



International Conference on Computational Heat and Mass Transfer-2015

Linear Stability of Double diffusive Convection of Hadley-Prats Flow with Viscous Dissipation in a Porous Media

N. Deepika^a, P. A. L. Narayana^a

^aDepartment of Mathematics, IIT Hyderabad, ODF Estate, Telangana, 502205, India

Abstract

Linear stability analysis of double diffusive convection in a horizontal fluid saturated porous layer has been carried out. Effects of viscous dissipation, horizontal mass flow are taken into account. Combined effects of horizontal mass flow, viscous dissipation may cause instability in the fluid system. To carry out linear stability analysis, disturbances are the form of longitudinal rolls, transverse rolls have been considered. In order to solve eigenvalue problem numerically, Runge-Kutta and shooting methods have been employed for the case of longitudinal rolls. Chebyshev-Tau method has been used for the case of transverse rolls. Critical wave number a_c , critical vertical thermal Rayleigh number Ra_c are evaluated for assigned values of flow governing parameters. For the onset of convection, physical explanation is given.

© 2015 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

Peer-review under responsibility of the organizing committee of ICCHMT – 2015

Keywords: Double diffusive convection - mass flow - viscous dissipation - porous medium

1. Introduction

It is noteworthy to study double diffusive convection in a fluid saturated porous media as it has several technical and geophysical applications. In the present article, linear stability analysis of double diffusive convection in a horizontal porous layer saturated with fluid has been carried out. Viscous heating contribution and horizontal mass flow are taken into account. Here the flow is induced by horizontal temperature and concentration gradients. It is an extension to the study of [1].

Heat convection in a horizontal porous layer induced by temperature differences between the boundaries was first studied by [2] and [3]. This particular problem is popularly known as Darcy- Bénard problem or Horton-Rogers-Lapwood problem. Heat convection in a porous layer with horizontal temperature gradients was studied by [4]. [5] investigated the horizontal mass flow effect on convection in a porous layer. All the improvements in this research area is incorporated in the books [6] and [7].

* Corresponding author. Tel.: 91 (40) 2301 7050

E-mail address: deepika.neela@gmail.com

Double diffusive convection can arise due to combined heat and mass transfer which are caused by buoyancy effects. Usually this type of convection arises in sea water flow, and mantle flow in the earth’s crust. Problem of double diffusive convection was studied in [8], [9], and [10]. In [11], nonlinear stability theory using energy method is developed to solve the problem of double diffusive convection.

Temperature profile becomes nonlinear when viscous dissipation is present. Article [12], was first to introduce dissipation term in the temperature equation and proved that for the case of natural convection, this effect becomes negligible. In general, most of the convection problems in a horizontal porous layer, imposed temperature differences between the boundaries, serves to cause the instability. Brinkman number measures the internal heat generation by the act of viscous dissipation to the temperature difference. Hence, viscous dissipation plays an important role for the problems with no external temperature difference across the plates, and it leads to an infinite Brinkman number. The viscous dissipation effect was investigated in many articles [13], [14], and [15].

2. Mathematical Formulation

A horizontal homogeneous porous layer which is saturated with fluid has been considered with characteristic length H . The Cartesian coordinates (x^*, y^*, z^*) , with z^* -axis vertically upwards and x^* -axis is in the net flow direction. Porous layer is bounded by horizontal plates $z^* = 0, z^* = H$. The vertical temperature difference is ΔT^* and concentration difference is ΔC^* . Boundary conditions at the plates are

$$T^*(x^*, y^*, 0, t^*) = T_0^* + \Delta T^* - \eta^* x^*, \quad C^*(x^*, y^*, 0, t^*) = C_0^* + \Delta C^* - \xi^* x^*,$$

$$T^*(x^*, y^*, H, t^*) = T_0^* - \eta^* x^*, \quad C^*(x^*, y^*, H, t^*) = C_0^* - \xi^* x^*, \tag{1}$$

where T^* be the temperature, C^* is the concentration, (x^*, y^*, z^*) are Cartesian coordinates. And η^* is uniform horizontal temperature gradient and ξ^* is concentration gradient. Dimensionless variables are defined as following

$$(x, y, z) = \frac{1}{H}(x^*, y^*, z^*), \quad \mathbf{v} = \frac{H}{\alpha} \mathbf{v}^*, \quad t = \frac{\alpha}{AH^2} t^*,$$

$$T = \frac{T^* - T_0^*}{\Delta T^*}, \quad C = \frac{C^* - C_0^*}{\Delta C^*}, \quad P = \frac{K}{\alpha\mu} P^*, \tag{2}$$

where t^* be the time, α be the thermal diffusivity, \mathbf{v}^* be the Darcy velocity, P^* be the pressure, K is the permeability of the porous medium, μ is the dynamic viscosity, and $A = \frac{(\rho c)_m}{(\rho c)_f}$ is the heat capacity ratio. Dimensionless parameters are defined as

$$Ge = \frac{\beta_T g H}{c}, \quad Le = \frac{\alpha}{D}, \quad Ra = \frac{\beta_T g \Delta T^* K H}{\nu \alpha},$$

$$Ra_H = \frac{\beta_T g \eta^* K H^2}{\nu \alpha}, \quad Sa = \frac{\beta_C g \Delta C^* K H}{\nu D}, \quad Sa_H = \frac{\beta_C g \xi^* K H^2}{\nu D},$$

where β_T and β_C are the thermal and solutal expansion coefficients, D is the mass diffusivity, ν is the kinematic viscosity, Ra and Ra_H are the vertical and horizontal thermal Rayleigh numbers and Sa and Sa_H are the vertical and horizontal solutal Rayleigh numbers, Ge is the Gebhart number, Le is the Lewis number. Non dimensional Governing equations are of the following form

$$\nabla \cdot \mathbf{v} = 0, \tag{3}$$

$$\mathbf{v} = -\nabla P + (RaT + \frac{1}{Le} SaC) \mathbf{k}, \tag{4}$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \nabla^2 T + \frac{Ge}{Ra} \mathbf{v} \cdot \mathbf{v}, \tag{5}$$

$$\frac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C = \frac{1}{Le} \nabla^2 C, \tag{6}$$

$$z = 0 : \quad w = 0, \quad T = 1 - \frac{Ra_H}{Ra} x, \quad C = 1 - \frac{Sa_H}{Sa} x,$$

$$z = 1 : \quad w = 0, \quad T = -\frac{Ra_H}{Ra}x, \quad C = -\frac{Sa_H}{Sa}x, \tag{7}$$

here \mathbf{k} is the unit vector. In the x -direction, it is assumed that a horizontal mass flow of with magnitude Pe i.e

$$\int_0^H u_s = Pe. \tag{8}$$

here u_s is basic state velocity in x -direction.

2.1. Steady-State solution

On evaluating the curl of Eq. (4), the following equations are obtained.

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \left(Ra \frac{\partial T}{\partial y} + \frac{1}{Le} Sa \frac{\partial C}{\partial y} \right), \tag{9}$$

$$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \left(Ra \frac{\partial T}{\partial x} + \frac{1}{Le} Sa \frac{\partial C}{\partial x} \right), \tag{10}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \tag{11}$$

By using boundary conditions Eq.(7), and from the Eqs.(3),(5),(6) and (9)-(11), the following steady state solution is obtained.

$$u_s(z) = \left(Ra_H + \frac{1}{Le} Sa_H \right) \left(z - \frac{1}{2} \right) + Pe, \quad v_s(z) = 0, \quad w_s(z) = 0, \tag{12}$$

$$T_s(x, z) = 1 - \frac{Ra_H}{Ra}x - z + \frac{(1-z)z}{24Ra} [12GePe^2 - 4GePeX + GeX^2 + 12PeRa_H - 2XRa_H + 2zX(4GePe - GeX + 2Ra_H) + 2z^2GeX^2], \tag{13}$$

$$C_s(x, z) = 1 - \frac{Sa_H}{Sa}x - z - \frac{(z-1)z}{2} \frac{LePeSa_H}{Sa} + Le \frac{Sa_H}{Sa} X \left(\frac{z^2}{4} - \frac{z^3}{6} + \frac{z}{12} \right), \tag{14}$$

where $X = Ra_H + \frac{1}{Le} Sa_H$.

In Eq. (12), basic state velocity in the x -direction which varying linearly with z represent superposition of Hadley flow and uniform flow. Eq.(13) and (14) indicates temperature and concentration fields in the fluid system varying in x, z directions. But when horizontal Rayleigh numbers Ra_H, Sa_H goes to zero, temperature and concentration gradients depends only on z .

In this article, Pe is the strength of the velocity in flow direction, Ge is measure of viscous dissipation effect. Ra, Ra_H , show buoyancy effects by cause of temperature differences whereas Sa, Sa_H represent buoyancy effects as a result of concentration differences.

2.2. Perturbation Analysis

In order to study linear stability analysis, small perturbations viz. $\mathbf{V}(x, y, z, t), \theta(x, y, z, t)$ and $\phi(x, y, z, t)$ are introduced.

$$\mathbf{v} = v_s + \epsilon \mathbf{V}, \quad T = T_s + \epsilon \theta, \quad C = C_s + \epsilon \phi, \tag{15}$$

where ϵ is small quantity and $\mathbf{V} = (U, V, W)$. By omitting δ^2 terms and sub. (15) in (3) and (5)-(7), the following perturbation equations are obtained

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0, \tag{16}$$

$$\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z} = \left(Ra \frac{\partial \theta}{\partial y} + \frac{1}{Le} Sa \frac{\partial \phi}{\partial y} \right), \tag{17}$$

$$\frac{\partial W}{\partial x} - \frac{\partial U}{\partial z} = \left(Ra \frac{\partial \theta}{\partial x} + \frac{1}{Le} Sa \frac{\partial \phi}{\partial x} \right), \tag{18}$$

$$\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = 0, \tag{19}$$

$$\frac{\partial \theta}{\partial t} - U \frac{Ra_H}{Ra} + u_s \frac{\partial \theta}{\partial x} + W \frac{\partial T_s}{\partial z} = 2 \frac{Ge}{Ra} u_s U + \nabla^2 \theta, \tag{20}$$

$$\frac{\partial \phi}{\partial t} + u_s \frac{\partial \phi}{\partial x} - \frac{Sa_H}{Sa} U + W \frac{\partial C_s}{\partial z} = \frac{1}{Le} \nabla^2 \phi, \tag{21}$$

$$z = 0, 1 : W = 0, \theta = 0, \phi = 0. \tag{22}$$

2.3. Transverse Rolls

Transverse rolls are rolls which axes are perpendicular to the basic flow direction. That is, disturbances lies in xz-plane.

$$U = U(x, z, t), V = 0, W = W(x, z, t), \theta = \theta(x, z, t), \phi = \phi(x, z, t), \tag{23}$$

Set up a stream function ψ such that

$$U = \frac{\partial \psi}{\partial z}, W = -\frac{\partial \psi}{\partial x}, \tag{24}$$

Substituting Eqs. (23)-(24) in (16)-(22), the following equations are obtained. And (16), (17), (19) satisfies identically.

$$\nabla^2 \psi = - \left(Ra \frac{\partial \theta}{\partial x} + \frac{1}{Le} Sa \frac{\partial \phi}{\partial x} \right), \tag{25}$$

$$\frac{\partial \theta}{\partial t} + u_s \frac{\partial \theta}{\partial x} - \frac{Ra_H}{Ra} \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x} \frac{\partial T_s}{\partial z} = \nabla^2 \theta + 2 \frac{Ge}{Ra} u_s \frac{\partial \psi}{\partial z}, \tag{26}$$

$$\frac{\partial \phi}{\partial t} + u_s \frac{\partial \phi}{\partial x} - \frac{Sa_H}{Sa} \frac{\partial \psi}{\partial z} - \frac{\partial \psi}{\partial x} \frac{\partial C_s}{\partial z} = \frac{1}{Le} \nabla^2 \phi, \tag{27}$$

$$z = 0, 1 : \psi = 0, \theta = 0, \phi = 0. \tag{28}$$

Solutions of eqs. (25)-(28) are of the form of

$$\begin{aligned} \psi(x, z, t) &= f(z) e^{i(ax)} e^{\lambda t}, \\ \theta(x, z, t) &= h(z) e^{i(ax)} e^{\lambda t}, \\ \phi(x, z, t) &= k(z) e^{i(ax)} e^{\lambda t}, \end{aligned} \tag{29}$$

where $\lambda = \lambda_R + \lambda_I$ is complex growth rate parameter, a is the wave number. Disturbances grows exponentially when $\lambda_I > 0$, which leads instability in the flow. Disturbances decays to 0 for $\lambda_I < 0$ which leads to stability in the flow. While $\lambda_I = 0$, is a case of principle of exchange of stabilities or marginal stability. It is interesting to study the case of $\lambda_I = 0$. By Substituting Eq. (29) in Eqs. (25)-(28), the following eigenvalue problem is obtained.

$$(D^2 - a^2)f + ia(Rah + \frac{1}{Le} Sak) = 0, \tag{30}$$

$$(D^2 - a^2)h - iau_s h + 2 \frac{Ge}{Ra} u_s Df + ia \frac{dT_s}{dz} f + \frac{Ra_H}{Ra} Df = \lambda h, \tag{31}$$

$$(D^2 - a^2)k - iaLeu_s k + Le \frac{Sa_H}{Sa} Df + ia \frac{dC_s}{dz} Lef = \lambda kLe, \tag{32}$$

$$z = 0, 1 : f = 0, h = 0, k = 0. \tag{33}$$

where

$$\gamma = \frac{\lambda_R}{a} - Pe. \tag{34}$$

$$F(z) = u_s - Pe = \left(Ra_H + \frac{1}{Le} S a_H \right) \left(z - \frac{1}{2} \right), \tag{35}$$

$$G(z) = \frac{\partial T_s}{\partial z} = -1 + \frac{1}{24Ra} [12GePe^2 + 12PeRa_H + GeX^2 - 2XRa_H - 4GePeX - 6z(4GePe^2 + 4PeRa_H + GeX^2 - 4GePeX - 2XRa_H) - 12z^2(XRa_H + 2GePeX - GeX^2) - 8z^3GeX^2], \tag{36}$$

$$H(z) = \frac{\partial C_s}{\partial z} = -1 + \frac{1}{2} Le \frac{S a_H}{S a} \left(Pe - \frac{X}{6} \right) + Le \frac{S a_H}{S a} z \left(\frac{X}{2} - Pe \right) - Le \frac{S a_H}{S a} \frac{X z^2}{2}, \tag{37}$$

where $X = Ra_H + \frac{1}{Le} S a_H$.

2.4. Longitudinal Rolls

Longitudinal rolls are rolls which axes are parallel to the basic flow direction. That is, disturbances lies in yz-plane.

$$U = 0, \quad V = V(y, z, t), \quad W = W(y, z, t), \quad \theta = \theta(y, z, t), \quad \phi = \phi(y, z, t), \tag{38}$$

Set up a stream function ψ such that

$$V = \frac{\partial \psi}{\partial z}, \quad W = -\frac{\partial \psi}{\partial y}, \tag{39}$$

Substituting Eqs. (38)-(39) in (16)-(22), the following equations are obtained.

$$\nabla^2 \psi = - \left(Ra \frac{\partial \theta}{\partial y} + \frac{1}{Le} S a \frac{\partial \phi}{\partial y} \right), \tag{40}$$

$$\frac{\partial \theta}{\partial t} - \frac{\partial \psi}{\partial y} \frac{\partial T_s}{\partial z} = \nabla^2 \theta, \tag{41}$$

$$\frac{\partial \phi}{\partial t} - \frac{\partial \psi}{\partial y} \frac{\partial C_s}{\partial z} = \frac{1}{Le} \nabla^2 \phi, \tag{42}$$

$$z = 0, 1 : \psi = 0, \theta = 0, \phi = 0. \tag{43}$$

Solutions of Eqs. (40)-(43) are of the form of

$$\begin{aligned} \psi(y, z, t) &= f(z) e^{i(ay)} e^{lt}, \\ \theta(y, z, t) &= h(z) e^{i(ay)} e^{lt}, \\ \phi(y, z, t) &= k(z) e^{i(ay)} e^{lt}, \end{aligned} \tag{44}$$

Substituting Eq.(44) into (40)-(43), the following eigenvalue problem is attained.

$$(D^2 - a^2)f + ia(Ra h + \frac{1}{Le} S a k) = 0, \tag{45}$$

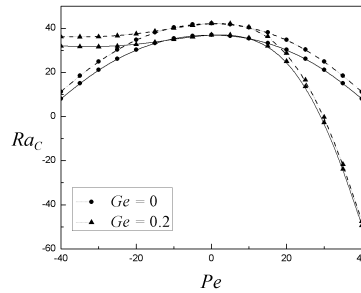


Fig. 1: Variation of Ra_C versus Pe for $Ra_H = 10, Le = 10, Sa = 5, Sa_H = 0$ and $Ge = 0, 0.2$.

$$(D^2 - a^2)h + ia \frac{dT_s}{dz} f = \lambda h, \tag{46}$$

$$(D^2 - a^2)k + iaLe \frac{dC_s}{dz} f = \lambda Lek, \tag{47}$$

$$z = 0, 1 : f = 0, h = 0, k = 0. \tag{48}$$

3. Results and discussion

For longitudinal rolls, Runge-Kutta and shooting methods have been employed to solve the eigenvalue problem (30)-(33). Whereas for transverse rolls, to solve the eigenvalue problem (45)-(48), Chebyshev-Tau method has been used. Through the discussion, dashed lines represent transverse rolls and solid lines represent longitudinal rolls.

Fig. 1 shows the response of critical thermal Rayleigh number Ra_C to Peclet number Pe for the parameters $Ra_H = 10, Le = 10, Sa = 5, Sa_H = 0$ with the comparison of $Ge = 0, 0.2$. Plots with $Ge = 0$ is symmetric about $Pe = 0$ axis. Transverse rolls are more unstable than longitudinal rolls. In the presence of viscous dissipation, upward throughflow is more prone instability comparing to the downward throughflow.

Fig. 2 display the behavior of Ra_C against horizontal Rayleigh number Ra_H for $Le = 10, Sa = 5, Sa_H = 0$ and $Ge = 0, 0.2$. In both the figures, small values of Ra_H upto 25, has stabilizing effect, but as Ra_H increases beyond this, destabilization effect takes place. For upward throughflow, flow with $Ge = 0.2$ is more unstable than the flow with $Ge = 0$ but an opposite observation is made for the downward throughflow.

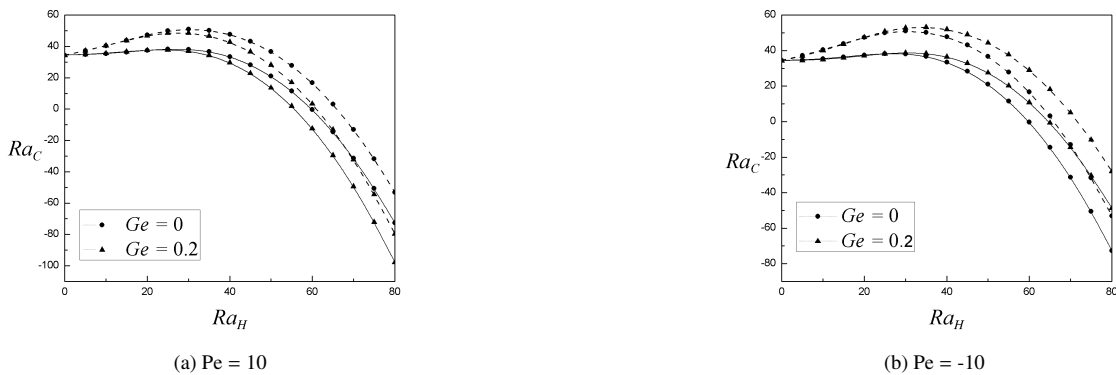


Fig. 2: Graph of Ra_C versus Ra_H for $Le = 10, Sa = 5, Sa_H = 0$ and $Ge = 0, 0.2$.

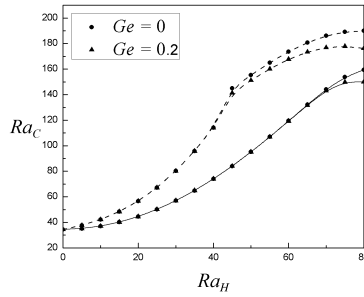


Fig. 3: Plot of Ra_C versus Ra_H for $Pe = 0, Le = 10, Sa = 5, Sa_H = 0$ and $Ge = 0, 0.2$.

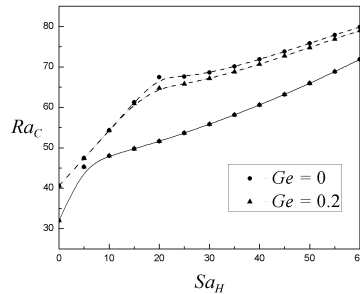


Fig. 4: Variation of Ra_C versus Sa_H for $Pe = 0, Le = 10, Sa = 10, Ra_H = 0$ and $Ge = 0, 0.2$.

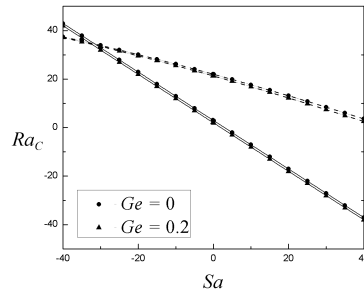


Fig. 5: Graph of Ra_C versus Sa for $Pe = 5, Le = 10, Ra_H = 10, Sa_H = 0$ and $Ge = 0, 0.2$.

Fig. 3 represents the plot of Ra_C versus Ra_H for values of $Pe = 0, Le = 10, Sa = 5, Sa_H = 0$ and $Ge = 0, 0.2$. Ra_H has stabilizing effect. When $Pe = 0$, effect of viscous dissipation is insignificant up to certain values of Ra_H (for longitudinal modes 70, for transverse modes 40). After this certain value flow with $Ge = 0.2$ is more unstable than the flow with $Ge = 0$.

Fig. 4 depicts graph of Ra_C to Sa_H for $Pe = 0, Le = 10, Sa = 10, Ra_H = 0$ and $Ge = 0, 0.2$. For both, longitudinal and transevre modes, Sa_H has stabilizing effect. There is no effect of viscous dissipation on longitudinal modes. But in the case of transverse modes, up to particular value of Sa_H , curves for $Ge = 0$ and $Ge = 0.2$ coincides. But after this value, $Ge = 0.2$ makes the flow more destable than $Ge = 0$.

Fig. 5 shows the response of Ra_C to Sa for $Pe = 5, Le = 10, Ra_H = 10, Sa_H = 0$ and $Ge = 0, 0.2$. As Sa increases from -40 to 40 , destabilization effect takes place. This effect of destabilization is stronger for longitudinal modes, weaker for transverse modes. In both cases, flow with $Ge = 0.2$ is more unstable than the flow with $Ge = 0$.

References

- [1] A. Barletta, D. A. Nield, Instability of Hadley-Prats flow with viscous heating in a horizontal porous layer, *Transp Porous Med.* 84 (2010) 241–256.
- [2] C. W. Horton, F. T. Rogers, Convection currents in a porous medium, *J. Appl. Phys.* 16 (1945) 367–370.
- [3] E. R. Lapwood, Convection of a fluid in a porous medium, *Proc. Cambridge Phil. Soc.* 44 (1948) 508–521.
- [4] J. E. Weber, Convection in a porous medium with horizontal and vertical temperature gradients, *Int. J. Heat Mass Transfer*, 17 (1974) 241–248.
- [5] M. Prats, The effect of horizontal fluid flow on thermally induced convection currents in porous mediums, *J. geophysical Res.* 71 (1966) 4835–4838.
- [6] K. Vafai, *Handbook of porous media*, Marcel Dekker, Inc. 2000.
- [7] D. A. Nield, A. Bejan, *Convection in Porous Media*, 4th Edition, Springer, 2013.
- [8] D. A. Nield, Onset of thermohaline convection in a porous medium, *Water Resources Research*, 4 (1968) 553–560.
- [9] D. A. Nield, D. M. Manole, J. L. Lage, Convection induced by inclined thermal and solutal gradients in a shallow horizontal layer of a porous medium, *J. Fluid Mech.* 257 (1993) 559–574.
- [10] M. Mamou, P. Vasseur, E. Bilgen, Double-diffusive convection instability in a vertical porous enclosure, *J. Fluid Mech.* 368 (1998) 263–289.
- [11] P. N. Kaloni, Z. Qiao, Nonlinear convection induced by inclined thermal and solutal gradients with mass flow, *Cont. Mech. Thermodyn.* 12 (2000) 185–194.
- [12] H. I. Ene, E. Sanchez-Palencia, On thermal equation for flow in porous media, *Int. J. Engng Sci.* 20 (1982) 623–630.
- [13] A. Barletta, M. Celli, D. A. S. Rees, The onset of convection in a porous layer induced by viscous dissipation: A linear stability analysis, *Int. J. Heat Mass Transf.* 52 (2009) 337–344.
- [14] A. Barletta, D. A. Nield, Thermosolutal convective instability and viscous dissipation effect in a fluid-saturated porous medium, *Int. J. Heat Mass Transf.* 54 (2011) 1641–1648.
- [15] D. A. Nield, A. Barletta, M. Celli, The effect of viscous dissipation on the onset of convection in an inclined porous layer, *J. Fluid Mech.* 679 (2011) 544–558.