

# **Oversampling of Fourier Coefficients : Theory and Applications**

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*by*

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## Approval Sheet

This thesis entitled **Oversampling of Fourier Coefficients: Theory and Applications** by **Diamond Raj Oraon** is approved for the degree of **Master of Science** from **IIT Hyderabad**.

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# Abstract

The objective behind the proposal studies around oversampling of Fourier coefficients and their applications.

I intend to study how oversampling of Fourier coefficients can be used for hiding messages. In the following report oversampling of Fourier coefficients has been discussed as providing room for storing or transmitting hidden information.

The scheme aims at transmitting an arbitrary signal and, simultaneously, embedding a hidden code. The most important feature of this scheme is that without the knowledge of exact oversampling parameter the hidden code cannot be retrieved. This parameter provide the key for retrieving the code.

# Chapter 1

## Fourier Analysis

Fourier analysis is the most important tool in the construction of the wavelet theory. This chapter relies on the well known theorems and formulas relating to Fourier series as well as on the basic understanding of Fourier transform on  $\mathbb{R}$ . In this chapter we give the account of the fundamentals of the Fourier analysis which play decisive role in all works of wavelets.

### 1.1 Fourier Series

Fourier series is mainly concerned with the periodic functions. Let the basic function space be  $L_o^2 := L^2(\mathbb{R}/2\pi)$ . The points in this space are measurable  $2\pi$ - periodic functions :

$$f(t + 2\pi) = f(t) \quad \forall t \in \mathbb{R},$$

for which the integral

$$\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt$$

is finite. The formula

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt$$

defines a scalar product on  $L_o^2$ . To this scalar product belongs the norm

$$\|f\| := \sqrt{\langle f, f \rangle} = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt \right)^{\frac{1}{2}}$$

and the distance function  $d(f, g) := \|f - g\|$ . With regard to this distance function, space  $L_o^2$  becomes a complete metric space, which means that Cauchy sequences of functions  $f_n \in L_o^2$  are automatically convergent to some point  $f \in L_o^2$ .  $L_o^2$  is also a vector space over  $\mathbb{C}$  and it is an example of a (complex) Hilbert space. Now define the functions

$$\mathbf{e}_k : t \mapsto e^{ikt} = \cos(kt) + i \sin(kt) \quad (k \in \mathbb{Z})$$

are  $2\pi$ - periodic, and because of

$$\langle \mathbf{e}_j, \mathbf{e}_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{i(j-k)t} dt = \begin{cases} 1 & (j = k) \\ \frac{1}{2\pi(j-k)} e^{i(j-k)t} \Big|_0^{2\pi} & (j \neq k) \end{cases}$$

The set  $\mathbf{e}_k$  forms an orthonormal system in  $L^2_o$  and hence is linearly independent. Any  $f \in L^2_o$  has Fourier coefficients

$$c_k := \hat{f}(k) := \langle f, \mathbf{e}_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt, \quad (k \in \mathbb{Z}) \quad (1.1)$$

where  $c_k$  is the  $k$ -th coordinate of  $f$  with respect to the orthonormal basis  $(\mathbf{e}_k | k \in \mathbb{Z})$ . The following so-called Riemann-Lebesgue lemma holds:

$$\lim_{k \rightarrow \pm\infty} c_k = 0$$

**Lemma 1** (Riemann-Lebesgue). *If  $f$  is  $L^1$  integrable) lebesgue integral of  $|f|$  is finite then the Fourier transform of  $f$  satisfies*

$$c_k := \hat{f}(k) = \int_0^{2\pi} f(t) e^{-ikt} dt \rightarrow 0 \quad \text{as} \quad |k| \rightarrow \infty.$$

But the central result of  $L^2_o$ - theory is Parseval's formula.

**Parseval Formula:** It says that the scalar product of any functions  $f$  and  $g \in L^2_o$  coincides with the "formal scalar product" of the corresponding coefficients vectors  $\hat{f}$  and  $\hat{g}$ : For arbitrary  $f$  and  $g \in L^2_o$ , the equality

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) \overline{\hat{g}(k)} = \langle f, g \rangle$$

is valid; in particular, one has  $\sum_{k=-\infty}^{\infty} |c_k|^2 = \|f\|^2$

**Proof:** As we know that the Fourier coefficients can be given by (1.1) we have:

$$c_k = \hat{f}(k) = \langle f, \mathbf{e}_k \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt \quad (k \in \mathbb{Z}).$$

Simialrly:

$$c_{-k} = \overline{\hat{g}(k)} = \overline{\langle g, \mathbf{e}_k \rangle} = \frac{1}{2\pi} \int_0^{2\pi} \overline{g(t)} e^{ikt} dt \quad (k \in \mathbb{Z}).$$

Now

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \hat{f}(k) \overline{\hat{g}(k)} &= \sum_{k=-\infty}^{\infty} \langle f, \mathbf{e}_k \rangle \overline{\langle g, \mathbf{e}_k \rangle} \\ &= \langle f, g \rangle \quad (f, g \in L^2_o). \end{aligned}$$



In particular if  $f = g$  then we have,

$$\sum_{k=-\infty}^{\infty} c_k c_{-k} = \langle f, f \rangle$$

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \|f\|^2 \quad (\text{proved}).$$

Now the Fourier coefficients of  $f$ , form the series

$$\sum_{k=-\infty}^{\infty} c_k \mathbf{e}_k \tag{1.2}$$

called the (formal) Fourier series of  $f$ . Occasionally we write

$$f(t) \rightsquigarrow \sum_{k=-\infty}^{\infty} c_k e^{ikt} \tag{1.3}$$

to express that the series(1.2) belongs to the given function  $f$ . The analogies between the geometries of  $L^2_o$  and of  $\mathbb{R}^n$  lead one to conjecture that the series (1.2) “represent” the function  $f$  in certain sense. In this regard we study the following:

The series (1.2) has sequence of partial sums:

$$s_N := \sum_{k=-N}^N c_k \mathbf{e}_k$$

and the partial sums be:

$$s_N(t) := \sum_{k=-N}^N c_k e^{ikt}$$

where  $s_N$  is nothing but the orthogonal projection of  $f$  onto the  $(2N + 1)$ - dimensional subspace

$$U_N := \text{span}(\mathbf{e}_N, \dots, \mathbf{e}_0, \dots, \mathbf{e}_{-N}) \subset L^2_o.$$

formed by all linear combinations of the  $\mathbf{e}_k$  having  $|k| \leq N$ . In particular  $s_N$  is orthogonal to  $f - s_N$  then by Pythagoras theorem we have

$$\|f - s_N\|^2 = \|f\|^2 - \|s_N\|^2 = \|f\|^2 - \sum_{k=-N}^N |c_k|^2.$$

On account of Parseval formula, we conclude that

$$\lim_{N \rightarrow \infty} \|f - s_N\|^2 = 0.$$

Thus the formal Fourier series of a function  $f \in L^2_o$  converges to  $f$  in the sense of the

$L^2$ - metric.

**Pointwise Convergence :** The pointwise convergence of  $s_N(t)$  to  $f(t)$  is given by the Carleson's theorem(1966).

**Carleson's theorem :** The partial sums  $s_N(t)$  of a function  $f \in L^2_o$  converge to  $f(t)$  for almost all  $t$ .

**Uniform Convergence :** Let the function  $f : \mathbb{R}/2\pi \rightarrow \mathbb{C}$  be continuous and of bounded variation. Then the partial sums  $s_N(t)$  of the Fourier series of  $f$  converge for  $n \rightarrow \infty$  uniformly on  $\mathbb{R}/2\pi$  to  $f(t)$ .

### Generalization

Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a periodic function with period  $L > 0$ , and suppose  $\int_0^L |f(x)|^2 dx < \infty$ . Then the formal Fourier series of  $f$  is given by

$$f(x) \rightsquigarrow \sum_{k=-\infty}^{\infty} c_k e^{2k\pi i x/L},$$
$$c_k := \frac{1}{L} \int_0^L f(x) e^{-2k\pi i x/L} dx$$

and the Parseval's formula appears as

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{L} \int_0^L |f(x)|^2 dx.$$

## 1.2 Fourier Transform

The Fourier transform  $\hat{f}$  of the function  $f \in L^1$  is given by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt \quad (\xi \in \mathbb{R}).$$

Here we will use the properties of the Fourier transform which are as follows:  
Now for any time signal  $f$  and arbitrary  $h \in \mathbb{R}$  the is defined by

$$T_h f(t) := f(t - h).$$

Let  $f \in L^1$  then the Fourier transform is given by  
(R1)

$$(T_h f)(\xi) = e^{-i\xi h} \hat{f}(\xi),$$

which may be expressed as follows: If  $f$  is translated by  $h$  to the right along the time axis, then Fourier transform  $\hat{f}$  picks up a factor  $e_{-h}$ .

Consider an arbitrary signal  $f \in L^1$  and modulate  $f$  with a pure oscillation  $\mathbf{e}_\omega, \omega \in \mathbb{R}$ ; that is to say, consider the function  $g(t) := e^{i\omega t} f(t)$ . Then the Fourier transform of  $g$  is given by

(R2)

$$(\widehat{\mathbf{e}_\omega f})(\xi) = \widehat{f}(\xi - \omega).$$

That is if the signal  $f$  is modulated with  $\mathbf{e}_\omega$ , then the graph of  $\widehat{f}$  is translated by  $\omega$  (to the right, if  $\omega > 0$ ) on the  $\xi$ -axis.

Now for any time signal  $f$  and for arbitrary  $a \in \mathbb{R}^*$  the function  $D_a f$  is defined by

$$D_a f(t) := f\left(\frac{t}{a}\right).$$

Then the Fourier transform is given by

(R3)

$$(\widehat{D_a f})(\xi) = |a| \widehat{f}(\xi) \quad (a \in \mathbb{R}^*).$$

If the graph of  $f$  is stretched horizontally by a factor  $a > 1$ , then the graph of  $\widehat{f}$  is compressed horizontally to the fraction  $\frac{1}{a} < 1$  of its original width; moreover it is scaled vertically by the factor  $|a|$ .

Let  $f$  be a  $C^1$  function and assume that  $f$  as well as  $f'$  are integrable, i.e., in  $L^1$  then the Fourier transform of the derivative can be given by

(R4)

$$\widehat{f'}(\xi) = i\xi \widehat{f}(\xi).$$

Consider an  $f \in L^1$  decaying for  $|t| \rightarrow \infty$  at least fast enough to make the integral  $\int |t| |f(t)| dt$  convergent. Denote the function  $t \rightarrow tf(t)$  by  $tf$  for short and assume  $tf \in L^1$ . then the Fourier transform is given by

(R5)

$$(\widehat{tf})(\xi) = i(\widehat{f})'(\xi).$$

### 1.3 The Heisenberg uncertainty principle

In context of signal processing and in particular time-frequency analysis one cannot simultaneously sharply localize a signal (function  $f$ ) in both the time-domain and the frequency-domain ( $\widehat{f}$ , its Fourier transform). This plays an important role in quantum mechanics, wherein the motion of particle is described by a certain function  $\psi \in \mathbf{S}$  in the following way:

$f_X(x) := |\psi(x)|^2$  as the probability density for the position  $X$  is taken as random variable,  $f_p(\xi) := |\widehat{\psi}(\xi)|^2$  is the corresponding density for its momentum. Here we have tacitly assumed  $\psi \in L^2$ , and, for the probabilistic interpretation,

$$\|\psi\|^2 = \int f_X(x) dx = 1$$

. The quantity

$$\int x^2 f_X(x) dx = \int x^2 \|\psi(x)\|^2 dx =: \|x\psi\|^2$$

is the expectation of the random variable  $X^2$  and consequently a measure for the horizontal spread of the function  $\psi$ . Similarly, the integral

$$\int \xi^2 f_P(\xi) d\xi = \int \xi^2 \|\hat{\psi}\|^2 d\xi =: \|\xi\hat{\psi}\|^2$$

can be regarded as a measure of the spread of  $\hat{\psi}$  over the  $\xi$ -axis. In terms of these quantities, the Heisenberg uncertainty principle can be formulated as follows:

**Theorem :** Let  $\psi$  be an arbitrary function in  $L^2$ . Then

$$\|x\psi\| \cdot \|\xi\hat{\psi}\| \geq \frac{1}{2} \|\psi\|^2, \quad (1.4)$$

the left-hand side is allowed to assume the value  $\infty$ . The equality sign is valid exactly for the constant multiples of the functions  $x \mapsto e^{-cx^2}$ ,  $c > 0$ .

**Proof :** If  $\|x\psi\| = \infty$  or  $\|\xi\hat{\psi}\| = \infty$  then nothing to prove as

$$\frac{1}{2} \|\psi\|^2 \leq \infty$$

which is true. In this case at least one of the two functions  $\psi$  and  $\hat{\psi}$  is definitely very spread out which is to say that both cannot be greater than  $\frac{1}{2} \|\psi\|^2$ , at least one has to be greater. So we can assume that left-hand side of (1.4) is finite and we will prove this inequality first for functions  $\psi \in \mathbf{S}$ . With respect to this additional hypothesis all convergence questions are moved out of the way; in particular, we have  $\lim_{x \rightarrow \pm\infty} x|\psi(x)|^2 = 0$ .

The Fourier transform  $\hat{\psi}$  may be eliminated from (1.4) by means of rule(R4) which is:

$$\hat{f}'(\xi) = i\xi\hat{f}(\xi)$$

and the Parseval's formula

$$\|\hat{f}\|^2 = \|f\|^2.$$

One has

$$\|\xi\hat{\psi}\| = \|\hat{\psi}'\| = \|\psi'\|$$

from which it follows that the stated inequality (1.4) is equivalent to

$$\|x\psi\| \cdot \|\psi'\| \geq \frac{1}{2} \|\psi\|^2. \quad (1.5)$$

Now by Schwarz' inequality we have

$$\|x\psi\| \cdot \|\psi'\| \geq |\langle x\psi, \psi' \rangle| \geq |\operatorname{Re} \langle x\psi, \psi' \rangle|. \quad (1.6)$$

Now the right hand side can be computed as follows:

$$2\operatorname{Re} \langle x\psi, \psi' \rangle = \langle x\psi, \psi' \rangle + \langle \psi', x\psi \rangle = \int x \left( \psi(x)\overline{\psi'(x)} + \psi'(x)\overline{\psi(x)} \right).$$

$$= x|\psi(x)|^2|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \|\psi(x)\|^2 dx = -\|\psi\|^2.$$

If we insert this on the right side of (1.6), the inequality (1.5) follows.

To finish the proof we have to get rid of assumption  $\psi \in \mathbf{S}$ . Since  $\mathbf{S}$  is dense in  $L^2$ , we get a sequence  $\psi_n$  which converges to  $\psi \in L^2$ . In (1.4) equality holds if and only if both  $\geq$  in (1.6) are in fact equalities. In the first place the two vectors  $x\psi$  and  $\psi' \in L^2$  are linearly independent. So there has to be a  $\mu + i\nu \in \mathbb{C}$  with

$$\psi'(x) = (\mu + i\nu)x\psi(x) \quad (x \in \mathbb{R}). \quad (1.7)$$

The solution of this differential equation are given by

$$\psi(x) := Ce^{(\mu+i\nu)x^2/2}, \quad C \in \mathbb{C}.$$

and such a  $\psi$  is an element of  $L^2$  if and only if  $\mu := -c$  is negative. For the second inequality in (1.6) to be equality,  $\langle x\psi, \psi' \rangle$  has to be real. Together with (1.7) we are led to the condition

$$\langle x\psi, \psi' \rangle = \langle x\psi, (\mu + i\nu)x\psi \rangle = (\mu - i\nu)\|x\psi\|^2 \in \mathbb{R},$$

so  $\nu$  has to be zero. **(Proved)**

According to this theorem, the two functions  $\psi, \hat{\psi}$  cannot simultaneously be sharply localized at  $x := 0, \xi := 0$ : At least one of the numbers  $\|x\psi\|^2$  and  $\|\xi\hat{\psi}\|^2$  is  $\geq \|\psi\|^2/2$ . Of course the same is true for an arbitrary pair  $(x_o, \xi_o)$  instead of  $(0,0)$ :

**Theorem :** For any  $\psi \in L^2$  and arbitrary  $x_o \in \mathbb{R}, \xi_o \in \mathbb{R}$  one has

$$\|(x - x_o)\psi\| \cdot \|(\xi - \xi_o)\hat{\psi}\| \geq \frac{1}{2}\|\psi\|^2.$$

Here  $\|(x - x_o)\|$  resp.  $\|(\xi - \xi_o)\|$  denotes the following quantities:

$$\left( \int (x - x_o)^2 \|\psi(x)\|^2 dx \right)^{\frac{1}{2}} \text{ respectively } \left( \int (\xi - \xi_o)^2 \|\hat{\psi}(\xi)\|^2 dx \right)^{\frac{1}{2}}$$

**Proof :** We bring the auxiliary function

$$g(t) := e^{-i\xi_o t} \psi(t + x_o)$$

into play and compute

$$\|g\|^2 = \int |\psi(t + x_o)|^2 dt = \|\psi\|^2,$$

$$\|tg\|^2 = \int t^2 |\psi(t + x_o)|^2 dt = \int (x - x_o)^2 |\psi(x)|^2 dx$$

Writing  $g$  in the form

$$g(t) = e^{-i\xi_o t} h(t), \quad h(t) := f(t + x_o),$$

and with the help of rules (R2) and (R1), we deduce that

$$\hat{g}(\tau) = \hat{h}(\tau + \xi_o) = e^{ix_o(\tau + \xi_o)} \hat{f}(\tau + \xi_o).$$

This implies

$$\|\tau g\|^2 = \int \tau^2 |\hat{f}(\tau + \xi_o)|^2 d\tau = \int (\xi - \xi_o)^2 |\hat{f}(\xi)|^2 d\xi.$$

If we now replace  $\|x\psi\|$ ,  $\|\xi\hat{\psi}\|$ , and  $\|\psi\|$  with  $\|tg\|$ ,  $\|\tau g\|$ , and  $\|g\|$ , we arrive at the stated formula. **(Proved)**

## 1.4 Shannon Sampling Theorem

**Qns:** Is it possible to reconstruct a time signal  $f$  from the discrete values  $(f(kT)|k \in \mathbb{Z})$  completely, i.e for all values of the continuous variable  $t$ ?

**Ans:** The Shannon sampling theorem gives the answer to this question.

**$\Omega$ - bandlimited signal:** A function  $f \in L^1$  is called  $\Omega$ -bandlimited if its Fourier transform  $\hat{f}$  vanishes identically for  $|\xi| > \Omega$  :

$$\hat{f}(\xi) \equiv 0 \quad (|\xi| > \Omega)$$

Shannon's theorem states that an  $\Omega$ -bandlimited function can be reconstructed completely from its values

$$(f(kT)|k \in \mathbb{Z}), \quad T := \pi/\Omega \quad (1)$$

sampled at the discrete points  $kT$ .

**Theorem:** Let the continuous function  $f : \mathbb{R} \rightarrow \mathbb{C}$  be  $\Omega$ -bandlimited and assume that  $f$  satisfies an estimate of the form

$$f(t) = O\left(\frac{1}{|t|^{1+\epsilon}}\right) \quad (t \rightarrow \pm\infty) \quad (2)$$

Let  $T := \pi/\Omega$ . Then

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT) \text{sinc}(\Omega(t - kT)) \quad (t \in \mathbb{R}) \quad (3)$$

The formal series appearing in (3) is called the cardinal series.

**Proof:** As we know that sinc-function is bounded on  $\mathbb{R}$ , the assumption (2) guarantees that the cardinal series is uniformly convergent on  $\mathbb{R}$  and represents a function  $\tilde{f}(t) = \sum_{k=-\infty}^{\infty} f(kT) \text{sinc}(\Omega(t - kT))$  that is continuous on all of  $\mathbb{R}$ . Since  $f$  is bandlimited, then  $\tilde{f}(t) = f(t)$  otherwise  $f \neq \tilde{f}$  but  $\tilde{f}(kT) = f(kT) \quad \forall k$  as

$\text{sinc}(k\pi) = \delta_{0k}$  implies that the function  $\tilde{f}$  automatically interpolates the given values  $(f(kT))_{|k \in \mathbb{Z}}$  even in cases where  $f$  is not bandlimited.

Now because of (2) the function  $f$  is in  $L^1 \cap L^2$  and has a continuous Fourier transform. Since  $\hat{f}$  vanishes for  $|\xi| > \Omega$ , it is in  $L^1$  as well as the right side of the inversion formula produces a continuous function  $t \mapsto \tilde{f}(t)$  which coincides with  $f$  almost everywhere, so is actually  $\equiv f$  :

$$f(t) = \frac{1}{\sqrt{2\pi}} \int \hat{f}(\xi) e^{i\xi t} d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \hat{f}(\xi) e^{i\xi t} d\xi \quad (t \in \mathbb{R}) \quad (4).$$

This equality in (4) is due to the assumption that  $\hat{f}$  vanishes identically outside of the interval  $[-\Omega, \Omega]$ . If this assumption is not fulfilled then we no longer have equality and the cardinal series will not represent  $f$ . Since  $\hat{f}$  is continuous, one has  $\hat{f}(-\Omega) = \hat{f}(\Omega) = 0$  and may say that on the  $\xi$ -interval  $[-\Omega, \Omega]$  the function  $\hat{f}$  coincides with a certain periodic function  $F$  of period  $2\Omega$  :

$$\hat{f}(\xi) \equiv F(\xi) \quad (-\Omega \leq \xi \leq \Omega). \quad (5)$$

This function  $F \in L^2(\mathbb{R}/(2\Omega))$  can be written into a Fourier series according to formula:

$$F(\xi) \rightsquigarrow \sum_{k=-\infty}^{\infty} c_k e^{2k\pi i \xi / (2\Omega)}, \quad (6)$$

and we know by Carleson's theorem that series converges for almost all  $\xi$  to the true function value  $F(\xi)$ . The coefficients  $c_k$  are computed as follows:

$$c_k = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\xi) e^{-2k\pi i \xi / (2\Omega)} d\xi = \frac{1}{2\Omega} \int \hat{f}(\xi) e^{-2k\pi i \xi / (2\Omega)} d\xi. \quad (7)$$

The equality in (7) is due to the same reason as on (4). On comparing this equality with (4) the last integral can be interpreted as  $f$ -value, so we get

$$c_k = \frac{\sqrt{2\pi}}{2\Omega} f(-k\pi/\Omega) = \frac{\sqrt{2\pi}}{2\Omega} f(-kT),$$

and formula (6) becomes

$$F(\xi) = \frac{\sqrt{2\pi}}{2\Omega} \sum_{k=-\infty}^{\infty} f(kT) e^{-ikT\xi} \quad (\text{almost all } \xi \in \mathbb{R}), \quad (8)$$

using (5) we may replace (4) with

$$f(t) = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \left( \sum_{k=-\infty}^{\infty} f(kT) e^{-ikT\xi} \right) e^{i\xi t} d\xi.$$

Because of (2), the series under the integral sign converges uniformly, and hence can be integrated term by term;

$$f(t) = \frac{1}{2\Omega} \sum_{k=-\infty}^{\infty} f(kT) \int_{-\Omega}^{\Omega} e^{i(t-kT)\xi} d\xi.$$

The last integral is computed as follows:

$$\begin{aligned} \int_{-\Omega}^{\Omega} e^{i(t-kT)\xi} d\xi &= \int_{-\Omega}^{\Omega} \cos((t-kT)\xi) d\xi \\ &= \frac{2}{(t-kT)} \sin(\Omega(t-kT)) \quad (t \neq kt) \\ &= 2\Omega \operatorname{sinc}(\Omega(t-kT)) \quad (t \in \mathbb{R}), \end{aligned}$$

so that we obtain the stated formula

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT) \operatorname{sinc}(\Omega(t-kT)) \quad (t \in \mathbb{R}) \quad \textbf{(Proved)}$$

The frequency  $\Omega := \pi/T$  is called the **Nyquist frequency** for the chosen sampling interval  $T$ .

The quantity  $T^{-1}$  represents the number of samples taken per unit of time and is called **Sampling rate**. The sampling rate  $T^{-1} := \Omega/\pi$  is called the **Nyquist rate** for the functions of bandwidth  $\Omega$ .

**Qns:** What can be said when the actual bandwidth  $\Omega'$  of the sampled function  $f$  is larger than the Nyquist frequency  $\Omega := \pi/T$  ?

**Ans:** It will create an aliasing. It will be clear in next topic.

### 1.4.1 Aliasing

Consider the function  $f$  that is only moderately undersampled. Take

$$\Omega < \Omega' < 3\Omega$$

and assume that  $\hat{f}(\xi) \equiv 0$  for  $|\xi| > \Omega'$ . Then we have

$$\begin{aligned} f(kT) &= \frac{1}{\sqrt{2\pi}} \int_{-\Omega'}^{\Omega'} \hat{f}(\xi) e^{ikT\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \left( \int_{-3\Omega}^{-\Omega} \hat{f}(\xi) e^{ikT\xi} d\xi + \int_{-\Omega}^{\Omega} \hat{f}(\xi) e^{ikT\xi} d\xi + \int_{\Omega}^{3\Omega} \hat{f}(\xi) e^{ikT\xi} d\xi \right). \end{aligned}$$

If we take the substitution

$$\xi := \xi' \pm 2\Omega \quad (-\Omega \leq \xi' \leq \Omega)$$

in the two exterior integrals on the right, then  $e^{ikt\xi} = e^{ikT\xi'}$  as  $(2\Omega T = 2\pi)$ , we obtain

$$f(kT) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \left( \hat{f}(\xi) + \hat{f}(\xi - 2\Omega) + \hat{f}(\xi + 2\Omega) \right) e^{ikT\xi} d\xi. \quad (9)$$



This brings into the continuous function  $g \in L^2$  whose Fourier transform is given by

$$\hat{g}(\xi) := \begin{cases} \hat{f}(\xi) + \hat{f}(\xi - 2\Omega) + \hat{f}(\xi + 2\Omega) & (-\Omega \leq \xi \leq \Omega) \\ 0 & (|\xi| > \Omega) \end{cases}$$

Because of (9), the function  $g$  satisfies

$$g(kt) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \hat{g}(\xi) e^{ikt\xi} d\xi = f(kT), \quad (k \in \mathbb{Z})$$

We found that  $g$  has same cardinal series as  $f$ , but  $g$  is, contrary to  $f$ , truly  $\Omega$ -bandlimited. This implies that the common cardinal series of  $f$  and  $g$  represents not  $f$  but  $g$ . Thus we conclude that if the true bandwidth  $\Omega'$  of  $f$  is larger than the Nyquist frequency  $\Omega := \pi/T$  then the high frequency parts of  $f$  are not simply filtered out by the cardinal series, but they appear to be afflicted with a frequency shift. The cardinal series produces an  $\Omega$ -bandlimited function  $g$  whose Fourier transform  $\hat{g}$  is given by second last equation.

## 1.4.2 Order of Convergence

The order of convergence of the Shannon sampling can be improved by oversampling the function  $f$ .

Let a sampling rate  $T^{-1}$  be given and let  $\Omega := \pi/T$  be the corresponding Nyquist frequency. We assume that the signals  $f$  taken into consideration are  $\Omega'$ -bandlimited for some  $\Omega' < \Omega$ . Let the auxiliary function  $q \in L^2$  be defined by giving its Fourier transform:

$$\hat{q}(\xi) := \begin{cases} 1 & (|\xi| \leq \Omega') \\ \frac{1}{2} \left( 1 - \sin \frac{\pi(2|\xi| - \Omega - \Omega')}{2(\Omega - \Omega')} \right) & (\Omega' \leq |\xi| \leq \Omega) \\ 0 & (|\xi| \geq \Omega) \end{cases}$$

Note that  $q$  is, apart from the parameter values  $\Omega$  and  $\Omega'$ , independent of  $f$ .

The signal  $f$  satisfies the assumptions of the Shannon sampling theorem, therefore (8) is valid and we may write

$$\hat{f}(\xi) = \frac{\sqrt{2\pi}}{2\Omega} \sum_{k=-\infty}^{\infty} f(kT) e^{-ikt\xi} \quad (-\Omega \leq \xi \leq \Omega)$$

Furthermore we know that  $\hat{f}(\xi)$  is identically zero for  $(-\Omega' \leq |\xi| \leq \Omega)$ . In the interval  $|\xi| \leq \Omega'$  we have  $\hat{q}(\xi) \equiv 1$ . Then we have

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \hat{f}(\xi) e^{it\xi} d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \hat{f}(\xi) \hat{q}(\xi) e^{it\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} \left( \sum_{k=-\infty}^{\infty} f(kT) e^{-ikt\xi} \right) \hat{q}(\xi) e^{it\xi} d\xi \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} f(kT) \int_{-\Omega}^{\Omega} \hat{q}(\xi) e^{i(t-kT)\xi} d\xi \quad .$$

Using the abbreviation

$$\frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \hat{q}(\xi) e^{is\xi} =: Q(s) \quad (10)$$

we rewrite the cardinal series as

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT) Q((t - kT)). \quad (11)$$

In order to judge the improvement in convergence we need a function  $Q$  independent of  $f$  in the explicit form. Since  $\hat{q}$  is even function, the integral (10) can be computed as:

$$\begin{aligned} Q(s) &= \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \hat{q}(\xi) \cos(s\xi) d\xi \\ &= \frac{1}{\Omega} \left( \int_0^{\Omega'} \cos(s\xi) d\xi + \int_{\Omega'}^{\Omega} \frac{1}{2} \left( 1 - \sin \frac{\pi(2|\xi| - \Omega - \Omega')}{2(\Omega - \Omega')} \right) \cos(s\xi) d\xi \right) \\ &= \frac{\pi^2}{2\Omega s} \frac{\sin(\Omega' s) + \sin \Omega s}{(\pi^2) - (\Omega - \Omega')^2 s^2}. \end{aligned}$$

from this, we deduce that

$$Q(s) = O\left(\frac{1}{|s|^{1+\epsilon}}\right) \quad (|s| \rightarrow \infty).$$

For the comparison of (11) and (3) we have to estimate the order of magnitude of the factor  $Q(t - kt)$  in (11) when  $|k| \rightarrow \infty$ . It is given by

$$\frac{2\pi^2}{(2\Omega) \cdot (|k|T) \cdot (\Omega/2)^2 (kT)^2} = \frac{4}{\pi} \frac{1}{|k|^3} \quad (\Omega' = \frac{1}{2}\Omega).$$

Here we used the relation  $\Omega T = \pi$ . The order of magnitude of the corresponding factor  $\text{sinc}(\Omega(t - kT))$  when  $|k| \rightarrow \infty$  is much larger, namely

$$\frac{1}{\pi} \frac{1}{|k|}$$

It follows that, using (3), we have to take several more terms into account as compared to (11) in order to guarantee the same level of precision.

# Chapter 2

## Frame Theory

### 2.1 Frame

**Def :** A family of vectors  $(\phi_i)_{i=1}^M$  in  $H^N$  is called a frame for  $H^N$  , if there exist constants  $0 < A \leq B < \infty$  such that

$$A\|x\|^2 \leq \sum_{i=1}^M |\langle x, \phi_i \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in H^N. \quad (2.1)$$

where the constants A and B are called the lower and upper bounds of the frame. If  $A = B$ , then  $(\phi_i)_{i=1}^M$  is called an **A-tight frame** .

### 2.2 Analysis Operator

**Def :** Let  $(\phi_i)_{i=1}^M$  be a family of vectors in  $H^N$  . Then the associated analysis operator  $T : H^N \mapsto l_2^M$  is defined by

$$Tx := (\langle x, \phi_i \rangle)_{i=1}^M, \quad x \in H^N.$$

The following lemma gives the basic property of the analysis operator.

**Lemma 1 :** Let  $(\phi_i)_{i=1}^M$  be a sequence of vectors in  $H^N$  with associated analysis operator  $T$  .

We have

$$\|Tx\|^2 = \sum_{i=1}^M |\langle x, \phi_i \rangle|^2 \quad \text{for all } x \in H^N.$$

Hence,  $(\phi_i)_{i=1}^M$  is a frame for  $H^N$  if and only if  $T$  is injective.

**Proof :** This is the immediate consequence of the definition of  $T$  and the frame property in (2.1).

## 2.3 Adjoint Operator

**Def :** The adjoint operator  $T^* : l_2^M \mapsto H^N$  of  $T$  is given by

$$T^*(a_i)_{i=1}^M = \sum_{i=1}^M a_i \phi_i.$$

**Proof :** For  $x = (a_i)_{i=1}^M$  and  $y \in H^N$ , we have

$$\langle T^*x, y \rangle = \langle x, Ty \rangle = \langle (a_i)_{i=1}^M, (\langle y, \phi_i \rangle)_{i=1}^M \rangle = \sum_{i=1}^M a_i \overline{\langle y, \phi_i \rangle} = \left\langle \sum_{i=1}^M a_i \phi_i, y \right\rangle.$$

Thus,  $T^*$  is as claimed.

## 2.4 Frame Operator

**Def :** Let  $(\phi_i)_{i=1}^M$  be a sequence of vectors in  $H^N$  with associated analysis operator  $T$ . Then the associated frame operator  $S : H^N \mapsto H^N$  is defined by

$$Sx := T^*Tx = \sum_{i=1}^M \langle x, \phi_i \rangle \phi_i, \quad x \in H^N.$$

**Lemma 2 :** Let  $(\phi_i)_{i=1}^M$  be a sequence of vectors in  $H^N$  with associated frame operator  $S$ . Then, for all  $x \in H^N$ ,

$$\langle Sx, x \rangle = \sum_{i=1}^m |\langle x, \phi_i \rangle|^2.$$

**Proof :** The proof follows directly from  $\langle Sx, x \rangle = \langle T^*Tx, x \rangle = \|Tx\|^2$  and Lemma 1.

Clearly, the frame operator  $S = T^*T$  is self adjoint and positive.

## Chapter 3

# Oversampling of fourier coefficients using frames

Let us represent a signal  $f(t)$ , which is defined for  $t \in [-T, T]$ , through its discrete Fourier expansion, i.e.,

$$f(t) = \frac{1}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi t}{T}}. \quad (3.1)$$

Since for  $t \in [-T, T]$  the complex exponentials in (3.1) constitute an orthonormal basis, the coefficients  $c_n$  in (3.1) are obtained as:

$$c_n = \frac{1}{\sqrt{2T}} \int_{-T}^T f(t) e^{-i \frac{n\pi t}{T}} dt. \quad (3.2)$$

Let us consider now the rescaling operation :  $t \rightarrow at$ , with  $a$  positive real number less than 1, and construct the function  $((\chi_T(t))\sqrt{2T})e^{i \frac{an\pi t}{T}}$ , with  $\chi_T(t)$  is defined as :  $\chi_T(t) = 1$  if  $t \in [-T, T]$  and zero otherwise. The new functions  $((\chi_T(t))\sqrt{2T})e^{i \frac{an\pi t}{T}}$ , are no longer a basis but a "tight frame" for the space of time limited signal with time-width  $2T$  (corresponding frame-bound begin  $a^{-1}$ ). Here, " $a$ " is the over sampling parameter and is the key for retrieving the hidden data.  $\{e^{i \frac{an\pi t}{T}}\} \subset L^2[-T, T]$  is a frame for some  $a \in (0, 1)$ . Then the coefficient  $c_n$  of the linear expansion,

$$f(t) = \frac{\chi_T(t)}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c_n e^{i \frac{an\pi t}{T}} \quad (3.3)$$

are not unique which means there exist infinitely many different sets of coefficients  $c_n$  which can produce an identical signal  $f$  by above linear super-position. A particular set of coefficients  $c_n$  is obtained as:

we have dilation function

$$D_a f(t) = f(t/a)$$

then Fourier coefficients

$$c_n = \frac{1}{\sqrt{2T}} \int_{-T}^T f(t/a) e^{-i \frac{n\pi t}{T}} dt.$$

By suitable change of variables we get

$$c_n = \frac{a}{\sqrt{2T}} \int_{-T}^T f(t) e^{-i \frac{an\pi t}{T}} dt. \quad (3.4)$$

Out of all possible sets of coefficients, the ones given by the above equation constitute the coefficients of minimum 2-norm .

Let us stress the cause for the nonuniqueness of the coefficients in the tight frame expansion. For  $a < 1$ , with the restriction  $t \in [-T, T]$ , the exponentials  $(\frac{1}{\sqrt{2T}})e^{i \frac{an\pi t}{T}}$  are not linearly independent, i.e., we can have the situation

$$\frac{1}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c'_n e^{i \frac{an\pi t}{T}} = 0 \quad \text{for} \quad \sum_{n=-\infty}^{\infty} |c'_n|^2 \neq 0$$

or, taking inner product of both sides with every  $(\frac{1}{\sqrt{2T}})e^{i \frac{am\pi t}{T}}$ ,

$$\frac{1}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c'_n \int_{-T}^T e^{-i \frac{am\pi t}{T}} e^{i \frac{an\pi t}{T}} dt = 0 \quad \text{for} \quad \sum_{n=-\infty}^{\infty} |c'_n|^2 \neq 0$$

which can be recast as:

$$G \vec{c}' = 0 \quad \text{for} \quad \|\vec{c}'\|^2 = \sum_{n=-\infty}^{\infty} |c'_n|^2 \neq 0.$$

The elements of  $G$  are given by

$$g_{m,n} = \frac{1}{\sqrt{2T}} \int_{-T}^T e^{-i \frac{am\pi t}{T}} e^{i \frac{an\pi t}{T}} dt = \frac{\sin a(m-n)\pi}{a(m-n)\pi}. \quad (3.5)$$

Notice that all vectors  $\vec{c}'$  satisfying  $G \vec{c}' = 0$  belong, by definition to  $\text{Null}(G)$ , the null space of  $G$ . All such vectors satisfy

$$f(t) = \frac{\mathcal{X}_T(t)}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c_n e^{i \frac{an\pi t}{T}} + \frac{\mathcal{X}_T(t)}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c'_n e^{i \frac{an\pi t}{T}} = \frac{\mathcal{X}_T(t)}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} c''_n e^{i \frac{an\pi t}{T}}, \quad (3.6)$$

where we have defined  $c''_n = c_n + c'_n$  with  $c_n$  as in (3.4) and  $c'_n$  the components of an arbitrary vector  $\vec{c}' \in \text{Null}(G)$ . Vectors  $\vec{c}''$  and  $\vec{c}'$  will, hereafter, be referred to as signal components and hidden code coefficients, respectively. The fact that all coefficients  $\vec{c}'' = \vec{c} + \vec{c}'$  reproduce an ideal signal as the coefficient  $\vec{c}$  provides us with the foundation to construct an encoding/decoding scheme for transmitting hidden information.

# Chapter 4

## Encoding-decoding system

Let  $f$  be the signal transmitted and we wish to transmit hidden code  $\vec{h}$  consisting of  $k$  numbers where  $k$  depends on the number of eigenvalues of  $G$  close to zero. Let  $a$  be fixed and consider that  $G$  is an  $M \times M$  matrix of elements given by

$$g_{m,n} = \frac{1}{2T} \int_{-T}^T e^{-i\frac{am\pi t}{T}} e^{i\frac{an\pi t}{T}} dt = \frac{\sin(a(m-n)\pi)}{a(m-n)\pi}.$$

Select  $k$ -eigenvectors of  $G$  corresponding to the zero (close) eigenvalues which are assumed to be orthonormal and construct vector  $\vec{c} \in \text{Null}(G)$  as follows:

$$\vec{c} = U \vec{h},$$

where  $U$  is an  $M \times K$  matrix, the columns of which are the  $k$ -selected eigenvectors.

**Claim :**  $\text{Null}(G) \neq 0$ .

**Proof :** Suppose  $\text{Null}(G) = 0$  then,  $G \vec{c} = 0 \quad \forall \vec{c} \Rightarrow \vec{c} = 0 \implies \|\vec{c}\| = 0$  which is a contradiction and we will not be able to transmit hidden code. Thus,  $\text{Null}(G) \neq 0$ .

### 4.1 Encoding

(i)  $f$  is given, compute  $\vec{c}$  signal coefficients by

$$\vec{c} = \frac{a}{\sqrt{2T}} \int_{-T}^T f(t) e^{-i\frac{an\pi t}{T}} dt.$$

(ii) Compute  $\vec{c}$  (hidden code) by

$$\vec{c} = U \vec{h}.$$

(iii) Add the coefficients  $\vec{c}$  and  $\vec{c}$  to construct

$$\vec{c}^{\#} = \vec{c} + \vec{c}.$$

## 4.2 Decoding

(i) Use the coefficients  $\vec{c}^H$  to recover the signal  $f(t)$  by

$$f(t) = \frac{\chi_T(t)}{\sqrt{2T}} \sum_{n=-\infty}^{\infty} \vec{c}^H e^{i \frac{an\pi t}{T}}.$$

(ii) Use the signal  $f$  to compute signal coefficients by

$$\vec{c} = \frac{a}{\sqrt{2T}} \int_{-T}^T f(t) e^{-i \frac{an\pi t}{T}} dt.$$

(iii) Now compute  $\vec{c}^H$  by

$$\vec{c}^H = \vec{c}^H - \vec{c}.$$

(iv) Compute matrix  $U$  using all eigenvectors of matrix  $G$  corresponding to eigenvalues less than previously specified tolerance parameter. Then the code is retrieved by

$$\vec{h} = U^* \vec{c}^H$$

where  $U^*$  is the transpose conjugate of  $U$ .

## 4.3 Example

Table 1

3 code numbers and their reconstruction for  $a = 0.5$  and  $a = 0.5 + 1 \times 10^{-13}$

Code numbers	$a = 0.5$	$a = 0.5 + 1 \times 10^{-13}$
0.37588560126315	0.37588560127432	-0.12821297976573
0.89859572240207	0.89859572239835	-0.07890926132092
0.42900148823994	0.42900148823850	1.42952335356611

Figure 4.1: Table 1

Consider the signal  $f(t) = t^3 \text{sinc}(t - 2)$ ,  $t \in [-4, 4]$ .

1. Corresponding to  $a = 1$ , 81 Fourier coefficients are used for good representation of the signal in the nonoversampling case.
2. If we consider  $a = 0.5$  a null space is created and  $K = 64$  eigenvectors corresponding to 64 smallest eigenvalues of matrix  $G$  are used to construct a code of 64 numbers.



3. The numbers, consisting of 15 digits, are taken randomly from  $(0, 1)$  interval. Table 1 gives three such numbers. The second column shows the corresponding reconstructed numbers. In order to assess the reconstruction of all numbers, let us denote as  $\vec{h}^r$  the reconstructed code and define the error of reconstruction as  $\delta^r = \|\vec{h} - \vec{h}^r\|$ . At the reconstruction stage the exact value of the oversampling parameter  $a$  is known, the error of the reconstruction is small ( $\delta^r = 5.1 \times 10^{-11}$ ).
4. Reconstruction is not possible if the exact value of  $a$  is not known. To show this, let us distort the value of  $a$  upto a very small number:  $1 \times 10^{-13}$ . this perturbation does not produce any detectable effect in the signal coefficients. But it produces enormous distortion to the eigenvectors of  $\text{Null}(G)$ . When we reconstruct the code, what we obtain has no relation with true code (see 3rd column of Table 1). The error of reconstruction in this case is  $\delta^r = 6.36$ .
5. Therefore the recovery of the code is only possible if the value of  $a$  is known to double precision, the key number for recovering the code is the value of the parameter.

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