Standard Model and Renormalization of Scalar Field

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Declaration

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Approval Sheet

This Thesis entitled Standard Model and Stability of Higgs Vacuum by Joydev Khatua is approved for the degree of Master in Physics from IIT Hyderabad

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Abstract

Standard Model has been a beautiful theory so far predicting the interactions among the elementary particles. Higgs mechanism generates the masses of the Standard Model particles The Standard Model(SM) is the backbone of elementary particle physics-not only does it provide a consistent framework for studding the interaction of quark and leptons,but it also gives predictions which have been extensively tested experimentally . In these notes ,I review the gauge theory and electroweak sector of the Standard Model,discuss the calculation of electroweak and summarize the status of Stability of the Higgs Vacuum.

Contents

Chapter 1

GAUGE THEORY

1.1 Introduction

From the beginning of our general physics class we are taught, unknowingly, the ideas of gauge theory and gauge invariance. In this paper we will discuss the uses of gauge theory and the meaning of gauge invariance.Finally Yang-Mills gauge theory will be addressed as an analogy to classical electrodynamics gauge transformation. we begin by defining a gauge theory as a field theory in which the equation of motion remains unchanged after a transformation . The gauge transformations leave the action and the classical equations of motion invariant and Gauge theories appear to describe three of the four fundamental forces of nature pretty well.Those three forces are electromagnetism, the weak force and the strong force.

1.1.1 Local Phase Invariance

What happens if one chooses a different phase at a different time space coordinate?
For this the field is changed then by $\psi(x) \rightarrow \psi\prime(x) = e^{iq\xi(x)}\psi(x)$ the effect of local transformation on $\partial_\mu\psi(x)$.

$$(\partial_\mu\psi(x))^{'} = \partial_\mu(e^{iq\xi(x)}\psi(x))$$

$$= e^{iq\xi(x)}(\partial_\mu\psi(x) + iq\partial_\mu\xi(x)\psi(x))$$

from the above equation it is clear that $\partial_\mu \psi(x)$ does not have the same transformation rule as ψ it self.To make Dirac Lagrangian $\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi$ invariant under local phase transformation construct a modified derivative transforming according to

$$D_\mu \psi(x) \longrightarrow (D_\mu \psi(x))^{'} \quad = \quad e^{iq\xi(x)} \psi(x)) \tag{1.1}$$

If we define $D_\mu \psi(x) = (\partial_\mu - iqA_\mu(x))\psi(x)$
the new quantity A_μ must have the transforming rule $A_\mu \longrightarrow (A_\mu)^{'} = A_\mu + \partial_\mu \xi(x)$

This new field $A_\mu \longrightarrow (A_\mu)^{'} = A_\mu + \partial_\mu \xi(x)$ called the gauge field. The replacement of $\partial_\mu \to D_\mu$ prescribe the form of interaction between the gauge field A_μ and matter . Introducing the covariant derivative into the Lagrangian shows that theory is no longer free but describes interactions the fermions and gauge field .

$$\mathcal{L}_\psi \quad = \quad \overline{\psi} i\gamma^\nu \psi - m\overline{\psi}\psi + iqA_\mu \overline{\psi}\gamma^\mu \psi \tag{1.2}$$

To arrive at the complete Lagrangian for quantum electrodynamics it remains only to add a kinetic term for the vector field,which describes the propagation of free photons. The field strength tensor can be used to write a gauge invariant Lagrangian for the gauge field itself $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$
The complete Lagrangian for quantum electrodynamics is

$$\mathcal{L}_{QED} = \mathcal{L}_{free} - \mathcal{J}^\mu A_\mu - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \tag{1.3}$$

The Lagrangian leads to Maxwell's equation and is manifestly invariant under local gauge . A photon mass term would have the form $\frac{1}{2}mA_\mu A^\mu$ Which violate the local gauge invariance because

$$A_\mu A^\mu \to (A_\mu + \partial_\mu \xi(x))(A^\mu + \partial^\mu \xi(x)) \neq A_\mu A^\mu$$

1.2 Summary of Local Phase Transformation

• It is possible for the Lagrangian for a (complex) Dirac field to be invariant under local U(1)transformation (phase rotation),in which the phase parameter depends on space-time.In order to accomplish this we include an interaction with a vector gauge boson which transforms under the local (gauge)transformation according to $A_\mu \longrightarrow (A_\mu)^{'} = A_\mu + \partial_\mu \xi(x)$

•This interaction is encoded by replacing the derivative ∂_μ by the covariant derivative $D_\mu.D_\mu\psi$ transforms under gauge transformation as $e^{iq\xi(x)}D_\mu\psi$

• The kinetic term for the gauge boson is $\frac{-1}{4}F_{\mu\nu}F^{\mu\nu}$ where F_μ is proportional to the commutator $[D_\mu, D_\nu]$ and is invariant under gauge transformations.

• The gauge boson must massless, since a term proportional to $A_\mu A^\mu$ is not invariant under gauge transformations and hence not include in the Lagrangian.

1.3 Non-Abelian Gauge Theories

Thus we find that local gauge invariance has led us to the existence of a massless photon.we have just seen in the previous discussion that electromagnetism possesses a local gauge invariance ,and that by imposing that local symmetry on a free -particle Lagrangian it is possible it is possible to construct an interesting theory of electrodynamics.In this case ,the “gauge” concept will be constructed so that the gauge bosons have self-interaction -as are observed among the gluons of QCD and the $W_+, W_-, Z\ and\ \gamma$ of the electroweak sector.However ,the gauge bosons will still be massless.(we will see how to give the$W_+, W_- and\ Z$ their observed masses in the Higgs chapter.)

1.4 Non-Abelian Transformation

We apply the ideas of the previous lecture to the case where the transformation do not commute with each other ,i.e. the group is “non-Abelion”.

The Lagrangian for free nucleons may be written in an obvious notation,

$$\mathcal{L} = \overline{p}(i\gamma^{\nu}\partial_{\mu} - m)p + \overline{n}(i\gamma^{\nu}\partial_{\mu} - m)n$$

In terms of the composite spinor $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$,the Lagrangian may be rewritten more compactly as

$$\mathcal{L} = \overline{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi$$

$$\psi(x) \rightarrow (\psi(x))^{'} = G(x)\psi(x)$$

with

$$G(x) = \exp(\frac{i\tau.\alpha(x)}{2})$$

.

Then the gradient transforms as $\partial_{\mu}\phi(x) \rightarrow G(\partial_{\mu}\phi(x)) + (\partial_{\mu}G)\psi(x)$.

To ensure the local gauge invariance of the theory we first introduce a gauge covariant derivative

$$\mathcal{D}_{\mu} = I\partial_{\mu} + igB_{\mu} \tag{1.4}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

serves as a reminder that the operators are 2×2 matrices in isospin space and g is play to role a strong- interaction coupling constant.The object B_{μ} is the 2×2 matrices defined by

$$B_{\mu} = \frac{1}{2}\tau.b_{\mu} = \frac{1}{2}\tau^{a}.b^{a}$$

where the three gauge fields are $b_{\mu} = (b_1, b_2, b_3)$ bold-face quantities denote isovectors , and the isospin index a runs from 1 to 3.The point of introducing the gauge field and the gauge covariant derivative is to obtain a generalization of the gradient that transforms as $\mathcal{D}_\mu\psi \rightarrow (\mathcal{D}^{'}_\mu\psi^{'}) = G\mathcal{D}_\mu\psi$.Now we want to know how field b_μ will transform if $\mathcal{D}^{'}_\mu\psi^{'} = G\mathcal{D}_\mu\psi$.

$$\begin{aligned}
So\mathcal{D}^{'}_\mu\psi^{'} &= (\partial_\mu + igB^{'}_\mu)\psi \\
&= (\partial_\mu + igB^{'}_\mu)G\psi \\
&= \partial_\mu(G\psi) + igB^{'}_\mu G\psi \\
&= (\partial_\mu G)\psi + G\partial_\mu\psi + igB^{'}_\mu G\psi
\end{aligned}$$

And $G\mathcal{D}_\mu\psi = G\partial_\mu\psi + igB_\mu G\psi$

by use transforming form of covariant derivative the field $b_\mu$transform as

$$igB^{'}_\mu G\psi = GigB_\mu\psi - (\partial_\mu)\psi)$$

this must hold for arbitrary values of the nucleon field ψ and finally field .

transforming as

$$B^{'}_\mu = G[B_\mu + \frac{i}{g}G^{-1}\partial_\mu G]G^{-1} \tag{1.5}$$

For the case of isospin gauge symmetry the meaning of eq. (1.5) is that B_μ transformed by an isospin rotation plus a gradient term.Now final invariant Lagrangian under local SU(2) gauge transformation is

$$\begin{aligned}
\mathcal{L} &= \overline{\psi}(i\gamma^\mu\mathcal{D} - m)\psi \\
&= \overline{\psi}(i\gamma^\mu(\partial_\mu + igB_\mu) - m)\psi \\
&= \overline{\psi}(i\gamma^\mu - m)\psi - \overline{\psi}\gamma^\mu gB_\mu\psi \\
&= \overline{\psi}(i\gamma^\mu - m)\psi - \frac{g}{2}b_\mu.\overline{\psi}\gamma^\mu\tau\psi \\
&= \mathcal{L}_0 - \frac{g}{2}b_\mu.\overline{\psi}\gamma^\mu\tau\psi
\end{aligned}$$

free Lagrangian plus an interacting term .

In analogy with electromagnetism ,we seek a field stress tensor $F_{\mu\nu} = \frac{1}{2}\mathcal{F}_{\mu\nu}.\tau = \frac{1}{2}\mathcal{F}^a\tau^a$

From which we construct a gauge invariant kinetic term $\mathcal{L}_{gauge} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$

We wish to find a field strength tensor that transforms under local gauge transformation as $F^{'}_{\mu\nu} = GF_{\mu\nu}G^{-1}$

This suggests that for the SU(2) gauge theory a candidate field-strength tensor is the form $F_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + ig[B_\nu, B_\mu]$ (2.1)

The Yang-Mills Lagrangian

$$\mathcal{L}_{YM} = \overline{\psi}(i\gamma^\mu \mathcal{D}_\mu - m)\psi - \frac{1}{2}tr(F_{\mu\nu}F^{\mu\nu}) \quad (1.6)$$

is therefore invariant under local gauge transformation . where the mass term $M^2 B_\mu B^\mu$ for the gauge field is incompatible with local gauge as in electromagnetism ,a common mass term for the nucleon is already permitted.

1.5 Summary of Non Abelian Gauge Theory

• A non-abelian gauge theory is one in which the Lagrangian is invariant under local transformations of a non-abelian group.

• This invariance is achieved by introducing a gauge boson for each generator of the group.The partial derivative in the Lagrangian for the fermion field is replaced by a covariant derivative as defined in eq.(1.4)

•The field strength $F^a_{\mu\nu}$ are obtained from the commutator of two covarient derivative. They transform as the adjoint representation under gauge transformation such that the quantity $F^a_{\mu\nu}F^{\mu\nu}_a$ is invariant .

•If we calculate the value of $F^a_{\mu\nu}F^{\mu\nu}_a$,we will see it contains terms which are cubic and quartic in the gauge bosons,indicating that these gauge bosons interact with each other.

Chapter 2

STANDARD MODEL

2.0.1 Introduction

An important feature of the Standard Model(SM) is that "it works": it is consistent with,or verified by ,all available data ,with no compelling evidence for physics beyond.Secondly it is a unified description ,in terms of "gauge theories" of all the interaction of known particles (except gravity). A gauge theory is one that possesses invariance under a set of "local transformation", i.e. transformation whose parameters are space-time dependent.

In order to apply such a gauge theory to weak interaction ,one considers particles which transform into each other under the weak interaction ,such as a u-quark and a d-quark,or an electron and a neutrino ,to be arranged in doublets of weak isospin .the three gauge bosons are interpreted as the W_{+}, W_{-} and Z bosons, that mediate weak interactions in the same way that the photon mediates electromagnetic interaction.

The difficulty in the case of weak interactions was that they are known to be short range,mediate by very massive vector boson, whereas Yang-Mills fields are required to be massless in order to preserve gauge invariance .The apparent paradox was solved by the application of the "Higgs mechanism".This is a prescription for breaking the gauge symmetry spontaneously. In this scenario one starts with a theory that possesses the required gauge invariance ,but where the ground state of the theory is not invariant under gauge transformation .The breaking of the invariance arises in the quantization of the theory,whereas the Lagrangian only contains terms which are invariant.one of the consequences of this is that the gauge bosons acquire a mass and the theory can thus be applied to weak interaction.

2.1 Spontaneous Symmetry Breaking

We have seen that in an unbroken gauge theory the gauge bosons must be massless. This is exactly what we want for QED (massless photon) . However, if we wish to extend the ideas of describing interactions by a gauge theory to the weak interactions,the symmetry must somehow be broken since the carriers of the weak interactions (W and Z bosons) are massive (weak interactions are very short range). We could simply break the symmetry by hand by adding a mass term for the gauge bosons, which we know violates the gauge symmetry.

In this chapter we will discuss a way to give masses to the W and Z, called “spontaneous symmetry breaking”, which maintains the renormalizability of the theory. In this scenario the Lagrangian maintains its symmetry under a set of local gauge transformations. On the other hand, the lowest energy state, which we interpret as the vacuum (or ground state), is not a singlet of the gauge symmetry. There is an infinite number of states each with the same ground-state energy and nature chooses one of these states as the “true” vacuum.

Spontaneous symmetry breaking is a phenomenon that is by far not restricted to gauge symmetries. It is a subtle way to break a symmetry by still requiring that the Lagrangian remains invariant under the symmetry transformation. However, the ground state of the symmetry is not invariant, i.e. not a singlet under a symmetry transformation.

In order to illustrate the idea of spontaneous symmetry breaking, consider a pen that is completely symmetric with respect to rotations around its axis. If we balance this pen on its tip on a table, and start to press on it with a force precisely along the axis we have a perfectly symmetric situation. This corresponds to a Lagrangian which is symmetric (under rotations around the axis of the pen in this case). However, if we increase the force, at some point the pen will bend (and eventually break). The question then is in which direction will it bend. Of course we do not know, since all directions are equal. But the pen will pick one and by doing so it will break the rotational symmetry. This is spontaneous symmetry breaking.

A better example can be given by looking at a point mass in a potential

$$V(\vec{r}) = \mu^2(\vec{r}.\vec{r} + \lambda(\vec{r}.\vec{r})^2 \tag{2.1}$$

This potential is symmetric under rotations and we assume $\lambda > 0$ (otherwise there would be no stable ground state). For $\mu^2 > 0$ the potential has a minimum at $\vec{r} = 0$, thus the point mass will simply fall to this point. The situation is more interesting if $\mu^2 < 0$,. For two dimensions the potential is shown in Fig. 2.1. If the point mass sits at $\vec{r} = 0$ the system is not in the ground state but the situation is completely symmetric. In order to reach the ground state, the symmetry has to be broken, i.e. if the point mass wants to roll down, it has to decide in which direction. Any direction is equally good, but one has to be picked. This is exactly what spontaneous symmetry breaking means. The Lagrangian (here the potential) is symmetric (here under rotations around the z-axis), but the ground state (here the position of the point mass once it rolled down) is not.

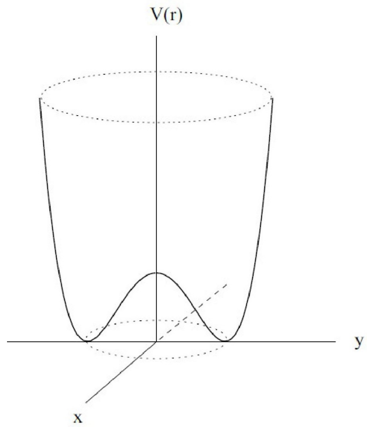


Figure 2.1: A potential that led to spontaneous symmetry breacking

2.2 Left and Right Handed Fermions

The weak interactions are known to violate parity. Parity non-invariant interactions for fermions can be constructed by giving different interactions to the "left-handed" and "righthanded" components defined in eq. (54. Thus, in writing down the Standard Model, we will treat the left-handed and right-handed parts separately.

A Dirac field ψ representing a fermion, can be expressed as the sum of a left -handed part,ψ_L , and a right -handed part ,ψ_R,

$$\psi = \psi_L + \psi_R$$

where

$$\psi_L = P_L \psi \quad with \quad P_L = \frac{1 - \gamma_5}{2}$$

$$\psi_R = P_R\psi \quad with \quad P_R = \frac{1+\gamma_5}{2}$$

P_L *and* P_R *are projection operator*, i.e

$$P_L P_L = P_L, P_R P_R = P_R \quad and \quad P_L P_R = 0 = P_R P_L$$

They project out the left-handed (negative) and right-handed (positive) chirality states of the fermion, respectively. This is the definition of chirality, which is a property of fermion fields, but not a physical observable.

The kinetic term of the Dirac Lagrangian and the interaction term of a fermion with a vector field can also be written as a sum of two terms, each involving only one chirality

$$\overline{\psi}\gamma^\mu\partial_\mu\psi = \overline{\psi_L}\gamma^\mu\partial_\mu\psi_R + \overline{\psi_R}\gamma^\mu A_\mu\psi_R$$

$$\overline{\psi}\gamma^\mu A_\mu\psi = \overline{\psi_L}\gamma^\mu A_\mu\psi_R + \overline{\psi_R}\gamma^\mu A_\mu\psi_R \tag{2.2}$$

On the other hand, a mass term mixes the two chiralities:

$$m\overline{\psi}\psi = m\overline{\psi_R}\psi_L + m\overline{\psi_L}\psi_R \tag{2.3}$$

we use

$$\gamma_5^2 = 1$$

$$\overline{\psi} = \psi^\dagger\gamma^0$$

$$\gamma^5\dagger = \gamma^5 \quad as \quad well \quad as \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$$

In the limit where the fermions are massless (or sufficiently relativistic), chirality becomes helicity, which is the projection of the spin on the direction of motion and which is a physical observable. Thus, if the fermions are massless, we can treat the left-handed and right-handed chiralities as separate particles of conserved helicity. We can understand this physically from the following simple consideration. If a fermion is massive and is moving in the positive z direction, along which its spin is having a positive component so that the helicity is positive in this frame, one can always boost into a frame in which the fermion is moving in the negative z direction, but with this spin component unchanged. In the new frame the helicity will hence be negative. On the other hand, if the particle is massless and travels with the speed of light, no such boost is possible, and in that case helicity/chirality is a good quantum number.

2.3 Symmetries and Particle Content

We have made all the preparations to write down a gauge invariant Lagrangian. We now only have to pick the gauge group and the matter content of the theory. It should be noticed that there are no theoretical reasons to pick a certain group or certain matter content. To match experimental observations we pick the gauge group for the Standard Model to be

$$U(1)_y \times Su(2) \times Su(3)$$

To indicate that the abelian $U(1)$ group is not the gauge group of QED but of hypercharge a subscript Y has been added. The corresponding coupling and gauge boson is denoted by $g^{'}$ and B_μ respectively.

The $Su(2)$ group has three generators ($T_a = \frac{\sigma_a}{2}$) the coupling is denoted by$g$ and three gauge bosons are denoted by $W^1_\mu W^2_\mu, W^3_\mu$ none of these gauge bosons (and neither B_μ) are physical particles. As we will see, linear combinations of these gauge bosons will make up the photon as well as the $W_\pm \;\; and \;\; Z \;\; Bosons$

Finally, the $Su(3)$ is the group of the strong interaction. The corresponding eight gauge bosons are the gluons. In this section we will concentrate on the other two groups, with one generation of fermions.

As matter content for the first family, we have

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}; u_R; d_R; l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}; e_R$$

Note also that the left- and right-handed fermion components have been given different weak interactions. The Standard Model is constructed this way, because the weak interactions are known to violate parity. The left-handed components form doublets under SU(2) whereas the right-handed components are singlets. This means that under SU(2) gauge transformations we have

$$e_R \rightarrow e_R^{'} = e_R$$

$$l_L \rightarrow l_L^{'} = e^{i\alpha^a T^a} l_L$$

Thus, the SU(2) singlets e_R, ν_R, u_R *amd* d_R are invariant under SU(2) transformations and do not couple to the corresponding gauge bosons $W_\mu^1, W_\mu^2, W_\mu^3$.
Since this separation of the electron into its left- and right-handed helicity only makes sense for a massless electron we also need to assume that the electron is massless in the exact SU(2) limit and that the mass for the electron arises as a result of spontaneous symmetry breaking in a similar way as the masses for the gauge bosons arise. We will come back to this later.

under $U(1)_Y$ gauge transformation the matter field transform as

$$\psi \rightarrow \psi^{'} = e^{i\beta(x)Y}\psi$$

where Y is the hypercharge of the particle under consideration. It is chosen to give the observed electric charge of the particles. The explicit values for the hypercharges of the particles listed in eq. (3.1) are as follows:

$$Y(l_L) = -\frac{1}{2}, \;\; Y(e_R) = -2, Y(\nu_R) = 0, Y(q_L) = \frac{1}{6}, Y(u_R) = \frac{4}{3}, Y(d_R) = -\frac{2}{3}$$

We have now listed all fermions that belong to the first family, together with their transformation properties under the various gauge transformations. However, since we ultimately want massive weak gauge bosons, we will have to break the $U(1)_Y \times SU(2)$ gauge group spontaneously, by introducing some type of Higgs scalar.

2.4 Kinetic Terms for the Gauge Bosons

Just as QED Lagrangian resulted from $U(1)_{em}$ local gauge invariance so we are led to electroweak Lagrangian by requiring $SU(2) \times U(1)_Y$ invarient form.For example electro-neutrino lepton pair we have

$$\mathcal{L} = \bar{l}_L \gamma^\mu [i\partial_\mu - g\frac{1}{2}\tau.W_\mu - g^{'}(-\frac{1}{2}B_\mu)]l_L + \bar{e}_R\gamma^\mu[i\partial_\mu - g^{'}(-)B_\mu]e_R - \frac{1}{4}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \quad (2.4)$$

where we have inserted the hypercharge value $Y_L = -1, Y_R = -2$

The Lagrangian $\mathcal{L}$ embodies the weak isospin and hypercharge interaction ,the final two terms are kinetic energy and self coupling of the B_μ field.Here $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is the hyparechage field strength.

Taken together the $SU(2) \times U(1)_Y$ transformations of the left and right handed component of $\updownarrow$ are

$$l_L \to l_L^{'} = e^{i\alpha(x).T + i\beta(x)Y} l_L$$

$$e_R \to e_R^{'} = e^{i\beta(x)Y} e_R \quad (2.5)$$

where the left handed fermions form isospin doublet l_L and right handed fermions form isospin isosinglet e_R. However $\mathcal{L}$ describes massless gauge bosons and massless fermions . Mass term such as $\frac{1}{2}M^2 B_\mu B^\mu$ and $-m\overline{\psi}\psi$ are gauge invariant and so can not be added .The electron mass term

$$-m\bar{e}e = -m_e\bar{e}[\frac{1}{2}(1-\gamma^5) + \frac{1}{2}(1+\gamma^5)]e$$

$$= -m_e[\bar{e}_R e_L + \bar{e}_L e_R]$$

Since the e_L is a member of an isospin doublet and e_R is a singlet ,this term manifestly breaks gauge invariance.

2.5 Choice Of The Higgs Field

We want to formulate the Higgs mechanism so that the $W^{\pm}$ and Z^0 become massive and the photon remains massless.To do this ,we introduce four real scalar field ϕ_i .We have add to $\mathcal{L}_1$ an $SU(2)\times U(1)$ gauge invariant for the scalar fields

$$\mathcal{L}_2 = |(i\partial_\mu - g\frac{1}{2}\tau.W_\mu - g^{'}\frac{Y}{2}B_\mu)\phi|^2 - V(\phi) \tag{2.6}$$

where $||^2 = ()^\dagger()$. To keep $\mathcal{L}_2$ gauge invariant ,the ϕ_i must belong to $SU(2)\times U(1)$ multiplates. The most economical choice is to arrange four fields in an isospin doublet with hypercharge Y=1:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad with \;\; \phi^+ = (\phi_1 + i\phi_2)/\sqrt{2}, \;\; \phi^0 = (\phi_3 + i\phi_4)/\sqrt{2}$$

It completes the specification of the standard (or minimal) model of electroweak interaction.It is also called the “Weinberg -salam model”.

To generate gauge boson masses,we use the familiar Higgs potential $V(\phi)$ with $\mu^2 < 0$ and $\lambda > 0$ and choose a vacuum expectation value, ϕ_0 , of $\phi(x)$. The appropriate choice is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$$

.

2.6 Breaking The Local Gauge Invariant $SU(2)_L \times U(1)_Y$ Symmetry

To break the $SU(2)_L \times U(1)_Y$ symmetry we follow the ingredients of Higgs Mechanism

1: Add an isospin doublet $\phi = \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$

The electric charge of the upper and lower component of the doublet are chosen such that the hyper-charge Y=+1

2.Add a potential $V(\phi)$ for the field that will break (spontaneously) the symmetry.

$$V(\phi) = \frac{\mu^2}{2}(\phi^\dagger \phi) + \frac{\lambda}{4}(\phi^\dagger \phi)^2 \tag{2.7}$$

with $\mu^2 < 0$

if we find minimum of the potential $V(\phi)$ the condition becomes $\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = v^2 = \sqrt{\frac{-\mu^2}{\lambda}}$

3.choose a vacuum: the vacuum we choose is that $\phi_1 = \phi_2 = \phi_3 = 0$ and $\phi_4 = v$

$$\text{perturbed vacuum}=\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix}$$

CHECKING WHICH SYMMETRIES ARE BROKEN IN A GIVEN VACUUM

$SU(2)_L$

$$\tau_1 \phi_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix} = +\frac{1}{\sqrt{2}} \begin{pmatrix} v+h \\ 0 \end{pmatrix} \neq 0 \quad \textit{Broken}$$

$$\tau_2\phi_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix} = -\frac{i}{\sqrt{2}} \begin{pmatrix} v+h \\ 0 \end{pmatrix} \neq 0 \;\; Broken$$

$$\tau_3\phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} v+h \\ 0 \end{pmatrix} \neq 0 \;\; Broken$$

$U(1)_Y$

$$Y\phi_0 = \mathcal{Y}_{\phi_0} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} v+h \\ 0 \end{pmatrix} \neq 0 \;\; Broken$$

This means that all 4 gauge boson W_1, W_2, W_3 *and* B acquire a mass through the higgs mechanism .

As photon remains mass less we are taking one generator from $SU(2)_L$ and other generator form $U(1)_Y$, such that Q=$\frac{1}{2}(\tau_3 + Y)$.

$U(1)_{EM}$

$$Q\phi_0 = \frac{1}{2}(\tau_3 + Y)\phi_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix} = 0 \; Unbroken$$

2.7 Breaking Of $SU(2)_L \times U(1)_Y SYMMETRY$: Looking a Bit Ahead

1. W_1 and W_2 mix and will form the massive W^+ and W^- gauge bosson.

2. W_3 and B mix to form massive Z and massless γ .

3.Remaining degree of freedom will form the mass of the scalar particle (Higgs boson).

2.8 The Higgs Part and Gauge Boson Masses

To obtain the masses for the gauge bosons we will only need to study the scalar part of the Lagrangian $\mathcal{L}_{scalar} = (D^\mu \phi)^\dagger D_\mu \phi - V(\phi)$

$$D_\mu \phi = (\partial_\mu + ig\frac{1}{2}\tau.\mathcal{W}_\mu + ig^{'}\frac{Y}{2}B_\mu)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2.8}$$

In the above equation only second and third term of the right hand side will contribute the masses of the gauge boson ,so we are interested only on those term.

$$
\begin{aligned}
Let\ us\ consider\ \mathcal{D}_\mu\phi &= \frac{1}{\sqrt{2}}[ig\frac{1}{2}\tau.\mathcal{W}_\mu + ig^{'}\frac{Y}{2}B_\mu]\begin{pmatrix} 0 \\ v \end{pmatrix} \\
&= \frac{i}{\sqrt{8}}[g(\tau_1 W_1 + \tau_2 W_2 + \tau_3 W_3) + g^{'} Y B_\mu]\begin{pmatrix} 0 \\ v \end{pmatrix} \\
&= \frac{i}{\sqrt{8}}[g(\begin{pmatrix} 0 & W_1 \\ W_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iW_2 \\ iW_2 & 0 \end{pmatrix} + \begin{pmatrix} W_3 & 0 \\ 0 & -W_3 \end{pmatrix})
\end{aligned}
$$

$$
+g^{'}\begin{pmatrix} Y_{\phi_0}B_\mu & 0 \\ 0 & Y_{\phi_0}B_\mu \end{pmatrix}]\begin{pmatrix} 0 \\ v \end{pmatrix}
$$

$$
\begin{aligned}
&= \frac{i}{\sqrt{8}}\begin{pmatrix} gW_3 + g\prime Y_{\phi_0}B_\mu & g(W_1 - iW_2) \\ g(W_1 + iW_2) & -gW_3 + g\prime Y_{\phi_0}B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} \\
&= \frac{iv}{\sqrt{8}}\begin{pmatrix} g(W_1 - iW_2) \\ -gW_3 + g\prime Y_{\phi_0}B_\mu \end{pmatrix}
\end{aligned}
$$

Now We can then also easily find out the complex conjugate of $\mathcal{D}_\mu\phi$

$$
(\mathcal{D}_\mu\phi)^\dagger = -\frac{iv}{\sqrt{8}}\begin{pmatrix} g(W_1 + iW_2), & (-gW_3 + g\prime Y_{\phi_0}B_\mu) \end{pmatrix}
$$

And we get the following expression for the kinetic part of the Lagrangian

$$
\mathcal{D}_\mu\phi)^\dagger\mathcal{D}_\mu\phi = \frac{v^2}{\sqrt{8}}[g^2(W_1^2 + W_2^2) + (-gW_3 + g\prime Y_{\phi_0}B_\mu)^2]
$$

we can rewrite W_1, W_2 term as W^+, W^- using $W^\mp = \frac{1}{\sqrt{2}}(W_1 \mp iW_2)$
Looking at the terms W_1, W_2 in the Lagrangian involving equation- (1) we see that

$$
g^2(W_1^2 + W_2^2) = g^2 W^+ W^-
$$

we can write $(-gW_3 + g_{\prime} Y_{\phi_0} B_\mu)^2$ this in the following matrix form

$$(-gW_3 + g_{\prime} Y_{\phi_0} B_\mu)^2 = \begin{pmatrix} W_3, & B_\mu \end{pmatrix} \begin{pmatrix} g^2 & -gg^{'} Y_{\phi_0} \\ -gg^{'} Y_{\phi_0} & g^{'2} \end{pmatrix} \begin{pmatrix} W_3 \\ B_\mu \end{pmatrix}$$

Again we know there is weak missing angle θ between $W_3,\ B_\mu\ with\ Z_\mu,\ A_\mu$

For Diagonalization of the above metrics doing rotation with angle θ .This angle related to coupling constant

$$\begin{aligned} e &= gsin\theta \\ e &= g^{'} cos\theta. \end{aligned}$$

Two eigen value and two eigen vector of the matrix P= $\begin{pmatrix} g^2 & -gg^{'} Y_{\phi_0} \\ -gg^{'} Y_{\phi_0} & g^{'2} \end{pmatrix}$ is

$$\begin{aligned} \lambda &= 0 \rightarrow \frac{1}{\sqrt{g^2 + g^{'2}}} \begin{pmatrix} g^{'} \\ g \end{pmatrix} \\ \lambda &= \sqrt{g^2 + g^{'2}} \rightarrow \begin{pmatrix} g \\ -g^{'} \end{pmatrix} \end{aligned}$$

So

$$\begin{pmatrix} W_3 \\ B_\mu \end{pmatrix} = \begin{pmatrix} \frac{g}{\sqrt{g^2+g^{'2}}} & \frac{g^{'}}{g^2+g^{'2}} \\ \frac{-g^{'}}{\sqrt{g^2+g^{'2}}} & \frac{g}{g^2+g^{'2}} \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$$

Consider matrix O= $\begin{pmatrix} \frac{g}{\sqrt{g^2+g^{'2}}} & \frac{g^{'}}{g^2+g^{'2}} \\ \frac{-g^{'}}{\sqrt{g^2+g^{'2}}} & \frac{g}{g^2+g^{'2}} \end{pmatrix}$

putting this value in equation no 2

$$= \begin{pmatrix} W_3, & B_\mu \end{pmatrix} \begin{pmatrix} g^2 & -gg^{'}Y_{\phi_0} \\ -gg^{'}Y_{\phi_0} & g^{'2} \end{pmatrix} \begin{pmatrix} W_3 \\ B_\mu \end{pmatrix}$$

$$= \begin{pmatrix} Z_\mu, & A_\mu \end{pmatrix} O^\dagger \begin{pmatrix} g^2 & -gg^{'}Y_{\phi_0} \\ -gg^{'}Y_{\phi_0} & g^{'2} \end{pmatrix} O \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$$

Middle three term in the above expression is the diagonalization of matrix P. So only diagonal term will be there which are the eigenvalue value of the matrix P and rest of the term will be zero.

$$= \begin{pmatrix} Z_\mu, & A_\mu \end{pmatrix} \begin{pmatrix} g^{'2} + g^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z_\mu, & A_\mu \end{pmatrix}$$

$$= (g^{'2} + g^2) Z_\mu^2 + 0 A_\mu^2$$

Rewriting Lagrangian in terms physical fields

$$\mathcal{D}_\mu \phi)^\dagger \mathcal{D}_\mu \phi = \frac{v^2}{8} (g^2 W^{+2} + g^2 W^{-^2} + (g^{'2} + g^2) Z_\mu^2 + 0 A_\mu^2) \tag{2.9}$$

2.9 Massive Charge And Neutral Gauge Bosons

As a general term for a massive gauge boson V has the form $\frac{1}{2} M_\mu V_\mu$. From equation we see that

$$M_{W_+} = M_{W_-} = \frac{1}{2} v g$$

$$M_Z = \frac{1}{2}\sqrt{g^2 + g^{'2}}$$

similar to Z boson we have for photon $\frac{1}{2}M_\gamma^2 = 0$

so photon mass $M_\gamma = 0$

MASS OF HIGGS BOSON

The contribution of mass term of the higgs boson we can get from potential term

$$V(\phi) = \frac{\mu^2}{2}(\phi^\dagger\phi) + \frac{\lambda}{4}(\phi^\dagger\phi)^2$$

Using small perturbation of the vacuum $\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$

If we calculate the $V(\phi_0)$, then the coefficient of h^2 is the mass of higgs boson ,which is

$$m_h = \sqrt{2vh}$$

2.10 Fermion Masses and Yukawa Couplings

We cannot have an explicit mass term for the quarks or electrons, since a mass term mixes left-handed and right-handed fermions and we have assigned these to different multiplets of weak SU(2). However, if an SU(2) doublet Higgs is introduced, there is a gauge invariant interaction that will look like a mass when the Higgs gets a vacuum expectation value("vev") . Such an interaction is called a Yukawa interaction.

An alternative feature of the standard model is that the same Higgs doublet which generate $W^{\pm}$ and Z masses is also sufficient to give masses to lepton and quarks.For example to generate the electron mass,we include the following $SU(2) \times U(1)$gauge invariant term in the Lagrangian

$$\mathcal{L} = -G_e[\begin{pmatrix} \overline{\nu_e} \; \overline{e} \end{pmatrix}_L \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R + \overline{e}_L \begin{pmatrix} \phi^- \; \overline{\phi^0} \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L] \tag{2.10}$$

The higgs doubet has exactly the requires $SU(2) \times U(1)$ quantum number to couple to $\overline{e}_L e_R$.

we introduce a scalar 'Higgs' field, which is a doublet under SU(2), singlet under SU(3) (no colour), and has a scalar potential

$$V(\phi) = -\frac{\mu^2}{2}\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2$$

This potential has a minimum at $\phi^\dagger\phi = \frac{1}{2}\mu^2/\lambda$.In the unitary gauge, this can be written as

$$< \phi >= \begin{pmatrix} 0 \\ v \end{pmatrix}$$

we spontaneously breaks the symmetry and substitute

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} o \\ v + h(x) \end{pmatrix}$$

the neutral higgs field $h(x)$ is the only remnant of the Higgs doublet after the spontaneous symmetry breaking has take place .on substituting ϕ in Lagrangian becomes

$$\mathcal{L} = -\frac{G_e v}{\sqrt{2}}(\overline{e}_L e_R + \overline{e}_R e_L) - \frac{G_e v}{\sqrt{2}}(\overline{e}_L e_R + \overline{e}_R e_L)h$$

we now choose G_e so that

$$m_e = \frac{G_e v}{\sqrt{2}}$$

and hence generate the electron mass

$$\mathcal{L} = -m_e\bar{e}e - \frac{m_e}{v}\bar{e}e$$

Now however that, since G_e is arbitrary actual mass of the electron is not predicted .Beside the mass term Lagrangian contains an interaction term coupling the higgs scalar to the electron.

The quark masses are generate in the same way.The only novel feature is that to generate a mass for the upper member of a quark doublet , we must construct a new higgs doublet form ϕ

$$\phi_c = -i\tau_2\phi^* = \begin{pmatrix} -\overline{\phi^0} \\ \phi_c^- \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix}$$

Due to special properties of SU(2) ϕ_c, transform identical as ϕ(but has opposite weak hypercharge to ϕ, namely Y=-1). It can therefore be used to construct a gauge invariant contribution to the Lagrangian

$$\begin{aligned} \mathcal{L} &= -G_d \begin{pmatrix} \bar{u}, & \bar{d} \end{pmatrix}_L \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} d_R - G_u \begin{pmatrix} \bar{u}, & \bar{d} \end{pmatrix}_L \begin{pmatrix} -\bar{\phi}^0 \\ \phi^- \end{pmatrix} u_R + hermitian \;\; conjugate \\ &= m_d\bar{d}d - m_u\bar{u}u - \frac{m_d}{v}\bar{d}dh - \frac{m_u}{v}\bar{u}uh \end{aligned}$$

we now choose G_d *and* G_u so that

$$m_d = \frac{G_d}{\sqrt{2}}v \;\; and \;\; m_u = \frac{G_u}{\sqrt{2}}v$$

checking ϕ_c transforms like ϕ or not .

we know $\phi_c = -i\tau_2\phi*$

$$\begin{aligned} \phi_c &= -i\tau_2\phi^{*'} \;\; where \;\; \phi' = e^{i\alpha.T}\phi \\ &= -i\tau_2 e^{-i\alpha.T^*}\phi^* \end{aligned}$$

we have $\tau_2 T_i^* \tau_2 = -T_i$

multiplying σ_2 from left and σ_2 from right hand side

we get

$$T_i^* \tau_2 = -\tau_2 T_i, \tau_2 T_i^* = -T_i \tau_2$$

$$\begin{aligned}
\phi_c^{'} &= -i\tau_2 e^{-i\alpha.T^*}\phi^* \\
&= -i\tau_2[1 - i\alpha_i.T_i^* - \frac{i}{2!}(T_i^*.\alpha_i)(T_j^*.\alpha_j) - \frac{i}{3!}(T_i^*.\alpha_i)(T_j^*.\alpha_j(T_k^*.\alpha_k) + ...] \\
&= -i[\tau_2 - i\alpha_i.\tau_2 T_i^* - \frac{1}{2!}(\tau_2 T_i^*.\alpha_i)(T_j^*.\alpha_j) - \frac{1}{3!}\tau_2 T_i^*.\alpha_i)(T_j^*.\alpha_j)(T_k^*.\alpha_k) + ...]
\end{aligned}$$

$\underline{second\ \ term}$:

$$\begin{aligned}
&= i\alpha_i.\tau_2 T_i^* \\
&= -i\alpha_i.T_i\tau_2 \ \ using \ \ \tau_2 T_i^* = -T_i\tau_2
\end{aligned}$$

$\underline{Thirdterm}$:

$$\begin{aligned}
&= \frac{1}{2!}(\tau_2 T_i^*.\alpha_i)(T_j^*.\alpha_j) \\
&= \frac{1}{2!}(-T_i.\tau_2.\alpha_i)(T_j^*.\alpha_j) \\
&= -\frac{1}{2!}(T_i.\alpha_i)(\tau_2 T_j^*.\alpha_j) \\
&= \frac{1}{2!}(T_i.\alpha_i)(T_j.\alpha_j)\tau_2
\end{aligned}$$

Fourth term:

$$
\begin{aligned}
&= \frac{i}{3!}(\tau_2 T_i^*.\alpha_i)(T_j^*.\alpha_j)(T_k^*.\alpha_k) \\
&= \frac{i}{3!}(-)(T_i.\alpha_i)(\tau_2 T_k^*.\alpha_k)(T_j^*.\alpha_j) \\
&= \frac{i}{3!}(T_i.\alpha_i)(T_k.\alpha_k)(\tau_2 T_j^*.\alpha_j) \\
&= -\frac{i}{3!}(T_i.\alpha_i)(T_k.\alpha_k)(T_j.\alpha_j)\tau_2
\end{aligned}
$$

Finally we are getting

$$\phi_c^{'} = [1 + (i\alpha_i.T_i) + \frac{i^2}{2!}(T_i.\alpha_i)(T_j.\alpha_j) + ...](-i\tau_2\phi^*)$$

So ϕ_c transforms as ϕ itself.

2.11 Standard Model: Final Lagrangian

To summarize the standard (Weinberg-salam)model,we gather together all the ingredients of the Lagrangian. The complete Lagrangian is:

$$\mathcal{L} = -\frac{1}{4}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \overline{L}\gamma^\mu(i\partial_\mu - g\frac{1}{2}\tau.W_\mu - g^{'}\frac{Y}{2}B_\mu)L + \overline{R}(i\partial_\mu - g^{'}\frac{Y}{2}B_\mu)R$$

$$+|(i\partial_\mu - g\frac{1}{2}\tau.W_\mu - g^{'}\frac{Y}{2}B_\mu)|^2 - V(\phi) - (G_1\overline{L}\phi R + G_2\overline{L}\phi_c R + hermitian \;\; conjugate)$$

L denotes a left-handed fermion (lepton or quark) doublet, and R denote a right- handed fermion singlet.

2.12 Summary

• Weak interactions are mediated by the SU(2) gauge bosons, which act only on the left-handed components of fermions.

•The (left-handed) neutrino and left-handed component of the electron form an SU(2) doublet, whereas the right-handed components of the electron and neutino are SU(2) singlets. Similarly for the quarks.

•There is also a weak hypercharge U(1)Y gauge symmetry. Both left- and right-handed quarks transform under this U(1)Y with a hypercharge. The left-handed leptons and the e_R also carry hypercharge, but the ν_R has no SM gauge interactions.

•In the symmetry limit (before spontaneous symmetry breaking) the fermions with SU(2) gauge interactions are massless.The spontaneous symmetry breaking mechanism which gives a vev to the scalar field also generates the fermion masses.

•The scalar multiplet that is responsible for the spontaneous symmetry breaking also carries weak hypercharge. As a result, one neutral gauge boson (the Z) acquires a mass, whereas its orthogonal superposition is the massless photon.

•The weak interactions proceed via the exchange of massive charged or neutral gauge bosons.

Chapter 3

RENORMALIZATION

3.1 Introduction

In this chapter we face the ultraviolet divergences that we have found in perturbative.These divergences are not simply a technical nuisance to be disposed of and forgotten.As we will explain, they parameterize the dependence on quantum fluctuation at short distance scale (or equivalently,high momenta). Historically, it took a long time to reach this understanding. In the 1930's, when the ultraviolet divergences were first discovered in quantum electrodynamics, many physicists believed that fundamental principles of physics had to be changed to eliminate the divergences. In the late 1940' s Bethe,Feynman,Schwinger,Tomonaga,and others proposed a program of "re-normalization" that gave finite and physically sensible results by absorbing the divergences into redefinitions of physical quantities. This leads to calculations that agree with experiment to 8 significant digits in QED, the most accurate calculations in all of science.

Even after the technical aspects of re normalization were understood, conceptual difficulties remained. It was widely believed that only a limited class of 're-normalizable' theories made physical sense. (The fact that general relativity is not re-normalizable in this sense was therefore considered a deep problem.) Also, there normalization program was viewed by many physicists as anaphora procedure justified only by the fact that it yields physically sensible results. This was changed by the profound work of K. Wilson in the 1970's, which laid the foundation for the modern understanding of re normalization. According to the present view, re-normalization is nothing more than parametrizing the sensitivity of low-energy physics to high-energy physics. This viewpoint allows one to make sense out of 'non-renormalizable' theories as effective field theories describing physics at low energies. We now understand that even 're-normalizable' theories are effective field theories in this sense, and this viewpoint explains why nature is (approximately) described by re-normalizable theories. This modern point of view is the one we will take in this chapter.

3.2 Renormalized Perturbation Theory

To obtain a finite result for an amplitude involving diver gents diagram .We are using the following procedure: compute the diagram using a regulator to obtain an expression that depends on the bare mass (m_0) ,the bare coupling constant (e_0) and some ultraviolet cutoff ($\bigwedge$).Then compute the physical mass (m) and the physical coupling constant (e) to whatever order is consistent with the rest of the calculation;these quantities also depend on $m_0, e_0 \ \ and \ \ \bigwedge$.

The above procedure always works in a renormalizable quantum field theory .However ,it can often be cumbersome ,especially at higher orders in perturbation theory.In this section we will develop an alternative procedure which works more automatically.We will do this for ϕ^4 theory. The Lagrangian for ϕ^4 theory is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 \tag{3.1}$$

We now write m_0 and λ_0, to emphasize that these are the bare values of the mass and coupling constant, not the values measured in experimentally.Since the theory is invariant under $\phi \to -\phi$, all amplitudes with an odd number of external legs vanish.

3.3 Divergences in ϕ^4 theory

In order to renormalize the theory, we need only to consider 3 possible diagrams. These are (along with their divergences)

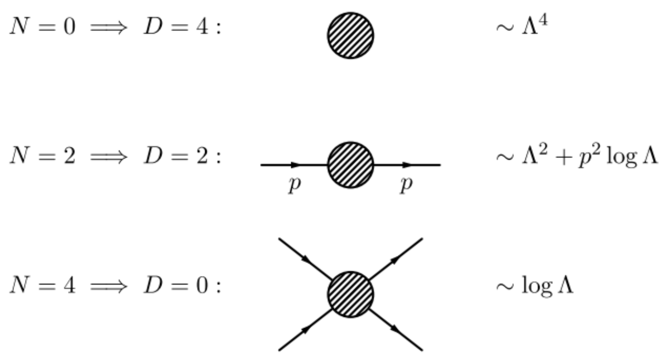


Figure 3.1: Divergences Diagrams

Therefore, there are 4 different types of inanities, each of which we must absorb into some parameter in the Lagrangian. Let us check that we have enough parameters in the bare Lagrangian.Ignoring the vacuum diagram ,these amplitude contains three infinite constant. Our goal is to absorb these constants into the three unobservable parameters of the theory:To accomplish this goal ,it is converting ,it is convenient to reformulate the perturbation expansion so that these unobservable quantities do not appear explicitly in the Feynman rules.To eliminate above three divergence we are introducing three parameter ,one parameter is bare mass ,and other two parameter is bare coupling constant,field strength parameter. First we will eliminate the shift in the field strength, rescaling the field :

$$\phi = Z^{1/2}\phi_r$$

.This transformation changes the values of correlation functions by a factor of $Z^{-1/2}$ for each field .The Lagrangian is much uglier after the rescaling :

$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu \phi_r)^2 - \frac{1}{2}m_0^2 Z\phi_r^2 - \frac{\lambda_0}{4!}Z^2\phi_r^4 \tag{3.2}$$

The bare mass and coupling constant still appear in $\mathcal{L}$,But they can be eliminated as follows .Define

$$\delta_Z = Z - 1, \delta_m = m_0^2 Z - m^2, \delta_\lambda = \lambda_0 Z^2 - \lambda$$

,

where m and λ are the physically measured mass and coupling constant.Then the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 + \frac{1}{2}\delta_Z(\partial_\mu \phi_r)^2 - \frac{1}{2}\delta_m \phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4 \tag{3.3}$$

.

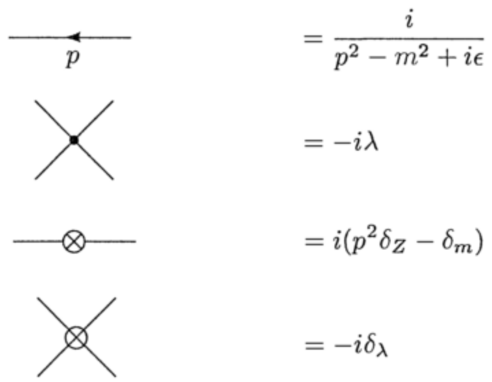


Figure 3.2: Feynman rules for ϕ^4 theory in renormalized perturbation theory.

The first line now looks like the familiar theory Lagrangian , but is written in terms of the physical mass and coupling.The terms in the second line,known as counterterms,have absorbed the infinite but unobservable shifts between the bare parameters and the physical parameters .It is tempting to say that we have "added" these counterterms to the Lagrangian ,but in fact we have merely split each term of the Lagrangian into two pieces.

The definition of δ_z, δ_m *and* δ_λ are not useful unless we give precise definitions of the physical mass and coupling constant.There is no obviously best definition of λ ,but a perfectly good definition would be obtained by setting λ equal to the magnitude of the scattering amplitude at zero momentum. Thus we have the two defining relations

These equation are called renormalization condition.(The first equation actually contains two con-

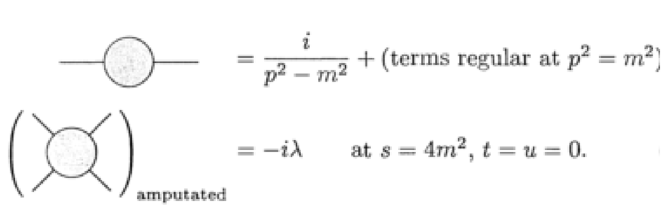


Figure 3.3: Renormalize condition

ditions, specifying both the location of the pole and its residue .)Our new Lagrangian eq. (3.3) gives a new set of Feynman rules,shown in figure (3.2).The propagator and the first vertex come from the first line of eq. (3.3) And are identical to the old rules except for the appearance of the physical mass and coupling in place of bare values.The counterterms in the second line of give two new vertices (also called counterterms).We can use these new Feynman rules to compute anu amplitude in ϕ^4 theory.The procedure is as follows Compute the desired amplitude as the sum of all possible diagrams created from the propagator and vertices.The loop integral in the diagrams will often diverge ,so one must introduce a regulator.The result of this computation will be a function of the three parameters as $\delta_z, \delta - m \;\; and \;\; \delta_\lambda$.Adjust these three parameters as necessary to maintains the renormalization condition .After the adjustment ,the expression for the amplitude should be finite and independent of the regulator.

3.4 Mandelstam Variables

Consider any kind of a 2 particles→ 2 particles process

The 4-momenta $p_1^\mu, p_2^\mu, p_1^{'\mu}, and \;\; p_2^{'\mu}$ of the 2 incoming and 2 outgoing particles are on -shell satisfy 8 constraints: the on-shell condition for each particle

$$p_1^2 = m_1^2, \;\; p_2^2 = m_2^2, \;\; p_1^{'2} = m_1^{'2}, \;\; p_2^{'2} = m_2^{'2}$$

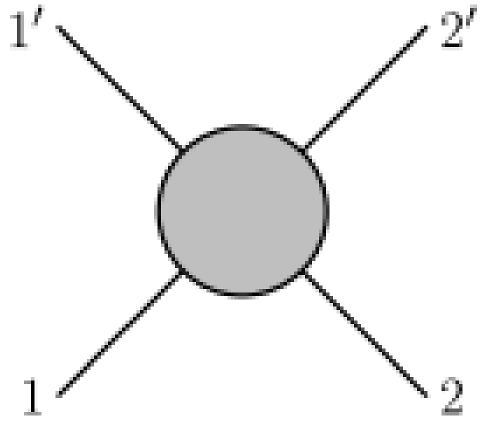


Figure 3.4: 2 particles $\rightarrow 2particle\ \ process$

and the net 4-momentum conservation

$$p_1 + p_2 - p_1^{'} - p_2^{'} = 0$$

.

consequently ,all Lorentz invariant combination of the 4 external may be expressed in terms of the particles masses and 3 Mandelstam's variable

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_1^{'} + p_2^{'})^2, \\ t &= (p_1 - p_1^{'})^2 = (p_2^{'} - p_2)^2, \\ u &= (p_1 - p_2^{'})^2 = (p_1^{'} - p_2^{'})^2. \end{aligned}$$

Moreover ,only 2 out of these 3 variable are independent while their sum has a fixed value

$$s + t + u = m_1^2 + m_2^2 + m_3^{'2} + m_2^{'2} \tag{3.4}$$

Indeed,

$$
\begin{aligned}
s+t+u &= (p_1+p_2)^2+(p_1-p_1^{'})^2+(p_1-p_2^{'})^2 \\
&= 3p_1^2+p_2^2+p_1^{'2}+p_2^{'2}+2(p_1p_2)-2(p_1p_1^{'})-2(p_1p_2^{'2}) \\
&= p_1^2+p_2^2+p_1^{'2}+p_2^{'2}+2p_1\times(p_1+p_2-p_1^{'}-p_2^{'}=0) \\
&= p_1^2+p_2^2+p_1^{'2}+p_2^{'2} \\
&= m_1^2+m_2^2+m_1^{'2}+m_2^{'2}.
\end{aligned}
$$

3.5 Dimensional Regularization

The idea of dimensional regularization is very simple to state: compute the Feynman diagram as an analytic function of the dimensionality of space -time,d.For sufficient small d ,any loop-momentum integral will converge and therefore the Ward identity can be proved.The final expression for any observable quantity should have a well -defined limit as $d \to 4$.

Let us do a practice calculation to see how this technically works.We consider spacetime to have one time dimension and (d-1) space dimensions. Then we can Wick-rotate Feynman integrals as before ,to give integrals over a d-dimensional Eucliden space .A typical example is

$$
\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+\triangle)^2} = \int \frac{d\Omega_d}{(2\pi)^d}.\int_0^\infty dl_E \frac{l_E^{d-1}}{(l_E^2+\triangle)^2}
$$

The first factor in above integration contains the area of a unit sphere in d dimensions. To compute it ,use the following trick:

$$
\begin{aligned}
(\sqrt{\pi})^d &= (\int dx e^{-x^2})^d = \int d^d x \exp(-\sum_{i=1}^{d} x_i^2) \\
&= \int d\Omega_d \int_0^\infty dx x^{d-1} e^{-x^2} = (\int d\Omega_d).\frac{1}{2}\int_0^\infty d(x^2)(x^2)^{\frac{d}{2}-1}e^{-x^2} \\
&= (\int d\Omega_d).\frac{1}{2}\Gamma\,(d/2)
\end{aligned}
$$

So the area of a d-dimentional unit sphere is

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(x)}$$

The second factor is

$$\int_0^\infty \frac{l^{d-1}}{(l^2+\triangle)^2} = \frac{1}{2}\int_0^\infty d(l^2)\frac{(l^2)^{d/2-1}}{l^2+\triangle})^2 = \frac{1}{2}(\frac{1}{\triangle})^{2-d/2}\int_0^1 dx x^{1-d/2}(1-x)^{d/2-1}$$

where we have substitute $x = \frac{\triangle}{(l^2+\triangle)}$ in the second line .using the definition of the beta funtion ,

$$\int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

we can easily evalute the integral over x. The final result for the d-dimensional integral is

$$int\frac{d^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+\triangle)^2} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}(\frac{1}{\triangle})^{2-d/2} \tag{3.5}$$

Since $\Gamma(Z)$ has isolated poles at z=o,-1,-2,.....,this integral has isolated poles at d=4,6,8,....TO find the behavior near d=4,define $\epsilon = 4-d$, and use the approximation

$$\Gamma(2-d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + O(\epsilon)$$

where $\gamma \approx .5772$ is the Euler -Mascheroni constant . The integral is then

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \triangle)^2} \to \frac{1}{(4\pi)^2}(\frac{2}{\epsilon} - \log \triangle - \gamma + O(\epsilon))$$

3.6 One -Loop structure of ϕ^4 Theory

To make more sense of the Renormalization procedure ,Let us carry it out explicitly at the one loop level.First the basic two -particle scattering amplitude

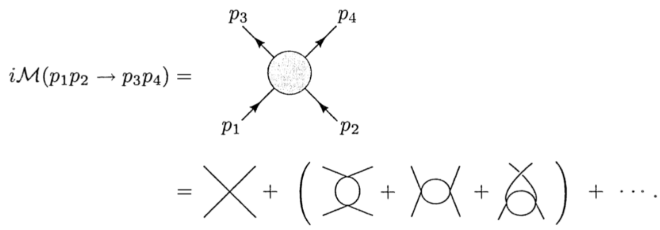


Figure 3.5: Contribution to first loop

The amplitude are

$$i\mathcal{M} = -i\lambda + \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k + p_1 - p_3)^2 - m^2} + \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k + p_1 - p_4)^2 - m^2}$$
$$+ \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k + p_1 + p_2)^2 - m^2}$$
$$= -i\lambda + (-i\lambda)^2[iV(s) + iV(t) + V(u)] - i\delta_\lambda$$

,if we define p=$p_1 + P_2$

$$V(p^2) = \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k + p)^2 - m^2} \tag{3.6}$$

This integral is of course infinite. We need to regulate this integral. We use dimensional regularization to do this.

According to our renormalization condition ,this amplitude should equal to $-i\lambda$ at $s = 4m^2$ and t=u=o.we must therefore set

$$\delta_\lambda = -\lambda^2[V(4m^2) + 2V(0)].$$

we can compute$V(p^2)$ explicitly using dimensional regularization .Introduce a Feynman parameter ,shift the integration variable ,rotate to Euclidean space ,and perform the momentum integral. we obtain

$$\begin{aligned} V(p^2) &= \frac{i}{2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 + 2xk.p + xp^2 - m^2]^2} && (3.7) \\ &= \frac{i}{2}\int_0^1 \int \frac{d^dk}{(2\pi)^d} \frac{1}{[l^2 + x(1-x)p^2 - m^2]^2} \quad (l = k + xp) && (3.8) \\ &= -\frac{1}{2}\int_0^1 \int \frac{d^dl_E}{(2\pi)^d} \frac{1}{[l_E^2 - x(1-x)p^2 + m^2]^2} \quad (l_E^0 = -il^0) && (3.9) \\ &= \frac{i}{2}\int_0^1 dx \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \frac{1}{[m^2 - x(1-x)p^2]^{2-d/2}} && (3.10) \end{aligned}$$

$$\to -\frac{1}{32\pi^2}\int_0^1 dx(\frac{2}{\epsilon} - \gamma + \log(4\pi) - log[m^2 - x(1-x)p^2]),$$

where $\epsilon = 4 - d$.Last line in the above equation we get by following way

$$\frac{1}{(4\pi)^{d/2}} \frac{1}{[m^2 - x(1-x)p^2]^{2-d/2}} = a^{\epsilon/2} \quad where \quad a = \frac{4\pi}{m^2 - x(1-x)p^2}$$

$$= a^{\epsilon/2}$$

$$= e^{\log a^{\epsilon/2}}$$

$$= e^{\frac{\epsilon}{2}\log a}$$

$$= 1 + \frac{\epsilon}{2}\log a + ...$$

Again

$$\Gamma\left(2-d/2\right)=\Gamma\left(\epsilon/2\right)=\frac{2}{\epsilon}-\gamma+O(\epsilon^2)$$

Multiplication of this two give

$$=-\frac{1}{2}\int_0^1\frac{dx}{(4\pi)^2}(\frac{2}{\epsilon}-\gamma+O(\epsilon^2))(1+\frac{\epsilon}{2}\log a+.......)$$

$$\rightarrow -\frac{1}{32\pi^2}\int_0^1 dx(\frac{2}{\epsilon}-\gamma+\log(4\pi)-log[m^2-x(1-x)p^2])$$

$$\delta_\lambda=\frac{(\lambda)^2}{2}\frac{\Gamma\left(2-d/2\right)}{(4\pi)^{d/2}}\int_0^1 dx(\frac{1}{[m^2-x(1-x)p^2]^{2-d/2}}+\frac{2}{[m^2]^{2-d/2}})$$

$$\rightarrow\frac{\lambda^2}{32\pi^2}\int_0^1 dx(\frac{6}{\epsilon}-3\gamma+3\log(4\pi)-\log[m^2-x(1-x)4m^2]-2\log[m^2]).$$

These expressions are divergent as $d\rightarrow 4$.But if we combine them according to (10.21),we obtain the finite (if rather complicated)result,

$$i\mathcal{M}=-i\lambda-\frac{i\lambda^2}{32\pi^2}\int_0^1 dx[\log(\frac{m^2-x(1-x)s}{m^2-x(1-x)4m^2})+\log(\frac{m^2-x(1-x)t}{m^2})+\log(\frac{m^2-x(1-x)u}{m^2})]$$

To determine δ_Z and δ_m we must compute the two-point function.Let us define $-iM(p^2)$ as the sum of all one -particle irreducible insertions into the propagator: Then the exact propagator can

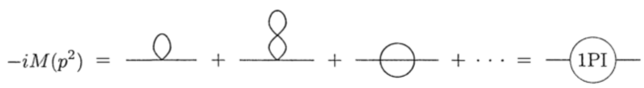


Figure 3.6: One particle -irreducible

be written as a geometric series and summed as

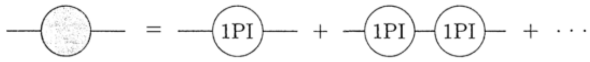


Figure 3.7: One particle -irreducible

$$= \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}(-iM^2)\frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}(-iM^2)\frac{i}{p^2 - m^2}(-iM^2)\frac{i}{p^2 - m^2} + ...$$

Taking common $\frac{i}{p^2-m^2}$

$$\begin{aligned}
&= \frac{i}{p^2 - m^2}[1 - \frac{iM^2 i}{p^2 - m^2} + \frac{(iM^2)^2 i^2}{(p^2 - m^2)^2} + ...] \\
&= \frac{i}{p^2 - m^2}[1 + \frac{iM^2 i}{p^2 - m^2}]^{-1} \\
&= \frac{i}{(p^2 - m^2)} \frac{1}{(1 + \frac{iM^2 i}{(p^2-m^2)})} \\
&= \frac{i}{p^2 - m^2 - M(p^2)}
\end{aligned}$$

The renormalization condition require that the pole in this full propagator occur at $p^2 = m^2$ and have residue 1.These two condition are equivalent ,respectively ,to

$$M(p^2)|_{p^2=m^2} = 0 \;\; and \;\; \frac{dM(p^2)}{dp^2}|_{p^2=m^2}$$

(To check the latter condition,expand M about $p^2 = m^2$)

Explicitly ,to one-loop,

$$-iM(p^2 = -i\lambda.\frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\frac{i}{(k^2 - m^2)} + i(p^2\delta_Z - \delta_m)$$

$$= \frac{i\lambda}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma\,(1 - d/2)}{(m^2)^{1-d/2}} + i(p^2\delta_Z - \delta_m)$$

Since the first term is independent of p^2,the result is rather trivial : setting

$$\delta_Z = 0 \quad and \quad \delta_m = -\frac{\lambda}{2(4\pi)^{d/2}} \frac{\Gamma\,(1 - d/2)}{(m^2)^{1-d/2}}$$

yields $M(p^2) = 0$ for all p^2 ,satisfying both of the condition.

Chapter 4

The Callen -Symanzik Equation

4.1 Introduction

We consider first the specific case of ϕ^4 theory in four dimensional ,where the coupling constant λ is dimensionless and the corresponding operator is marginal .For simplicity ,we will also assume that the mass term m^2 has been adjust to zero ,so that the theory sits just at its critical point.We will perform this analysis in Minkowski space ,using spacelike reference momenta.However ,the analysis would be essential identical if carried out in Euclidean space .If we wish to consider renormalization group prediction at timelike momenta,we must considered the possibilities of new singularities which make the analysis more complicated.

4.2 Renormalization conditions

To define the theory properly ,we must specify the renormalization condition.In previous chapter We used a natural set of renormalization condition for ϕ^4 theory,defined in terms of the physical mass m.However ,in a theory where m=0,these conditions cannot be used because they lead to singularities in counterterms.(consider,for example the limit $m^2 \to 0$) of equ.(10.24).To avoid such singularities ,we choose an arbitary momentum scale M and impose the renormalization condition at a spacelike momenta p with $p^2 = -M^2$:

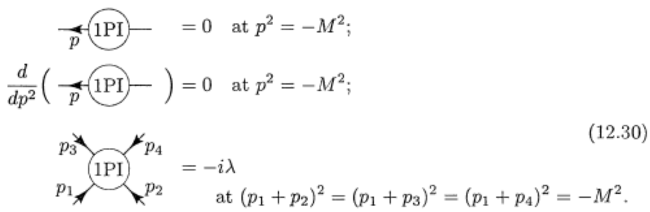


Figure 4.1: renormalization condition

The parameter M is called the renormalization scale.These condition define the values of the two -and four point Green's function at a certain point and in the process,remove all ultraviolet divergences.Speacking loosely ,we say that we are "defining the theory at the scale M".These new renormalization conditions take some getting used to ,The second condition ,in particular ,implies that the two-point Greens function has a coefficient of 1 at the unphysical momentum $p^2 = -M^2$,rather than on shell (at $p^2 = 0$):

$$< \Omega|\phi(p)\phi(-p)|\Omega >= \frac{i}{p^2} \quad at \quad p^2 = -M^2$$

Here ϕ is the renormalized field ,related to the bare field ϕ_0 by a scale factor that we again call Z:

$$\phi = Z^{-1/2}\phi_0$$

.

This Z,however is not the residue of the physical pole in the two point Green's function of bare fields.Insted ,we now have

$$< \Omega|\phi_0(p)\phi_0(-p)|\Omega >= \frac{iZ}{p^2} \quad at \quad p^2 = -M^2.$$

The Feynman rules for renormalized perturbation theory are the same as in previous chapter,with the same relation between Z and the counterterm $\delta_Z = Z - 1$.

Now ,however ,the counterterms δ_Z ans δ_λ must be adjusted to maintain the new conditions (12.30).

The renormalization condition in (12.30) holds the physical mass of the scalar field at zero.we sae in previous chapter that ,in ϕ^4 theory,the one -loop propagator correction is momentum-independent and is completely canceled by the mass renormalization counterterm.At two level order ,how ever ,the situation becomes more complicated and the propagator corrections require both mass and field strength renormalization.

In the renormalization condition (12.30) the renormalization scale M is arbitrary,We could just as well have defined the same theory at a different scale $M^{'}$. By "the same theory", we mean a theory whose bare Green's functions,

$$< \Omega|T\phi_0(x_1)\phi_0(x_2)..................\phi_0(x_n)|\Omega >,$$

are given by the same function of the bare coupling constant λ_0 and the cutoff $\bigwedge$.These function make no reference to M.The dependence on M enters only when we remove the cutoff dependence b by rescaling the fields and eliminating λ_0 in favor of the renormalized coupling λ.The renormalized Green's functions are numerically equal to the bare Green's functions,up to a rescaling by powers of the field strength renormalization Z:

$$< \Omega|T\phi(x_1)\phi(x_2)............\phi(x_n)|\Omega >= Z^{-n/2} < \Omega|T\phi_0(x_1)\phi_0(x_2)..................\phi_0(x_n)|\Omega > \quad (4.1)$$

The renormalised Green's functions could be defined equally well at another scale $M^{'}$,using a new renormalized coupling $\lambda^{'}$ and a new rescaling factor $Z^{'}$.

Let us write more explicitly the effect of an infinitesimal shift of M.Let

$$G^{(n)}(x_1, ..., x_n) =< \Omega|T\phi(x_1)...............\phi(x_n)|\Omega >_{connected} .$$

Now suppose that we shift M and δM.There is a corresponding shift in the coupling constant and the field strength such that the bare Green's functions remain fixed:

$$M \to M + \delta M$$

,

$$\lambda \to \lambda + \delta\lambda$$

$$\phi \to (1 + \delta\eta)\phi$$

.

Then the shift in any renormalized Green's function is simply that induced by the field rescaling ,

$$G^n \to (1 + n\delta\eta)G^n.$$

If we think of G^n as a function of M and λ,we can write this transformation as

$$dG^n = \frac{\partial G^(n)}{\partial M}\delta M + \frac{\partial G^(n)}{\partial \lambda}\delta\lambda = n\delta\eta G^{(n)}.$$

Rather than writing this relation in terms of $\delta\lambda$ and $\delta\eta$, it is conventional to define the dimensionless parameters

$$\beta \equiv \frac{M}{\delta M}\delta\lambda; \gamma \equiv -\frac{M}{\delta M}\delta\eta.$$

Making these substitution in above equation and multiplying through by $M/\delta M$,
we obtain

$$[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + n\gamma]G^{(n)}(x_1, ...x_n; M, \lambda) = 0. \tag{4.2}$$

The parameters β and γ are the same for every n,and must be independent of the x_i.Since the Green's function G^n is renormalized ,β and γ cannot depend on the cutoff,and hence ,by dimensional analysis,these functions cannot depend on M.Therefore they are functions only of the dimensionless variable λ.we conclude that any Green's function of massless ϕ^4 theory must satisfy

$$[M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+n\gamma(\lambda)]G^{(n)}(x_i;M,\lambda)=0.$$

This relation is called the Callan-Symanzik equation.It asserts that there exist two-universal functions $\beta(\lambda)$ and $\gamma(\lambda)$,related to the shifts in the coupling constant and field strength,that compensate for the shift in the renormalization scale M.

4.3 Computation of β and γ

Before we work out the implication of the callan-symanzik equation ,let us look more closely at the functions β γ that appear in it.From their definitions ,we see that they are proportional to the shift in the coupling constant and the shift in the field normalization,respectively ,when the renormalization scale M is increased.The behavior of the coupling constant as a function of M is of particular interest,since it determine the strength of the interaction and the conditions under which perturbation theory is valid.

The easiest way to compute the Callan-Symanzik function is to begin with explicit perturbative expressions for some conveniently chosen Green's functions.If we insist that these expressions satisfy the callan-Symanzik equation,we will obtain equation that can be solved for β and γ.Because the M dependence of a renormalized Green's function originates in the counter-terms that cancel its logarithmic divergences, we will find that theβ and γ function are simply related to these counterterms,or equivalently to the coefficient of the divergent logarithms.At one -loop order ,however ,the expressions for β and γ are simple and unambiguous.

as a first example ,let us calculate the one -loop contributions to $\beta(\lambda)$ and $\gamma(\lambda)$ in massless ϕ^4 theory.We can simplify the analysis by working in momentum space rather than coordinate space.Our strategy will be to apply the Callen-Symanzik equation to the diagrammatic expression for the two and four -point Green's functions.The two point function is given by In massless ϕ^4 theory ,the one -loop propagator correction is completely canceled by the mass counterterm.Then the first nontrivial

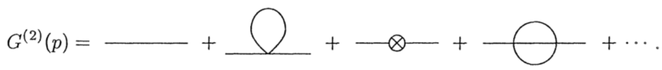


Figure 4.2: Two-point function

correction to the propagator comes from the two-loop diagram and its counterterms, and is of order λ^2.Meanwhile ,the four -point function is given by where we have omitted the canceled one-loop

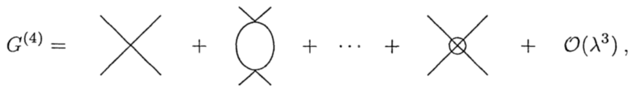


Figure 4.3: Four-point function

propagator correction to the external legs.The diagrams of order λ^3 include nonvanishing two loop propagator corrections to the external legs.To calculate β,we apply the callan-Symanzik equation to the four-point function:

$$[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial \lambda} + 4\gamma(\lambda)]G^{(4)}(p_1, ...p_4) = 0$$

Borrowing our result from previous section ,we can write $G^{(4)}$ as

$$G^{(4)} = [-i\lambda + (-i\lambda)^2[iV(s) + iV(t) + V(u)] - i\delta_\lambda]. \prod_{i=1,..4.} \frac{i}{p_i^2} \tag{4.3}$$

where $V(s)$ represents thge loop integral that we know from previous section.Our renormalization condition (12.30) requires that the correction terms cancel at $s = u = t = -M^2$. The order λ^2 vertex counterterm is therefore

$$\delta_\lambda = (-i\lambda)^2.3V(-M^2) = \frac{3\lambda^2}{2(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma\,(2 - d/2)}{(x(1-x)M^2)^{2-D/2}}.$$

The last expression follows from setting m=0 and $p^2 = -M^2$. In the limit as $d \to 4$, then

$$\delta_\lambda = \frac{3\lambda^2}{2(4\pi)^2}[\frac{1}{2 - d/2} - \log M^2 + finite] \tag{4.4}$$

where the finite terms are independent of M.This counterterm gives $G^{(4)}$ its M dependence:

$$M\frac{\partial G^{(4)}}{\partial M} = \frac{3i\lambda^2}{(4\pi)^2} \tag{4.5}$$

Then the the Callen-Symanzik equation can be satisfied to order λ^2 only if the β function of ϕ^4 theory is given by

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3) \tag{4.6}$$

.Next consider the Callan-Symanzik equation for two the two point function:

$$[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial \lambda} + 2\gamma(\lambda)]G^{(2)}(p) = 0$$

and

$$G^2 = \frac{i}{p^2} + (-i\lambda)\frac{1}{2}\int \frac{d^dk}{(2\pi^2)}\frac{i}{p^2} + i(p^2\delta_z - \delta_m) \;\; upto \;\; \lambda \;\; order$$

Using following renormalization condition The value of $\delta_Z = 0, \;\; so \;\; \delta\eta = 0$. As we know the defini-

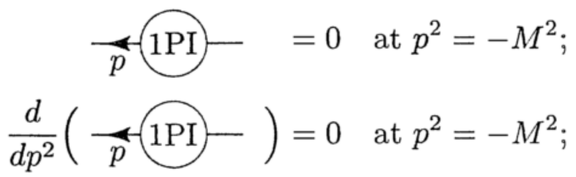


Figure 4.4: Renormalization condition

tion of $\gamma \equiv -\frac{M}{\delta M}\delta\eta which \;\; is \;\; zero \;\; from \;\; the \;\; above \;\; condition.$Thus the γ is zero to this order:

$$\gamma = 0 + O(\lambda^2) \tag{4.7}$$

4.4 Vacuum stability

So far we get the value $\beta(\lambda)$ function $\beta(\lambda) = \frac{3\lambda^2}{16\pi^2}$

Again

$$\beta(\lambda) = M\frac{M}{\delta M}\delta\lambda = \frac{d\lambda}{d(\log M}) = \frac{3\lambda^2}{16\pi^2} \quad (4.8)$$

From the above equation it is clear that the value of coupling constant depend on the energy scale . Also there is interaction of the Higgs boson with fermions and gauge bosons.For this interaction this eq will be modified. This modified form is

$$\frac{d\lambda}{dt} = \frac{1}{16\pi^2}[12\lambda^2 + 12\lambda g_t^2 - 12g_t^4 - \frac{3}{2}\lambda(3g^2 + g^{'2}) + \frac{3}{16}(2g^4 + (g^2 + g^{'2})^2)]. \quad (4.9)$$

If we solve this differential equation using mass of the Higgs boson,we will get a relation between λ with energy .we know $V(\phi) = \frac{\mu^2}{2}(\phi^\dagger\phi) + \frac{\lambda}{4}(\phi^\dagger\phi)^2$.In the following graph with condition $\mu^2 < 0$ and $\lambda > 0$ is bound from bellow and for $\lambda < 0$,$\mu^2 < 0$ graph is unbound.

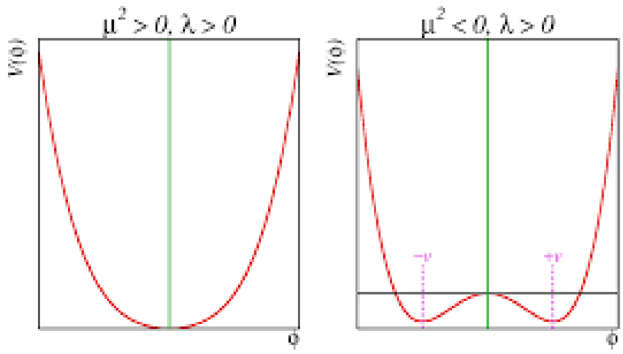
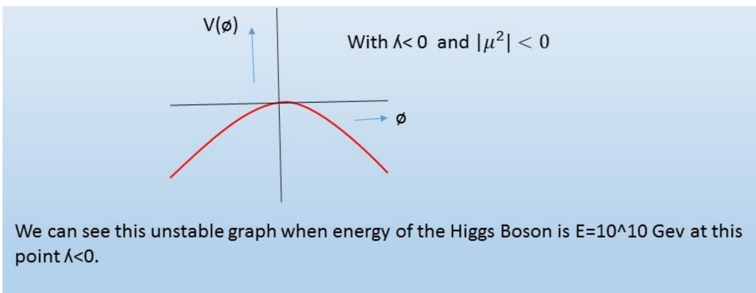


Figure 4.5: Higgs potential graph with different condition



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