STINESPRING'S THEOREM FOR MAPS ON HILBERT C^*-MODULES

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This Thesis entitled **Stinespring's Theorem for Maps on Hilbert C*-modules** by **Shefali Gupta** is approved for the degree of Master of Science from IIT Hyderabad.

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Abstract

Stinespring's representation theorem is a fundamental theorem in the theory of completely positive maps. It is a structure theorem for completely positive maps from a C^*-algebra into the C^*-algebra of bounded operators on a Hilbert space. This theorem provides a representation for completely positive maps, showing that they are simple modifications of $^*-$homomorphisms. One may consider it as a natural generalization of the well-known Gelfand-Naimark-Segal thoerem for states on C^*-algebras. Recently, a theorem which looks like Stinesprings theorem was presented by Mohammad B. Asadi in for a class of unital maps on Hilbert C^*-modules. This result can also be proved by removing a technical condition of Asadis theorem. The assumption of unitality on maps under consideration can also be remove. This result looks even more like Stinesprings theorem.

Contents

Chapter 1

Basics of Banach algebras

1.1 Banach Algebras

In this section, we will provide the basic information about Banach algebras and of spectral theory. The most important of these are the non-emptiness of the spectrum, Beurling's spectral radius formula and the Gelfand reprsentation theory. Thoroughout this thesis, the field for all vector spaces and algebras is the complex field $\mathbb{C}$.

Definition 1 (Algebra)**.** *Let $\mathcal{A}$ be a non-empty set. Then, $\mathcal{A}$ is said to be an algebra, if*

1. $(\mathcal{A}, +, \cdot)$ *is a vector space over the field* $\mathbb{F}$.
2. $(\mathcal{A}, +, *)$ *is a ring, and*
3. $(\alpha \cdot a) * b = \alpha \cdot (a * b) = a * (\alpha \cdot b) \forall \alpha \in \mathbb{F}, \forall a, b \in \mathcal{A}$.

we will write ab instead of $a * b$ and αa in place of $\alpha \cdot a$.

Definition 2. *An algebra $\mathcal{A}$ is said to be commutative, if the ring $(\mathcal{A}, +, *)$ is commutative.*

Definition 3. *An Algebra $\mathcal{A}$ is said to be unital, if $(\mathcal{A}, +, *)$ has a unit. Let $\mathcal{A}$ be unital algebra and $a \in \mathcal{A}$. If there exist $b \in \mathcal{A}$ such that $ab = ba = 1$, then b is called inverse of a.*

Remark 1. *The unit element in an algebra is unique, denoted by 1. Also, if $a \in \mathcal{A}$ has an inverse, then it is unique, denoted by a^{-1}.*

Definition 4. *Let $\mathcal{A}$ be an algebra and $\mathcal{B} \subset \mathcal{A}$. Then $\mathcal{B}$ is said to be a subalgebra of $\mathcal{A}$, if $\mathcal{B}$ itself is an algebra with respect to the operations of $\mathcal{A}$.*

Definition 5 (Normed Algebra)**.** *If $\mathcal{A}$ is an algebra and $||\cdot||$ is a norm on $\mathcal{A}$ satisfying*

$$||ab|| \leq ||a||||b||, \textit{ for all } a, b \in \mathcal{A},$$

then $||\cdot||$ is called an algebra norm and $(\mathcal{A}, ||\cdot||)$ is called a normed algebra. A complete normed algebra is called a Banach algebra.

Remark 2. *In a normed algebra $\mathcal{A}$, if $(a_n) \subset \mathcal{A}$, $(b_n) \subset \mathcal{A}$ such that $a_n \to a$, and $b_n \to b$ then $a_n b \to b$, $ba_n \to ba$ and $a_n b_n \to ab$ as $n \to \infty$. Thus, multiplication is left, right and jointly continuous.*

Proof. It can be proved using the norm condition on algebra. □

Remark 3. *Assume $||1|| = 1$.*

Now, let us see, some examples of algebras, Banach algebras, subalgebras and unital commutative Banach algebras.

Example 1 (Function Algebras)**.** *In these examples we consider algebras of functions.*

1. *Let $A = \mathbb{C}$. Then with respect to the usual addition, multiplication of complex numbers and the modulus, $\mathcal{A}$ is a commutative, unital Banach algebra.*

2. *Let $\mathcal{A} = C(K)$, where K is compact Hausdorff space. For $f, g \in C(K)$, define*

$$(f+g)(x) := f(x) + g(x)$$

$$(\alpha f)(x) := \alpha f(x)$$

$$fg(x) := f(x)g(x)$$

$$||f||_\infty := \sup\{|f(t)| : t \in K\}.$$

Then $\mathcal{A}$ is commutative unital Banach algebra with unit $f(x) = 1$, for all $x \in K$.

3. *Let $S \neq \emptyset$ and $B(S) = \{f : S \to \mathbb{C} : f \text{ is bounded}\}$. Then, $B(S)$ is a commutative unital Banach algebra with unit $f(x) = 1$, for all $x \in K$.*

4. *Let Ω be a locally compact Hausdorff space and $\mathcal{A} = C_b(\Omega) := \{f \in C(\Omega) : f$ is bounded$\}$. Then $\mathcal{A}$ is unital commutative Banach algebra with unit $f = 1$.*

5. *Let Ω be a locally compact Hausdorff space and $\mathcal{A} = C_0(\Omega) := \{f \in C(\Omega) : f$ vanishes at $\infty\}$. Then $\mathcal{A}$ is a commutative Banach algebra and $\mathcal{A}$ is unital iff Ω is compact.*

6. *Let Ω be a locally compact Hausdorff space and $\mathcal{A} = C_c(\Omega) := \{f \in C(\Omega) : f$ has compact support$\}$. Then $\mathcal{A}$ is commutative normed algebra but not Banach algebra and it is unital iff Ω is compact.*

7. *Let $X = [0,1]$. Then $C'[0,1] \subset C[0,1]$ is an algebra but $(C'[0,1], ||\cdot||_\infty)$ is not complete. Now define,*

$$||f|| := ||f||_\infty + ||f'||_\infty, \quad f \in C'[0,1].$$

Then $(C'[0,1], ||\cdot||)$ is a commutative unital Banach algebra with unit $f(x) = 1$, for all $x \in K$.

8. *Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Consider, $\mathcal{A}(\mathbb{D}) := \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}}$ is analytic$\}$. Then $\mathcal{A}(\mathbb{D})$ is a closed subalgebra of $C(\overline{\mathbb{D}})$. Hence it is a commutative, unital Banach algebra, known as the Disc algebra.*

Example 2 (Operator Algebras). *Here we consider algebras whose elements are operators on a Banach space. In this case the multiplication is the composition of operators.*

1. *Let $\mathcal{A} = M_n(\mathbb{C}), (n \geq 2)$ with matrix addition, matrix multiplication and Frobenius norm defined by*
$$||A|| = \sqrt{\sum_{i,j=1}^{\infty} |a_{ij}|^2}$$
is a non-commutative unital Banach algebra.

2. *Let X be a Banach space, then B(X), the space of all bounded linear opeartors with composition of operators as multiplication and with respect to the opeartor norm, $B(X)$ is non-commutative unital Banach algebra with identiy operator being the unit in $B(X)$.*

3. *Let X be Banach space and $K(X) := \{T \in B(X) : T \text{ is compact}\}$. Then $K(X)$ is a closed subalgebra of $B(X)$. Hence, $K(X)$ is non-commutative Banach algebra and it is unital iff dim$(X) < \infty$.*

   Recall that $T \in B(X)$ is compact iff $\overline{T(B)}$ is compact for every bounded subset B of X.

4. *Let $\mathcal{A} = M_n(\mathbb{C})$. Define*
$$||A|| = \max_{1 \leq i,j \leq n} |a_{ij}|$$
is a norm on $M_n(\mathbb{C})$ but not an algebra norm.

5. *Let $\mathcal{A} = M_2(\mathbb{C})$ and $\mathcal{B} := \left\{ \begin{bmatrix} x & x \\ x & x \end{bmatrix} : x \in \mathbb{C} \right\}$. Then $\mathcal{A}$ is a unital non-commutative Banach algebra with unit $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, while $\mathcal{B}$ is non-commutatice unital Banach sub-algebra of $\mathcal{A}$ with unit $= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.*

Definition 6. *Let $I \subset \mathcal{A}$. Then I is called an ideal (two sided) if*

1. *I is a subspace of $\mathcal{A}$ over $\mathbb{F}$.*

2. *I is an ideal in the ring $\mathcal{A}$.*

An Ideal I is said to be maximal, if $I \neq \{0\}$ and $I \neq \mathcal{A}$ and J is any ideal of $\mathcal{A}$ such that $I \subset J$, then either $J = I$ or $J = \mathcal{A}$.

Remark 4. *Every ideal is a subalgebra but a subalgebra need not be an ideal.*
Let $A := \{(a_{ij}) \in M_n(\mathbb{C}) : a_{ij} = 0, i \neq j\}$. A is not an ideal of $M_n(\mathbb{C})$, but it is subalgebra.

Example 3. *1. Let H be a complex Hilbert space, then $K(H)$ is closed ideal of $B(H)$.*

2. *Let K be a compact, Hausdorff space and F be a closed subset of K. Then*

$$I_F := \{f \in C(K) : f|_F = 0\}$$

is a closed ideal.

3. *$M_n(\mathbb{C})$ has no ideal, for all $n \in \mathbb{N}$.*

Throughout we assume that $\mathcal{A}$ to be a complex unital Banach algebra. Now, we discuss the definition of Multiplicative group and its properties.

Definition 7. *Let $G(\mathcal{A}) := \{a \in \mathcal{A} : a \text{ is invertible in } \mathcal{A}\}$. Since, $1 \in G(\mathcal{A})$, so $G(\mathcal{A}) \neq \emptyset$ and $0 \notin G(\mathcal{A})$, hence the set $G(\mathcal{A})$ is a multiplicative group.*

Example 4.

1. *Let $A = C(K)$, where K is compact Hausdorff space. Then $G(\mathcal{A}) = \{f \in C(K) : f(t) \neq 0 \text{ for each } t \in K\}$.*
2. *Let $\mathcal{A} = M_n(\mathbb{C})$. Then $G(\mathcal{A}) = \{A \in M_n(\mathbb{C}) : det(A) \neq 0\}$.*

Recall that if $z \in \mathbb{C}$ such that $||z|| < 1$, then $(1 - z)$ is invertible and $(1 - z)^{-1} = \sum_{n=0}^{\infty} z^n$. This result can be generalized to the elements of Banach algebra.

Lemma 1. *If $a \in \mathcal{A}$ with $||a|| < 1$. Then*

$$1 - a \in G(\mathcal{A}) \text{ and } (1 - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Furthermore,

$$||(1 - a)^{-1}|| \leq \frac{1}{1 - ||a||}.$$

This inequality is called Neumann's Inequality.

Proof. We can prove that this series $\sum_{n=0}^{\infty} a^n$ is convergent using the result that the absolute conevrgence implies convergence of the series iff the space is complete. Let $y = \sum_{n=0}^{\infty} a_n$. The sequence of partial sums criteria and the result for $||a|| < 1$, $a^n \to 0$, as $n \to \infty$ implies

$$(1 - a)y = 1 = y(1 - a).$$

Hence, $1 - a \in G(\mathcal{A})$ and $(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n$. Also,

$$||(1 - a)^{-1}|| = ||\lim_{N\to\infty} \sum_{n=0}^{N} a^n|| \leq \lim_{N\to\infty} \sum_{n=0}^{N} ||a||^n = \sum_{n=0}^{\infty} ||a||^n = \frac{1}{1 - ||a||}. \qquad \square$$

Corollary 1.

1. *Let $\mathcal{A}$ be unital Banach algebra and $a \in \mathcal{A}$. Let $\lambda \in \mathbb{C}\backslash\{0\}$ such that $||a|| < |\lambda|$, then*

$$\lambda.1 - a \in G(\mathcal{A}), \text{ and } (\lambda.1 - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}.$$

Furthermore,

$$||(\lambda.1 - a)^{-1}|| \leq \frac{1}{|\lambda| - ||a||}.$$

2. *Let $a \in \mathcal{A}$ such that $||1-a|| < 1$, then*

$$a \in G(\mathcal{A}) \text{ and } a^{-1} = \sum_{n=0}^{\infty}(1-a)^n.$$

Proposition 1. *The set $G(\mathcal{A})$ is open in $\mathcal{A}$.*

Proof. Let $a \in G(\mathcal{A})$ and $B := \{b \in \mathcal{A} : ||a-b|| < \frac{1}{||a^{-1}||}\}$. Now,

$$||1-a^{-1}b|| \leq ||a^{-1}||.||a-b|| < ||a^{-1}||.\frac{1}{||a^{-1}||} = 1.$$

Thus, $a^{-1}b \in G(\mathcal{A})$. Hence, $b = a.a^{-1}b \in G(\mathcal{A})$. □

Proposition 2. *The map $a \to a^{-1}$ is continuous on $G(\mathcal{A})$.*

Proof. Let $(a_n) \subset G(\mathcal{A})$ such that $a_n \to a \in G(\mathcal{A})$.

$$||a_n^{-1} - a^{-1}|| = ||a_n^{-1}(a-a_n)a^{-1}|| \leq ||a_n^{-1}||.||a-a_n||.||a^{-1}||.$$

Corollary 1, implies $||a_n^{-1}|| \leq M$. Hence, $a_n^{-1} \to a^{-1}$ as $n \to \infty$. □

Definition 8 (Topological Group)**.** *A topological group G is a topological space and group such that the group operations of product:*

$$G \times G \to G : (x, y) \mapsto xy$$

and taking inverses

$$G \to G : x \mapsto x^{-1}$$

are continuous functions.

Remark 5. *As the maps $(a, b) \to ab$ from $G(\mathcal{A}) \times G(\mathcal{A})$ into $G(\mathcal{A})$ and $a \to a^{-1}$ from $G(\mathcal{A})$ into $G(\mathcal{A})$ are continuous, hence we can conclude that $G(\mathcal{A})$ is a topological group.*

In this section we define the concept of spectrum and resolvent of an element in a Banach algebra.

Definition 9. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. The resolvent $\rho_{\mathcal{A}}(a)$ of a with respect to $\mathcal{A}$ is defined by*

$$\rho_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : (\lambda.1 - a) \in G(\mathcal{A})\}.$$

The spectrum $\sigma_{\mathcal{A}}(a)$ of a with respect to $\mathcal{A}$ is defined by $\sigma_{\mathcal{A}}(a) = \mathbb{C} \backslash \sigma_{\mathcal{A}}(a)$. That is

$$\sigma_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : (a - \lambda.1) \notin G(\mathcal{A})\}.$$

Example 5. 1. *Let $f \in C(K)$ for some compact Hausdorff space K. Then $\sigma(f) = range(f)$.*

2. *Let $A \in M_n(\mathbb{C})$. Then $\sigma(A) = \{\lambda \in \mathbb{C} : \lambda$ is an eigenvalue of $A\}$.*

Now, we will show spectrum of an element in a Banach algebra is non-empty.

Theorem 1. *Let $\mathcal{A}$ be unital Banach algebra and $a \in \mathcal{A}$. Then $\sigma(a) \neq \emptyset$.*

Proof. Let $a \in \mathcal{A}$.

1. If $a \notin G(\mathcal{A})$, then $0 \in \sigma(a)$. Hence, $\sigma(a) \neq \phi$

2. Suppose, $a \in G(\mathcal{A})$ and $\sigma(a) = \emptyset$. Hence, $\rho(a) = \mathbb{C}$. Let, $\phi \in \mathcal{A}^*$and $f : \mathbb{C}(= \rho(a)) \to \mathbb{C}$ defined by

$$f(\lambda) = \phi((a - \lambda.1)^{-1}).$$

We can show that f is entire and bounded and $|f(\lambda)| \to 0$ as $|\lambda| \to \infty$. Then, by Liouville theorem, $f = 0$. That is, $\phi((a - \lambda.1)^{-1})) = 0, \forall\ \phi \in \mathcal{A}^*$. Hence, by Hahn Banach theorem, $(a - \lambda.1)^{-1} = 0$, a contradiction. □

Remark 6. 1. *If $\lambda \in \mathbb{C}$ such that $|\lambda| > ||a||$, then $(a - \lambda.1) \in G(\mathcal{A})$. Hence $\sigma(a) \subset \{z \in \mathbb{C} : |z| \leq ||a||\}$. Hence $\sigma(a)$ is bounded subset of $\mathbb{C}$.*
$\sigma(a)$ is closed, since the map $f : \mathbb{C} \to \mathcal{A}$ given by $f(\lambda) = (a - \lambda.1)$ is continuous and $\rho(a) = f^{-1}(G(\mathcal{A}))$.

2. *Let $\phi \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Define $g : \rho(a) \to \mathbb{C}$ by*

$$g(\lambda) = \phi((\lambda.1 - a)^{-1}).$$

Then g is analytic in $\rho(a)$.

Theorem 2 (Gelfand-Mazur Theorem)**.** *Every Banach division algebra $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.*

Proof. If $a \in \mathcal{A}$, then $\sigma(a) \neq \emptyset$. Let $\lambda \in \sigma(a)$. Thus, $\lambda.1 - a \notin G(\mathcal{A})$ and $\mathcal{A}\backslash\{0\} = G(\mathcal{A})$. Hence, $\lambda.1 - a = 0$ i.e. $\lambda.1 = a$. Define $f : \mathbb{C} \to \mathcal{A}$ by $f(\lambda) = \lambda.1$. Then, f is isometrically isomorphism. □

Proposition 3. *Let $\mathcal{A}$ be a unital Banach algebra and $a, b \in \mathcal{A}$. Then $1 - ab \in G(\mathcal{A}) \iff 1 - ba \in G(\mathcal{A})$.*

Proof. Let $(1-ab) \in G(\mathcal{A})$. Let $c = 1+b(1-ab)^{-1}a$, then $c(1-ba) = 1 = (1-ba)c$. □

Corollary 2. *Let $\mathcal{A}$ be a unital Banach algebra and $a, b \in \mathcal{A}$. Then $\sigma(ab)\backslash\{0\} = \sigma(ba)\backslash\{0\}$.*

Definition 10 (Spectral Radius)**.** *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then the spectral radius of a is defined by $r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}$.*

From Remark 6, it follows that $0 \leq r(a) \leq ||a\|$.

Example 6. 1. *Let $\mathcal{A} = C(K)$ and $f \in \mathcal{A}$. Then $r(f) = \sup\{|\lambda| : \lambda \in range(f)\} = ||f||_\infty$.*

2. *Let $T \in B(H)$ be normal. Then $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} = ||T||$.*

3. *Let $\mathcal{A} = M_n(\mathbb{C})$. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Since A is nilpotent, $\sigma(A) = \{0\}$ and hence $r(A) = 0$. But $||A|| = 1$.*

4. *Let $\mathcal{A} = C^{'}[0,1]$ and $f \in \mathcal{A}$. Define $||f|| = ||f||_\infty + ||f^{'}||_\infty$. Let $g : [0,1] \to \mathbb{C}$ be the inclusion map. Then $r(g) = 1$ and $||g|| = 2$. -*

Proposition 4. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$.Then,*

1. *If $a \in G(\mathcal{A})$, then $\sigma(a^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(a)\}$.*

2. *$\sigma(a+1) = \{\lambda + 1 : \lambda \in \sigma(a)\}$.*

3. *$\sigma(a^2) = (\sigma(a))^2 = \{\lambda^2 : \lambda \in \sigma(a)\}$.*

4. *$r(a^n) = (r(a))^n, \forall\, n \in \mathbb{N}$.*

5. *If p is a polynomial with complex coefficients, then $\sigma(p(a)) = p(\sigma(a)) = \{p(\lambda) : \lambda \in \sigma(a)\}$.*

6. *$r(ab) = r(ba)$, where $b \in \mathcal{A}$.*

Theorem 3 (Gelfand spectral radius formula: Beurling's proof)**.** *Let $\mathcal{A}$ be a complex unital Banach algebra and $a \in \mathcal{A}$. Then $r(a) = \lim\limits_{n\to\infty} ||a^n||^{\frac{1}{n}}$.*

Proof. Let $\lambda \in \sigma(a)$. Then $\lambda^n \in \sigma(a^n)$. Then, $|\lambda^n| \leq r(a^n) \leq ||a^n||$. This implies, $|\lambda| \leq ||a^n||^{\frac{1}{n}}$. Hence, $r(a) \leq \liminf ||a^n||^{\frac{1}{n}}$. Let $\triangle := \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{r(a)}\}$. Suppose, $\lambda \in \triangle$. Then $1 - \lambda a \in G(\mathcal{A})$.

Suppose $\phi \in \mathcal{A}^*$. Define, $f : \triangle \to \mathbb{C}$ by $f(\lambda) = \phi((1-\lambda a)^{-1})$. Then f is analytic in $\triangle$ and hence f has Maclaurian series expansion, $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$. For $|\lambda| < ||a||^{-1} < (r(a))^{-1}$, $(1 - \lambda a) \in G(\mathcal{A})$. This implies, $(1-\lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. Thus, $f(\lambda) = \sum_{n=0}^{\infty} \phi(a^n)\lambda^n$.

On comparing, $\alpha_n = \phi(a^n)$, for all $n \geq 0$. Hence the sequence $(\phi(a^n\lambda^n))$ converging to 0, for each $\lambda \in \triangle$, therefore it is bounded and this is hold for all $\phi \in \mathcal{A}^*$, by the uniform boundedness theorem $\lambda^n a^n$ is a bounded sequence. Hence, there exist $M > 0$ such that $||\lambda^n a^n|| \leq M$, for all $n > 0$, Therefore $||a^n||^{\frac{1}{n}} \leq M^{1/n}/|\lambda|$, for $\lambda \neq 0$. This implies

$$\limsup ||a^n||^{\frac{1}{n}} \leq 1/|\lambda| < r(a).$$

Therefore, $r(a) = \lim\limits_{n\to\infty} ||a^n||^{\frac{1}{n}}$. □

1.2 The Gelfand Map

Definition 11 (Multiplicative Functionals)**.** *Let $\mathcal{A}$ be a unital Banach algebra and $\phi \in \mathcal{A}^*$, then ϕ is multiplicative functional, if $\phi(ab) = \phi(a)\phi(b), \forall a, b \in \mathcal{A}$.*

If ϕ is non-zero, then $\phi(1) = 1$.
The set of all multiplicative functionals on $\mathcal{A}$ is denoted by $\mathcal{M}_{\mathcal{A}}$.

Example 7. *1. Let $\mathcal{A} = C(K)$, where K is compact Hausdorff space. Let $x \in K$. Define, $\phi_x : \mathcal{A} \to \mathbb{C}$ by $\phi_x(f) = f(x)$. Then ϕ_x is multiplicative. Infact, we can show that all multiplicative linear functionals on $C(K)$ are of this form only.*

2. For $\mathcal{A} = M_n(\mathbb{C}), \mathcal{M}_{\mathcal{A}} = \emptyset$. Because, the trace is the only linear functional on $M_n(\mathbb{C})$. But this is not multiplicative. Hence $\mathcal{A}$ has no multiplicative linear functional.

Proposition 5. *Let $\phi \in \mathcal{M}_{\mathcal{A}}$. Then $||\phi|| = 1$.*

Proof. Since $\phi(1) = 1$, $||\phi|| \geq |\phi(1)| = 1$. Thus $||\phi|| \geq 1$. Suppose, $||\phi|| > 1$. Then there exist $a_1 \in \mathcal{A}$ with $||a_1|| = 1$ and $|\phi(a_1)| > 1$. Let $a = a_1 - \phi(a_1)$. Then $\phi(a_1) = 0$ and $||\frac{a_1}{\phi(a_1)}|| < 1$. Thus, $1 - \frac{a_1}{\phi(a_1)} \in G(\mathcal{A})$, implies that $a \in G(\mathcal{A})$. Hence, $\ker \phi$ contains invertible elements, which is a contradiction. Thus $||\phi|| = 1$.

This shows, ϕ is continuous on $\mathcal{A}$. □

Proposition 6. *$\mathcal{M}_{\mathcal{A}}$ is compact and Hausdorff subset of $\mathcal{A}^*$.*

Recall that if X is a Banach space and $f : X \to \mathbb{C}$ be linear, then $f \in X^*$ if and only if $\ker(f)$ is closed . In this case, $X/\ker(f) \cong \mathbb{C}$. We have a similar kind of result for multiplicative functionals on a Banach algebra.

Proposition 7. *Let $\mathcal{A}$ be unital commutative Banach algebra. Then, there is a one-one correspondence between $\mathcal{M}_{\mathcal{A}}$ and the maximal ideals of $\mathcal{A}$.*

Remark 7. *If $\mathcal{A}$ is non commutative, then $\mathcal{M}_{\mathcal{A}}$ may be trivial. For example $\mathcal{A} = M_n(\mathbb{C}), n \geq 2$ has no non trivial ideals and hence no multiplicative functionals on it. But if the Banach algebra is commutative, then it has non trivial maximal ideals and hence non zero multiplicative functionals.*

Now, the following theorem characterizes all the multiplicative functionals on complex Banach algebras.

Theorem 4 (Gleason-Kahane-Zelazko Theorem)**.** *Let $\mathcal{A}$ be unital Banach algebra and $\phi \in \mathcal{A}^*$. Then $\phi \in \mathcal{M}_{\mathcal{A}}$ iff $\phi(1) = 1$, and $\phi(a) \neq 0$ for all $a \in G(\mathcal{A})$.*

Definition 12 (Gelfand Map)**.** *Let $\mathcal{A}$ be unital Banach algebra. Then, $\Gamma : \mathcal{A} \to C(\mathcal{M}_{\mathcal{A}})$ defined by, $\Gamma(a)(\phi) = \phi(a)$, for all $\phi \in \mathcal{A}$ is called Gelfand map.*

Now, we discuss the properties of Gelfand map.

Proposition 8. *Let Γ be the Gelfand map defined on a unital commutative Banach algebra $\mathcal{A}$. Then*

1. *Γ is linear.*
2. *Γ is multiplicative.*
3. *Γ is bounded and $||\Gamma|| = \sup\{||\Gamma a||_\infty : a \in \mathcal{A}, ||a|| \leq 1\} \leq 1$.*
4. *$\Gamma(1) = 1$.*

5. Γ *maps invertible elements of* $\mathcal{A}$ *into the invertible elements of* $C(\mathcal{M}_{\mathcal{A}})$.

6. Γ *preserves spectrum, i.e.* $\sigma(a) = \sigma(\Gamma a)$. *And* $r(a) = ||\Gamma a||_{\infty}$.

Theorem 5 (Spectral Mapping Theorem)**.** *Let* $\mathcal{A}$ *be a unital Banach algebra,* $x \in \mathcal{A}$*. If* $f(z)$ *is analytic in a neighbourhood of* $\mathbb{D}$*, then*

$$\sigma(f(x)) = f(\sigma(x)) = \{f(\lambda) : \lambda \in \sigma(x)\}.$$

Chapter 2

C^*-Algebras

This chapter begins with the study of C^*-algebras. As Hilbert spaces has rich structure compared with general Banach spaces, the same is true for the C^*-algebras as compared with Banach algebras. The main results of this chapter are the Gelfand-Naimark Representation Theorem for Commutative C^*-algebras and Gelfand-Naimark-Segal Representation Theorem for non-commutative C^*-algebras.

2.1 C^*-algebras

Definition 13 ($*$-algebra)**.** *A complex algebra $\mathcal{A}$ together with involution operation $* : \mathcal{A} \to \mathcal{A}$ maps $a \mapsto a^*$, forms an $*$-algebra, if*

1. $(a+b)^* = a^* + b^*, \forall\, a, b \in \mathcal{A}$.
2. $(\alpha a)^* = \overline{\alpha} a^*, \forall\, \alpha \in \mathbb{C}, a \in \mathcal{A}$.
3. $(ab)^* = b^* a^*, \forall\, a, b \in \mathcal{A}$.
4. $(a^*)^* = a, \forall\, a \in \mathcal{A}$.

The element a^* is called the *adjoint* of a.

Definition 14 (C^*-Algebra)**.** *Let $\mathcal{A}$ be a $*$ algebra which is also a normed algebra. If norm on $\mathcal{A}$ satifies C^*-condition:*

$$||a^*a|| = ||a||^2, \forall\, a \in \mathcal{A},$$

then it is called C^*-norm. If with C^*-norm $\mathcal{A}$ is complete, then $\mathcal{A}$ is called C^*-Algebra.

Definition 15 (C^*-subalgebra)**.** *A closed subalgebra of a C^*-algbera which is closed under involution is called C^*-subalgebra.*

Properties 1. *Let $\mathcal{A}$ be a C^*-algebra, then*

1. *$*$ is an isometry, i.e. $\|x^*\| = \|x\|$, for all $x \in \mathcal{A}$.*

   Proof. For every $x \in \mathcal{A}$, $||x||^2 = ||xx^*|| \leq ||x||.||x^*||$. This implies, $||x|| \leq ||x^*||$. Replace x by x^*, we get $||x^*|| \leq ||x||$ and hence $||x|| = ||x^*||$. □

2. $1^* = 1$, *where 1 is the multiplicative unit in* $\mathcal{A}$.

   Proof. For every $x \in \mathcal{A}$, $1^*x = (x^*1)^* = (x^*)^* = x$, and $x1^* = (1x^*)^* = (x^*)^* = x$. But uniqueness of the unit in an algebra implies $1^* = 1$. $\square$

3. $x \in G(\mathcal{A})$ *iff* $x^* \in G(\mathcal{A})$ *and* $(x^*)^{-1} = (x^{-1})^*$.

   Proof. Let $x \in G(\mathcal{A})$. Then $x^*(x^{-1})^* = (x^{-1}x)^* = 1^* = 1$ and $(x^{-1})^*x^* = 1$. This implies, $x^* \in G(\mathcal{A})$ and $(x^*)^{-1} = (x^{-1})^*$.
   Similarly, we can show that if x^* is invertible, then x is invertible. $\square$

4. $\sigma(x^*) = \overline{\sigma(x)} = \{\overline{\lambda} : \lambda \in \sigma(x)\}$, *and* $r(x^*) = r(x)$.

   Proof. Let $\lambda \in \sigma(x)$. Then $\lambda.1 - x \notin G(\mathcal{A})$. From (3), $\lambda.1 - x^* = (\overline{\overline{\lambda}.1 - x^*}) \notin G(\mathcal{A})$. Hence, $\overline{\lambda} \in \sigma(x^*)$ and $\sigma(x) \subseteq \overline{\sigma(x^*)}$. Similarly, we can prove that $\overline{\sigma(x^*)} \subseteq \sigma(x)$.

   Finally, we have $r(x^*) = \sup\{|\lambda| \ : \ \lambda \in \sigma(x^*)\} = \sup\{|\lambda| \ : \ \lambda \in \sigma(x)\} = r(x)$. $\square$

Example 8. 1. *Let $\mathcal{A} = \mathbb{C}$ with $z^* = \overline{z}$ is a C^*-algebra.*

2. *Let $\mathcal{A} = C(K)$, where K is compact Hausdorff space. Clearly, $C(K)$ is commutative unital Banach algebra w.r.t pointwise addition, multiplication and sup norm. If we define $f^* = \overline{f}$, then $C(K)$ is commutative C^*-algebra. In fact, we will see later that every commuattive C^*-algebra is in this form only.*

3. *Let $\mathcal{A} = B(H)$ with the adjoint operation as an involution is a C^*-algebra. Later, we will see every C^*-algebra can be thought as a C^*-subalgebra of $B(H)$.*

4. *$\mathcal{A} = C^{'}[0,1]$ with $||f|| = ||f||_\infty + ||f^{'}||_\infty$ is Banach algebra. But it is not C^*-algebra w.r.t $f^*(z) = \overline{f(z)}$. Because $f(z) = z$ is not closed under involution.*

Definition 16. *Suppose $\mathcal{A}$ is a C^*-algebra and $x \in \mathcal{A}$. Then,*

1. *x is self-adjoint if $x^* = x$.*

2. *x is unitary if $x^*x = xx^* = 1$, or equivalently $x^* = x^{-1}$.*

3. *x is normal if $xx^* = x^*x$.*

4. *x is positive if $x = y^*y$ for some $y \in \mathcal{A}$.*

5. *x is a projection if $x^* = x = x^2$.*

Remark 8. 1. *Projection $\implies$ positive $\implies$ self-adjoint $\implies$ normal.*

2. *Unitary $\implies$ Normal.*

3. *We write $x \geq 0$ iff x is positive.*

4. *Every unitary element is a unit vector.*

5. *A subset of a C^*-algebra is said to be self-adjoint if it contains all the adjoints of its elements.*

6. *Every element in a C^*-algebra can be uniquely written as a $x_1 + ix_2$, where $x_1 = \frac{x+x^*}{2}$ and $x_2 = \frac{x-x^*}{2i}$ are self adjoint elements.*

Theorem 6 (Spectral Radius Formula)**.** *If x is normal in C^*-algebra $\mathcal{A}$, then $r(x) = ||x||$.*

Proof. First assume that x is self-adjoint i.e. $x = x^*$. In this case,

$$||x^2|| = ||xx^*|| = ||x||^2.$$

Then by Induction, $||x^{2^n}|| = ||x||^{2^n}, \quad n \in \mathbb{N}$. By the Spectral Radius formula, $r(x) = \lim_{n\to\infty} ||x^{2^n}||^{\frac{1}{2^n}} = ||x||$.

Now, assume x is normal. In this case,

$$\begin{aligned} r(x)^2 \le ||x||^2 = ||xx^*|| = r(xx^*) &= \lim_{n\to\infty} ||(xx^*)^n||^{\frac{1}{n}} \\ &\le \lim_{n\to\infty} (||(x^*)^n||^{\frac{1}{n}} ||x^n||^{\frac{1}{n}}) = r(x^*)r(x) = (r(x))^2. \end{aligned}$$

This implies $r(x) = ||x||$. □

Consequences:

1. In C^*-algebra, the spectrum of a normal element contains at least one point on the circle $|z| = ||x||$.

2. If x is normal and $x^n = 0$, then $x = 0$.

3. There is atmost one norm on the *-algebra which make it as a C^*-algebra.

   Proof. Let $|| \cdot ||_1$ and $|| \cdot ||_2$ be two norms on a *-algebra making it a C^*-algebra. Then

   $$||a||_1{}^2 = ||aa^*||_1 = r(aa^*) = \sup\{|\lambda| \ : \ \lambda \in \sigma(aa^*)\} = ||aa^*||_2 = ||a||_2{}^2$$

   Thus, $||a||_1 = ||a||_2$, for all $a \in \mathcal{A}$. Hence $|| \cdot ||_1 = || \cdot ||_2$. □

Proposition 9. *Suppose x is in C^*-algebra $\mathcal{A}$.*

1. *If x is unitary, then $\sigma(x) \subset \{z \in \mathbb{C} \mid |z| = 1\}$.*

2. *If x is self-adjoint, then $\sigma(x) \subset \mathbb{R}$.*

3. *If x is projection and $x \neq \{0, 1\}$, then $\sigma(x) = \{0, 1\}$.*

Proof. 1. Since x is unitary, x is normal and $r(x) = ||x|| = 1$. Hence,

$$\sigma(x) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \le 1\}.$$

Now, $\sigma(x^*) = \sigma(x^{-1}) = \{\lambda^{-1} \in \mathbb{C} \mid \lambda \in \sigma(x)\} \subset \{\lambda \in \mathbb{C} \mid |\lambda| \ge 1\}$
Since, $\sigma(x^*) = \overline{\sigma(x)}$, this implies that $\sigma(x) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \ge 1\}$.
Hence $\sigma(x) \subset \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.

2. Let x is self-adjoint and $y = \exp(ix)$. By the power series multiplication, $yy^* = y^*y = 1$, so y is unitary. So, $\sigma(y) \subset \{z \in \mathbb{C} \mid |z| = 1\}$ and by the spectral mapping theorem, $\sigma(y) = \{\exp(i\lambda) : \lambda \in \sigma(x)\}$. This clearly implies that $\sigma(x) \subset \mathbb{R}$.

3. Since, x is a projection, so $x = x^*$, implies that $\sigma(x) \subset \mathbb{R}$.

   Now, $\sigma(x - x^2) = \{\lambda - \lambda^2 \ : \ \lambda \in \sigma(x)\}$ and $\sigma(0) = 0$, implies that $\sigma(x) = \{0, 1\}$. □

2.2 Commutative C^*-algebras

Let $\mathcal{A}$ be a C^*-algebra. If $\mathcal{A}$ is commutative ring, then $\mathcal{A}$ is called commuatative C^*-algebra. For example, $C(K)$ is commuattive C^*-algebra, where K is compact Hausdorff space. In this section, we will show that every commutative C^*-algebra is isometrically isomorphic to $C(K)$ for some suitable compact Hausdorff space K. This result is called the Gelfand Naimark Representation of Commutative C^*-algebras.

Definition 17. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras, then $\Phi : \mathcal{A} \to \mathcal{B}$ is called a C^*-homomorphism, if*

1. $\Phi(\alpha x + \beta y) = \alpha\Phi(x) + \beta\Phi(x)$, *for all $x, y \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$.*

2. $\Phi(xy) = \Phi(x)\Phi(y)$, *for all $x, y \in \mathcal{A}$.*

3. $\Phi(x^*) = \Phi(x)^*$, *for all $x \in \mathcal{A}$.*

4. *If $1_{\mathcal{A}} \in \mathcal{A}$ and $1_{\mathcal{B}} \in \mathcal{B}$, then $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$.*

If Φ is 1-1, then Φ is a C^*-isomorphism. Two C^*-algebras are C^*-isomorphic if there exists a C^*-isomorphism from one onto the other.

Remark 9. *If Φ is a C^*-homomorphism, then $\ker(\Phi)$ is self-adjoint ideal in $\mathcal{A}$ and $\Phi(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}$.*

Example 9.

1. *Let $\mathcal{A}$ be a C^*-algebra and $u \in \mathcal{A}$ be unitary. Define, $\Phi : \mathcal{A} \to \mathcal{A}$ by $\Phi(x) = uxu^*$. Then Φ is C^*-homorphism.*

2. *The Gelfand map is a C^*-isomorphism.*

3. *Let $\mathcal{A} = B(H)$ and $\mathcal{B} = M_2(B(H))$. Suppose $(T_{ij})^* = (T^*_{ji})$, so $\mathcal{B}$ forms an $*$-algebra. Define $\Phi : \mathcal{A} \to \mathcal{B}$ by*

$$\Phi(T) = \begin{pmatrix} T & 0 \\ 0 & T. \end{pmatrix}$$

   Then Φ is C^*-homomorphism.

Proposition 10. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital Banach algebras, $\Phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism with $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. Then*

1. *$a \in G(\mathcal{A})$ implies $\Phi(a) \in G(\mathcal{B})$.*

2. *If Φ is an onto isomorphism, then the converse of* (1) *also holds.*

Corollary 3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ be C^*-homorphism. Then for all $x \in \mathcal{A}, \sigma(\Phi(x)) \subset \sigma(x)$ and $||\Phi(x)|| \leq ||x||$ (i.e. Φ is a contraction).*

Proof. Let $x \in \mathcal{A}$ and $\lambda \in \sigma(\Phi(x))$. That is $\lambda.1_{\mathcal{B}} - \Phi(x) \notin G(\mathcal{B})$, hence $\Phi(\lambda.1_{\mathcal{A}} - x) \notin G(\mathcal{B})$. From Proposition 2.2.4, we have $\lambda.1_{\mathcal{A}} - x \notin G(\mathcal{A})$. Hence $\lambda \in \sigma(x)$. Thus $\sigma(\Phi(x)) \subset \sigma(x)$. Hence

$$||x||^2 = ||xx^*|| = r(xx^*) \geq r(\Phi(xx^*)) = r((\Phi(x)^*\Phi(x)) = ||\Phi(x)^*\Phi(x)|| = ||\Phi(x)||^2.$$

Hence $||x|| \geq ||\Phi(x)||$. □

Note 1. *1. Since, $||\Phi|| \leq 1$, therefore Φ is bounded, hence continuous.*

2. *Φ is contractive.*

3. *If $\Phi : \mathcal{A} \to \mathbb{C}$, then Φ is multiplicative functional and $||\Phi|| = 1$.*

Corollary 4. *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be an onto C^*-isomorphism. Then $\sigma(\Phi(x)) = \sigma(x)$ and $||\Phi(x)|| = ||x||, \ \forall\, x \in \mathcal{A}$.*

Proof. This is because Φ^{-1} is also a C^*-isomorphism. □

In fact, we will prove later that this corollary also holds when Φ is only a C^*-homorphism from $\mathcal{A}$ into $\mathcal{B}$.

We shall now completely determine the commutative C^*-algebras using the Gelfand map. This result can be thought of as a preliminary form of the Spectral Theorem. It allows us to construct the functional calculus, a very useful tool in the analysis of non-commutative C^*-algebras.

Theorem 7 (Stone-Weierstrass Theorem:Complex Version)**.** *Let K be a compact Hausdorff space and let $\mathcal{A}$ be a closed self-adjoint subalgebra of $C(K, \mathbb{C})$ which contains the constant functions and separates the points of K. Then $\mathcal{A} = C(K)$.*

Theorem 8 (Gelfand-Naimark Representation Theorem)**.** *Every commutative C^*-algebra is C^*-isomorphic to $C(K)$ for some compact Hausdorff space K. Specifically, for a commutative C^*-algebra $\mathcal{A}$, the Gelfand transform is a C^*-isomorphism from $\mathcal{A}$ onto $C(\mathcal{M}_{\mathcal{A}})$.*

Proof. Let $\mathcal{A}$ be a commutative C^*-algebra. As we know, for a commuative Banach algebra $\mathcal{A}$, $\mathcal{M}_{\mathcal{A}} \neq \emptyset$, hence $C(\mathcal{M}_{\mathcal{A}}) \neq \{0\}$.

Let $\Gamma : \mathcal{A} \to C(\mathcal{M}_{\mathcal{A}})$ by $\Gamma(a)(\phi) = \phi(a)$ be the Gelfand transform. We know that Γ is an algebraic homomorphism. We need to show that Γ is 1-1, onto and preserves the involutions.

Let $x \in \mathcal{A}$, then $x = x_1 + ix_2$, where $x_1 = (\frac{x+x^*}{2i})$ and $x_2 = (\frac{x-x^*}{2})$. Clearly, x_1 and x_2 are self-adjoint and $x^* = x_1 - ix_2$. And, for $k = 1, 2$; we have $range(\Gamma(x_k)) = \sigma(\Gamma(x_k)) = \sigma(x_k) \subset \mathbb{R}$. Thus, $\Gamma(x_1)$ and $\Gamma(x_2)$ are real valued functions in $C(\mathcal{M}_{\mathcal{A}})$. Therefore, $\Gamma(x^*) = \Gamma(x_1 - ix_2) = \Gamma(x_1) - i\Gamma(x_2) = \overline{\Gamma(x_1 + ix_2)} = (\Gamma(x))^*$, so Γ preserves the involutions.

By the spectral radius formula for normal elements,

$$||\Gamma(x)||_\infty^2 = ||\Gamma(x).\Gamma(x)^*||_\infty = ||\Gamma(xx^*)||_\infty = r(xx^*) = ||xx^*|| = ||x||^2,$$

implying Γ is an isometry, hence Γ is one-one and $range(\Gamma(x))$ is closed subalgebra of $C(\mathcal{M}_\mathcal{A})$.

Since, $\Gamma(1_\mathcal{A}) = 1, C(\mathcal{M}_\mathcal{A})$ contain constant functions. Let $\phi_1, \phi_2 \in \mathcal{M}_\mathcal{A}$ be such that $\phi_1 \neq \phi_2$ i.e. there exist $a \in \mathcal{A}$ such that $\phi_1(a) \neq \phi_2(a)$. It means $\Gamma(a)(\phi_1) \neq \Gamma(a)(\phi_2)$. Thus, there exist a non-zero continuous function $\Gamma(a) \in C(\mathcal{M}_\mathcal{A})$ such that it separates ϕ_1 and ϕ_2 in $\mathcal{M}_\mathcal{A}$. So, by the Stone-Weierstrass theorem, Γ is onto. □

So, this theorem says that the study of commutative C^*-algebras is equivalent to the study of their maximal ideal spaces. Hence the commutative C^*-algebra theory is the theory of commutative topology.

Proposition 11. *Suppose $\mathcal{B}$ is a unital C^*-subalgebra of a C^*-algebra $\mathcal{A}$. Let $1_\mathcal{A} = 1_\mathcal{B}$ and $x \in \mathcal{B}$. Then $\sigma_\mathcal{A}(x) = \sigma_\mathcal{B}(x)$.*

2.3 The Spectral Theorem and Applications

Let $\mathcal{A}$ be a C^*-algebra and $S \subset \mathcal{A}$. The C^*-algebra generated by S, denoted by $\mathcal{A}[S]$, is the smallest C^*-subalgebra of $\mathcal{A}$ containing S. It is the intersection of all C^*-subalgebras of $\mathcal{A}$ containg S. In particular, $\mathcal{A}[x]$ is the C^*-subalgebra generated by x, x^* and 1 and it is equal to $\overline{span}\{p(x, x^*)\}$, where p is a polynomial in two variables.

Proposition 12. 1. *If x is normal in a C^*-algebra $\mathcal{A}$, then $\mathcal{A}[x]$ is commutative.*

2. *Let $y \in \mathcal{A}$. Then $y \in \mathcal{A}[x]$ iff y can be approximated in norm by polynomials in x and x^*.*

Proof. 1. Since, every element of $\mathcal{A}[x]$ is linear combinations of polynomials in x, x^* and $(x^n)^* = (x^*)^n$, $x^m(x^*)^n = (x^*)^n x^m$, for every $m, n \in \mathbb{N}$. So, we can easily show that $\mathcal{A}[x]$ is commutative.

2. Follows by the definition of $\mathcal{A}[x]$.

□

Now we will prove the spectral theorem for a normal element in C^*-algebra. It says that the maximal ideal space of $\mathcal{A}[x]$ is homeomorphic to $\sigma(x)$.

Theorem 9 (The Spectral Theorem)**.** *Let $\mathcal{A}$ be a C^*-algebra and $x \in \mathcal{A}$ be a normal element. Then the maximal ideal space of $\mathcal{A}[x]$ is homeomorphic to $\sigma(x)$. Moreover, the Gelfand transform Γ on $\mathcal{A}[x]$ has the property that*

$$\Gamma(p(x, x^*)) = p(z, \overline{z})$$

for every polynomial p of two variables.

Proof. Let $\mathcal{M}$ be the maximal ideal space of $\mathcal{A}[x]$ and $\Gamma : \mathcal{A}[x] \to C(\mathcal{M})$ be the Gelfand transform. Since, x is normal, so $\mathcal{A}[x]$ is commutative. Hence $\mathcal{A}[x]$ is isomorphic to $C(\mathcal{M})$, by the Gelfand-Naimark theorem.

Now define, $\Phi : \mathcal{M} \to \sigma(x)$ by

$$\Phi(\varphi) = \Gamma(x)(\varphi) = \varphi(x).$$

Clearly, $\varphi(x) \in \sigma(x)$ because $\varphi(x) \in range(\Gamma(x)) = \sigma(\Gamma(x)) = \sigma_{\mathcal{A}[x]}(x) = \sigma(x)$, from corolarry 2.2.7. And, we can see that Φ is well-defined.

Let $\lambda \in \sigma(x) = range(\Gamma(x))$. Then there exist $\varphi \in \mathcal{M}$ such that $\lambda = \varphi(x)$ and $\Phi(\varphi) = \lambda$. Hence, Φ is surjective.

Suppose, $\Phi(\varphi_1) = \Phi(\varphi_2)$ or $\varphi_1(x) = \varphi_2(x)$. We Claim that $\varphi_1 = \varphi_2$ in $\mathcal{M}$. Since, for $k = 1, 2$

$$\varphi_k(x^*) = \Gamma(x^*)(\varphi_k) = \overline{\Gamma(x)(\varphi_k)} = \overline{\varphi_k(x)} = (\varphi_k(x))^*.$$

We have $\varphi_1(x^*) = \varphi_2(x^*)$ and $\varphi_1(p(x, x^*)) = \varphi_2(p(x, x^*))$. This shows that $\varphi_1 = \varphi_2$ on a dense subspace. So, Φ is injective.

If $\varphi_\alpha \to \varphi$ in $\mathcal{M}$, then $\varphi_\alpha(y) \to \varphi(y)$ for every $y \in \mathcal{A}[x]$. In particular, $\varphi_\alpha(x) \to \varphi(x)$. Thus $\Phi(\varphi_\alpha) \to \Phi(\varphi)$ and Φ is continuous. Being a bijection continuous function from a compact Hausdorff space to another compact Hausdorff space, Φ must be a homeomorphism.

Since, $\mathcal{A}[x] \xrightarrow{\Gamma} C(\mathcal{M}) \xrightarrow{\eta} c(\sigma(x))$ by

$$\eta \circ \Gamma(x) = \Gamma x \circ \Phi^{-1},$$

where $\Gamma x \circ \Phi^{-1} : \sigma(x) \to \mathbb{C}$ satisfies $\Gamma x \circ \Phi^{-1}(\varphi(x)) = \varphi(x)$. Thus $\eta \circ \Gamma(x)(z) = z$ for all $z \in \sigma(x)$ and $\eta \circ \Gamma(x^*)(z) = \overline{z}$. Since $\eta \circ \Gamma$ is C^*-homomorphism, $\eta \circ \Gamma(p(x, x^*))(z) = p(z, \overline{z})$ for every polynomial p of two variables. $\square$

Since, for a normal element, we can identify the maximal ideal space of $\mathcal{A}[x]$ with $\sigma(x)$. Under the appropriate composition map, this identification the Gelfand transform become

$$\Gamma : \mathcal{A}[x] \to C(\sigma(x))$$

is a surjective C^*-isomorphism. This allows us to define $f(x)$ for every function $f \in C(\sigma(x))$.

Definition 18 (The Continuous Functional Calculus)**.** *Suppose x is normal in a C^*-algebra $\mathcal{A}$. For every $f \in C(\sigma(x))$ we define*

$$f(x) = \Gamma^{-1}(f).$$

The mapping $f \mapsto f(x)$ from $C(\sigma(x))$ onto $\mathcal{A}[x]$ is called the continuous functional calculus for x.

In the following theorem, we list some of the basic properties of the continuous functional calculus.

Theorem 10. *Suppose x is normal in a C^*-algebra $\mathcal{A}$. The continuous functional calculus for x has the following properties:*

1. *$f \mapsto f(x)$ is a C^*-isomorphism from $C(\sigma(x))$ onto $\mathcal{A}[x]$.*

2. *If $f(z) = p(z, \overline{z})$, then $f(x) = p(x, x^*)$. In particular, if $f(z) = z$ is the identity function on $\sigma(x)$, we have $f(x) = x$.*

   Proof. Proof of (1) and (2) are trivial from the spectral theorem. □

3. *$\sigma(f(x)) = f(\sigma(x))$, for all $f \in C(\sigma(x))$.*

   Proof. Since, Γ^{-1} is C^*-isomorphism, therefore

$$\sigma(f(x)) = \sigma(f) = range(f) = f(\sigma(x)).$$

□

4. *If $\Phi : \mathcal{A}[x] \to \mathcal{B}$ is a C^*-homomorphism, then*

$$\Phi(f(x)) = f(\Phi(x))$$

   for all $f \in C(\sigma(x))$.

2.3.1 The continuous functional calculus

Now, we will derive some consequences of the spectral theorem and the continuous functional calculus for the normal elements in a C^*-algebra.

Theorem 11. *Suppose x is normal in a C^*-algebra $\mathcal{A}$. Then*

1. *x is self-adjoint iff $\sigma(x) \subset \mathbb{R}$.*

   Proof. If x is self-adjoint, we already proved that $\sigma(x) \subset \mathbb{R}$. Conversly, let $\sigma(x) \subset \mathbb{R}$ and $f(t) = t$, for all $t \in \sigma(x)$. Then the inverse image of f under the continuous functional calculus is $f(x) = x$. Since, f is real-valued, therefore $\overline{f(t)} = t = f(t)$ and it's inverse image is $f^*(x) = x^*$. Therefore $x = x^*$. □

2. *x is unitary iff $\sigma(x) \subset \partial\mathbb{D}$.*

   Proof. We already proved that in first section that if x is unitary, then $\sigma(x) \subset \partial\mathbb{D}$.

   Conversly, assume $\sigma(x) \subset \partial\mathbb{D}$ and $f(t) = t$, for all $t \in \sigma(x)$. This implies $\overline{f(t)} = \overline{t}$ and inverse image of f and $\overline{f}$ are $f(x) = x$ and $f^*(x) = x^*$ respectively. Since, $f(t)\overline{f(t)} = t\overline{t} = |t|^2 = 1$, therfore, $xx^* = 1$ and $x^*x = 1$. Hence x is unitary. □

3. *x is a projection iff $\sigma(x) \subset \{0, 1\}$.*

   Proof. Forward implication, we proved in first section. Now assume $\sigma(x) \subset \{0, 1\}$. Clearly, $x = x^*$ from (1). Now let $f(t) = t$, for all $t \in \sigma(x)$. This implies $f^2(t) = t = f(t)$, so from continuous function calculus $x^2 = x$, hence x is projection. □

Example 10. *Let $\mathcal{A} = B(H)$ and $z \in \mathbb{C}$. Define $T : H \to H$ by $Tx = zx$, for all $x \in H$. Clearly, $T^*x = \overline{z}x$, T is normal in $B(H)$ and $\sigma(T) = \{z\}$. Then*

1. *T is self-adjoint iff $z \in \mathbb{R}$.*
2. *T is unitary iff $|z| = 1$.*
3. *T is positive iff $z \in \mathbb{R}^+$.*
4. *T is projection iff $z \in \{0, 1\}$.*
5. *For $z \in \mathbb{C}$ such that imaginary part of $z \neq 0$, then T is normal but not self-adjoint.*
6. *If $z \in \mathbb{R}^-$, then T is self-adjoint but not positive.*

As we know, for every positive real number, there exist a unique positive square root. Similarly, in a C^*-algebra $\mathcal{A}$, positive element has unique positive square root in $\mathcal{A}$.

Theorem 12. *Let $\mathcal{A}$ be a C^*-algebra and $x \in \mathcal{A}$ such that $x \geq 0$. There exists a unique positive $y \in \mathcal{A}$ such that $y^2 = x$.*

Proof. Since, x is positive, therefore $\sigma(x) \subset \mathbb{R}^+$. Define, $f(t) = \sqrt{t}$, for all $t \in \sigma(x)$. So, it's inverse image is $f(x) = \sqrt{x} = y$ (say,) and $f^2(t) = t$. Therefore by the continuous functional calculus, $y^2 = x$ and $\sigma(y) = \{\sqrt{\lambda} : \lambda \in \sigma(x)\} \subset \mathbb{R}^+$, implies $y \geq 0$.

To prove the uniqueness of y, let there exist positive $y_1 \in \mathcal{A}$ such that $y_1^2 = x$. We claim that $y_1 = y$. Let (p_n) be a sequence of polynolmials converging uniformly to $f(t) = \sqrt{t}$ on $\sigma(x)$. Let $q_n(t) = p_n(t^2)$. Since, $\sigma(y_1) \subset \mathbb{R}^+$ and

$$\sigma(x) = \sigma(y_1^2) = \{t^2 : t \in \sigma(y_1)\},$$

we have

$$\lim_{n\to\infty} q_n(t) = \lim_{n\to\infty} p_n(t^2) = f(t^2) = t,$$

for all $t \in \sigma(y_1)$. By the continuous functional calculus,

$$y_1 = \lim_{n\to\infty} q_n(y_1) = \lim_{n\to\infty} p_n(y_1^2) = \lim_{n\to\infty} p_n(x) = f(x) = y.$$

Therefore, $y_1 = y$. □

The positive element y in the theorem above is called the positive square root of x and will be denoted by $\sqrt{x}$ or $x^{\frac{1}{2}}$. And, for every $x \in \mathcal{A}$, define $|x| := \sqrt{x^*x}$ is called the modulus of x. By the above theorem, it is clear that $|x| \geq 0$, for all $x \in \mathcal{A}$.

Theorem 13. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ be an injective C^*-homomorphism, then for all $x \in \mathcal{A}$, $||\Phi(x)|| = ||x||$ and $\sigma(\Phi(x)) = \sigma(x)$.*

Proof. We first assume that $x = x^*$. It is clear that $\sigma(\Phi(x)) \subseteq \sigma(x)$. Assume that the inclusion is strict that is $\sigma(\Phi(x)) \subset \sigma(x)$. Then by the Urysohn's lemma, there exist non-zero $f \in C(\sigma(x))$ such that $f(t) = 0$, for all $t \in \sigma(\Phi(x))$. Then by the continuous functional calculus, $f(x) \neq 0$ and $\Phi(f(x)) = f(\Phi(x)) = 0$, a contradiction to the fact that Φ is injective. Thus $\sigma(\Phi(x)) = \sigma(x)$ and $r(x) = r(\Phi(x))$. By the spectral radius formula for normal elements, $||x|| = r(x) = r(\Phi(x)) = ||\Phi(x)||$ (since, $\Phi(x)$ is also normal).

For general x, we have $||x||^2 = ||xx^*|| = ||\Phi(xx^*)|| = ||\Phi(x)||^2$. □

Above theorem shows that Φ is an isometry, hence $range(\Phi)$ is closed. But, in the case of C^*-algebras, the image of any C^*-homomorphism is always closed.

Theorem 14. *Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ be a C^*-homomorphism, then $range(\Phi)$ is closed in $\mathcal{B}$.*

Proof. Let $y \in \overline{range(\Phi)} \subset \mathcal{B}$. Then there exist $(x_n) \subset \mathcal{A}$ such that $||\Phi(x_n) - y|| \to 0$ as $n \to \infty$. We claim that there exist $x \in \mathcal{A}$ such that $y = \Phi(x)$. By expressing y and x_n in terms of their real and imaginary parts, we may as well assume that y and x_n are all self-adjoint. Passing to a subsequence if necessary, we may assume that

$$||\Phi(x_{n+1}) - \Phi(x_n)|| < \frac{1}{2^n}, \quad n \geq 1.$$

Define, $f_n(t) = \begin{cases} \frac{1}{2^n}, & t \geq \frac{1}{2^n} \\ t, \quad \frac{1}{2^n} < & t < \frac{1}{2^n} \\ -\frac{1}{2^n}, & t \leq -\frac{1}{2^n}. \end{cases}$

Since $\Phi(x_{n+1}) - \Phi(x_n)$ is self-adjoint and has norm less than $\frac{1}{2^n}$, $f_n(t) = t$, for all $t \in \sigma(\Phi(x_{n+1}) - \Phi(x_n))$. So, by the continuous functional calculus,

$$\begin{aligned} \Phi(x_{n+1}) - \Phi(x_n) &= f_n(\Phi(x_{n+1}) - \Phi(x_n)) \\ &= f_n(\Phi(x_{n+1} - x_n)) \\ &= \Phi(f_n(x_{n+1} - x_n)). \end{aligned}$$

Also by the continuous functional calculus,

$$||f_n(x_{n+1} - x_n)|| = \sup\{|f(t)| \ : \ t \in \sigma(x_{n+1} - x_n)\} \leq \frac{1}{2^n}.$$

And,

$$\sum_{n=1}^{\infty} ||f_n(x_{n+1} - x_n)|| < \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

This shows that $\sum_{n=1}^{\infty} f_n(x_{n+1} - x_n)$ is absolutely convergent and since $\mathcal{A}$ is complete, we have $\sum_{n=1}^{\infty} f_n(x_{n+1} - x_n)$ is convergent.
Let

$$x = x_1 + \sum_{n=1}^{\infty} f_n(x_{n+1} - x_n).$$

Then $x \in \mathcal{A}$ and

$$\begin{aligned} \Phi(x) &= \Phi(x_1) + \sum_{n=1}^{\infty} \Phi(f_n(x_{n+1} - x_n)) \\ &= \Phi(x_1) + \sum_{n=1}^{\infty} (\Phi(x_{n+1}) - \Phi(x_n)) \\ &= \lim_{n \to \infty} \Phi(x_n) \\ &= y. \end{aligned}$$

This completes the proof. □

Theorem 15 (Positive and Negative Parts of an Element)**.** *Suppose x is self-adjoint in a C^*-algebra $\mathcal{A}$. Then there exist unique self-adjoint elements x^+ and x^- in $\mathcal{A}$ such that*

1. $x = x^+ - x^-$.
2. $|x| = x^+ + x^-$.
3. $\sigma(x^+), \sigma(x^-) \subset \mathbb{R}^+$.
4. $x^+x^- = x^-x^+ = 0$.
5. $||x|| = \max\{||x^+||, ||x^-||\}$.

Now we will show that a normal element in a C^*-algebra is positive if and only if its spectrum is contained on $\mathbb{R}^+$.

Theorem 16. *Suppose x is normal in a C^*-algebra $\mathcal{A}$. Then x is positive if and only if $\sigma(x) \subset \mathbb{R}^+$.*

2.4 States

Definition 19. *Suppose $\mathcal{A}$ is a C^*-algebra and $\varphi \in \mathcal{A}^*$. Then*

1. *φ is Hermitian or self-adjoint, if for all $x \in \mathcal{A}$, $\varphi(x^*) = \overline{\varphi(x)}$.*
2. *φ is positive, if $\varphi(x) \geq 0$, whenever $x \geq 0$.*
3. *φ is a state, if φ is positive and $\varphi(1) = 1$.*

Proposition 13. *Let φ be a linear functional on a C^*-algebra $\mathcal{A}$. Then φ is Hermitian iff $\varphi(x) \in \mathbb{R}$ for every self-adjoint $x \in \mathcal{A}$..*

Proof. If φ is Hermitian and $x \in \mathcal{A}$ be self-adjoint, then $\varphi(x) = \varphi(x^*) = \overline{\varphi(x)}$. Thus $\varphi(x) \in \mathbb{R}$.

Conversly, let for all self-adjoint $x \in \mathcal{A}$, $\varphi(x) \in \mathbb{R}$. Then $\varphi(x^*) = \varphi(x_1 - ix_2) = \varphi(x_1) - i\varphi(x_2) = \overline{\varphi(x)}$. Here x_1 and x_2 are self-adjoint, so $\varphi(x_1)$ and $\varphi(x_2)$ are real. □

Proposition 14. *Every positive linear functionals on a C^*-algebra is Hermitian.*

Proof. Suppose φ is a positive linear functional on a C^*-algebra $\mathcal{A}$ and x is self-adjoint in $\mathcal{A}$. Since $|x| + x$ and $|x| - x$ are positive in $\mathcal{A}$ by the continuous functional calculus, it follows that

$$\varphi(|x| + x) \geq 0, \quad \varphi(|x| - x) \geq 0.$$

Hence

$$\varphi(x) = \frac{1}{2}(\varphi(|x| + x) - \varphi(|x| - x)$$

is real, so that φ is Hermitian by Proposition 2.4.2. □

Example 11. *1. Let $\mathcal{A} = M_n(\mathbb{C})$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is given by $\varphi(A) = \frac{1}{n}trace(A)$. Then φ is state.*

2. *Let $\mathcal{A} = B(H)$, where H is a Hilbert space and $x \in H$ with $||x|| = 1$. Define $\varphi : B(H) \to \mathbb{C}$ by $\varphi(A) = \langle Ax, x\rangle$. Then φ is a state, called vector state.*

3. *Let $\mathcal{A} = C(K)$, where K is comapct Hausdorff space. Let $x \in K$ and $\varphi : C(K) \to \mathbb{C}$ is defined as $\varphi(f) = f(x)$. Then φ is state.*

4. *Infact, by the Riesz representation theorem for bounded linear functional F on $C(K)$,*
$$F(f) = \int_K f \, d\mu, \quad f \in C(K),$$
where μ is finite regular complex Borel measure on K. Then, F is positive.

Proposition 15. *If φ is a Hermitian linear functional on a C^*-algebra $\mathcal{A}$, then*
$$||\varphi|| = \sup\{\varphi(x) \ : \ x^* = x, ||x|| \leq 1\}.$$

Proposition 16. *For every positive linear functional φ on a C^*-algebra $\mathcal{A}$ we have*
$$|\varphi(x^*y)|^2 \leq \varphi(x^*x)\varphi(y^*y), \quad x, y \in \mathcal{A}.$$

Proof. Since, φ is a positive linear functional on $\mathcal{A}$. Define
$$\langle x, y\rangle := \varphi(y^*x), \text{ for all } x, y \in \mathcal{A},$$
defines a semi-inner product on the linear space $\mathcal{A}$. The desired result is then a consequence of the Cauchy-Schwarz inequality. □

Theorem 17 (Characterization of Positivity)**.** *Suppose φ is a linear functional on a C^*-algebra $\mathcal{A}$. Then φ is positive iff φ is bounded with $||\varphi|| = \varphi(1)$.*

Corollary 5. *A linear functional on a C^*-algebra $\mathcal{A}$ is a state if and only if $||\varphi|| = \varphi(1) = 1$.*

2.4.1 The State Space

First, we recall the notion of the weak and the weak-star topologies on a normed linear space.

Definition 20. *Let X be a set and $\mathcal{F}$ be a family of functions from X into a topological space Y. The weak topology on X induced by $\mathcal{F}$ is the smallest topology on X which makes every function in $\mathcal{F}$ continuous. Thus a net (x_α) in X converges to some $x \in X$ in this weak topology if and only if $(f(x_\alpha))$ converges to $f(x)$ for every $f \in \mathcal{F}$.*

Definition 21 (Weak-Topology and Weak-star Topology)**.** *Let X be a Banach space and X^* be the Banach dual of X. By the weak topology on X, we mean the weak topology induced by the family of a bounded linear functionals on X.*
The weak-star topology (or W^*-topology) is simply the weak topology on X^* induced by the family $\mathcal{F} = \{f_x : x \in X\}$, where $f_x : X^* \to \mathbb{C}$ by $f_x(\phi) = \phi(x)$. Thus a net $(\phi_\alpha) \subset X^*$ converges to $\phi \in X^*$ in W^*-topology iff $(\phi_\alpha(x))$ converges to $\phi(x)$ for every $x \in X$.

Remark 10. *If X is a Banach space, then both the weak topology on X and weak-star topology on X^* are Hausdorff.*

Theorem 18 (Banach Alaoglu's Theorem)**.** *Suppose X is a Banach space and X^* is its dual. Let $(X^*)_1$ be the closed unit ball in X^*. Then $(X^*)_1$ is compact in the W^*-topology.*

Now, let $\mathcal{A}$ be a C^*-algebra, the $S(\mathcal{A})$ denote the space of all the states on $\mathcal{A}$. We topologize $S(\mathcal{A})$ with the weak-star topology inherited from $\mathcal{A}^*$ and $S(\mathcal{A})$ is a subset of closed unit ball in $\mathcal{A}^*$. Now, we will prove that the state space is always non-empty.

Theorem 19. *Let $\mathcal{A}$ be a C^*-algebra and $x \in \mathcal{A}$. Then for every $\lambda \in \sigma(x)$, there exist $\varphi \in S(\mathcal{A})$ such that $\varphi(x) = \lambda$.*

Proof. Let $x \in \mathcal{A}$ and $\lambda \in \sigma(x)$. Consider

$$M = \{ax + b1 : a, b \in \mathbb{C}\}.$$

Then M is a subspace of $\mathcal{A}$. Define $\varphi_0 : M \to \mathbb{C}$ by

$$\varphi_0(ax + b1) = a\lambda + b.$$

Clearly, φ_0 is linear with $\varphi_0(x) = \lambda, \varphi_0(1) = 1$.
For $a, b \in \mathbb{C}$, $|a\lambda + b| \leq r(ax + b1) \leq ||ax + b1||$. Therefore

$$||\varphi_0|| = \sup\{|\varphi_0(ax + b1)| : ||ax + b1|| \leq 1\} = 1.$$

So, by Hahn-Banach Extension theorem, φ_0 extends to a bounded linear functional φ on $\mathcal{A}$ with $||\varphi|| = 1$. Thus the linear functional φ satifies $\varphi(x) = \lambda$ and is a state from Corollary(5). □

Proposition 17. *$S(\mathcal{A})$ is a convex, compact and Hausdorff space.*

Proof. Let $\varphi_1, \varphi_2 \in S(\mathcal{A}),\ t \in (0, 1,)$ and $\varphi = t\varphi_1 + (1 - t)\varphi_2$. Clearly, φ is linear and

$$\begin{aligned}\sup\{|\varphi(x)| \ : \ ||x|| = 1\} &= \sup\{|t\varphi_1(x)| + (1 - t)\varphi_2(x) \ ; \ ||x|| = 1\}\\ &\leq t||\varphi_1|| + (1 - t)||\varphi_2|| = 1.\end{aligned}$$

Thus, $||\varphi|| \leq 1$. Also $\varphi(1) = 1$ which implies $||\varphi|| = \varphi(1) = 1$, this shows that $\varphi \in S(\mathcal{A})$ and hence $S(\mathcal{A})$ is convex.
Clearly, $S(\mathcal{A})$ is Hausdorff and

$$S(\mathcal{A}) = \{\varphi \in (\mathcal{A}^*)_1 \ : \ \varphi(x) \geq 0, x \geq 0, \varphi(1) = 1\}$$

is closed in $(\mathcal{A}^*)_1$. So, $S(\mathcal{A})$ must be compact in the weak star topology. □

The following theorem shows that the state space on a C^*-algebra is not only non-empty, it is also large enough to reveal many properties of an element in the algebra.

Theorem 20. *Let $\mathcal{A}$ be a C^*-algebra and $x \in \mathcal{A}$. Then*

1. *$x = 0$ iff $\varphi(x) = 0$ for all $\varphi \in S(\mathcal{A})$.*
2. *$x = x^*$ iff $\varphi(x) \in \mathbb{R}$ for all $\varphi \in S(\mathcal{A})$.*
3. *$x \geq 0$ iff $\varphi(x) \geq 0$ for all $\varphi \in S(\mathcal{A})$.*
4. *If x is normal, then $||x|| = |\varphi(x)|$ for some $\varphi \in S(\mathcal{A})$.*

2.4.2 Pure States

First we recall the Krein-Milman theorem

Definition 22. *A vector space X together with a Hausdorff topology such that the vector addition and scalar multiplication are both continuous is called topologiacal vector space.*

A topological vector space X is called locally convex if the origin of X has a local base whose elements are convex.

Definition 23. *Suppose X is a locally convex topological vector space and S is a convex subset of X. Then $x \in S$ is called an extreme point of S if x cannot be written as $x = tx_1 + (1-t)x_2$, with $t \in (0,1)$ and x_1, x_2 are different points in S.*

The following theorem assures the existence of extreme points for compact convex subsets of a locally conves space.

Theorem 21 (Krein-Milman's Theorem)**.** *Let S be a compact convex set in a locally convex space X. Then the set E of extreme points of S is non-empty and S is the closed convex hull of E i.e. S is closure of the set $\{\sum_{i=1}^{n} a_i x_i : x_i \in E, a_i \geq 0, \sum_{i=1}^{n} a_i = 1, n \in \mathbb{N}\}$.*

Since $\mathcal{A}^*$ is locally convex space and the state space $S(\mathcal{A})$ is convex and weak-star compact subset, then by Krein-Milman's theorem $S(\mathcal{A})$ has extreme points and it is the weak-star closed convex hull of the set $P(\mathcal{A})$ of its extreme points. Elements of $P(\mathcal{A})$ are called Pure states of $\mathcal{A}$.

The following result shows that the set of pure states of a C^*-algebra is also sufficiently large enough to describe the C^*-algbera.

Theorem 22. *Let $\mathcal{A}$ be a C^*-algebra and $x \in \mathcal{A}$, then*

1. *$x = 0$ iff $\varphi(x) = 0$ for all $\varphi \in P(\mathcal{A})$.*
2. *$x = x^*$ iff $\varphi(x) \in \mathbb{R}$ for all $\varphi \in P(\mathcal{A})$.*
3. *$x \geq 0$ iff $\varphi(x) \geq 0$ for all $\varphi \in P(\mathcal{A})$.*
4. *If x is normal, then $||x|| = |\varphi(x)|$ for some $\varphi \in P(\mathcal{A})$.*

2.5 The G-N-S Construction

In this section we will show that every C^*-algebra is C^*-isomorphic to a C^*-subalgebra of $B(H)$ for some Hilbert space H. The proof is constructive. This construction is usually called the G-N-S (standing for Gelfand, Neumark, and Segal) construction.

Proposition 18. *Let $\mathcal{A}$ be a C^*-algebra and $\varphi \in S(\mathcal{A})$. Let*

$$L_\varphi = \{x \in \mathcal{A} : \varphi(x^*x) = 0\}.$$

Then L_φ is a closed left ideal in $\mathcal{A}$ and $\varphi(x^*y) = 0$, whenever x or y is in L_φ. Thus, $L_\varphi = \{x \in \mathcal{A} : \varphi(x^*y) = 0, \ \forall\, y \in \mathcal{A}\}$.

Proof. If $x \in L_\varphi$ then Proposition 2.4.6 implies that $\varphi(x^*y) = 0$. From this we can conclude that L_φ is subalgebra of $\mathcal{A}$. Let $y \in \mathcal{A}, x \in L_\varphi$, then $\varphi((yx)^*yx) = \varphi(x^*y^*yx) \leq \varphi(x^*||y||.1x) = 0$. This implies, $yx \in L_\varphi$, hence L_φ is a left ideal of $\mathcal{A}$ and it is easy to check that L_φ is closed in $\mathcal{A}$. □

Let $x, y \in \mathcal{A}$, then $x \sim y$ iff $x - y \in L_\varphi$. This is an equivalence relation and suppose $H_\varphi^\circ = \mathcal{A}/L_\varphi$ is the quotient space and $[x]$ represents the coset of x in H_φ°. Define an inner product on H_φ° by

$$\langle [x], [y] \rangle = \varphi(y^*x), \quad x, y \in \mathcal{A}.$$

We can see that $(H_\varphi^\circ, \langle , \rangle)$ forms an Inner Product Space. Let H_φ be the completion of H_φ° with respect to this inner product and H_φ° is dense in H_φ. Now, we will define a bounded linear tranformation on H_φ°.

Proposition 19. *Let $\mathcal{A}$ be a C^*-algebra, $\varphi \in S(\mathcal{A})$ and $x \in \mathcal{A}$. Define $T_x : H_\varphi^\circ \to H_\varphi^\circ$ by*

$$T_x([y]) = [xy], \quad y \in \mathcal{A}.$$

Then T_x is well-defined bounded linear operator and extends on H_φ with $||T_x|| \leq ||x||$.

Proof. Since L_φ is left ideal, we can check that T_x is well-defined and linear. It remains to show that $||T_x[y]|| \leq ||x||||[y]||$. So,

$$\begin{aligned} ||T_x([y])||^2 = ||[xy]||^2 = \langle [xy], [xy] \rangle &= \varphi((xy)^*xy) \\ &= \varphi(y^*x^*xy) \\ &\leq \varphi(y^*||x||^2.1y) \\ &= ||x||^2\varphi(y^*y) \\ &= ||x||^2||[y]||^2. \square \end{aligned}$$

Theorem 23. *Let $\mathcal{A}$ be a C^*-algebra and $\varphi \in S(\mathcal{A})$. Then the mapping $\Phi_\varphi : \mathcal{A} \to B(H_\varphi)$ defined by $\Phi_\varphi(x) = T_x,\ \ x \in \mathcal{A}$ is a C^*-homomorphism.*

Proof. For $x_1, x_2, y \in \mathcal{A}$ and $a, b \in \mathbb{C}$. Then we have

$$\begin{aligned} \Gamma_{ax_1+bx_2}([y]) &= [ax_1y + bx_2y] \\ &= a[x_1y] + b[x_2y] \\ &= a\Gamma_{x_1}([y]) + b\Gamma_{x_2}([y]). \end{aligned}$$

Since, H_φ° is dense in H_φ, we have

$$\Phi(ax_1 + bx_2) = a\Phi(x_1) + b\Phi(x_2),$$

so that Φ is linear. Similarly,

$$\Gamma_{x_1x_2}([y]) = [x_1x_2y] = \Gamma_{x_1}([x_2y]) = \Gamma_{x_1}\Gamma_{x_2}([y]),$$

implies that

$$\Phi(x_1x_2) = \Phi(x_1)\Phi(x_2),$$

so that Φ is multiplicative. It is clear that $\Phi(1) = 1$. For y_1, y_2 and $x \in \mathcal{A}$ we have

$$\begin{aligned}\langle T_{x^*}([y_1]), [y_2]\rangle &= \varphi(y_2^* x^* y_1)\\ &= \varphi((xy_2)^* y_1)\\ &= \langle [y_1], [xy_2]\rangle\\ &= \langle T_x^*([y_1]), [y_2]\rangle\end{aligned}$$

This shows that $\Phi(x^*) = \Phi(x)^*$. □

2.5.1 Review on Direct Sums of Hilbert spaces

Suppose $\{H_\alpha\}$ is a family of Hilbert spaces, then

$$H = \oplus_{\alpha\in\Lambda} H_\alpha = \{x = \{x_\alpha\}_{\alpha\in\Lambda} \ : \ ||x||^2 = \sum_{\alpha\in\Lambda} ||x_\alpha||^2 < \infty\}.$$

Define an inner product on H as

$$\langle x, y\rangle = \sum_{\alpha\in\Lambda} \langle x_\alpha, y_\alpha\rangle.$$

Suppose $\{T_\alpha\}$ is a family of bounded linear operators on H_α for each $\alpha \in \Lambda$. Define

$$\oplus_{\alpha\in\Lambda} T_\alpha : \oplus_{\alpha\in\Lambda} H_\alpha \to \oplus_{\alpha\in\Lambda} H_\alpha$$

by

$$\oplus_{\alpha\in\Lambda} T_\alpha(\{x_\alpha\}) = \{T_\alpha x_\alpha\}.$$

Then $\oplus_{\alpha\in\Lambda} T_\alpha$ is linear on $\oplus_{\alpha\in H_\alpha} H_\alpha$ with

$$|| \oplus_{\alpha\in\Lambda} T_\alpha|| = \sup\{||T_\alpha||_{H_\alpha} : \alpha \in \Lambda\}.$$

A representation of a C^*-algebra $\mathcal{A}$ is a pair $(H_\varphi, \Phi_\varphi)$ where H_φ is a Hilbert space and $\Phi_\varphi : \mathcal{A} \to B(H_\varphi)$ is a C^*-homomorphism. To define its universal represenattion, we take the direct sum of all the representations $(H_\varphi, \Phi_\varphi)$, where $\varphi \in S(\mathcal{A})$.

2.5.2 A Representation of C^*-algebras

Theorem 24 (Universal GNS Representation of C^*-algebras)**.** *Let $\mathcal{A}$ be a unital C^*-algebra, then it has a faithful representation i.e. there exist a Hilbert space H such that $\mathcal{A}$ is injective C^*-homomorphic to $B(H)$.*

Proof. Let $S(\mathcal{A})$ be the state space and

$$H = \oplus_{\varphi\in S(\mathcal{A})} H_\varphi$$

and define $\Phi : \mathcal{A} \to B(H)$ by

$$\Phi(x) = \oplus_{\varphi\in S(\mathcal{A})} T_x^\varphi,$$

where $T_x^\varphi : H_\varphi \to H_\varphi$ is defined by

$$T_x^\varphi([y]) = [xy], \quad y \in \mathcal{A}.$$

By Theorem (18) and the properties about direct sums of operators, the mapping Φ is a C^*-homomorphism. Next we show that Φ is one-to-one.

Suppose $\Phi(x) = 0$. Then $T_x^\varphi = 0 \quad \forall \varphi \in S(\mathcal{A})$. It follows that $T_x^\varphi([y]) = 0, \quad y \in \mathcal{A} \;\; \varphi \in S(\mathcal{A})$ i.e. $[xy] = 0, \;\; y \in \mathcal{A} \;\; \varphi \in S(\mathcal{A})$. This implies that $xy \in L_\varphi$, or

$$\varphi((xy)^*(xy)) = 0, \;\; y \in \mathcal{A} \;\; \varphi \in S(\mathcal{A}).$$

By Theorem (15), we have $(xy)^*(xy) = 0$. This implies $||xy||^2 = ||(xy)^*(xy)|| = 0$, and hence $xy = 0$, for all $y \in \mathcal{A}$. In particular, $xx^* = 0$. This implies $x = 0$. Therefore, Φ is injective. This shows that $\mathcal{A}$ is C^*-isomorphic to $range(\Phi)$ which is C^*-subalgebra of $B(H)$. □

Note 2. *In proving Theorem (19) we do not need the whole state space $S(\mathcal{A})$. It suffices to use the space $P(\mathcal{A})$ of pure states when forming the direct sum H.*

Example 12. *Let H be a Hilbert space and $H^{(n)}$ denote the orthogonal sum of n copies of H. We can check that $M_n(B(H))$ and $B(H^{(n)})$ forms *-algebras, where the involution on $M_n(B(H))$ is given by $(a_{ij})^* = (a_{ji}^*)$.*
Define, $\Phi : M_n(B(H)) \to B(H^{(n)})$ by

$$\Phi(T)(x_1, \cdots, x_n) = (\sum_{j=1}^{n} T_{1j}(x_j), \cdots, \sum_{j=1}^{n} T_{nj}(x_j)),$$

where $x = (x_1, \cdots, x_n) \in H^{(n)}$. Then Φ is *-isomorphism. We call Φ the canonical *-isomorphism of $M_n(B(H))$ onto $B(H^{(n)})$. If v is an operator in $B(H^{(n)})$ such that $\Phi(u) = v$, where $u \in M_n(B(H))$, we call u the operator matrix of v. We can define a norm on $M_n(B(H))$ making it a C^*-algebra by setting $||u|| = ||\Phi(u)||$.

Chapter 3

Positive Maps

Before turning our attention to Stinespring theorems, we begin with some results on positive maps. Let $\mathcal{A}$ be a unital C^*-algebra and S is a subset of $\mathcal{A}$, then define

$$S^* := \{a : a^* \in S\},$$

and we call S self-adjoint when $S = S^*$.

Definition 24 (Operator System)**.** *If S is a self-adjoint subspace of $\mathcal{A}$ containing 1, then S is called an operator system.*

First, let us discuss some examples of operator system:

Example 13. *$\mathcal{A}$ unital C^*-algbera is an operator system.*

Example 14. *Let M be a subspace of $\mathcal{A}$. Then $S := M + M^* + \mathbb{C}.1$ is an operator system.*

Example 15. *Let H be an infinite dimensional Hilbert space. From example* (13) *$\mathcal{B}(H)$ is an operator system. Consider $K(H) := \{T \in \mathcal{B}(H) : T \text{ is compact}\}$ a subspace of $\mathcal{B}(H)$. Since, $1 \notin K(H)$, therefore $K(H)$ is not an operator system.*

Example 16. *Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $C(\overline{\mathbb{D}}) := \{f : \overline{D} \to \mathbb{C} | f \text{ is continuous on} \overline{\mathbb{D}}\}$. Clearly, $C(\overline{\mathbb{D}})$ is an operator system with $f^*(z) := \overline{f(z)}$. Suppose, $\mathcal{A}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic on } \mathbb{D}\}$ is a closed subspace of $C(\overline{\mathbb{D}})$. and $f(z) := z, \forall z \in \mathbb{D}$. Then $f \in \mathcal{A}(\mathbb{D})$ but $f^* \notin \mathcal{A}(\mathbb{D})$. Therefore, $\mathcal{A}(\mathbb{D})$ is not an operator system.*

Example 17. *If $f^*(z) := \overline{f(\overline{z})}$, then $\mathcal{A}(\mathbb{D})$ forms an operator system.*

Note 3. *Let S be an operator system and h be self-adjoint in S, then we can write h as the difference of two positive elements in S, i.e.*

$$h = \frac{1}{2}(||h||.1 + h) - \frac{1}{2}(||h||.1 - h),$$

where $\frac{1}{2}(||h||.1 + h), \frac{1}{2}(||h||.1 - h)$ are positive in S.

Definition 25 (Positive Map)**.** *Suppose S is an operator system, $\mathcal{B}$ a C^*-algebra and $\phi : S \to \mathcal{B}$ is a linear map, then ϕ is called a positive map, if it maps positive elements of S to positive elements of $\mathcal{B}$.*

Example 18. *Let H be a Hilbert space and $X \in \mathcal{B}(H)$. Define $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ by*

$$\phi(T) = X^*TX, \text{ for all } T \in \mathcal{B}(H).$$

Then ϕ is positive, since if $T \geq 0$ in $\mathcal{B}(H)$, then $\langle Tx, x\rangle \geq 0$, for all $x \in H$. Hence,

$$\langle \phi(T)x, x\rangle = \langle X^*TXx, x\rangle = \langle TXx, Xx\rangle \geq 0, \text{ for all } x \in H.$$

Therefore $\phi(T)$ is positive.

Example 19. *Let K be a compact Hausdorff space and $\phi : C(K) \to \mathbb{C}$ is defined by*

$$\phi(f) = \int_K f(x)d\mu(x)$$

where, μ is Borel measure. Then by the Riesz representation theorem, ϕ is positive functional.

Example 20. *The Map $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ given by*

$$\phi(A) = \frac{1}{n} Trace(A) \cdot I$$

is a positive map.

Note 4. *If p is positive, then $0 \leq p \leq ||p||.1$ This is because, $\sigma(||p||.1 - p) = \{||p|| - \lambda : \lambda \in \sigma(p)\} \subseteq \mathbb{R}^+$, since $|\lambda| \leq ||p||$.*

Proposition 20. *Let S be an operator system and $\phi : S \to \mathbb{C}$ be a positive linear functional, then $||\phi|| = \phi(1)$.*

Proof. Let $a \in S$. Since, $\phi(a) = |\phi(a)|e^{\iota\theta}$, for some $\theta \in \mathbb{R}$. We have $|\phi(a)| = \lambda\phi(a) = \phi(\lambda a)$, where $\lambda = e^{-\iota\theta}$. Similarly, $|\phi(a)| = \overline{\phi(\lambda a)} = \phi((\lambda a)^*)$. Now,

$$\begin{aligned}
|\phi(a)| &= \frac{1}{2}\left(\phi(\lambda a) + \phi((\lambda a)^*)\right) \\
&= \phi\left(\frac{\lambda a + (\lambda a)^*}{2}\right) \\
&= \phi(Re(\lambda a)), && \text{where } Re \text{ represents real part.} \\
&\leq \phi(||\lambda a||.1) && \text{(from Note (4) and } \phi \text{ is positive)} \\
&= ||a|| \cdot \phi(1) && \text{(since, } |\lambda| = 1)
\end{aligned}$$

This implies, $||\phi|| \leq \phi(1)$, hence $||\phi|| = \phi(1)$. □

Thus, for positive linear functional ϕ, $||\phi|| = \phi(1)$. But, when the range of positive linear map is C^*-algebra then the situation will be different.

Note 5. *If $0 \leq a \leq b$, then $0 \leq ||a|| \leq ||b||$.*

Proof. From Note(4), $0 \leq a \leq b \leq ||b||.1$. Let $\lambda \in \sigma(a)$, then $||b|| - \lambda \geq 0$. This implies $|\lambda| \leq ||b||$. Since, $a \geq 0$, therefore it is self-adjoint. Hence spectral radius, $r(a) = ||a||$. Thus, $||a|| \leq ||b||$, because $r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$. □

Remark 11. *If $a, b \in \mathcal{A}$ such that $a = a^*$ and $b = b^*$, then Note* (5) *need not be true.*
Let $a = -5.1$ and $b = -3.1$. Clearly, $a \leq b$, but $||a|| \nleq ||b||$.

Note 6. *If a, b are two positive elements, then*

$$||a - b|| \leq \max\{||a||, ||b||\}.$$

Proof. From the Gelfand Representation for C^*-algebras, $\mathcal{A} \subseteq \mathcal{B}(H)$, for some Hilbert space H. So, we treat $a, b \in \mathcal{A}$ as positive operators of $\mathcal{B}(H)$. Now, using the fact that for any two positive real numbers t_1, t_2, we have $|t_1 - t_2| \leq \max\{t_1, t_2\}$, we obtain

$$\begin{aligned} ||a - b|| &= \sup_{||x||=1} \{|\langle (a-b)x, x\rangle| : x \in H\} \\ &= \sup_{||x||=1} \{|\langle ax, x\rangle - \langle bx, x\rangle| : x \in H\} \\ &\leq \sup_{||x||=1} \{\max\{\langle ax, x\rangle, \langle bx, x\rangle\} : x \in H\} \\ &= \max\{\sup_{||x||=1} \{\langle ax, x\rangle\}, \sup_{||x||=1} \{\langle bx, x\rangle\}\} \\ &= \max\{||a||, ||b||\}. \end{aligned}$$

Thus $||a - b|| \leq \max\{||a||, ||b||\}$. □

Proposition 21. *Let S be an operator system and let $\mathcal{B}$ be a C^*-algebra. If $\phi : S \to \mathcal{B}$ is a positive map, then $||\phi|| \leq 2||\phi(1)||$.*

Proof. First, note that if p is positive in S, then $0 \leq p \leq ||p||.1$ (from Note (4)) and so, $0 \leq \phi(p) \leq ||p||.\phi(1)$ since ϕ is positive. Hence from Note (5), $||\phi(p)|| \leq ||\phi(1)|| \cdot ||p||$.

If h is self-adjoint in S, then using the decomposition of h as a difference of two positive elements (refer Note(3)), i.e.

$$\begin{aligned} h &= \frac{1}{2}(||h||.1 + h) - \frac{1}{2}(||h||.1 - h) \\ \phi(h) &= \frac{1}{2}\phi(||h||.1 + h) - \frac{1}{2}\phi(||h||.1 - h) \end{aligned}$$

We can express $\phi(h)$ as a difference of two positive elements of $\mathcal{B}$. Thus,

$$\begin{aligned} ||\phi(h)| &\leq \frac{1}{2}\max\{||\phi(||h||.1 + h)||, ||\phi(||h||.1 - h)||\} \qquad (\text{ from Note(6)}) \\ &\leq ||\phi(1)|| \cdot ||h||. \end{aligned}$$

Finally, if a is an arbitrary element of S, then $a = h + \imath k$ with $||h||, ||k|| \leq ||a||, h = h^*, k = k^*$. So,

$$||\phi(a)|| \leq ||\phi(h)|| + ||\phi(k)|| \leq 2||\phi(1)|| \cdot ||a||.$$

Hence, $||\phi|| \leq 2||\phi(1)||$. □

An example of Arveson shows that 2 is the best constant in Proposition (21).

Example 21. *Let $\mathbb{T}$ denote the unit circle in complex plane, $C(\mathbb{T})$ the continuous functions on $\mathbb{T}$, z the coordinate function, and $S \subseteq C(\mathbb{T})$ the subspace spanned by $1, z$, and $\overline{z}$.*

We define $\phi : S \to \mathcal{M}_2(\mathbb{C})$ by

$$\phi(a + bz + c\overline{z}) = \begin{pmatrix} a & 2b \\ 2c & a \end{pmatrix}.$$

An element, $a1 + bz + c\overline{z}$ of S is postive if and only if $c = \overline{b}$ and $a \geq 2|b|$. A self-adjoint element of $\mathcal{M}_2(\mathbb{C})$ is positve if and only if its diagonal entries and its determinant are non-negative real numbers. Combining these two facts it is clear that ϕ is a positve map. However,

$$2||\phi(1)|| = 2 = ||\phi(z)|| \leq ||\phi||,$$

so that $||\phi|| = 2||\phi(1)||$.

Now we are going to show that if $S = C(X)$, where X is compact Hausdorff space, then $||\phi|| = ||\phi(1)||$.

Lemma 2. *Let $\mathcal{A}$ be a unital C^*-algebra and $p_i, i = 1, 2, \cdots, n$ be positive elements of $\mathcal{A}$ such that*

$$\sum_{i=1}^{n} p_i \leq 1.$$

If $\lambda_i \in \mathbb{C}, i = 1, 2, \cdots, n$ with $|\lambda_i| \leq 1$, then

$$\left|\left|\sum_{i=1}^{n} \lambda_i p_i\right|\right| \leq 1.$$

Proof. Note that

$$\begin{bmatrix} \sum_{i=1}^{n} \lambda_i p_i & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} p_1^{1/2} & \cdots & p_n^{1/2} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \cdot \begin{bmatrix} p_1^{1/2} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ p_n^{1/2} & 0 & \cdots & 0 \end{bmatrix}.$$

The norm of the matrix on the left is $|| \sum_{i=1}^{n} \lambda_i p_i||$, while each of the three matrices on the right can be easily seen to have norm less than 1, by using the fact that $||a^*a|| = ||aa^*|| = ||a||^2$. □

Theorem 25. *Let $\mathcal{B}$ be a unital C^*-algebra and let X be compact Hausdorff space. Let $\phi : C(X) \to \mathcal{B}$ be a positive map. Then $||\phi|| = ||\phi(1)||$.*

Proof. By scaling, we may assume that $\phi(1) \leq 1$. Let $f \in C(X), ||f|| \leq 1$, and let $\epsilon > 0$ be given. Since, X is compact and ϕ is continuous, $f(X)$ is compact. Hence, for any $x \in X$, there exist $i \in \{1, \cdots, n\}, n \in \mathbb{N}$ such that $|f(x) - f(x_i)| < \epsilon$.

Let $\{p_i\}$ be non-negative continuous functions satisfying $\sum_{i=1}^{n} p_i = 1$ and $p_i(x) = 0$ for $|f(x) - f(x_i)| > \epsilon, i = 1, 2, \cdots, n$. Set $\lambda_i = f(x_i)$. Note that if $p_i(x) \neq 0$, for some $i = 1, 2, \cdots, n$, then $|f(x) - \lambda_i| < \epsilon$. Hence for any $x \in X$,

$$\begin{aligned}|f(x) - \sum_{i=1}^{n} \lambda_i p_i(x)| &= |\sum_{i=1}^{n}(f(x) - \lambda_i)p_i(x)| \\ &\leq |f(x) - \lambda_i|p_i(x) \\ &< \sum_{i=1}^{n} \epsilon.p_i(x) = \epsilon.\end{aligned}$$

Since, $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$, therefore $\sum_{i=1}^{n} \phi(p_i) = \phi(\sum_{i=1}^{n} p_i) = \phi(1) \leq 1$ and $|\lambda_i| = |f(x_i)| \leq ||f|| \leq 1$. By Lemma (2), $|| \sum_{i=1}^{n} \lambda_i \phi(p_i)|| \leq 1$. Now

$$\begin{aligned}||\phi(f)|| \leq ||\phi(f - \sum_{i=1}^{n} \lambda_i p_i)|| + || \sum_{i=1}^{n} \lambda_i \phi(p_i)|| \\ < ||\phi||.||f - \sum_{i=1}^{n} \lambda_i p_i|| + 1 \\ < ||\phi||.\epsilon + 1, \text{ for all } \epsilon > 0.\end{aligned}$$

This implies, $||\phi|| \leq 1$. □

Proposition 22. *Let S be an operator system and let $\phi : S \to C(X)$, where $C(X)$ denotes the continuous functions on a compact Hausdorff space X. If ϕ is positive, then $||\phi|| = ||\phi(1)||$.*

Proof. For $x \in X$, define $\pi_x : C(X) \to \mathbb{C}$ by $\pi_x(f) = f(x)$. Clearly, π_x is positive, hence $\pi_x \circ \phi : S \to \mathbb{C}$ is positive. Now for $a \in S$,

$$\begin{aligned}|\phi(a)(x)| &= |\pi_x \circ \phi(a)| \\ &\leq ||\pi_x \circ \phi(1)||.||a|| = ||\phi(1)(x)||.||a|| \\ &\leq ||\phi(1)||.||a||.||x||\end{aligned}$$

Thus $||\phi(a)|| \leq ||\phi(1)||.||a||$, for all $a \in S$. Hence $||\phi|| = ||\phi(1)||$. □

Theorem 26. *Let T be an operator on a Hilbert space H with $||T|| \leq 1$ and let $S \subseteq C(T)$ be the operator system defined by*

$$S = \{p(e^{\iota\theta}) + q(e^{\iota\theta}) : p, q \textit{ are polynomials } \}.$$

Then the map $\phi : S \to B(H)$ defined by $\phi(p + \bar{q}) = p(T) + q(T)^*$ is positive.

Proposition 23. *Let S be an operator system, $\mathcal{B}$ be a C^*-algebra and $\phi : S \to \mathcal{B}$ be a positive map. Then ϕ extends to a positive map on the norm closure of S.*

Proof. Positivity of ϕ implies that it is bounded, hence, there exist unique map $\phi' : \overline{S} \to \mathcal{B}$ such that $||\phi'|| = ||\phi||$ and $\phi'|_S = \phi$. Now claim is ϕ' is positive on $\overline{S}$.

Let p be positive element in $\overline{S}$. Then there exist a sequence (x_n) in S such that $||x_n - p|| \to 0$ as $n \to \infty$ and $||p - x_n^*|| = ||(p - x_n)^*|| = ||p - x_n||$. Let $h_n = \frac{1}{2}(x_n + x_n^*)$, then

$$||p - h_n|| = ||p - \frac{1}{2}(x_n + x_n^*)|| \leq \frac{1}{2}\left(||p - x_n|| + ||p - x_n^*||\right) \to 0 \text{ as } n \to \infty$$

Let $\epsilon > 0$ be given. Therefore, there exist $N_\epsilon \in \mathbb{N}$ such that $||p - h_n|| < \epsilon/2$, for all $n \geq N_\epsilon$. Since, $h_n + \epsilon.1$ are elements of C^*-algebra, so they can be treated as operators in $\mathcal{B}(H)$, for some Hilbert space H. Let $x \in H$. Then

$$\langle (h_n + \epsilon.1)x, x\rangle = \langle (h_n + \epsilon.1 - p - \epsilon.1)x, x\rangle + \langle (p + \epsilon.1)x, x\rangle.$$

Now, $\langle (p+\epsilon.1)x, x\rangle = \langle px, x\rangle + \epsilon||x||^2 \geq \epsilon||x||^2$, since p is positive in S. And, from the Cauchy-Schwartz inequality, we have

$$\begin{aligned} |\langle (h_n + \epsilon.1 - p - \epsilon.1)x, x\rangle| &\leq ||h_n + \epsilon.1 - p - \epsilon.1||.||x||^2 \\ &= ||h_n - p||.||x||^2 \\ &< \frac{\epsilon}{2}.||x||^2, \qquad \text{for all } n \geq N_\epsilon. \end{aligned}$$

Therefore, $\langle (h_n + \epsilon.1)x, x\rangle \geq (\epsilon - \frac{\epsilon}{2})||x||^2 = \frac{\epsilon}{2}||x||^2$. This implies, $h_n + \epsilon.1$ are positive in S, hence $\phi(h_n + \epsilon.1)$ for all $n \in \mathbb{N}$, since ϕ is positive. Therefore, $\phi'(p + \epsilon.1) = \lim_{n\to\infty} \phi(h_n + \epsilon.1) \geq 0$ in $\mathcal{B}$, for all $\epsilon > 0$ i.e. ϕ' is positive. □

Proposition 24. *Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}$ with $||a|| \leq 1$. Then there is a unital positive map $\phi : C(\mathbb{T}) \to \mathcal{A}$ with $\phi(p) = p(a)$.*

Proof. By GNS representation for C^*-algebras, $\mathcal{A} \subseteq \mathcal{B}(H)$, for some H. Let $a \in \mathcal{A}$ with $||a|| \leq 1$ and $S = \{p(e^{\iota\theta}) + \overline{q(e^{\iota\theta})} : p, q \text{ are polynomials}\}$. Clearly, S is an operator system. Define, $\phi' : S \to \mathcal{B}(H)$ by

$$\phi'(p + q) = p(a) + q(a)^*.$$

By Theorem(31), ϕ' is positive and unital. Since, polynomials is dense in $C(\mathbb{T})$, there exist a unital positive map $\phi : C(\mathbb{T}) \to \mathcal{B}(H)$ such that $||\phi|| = ||\phi'||$ and ϕ' is the restriction of ϕ. From definition, $\phi(p) = p(a)$. □

Corollary 6. *Let $\mathcal{B}, \mathcal{C}$ be two unital C^*-algebras, let $\mathcal{A}$ be a unital subalgebra of $\mathcal{B}$.If $\phi : \mathcal{A} + \mathcal{A}^* \to \mathcal{C}$ is positive, then $||\phi(a)|| \leq ||\phi(1)|| \cdot ||a||$ for all $a \in \mathcal{A}$.*

Proof. Let a be in $\mathcal{A}$, $||a|| \leq 1$ and $S = \mathcal{A} + \mathcal{A}^*$. By Proposition (21) and Proposition (23), we may extend ϕ to a positive map on the closure $\overline{S}$ of S. As remarked above, there is a positive map $\psi : C(\mathbb{T}) \to \mathcal{B}$ with $\psi(p) = p(a)$. Since $\mathcal{A}$ is an algebra, the range of ψ is actually contained in $\overline{S}$. Clearly, the composition of positive maps is positive, so by Theorem 2.4, $||\phi(a)|| = ||\phi \circ \psi(e^\iota\theta)|| \leq ||\phi \circ \psi(1)|| \cdot ||e^\iota\theta|| = ||\phi(1)||$. □

If $\phi(1) = 1$, in the above, then ϕ is a contraction on $\mathcal{A}$.

Corollary 7. *Let $\mathcal{A}, \mathcal{B}$ be two unital C^*-algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a positive map. Then $||\phi|| = ||\phi(1)||$.*

So far we have concentrated on positive maps without indicating how positive maps arise. We end this chapter with two such results.

Lemma 3. *Let $\mathcal{A}$ be a C^*-algebra, $S \subset \mathcal{A}$ an operator system, and $f : S \to \mathbb{C}$ be a linear functional with $f(1) = 1$, $||f|| = 1$. If a is a normal element of $\mathcal{A}$ and $a \in S$, then $f(a)$ will lie in the closed convex hull of the spectrum of a.*

Proof. Suppose not. Note that the convex hull of a compact set $\sigma(a)$ is the intersection of all closed discs containing the set $\sigma(a)$. Thus, there will exist a λ and $r > 0$ such that $|f(a) - \lambda| > r$, while spectrum of a satisfies $\sigma(a) \subseteq \{z : |z - \lambda| \leq r\}$. But then $\sigma(a\lambda.1) \subseteq \{z : |z| \leq r\}$, and since the norm and spectral radius agree for normal elements, we have $||a - \lambda.1|| \leq r$. But,

$$r < |f(a) - \lambda| = |f(a - \lambda.1)| \leq ||f||.||a - \lambda.1|| \leq r.$$

This contradiction completes the proof. □

Since the convex hull of the spectrum of a positive operator is contained in the non-negative reals, we see that Lemma (3) implies that such an f must be positive.

Proposition 25. *Let S be an operator system, $\mathcal{B}$ a unital C^*-algebra, and $\phi : S \to \mathcal{B}$ a unital, contraction. Then ϕ is positive.*

Proof. Since $\mathcal{B}$ can be represented on a Hilbert space, we may, without loss of generality, assume that $\mathcal{B} \subseteq \mathcal{B}(H)$ for some Hilbert space H. Fix x in H, $||x|| = 1$. Setting $f(a) = \langle\phi(a)x, x\rangle$, we have that $f(1) = 1, ||f|| \leq ||\phi|||$. By Lemma (3), if a is positive, then $f(a)$ is positive and consequently, since x was arbitrary, $\phi(a)$ is positive. □

Proposition 26. *Let $\mathcal{A}$ be a unital C^*-algebra and let M be a subspace of $\mathcal{A}$ containing 1. If $\mathcal{B}$ is a unital C^*-algebra and $\phi : M \to \mathcal{B}$ is a unital contraction, then the map $\phi' : M + M^* \to \mathcal{B}$ given by*

$$\phi'(a + b^*) = \phi(a) + \phi(b)^*,$$

is well-defined and is the unique positive extension of ϕ to $M + M^*$.

Proof. First we will prove that ϕ' well defined. For this, it is enough to show that if a and a^* belong to M, then $\phi(a^*) = \phi(a)^*$. For this, set

$$S_1 := \{a : a \in M \text{ and } a^* \in M\} = M \cap M^*.$$

Then S_1 is an operator system and ϕ is unital, contractive map on S_1 and hence positive by Proposition (25). Hence ϕ is self-adjoint on S_1, thus if $a \in S_1$, then $\phi(a^*) = \phi(a)^*$.

Now we will claim that ϕ' is positive. By GNS theorem, $\mathcal{B} \subset \mathcal{B}(H)$, for some Hilbert space H. Let $x \in H$ with $||x|| = 1$ and $a \in \mathcal{A}$ be positive element. Set $\rho'(a) = \langle\phi'(a)x, x\rangle$. Let $\rho : M \to \mathbb{C}$ by

$$\rho(a) = \langle\phi(a)x, x\rangle.$$

Therefore,

$$|\rho(a)| = |\langle\phi(a)x, x\rangle| \leq ||\phi(a)|| \leq 1.||a||.$$

This implies $||\rho|| \leq 1$ and $\rho(1) = 1$. Therefore $||\rho|| = 1$. So, by Hahn Banach extension theorem, there exist $\rho_1 : M + M^* \to \mathbb{C}$ such that $||\rho_1|| = ||\rho|| = 1 = ||\rho_1(1)||$ and $\rho_1|_M = \rho$. Thus, ρ_1 is positive and hence Hermitian.

$$\begin{aligned}
\rho_1(a + b^*) &= \rho_1(a) + \rho_1(b^*) \\
&= \rho_1(a) + \rho_1(b)^* \\
&= \rho(a) + \rho(b)^* \\
&= \rho(a) + \overline{\rho(b)} \\
&= \langle \phi(a)x, x \rangle + \langle x, \phi(b)x \rangle \\
&= \langle \phi(a)x, x \rangle + \langle \phi(b)^* x, x \rangle \\
&= \langle \phi(a) + \phi(b)^* x, x \rangle \\
&= \langle \phi'(a + b^*)x, x \rangle \\
&= \rho'(a + b^*).
\end{aligned}$$

Thus $\rho_1 = \rho'$, hence ρ' is positive functional. This shows that $\phi'(a)$ is positive in $\mathcal{B}$ that is ϕ' is positive linear map. $\square$

Chapter 4

Completely Positive Maps

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, S an operator system and M be a subspace of $\mathcal{A}$. Then clearly, $\mathcal{M}_n(M)$ can be regarded as a subspace of $\mathcal{M}_n(\mathcal{A})$. Let $\phi : S \to \mathcal{B}$ be a linear map. Then define a linear map $\phi_n : \mathcal{M}_n(S) \to \mathcal{M}_n(\mathcal{B})$ by

$$\phi_n((a_{i,j})) = (\phi(a_{i,j})).$$

Definition 26. *We call ϕ $n-$positive if ϕ_n is positive and we call ϕ completely positive if ϕ is $n-$positive for all n.*

Definition 27. *We call ϕ completely bounded if $\sup_n ||\phi_n||$ is finite (or, the sequence (ϕ_n) is uniformly bounded) and set*

$$||\phi||_{cb} = \sup_n ||\phi_n||.$$

Note that $||\cdot||_{cb}$ is a norm on the space of completely bounded maps.

Definition 28. *We say ϕ is completely isometric and completely contractive, if ϕ_n is isometric for all n and $||\phi||_{cb} \leq 1$, respectively.*

Remark 12. *If ϕ is $n-$positive, then ϕ is $k-$positive for $k \leq n$. Also, $||\phi_k|| \leq ||\phi_n||$ for $k \leq n$.*

Let us begin the study of completely positive maps with some examples.

Example 22. *Let $\pi : \mathcal{A} \to \mathcal{B}$ be a $*-$homomorphism, then π is completely positive and completely contractive. Since, π_n is also $*-$homomorphism, for all n, so it maps positive elements to positive elements of $\mathcal{B}$ and $||\pi_n|| \leq 1$, for all n.*

Example 23. *Let $x, y \in \mathcal{A}$ and define $\phi : \mathcal{A} \to \mathcal{A}$ by*

$$\phi(a) = xay.$$

Then ϕ is completely bounded. Since, $\phi_n((a_{i,j})) = (xa_{i,j}y)$, we have

$$||\phi_n((a_{i,j}))|| = \left|\left| \begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & x \end{bmatrix} \cdot \begin{bmatrix} a_{i,j} \end{bmatrix} \cdot \begin{bmatrix} y & 0 & \cdots & 0 \\ 0 & y & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & y \end{bmatrix} \right|\right|.$$

This implies, $||\phi_n|| \leq ||x||.||y||$, *for all* $n \in \mathbb{N}$. *Thus* ϕ *is completely bounded and* $||\phi||_{cb} \leq ||x||.||y||$.

If $x = y^*$, *then* ϕ *is completely positive. This is because if* $(a_{i,j})$ *is positive in* $\mathcal{M}_n(\mathcal{A})$, *then there exist* $B \in \mathcal{M}_n(\mathcal{A})$ *such that* $(a_{i,j}) = B^*B$. *Hence* $\phi_n((a_{i,j})) = (BD)^*BD$, *where* $D = \begin{bmatrix} y & 0 & \cdots & 0 \\ 0 & y & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & y \end{bmatrix}$ *which is positive in* $\mathcal{M}_n(\mathcal{B})$.

Example 24. *Let* H_1 *and* H_2 *be Hilbert spaces and* $V_i \in \mathcal{B}(H_1, H_2), i = 1, 2$ *and* $\pi : \mathcal{A} \to \mathcal{B}(H_2)$ *is* $*-$*homomorphism. Define* $\phi : \mathcal{A} \to \mathcal{B}(H_1)$ *by*

$$\phi(a) = V_2^* \pi(a) V_1, \text{ for all } a \in \mathcal{A}.$$

Then ϕ *is completely bounded and* $||\phi||_{cb} \leq ||V_2||.||V_1||$.

If $V_1 = V_2$, *then* ϕ *is completely positive.*

In this chapter we study some of the elementary properties of completely positive maps, some theorems about when positive maps are automatically completely positive maps and about Tensor and Schur products. At the end, we will see Stinespring dilation theorem for completely positive maps.

Lemma 4. *Let* $\mathcal{A}$ *be a* C^*-*algebra with unit and let* $a, b \in \mathcal{A}$. *Then:*

1. $||a|| \leq 1$ *if and only if* $\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}$ *is positive in* $\mathcal{M}_2(\mathcal{A})$.
2. $\begin{pmatrix} 1 & a \\ a^* & b \end{pmatrix}$ *is positive in* $\mathcal{M}_2(\mathcal{A})$ *if and only if* $a^*a \leq b$.

Proof. By GNS construction, $\mathcal{A} \subset \mathcal{B}(H)$, for some Hilbert space H, so elements of $\mathcal{A}$ can be treated as operators in $\mathcal{B}(H)$.

If $||a|| \leq 1$, then for any $x, y \in H$, it follows that

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle &= \langle x, x \rangle + \langle ay, x \rangle + \langle x, ay \rangle + \langle y, y \rangle \\ &\geq ||x||^2 - 2||a||.||y||.||x|| + ||y||^2 \geq 0. \end{aligned}$$

Conversely, if $||a|| > 1$, then there exist unit vectors x and y such that $\langle ay, x \rangle < -1$ and the above inner product will be negative.

The proof of second, is similar to that of the first. □

Proposition 27. *Let* S *be an operator system,* $\mathcal{B}$ *a* C^*-*algebra with unit, and* $\phi : S \to \mathcal{B}$ *a unital,* $2-$*positive map. Then* ϕ *is contractive.*

Proof. Let $a \in S, ||a|| \leq 1$, then

$$\phi \begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} = \begin{pmatrix} 1 & \phi(a) \\ \phi(a^*) & 1 \end{pmatrix}$$

is positive and hence $||\phi(a)|| \leq 1$, from Lemma (4). □

Proposition 28. *(Schwarz Inequality for 2-positive maps).*
Let $\mathcal{A}, \mathcal{B}$ be unital C^*-algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a unital 2-positive map, then $\phi(a)^*\phi(a) \leq \phi(a^*a)$ for all $a \in \mathcal{A}$.

Proof. We have that $\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ a^* & a^*a \end{pmatrix} \geq 0$ and hence

$$\begin{pmatrix} 1 & \phi(a) \\ \phi(a^*) & \phi(a^*a) \end{pmatrix} \geq 0.$$

By Lemma (4), we have the result. □

Proposition 29. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with unit, let M be a subspace of $\mathcal{A}$, $1 \in M$, and let $S = M + M^*$. If $\phi : M \to \mathcal{B}$ is unital and 2-contractive (i.e., $||\phi||_2 \leq 1$), then the map $\phi' : S \to \mathcal{B}$ given by*

$$\phi'(a + b^*) = \phi(a) + \phi(b)^*$$

is 2-positive and contractive.

Proof. Since ϕ is contractive, ϕ' is well-defined by Proposition (26). Note that $\mathcal{M}_2(S) = \mathcal{M}_2(M) + \mathcal{M}_2(M)^*$ and that $(\phi')_2 = (\phi_2')$. Since ϕ_2 is contractive, again by Proposition (26), ϕ_2' is positive and so ϕ' is contractive by Proposition (31). □

Proposition 30. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with unit, let M be a subspace of $\mathcal{A}$, $1 \in M$, and let $S = M + M^*$. If $\phi : M \to \mathcal{B}$ is unital and completely contractive, then $\phi' : S \to \mathcal{B}$ is completely positive and completely contractive.*

Proof. We have that ϕ_n' is positive since ϕ_n is unital and contractive and ϕ_n' is contractive since $\phi_{2n}' = (\phi_n')_2$ is positive. □

4.1 Tensor Products and Schur products

We study tensor products from a purely algebraic viewpoint. We define tensors as functionals that act on bilinear forms.

Let X, Y, Z be vector spaces over the field $\mathbb{C}$. A map $A : X \times Y \to Z$ is bilinear if it is linear in each variable, that is,

$$A(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 A(x_1, y) + \alpha_2 A(x_2, y) \text{ and}$$
$$A(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 A(x, y_1) + \beta_2 A(x, y_2)$$

for all $x_i, x \in X, y_i, y \in Y$ and all scalars $\alpha_i, \beta_i, i = 1, 2$. We write $B(X \times Y, Z)$ for the vector space of bilinear mappings from the product $X \times Y$ into Z; when Z is the scalar field we denote the corresponding space of bilinear forms simply by $B(X \times Y)$.

For $x \in X, y \in Y$, $x \otimes y$, we call x tensor y is defined as

$$(x \otimes y)(A) = A(x, y),$$

for each bilinear form A on $X \times Y$. The tensor product $X \otimes Y$ is the subspace of the algebraic dual of $B(X \times Y)$ spanned by $x \otimes y$. That is, a tensor $u \in X \otimes Y$ has the form

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i,$$

where n is a natural number, $\lambda_i \in \mathbb{K}, x_i \in X$ and $y_i \in Y$. The representation of u is not unique. If $u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$ is a tensor and A a bilinear form, then the action of u on A is given by

$$u(A) = \sum_{i=1}^{n} \lambda_i A(x_i, y_i).$$

Now, we see tensor product of matrices.

Let $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ so that A and B can be thought of as linear transformations on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then $A \otimes B$ is the linear transformations on $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$, which is defined by setting

$$A \otimes B(x \otimes y) = Ax \otimes By.$$

We can write $A \otimes B = (A \otimes I)(I \otimes B)$ and $||A \otimes B|| = ||A|| \cdot ||B||$.

Now we will study about Schur products of matrices. If $A = (a_{i,j}), B = (b_{i,j})$ are elements of $M_n(\mathbb{C})$, the we define the Schur product by

$$A \circ B = (a_{i,j} \cdot b_{i,j}).$$

Now let A and B be in $M_n(\mathbb{C})$ and $\{e_i : i = 1, 2, \cdots, n\}$ be the standard basis of $\mathbb{C}^n$. Define an isometry $V : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$ by

$$V(e_i) = e_i \otimes e_i.$$

Then

$$V^*(A \otimes B)V = A \circ B.$$

To see this, note that

$$\begin{aligned}
\langle V^*(A \otimes B)Ve_j, e_i\rangle &= \langle A \otimes B(e_j \otimes e_j), (e_i \otimes e_i)\rangle \\
&= \langle Ae_j, e_i\rangle \cdot \langle Be_j, e_i\rangle \\
&= a_{i,j} \cdot b_{i,j} = \langle A \circ Be_j, e_i\rangle.
\end{aligned}$$

For fixed A, this gives rise to a linear map, $S_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, defined by

$$S_A(B) = A \circ B.$$

Thus, $||S_A(B)|| = ||V^*(A \otimes B)V|| \leq ||A|| \cdot ||B||$ and so $||S_A|| \leq ||A||$. Now the following characterizes when a Schur product map S_A is completely positive.

Theorem 27. *Let $A = (a_{i,j}) \in M_n(\mathbb{C})$. Then the following are equivalent:*

1. *A is positive.*

2. $S_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *is positive.*

3. $S_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *is completely positive.*

Proof. Clearly, $(3) \implies (1)$. Note that $J = (1)_{n\times n}$ is positive and that $S_A(J) = A$. Hence, $(2) \implies (1)$. It remains to prove that $(1) \implies (3)$.

First note that if A and B are positive, then $A\otimes B$ is positive . This is because $A\otimes B = (A^{1/2}\otimes B^{1/2})^2$. Let $B = (B_{i,j}) \in M_k(M_n(\mathbb{C}))$ be positive, then $B = (X_{i,j})^*(X_{i,j})$, for some $X_{i,j} \in \mathcal{M}_n(\mathbb{C})$. Observe that

$$(S_A)_k(B) = (V^*(A\otimes B_{i,j})V) = Y^*Y,$$

where $Y = ((A^{1/2}\otimes X_{i,j})V)$. $\square$

4.2 Stinespring's Dilation Theorem

Stinespring's representation theorem is a fundamental theorem in the theory of completely positive maps. It is a structure theorem for completely positive maps from a C^*-algebra into the C^*-algebra of bounded operators on a Hilbert space. This theorem provides a representation for completely positive maps, showing that they are simple modifications of $*-$homomorphisms. One may consider it as a natural generalization of the well-known Gelfand-Naimark-Segal theorem for states on C^*-algebras.

Theorem 28. *(Stinespring's Dilation Theorem) Let $\mathcal{A}$ be a unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}(H)$ be a completely positive map. Then there exist a Hilbert space K, a unital $*-$homomorphism $\pi : \mathcal{A} \to \mathcal{B}(K)$, and $V \in \mathcal{B}(H, K)$ with $||V||^2 = ||\phi(1)||$ such that*

$$\phi(a) = V^*\pi(a)V, \quad \textit{for all } a \in \mathcal{A}.$$

Proof. Let $\mathcal{A}$ be a unital C^*-algebra, $\phi : \mathcal{A} \to \mathcal{B}(H)$ be a completely positive map and H be a Hilbert space. The proof of this theorem is constructive and it follows the same steps as in the proof of Gelfand-Naimark-Segal construction for C^*-algebras.

1. First, construct a Hilbert space K, which is the completion of an inner product of quotient space.

2. Define a bounded linear map $\pi(a)$ on k, and a map $\pi : \mathcal{A} \to \mathcal{B}(K)$ defined as $a \mapsto \pi(a)$, which is unital $*-$homomorphism.

3. Construct a bounded operator $V : H \to K$ with $||\phi(1)|| = ||V||^2$ such that

$$\phi(a) = V^*\pi(a)V, \text{ for all } a \in \mathcal{A}.$$

Step 1. Consider the algebraic tensor product $\mathcal{A}\otimes H$. Define a map $\langle\cdot,\cdot\rangle : (\mathcal{A}\otimes H)\times(\mathcal{A}\otimes H) \to \mathbb{C}$ by

$$\langle a\otimes x, b\otimes y\rangle = \langle\phi(b^*a)x, y\rangle_H, \text{ and}$$

$$\left\langle \sum_{j=1}^n a_j\otimes x_j, \sum_{i=1}^n b_i\otimes y_i \right\rangle = \left\langle \phi_n((b_i^*a_j)) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle_{H^{(n)}},$$

where $\langle \cdot, \cdot \rangle_H$ is an inner product on H, $(b_i^* a_j) \in M_n(\mathcal{A})$, and $\langle \cdot, \cdot \rangle_{H^{(n)}}$ denotes the inner product on the direct sum of n copies of H, $H^{(n)}$, given by

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \langle x_1, y_1 \rangle_H + \cdots + \langle x_n, y_n \rangle_H.$$

Clearly, from the definition, $\langle \cdot, \cdot \rangle$ is well-defined. Let $\alpha, \beta \in \mathbb{C}$, then

$$\begin{aligned}
&\left\langle \sum_{j=1}^{n} \alpha_j a_j \otimes x_j, \sum_{i=1}^{n} \beta_i b_i \otimes y_i \right\rangle \\
&= \left\langle \phi_n(((\beta_i b_i)^* \alpha_j a_j)) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle_{H^{(n)}} \\
&= \left\langle \left(\phi(\bar{\beta}_i b_i^* \alpha_j a_j)\right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle_{H^{(n)}} \\
&= \left\langle \begin{pmatrix} \sum_{j=1}^{n} \phi(\bar{\beta}_1 b_1^* \alpha_j a_j) x_j \\ \vdots \\ \sum_{j=1}^{n} \phi(\bar{\beta}_1 b_n^* \alpha_j a_j) x_j \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle_{H^{(n)}} \\
&= \left\langle \sum_{j=1}^{n} \phi(\bar{\beta}_1 b_1^* \alpha_j a_j) x_j, y_1 \right\rangle_H + \cdots + \left\langle \sum_{j=1}^{n} \phi(\bar{\beta}_n b_n^* \alpha_j a_j) x_j, y_n \right\rangle_H \\
&= \bar{\beta}_1 \sum_{j=1}^{n} \alpha_j \langle \phi(b_1^* a_j) x_j, y_1 \rangle_H + \cdots + \bar{\beta}_n \sum_{j=1}^{n} \alpha_j \langle \phi(b_n^* a_j) x_j, y_n \rangle_H \\
&= \sum_{i=1}^{n} \bar{\beta}_i \sum_{j=1}^{n} \alpha_j \langle \phi(b_i^* a_j) x_j, y_i \rangle_H \\
&= \sum_{i=1}^{n} \bar{\beta}_i \sum_{j=1}^{n} \alpha_j \langle a_j \otimes x_j, b_i \otimes y_i \rangle.
\end{aligned}$$

Thus, $\langle \cdot, \cdot \rangle$ is linear in first variable and conjugate linear in the second variable.

Consider,

$$
\begin{aligned}
\left\langle \sum_{j=1}^{n} a_j \otimes x_j, \sum_{i=1}^{n} b_i \otimes y_i \right\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle a_j \otimes x_j, b_i \otimes y_i\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle \phi(b_i^* a_j) x_j, y_i\rangle_H \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle \phi((a_j^* b_i)^*) x_j, y_i\rangle_H \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle \phi(a_j^* b_i)^* x_j, y_i\rangle_H \quad (\text{ since } \phi \text{ is hermitian}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle x_j, \phi(a_j^* b_i) y_i\rangle_H \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{\langle \phi(a_j^* b_i) y_i, x_j\rangle_H} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{\langle b_i \otimes y_i, a_j \otimes x_j\rangle} \\
&= \overline{\left\langle \sum_{i=1}^{n} b_i \otimes y_i, \sum_{j=1}^{n} a_j \otimes x_j \right\rangle}.
\end{aligned}
$$

This shows that $\langle \cdot, \cdot \rangle$ is conjugate symmetry. Now,

$$
\begin{aligned}
\left\langle \sum_{j=1}^{n} a_j \otimes x_j, \sum_{i=1}^{n} a_i \otimes x_i \right\rangle &= \left\langle \phi_n((a_i^* a_j)) \begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix} \right\rangle_{H^n} \\
&= \left\langle \phi_n(a^* a) \begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \cdot \\ x_n \end{pmatrix} \right\rangle_{H^n}, \quad \text{where } a = (a_1, \cdots, a_n)_{1\times n} \\
&\geq 0.
\end{aligned}
$$

Since, $a^* a \geq 0$ in $\mathcal{M}_n(\mathcal{A})$ and ϕ is completely positive, therefore $\phi_n(a^* a) \geq 0$ in $\mathcal{B}(H^n)$, and eq. 1 holds for all $x \in H$. Also if $a \otimes x = 0$, then $\langle a \otimes x, a \otimes x \rangle = 0$. Hence, $\langle \cdot, \cdot \rangle$ is positive semi-definite on $\mathcal{A} \otimes H$.

Let $\mathcal{N} = \{u \in \mathcal{A} \otimes H : \langle u, u \rangle = 0\}$. Clearly, $\mathcal{N}$ is closed subspace of $\mathcal{A} \otimes H$. Since positive semi-definite inner products satisfy the Cauchy-Schwartz inequality,

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \cdot \langle v, v \rangle$$

Therefore, $\mathcal{N} = \{u \in \mathcal{A} \otimes H : \langle u, v \rangle = 0, \text{ for all } v \in \mathcal{A} \otimes H\}$. Consider the quotient space $\mathcal{A} \otimes H/\mathcal{N}$ and define map $\langle \cdot, \cdot \rangle$ on $\mathcal{A} \otimes H/\mathcal{N}$ by

$$\langle u + \mathcal{N}, v + \mathcal{N} \rangle := \langle u, v \rangle.$$

Clearly, the space $(\mathcal{A} \otimes H/\mathcal{N}, \langle \cdot, \cdot \rangle)$ will become an inner product space. Let K denote the Hilbert space that is the completion of the inner product space $\mathcal{A} \otimes H/\mathcal{N}$.

Step 2. Let $a \in \mathcal{A}$. Define a linear map $\pi(a) : \mathcal{A} \otimes H \to \mathcal{A} \otimes H$ by

$$\pi(a) \left(\sum_{i=1}^{n} a_i \otimes x_i \right) = \sum_{i=1}^{n} (aa_i) \otimes x_i.$$

Note that $a^*a \leq ||a^*a||.1$, this implies that $||a^*a||.1 - a^*a \geq 0$. Let $c^*c = ||a^*a||.1 - a^*a$, for some $c \in \mathcal{A}$. We want to show that $(a_i^*a^*aa_j)_{n\times n} \leq ||a^*a|| \cdot (a_i^*a_j)_{n\times n}$. Thus

$$\begin{aligned} ||a^*a|| \cdot (a_i^*a_j)_{n\times n} - (a_i^*a^*aa_j)_{n\times n} &= (a_i^*(||a^*a|| - a^*a)a_j)_{n\times n} \\ &= (a_i^*c^*ca_j)_{n\times n} \\ &= e^*e \qquad \text{where, } e = (ca_1, \cdots, ca_n)_{1\times n} \\ &\geq 0. \end{aligned}$$

Consider,

$$\begin{aligned} \left\| \pi(a) \left(\sum_{k=1}^{n} a_k \otimes x_k \right) \right\|^2 &= \left\langle \pi(a) \left(\sum_{j=1}^{n} a_j \otimes x_j \right), \pi(a) \left(\sum_{i=1}^{n} a_i \otimes x_i \right) \right\rangle \\ &= \left\langle \sum_{j=1}^{n} (aa_j) \otimes x_j, \sum_{i=1}^{n} (aa_i) \otimes x_i \right\rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \phi(a_i^*a^*aa_j)x_j, x_i \rangle_H \\ &\leq ||a^*a|| \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \phi(a_i^*a_j)x_j, x_i \rangle_H \qquad \text{since, } \phi \text{ is positive.} \\ &= ||a||^2 \cdot \left\langle \sum_{j=1}^{n} a_j \otimes x_j, \sum_{i=1}^{n} a_i \otimes x_i \right\rangle \\ &= ||a||^2 \cdot \left\| \sum_{k=1}^{n} a_k \otimes x_k \right\|^2 . \end{aligned}$$

Thus $||\pi(a)|| \leq ||a||$, $\pi(a)$ leaves $\mathcal{N}$ invariant and, consequently, induces a quotient linear transformation on $\mathcal{A} \otimes H/\mathcal{N}$, which we still denote by $\pi(a)$. The above inequality also shows that $\pi(a)$ is bounded with $||\pi(a)|| \leq ||a||$. Thus, $\pi(a)$ extends to a bounded linear operator on K, which we still denote by $\pi(a)$.

Define a linear map $\pi : \mathcal{A} \to \mathcal{B}(K)$ by

$$a \mapsto \pi(a).$$

We are claiming that π is a unital $*-$homomorphism. Clearly $\pi(1)$ is identity operator in

$\mathcal{B}(K)$, hence π is unital. Now let $a_1, a_2 \in \mathcal{A}$, then

$$\begin{aligned}
\pi(a_1a_2)\left(\sum_{i=1}^{n}(a_i)\otimes x_i\right) &= \sum_{i=1}^{n}(a_1a_2a_i)\otimes x_i \\
&= \pi(a_1)\left(\sum_{i=1}^{n}(a_2a_i)\otimes x_i\right) \\
&= \pi(a_1)\cdot\pi(a_2)\left(\sum_{i=1}^{n}a_i\otimes x_i\right).
\end{aligned}$$

Thus $\pi(a_1a_2) = \pi(a_1)\cdot\pi(a_2)$, and for π to be $*-$preserving, we need to show that $\pi(a^*) = \pi(a)^*$. For $a_j, b_i \in \mathcal{A}, x_i, y_j \in H$, we have

$$\begin{aligned}
\left\langle \pi(a)\left(\sum_{j=1}^{n}a_j\otimes x_j\right), \sum_{i=1}^{n}b_i\otimes y_i\right\rangle &= \left\langle \sum_{j=1}^{n}(aa_j)\otimes x_j, \sum_{i=1}^{n}b_i\otimes y_i\right\rangle \\
&= \sum_{i,j=1}^{n}\langle aa_j\otimes x_j, b_i\otimes y_i\rangle \\
&= \sum_{i,j=1}^{n}\langle \phi(b_i^*aa_j)x_j, y_i\rangle_H \\
&= \sum_{i,j=1}^{n}\langle \phi((a^*b_i)^*a_j)x_j, y_i\rangle_H \\
&= \sum_{i,j=1}^{n}\langle a_j\otimes x_j, (a^*b_i)\otimes y_i\rangle \\
&= \left\langle \sum_{j=1}^{n}a_j\otimes x_j, \sum_{i=1}^{n}(a^*b_i)\otimes y_i\right\rangle \\
&= \left\langle \sum_{j=1}^{n}a_j\otimes x_j, \pi(a^*)\left(\sum_{i=1}^{n}b_i\otimes y_i\right)\right\rangle.
\end{aligned}$$

This implies $\pi(a)^* = \pi(a^*)$, for all $a \in \mathcal{A}$. Thus π is a unital $*-$homomorphism from $\mathcal{A}$ into $\mathcal{B}(K)$.

Step 3. Existence of V. Now define $V : H \to K$ via

$$V(x) = 1\otimes x + \mathcal{N},$$

then V is bounded, since

$$||Vx||^2 = \langle 1\otimes x, 1\otimes x\rangle = \langle \phi(1)x, x\rangle_H \le ||\phi(1)||\cdot||x||^2.$$

Indeed, it is clear that $||V||^2 = \sup\{\langle\phi(1)x, x\rangle_H : ||x|| \le 1\} = ||\phi(1)||$.

To complete the proof, we only need observe that

$$\langle V^*\pi(a)Vx, y\rangle_H = \langle \pi(a)1\otimes x, 1\otimes y\rangle_K = \langle \phi(a)x, y\rangle_H,$$

for all x and y, and so $V^*\pi(a)V = \phi(a)$. $\square$

Remark 13. *Any map of the form $\phi(a) = V^*\pi(a)V$ is completely positive map.*

Remark 14. *Stinespring's theorem is really the natural generalization of the Gelfand-Naimark-Segal representation of states. Indeed, if $H = \mathbb{C}$ is one-dimensional so that $\mathcal{B}(\mathbb{C}) = \mathbb{C}$, then an isometry $V : \mathbb{C} \to K$ is determined by $V1 = x$ and we have*

$$\phi(a) = \phi(a)1 \cdot 1 = V^*\pi(a)V1 \cdot 1 = \langle \pi(a)V1, V1\rangle_K = \langle \pi(a)x, x\rangle.$$

Infact, re-reading the above proof with $H = \mathbb{C}$ and $\mathcal{A} \otimes \mathbb{C} = \mathcal{A}$, one will find a proof of the GNS representation of states.

Remark 15. *Note that if H and $\mathcal{A}$ are separable, then the space K constructed above will be separable as well. Similarly, if H and $\mathcal{A}$ are finite dimensional, then K is finite dimensional.*

We now turn our attention to considering the uniqueness of the Stinespring representation. We shall call a triple (π, V, K) as obtained in Stinesprings Theorem a Stinespring representation for ϕ. If L is a subset of a Hilbert space, then $[L] := \overline{span}(L)$. Given a Stinespring representation (π, V, K), let $K_1 = [\pi(\mathcal{A})VH]$. Since, K_1 reduces $\pi(\mathcal{A})$ so that the restriction of π to K_1 defines a $*-$homomorphisms, $\pi_1 : \mathcal{A} \to \mathcal{B}(K_1)$.

Clearly, $VH \subset K_1$, so we have that $\phi(a) = V^*\pi_1(a)V$, i.e. that (π_1, V, K_1) is also a Stinespring representation. Whenever the space of the representation enjoys this additional property, we call the triple a minimal Stinespring representation. The following result summarizes the importance of this minimality condition. It states that two minimal representation for ϕ are unitarily equivalent.

Theorem 29. *Let $\mathcal{A}$ be a C^*-algebra, let $\phi : \mathcal{A} \to \mathcal{B}(H)$ be completely positive and let $(\pi_i, V_i, K_i), i = 1, 2,$ be two minimal Stinespring representations for ϕ. Then there exists a unitary map $U : K_1 \to K_2$ satisfying $UV_1 = V_2$ and $U\pi_1 = \pi_2 U$.*

Proof. Define a map $U' : span\{\pi_1(\mathcal{A})V_1H\} \to span\{\pi_2(\mathcal{A})V_2H\}$ by

$$U'\left(\sum_{i=1}^{n} \pi_1(a_i)V_1H_i\right) = \sum_{i=1}^{n} \pi_2(a_i)V_2h_i.$$

We will show that U' yields a well-defined isometry.

$$\begin{aligned}
\left|\left|\sum_{i=1}^{n}\pi_1(a_i)V_1h_i\right|\right|^2 &= \sum_{i,j=1}^{n}\langle\pi_1(a_j)V_1h_j,\pi_1(a_i)V_1h_i\rangle \\
&= \sum_{i,j=1}^{n}\langle\pi_1(a_i)^*\pi_1(a_j)V_1h_j,V_1h_i\rangle \\
&= \sum_{i,j=1}^{n}\langle\pi_1(a_i^*a_j)V_1h_j,V_1h_i\rangle \\
&= \sum_{i,j=1}^{n}\langle V_1^*\pi_1(a_i^*a_j)V_1h_j,h_i\rangle \\
&= \sum_{i,j=1}^{n}\langle\phi_1(a_i^*a_j)h_j,h_i\rangle \\
&= \sum_{i,j=1}^{n}\langle V_2^*\pi_2(a_i^*a_j)V_2h_j,h_i\rangle \\
&= \left|\left|\sum_{i=1}^{n}\pi_2(a_i)V_2h_i\right|\right|^2,
\end{aligned}$$

so U' is onto-isometric and consequently well-defined. By the Hahn-Banach extension theorem, we can extend U' to $U : K_1 \to span\{\pi_2(\mathcal{A})V_2H\}$ with $||U|| = ||U'||$. Therefore, U' is also isometry. Let $y \in K_2$, then there exist a sequence $(y_n) \subset span\{\pi_2(\mathcal{A})V_2H\}$ such that $y_n \to y$ as $n \to \infty$. Since, U is onto map, so there exist a sequence $(x_n) \subset K_1$ such that $Ux_n = y_n$. Now, for $m, n \in \mathbb{N}$

$$||x_n - x_m|| = ||Ux_n - Ux_m|| = ||y_n - y_m|| \to 0 \text{ as } n \to \infty.$$

Therefore, (x_n) is cauchy in K_1 and K_1 being complete, hence $x_n \to x$ as $n \to \infty$, for some $x \in K_1$. Thus, $Ux = y$ and we can extend $U : K_1 \to K_2$ as onto isometry map. Hence, U is unitary. And,

$$UV_1h = U(\pi_1(1)V_1h) = \pi_2(1)V_2h = V_2h, \text{ for all } h \in H.$$

Hence, $UV_1 = V_2$. $\square$

Chapter 5

Hilbert C*-Modules

Definition 29. *Let $\mathcal{A}$ be a C^*-algebra. An inner-product $\mathcal{A}-$module is a linear space E which is a right $\mathcal{A}-$module (with compatible scalar multiplication: $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for $x \in E$, $a \in \mathcal{A}, \lambda \in \mathbb{C}$), together with a map $(x, y) \to \langle x, y\rangle : E \times E \to \mathcal{A}$ such that for all $x, y, z \in E, a \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$*

1. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.
2. $\langle xa, y\rangle = \langle x, y\rangle a$.
3. $\langle y, x\rangle = \langle x, y\rangle^*$.
4. $\langle x, x\rangle \geq 0$; *if* $\langle x, x\rangle = 0$, *then* $x = 0$.

Note 7. *Inner-product defined above is linear in first variable and conjugate linear in second variable.*

Note 8. *Ordinary inner-product spaces and Hilbert spaces are inner-product $\mathbb{C}-$module.*

Remark 16. *If E satisfies all the conditions for an inner-product $\mathcal{A}-$module except for the second part of* 4. *then we call E a semi-inner-product $\mathcal{A}-$module.*

Proposition 31. *(Cauchy-Schwarz Inequality)*
If E is a semi-inner-product $\mathcal{A}-$module and $x, y \in E$. Then

$$\langle y, x\rangle\langle x, y\rangle \leq ||\langle x, x\rangle|| \cdot \langle y, y\rangle.$$

Proof. Suppose, without loss of generality, $||\langle x, x\rangle|| = 1$. For $a \in \mathcal{A}$, we have

$$\begin{aligned} 0 &\leq \langle xa - y, xa - y\rangle \\ &= a^*\langle x, x\rangle a - \langle y, x\rangle a - a^*\langle x, y\rangle + \langle y, y\rangle \\ &\leq a^*||\langle x, x\rangle||a - \langle y, x\rangle a - a^*\langle x, y\rangle + \langle y, y\rangle \\ &= a^*a - \langle y, x\rangle a - a^*\langle x, y\rangle + \langle y, y\rangle. \end{aligned}$$

Put $a = \langle x, y\rangle$, to get the required inequlaity. $\square$

Remark 17. *For $x \in E$, we write $||x|| = ||\langle x, x\rangle||^{1/2}$. It follows from Proposition* (31) *that $||\langle x, y\rangle|| \leq ||x||.||y||$. And, this define a norm on E.*

Definition 30. *An inner-product $\mathcal{A}-$module which is complete with respect to its norm is called a Hilbert $\mathcal{A}-$module, or a Hilbert C^*-module over the C^*-algebra $\mathcal{A}$.*

Example 25. *If $\mathcal{A}$ is a C^*-algebra, then $\mathcal{A}$ is itself a Hilbert $\mathcal{A}-$module if we define $\langle a, b\rangle = a^*b$, for $a, b \in \mathcal{A}$.*

Example 26. *If J is a closed right ideal in $\mathcal{A}$ then J is a sub-module of $\mathcal{A}$ and therefore a Hilbert $\mathcal{A}-$module.*

Example 27. *If $\{E_i\}$ is a finite set of Hilbert $\mathcal{A}-$modules, then the direct sum $\otimes_{i=1}^n E_i$ forms a Hilbert $\mathcal{A}-$module, if we define $\langle x, y\rangle = \sum_{i=1}^n \langle x_i, y_i\rangle$, where $x = (x_i)$ and $y = (y_i)$.*

Example 28. *$\mathcal{B}(H_1, H_2)$ is a Hilbert $\mathcal{B}(H_1)-$module for any two Hilbert spaces H_1, H_2 with the following operations:*

1. *module map: $(T, S) \mapsto TS : \mathcal{B}(H_1, H_2) \times \mathcal{B}(H_1) \to \mathcal{B}(H_1, H_2)$*

2. *inner product: $\langle T, S\rangle \mapsto T^*S : \mathcal{B}(H_1, H_2) \times \mathcal{B}(H_1, H_2) \to \mathcal{B}(H_1)$.*

Definition 31. *Let $\phi : \mathcal{A} \to \mathcal{B}(H)$ be linear. Then ϕ is said to be a morphism if it is a $*-$ homomorphism and nondegenerate (i.e., $\phi(\mathcal{A})H = H$).*

Definition 32. *A map $\Phi : E \to \mathcal{B}(H_1, H_2)$ is said to be a*

1. *$\phi-$map if $\langle \Phi(x), \Phi(y)\rangle = \phi(\langle x, y\rangle)$ for all $x, y \in E$.*

2. *$\phi-$morphism if Φ is a $\phi-$map and ϕ is a morphism.*

3. *$\phi-$representation if Φ is a $\phi-$morphism and ϕ is a representation.*

Note 9. *A $\phi-$morphism Φ is linear and satisfies $\Phi(xa) = \Phi(x)\phi(a)$ for every $x \in E$ and $a \in \mathcal{A}$.*

Several module versions of Stinespring theorem can be found in the literature. Typically they are structure theorems for completely positive maps in more general context. The result we are going to consider here are for $\phi-$maps.

A theorem similar to first Stinesprings theorem was presented by Mohammad B. Asadi for a class of unital maps on Hilbert C^*-modules. This result can also be proved by removing a technical condition of Asadis theorem. The assumption of unitarily on maps under consideration can also be removed. Further we will see the uniqueness up to unitary equivalence for minimal representations, which is an important ingredient of structure theorems like GNS theorem and Stinesprings theorem. Now the result looks even more like Stinesprings theorem.

Theorem 30. *(Mohammad B. Asadi).*
If E is a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$, $\phi : \mathcal{A} \to \mathcal{B}(H_1)$ is a completely positive map with $\phi(1) = 1$ and $\Phi : E \to \mathcal{B}(H_1, H_2)$ is a $\phi-$map with the additional property $\Phi(x_o)\Phi(x_o)^* = I_{H_2}$, for some $x_o \in E$, where H_1, H_2 are Hilbert spaces, then there exist Hilbert spaces K_1, K_2, isometries $V : H_1 \to K_1, W : H_2 \to K_2$, a $*-$homomorphism $\rho : \mathcal{A} \to \mathcal{B}(K_1)$ and a $\rho-$representation $\Psi : E \to \mathcal{B}(K_1, K_2)$ such that

$$\phi(a) = V^*\rho(a)V, \qquad \Phi(x) = W^*\Psi(x)V, \text{ for all} x \in E, a \in \mathcal{A}.$$

5.1 Stinespring's theorem for maps on Hilbert C*-modules

In this Section we strengthen Asadis theorem for a $\phi-$map Φ and discuss the minimality of the representations.

Theorem 31. *Let $\mathcal{A}$ be a unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}(H_1)$ be a completely positive map. Let E be a Hilbert $\mathcal{A}-$module and $\Phi : E \to \mathcal{B}(H_1, H_2)$ be a $\phi-$map. Then there exist a pair of triples $((\pi, V, K_1), (\Psi, W, K_2)$, where*

1. *K_1 and K_2 are Hilbert spaces;*

2. *$\pi : \mathcal{A} \to \mathcal{B}(K_1)$ is a unital *-homomorphism and $\Psi : E \to \mathcal{B}(K_1, K_2)$ is a $\pi-$morphism;*

3. *$V : H_1 \to K_1$ and $W : H_2 \to K_2$ are bounded linear operators;*

such that

$$\phi(a) = V^*\pi(a)V, \textit{ for all } a \in \mathcal{A} \textit{ and } \Phi(x) = W^*\Psi(x)V, \textit{ for all } x \in E.$$

Proof. We prove the theorem in two steps.

Step 1. Existence of π, V and K_1 : Since ϕ is a completely positive map, by the Stinespring's dilation theorem, there exist a minimal Stinespring representation (π, V, K_1) for ϕ, where $K_1 = \overline{span\{\pi(\mathcal{A})VH_1\}}$.

Step 2. Existence of Ψ, W and K_2 : Let $K_2 = \overline{span\{\Phi(E)H_1\}}$. Define, $\Psi : E \to \mathcal{B}(K_1, K_2)$ as follows:
For $x \in E$, define $\Psi(x) : span\{\pi(\mathcal{A})VH_1\} \to span\{\Phi(E)H_1\}$ by

$$\Psi(x)\left(\sum_{j=1}^{n}\pi(a_j)Vh_j\right) = \sum_{j=1}^{n}\Phi(xa_j)h_j, \qquad a_j \in \mathcal{A}, h_j \in H_1, j = 1, 2, \cdots, n.$$

First we claim that $\Psi(x)$ is well-defined. Consider,

$$\begin{aligned}
\left\|\Psi(x)\left(\sum_{j=1}^{n}\pi(a_j)Vh_j\right)\right\|^2 &= \left\langle \Psi(x)\left(\sum_{j=1}^{n}\pi(a_j)Vh_j\right), \Psi(x)\left(\sum_{i=1}^{n}\pi(a_i)Vh_i\right)\right\rangle_{H_2} \\
&= \left\langle \sum_{j=1}^{n}\Phi(xa_j)h_j, \sum_{i=1}^{n}\Phi(xa_i)h_i \right\rangle_{H_2} \\
&= \sum_{i,j=1}^{n}\langle \Phi(xa_j)h_j, \Phi(xa_i)h_i\rangle_{H_2} \\
&= \sum_{i,j=1}^{n}\langle \Phi(xa_i)^*\Phi(xa_j)h_j, h_i\rangle_{H_1} \\
&= \sum_{i,j=1}^{n}\langle \phi(\langle xa_i, xa_j\rangle_E)h_j, h_i\rangle_{H_1} \\
&= \sum_{i,j=1}^{n}\langle \phi(a_i^*\langle x, x\rangle_E a_j)h_j, h_i\rangle_{H_1} \\
&= \sum_{i,j=1}^{n}\langle V^*\pi(a_i^*\langle x, x\rangle_E a_j)Vh_j, h_i\rangle_{H_1} \quad (\because \phi(a) = V^*\pi(a)V) \\
&= \sum_{i,j=1}^{n}\langle \pi(a_i^*)\pi(\langle x, x\rangle_E)\pi(a_j)Vh_j, Vh_i\rangle_{K_1} \quad (\pi \text{ is } *-\text{homm}) \\
&= \sum_{i,j=1}^{n}\langle \pi(a_i)^*\pi(\langle x, x\rangle_E)\pi(a_j)Vh_j, Vh_i\rangle_{K_1} \\
&= \left\langle \pi(\langle x, x\rangle_E)\left(\sum_{j=1}^{n}\pi(a_j)Vh_j\right), \left(\sum_{i=1}^{n}\pi(a_i)Vh_i\right)\right\rangle_{K_1} \\
&\leq ||\pi(\langle x, x\rangle_E)|| \cdot \left\|\sum_{j=1}^{n}\pi(a_j)Vh_j\right\|^2 \quad (\text{ C-S inequality}) \\
&\leq ||x||^2 \cdot \left\|\sum_{j=1}^{n}\pi(a_j)Vh_j\right\|^2 \qquad (\text{ since } ||\pi|| \leq 1).
\end{aligned}$$

This shows that if $\sum_{i=1}^{n}\pi(a_i)Vh_i = 0$, then $\Psi(x)\left(\sum_{i=1}^{n}\pi(a_i)Vh_i\right) = 0$, hence $\Psi(x)$ is well-defined. Moreover, $\Psi(x)$ is bounded and $||\Psi(x)|| \leq ||x||$.

Since, $span\{\pi(\mathcal{A})VH_1\}$ is dense in K_1, $\Psi(x)$ can be extended uniquely to whole of K_1, call the extension $\Psi(x)$ itself.

Clearly, π is a morphism. Now we show that Ψ is $\pi-$map.

$$
\begin{aligned}
&\left\langle \Psi(x)^*\Psi(y)\left(\sum_{j=1}^{n}\pi(b_j)Vh_j\right), \sum_{i=1}^{m}\pi(a_i)Vg_i\right\rangle_{K_1} \\
&= \left\langle \Psi(y)\left(\sum_{j=1}^{n}\pi(b_j)Vh_j\right), \Psi(x)\left(\sum_{i=1}^{m}\pi(a_i)Vg_i\right)\right\rangle_{K_1} \\
&= \left\langle \sum_{j=1}^{n}\Phi(yb_j)h_j, \sum_{i=1}^{m}\Phi(xa_i)g_i\right\rangle_{H_2} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\langle \Phi(xa_i)^*\Phi(yb_j)h_j, g_i\rangle \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\langle \phi(\langle xa_i, yb_j\rangle)h_j, g_i\rangle \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\langle \phi(a_i^*\langle x, y\rangle b_j)h_j, g_i\rangle \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\langle V^*\pi(a_i^*\langle x, y\rangle b_j)Vh_j, g_i\rangle \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\langle \pi(a_i)^*\pi(\langle x, y\rangle)\pi(b_j)Vh_j, Vg_i\rangle \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\langle \pi(\langle x, y\rangle)\pi(b_j)Vh_j, \pi(a_i)Vg_i\rangle \\
&= \left\langle \pi(\langle x, y\rangle)\left(\sum_{j=1}^{n}\pi(b_j)Vh_j\right), \sum_{i=1}^{m}\pi(a_i)Vg_i\right\rangle.
\end{aligned}
$$

This shows that $\langle \Psi(x), \Psi(y)\rangle = \Psi(x)^*\Psi(y) = \pi(\langle x, y\rangle)$, for all $x, y \in E$ on the dense set $span\{\pi(\mathcal{A})VH_1\}$ and hence they are equal on K_1. Note that $K_2 \subset H_2$.

Let $W := P_{K_2}$, the orthogonal projection onto K_2. Then $W^* : K_2 \to H_2$ is the inclusion map. Hence $WW^* = I_{K_2}$. That is W is a co-isometry.

Now for $x \in E$ and $h \in H_1$, we have

$$W^*\Psi(x)Vh = \Psi(x)Vh = \Psi(x)(\rho(1)Vh) = \Phi(x)h.$$

□

Definition 33. *We say a pair $((\pi, V, K_1), (\Psi, W, K_2)$ is a Stinespring representation for (ϕ, Φ) if the conditions $1-3$ of Theorem(31) are satisfied. Such a representation is said to be minimal if*

1. $K_1 = [\pi(\mathcal{A})VH_1]$, *and*

2. $K_2 = [\Psi(E)VH_1]$.

Note 10. *The pair* $((\pi, V, K_1), (\Psi, W, K_2))$ *obtained in the proof of Theorem* (31) *is a minimal representation for* (ϕ, Φ).

Theorem 32. *Let* ϕ *and* Φ *be as in Theorem* (31). *Assume that* $((\pi, V, K_1), (\Psi, W, K_2))$ *and* $((\pi', V', K_1'), (\Psi', W', K_2'))$ *are minimal representations for* (ϕ, Φ). *Then there exists unitary operators* $U_1 : K_1 \to K_1'$ *and* $U_2 : K_2 \to K_2'$ *such that*

1. $U_1 V = V', U_1\pi(a) = \pi'(a)U_1,$ *for all* $a \in \mathcal{A}$, *and*

2. $U_2 W = W', U_2\Psi(x) = \Psi'(x)U_1,$ *for all* $x \in E$.

That is, the following diagram commutes, for $a \in \mathcal{A}$ *and* $x \in E$:

$$\begin{tikzcd}
H_1 \arrow[r, "V"] \arrow[dr, "V'"'] & K_1 \arrow[r, "\pi(a)"] \arrow[d, "U_1"] & K_1 \arrow[r, "\Psi(x)"] \arrow[d, "U_1"] & K_2 \arrow[d, "U_2"] & H_2 \arrow[l, "W"'] \arrow[dl, "W'"] \\
 & K_1' \arrow[r, "\pi'(a)"'] & K_1' \arrow[r, "\Psi'(x)"'] & K_2' &
\end{tikzcd}$$

Proof. Define $U_1 : span(\pi(\mathcal{A})VH_1) \to span(\pi'(\mathcal{A})V'H_1)$ by

$$U_1\left(\sum_{j=1}^{n} \pi(a_j)Vh_j\right) := \sum_{j=1}^{n} \pi'(a_j)V'h_j, \quad a_j\mathcal{A}, h_j \in H_1, j = 1, \cdots, n, n \geq 1,$$

which can be seen to be an onto isometry and the unitary extension of this is the required map $U_1 : K_1 \to K_2$, refer Theorem (29).

Now define $U_2 : span(\Psi(E)VH_1) \to span(\Psi'(E)V'H_1)$ by

$$U_2\left(\sum_{j=1}^{n} \Psi(x_j)Vh_j\right) := \sum_{j=1}^{n} \Psi'(x)V'h_j, \quad x_j \in E, h_j \in H_1, j = 1, \cdots, n, n \geq 1.$$

Consider

$$\begin{aligned}
\left\|\sum_{j=1}^{n} \Psi'(x_j)V'h_j\right\|^2 &= \sum_{i,j=1}^{n} \langle h_j, V'^*\pi'(\langle x_j, x_i\rangle)V'h_i\rangle \\
&= \sum_{i,j=1}^{n} \langle h_j, V^*\pi(\langle x_j, x_i\rangle)Vh_i\rangle \\
&= \left\|\sum_{j=1}^{n} \Psi(x_j)Vh_j\right\|^2.
\end{aligned}$$

Thus U_2 is well defined and is an isometry and can be extended to whole of K_2, call the extension U_2 itself,and being onto it is a unitary.

Since $((\pi, V, K_1), (\Psi, W, K_2)$ and $((\pi', V', K_1'), (\Psi', W', K_2')$ are representation for (ϕ, Φ), it follows that

$$\Phi(x) = W^*\Psi(x)V = W'\Psi'(x)V' = W'^*U_2\Psi(x)V$$

and hence $(W^* - W'U_2)\Psi(x)V = 0$. Since $[\Psi(E)VH_1] = K_2$, it follows that $U_2W = W'$. As Ψ is a $\pi-$morphism and Ψ' is a $\pi'-$morphism, it can be shown that

$$U_2\Psi(x)\left(\sum_{j=1}^{n}\pi(a_j)Vh_j\right) = \Psi'(x)U_1\left(\pi(a_j)Vh_j\right)'$$

for all $x \in E, a_j \in \mathcal{A}, h_j \in H_1, j = 1, \cdots, n, n \geq 1$, concluding $U_2\Psi(x) = \Psi'(x)U_1$. □

Example 29. *Let* $\mathcal{A} = \mathcal{M}_2(\mathbb{C}), H_1 = \mathbb{C}^2, H_2 = \mathbb{C}^*$ *and* $E = \mathcal{A} \oplus \mathcal{A}$. *Let* $D = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$. *Define* $\phi : \mathcal{A} \to \mathcal{B}(H_1)$ *by*

$$\phi(A) = D \circ A, \quad \textit{for all } A \in \mathcal{A};$$

where $\circ$ *denote the Schur product. As* D *is positive,* ϕ *is acompletely positve map (see Theorem* (27)*). Let* $D_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ *and* $\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$. *Let* $K_1 = \mathbb{C}^4$ *and* $K_2 = H_2$. *Define* $\Phi : E \to \mathcal{B}(H_1, H_2)$ *and* $\Psi : E \to \mathcal{B}(k_1, K_2)$ *by*

$$\Phi(A_1 \oplus A_2) = \begin{pmatrix} \frac{\sqrt{3}}{\sqrt{2}}A_1D_1 \\ \frac{\sqrt{3}}{\sqrt{2}}A_2D_1 \\ \frac{\sqrt{3}}{\sqrt{2}}A_1D_2 \\ \frac{\sqrt{3}}{\sqrt{2}}A_2D_2 \end{pmatrix}, \quad \Psi(A_1 \oplus A_2) = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \\ 0 & A_1 \\ 0 & A_2 \end{pmatrix}, \textit{ for all } A_1, A_2 \in \mathcal{A}.$$

Clearly, Φ *is a* $\phi-$*map.*

Define $V : H_1 \to K_1$ *and* $\pi : \mathcal{A} \to \mathcal{B}(K_1)$ *by*

$$V = \begin{pmatrix} \frac{\sqrt{3}}{\sqrt{2}}D_1 \\ \frac{1}{\sqrt{2}}D_2 \end{pmatrix}, \quad \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \textit{for all } A \in \mathcal{A}.$$

Clearly Ψ *is a* $\pi-$*morphism and* $\Phi(A_1 \oplus A_2) = W^*\Psi(A_1 \oplus A_2)V$, *where* $W = I_{H_2}$. *This example illustrates the Theorem* (31).

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