}|\boldsymbol{s}_i)$ can be easy to analyse

Conditional Error Probability when $M = 2$

$\mathcal{E}|\boldsymbol{s}_1$: $\boldsymbol{s}_1$ is transmitted $\boldsymbol{r} = \boldsymbol{s}_1 + \boldsymbol{n}$, and $\|\boldsymbol{r} - \boldsymbol{s}_1\|^2 > \|\boldsymbol{r} - \boldsymbol{s}_2\|^2$

$$
\begin{aligned}
(\mathcal{E}|\boldsymbol{s}_1) : &\|\boldsymbol{s}_1 + \boldsymbol{n} - \boldsymbol{s}_1\|^2 > \|\boldsymbol{s}_1 + \boldsymbol{n} - \boldsymbol{s}_2\|^2 \\
&\Leftrightarrow \|\boldsymbol{n}\|^2 > \langle \boldsymbol{s}_1 - \boldsymbol{s}_2 + \boldsymbol{n}, \boldsymbol{s}_1 - \boldsymbol{s}_2 + \boldsymbol{n} \rangle \\
&\Leftrightarrow \|\boldsymbol{n}\|^2 > \langle \boldsymbol{s}_1 - \boldsymbol{s}_2, \boldsymbol{s}_1 - \boldsymbol{s}_2 \rangle + \langle \boldsymbol{s}_1 - \boldsymbol{s}_2, \boldsymbol{n} \rangle + \langle \boldsymbol{n}, \boldsymbol{s}_1 - \boldsymbol{s}_2 \rangle + \langle \boldsymbol{n}, \boldsymbol{n} \rangle \\
&\Leftrightarrow \|\boldsymbol{n}\|^2 > \|\boldsymbol{s}_1 - \boldsymbol{s}_2\|^2 + 2\langle \boldsymbol{n}, \boldsymbol{s}_1 - \boldsymbol{s}_2 \rangle + \|\boldsymbol{n}\|^2 \\
&\Leftrightarrow \langle \boldsymbol{n}, \boldsymbol{s}_1 - \boldsymbol{s}_2 \rangle < -\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|^2}{2} \\
&\Leftrightarrow \left\langle \boldsymbol{n}, \frac{\boldsymbol{s}_1 - \boldsymbol{s}_2}{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|} \cdot \sqrt{\frac{2}{N_o}} \right\rangle < -\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|^2}{2} \cdot \frac{1}{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|} \cdot \sqrt{\frac{2}{N_o}} \\
&\Leftrightarrow \left\langle \boldsymbol{n}, \frac{\boldsymbol{s}_1 - \boldsymbol{s}_2}{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|} \cdot \sqrt{\frac{2}{N_o}} \right\rangle < -\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|}{\sqrt{2N_o}}
\end{aligned}
$$

Error Probability when $M = 2$

- $Z = \left\langle \boldsymbol{n}, \frac{\boldsymbol{s}_1 - \boldsymbol{s}_2}{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|} \cdot \sqrt{\frac{2}{N_o}} \right\rangle$ is Gaussian with zero mean and variance

$$\frac{N_o}{2} \left\| \frac{\boldsymbol{s}_1 - \boldsymbol{s}_2}{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|} \cdot \sqrt{\frac{2}{N_o}} \right\|^2 = \frac{N_o}{2} \cdot \frac{2}{N_o} \left\| \frac{\boldsymbol{s}_1 - \boldsymbol{s}_2}{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|} \right\|^2 = 1$$

- $P(\mathcal{E}|\boldsymbol{s}_1) = P\left(Z < -\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|}{\sqrt{2N_o}}\right) = Q\left(\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|}{\sqrt{2N_o}}\right)$
- $P(\mathcal{E}|\boldsymbol{s}_2) = Q\left(\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|}{\sqrt{2N_o}}\right)$

$$\boxed{P(\mathcal{E}) = \frac{P(\mathcal{E}|\boldsymbol{s}_1) + P(\mathcal{E}|\boldsymbol{s}_2)}{2} = Q\left(\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|}{\sqrt{2N_o}}\right)}$$

- Error probability decreasing function of distance $\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|$

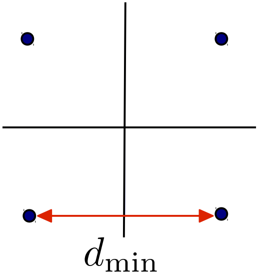
Bound on Error Probability when $M > 2$

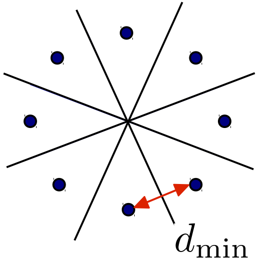
Scenario

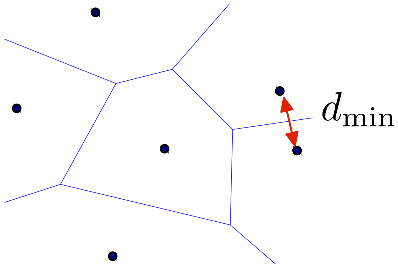
Let $\mathscr{C} = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_M\} \subset \mathbb{R}^N$ be a modulation/coding scheme with

- $P(\boldsymbol{s}_1) = \cdots = P(\boldsymbol{s}_M) = 1/M$, and
- detected using the nearest neighbor decoder

- Minimum distance
  $d_{\min}$ = smallest Euclidean distance between any pair of vectors in $\mathscr{C}$
$$d_{\min} = \min_{i \neq j} \|\boldsymbol{s}_i - \boldsymbol{s}_j\|$$
- Observe that $\|\boldsymbol{s}_i - \boldsymbol{s}_j\| \geq d_{\min}$ for any $i \neq j$
- Since $Q(\cdot)$ is a decreasing function
$$Q\left(\frac{\|\boldsymbol{s}_i - \boldsymbol{s}_j\|}{\sqrt{2N_o}}\right) \leq Q\left(\frac{d_{\min}}{\sqrt{2N_o}}\right) \text{ for any } i \neq j$$
- Bound based only on $d_{\min} \Rightarrow$ Simple calculations, not tight, intuitive


$d_{\min}$


$d_{\min}$


$d_{\min}$

Union Bound on Conditional Error Probability

Assume that $\boldsymbol{s}_1$ is transmitted, i.e., $\boldsymbol{r} = \boldsymbol{s}_1 + \boldsymbol{n}$. We know that

$$P(\, \|\boldsymbol{r} - \boldsymbol{s}_j\| < \|\boldsymbol{r} - \boldsymbol{s}_j\| \mid \boldsymbol{s}_1 \,) = Q\left(\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_j\|}{\sqrt{2N_o}} \right)$$

Decoding error occurs if $\boldsymbol{r}$ is closer some $\boldsymbol{s}_j$ than $\boldsymbol{s}_1$, $j = 2, 3, \ldots, M$

$$P(\mathcal{E}|\boldsymbol{s}_1) = P(\boldsymbol{r} \notin D_1 \,|\, \boldsymbol{s}_1) = P\left(\bigcup_{j=2}^{M} \|\boldsymbol{r} - \boldsymbol{s}_j\| < \|\boldsymbol{r} - \boldsymbol{s}_1\| \mid \boldsymbol{s}_1 \right)$$

From union bound $P(A_2 \cup \cdots \cup A_M) \leq P(A_2) + \cdots + P(A_M)$

$$P(\mathcal{E}|\boldsymbol{s}_1) \leq \sum_{j=2}^{M} P(\, \|\boldsymbol{r} - \boldsymbol{s}_j\| < \|\boldsymbol{r} - \boldsymbol{s}_j\| \mid \boldsymbol{s}_1 \,) = \sum_{j=2}^{M} Q\left(\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_j\|}{\sqrt{2N_o}} \right)$$

Union Bound on Error Probability

Since $Q(\cdot)$ is a decreasing function and $\|\boldsymbol{s}_1 - \boldsymbol{s}_j\| \geq d_{\min}$

$$P(\mathcal{E}|\boldsymbol{s}_1) \leq \sum_{j=2}^{M} Q\left(\frac{\|\boldsymbol{s}_1 - \boldsymbol{s}_j\|}{\sqrt{2N_o}}\right) \leq \sum_{j=2}^{M} Q\left(\frac{d_{\min}}{\sqrt{2N_o}}\right)$$

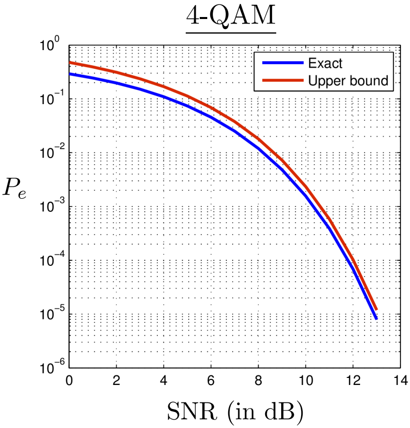
$$\boxed{P(\mathcal{E}|\boldsymbol{s}_1) \leq (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_o}}\right)}$$

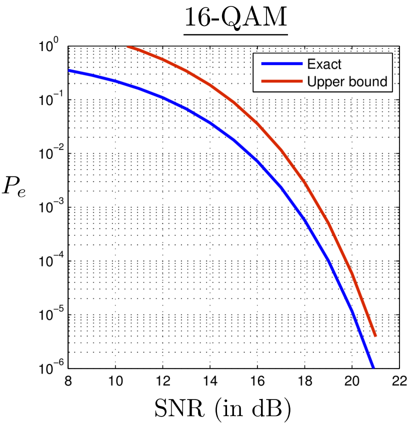
Upper bound on average error probability $P(\mathcal{E}) = \sum_{i=1}^{M} P(\boldsymbol{s}_i)P(\mathcal{E}|\boldsymbol{s}_i)$

$$\boxed{P(\mathcal{E}) \leq (M-1)Q\left(\frac{d_{\min}}{\sqrt{2N_o}}\right)}$$

Note

- Exact P_e (or good approximations better than the union bound) can be derived for several constellations, for example PAM, QAM and PSK.
- Chernoff bound can be useful: $Q(a) \leq \frac{1}{2}\exp(-a^2/2)$ for $a \geq 0$
- Union bound, in general, is loose.

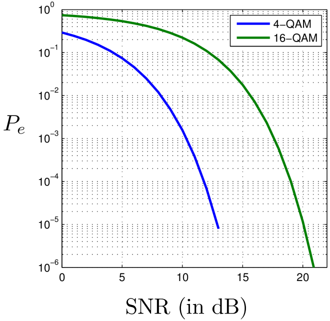




- Abscissa is $[\mathrm{SNR}]_{\mathrm{dB}} = 10\log_{10}\mathrm{SNR}$
- The union bound is a reasonable approximation for large values of SNR

Performance of QAM and FSK

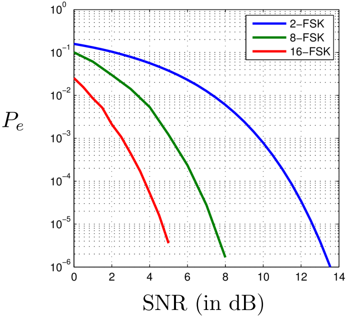
M-QAM



$$N = 2$$

$$\eta = \log_2 M$$

M-FSK



$$N = M$$

$$\eta = \frac{2 \log_2 M}{M}$$

$$\eta = \frac{2 \log_2 M}{N}$$

Performance of QAM and FSK

Probability of Error $P_e = 10^{-5}$

Modulation/Code	Spectral Efficiency η (bits/sec/Hz)	Signal-to-Noise Ratio SNR (dB)
16-QAM	4	20
4-QAM	2	13
2-FSK	1	12.6
8-FSK	$^3/_4$	7.5
16-FSK	$^1/_2$	4.6

How good are these modulation schemes ?
What is the best trade-off between SNR and η ?

Capacity of the (Vector) Gaussian Channel

Let the maximum allowable power be P and noise variance be $N_o/2$.

$$\mathrm{SNR} = \frac{P}{N_o/2} = \frac{2P}{N_o}$$

What is the highest η achievable while ensuring that P_e is small?

Theorem

Given an $\epsilon > 0$ and any constant η such that $\eta < \log_2(1 + \mathrm{SNR})$, there exists a coding scheme with $P_e \leq \epsilon$ and spectral efficiency at least η.

Conversely, for any coding scheme with $\eta > \log_2(1 + \mathrm{SNR})$ and M sufficiently large, P_e is close to 1.

$C(\mathrm{SNR}) = \log_2(1 + \mathrm{SNR})$ is the **capacity** of the Gaussian channel.

How Good/Bad are QAM and FSK?

Least SNR required to communicate reliably with spectral efficiency η is

$$\mathrm{SNR}^*(\eta) = 2^\eta - 1$$

Probability of Error $P_e = 10^{-5}$

Modulation/Code	η	SNR (dB)	$\mathrm{SNR}^*(\eta)$
16-QAM	4	20	11.7
4-QAM	2	13	4.7
2-FSK	1	12.6	0
8-FSK	$3/4$	7.5	-1.7
16-FSK	$1/2$	4.6	-3.8

How to Perform Close to Capacity?

- We need P_e to be small at a fixed finite SNR
  - $d_{\min}$ must be large to ensure that P_e is small
- It is necessary to use coding schemes in high dimensions $N \gg 1$
  - Can ensure that $d_{\min} \approx$ constant $\times \sqrt{N}$
- If N is large it is possible to 'pack' vectors $\{\boldsymbol{s}_i\}$ in $\mathbb{R}^N$ such that
  - Average power is at the most P
  - $d_{\min}$ is large
  - η is close to $\log_2(1 + \mathrm{SNR})$
  - P_e is small
- A large N implies that $M = 2^{\eta N/2}$ is also large.
  - We must ensure that such a large code can be encoded/decoded with practical complexity

Several known coding techniques

$\eta > 1$: Trellis coded modulation, multilevel codes, lattice codes, bit-interleaved coded modulation, etc.

$\eta < 1$: Low-density parity-check codes, turbo codes, polar codes, etc.

Thank You!