

T-subnorms with strong associated negation : Some Properties

Balasubramaniam Jayaram
Department of Mathematics,
Indian Institute of Technology Hyderabad,
Yeddumailaram - 502 205, INDIA
jbala@iith.ac.in

Summary

In this work we investigate t-subnorms M that have strong associated negation. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeaness, conditional cancellativity, left-continuity, nilpotent elements, etc. In particular, we show that under this setting many of these properties are equivalent.

Keywords: T-norms, t-subnorms, Archimedeaness, conditional cancellativity, left-continuity, residual implications.

1 Introduction

The theory of triangular norms and triangular subnorms have been well studied and their applications well-established. Many algebraic and analytical properties of these operations, viz., Archimedeaness, conditional cancellativity, left-continuity, etc., have been studied and their inter-relationships shown (see for instance, KLEMENT et al. [4]).

Yet another way of categorizing t-subnorms is as follows: Given a t-subnorm M , one can obtain its associated negation n_M (see Definitions 2.2 and 2.4 below). Note that n_M is usually not a fuzzy negation, i.e., $n_M(1) \geq 0$. However, we can broadly consider two sub-classes of t-subnorms based on whether their associated negation n_M is strong or not.

In this work, we study the class of t-subnorms whose associated negation n_M is strong. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate some particular classes of these

and study the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeaness, conditional cancellativity, left-continuity, etc. In particular, we show that under this setting many of these properties are equivalent.

2 Preliminaries

Definition 2.1. A fuzzy negation is a function $N: [0, 1] \rightarrow [0, 1]$ that is non-increasing and such that $N(1) = 0$ and $N(0) = 1$. Further, it is said to be strong or involutive, if $N \circ N = id_{[0,1]}$.

Definition 2.2. A t-subnorm is a function $M: [0, 1]^2 \rightarrow [0, 1]$ such that it is monotonic non-decreasing, associative, commutative and $M(x, y) \leq \min(x, y)$ for all $x, y \in [0, 1]$, i.e., 1 need not be the neutral element.

Definition 2.3. Let M be a t-subnorm.

(i) If 1 is the neutral element of M , then it becomes a t-norm. We denote a t-norm by T in the sequel.

(ii) M is said to satisfy the Conditional Cancellation Law if, for any $x, y, z \in (0, 1]$,

$$M(x, y) = M(x, z) > 0 \text{ implies } y = z. \quad (\text{CCL})$$

Alternately, (CCL) implies that on the positive domain of M , i.e., on the set $\{(x, y) \in (0, 1]^2 \mid M(x, y) > 0\}$, M is strictly increasing.

(iii) M is said to be Archimedean, if for all $x, y \in (0, 1)$ there exists an $n \in \mathbb{N}$ such that $x_M^{[n]} < y$;

(iv) An element $x \in (0, 1)$ is a nilpotent element of M if there exists an $n \in \mathbb{N}$ such that $x_M^{[n]} = 0$;

(v) A t-norm T is said to be nilpotent, if it is continuous and if each $x \in (0, 1)$ is a nilpotent element of T .

Definition 2.4. Let M be any t-subnorm and $x, y \in [0, 1]$.

- The residual implication I_M of M is given by

$$I(x, y) = \sup \{t \in [0, 1] \mid M(x, t) \leq y\}. \quad (1)$$

- The associated negation n_M of M is given by

$$n_M(x) = \sup \{t \in [0, 1] \mid M(x, t) = 0\}. \quad (2)$$

Clearly, n_M is a non-increasing function. Note that though $n_M(0) = 1$, it need not be a fuzzy negation, since $n_M(1)$ can be greater than 0. Hence, only in the case n_M is a fuzzy negation we call n_M the *natural negation* of M in this work. However, many results hold even if $n_M(1) > 0$ and hence to preserve this generality in such situations we term n_M as the *associated negation*.

For instance, the following result is true even when $n_M(1) > 0$.

Lemma 2.5 (cf. [1], Proposition 2.3.4). *Let M be any t -subnorm and n_M its associated negation. Then we have the following:*

- (i) $M(x, y) = 0 \implies y \leq n_M(x)$.
- (ii) $y < n_M(x) \implies M(x, y) = 0$.
- (iii) If M is left-continuous then $y = n_M(x) \implies M(x, y) = 0$, i.e., the reverse implication of (i) also holds.

Lemma 2.6. *Let M be any t -subnorm with n_M being a natural negation with e as its fixed point, i.e., $n_M(e) = e$. Then*

- (i) Every $x \in (0, e)$ is a nilpotent element; in fact, $x_M^{[2]} = 0$ for all $x \in [0, e)$.
- (ii) In addition, if M is either conditionally cancellative or left-continuous, then e is also a nilpotent element.

Proof. (i) By definition,

$$n_M(e) = \sup \{t \in [0, 1] \mid M(e, t) = 0\} = e,$$

implies that $M(e, e^-) = 0$, from whence we get $M(x, x) \leq M(e, e^-) = 0$ for all $x \in [0, e)$. In other words, $x_M^{[2]} = 0$ for all $x \in [0, e)$.

- (ii) If M is conditionally cancellative, then $M(e, e) = x < e$ and from (ii) above we have $M(x, x) = 0$. Now,

$$e_M^{[4]} = M(M(e, e), M(e, e)) = M(x, x) = 0.$$

If M is left-continuous, then $n_M(e) = \max \{t \in [0, 1] \mid M(e, t) = 0\} = e$, i.e., $e \in \{t \in [0, 1] \mid M(e, t) = 0\}$ and hence $M(e, e) = 0$, i.e., e is also a nilpotent element. \square

Remark 2.7. (i) In the case n_M is a strong natural negation we can show that if M is conditionally cancellative then every $x \in (0, 1)$ is also a nilpotent element, see Remark 5.8(ii).

- (ii) Note that without any further assumptions, the set of nilpotent elements need not be the whole of $(0, 1)$. For instance, for the nilpotent minimum t -norm

$$T_{\mathbf{NM}}(x, y) = \begin{cases} 0, & \text{if } x + y \leq 1, \\ \min(x, y), & \text{otherwise,} \end{cases} \quad x, y \in [0, 1],$$

which is left-continuous but not conditionally cancellative, its set of nilpotent elements is $(0, .5]$, while its set of zero divisors is $(0, 1)$.

However, Theorem 6.1 gives an equivalence condition for the whole of $(0, 1)$ to be the set of nilpotent elements under a suitable condition on n_M .

3 T-subnorms with strong associated negation = T-norms

There are works showing that some classes of t -subnorms M whose associated negations n_M are involutive do become t -norms. Jenei [3] showed it for the class of left-continuous M , while Jayaram [2] did the same for conditionally cancellative M . The main result of this section shows that the above results are true in general, i.e., any t -subnorm with a strong natural negation is a t -norm.

The following result was firstly proven by Jenei in [3]. However, we give a very simple proof of this result without resorting to the rotation-invariance property.

Theorem 3.1 (Jenei, [3], Theorem 3). *If M is a left-continuous t -subnorm with n_M being strong, then M is a t -norm.*

Proof. Firstly, note that if M is a left-continuous t -subnorm, then its residual satisfies the exchange principle, i.e.,

$$I_M(x, I_M(y, z)) = I_M(y, I_M(x, z)).$$

It follows from the fact that the neutral element of M does not play any role in the proof, see, for instance the proof given for Theorem 2.5.7 in [1].

If n_M is strong, then for every $y \in [0, 1]$ there exists $y' \in [0, 1]$ such that $n_M(y) = y'$. Now,

$$\begin{aligned} I_M(1, y') &= I_M(1, I_M(y, 0)) \\ &= I_M(y, I_M(1, 0)) = I_M(y, 0) = y'. \end{aligned}$$

Thus, for all $y' \in [0, 1]$,

$$I_M(1, y') = \max \{t \mid M(1, t) \leq y'\} = y' \implies M(1, y') = y'. \quad \square$$

Theorem 3.2 (Jayaram [2], Theorem 4). *Let M be any conditionally cancellative t -subnorm. If n_M is a strong natural negation then M is a t -norm.*

Now, we prove the main result of this section which shows that the above results are true in general.

Theorem 3.3. *Let M be any t -subnorm with n_M being a strong natural negation. M is a t -norm.*

Proof. Note, firstly, that since $n_M(x) = \sup\{t \in [0, 1] \mid M(x, t) = 0\}$, is a strong negation, we have that $n_M(z) = 1 \iff z = 0$ and $n_M(z) = 0 \iff z = 1$. Equivalently, $M(1, z) = 0 \iff z = 0$.

On the contrary, let us assume that $M(1, x) = x' \lesssim x$ for some $x \in (0, 1]$. Since n_M is strong, the following are true:

- (i) $n_M(x') > n_M(x)$
- (ii) if $p > n_M(x)$ then $M(x, p) > 0$,
- (iii) there exists a $y \in (0, 1)$ such that $n_M(x') > y > n_M(x)$ and $M(y, x) = q > 0$ while $M(y, x') = 0$.

Now, by associativity we have

$$\left. \begin{array}{l} M(y, M(x, 1)) = M(y, x') = 0 \\ M(M(y, x), 1) = M(q, 1) \end{array} \right\} \implies M(q, 1) = 0,$$

a contradiction. Thus $M(1, x) = x$ for **all** $x \in [0, 1]$ and hence we have the result. \square

In the following sections, we deal with t -subnorms whose associated negations are strong, or equivalently t -norms whose associated negations are strong. We discuss the inter-relationships between the different algebraic and analytical properties for this subclass of t -norms; in particular, Archimedeaness, Conditional Cancellativity, (Left-)continuity and Nilpotence that are relevant to our context. We begin with listing out some established results and go on to present some new ones.

4 Continuity and Nilpotence

Let T be a t -norm and n_T a strong negation. The following result, whose proof is straight-forward, shows the equivalence between continuity and nilpotence:

Theorem 4.1 (KLEMENT et al. [4]). *Let T be a t -norm with n_T being strong. Then the following are equivalent:*

- (i) T is continuous.
- (ii) T is a nilpotent t -norm.

Further, we know that every nilpotent t -norm is both Archimedean and Conditionally cancellative, since every nilpotent t -norm is isomorphic to the Łukasiewicz t -norm and Archimedeaness and Conditionally cancellativity are preserved under isomorphism, see [4], Examples 2.14(iv) and 2.15(v). Trivially, every nilpotent t -norm is also left-continuous.

5 Conditional Cancellativity, Left Continuity and Nilpotence

Recently, in Jayaram [2], the following problem of U.Höhle, given in KLEMENT et al. [5] has been solved. Further it was shown that it characterizes the set of all conditionally cancellative t -subnorms.

(U.Höhle, [5], Problem 11) Characterize all left-continuous t -norms T which satisfy

$$I_T(x, T(x, y)) = \max(n_T(x), y), \quad x, y \in [0, 1]. \quad (3)$$

where I_T, n_T are as given in (1) and (2) with $M = T$.

Theorem 5.1 (Jayaram [2], Theorem 1). *Let M be any t -subnorm, not necessarily left-continuous. Then the following are equivalent:*

- (i) The adjoint pair (I, M) satisfies (3).
- (ii) M is a Conditionally Cancellative t -subnorm.

Remark 5.2. *The following statements follow from Theorem 5.1 with $M = T$, a t -norm:*

- (i) If a (right) continuous T satisfies (3) along with its residual then T is necessarily Archimedean, see [4], Proposition 2.15(ii).
- (ii) However, if a left-continuous T satisfies (3) along with its residual then T need not be Archimedean and hence not continuous. An example is Hajék's t -norm or the following t -norm $T_{\mathbf{OY}}$ of Ouyang et al [7], Example 3.4, which is a (CCL) t -norm (and hence a t -subnorm too) that is left-continuous but not continuous at $(0.5, 0.5)$ and hence is not Archimedean (see Figure 1(a)):

$$T_{\mathbf{OY}}(x, y) = \begin{cases} 2(x - 0.5)(y - 0.5) + 0.5, & \text{if } (x, y) \in (0.5, 1]^2 \\ 2y(x - 0.5), & \text{if } (x, y) \in (0.5, 1] \times [0, 0.5] \\ 2x(y - 0.5), & \text{if } (x, y) \in [0, 0.5] \times (0.5, 1] \\ 0, & \text{otherwise} \end{cases}.$$

Theorem 5.3 (Jenei, [3], Theorem 2). *Let T be a left-continuous t -norm with n_T being strong. Then the following are equivalent:*

- (i) T is a conditionally cancellative t -norm.
- (ii) T is a nilpotent t -norm.

In fact, for a conditionally cancellative t -subnorm M we can give an equivalent condition for it to be left-continuous.

Theorem 5.4. *Let M be a (CCL) t -subnorm. Then the following are equivalent:*

- (i) $M(x, n_M(x)) = 0, \quad x \in [0, 1]$.
- (ii) M is left-continuous.

Proof. (i) \implies (ii): Let $M(x, n_M(x)) = 0$, for all $x \in [0, 1]$. On the contrary, let us assume that M is non-left-continuous. Then there exist $x_0 \in (0, 1)$, $y_0 \in (0, 1)$ and an increasing sequence $(x_n)_{n \in \mathbb{N}}$, where $x_n \in [0, 1)$, such that $\lim_{n \rightarrow \infty} x_n = x_0$, but

$$\lim_{n \rightarrow \infty} M(x_n, y_0) = M(x_0^-, y_0) = z' < z_0 = M(x_0, y_0).$$

Observe that

$$I_M(y_0, z') = \sup\{t \in [0, 1] \mid M(y_0, t) \leq z'\} = x_0, \quad (4)$$

since from the monotonicity of M we have $M(y_0, x_n) \leq z'$ for every $n \in \mathbb{N}$ and $M(y_0, x_0) = z_0 > z'$. Since M is (CCL), we have

$$I_M(y_0, z') = I_M(y_0, M(y_0, x_0^-)) = \max(n(y_0), x_0^-).$$

Now, we have two cases. On the one hand, if $I_M(y_0, z') = x_0^- \leq x_0$, then it is a contradiction to (4). On the other hand, if $I_M(y_0, z') = n(y_0)$, then this implies that $n(y_0) = x_0$ from (4) and hence

$$M(x_0, y_0) = M(y_0, n(y_0)) = z_0 = 0,$$

by the hypothesis and hence there does not exist any $z' < z_0$ and hence M is left-continuous.

- (ii) \implies (i): Follows from Lemma 2.5(iii). □

Remark 5.5. *In Theorem 5.4 we do not need n_M to be a negation, i.e., $n_M(1) \geq 0$. Consider the following t -subnorm $M_{\mathbf{P}_f}$ (cf. Example 3.15 of [4], see Figure 1(b)) which is a left-continuous (CCL) but $n_{M_{\mathbf{P}_f}}$ is not a negation since $n_{M_{\mathbf{P}_f}}(1) = 0.2$*

$$M_{\mathbf{P}_f} = \begin{cases} 0.2 + \frac{3(x-0.2)(y-0.2)}{4}, & \text{if } (x, y) \in (0.2, 1]^2 \\ 0, & \text{otherwise} \end{cases}.$$

Theorem 5.6. *Let M be a (CCL) t -subnorm whose n_M is strong. Then M is left-continuous.*

Proof. If possible, let $M(x_0, n(x_0)) = p > 0$ for some $x_0 \in (0, 1)$. Since M is (CCL), we have $M(1^-, x_0) < x_0$ and hence by associativity we have

$$\begin{aligned} M(1^-, M(x_0, n(x_0))) &= M(1^-, p) \\ M(M(1^-, x_0), n(x_0)) &= 0 \end{aligned}$$

from whence it follows $M(1^-, p) = 0$, i.e., $n(p) = 1$, a contradiction to the fact that n_M is strong. Thus $p = 0$ and the result follows from Theorem 5.4. □

Theorem 5.7. *Let M be a t -subnorm such that n_M is strong. Then the following are equivalent:*

- (i) M is conditionally cancellative.
- (ii) M is a nilpotent t -norm.

Proof. If M satisfies (CCL) then M is left-continuous, from Theorem 5.6 and now, using Theorem 5.3 we have the result. □

Remark 5.8. (i) *The nilpotent minimum t -norm $T_{\mathbf{nM}}$ is an example of a t -subnorm M whose n_M is involutive and M satisfies (LEM) with n_M but is not conditionally cancellative and hence is not a nilpotent t -norm.*

- (ii) *In the case n_M is a strong natural negation, from Theorem 5.6 we see that conditionally cancellativity is equivalent to left-continuity and from Theorem 5.7 that every $x \in (0, 1)$ is a nilpotent element.*

6 Archimedeaness , Left Continuity and Nilpotence

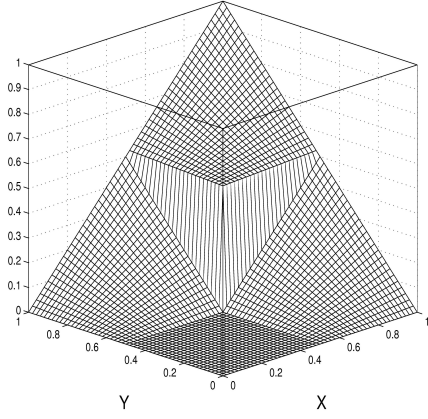
We begin with a result that shows that if n_M is strong, then the Archimedeaness is equivalent to every element $x \in (0, 1)$ being nilpotent. However, unless M is also left-continuous, M is not a nilpotent t -norm.

Theorem 6.1. *Let M be any t -subnorm such that n_M is not completely vanishing, i.e., there exists $z \in (0, 1)$ such that $n_M(z) > 0$. The following are equivalent:*

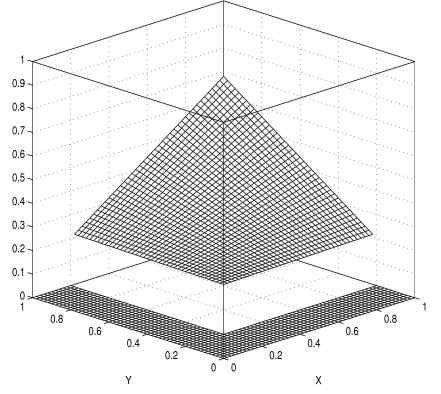
- (i) Every $x \in (0, 1)$ is a nilpotent element.
- (ii) M is Archimedean.

Proof. (i) \implies (ii): Follows from Proposition 2.15 (iv) in KLEMENT et al. [4].

- (ii) \implies (i): Let M be any Archimedean t -subnorm such that n_M is not completely vanishing, i.e., there exist $z \in (0, 1)$ such that $n_M(z) > 0$. By Lemma 2.5(ii) we see that for any $0 < z' < n_M(z)$ we have $M(z', z) = 0$.



(a) T_{OY}



(b) M_{P_f}

Figure 1: A t-norm and a t-subnorm that are conditionally cancellative

For any $x \in [0, 1)$, by the Archimedeaness of M , there exists an $n, p \in \mathbb{N}$ such that $x_M^{[n]} < z'$ and $x_M^{[p]} < z$ from whence we have that

$$x_M^{[n+p]} = M(x_M^{[n]}, x_M^{[p]}) \leq M(z', z) = 0.$$

□

Corollary 6.2. *Let M be any t-subnorm such that n_M is a strong negation. Then the following are equivalent:*

- (i) *Every $x \in (0, 1)$ is a nilpotent element.*
- (ii) *M is Archimedean.*

The following result is due to Kolesárová [6]:

Theorem 6.3. *Let T be any Archimedean t-norm. Then the following are equivalent:*

- (i) *T is left-continuous.*
- (ii) *T is continuous.*

Corollary 6.4. *A left-continuous Archimedean t-subnorm M whose n_M is involutive is a nilpotent t-norm.*

Proof. From Theorem 3.1 we see that M is a left-continuous t-norm. From Theorem 6.3, since M is Archimedean it is continuous. Also by Theorem 6.1, we have that every $x \in (0, 1)$ is a nilpotent element. Thus T is nilpotent, i.e., isomorphic to T_L . □

Remark 6.5. (i) *Note that there exist left-continuous Archimedean t-subnorms M that*

are not continuous and hence their n_M is not involutive. For instance, consider the t-subnorm

$$M(x, y) = \begin{cases} x + y - 1, & \text{if } x + y > \frac{3}{2}, \\ 0, & \text{otherwise,} \end{cases} \quad x, y \in [0, 1].$$

- (ii) *The nilpotent minimum t-norm T_{\min} is an example of a left-continuous t-subnorm M whose n_M is involutive but is not Archimedean and hence is not a nilpotent t-norm.*
- (iii) *However, it is not clear whether there exists any non-nilpotent Archimedean t-subnorm M whose n_M is involutive. Clearly such t-(sub)norms are not left-continuous.*

Problem 1. Does there exist any non-nilpotent Archimedean t-subnorm M whose n_M is involutive. In other words, is an Archimedean t-subnorm M whose n_M is involutive necessarily left-continuous?

7 Archimedeaness and Conditional Cancellativity

In general, there does not exist any inter-relationships between Archimedeaness and conditional cancellativity, as the following examples show.

Example 7.1. (i) *The Ouyang t-norm T_{OY} is an example of a t-(sub)norm which is not Archimedean but is both left-continuous and conditionally cancellative.*

- (ii) *The following t-norm is neither Archimedean nor left-continuous but is conditionally cancellative:*

$$T(x, y) = \begin{cases} 0, & \text{if } xy \leq \frac{1}{2} \& \max(x, y) < 1 \\ xy, & \text{if } xy > \frac{1}{2} \\ \min(x, y), & \text{otherwise} \end{cases}.$$

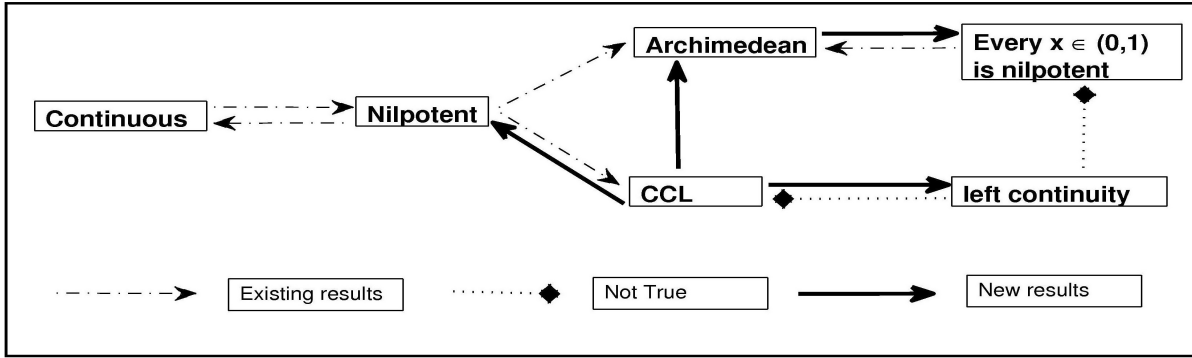


Figure 2: A Summary of the results available so far when n_T is strong

(iii) The following t -subnorm is Archimedean and continuous, but not conditionally cancellative:

$$M(x, y) = \max(0, \min(x + y - 1, x - a, y - a, 1 - 2a)),$$

where $a \in (0, 0.5)$. For instance, with $a = 0.25$ we have $M(0.75, 0.75) = M(0.75, 0.8) = 0.5$.

(iv) The nilpotent minimum T_{nM} , whose n_M is strong, is neither Archimedean nor conditionally cancellative, but is left-continuous.

In the case, when n_M is strong we have the following partial implication.

Lemma 7.2. Let M be any t -subnorm whose n_M is strong. If M is conditionally cancellative then M is Archimedean.

Proof. From Theorem 5.7, we have that if M is conditionally cancellative then M is a nilpotent t -norm from whence it follows that M is Archimedean. \square

Problem 2. Does there exist any Archimedean t -subnorm M whose n_M is involutive but is not conditionally cancellative? In other words, is an Archimedean t -subnorm M whose n_M is involutive necessarily conditionally cancellative?

8 Concluding Remarks

In this work, we have shown that t -subnorms whose associated negations are strong are necessarily t -norms. Further, we have studied the inter-relationships between some algebraic and analytical properties of such t -(sub)norms. Figure 2 gives a pictorial summary of the results that exist so far.

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