# Johnson's Theorem

**Varun Mishra**

*Under the supervision of* ***Dr. Daniel Sukumar***

A Thesis Submitted to
**Indian Institute of Technology Hyderabad**
In Partial Fulfillment of the Requirements for
The Degree of Master of science
**Department of Mathematics**

# Approval Sheet

This thesis entitled **Johnson's Theorem** by Varun Mishra is approved for the degree of Master of Science from Indian Institute of Technology Hyderabad.

4/5/2016.

Dr. Daniel Sukumar
(Supervisor)
Department of Mathematics
Indian Institute of Technology Hyderabad

# Declaration

I declare that this written submission represents my ideas in my own words, and where ideas or words of others have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea,data,fact and source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

Varun 3/5/2016.

Varun Mishra

Roll No. MA14MSCST11009

# Contents

# Chapter 1

# Tensor Product

## 1.1 Bilinear forms

Let $X, Y$ and $Z$ be normed spaces over the field $\mathbb{F}$. A mapping $\phi : X \times Y \to Z$ is said to be bilinear if

1. for each $y \in Y$, the mapping $x \to \phi(x, y)$ is linear,
2. for each $x \in X$, the mapping $y \to \phi(x, y)$ is linear as well.

When $Z = \mathbb{F}$, such a mapping is called bilinear form or bilinear functional.
A bilinear form is said to be bounded if there exist $M > 0$ such that

$$||\phi(x, y)|| \leq M||x||||y||$$

The norm of $\phi$, $||\phi||$ is then defined as

$$||\phi|| := sup\{||\phi(x, y)|| : ||x|| \leq 1, ||y|| \leq 1\}$$

When $X, Y$ and $Z$ are Banach Spaces, each separately continuous bilinear mapping, $\phi : X \times Y \to Z$ is bounded, i.e. let for each fix $x$, $\phi_x(y) \mapsto \phi(x, y)$ is continuous, i.e. for each $x$ there exists $M_x$, such that $||\phi_x|| \leq M_x$. Similarly we have $\phi_y$, these are continuous linear map from Banach spaces. So from Uniform Boundedness Theorem, we have a bound $M$ such that $||\phi_x|| \leq M$.

$$||\phi(x, y)|| \leq ||\phi_x|| \ ||y|| \leq M \ ||x|| \ ||y||,$$

which gives that $\phi$ is bounded. The set of all bounded bilinear mappings from $X \times Y$ to $Z$ is denoted by $BL(X, Y; Z)$[1].

[1]BL(X) will denote all bounded linear map from $X$ to $X$ and L(X) will denote all linear map from $X$ to $X$.

**Proposition 1.1.1.** *$BL(X,Y;Z)$ is a normed space with the usual pointwise operations, and is a Banach space if $Z$ is a Banach space.*

*Proof.* Suppose $(\phi_n)$ is a cauchy sequence in $BL(X,Y;Z)$, which provides us that $\forall\ \epsilon > 0$, we can have a $N \in \mathbb{N}$, such that $\forall\ n, m > N$, $||\phi_n - \phi_m|| < \epsilon/2||(x,y)||$, Now for fixed $(x,y) \in X \times Y$, we have

$$|\phi_n(x,y) - \phi_m(x,y)| \leq ||\phi_n - \phi_m||\ ||(x,y)|| < \epsilon/2$$

which immediately implies that $(\phi_n(x,y)) \subseteq Z$ is cauchy. As $Z$ is complete $(\phi_n(x,y))$ is a convergent sequence for every $(x,y) \in X \times Y$, let say converging to $z$. Define a function $\phi : X \times Y \to Z$ pointwise as $(x,y) \mapsto \lim\limits_{n\to\infty} \phi_n(x,y)$.
Now we claim that $(\phi_n) \to \phi$, and $\phi \in BL(X,Y;Z)$
*Claim* (1) : *$\phi$ is bilinear* :

$$\begin{aligned}\phi(x+y,z) &= \lim_{n\to\infty} \phi_n(x+y,z) \\ &= \lim_{n\to\infty} \phi_n(x,z) + \lim_{n\to\infty} \phi_n(y,z) \\ &= \phi(x,z) + \phi(y,z).\end{aligned}$$

Similar calculation for other variable simply implies that $\phi$ is bilinear.
*Claim* (2) : $(\phi_n) \to \phi$ : We have that for every $(x,y)$, $\phi_n(x,y) \to \phi(x,y)$, so $|\phi(x,y) - \phi_n(x,y)| < \epsilon/2$ and as $(\phi_n(x,y))$ is cauchy sequence so we have, $m \in \mathbb{N}$, such that

$$\begin{aligned}|\phi(x,y) - \phi_n(x,y)| &= |\phi(x,y) - \phi_m(x,y) + \phi_m(x,y) - \phi_n(x,y)| \\ &\leq |\phi(x,y) - \phi_m(x,y)| + |\phi_m(x,y) - \phi_n(x,y)| \leq \epsilon,\end{aligned}$$

Which simply gives that,

$$||\phi - \phi_n|| \leq \epsilon.$$

Thus, $(\phi_n)$ converges to $\phi$.
*Claim* (3) : $\phi \in BL(X,Y;Z)$ : $\phi$ is bounded.

$$\begin{aligned}|\phi(x,y)| &\leq |\phi_n(x,y)| + |\phi(x,y) - \phi_n(x,y)| \\ &\leq k + \epsilon \qquad (||\phi_n|| < k)\end{aligned}$$

This inequality holds for each $(x,y) \in X \times Y$, thus, $\phi \in BL(X,Y;Z)$. Hence, we proved that $BL(X,Y;Z)$ is complete.

□

The above remarks can be extended to the case of n-linear mappings from $X_1 \times X_2 \times \cdots \times X_n$ to $Z$. The corresponding space of bounded n-linear mappings is denoted by $BL(X_1, X_2, \ldots, X_n; Z)$. For the case $X_1 = \cdots = X_n = X$, we write this more simply as $BL^n(X; Z)$.

## 1.2 Tensor Product and Linearization

In this section we will define the tensor product as functionals that act on bilinear forms. We will see how tensor product helps to linearize the bilinear forms. We will work with vector spaces over the field $\mathbb{F}$ which can be either $\mathbb{R}$ or $\mathbb{C}$. We denote the algebraic dual of a vector space $X$ by $X'$ and define as $X' := \{f : X \to \mathbb{F},$ and f is linear$\}$ while topological dual by $X^*$ and define it as $X^* := \{f : X \to \mathbb{F},$ f is linear and bounded$\}$.

**Definition 1.2.1.** *Let $X, Y$ be normed spaces over $\mathbb{F}$ with dual spaces $X^*, Y^*$. Given $x \in X, y \in Y$, $x \otimes y$ is the element of $BL(X^*, Y^*; \mathbb{F})$ defined by*

$$x \otimes y(f, g) = f(x)g(y) \qquad (f \in X^*, g \in Y^*).$$

The algebraic tensor product of $X$ and $Y$, $X \otimes Y$, is then defined to be the linear span of $\{x \otimes y : x \in X, y \in Y\}$ in $BL(X^*, Y^*; \mathbb{F})$. Thus a typical element in $X \otimes Y$ will look like

$$u = \sum_{i=1}^{n} \alpha_i \ (x_i \otimes y_i) \qquad n \in \mathbb{N},\ \alpha_i' s \in \mathbb{F}$$

It's important to note that this representation is not unique. In general there are many ways to write a given tensor. The action of a tensor on elements of $X^* \times Y^*$ is independent of representation of tensor. The tensor $x \otimes y$ is called elementary tensor.
Now it is easy to check that tensor product is bilinear, i.e.

1. $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,$
2. $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y,$
3. $\alpha \ (x \otimes y) = \alpha x \otimes y = x \otimes \alpha y$
4. $0 \otimes y = x \otimes 0 = 0$

The primary purpose of tensor products is to linearize the bilinear mappings. In this section we will show that given each bilinear mapping

$$\phi : X \times Y \to Z$$

there exist a unique linear mapping $\sigma : X \otimes Y \to Z$ such that $\phi = \sigma o \tau$; $\tau : X \times Y \to X \otimes Y$ be the bilinear mapping defined by

$$\tau(x, y) = x \otimes y \qquad (x \in X, y \in Y).$$

This is key property of algebraic tensor product. We will further show that actually $(X \otimes Y)^*$ is in one-one correspondence with $BL(X, Y; \mathbb{F})$.

**Lemma 1.2.2.** *Given $u \in X \otimes Y$, there exist linearly independent sets $\{x_i\}, \{y_i\}$ such that $u = \sum_{i=1}^n x_i \otimes y_i$*

*Proof.* Let $u = \sum_{i=1}^n x_i \otimes y_i$ is the minimal representation for $u$ in the sense that $n$ is as small as possible. Suppose $\{y_i\}$ is linearly dependent, then without loss of generality assume that $y_n = \sum_{i=1}^{n-1} c_i y_i$, then

$$\begin{aligned} u &= \sum_{i=1}^{n-1} x_i \otimes y_i + x_n \otimes y_n \\ &= \sum_{i=1}^{n-1} x_i \otimes y_i + x_n \otimes \left( \sum_{i=1}^{n-1} c_i y_i \right) \\ &= \sum_{i=1}^{n-1} (x_i + x_n) \otimes c_i y_i \qquad \textit{(tensor products is bilinear)} \end{aligned}$$

Now the independency of set $\{x_i + x_n\}_{i=1}^{n-1}$ contradicts the minimality of $n$ in representation for $u$. It follows that $\{y_i\}$ is linearly independent, and a similar argument applies to $\{x_i\}$.

□

Now, when can we say two tensors are equal, this question reduces to the following: How to determine the $\sum_{i=1}^n x_i \otimes y_i$ is a representation of the zero tensor? In principle this can be determined by evaluating

$$\sum_{i=1}^n x_i \otimes y_i(f, g) \;\; \textit{for each} \;\; f \in X^* \;\; \textit{and} \;\; g \in Y^*.$$

Now we will see some equivalent conditions, which will pave our way to determine a zero tensor.

**Proposition 1.2.3.** *The following are equivalent for $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$;*

1. $u = 0$,
2. $\sum_{i=1}^n f(x_i) g(y_i) = 0$ *for every* $f \in X^*, g \in Y^*$,
3. $\sum_{i=1}^n f(x_i) y_i = 0$ *for every* $f \in X^*$,
4. $\sum_{i=1}^n x_i g(y_i) = 0$ *for every* $g \in Y^*$,

*Proof.* (1) $\implies$ (2).
As $u \in BL(X^*, Y^*; \mathbb{F})$, if $u = 0$ implies $u(f, g) = 0$, i.e.

$$\sum_{i=1}^{n} x_i \otimes y_i(f, g) = 0 \qquad (\forall\ f \in X^*, g \in Y^*)$$

$$\sum_{i=1}^{n} f(x_i)g(y_i) = 0 \ \ for\ every\ f \in X^*, g \in Y^*.$$

(2) $\implies$ (3).
If $\sum_{i=1}^{n} f(x_i)g(y_i) = 0 \ \ for\ every\ f \in X^*,\ g \in Y^*$, We have

$$g\left(\sum_{i=1}^{n} f(x_i)y_i\right) = 0 \ \ for\ every\ g \in Y^*,$$

Which gives, as the consequence of Hahn Banach theorem,

$$\sum_{i=1}^{n} f(x_i)y_i = 0 \ \ for\ every\ f \in X^*$$

Similarly (2) $\implies$ (4).
(3) $\implies$ (2).
If $\sum_{i=1}^{n} f(x_i)y_i = 0\ for\ every\ f \in X^*$,

$$\sum_{i=1}^{n} f(x_i)g(y_i) = g\left(\sum_{i=1}^{n} f(x_i)y_i\right) = 0 \ \ for\ every\ f \in X^*, g \in Y^*$$

Similarly (4) $\implies$ (2).
Now we will prove that (4) $\implies$ (1).
Suppose that $u = \sum_{i=1}^{n} x_i \otimes y_i$ and $\sum_{i=1}^{n} x_i g(y_i) = 0\ for\ every\ g \in Y^*$, now

$$u(f, g) = \sum_{i=1}^{n} f(x_i)g(y_i) \qquad (\forall\ f \in X^*, g \in Y^*)$$

$$= f\left(\sum_{i=1}^{n} x_i g(y_i)\right) = 0 \qquad (as\ f\ is\ linear\ and\ \sum_{i=1}^{n} x_i g(y_i) = 0)$$

As this holds for every $(f, g) \in X^* \times Y^*$, we have $u = 0$.

□

We have following characterization for zero tensor, which will be used frequently.

**Lemma 1.2.4.** *Let* $\sum_{i=1}^{n} x_i \otimes y_i = 0$*, where* $\{x_i\}$ *is linearly independent set. Then* $y_i = 0 \quad (i = 1, 2, 3, \dots)$

*Proof.* Given $\sum_{i=1}^{n} x_i \otimes y_i = 0$, implies that

$$\begin{aligned} \sum_{i=1}^{n} x_i \otimes y_i(f, g) = 0 & \qquad (\forall f \in X^*, g \in Y^*) \\ \sum_{i=1}^{n} f(x_i) g(y_i) = 0 & \\ f\left(\sum_{i=1}^{n} g(y_i) x_i\right) = 0 & \qquad (\forall f \in X^*) \end{aligned}$$

from Hahn-Banach Theorem,[2] we have if $\forall f \in X^*, f(x) = 0$ implies that $x = 0$[3], so we have

$$\begin{aligned} \sum_{i=1}^{n} g(y_i) x_i = 0 & \qquad (g \in Y^*) \\ g(y_i) = 0 & \qquad (as\ x_i's\ are\ linearly\ independent)(g \in Y^*, i = 1, 2, \dots n) \\ y_i = 0 & \qquad (i = 1, 2, \dots n) \end{aligned}$$

□

**Proposition 1.2.5.** *Let* $\{x_1, x_2, \cdots, x_n\}$*,* $\{y_1, y_2, \cdots, y_m\}$ *be linearly independent subsets of* $X, Y$ *respectively. Then* $\{x_i \otimes y_j : i = 1 \cdots n,\ j = 1 \cdots m\}$ *is a linearly independent subset of* $X \otimes Y$*.*

*Proof.* suppose that $\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} x_i \otimes y_j = 0$. As we know that tensor product is bilinear so,

$$\begin{aligned} \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} x_i \otimes y_j = 0 & \\ \sum_{i=1}^{m} x_i \otimes \left(\sum_{j=1}^{n} \alpha_{ij} y_j\right) = 0 & \\ \sum_{j=1}^{n} \alpha_{ij} y_j = 0 & \qquad (for\ a\ fixed\ i)(using\ lemma\ 1.2.3) \\ \alpha_{ij} = 0 & \qquad (i = 1, 2, \dots, m,\ j = 1, 2, \dots, n) \end{aligned}$$

[2] Let $f$ be a bounded linear functional on a subspace $Z$ of a normed space $X$. Then there exists a bounded linear functional $g$ on $X$ which is an extension of $f$ to $X$ and has the same norm, $||g||_X = ||f||_Z$.

[3] For every x in anormed space X we have $||x|| = sup|f(x)|$ where supremum is taken over all bounded functional with norm 1

□

**Corollary 1.2.6.** *If $\{e_i, i \in I\}$ and $\{f_j, j \in J\}$ are the bases of $X$ and $Y$ respectively, then $\{e_i \otimes f_j : (i,j) \in I \times J\}$ is a basis for $X \otimes Y$.*

Now we are done with ground work to state the centerpiece of this section. A theorem which linearizes the bilinear maps with the help of tensor product.

**Theorem 1.2.7.** *Given a bilinear mapping $\phi : X \times Y \to Z$, there exist a unique linear mapping $\sigma : X \otimes Y \to Z$ such that*

$$\sigma(x \otimes y) = \phi(x, y) \qquad (x \in X, y \in Y).$$

*Proof.* Define a linear mapping $\sigma : X \otimes Y \to Z$ as,

$$\sigma\left(\sum_{r=1}^{k} x_r \otimes y_r\right) = \sum_{r=1}^{k} \phi(x_r, y_r),$$

To show this mapping is well defined, it suffices to show that if $\sum_{i=r}^{k} x_r \otimes y_r = 0$, then $\sum_{r=1}^{k} \phi(x_r, y_r) = 0$. To see this, let $\{a_i\}$ and $\{b_j\}$ be the bases for linear span of $\{x_r\}$, $\{y_r\}$ respectively, and let

$$x_r = \sum_i \alpha_{ir} a_i \qquad y_r = \sum_j \beta_{jr} b_j$$

We now have

$$\sum_i \sum_j \sum_r \alpha_{ir}\beta_{jr} a_i \otimes b_j = 0 \qquad (Tensor\ product\ is\ bilinear)$$

As we have $\{a_i\}$ and $\{b_j\}$ are linearly independent sets, proposition (1.2.4) tells us that $\sum_r \alpha_{ir}\beta_{jr} = 0$. Therefore

$$\sum_r \phi(x_r, y_r) = \sum_i \sum_j \sum_r \alpha_{ir}\beta_{jr} \phi(a_i, y_j) = 0.$$

Thus we got desired well defined linear mapping $\sigma$ of $X \otimes Y$ into $Z$.

□

The above situation is illustrated as follows

$$\begin{array}{ccc} X \times Y & \xrightarrow{\phi} & Z \\ \downarrow \tau & \nearrow \sigma & \\ X \otimes Y & & \end{array}$$

The special bilinear mapping $\tau : X \times Y \to X \otimes Y$ given as $(x, y) \mapsto x \otimes y$ acts as a "universal" bilinear mapping as any bilinear mapping $\phi$ on $X \times Y$ factors through this one via a linear mapping $\sigma$ on tensor product $X \otimes Y$, i.e. $\phi = \sigma o \tau$.

## 1.3 Norms on Tensor Product

Now we will define the norms on tensor product of two Banach spaces called as projective and injective (weak) norm. The projective tensor product linearizes bounded linear mapping just as the algebraic tensor product linearizes the bilinear mappings. The projective tensor product derives it's name from the fact that it behaves well with construction of quotient spaces.

### 1.3.1 Injective tensor norm

As the elements of the tensor product can be viewed as the bilinear form on the product $X' \otimes Y'$ of the algebraic duals. Now if $\sum_{i=1}^{n} x_i \otimes y_i$ is the representation of tensor $u$, then associated bilinear form $B_u$ is given as

$$B_u(f, g) = \sum_{i=1}^{n} f(x_i)g(y_i).$$

Now the restriction of $B_u$ to the product $X^* \times Y^*$ of the dual spaces is bounded and so we have a canonical algebraic embedding of $X \otimes Y$ into $BL(X^*,\ Y^*;\ \mathbb{F})$. The injective norm on $X \otimes Y$ is norm induced by this embedding.

**Definition 1.3.1.1.** *Let $X$ and $Y$ be linear spaces, then $X \otimes Y$ inherits norm from $BL(X^*,\ Y^*;\ \mathbb{F})$. This norm is called the injective (weak) tensor norm and denoted by $w$,*

$$w(u) = sup \left\{ \left| \sum_i f(x_i)g(y_i) \right| : ||f|| \leq 1, ||g|| \leq 1 \right\},$$

*where $u = \sum_i x_i \otimes y_i$.*

It is easy to check that $w$ is actually a norm.
First we observe that the action of an element $u$ of $X \otimes Y$ on $(f, g)$ for given $f \in X^*,\ g \in Y^*$, is independent of representation of $u$. Now consider $u \in X \otimes Y$, then $w(u) \geq 0$ as it is the supremum of positive scalars. Take $u = \sum_i x_i \otimes y_i$ with it's minimal representation, then

if $u = 0$ implies $w(u) = 0$, as we have $u = a \otimes 0,\ (a \in X)$ as one of the representation for zero tensor, so $u(f,g) = f(a)g(0) = 0\ for\ all\ (f,g) \in X^* \times Y^*$ which gives us $w(u) = 0$.
Consider $w(u) = 0$, implies

$$\sum_i f(x_i)g(y_i) = 0 \quad (\forall\ (f,g) \in X^* \times Y^*,\ with\ ||f|| \leq 1,\ ||g|| \leq 1)$$

$$f\left(\sum_i x_i\ g(y_i)\right) = 0 \quad (\forall\ (f,g) \in X^* \times Y^*)$$

As the consequence of Hahn-Banach theorem, we have that if $h(x) = 0,\ \forall\ h \in X^*,\ x = 0$. So,

$$\sum_i x_i\ g(y_i) = 0 \qquad (\forall\ g \in Y^*)$$

We have that $\{x_i\}$ linearly independent implying that,

$$g(y_i) = 0 \qquad (\forall\ i\ and\ \forall\ g \in Y^*),$$

Again by Hahn-Banach theorem, we have, $y_i = 0$ for all $i$, so $u = 0$.
Now to verify triangle inequality, assume that $u = \sum_i x_i \otimes y_i$ and $v = \sum_j u_i \otimes v_i$. Then for $f \in X^*,\ g \in Y^*$ such that $||f|| \leq 1,\ ||g|| \leq 1$,

$$\left|\sum_i f(x_i)\ g(y_i) + \sum_j f(u_j)\ g(v_j)\right| \leq w(u) + w(v),$$

Thus, $w(u+v) \leq w(u) + w(v)$. Hence we are done proving $w$ is a norm.

**Proposition 1.3.1.2.** *Let $x \otimes y$ be elementary tensor, then $w(x \otimes y) = ||x||||y||$.*

*Proof.* Let $u = x \otimes y$ be an elementary tensor. Then

$$w(u) := sup\{|f(x)\ g(y)| : ||f|| \leq 1,\ ||g|| \leq 1\}.$$

So, for $(f,g) \in X^* \times Y^*$, such that $||f|| \leq 1,\ ||g|| \leq 1$ we will have $|u(f,g)| \leq ||x||\ ||y||$, which directly gives us that, $w(x \otimes y) \leq ||x||\ ||y||$.
Now to prove other side inequality, we will again use one of the important consequence of Hahn-Banach theorem. For a given $f \in X^*,\ g \in Y^*$ with unit norm we have,

$$|f(x)g(y)| \leq w(x \otimes y),$$

$$\sup_{f \in X^*} |f(x)| \sup_{g \in Y^*} |g(y)| \leq w(x \otimes y),$$

$$||x||\ ||y|| \leq w(x \otimes y).$$

and we are done.

∎

The completion of $X \otimes Y$ with respect to injective norm, i.e. closure of $X \otimes Y$ in $BL(X^* \times Y^*; \mathbb{F})$ is called injective (weak) tensor product of $X$ and $Y$. The weak tensor product of $X$ and $Y$ will be denoted by $X \otimes_w Y$.

### 1.3.2 Projective tensor norm

**Definition 1.3.2.1.** *Let $X$ and $Y$ be Banach spaces, then projective tensor norm $p$ on $X \otimes Y$ is defined as*

$$p(u) = inf\left\{\sum_i ||x_i||||y_i|| : u = \sum_i x_i \otimes y_i\right\},$$

*where the infimum is taken over all (finite) representation of $u$.*

**Proposition 1.3.2.2.** *Let $X$ and $Y$ be normed linear spaces over the field $\mathbb{F}$, then*

1. *$p$ is a norm $X \otimes Y$,*
2. *$p(u) \geq w(u) \qquad (u \in X \otimes Y)$,*
3. *$p(x \otimes y) = ||x||||y|| \qquad (x \in X, y \in Y)$.*

*Proof.* (1) First we will show that $p(\lambda u) = |\lambda| p(u)$. This is obvious when $\lambda = 0$. So suppose that $\lambda \neq 0$ and $u = \sum_{i=1}^n x_i \otimes y_i$ is a representation of $u$, then $\lambda u = \sum_{i=1}^n \lambda x_i \otimes y_i$ and so we have $p(\lambda u) \leq \sum_{i=1}^n ||\lambda x_i||||y_i|| = |\lambda| \sum_{i=1}^n ||x_i||||y_i||$. Since this holds for every representation of u, it follows that $p(\lambda u) \leq |\lambda| p(u)$. In same way we have $p(u) = p(\lambda^{-1}\lambda u) \leq |\lambda^{-1}| p(\lambda u)$, giving $|\lambda| p(u) \leq p(\lambda u)$. Therefore $p(\lambda u) = |\lambda| p(u)$.

Now to prove that it satisfies triangle inequality, let $u, v \in X \otimes Y$ and let $\epsilon > 0$ . It follows from definition that we may choose $u = \sum_{i=1}^n x_i \otimes y_i$ and $v = \sum_{j=1}^m w_j \otimes z_j$ such that $\sum_{i=1}^n ||x_i||||y_i|| \leq p(u) + \epsilon/2$ and $\sum_{j=1}^m ||w_j||||z_j|| \leq p(v) + \epsilon/2$. Then $\sum_{i=1}^n x_i \otimes y_i + \sum_{j=1}^m w_j \otimes z_j$ is the representation of $u + v$ and so

$$p(u+v) \leq \sum_{i=1}^n ||x_i||||y_i|| + \sum_{j=1}^m ||w_j||||z_j|| \leq p(u) + p(v) + \epsilon$$

Since this holds for every $\epsilon > 0$, we have $p(u+v) \leq p(u) + p(v)$

Now suppose that $p(u) = 0$, Then for every $\epsilon > 0$ there is a representation $\sum_{i=1}^n x_i \otimes y_i$ such that $\sum_{i=1}^n ||x_i||||y_i|| < \epsilon$. Hence for every $f \in X^*, g \in Y^*$, such that $||f|| \leq 1, ||g|| \leq 1$ we have

$$\left|\sum_{i=1}^n f(x_i) g(y_i)\right| \leq ||f||||g|| \sum_{i=1}^n ||x_i||||y_i||$$

$$\left|\sum_{i=1}^{n} f(x_i)g(y_i)\right| \leq \epsilon$$

Since the value of the sum $|\sum_{i=1}^{n} f(x_i)g(y_i)|$ is independent of the representation of $u$, it follows that $\sum_{i=1}^{n} f(x_i)g(y_i) = 0$. So from proposition (1.2.2) it follows that $u = 0$.

(2) Given $u = \sum_{i=1} x_i \otimes y_i$, and given $f \in X^*, g \in Y^*$, we have

$$|u(f,g)| = \left|\sum_i f(x_i)g(y_i)\right| \leq ||f||||g|| \sum_i ||x_i||||y_i||,$$

and hence $w(u) \leq p(u)$.

(3) Given $x \in X, y \in Y$, we have

$$||x||||y|| = w(x \otimes y) \leq p(x \otimes y),$$

and $p(x \otimes y) \leq ||x||||y||$ is clear from the definition of $p$. Thus we have $p(x \otimes y) = ||x||||y||$.
□

We shall denote the completion of $(X \otimes Y, p)$ by $X \otimes_p Y$ and call the projective tensor product of $X$ and $Y$. If $X$ and $Y$ are finite dimensional then $(X \otimes_p Y, p)$ is complete.

**Proposition 1.3.2.3.** *$X \otimes_p Y$ can be represented as the linear subspace of $BL(X^*, Y^*; \mathbb{F})$ consisting of all elements of the form $u = \sum_{i=1}^{\infty} x_i \otimes y_i$ where $\sum_{i=1}^{\infty} ||x_n||||y_n|| < \infty$. Moreover $p(u)$ is the infimum of the sums $\sum_{i=1}^{\infty} ||x_n||||y_n||$ over all such representation of $u$.*

*Proof.* Let $a \in X \otimes Y$ and $\epsilon > 0$. Since $X \otimes Y$ is dense in $X \otimes_p Y$. Then $a = lim \lambda_m$ for a sequence $(\lambda_m)$ in $X \otimes Y$. Pick out a subsequence $(a_m)$ of $(\lambda_m)$ such that $p(a - a_m) \leq \epsilon/2^{m+3}$ for $m \in \mathbb{N}$.
we want to write $a$ as a sum of elementary tensors. Choose $x_i, y_i$ $i = 1, 2, ..., n$ from $X, Y$ respectively, such that $a_1 = \sum_{i=1}^{n} x_i \otimes y_i$ and

$$p(a_1) \leq \sum_{i=1}^{n} ||x_i||||y_i|| \leq p(a_1) + \epsilon/2^4 \leq p(a) + p(a - a_1) + \epsilon/2^4 \leq p(a) + \epsilon/2^3.$$

Since $p(a_2 - a_1) \leq p(a - a_2) + p(a - a_1) \leq \epsilon/2^5 + \epsilon/2^4 < \epsilon/2^3$, we can choose $x_i \in X, y_i \in Y$, $i = n_1 + 1, , ..., n_2$ such that $(a_2 - a_1) = \sum_{i=n_1+1}^{n_2} x_i \otimes y_i$ and

$$p(a_2 - a_1) \leq \sum_{i=n_1+1}^{n_2} ||x_i||||y_i|| \leq p(a_2 - a_1) + (\epsilon/2^3 - p(a_2 - a_1)) = \epsilon/2^3.$$

Since $p(a_3 - a_2) \leq p(a - a_3) + p(a - a_2) \leq \epsilon/2^6 + \epsilon/2^5 < \epsilon/2^4$. We can then choose $x_i \in X, y_i \in Y$ $i = n_2 + 1, ..., n_3$, such that $a_3 - a_2 = \sum_{i-n_2+1}^{n_3} x_i \otimes y_i$ and

$$p(a_3 - a_2) \leq \sum_{i=n_2+1}^{n_3} ||x_i||||y_i|| \leq p(a_3 - a_2) + (\epsilon/2^4 - p(a_3 - a_2)) = \epsilon/2^4.$$

Continuing expanding, choosing $x_i \in X, y_i \in Y$ and estimating difference for $n \geq 4$. Now, let $b_1 = a_1, b_{m+1} = a_{m+1} - a_m$ for $m \in \mathbb{N}$, so that $a = \sum_{m=1}^{\infty} b_m = \sum_{i=1}^{\infty} x_i \otimes y_i$.
and

$$\sum_{i=1}^{\infty} ||x_i||||y_i|| \leq p(a) + \epsilon/2 < \infty.$$

So $\sum_{i=1}^{\infty} x_i \otimes y_i$ converges absolutely. Let $\pi(a)$ denote the above mentioned infimum over all representations of $a$; it follows from above equation that $\pi(a) \leq p(a)$. But

$$p(a) \leq \lim_{N\to\infty} \sum_{i=1}^{N} p(x_i \otimes y_i) = \sum_{i=1}^{\infty} ||x_i||||y_i||,$$

so equality holds. Finally, if $a = \sum_{i=1}^{\infty} x_i \otimes y_i$ where $\sum_{i=1}^{\infty} ||x_i||||y_i|| < \infty$, then the sum converges in $BL(X^*, Y^*; \mathbb{F})$ and letting $a_N = \sum_{i=1}^{N} x_i \otimes y_i$, it is obvious that $p(a - a_N) \to 0$ for $N \to \infty$.

□

We have seen that the tensor product $X \otimes Y$ linearizes bilinear mappings on $X \times Y$ . We now add norms to this picture. We identify the dual space of $X \otimes_p Y$. Given $F \in (X \otimes_p Y)'$, let $\phi_F$ be the bilinear mapping on $X \times Y$ defined by

$$\phi_F(x, y) = F(x \otimes y) \qquad (x \in X, y \in Y).$$

**Proposition 1.3.2.4.** *The mapping $F \to \phi_F$ is an isometric linear isomorphism of $(X \otimes Y)'$ onto $BL(X, Y; \mathbb{F})$.*

*Proof.* Given $F \in (X \otimes_p Y)'$, we clearly have $\phi_F \in BL(X, Y; \mathbb{F})$ and

$$||\phi_F|| = sup\{|\phi_F(x, y)| : ||x|| \leq 1, ||y|| \leq 1\}$$

$$Since \qquad |\phi_F(x, y)| = |F(x \otimes y)| \leq ||F||$$

$$we\ have \quad ||\phi_F|| \leq ||F||.$$

Conversely let $\phi \in BL(X, Y; \mathbb{F})$, by theorem 1.2.7 there exists unique linear functional $F$ on $X \otimes Y$ such that

$$F(x \otimes y) = \phi(x, y) \qquad (x \in X, y \in Y),$$

Then

$$\left|F\left(\sum_i x_i \otimes y_i\right)\right| = \left|\sum_i \phi(x_i, y_i)\right| \leq ||\phi|| \sum_i ||x_i||||y_i||,$$

And so $|F(u)| \leq ||\phi||p(u)$ $(u \in X \otimes Y)$. Therefore $F$ has a unique extension to an element $\tilde{F}$ of $(X \otimes_p Y)'$, $||\tilde{F}|| = ||F|| \leq ||\phi||$, and $\phi = \phi_{\tilde{F}}$.

$\square$

**Definition 1.3.3.** *Given normed spaces $X$, $Y$, a norm $\alpha$ on $X \otimes Y$ is said to be a cross norm if*

$$\alpha(x \otimes y) = ||x||||y|| \qquad (x \in X, y \in Y).$$

The projective and injective tensor norms on $X \otimes Y$ are cross norms. In fact projective tensor norm is the largest cross norm on $X \otimes Y$; suppose $\alpha$ is a cross norm on $X \otimes Y$ and $u = \sum_i x_i \otimes y_i$, then

$$\alpha(u) \leq \sum_i \alpha(x_i \otimes y_i) = \sum_i ||x_i||||y_i||,$$

and so $\alpha(u) \leq p(u)$.

Now we will conclude this section by following proposition which shows how the tensor product of subspaces of bounded bilinear maps can be embedded in tensor product of spaces, and what is the behavior of induced norm.

**Proposition 1.3.4.** *Let $A$, $B$ be linear subspaces of $BL(X)$, $BL(Y)$ respectively. Then $A \otimes B$ can be embedded in $BL(X \otimes_p Y)$ and the induced norm on $A \otimes B$ is a cross norm.*

*Proof.* Given $S \in A$, $T \in B$, we have bilinear map $\phi \in BL(X,\ Y; X \otimes Y)$, such that $\phi(x, y) = Sx \otimes Ty$, further from theorem 1.2.7, there exists unique linear operator $S \square T$ on $X \times Y$ such that

$$S \square T(x \otimes y) = Sx \otimes Ty \qquad (x \in X, y \in Y)$$

Suppose $u = \sum_i x_i \otimes y_i$ then,

$$p\left(\sum_i S(x_i) \otimes T(y_i)\right) \leq \sum_i ||S(x_i)||||T(y_i)|| \leq ||S||||T|| \sum_i ||x_i||||y_i||,$$

This implies that,

$$||S \square T|| \leq ||S||||T||.$$

Thus $S \square T$ can be extended to an element of $BL(X \otimes_p Y)$.
Moreover,

$$\begin{aligned}
||S\Box T|| &\geq sup\{p(S\Box T(x \otimes Y) : p(x \otimes y) = 1)\} \\
&\geq sup\{||Sx||||Ty|| : ||x|| = 1, ||y|| = 1\} \\
&= ||S||||T||.
\end{aligned}$$

So we have $||S\Box T|| = ||S||||T||$.

Now $f(S, T) = S\Box T$ is clearly a bilinear mapping on $A \times B$, so from theorem 1.2.7 we have a unique linear mapping $\sigma$ on $A \otimes B$ $\sigma : A \otimes B \to BL(X \otimes_p Y)$ such that $\sigma(S \otimes T) = S\Box T$. Now we will prove that $\sigma$ is injective.

Let $S\Box T = 0$. We will prove that $S \otimes T = 0$, consider $u = \sum_i x_i \otimes y_i$ with minimal representation, then

$$\begin{aligned}
& S\Box T(u) = 0, \qquad (\forall\ u \in X \otimes_p Y) \\
\implies & \sum_i S(x_i) \otimes T(y_i) = 0, \qquad (\forall\ u = \sum_i x_i \otimes y_i \in X \otimes_p Y) \\
\implies & \sum_i (Sx_i \otimes Ty_i)(f, g) = 0, \qquad (\forall\ (f, g) \in X^* \times Y^*) \\
\implies & \sum_i f(Sx_i)\ g(Ty_i) = 0, \qquad (\forall\ (f, g) \in X^* \times Y^*) \\
\implies & f\left(\sum_i Sx_i\ g(Ty_i)\right) = 0, \qquad (\forall\ f \in X^*,\ g \in Y^*) \\
\implies & \sum_i Sx_i\ g(Ty_i) = 0, \qquad (\forall\ g \in Y^*) \\
\implies & S\left(\sum_i x_i\ g(Ty_i)\right) = 0, \qquad (\forall\ g \in Y^*,\ u \in X \otimes_p Y) \\
\implies & S = 0, \\
\implies & S \otimes T = 0.
\end{aligned}$$

Hence we are done. $\Box$

Now the most awaited terminology comes into picture i.e. Banach algebra and tensor product of Banach algebras, which will be used frequently. Banach algebra indicates completion of algebra. So first we define what is an algebra.

## 1.4 Tensor Product of Banach Algebras

**Definition 1.4.1.** *Let* $(X, +, .)$ *be a vector space over the field* $\mathbb{F}$*, define a map* $* : X \times X \to X$*, called vector multiplication, such that,*

1. $x * (y * z) = (x * y) * z \qquad (\forall\ x, y, z \in X)$
2. $x * (y + z) = (x * y) + (x * z) \qquad (\forall\ x, y, z \in X)$
3. $\alpha.(x * y) = (\alpha.x) * y = x * (\alpha.y) \qquad (\forall\ x, y \in X,\ \alpha \in \mathbb{F})$.

*Then* $(X, +, ., *)$ *is called an algebra. If* $|| \ ||$ *is norm on* $X$*, satisfying*

$$||x * y|| \le ||x||||y|| \qquad \forall\ x, y \in X,$$

*then* $(X, +, ., *, || \ ||)$ *is called normed algebra.*
*If the space is complete with respect to this norm then* $X$ *is called Banach algebra.*

We say a Algebra is commutative if vector multiplication is commutative i.e. $x*y = y*x$ for all $x, y \in X$. If $X$ has identity with respect to vector multiplication i.e. there exists an element $e$ such that $e * x = x * e = x$ for all $x \in X$, then $X$ is said to be algebra with identity. Moreover if $||e|| = 1$, $x$ is called unital algebra. Now we will look at some examples of algebras to get handy with these concepts.

**Examples 1.4.2.** *1.* $(\mathbb{Q}, +, *, | \ |)$ *is a normed algebra over field* $\mathbb{Q}$ *with vector product defined as usual multiplication of rational numbers.*

*2.* $\mathbb{R}$, $\mathbb{C}$ *are commutative Banach algebra with identity.*

*3.* $l_1$ *is Banach algebra without identity, vector multiplication defined as*

$$(a_1, a_2, \dots) * (b_1, b_2, \dots) = (a_1b_1, a_2b_2, \dots).$$

*It's easy to verify with this multiplication* $l_1$ *space is Banach algebra.*

*4.* $(M_n(\mathbb{C}), +, *, \circ, || \ ||)$ *is non-commutative Banach algebra with vector multiplication defined as usual matrix multiplication and norm induced by* $\mathbb{R}^{n^2}$

Now we will go for our purpose that is tensor product of Banach algebras. We are skipping results in algebras. We will prove and see the consequences as we need them in our way.

**Proposition 1.4.3.** *Let* $A$*,* $B$*, be normed algebras over* $\mathbb{F}$ *There exists a unique product on* $A \otimes B$*, with respect to which* $A \otimes B$ *is an algebra and*

$$(a \otimes b)(c \otimes d) = (ac \otimes bd) \qquad (a, c \in A,\ b, d \in B).$$

*Moreover projective tensor norm on* $A \otimes B$ *is an algebra norm.*

*Proof.* Given $a \in A,\ b \in B$, by theorem 1.2.7 there exists a unique linear operator $\lambda_{(a,b)} : A \otimes B \to A \otimes B$. such that,

$$\lambda_{(a,b)}(c \otimes d) = ac \otimes bd \qquad (c \in A,\ d \in B).$$

Now the mapping $(a, b) \mapsto \lambda_{(a,b)}$, is bilinear by the fact that tensor product is bilinear, and so again by theorem 1.2.7 there exists a unique linear mapping $\sigma : A \otimes B \to L(A \otimes B)$, such that

$$\sigma(a \otimes b) = \lambda_{(a,b)} \qquad (a \in a,\ b \in B).$$

Now the required product on $A \otimes B$ is given as follows.

$$(u, v) \mapsto \sigma(u)v \qquad (u, v \in A \otimes B).$$

i.e. if $u = \sum_i a_i \otimes b_i,\ and\ v = \sum_j c_j \otimes d_j$ then $uv = \sum_i \sum_j a_i c_j \otimes b_i d_j$.

By using bilinear properties of tensor product, it is easy to verify the algebra properties. Now we will prove that projective tensor norm is algebra norm, Let $u, v \in A \otimes B$, $u = \sum_{i=1}^{n} a_i \otimes b_i,\ and\ v = \sum_{j=1}^{m} c_j \otimes d_j$, then

$$uv = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i c_j \otimes b_i d_j,$$

As $A$ and $B$ are normed algebras, so we have,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} ||a_i c_j||\ ||b_i d_j|| \leq \sum_{i=1}^{n} ||a_i||\ ||b_i|| \sum_{j=1}^{m} ||c_j||\ ||d_j||$$

$$\implies p(uv) \leq p(u)p(v).$$

We are done. $\square$

# Chapter 2

# Abstract Harmonic Analysis

As we are already familiar with various properties of Lebesgue measure on real line (for more information refer [Roy-Fitz] and [Cohn]). One of the very strong and useful property is that $\lambda(A+x) = \lambda(A)$ for any $A$ *in* $\mathcal{B}(\mathbb{R})$ and $x$ in $\mathbb{R}$, this property is so called translation invariant. In this chapter first we will introduce the concept of topological groups, we will focus only particularly on locally compact groups. In next section we will see that every locally compact group possesses a unique (upto constant multiple) invariant measure, a Haar measure. Which will lead us to define algebra structure on $L^1(G)$ and $M(G)$ (Set of all finite complex valued regular borel measure (see appendix B) on $G$) with the help of convolution. Now we will begin by defining topological groups.

## 2.1 Locally Compact Group

**Definition 2.1.1.** *A topological group $G$ is a group equipped with a topology $\tau$ with respect to which the group operations are continuous, i.e. $(x, y) \to xy$ is continuous from $G \times G$ to $G$ and $x \to x^{-1}$ is continuous from $G$ to $G$.*

If $G$ is a topological group we denote the unit element of $G$ by $e$. Now we define some terminology which will be used frequently. Let $A,\ B \subseteq G$

$$Ax := \{yx : y \in A\}, \qquad xA := \{xy : y \in A\} \qquad A^{-1} := \{y^{-1} : y \in A\},$$

and

$$AB := \{xy : x \in A,\ y \in B\}.$$

Now we will prove some basic results in theory of topological groups that have a large impact on further development of theory.

**Examples 2.1.2.** *1. Any group with the discrete or the indiscrete topology.*

*2.* $\mathbb{R}$ *with addition and* $\mathbb{R}^* := \mathbb{R}\ 0 or \mathbb{C}^*$ *with multiplication are topological group with usual topology.*

**Proposition 2.1.3.** *Let $G$ be a topological group.*

*a. The topology of $G$ is invariant under translation and inversion, that is, if $U$ is an open set then so are $xU$, $Ux$, and $U^{-1}$ for any $x \in G$. Moreover, if $U$ is open then so are $AU$ and $UA$ for any $A \subseteq G$*

*b. Let $U$ be a nbd*[1] *of $e$, then there exists a nbd $V$ of $e$ such that $V^2 \subseteq U$. Moreover $V$ can be chosen symmetric i.e. $V = V^{-1}$.*

*c. If $H$ is a subgroup of $G$, so is $\overline{H}$*

*d. Every open subgroup of $G$ is closed.*

*e. Product of two compact subsets is compact.*

*f. For any nbd $U$ of $e$ we have $\overline{U} \subseteq U^2$.*

*Proof.* a. This follows immediately from the fact that the left translation, $L_g$, right translation $R_g$ and the inverse map $i$ are homeomorphism. As $L_g$ is the composition of two continuous maps,

$$G \to g \times G \to G, \qquad x \mapsto (g, x) \mapsto gx$$

thus $L_g$ is continuous. The inverse map of $L_g$ is $L_{g^{-1}}$ which is again continuous. Hence $L_g$ is homeomorphism. Similar argument shows that the right translation $R_g$ is homeomorphism. Since $i$ is continuous and $i^{-1} = i$ so it is a homeomorphism.
So we are done with showing that for any open set $U$ and $x \in G$ the sets $xU,\ Ux\ and\ U^{-1}$ are open. Now as

$$AU = \bigcup_{a \in A} aU,\ and\ UA = \bigcup_{a \in A} Ua$$

which is the arbitrary union of open set. So, $UA,\ and\ AU$ are open.

b. As the multiplication map is continuous at $e$ so for any nbd $U$ of $e$ the inverse image of $U$ under multiplication map gives us nbd of $e$ in product topology on $G \times G$, i.e. there exists $W_1$ and $W_2$ open set in $G$, such that $W_1 \times W_2$ is a nbd of $e$. Moreover we have $W_1 \times W_2 \subseteq U$ i.e. $W_1 W_2 \subseteq U$. Now take $V = W_1 \cap W_2$, it is clear that $V^2 \subset U$. Moreover as $W_1$ is nbd of $e$ so as $W_1^{-1}$, so the symmetric set $V$ can be chosen as $V = W_1 \cap W_2 \cap W_1^{-1} \cap W_2^{-1}$.

[1] A set $N$ of $G$ is said to be nbd (abbrevation for neighborhood) of $x \in G$ if there exists an $W \in \tau_G$ (*topology on* G) containig $x$ and contained in $N$. In this context we will assume nbd to be open untill and unless stated otherwise.

c. We will show that $\overline{H}$ is closed under operation of multiplication and inverse, this will imply that $\overline{H}$ is also a subgroup. Let $x,\ y \in \overline{H}$, so there are nets $\{x_\alpha\}$ and $\{y_\beta\}$ in $H$ converging to $x,\ y$. Then $x_\alpha y_\beta \to xy$ and $x_\alpha^{-1} \to x^{-1}$, as multiplication and inverse mapping are continuous. So $xy$ and $x^{-1}$ are in $\overline{H}$

d. Let $H$ be an open subgroup of $G$. Then $xH$ is open for each $x \in G$ and since

$$H = G \setminus \bigcup_{x \in H^c} xH,$$

which simply imply that $H$ is closed in $G$.

e. Let $K$ and $L$ be compact set the $KL$ is the image of the $K \times L$ under continuous map, multiplication. Hence as the continuous image of compact set is compact so $KL$ is compact.

f. If $x \in \overline{U}$ then $xU^{-1}$ is a neighborhood of $x$. It follows that $xU^{-1} \cap U \neq \phi$ as being a limit point of $U$, intersection of $U$ with any nbd of $x$ is non-empty and hence we have that there exist some $t$ and $z$ in $U$ such that $xz^{-1} = t$, i.e. $x = tz$. Hence $x \in U^2$ which shows that $\overline{U} \subseteq U^2$.

□

**Proposition 2.1.4.** *Every topological group is regular*[2]*, and if $G$ is $T_0$*[3]*, then $G$ is $T_1$*[4]*. Moreover it is $T_3$*[5]*.*

*Proof.* For any nbd $U$ of $e$ from 2.1.3, there exist nbd $V$ of $e$, such that $\overline{V} \subseteq U$, hence any topological group is regular at $e$. Now as translation is homeomorphism so $G$ is regular at any point $x$. Now suppose $G$ is $T_0$, and $g,\ h \in G$. As $G$ is $T_0$ so there exist nbd $U$ of $e$ such that

$$h \notin Ug^{-1}$$
$$hg^{-1} \notin U$$
$$g \notin U^{-1}h,$$

hence $G$ is $T_1$.

□

**Proposition 2.1.5.** *Let $X$ be a Hausdorff space and suppose that there exists a chain of relatively compact open sets $U_1 \subseteq U_2 \subseteq \cdots$ such that $X = \bigcup_{i=1}^{\infty} U_i$. Then $X$ is a lindelof space.*

[2] A space $X$ is said to be regular if for given any $x \in X$ and a nbd $U$ of $x$, we have a nbd $V$ of $x$, such that $\overline{V} \subseteq U$

[3] $X$ is said to be $T_0$ if for given $x,\ y \in X$ there exist open set $U$ containing one of the point but not other.

[4] $X$ is said to be $T_1$ if for every pair of distinct points, each has a nbd not containing the other.

[5] X is said to be $T_3$ if it is both $T_1$ and regular.

*Proof.* Let $\theta$ be an open cover of $X$. Define $U_0 := \phi$; and let $n \in \mathbb{N}$. Clearly $\overline{U_n} \backslash U_{n-1}$ is a closed subset of $\overline{U_n}$ and thus also compact since $\overline{U_n}$ is compact. Therefore there exists a finite subcover $\theta_n$ of $\theta$ such that $\theta_n$ covers $\overline{U_n} \backslash U_{n-1}$. It follows that $\bigcup_{n=1}^{\infty} \theta_n$ is a countable subcover of $\theta$ that covers $X$. □

**Definition 2.1.6** (Locally compact group)**.** *A locally compact group is a topological group $G$ for which the underlying topology is locally compact and Hausdorff, i.e. for each $g \in G$ there exist open nbd $N_x$ whose closure is compact.*

**Proposition 2.1.7.** *If $H$ is a closed subgroup of a locally compact group $G$. Then $H$ is locally compact.*

*Proof.* Let $U$ be relatively compact nbd of $e$. Now $U \cap H$ is nbd of $e$ in $H$, now as $H$ is closed $\overline{U \cap H} \subseteq H$ is closed in $H$. Also $\overline{U \cap H} \subseteq \overline{U}$, so $\overline{U \cap H}$ is compact. Hence $H$ is locally compact. □

**Proposition 2.1.8.** *Let $G$ be a locally compact group. Then $G$ has a subgroup $H$ which is both open and closed such that $H = \cup_{i=1}^{\infty} U_i$ where $U_1 \subseteq U_2 \subseteq \cdots$ is a chain of relatively compact*[6] *open sets.*

*Proof.* As $G$ is locally compact, let $V$ be relatively compact nbd of $e$, so from 2.1.3 we have nbd $U$ of $e$ such that $U = U^{-1}$ and,

$$\overline{U} \subseteq U^2 \subseteq V \subseteq \overline{V},$$

so $\overline{U}$ is compact as $\overline{V}$ is compact. Now as $U$ is open so are $U^n = U.U.\ldots.U$ (n-times), $\forall n \in \mathbb{N}$. Take $H = \bigcup_{i=1}^{\infty} U^n$ is an open subgroup of $G$. As for any $x \in H$ there exist some $n$ such that $x \in U^n$ then $x \in U^n \subseteq U^{n+1}$. Thus $H$ is open and also closed from proposition (2.0.6). Again from proposition 2.1.3 $\overline{V}^n$ are compact for each $n \in \mathbb{N}$, and $\overline{U}^n \subseteq \overline{V}^n$ so compact, and hence we have chain of relatively compact open sets.

□

**Corollary 2.1.9.** *Let $G$ be locally compact group. Then $G$ has a subgroup which is open, closed and $\sigma-$compact (i.e. it is countable union of compact sets.)*

*Proof.* As for any nbd $U$ of $e$, we have $U^n \subset (\overline{U})^n \subset U^{2n}$ for each $n$ in $\mathbb{N}$. If $U$ is relatively compact the $(\overline{U})^n$ is also compact. So take $U$ and $H$ as in proof of proposition 2.1.8, then

$$H = \bigcup_{n=1}^{\infty} (\overline{U})^n.$$

Hence $H$ is $\sigma-$compact. □

[6] A relatively compact subset $Y$ of a topological space $X$ is a subset whose closure is compact.

**Remark 2.1.10.** *Every regular lindelof space is normal.*

**Corollary 2.1.11.** *Every locally compact group has an open, closed subgroup which is topological normal.*

**Lemma 2.1.12.** *Let $G$ be a topological group and $H$ an open subgroup of $G$ which is topologically normal. Then $G$ is normal.*

*Proof.* Let $A := \{a \in G | aH$ form disjoint coset of $H\}$, as $H$ is open which implies $aH$ is open for each $a \in A$. Let $C_1$ and $C_2$ be two closed sets of $G$ then $C_1^c$ is open and so $C_1^c \cup aH$ which gives $C_1 \cap aH$ is closed in $aH$. Similarly $C_2 \cap aH$ is closed in $aH$. As $H$ is normal, so there exists open sets $U_a$ and $V_a$ in $aH$ such that $C_1 \cap aH \subseteq U_a$ and $C_2 \cap aH \subseteq V_a$. Now take $U = \cup_{a \in A} U_a$ and $V = \cup_{a \in A} V_a$. As $U_a$ is open in $aH$ and $aH$ is itself open imply that $U_a$ is open $G$. Thus $U$ and $V$ are open in $G$, and $C_1 \subseteq U$, $C_2 \subseteq V$, moreover as for each $a \in A$, $U_a \cap V_a = \phi$ gives that $U \cap V = \phi$. Hence we find two open disjoint set of $G$ for given two closed set thus $G$ is normal. □

**Corollary 2.1.13.** *Every locally compact group is normal.*

## 2.2 Haar Measure

In first section we looked at some basic properties of locally compact groups. Now in this section we will see that on such groups there exist a natural measure invariant under translation called Haar measure.

**Definition 2.2.1.** *Let $G$ be a topological group, and let $f$ be a real or complex-valued function on $G$. Then $f$ is left uniformly continuous if for each positive number $\epsilon$ there is an open neighborhood $U$ of $e$ such that $|f(x) - f(y)| < \epsilon$ holds whenever $x$ and $y$ belong to $G$ and satisfy $y \in xU$. Likewise, $f$ is right uniformly continuous if for each positive number $\epsilon$ there is an open neighborhood $U$ of $e$ such that $|f(x) - f(y)| < \epsilon$ holds whenever $x$ and $y$ belong to $G$ and satisfy $y \in Ux$.*
*Or equivalently, a function $f$ on $G$ is said to be left uniformly continuous if, whenever $(x_\alpha)_\alpha$ is a net in $G$ converging to $x \in G$ then*

$$\sup_{y \in G} |L_{x_\alpha} f(y) - L_x f(y)| \to 0.$$

*where for $x$ and $y$ in $G$, $L_x f(y) = f(xy)$. Similarly for right uniformly continuous we have $R_x f(y) = f(yx)$.*

**Remark 2.2.2.** *It is enough to consider the nets converging to $e$, as if $x_\alpha$ converges to $x$, it implies that $x_\alpha x^{-1}$ converges to $e$.*

**Note 2.2.3.** *1. We will consider our locally compact group to be $\sigma$-finite.*

2. $C_c(G) := \{f : G \to \mathbb{C}|\ f$ *is continuous with compact support*$\}$

3. $L^1(G) := \{f : G \to \mathbb{C}|\ f$ *is measurable and integrable*$\}$

**Proposition 2.2.4.** *Let $G$ be a locally compact group and let $f$ belongs to $C_c(G)$. Then $f$ is left and right uniformly continuous.*

*Proof.* Let $f$ belongs to $C_c(G)$, $K$ be support of $f$ and let $\epsilon$ be a positive real number. For each $x$ in $K$, there exist a neighborhood $U_x$ of $e$ such that $|f(z) - f(x)| < \epsilon/2$ for all $z$ in $xU_x$, such a neighborhood exist by continuity of $f$. Now by 2.1.3, we can have, for every $x$ in $K$, $V_x$ the symmetric neighborhood of $e$ such that $V_x^2$ is contained in $U_x$. Then $\{xV_x\}_{x\in K}$ is an open cover of $K$, so for some $\{x_1, x_2, ..., x_n\}$ the subcovering $\{x_iV_{x_i}\}_{i=1}^n$ is an open cover of $K$. Let $V = \cap_{i=1}^n V_{x_i}$ then $V$ is symmetric neighborhood of $e$. Let $x$ in $G$ and $y$ in $V$ be arbitrary.
If $x$ is in $K$, it implies $x \in x_iV_{x_i} \subset x_iU_{x_i}$ for some $i$, then $xy \in xV_{x_i}V \subset xV_{x_i}^2 \subset x_iU_{x_i}$ and so,

$$|R_yf(x) - f(x)| \leq |f(xy) - f(x_i)| + |f(x_i) - f(x)| < \epsilon$$

If $xy \in K$, then we have that $xy \in x_iV_{x_i} \subset x_iU_{x_i}$ for some $i$. Since $V$ is symmetric it implies for $x \in x_iV_{x_i}y^{-1} \subset x_iV_{x_i}V \subset x_iV_{x_i}^2 \subset x_iU_{x_i}$. Hence,

$$|R_yf(x) - f(x)| \leq |f(xy) - f(x_i)| + |f(x_i) - f(x)| < \epsilon.$$

If both $x$ *and* $xy$ are not in $K$ then $f(x) = f(xy) = 0$, so we have the inequality.
Thus we have for any net $(y_\alpha)_\alpha$ in $G$ converging to $e$, $\sup_{y\in G} |R_{y_\alpha}f(x) - f(x)| \to 0$. Hence $f$ if right uniformly continuous. Similarly for left uniformly continuous.

□

**Corollary 2.2.5.** *Let $G$ be locally compact group with regular Borel measure $\mu$ and let $f$ be in $C_c(G)$. Then the functions*

$$x \mapsto \int_G L_xf\ d\mu \quad and \quad y \mapsto \int_G R_xf\ d\mu$$

*are continuous.*

*Proof.* Let $(x_\alpha)_\alpha \to x$ be a net in $G$. Let $K$ be the support of $f$ and let $W$ be the compact neighborhood of $x$. Now as $L_x$ and $f$ are continuous functions, so we have that the composition $L_xf$ is also continuous with compact support $W^{-1}K$. Let $\alpha_0$ be an index such that $x_\alpha$ belongs to $W$ for all $\alpha \geq \alpha_0$. Then it follows that

$$|\int_G L_{x_\alpha}f\ d\mu - \int_G L_xf\ d\mu| \leq \int_G |L_{x_\alpha}f - L_xf|d\mu\ \leq \mu(W^{-1}K)\ sup_{y\in G}|L_{x_\alpha}(y) - L_xf(y)|,$$

Since $f$ belongs to $C_c(G)$ so from 2.2.4 it is left uniformly continuous and so we have desired result. □

**Definition 2.2.6.** *Let $G$ be locally compact group and let $\mu$ be a non-zero regular Borel measure on $G$. Then $\mu$ is left Haar measure if it is invariant under left translations, i.e.*

$$\mu(xA) = \mu(A) \qquad \forall\ x \in G \text{ and } A \in \mathcal{B}(G).$$

*Likewise mu is right Haar measure if for all $x$ in $G$ and $A$ in $\mathcal{B}(G)$,*

$$\mu(Ax) = \mu(A).$$

**Remark 2.2.7.** *As we know for $f : X \to Y$, where $f$ is continuous and $X$, $Y$ be Hausdorff topological space then $f$ is Borel measurable, also for topological group $G$, with arbitrary $a, x$ in $X$, $x \mapsto ax$ and $x \mapsto xa$ are homeomorphisms. So the expression $\mu(xa)$ and $\mu(ax)$ are meaningful.*

**Example 2.2.8.**

1. *Lebesgue measure on $\mathbb{R}$.*
2. *Counting measure on $(G, \tau_{dis.})$*

Haar measure is special in the sense that it gives us freedom of translation. We will see that every locally compact group possesses a left Haar measure, moreover it is unique upto constant multiple. We will look at few properties of Haar measures, plus the relationship between left and right Haar measures.

**Theorem 2.2.9.** *Let $G$ be locally compact group then $G$ has a left Haar measure on it.*

for proof of the above theorem refer Theorem 9.2.2 of [Cohn]
Now let fix our notations, now onwards $G$ will denote the $\sigma-$finite locally compact group and $m_G$ will denote the fix left Haar measure on $G$.

**Remark 2.2.10.** *Let $f$ be borel measurable function then*

$$\int_G L_x f \ dm_G = \int_G f \ dm_G.$$

*Proof.* If $f$ is characteristic function of Borel set $A$, then

$$\int L_x f \ dm_G = m_G(xA) = m_G(A) = \int f \ dm_G,$$

similarly for simple functions. Now let $f$ be any measurable function then as $f = f^+ + f^-$ and by simple approximation theorem there exists sequence $(\phi_n)$ *and* $(\psi_n)$ of positive and increasing simple functions such that $\phi_n$ converges to $f^+$ and $\psi_n$ converges to $f^-$ as $n \to \infty$. then by linearity and monotone convergence theorem we have desired result. $\square$

**Proposition 2.2.11.** *Let $U$ be a non-empty open set in $G$ and $f$ be in $C_c^+(G)$ and non-zero. Then $m_G(U) > 0$ and $\int f \ \ dm_G > 0$.*

*Proof.* As $m_G$ is regular Borel measure there exist a compact set $K$ such that $m_G(K) > 0$. Now let $y$ in $U$, then $e \in y^{-1}U$ which implies that $\{xy^{-1}U\}_{x\in G}$ is an open cover of $K$. That is there exists $x_1, x_2, ..., x_n$ such that $\{x_iU\}_{i=1}^n$ covers $K$. So as $m_G$ is left Haar measure we have,

$$m_G(K) \leq \sum_{i=1}^{n} m_G(x_iU) = nm_G(U).$$

This implies $m_G(U) > 0$.
Since $f \in C_c^+G$, there exists non-empty open set $U$ and positive real number $\epsilon$ such that $f > \epsilon\chi_U$. This implies that

$$\int f \ dm_G > \int \epsilon\chi_U \ dm_G = \epsilon \ m_G(U) > 0.$$

So, $\int f \ dm_G > 0$. Such an open set exists as if for all open set $U$ and for all $\epsilon > 0$, $f \leq \epsilon\chi_U$. Then $f(x) \leq \epsilon$ implying that $f = 0$. That is a contradiction. □

Theorem 2.2.9 gives the existence of left Haar measure on locally compact group $G$. Now we will prove the uniqueness of such measures upto constant multiple. That is,

**Theorem 2.2.12.** *Let $\nu$ be left Haar measure on $G$, then $\nu = cm_G$ for some $c > 0$.*

*Proof.* Let $g$ be in $C_c^+(G)$, then for all $x$ in $G$, $R_xg$ belongs to $C_c^+(G)$. So from 2.2.11 and 2.2.5 $x \mapsto \int R_xg \ d\nu$ is a continuous function with positive values.
Now for any $f$ in $C_c(G)$, define $h : G \times G \to \mathbb{C}$, as

$$h(x, y) = \frac{f(x)g(yx)}{\int_G R_xg \ d\nu}.$$

$h$ is continuous function. Now let $K$ and $L$ denote the support of $f$ and $g$ respectively. Then $supp(h) \subset K \times LK^{-1}$ which gives that $h$ belongs to $C_c(G \times G)$.

$$\begin{aligned}
\int_G \int_G h(x, y) \ d(m_G)d\nu &= \int_G \int_G \frac{f(x)g(yx)}{\int_G R_xg(t) \ d\nu} m_G(dx)\nu(dy) \\
&= \int_G \int_G \frac{f(x)g(yx)}{\int_G g(tx)\nu(dt)} m_G(dx)\nu(dy) \\
&= \int_G f(x) \frac{\int_G g(yx)\nu(dy)}{\int_G g(tx)\nu(dt)} m_G(dx) \\
&= \int_G f(x) \ d(m_G).
\end{aligned}$$

On the other hand, by using property of left invariance of Haar measure and by changing the order of integration using Fubini's theorem, we have

$$\begin{aligned}
\int_G\int_G h(x,y)\ m_G(dx)\nu(dy) &= \int_G\int_G \frac{f(x)g(yx)}{\int_G g(tx)\nu(dt)} m_G(dx)\nu(dy) \\
&= \int_G\int_G \frac{f(y^{-1}x)g(x)}{\int_G g(ty^{-1}x)\nu(dt)} m_G(dx)\nu(dy) && \text{(replace x with } y^{-1}\text{x)} \\
&= \int_G\int_G \frac{f((xy)^{-1}x)g(x)}{\int_G g(t(xy)^{-1}x)\nu(dt)} \nu(dy)m_G(dx) && \text{(replace y with xy)} \\
&= \int_G\int_G \frac{f(y^{-1}x)g(x)}{\int_G g(ty^{-1})\nu(dt)} \nu(dy)m_G(dx) \\
&= \int_G g(x)m_g(dx)\int_G \frac{f(y^{-1})\nu(dy)}{\int_G g(ty^{-1})\nu(dt)}
\end{aligned}$$

So from these two observations, we have

$$\int_G f(x)\ dm_G = \int_G g\ dm_G \int_G \frac{f(y^{-1})\nu(dt)}{\int_G g(ty^{-1}\nu)dt},$$

which implies,

$$\frac{\int_G f\ dm_G}{\int_G g\ dm_G} = \frac{\int_G f\ d\nu}{\int_G g\ d\nu}$$

so by fixing $c = \frac{\int_G g\ d\nu}{\int_G g\ dm_G}$, for all $f$ in $C_c^+(G)$ we have,

$$\int_G f\ d\nu = \int_G f\ dm_G$$

Direct application of Ries'z Representation theorem gives that $\nu = cm_G$ □

Here we showed that on a locally compact group we have unique Haar measure whether it is left or right. So from this we can say that Lebesgue measure is actually the only (obviously upto constant multiple) Haar measure on $\mathbb{R}$. Now as we proved that composition of functions with left translation function does not affect the integration value of function with respect to left Haar measure, i.e. $\int_G L_x f\ dm_G = \int_G f\ dm_G$. Now the question is what about composition with right translation function?

**Proposition 2.2.13.** *Let $G$ be a locally compact group. Then there is a continuous homomorphism $\Delta$ from $G$ to $\mathbb{R}_\times$, where $\mathbb{R}_\times = (\mathbb{R}_+^*, \cdot\ )$, such that*

$$\int_G R_y f\ dm_G = \Delta(y^{-1})\int_G f\ dm_G \qquad (f \in L^1(G),\ y \in G)$$

*Proof.* for $x$ in $G$ define measure $m_G^x$ as,

$$m_G^x(E) = m_G(Ex),$$

for all $E$ in $\mathcal{B}(G)$. Now we will prove that this measure is regular measure.
Since $R_x$ is homeomorphism and $m_G$ is regular we have that $m_G^x(K)$ is finite for every compact set $K$. Now if $U$ is any open set and $U$ contains the compact set $K$ then also $Ux$ contains $Kx$. Then

$$\begin{aligned} m_G^x(U) &= m_G(Ux) \\ &= sup\{m_G(Kx) : \; K \text{ is compact and contained in } U\} \\ &= sup\{m_G^x(K) : \; K \text{ is compact and contained in } U\} \end{aligned}$$

which implies $m_G^x$ is inner regular.
Similarly we can proceed for proving $m_G^x$ is outer regular. Now observe that for every Borel set $E$,

$$m_G^x(yE) = m_G(yEx) = m_G(Ex) = m_G^x(E)$$

this gives that $m_G^x$ is left Haar measure on $G$.
Theorem 2.2.12 implies that there exist a constant real number $\Delta(x)$ depending on $x$ such that

$$m_G^x = \Delta(x)m_G.$$

Let $n_G$ be any other left Haar measure on $G$ then for some real number $c > 0$ $n_G = cm_G$ which implies,

$$n_G^x = cm_G^x = c\Delta(x)m_G = \Delta(x)n_G.$$

This observation leads us to conclusion that $\Delta(x)$ does not depend on measure, it is determined by group $G$. Furthermore let $U$ in $\mathcal{B}(G)$,

$$\Delta(xy)m_G(U) = m_G(Uxy) = \Delta(y)m_G(Ux) = \Delta(y)\Delta(x)m_G(U)$$

this implies, $\Delta(xy) = \Delta(x)\Delta(y)$ i.e. $\Delta$ is a homeomorphism.
Now as it is easy to see that $\chi_U(xy) = \chi_{Uy^{-1}}(x)$, it gives that

$$\int_G \chi_U(xy)\; m_G(dx) = m_G(Uy^{-1}) = \Delta(y^{-1})m_G(U) = \Delta(y^{-1}) \int_G \chi_U(x)\; m_G(dx),$$

similarly as in remark 2.2.10 by using monotone convergence theorem and linearity of integration we have above result for any $f$ in $L^1(G)$ i.e.

$$\int R_y f \; dm_G = \Delta(y^{-1}) \int f \; dm_G$$

Since $y \mapsto \int R_y f\ dm_G$ is continuous for $f$ in $C_c(G)$ and also the inverse map is continuous, so we have that

$$\Delta : G \to \mathbb{R}_\times$$

defined as

$$y \mapsto \frac{\int R_{y^{-1}} f\ dm_G}{\int f\ dm_G}$$

is continuous. Hence we are done.

□

**Remark 2.2.14.** *1. A locally compact group $G$ is called unimodular if $\Delta = 1$ i.e. if left Haar measure are right invariant. Clearly abelian group are unimodular.*

*2. If $G$ is compact $\Delta(G)$ is bounded.*

*3. Let $\mu$ be a left Haar measure, then $\hat{\mu} = \mu(A^{-1})$ is right Haar measure.*

**Proposition 2.2.15.** *$m_G$ is finite iff $G$ is compact.*

*Proof.* If $G$ is compact then from regularity property of $m_G$, we get that $m_G(G)$ is finite. Let $m_G$ is finite and $K$ be a compact set such that $m_G(K) > 0$, we can get such a compact set as $m_G$ is regular. As $m_G$ is finite which implies there is an upper bound for the lengths of those finite sequences $(x_i)_{i=1}^n$ for which the sets $x_iK$ are disjoint and no choice of $x_{n+1}$ $x_iK$ are disjoint for $i = 1, 2, ..., n+1$. So for $x$ in $G$, $xk \cup (\cap_{i=1}^n x_iK \neq \phi)$ which gives that $x$ belongs to $\cup(\cap_{i=1}^n x_iK)K^{-1}$. So $G = \cup(\cap_{i=1}^n x_iK)K^{-1}$ hence $G$ is compact as the finite union of compact sets.

□

As we defined a new measure $\hat{\mu}$ in remark 2.2.14, now we want to see that for left Haar measure $m_G$ how that $\hat{m_G}$ is related with. We have following proposition which give the relation between these two and hence any two left and right Haar measure as they (left or right) differ by just a constant.

**Proposition 2.2.16.** *If $\Delta$ is defined as above in proposition 2.2.13, then*

$$\hat{m_G}(A) = \int_A \Delta(x^{-1}) m_G(dx) \qquad (A \in \mathcal{B}(G)).$$

*Proof.* Define, $\nu : \mathcal{B}(G) \to [0, \infty]$ as,

$$\nu(A) = \int_A \Delta(x^{-1}) m_G(dx).$$

Now we will claim that actually $\nu = \hat{m_G}$, to claim that we have to first show that $\nu$ is regular and right Haar measure. $\nu$ is regular (refer Appendix B). Now,

$$\begin{aligned}
\nu(Ay) &= \int_{Ay} \Delta(x^{-1}) m_G(dx) \\
&= \int \chi_{Ay}(x)\Delta(x^{-1}) m_G(dx) \\
&= \int \chi_{Ay}(x)\Delta(y^{-1})\Delta((xy^{-1})^{-1}) m_G(dx) \\
&= \Delta(y^{-1}) \int_G R_{y^{-1}}\chi_A(x)\Delta((xy^{-1})^{-1}) m_G(dx) \\
&= \Delta(y^{-1}) \int_G R_{y^{-1}}(\chi_A(x)(\Delta(x))^{-1}) m_G(dx) \\
&= \Delta(y^{-1})[\Delta(y) \int \chi_A(x)\Delta(x^{-1}) m_G(dx)] \\
&= \nu(A)
\end{aligned}$$

Hence $\nu$ is right Haar measure. So from Uniqueness of Haar measure we have some $c > 0$ such that,

$$\nu = c\hat{m_G}.$$

As $\Delta(e) = 1$ and it is continuous so from topological properties of $G$ we have a symmetric neighbourhood $U$ of $e$ such that for all $\epsilon > 0$,

$$|\Delta(x) - 1| < \epsilon,$$

for all $x$ in $U$.
This gives that

$$\begin{aligned}
c &= \frac{\nu(U)}{\hat{m_G}(U)} = \frac{\nu(U)}{m_G(U^{-1})} \\
&= \frac{1}{m_G(U)} \int_U \Delta(x^{-1}) m_G(dx) \\
&= \frac{1}{m_G(U)} \left[ \int_G (\Delta(x^{-1} - 1)) m_G(dx) + \int_G 1.m_G(dx) \right] \\
&< \frac{m_G(U)}{m_G(U)}(1+\epsilon)
\end{aligned}$$

similarly we have $c > 1 + \epsilon$ for all $\epsilon > 0$. Hence $c = 1$ which implies that $\nu = \hat{m_G}$.

$\square$

**Corollary 2.2.17.** *All left and right Haar measure are pairwise equivalent. In the sense that if $\mu$ is left Haar measure and $\nu$ is right Haar measure then for $A$ in $\mathcal{B}(G)$*

$$\mu(A) = 0 \iff \nu(A) = 0.$$

**Proposition 2.2.18.** *Let $f$ be in $L^1(G)$ then the mappings $G \mapsto L^1(G)$ defined as,*

$$x \mapsto L_x f \quad and \quad y \mapsto R_y f$$

*are continuous.*

*Proof.* We will first show the continuity at $e$ for $g$ in $C_c(G)$ then will extend it to $L^1(G)$ for any $x$ in $G$. Let $(y_\alpha)_{\alpha\in\Lambda}$ be a net converging to $e$ and let $V$ be compact symmetric neighbourhood of $e$ and $g$ be an element of $C_c(G)$. Define $K = (supp\ g)\ V \cup V(supp\ g)$, $K$ is compact. Observe that $L_x f$ and $R_x f$ are supported by $K$. Let $\alpha_0 \in \Lambda$ be such that $y_\alpha$ in $V$ for all $\alpha \geq \alpha_0$. Since $g$ belongs to $C_c(G)$ so $g$ is left uniformly continuous hence

$$\begin{aligned} ||L_{y_\alpha} g - g||_1 &= \int |L_{y_\alpha} g - g| m_G(dx) \\ &\leq ||L_{y_\alpha} g - g||_\infty m_G(K) \to 0. \end{aligned}$$

Similarly $||R_{y_\alpha} g - g||_1 \to 0$.
Now let $f$ belongs to $L^1(G)$. As we know that $||L_y f||_1 = ||f||_1$ and for right translation function, $||R_y f||_1 = \Delta(y^{-1})||f||_1 < C||f||_1$ where $C := max\{\Delta(y^{-1}) : y \in V\}$. We know that $\overline{C_c(G)} = L^1(G)$. So we have that for all $\epsilon > 0$, there exists $g$ in $C_c(G)$ such that $||f - g||_1 < \epsilon$. So for $\alpha \geq \alpha_0$,

$$\begin{aligned} ||R_{y_\alpha} f - f||_1 &\leq ||R_{y_\alpha} f - R_{y_\alpha} g||_1 + ||R_{y_\alpha} g - g||_1 + ||g - f||_1 \\ &\leq C||f - g||_1 + ||f - g||_1 + ||R_{y_\alpha} g - g||_1 \\ &\leq (C + 1)\epsilon + ||R_{y_\alpha} g - g||_1 \\ &\to 0 \qquad\qquad as\ R_{y_\alpha} g \to g\ for\ g \in C_c(G) \end{aligned}$$

Similarly $||L_{y_\alpha} f - f||_1 \to 0$, which implies that $x \mapsto L_x f$ is continuous at $e$.
For $x \in G$ if $(x_\alpha)_{\alpha\in\Lambda} \to x$ then $x^{-1}x_\alpha \to e$ and $x_\alpha x^{-1} \to e$ and we already know that

$$||L_{x^{-1}x_\alpha} f - f||_1 \to 0 \quad and \quad ||R_{x^{-1}x_\alpha} f - f||_1 \to 0.$$

Hence

$$\begin{aligned} ||L_{x_\alpha} f - L_x f||_1 &= \int |L_{x_\alpha} f - L_x f| m_G(dy) \\ &= \int |f(x_\alpha y) - f(xy)| m_G(dy) \\ &= \int |f(x\alpha x^{-1} y) - f(y)| m_G(dy) \\ &= ||L_{x_\alpha x^{-1}} f - f||_1 \to 0. \end{aligned}$$

also

$$
\begin{aligned}
||R_{x_\alpha} f - R_x f||_1 &= \int |f(yx_\alpha) - f(yx)| m_G(dy) \\
&= \int |f(yx^{-1}x_\alpha) - f(y)| m_G(d(yx^{-1})) \\
&= \int |f(yx^{-1}x_\alpha) - f(y)| m_G^{x^{-1}}(dy) \\
&= \Delta \int |f(yx^{-1}x_\alpha) - f(y)| m_G(dy) \\
&= \Delta(x^{-1}) ||R_{x^{-1}x_\alpha} f - f||_1 \to 0.
\end{aligned}
$$

Hence Proved. □

## 2.3 The Algebras: Group algebra and Measure Algebra

In this section we will try to define convolution as algebra multiplication in $L^1(G)$. For that we have to check whether it is well defined or not. So firstly let start with some results which will be helpful while proving our main results.

**Lemma 2.3.1.** *Let $f$ be in $L^1(G)$. Then $A := \{x \in G | f(x) \neq 0\}$ is contained in a $\sigma-$compact set.*

*Proof.* Let $A_n := \{x \in X | f(x) \geq 1/n\}$, as $f$ belongs to $L^1(G)$ implies that $f$ is integrable, which implies that $m_G(A_n)$ is finite infact there exist some $M > 0$, such that,

$$
\begin{aligned}
M &> \int_G |f(x)| \; m_g(dx) \\
&\geq \int_{A_n} |f(x)| \; m_G(dx) \\
&\geq \frac{1}{n} \; m_G(A_n)
\end{aligned}
$$

So we get here that $A = \bigcup_{n \in \mathbb{N}} A_n$ is $\sigma-$finite. Now as for each $n$ $A_n$ is measurable so from regularity of measure there exist open set $U_n$ containing $A_n$ and with finite measure. Let $H$ be open $\sigma-$compact subgroup of $G$ (see corollary 2.1.9).
Define $J := \{x \in G | xH \text{ is pairwise disjoint}\}$, take $U_n = \bigcup_{x \in J}(xH \cap U_n)$ as each $xH \cap U_n$ is open so from proposition 2.2.11, $m_G(xH \cap U_n) > 0$. Since $m_G(U_n) = \sum_{x \in J} m_G(xH \cap U_n)$ is finite, it implies that there exists countable set $J'$ of $J$ such that $U_n = \bigcup_{x \in J'}(xH \cap U_n)$. We have that $U_n \subset \bigcup_{x \in J'} xH$ which further implies that $A \subset \bigcup_{n \in \mathbb{N}} \bigcup_{x \in J'} xH$, since each $xH$ is $\sigma-$compact so as $A$. □

**Lemma 2.3.2.** *For a locally compact group $G$, the mapping $F : G \times G \to G \times G$ defined as $F(x, y) = (x, xy)$ is a measure preserving homeomorphism. That is*

$$(m_G \times m_G)(F(A)) = (m_G \times m_G)(A).$$

*Proof.* It is easy to see that $F$ is one-one and onto. As the inverse of $F$ is defined as $F^{-1}(x, y) = (x, x^{-1}y)$ so by continuity of group operations $F$ and $F^{-1}$ is continuous. To prove measure preserving we will first look for the open sets then by using regularity of measure extend it to any measurable set. Let $U$ be an open set of $G \times G$, $x \in G$.

$$\begin{aligned} s \in (F(U))_x &\iff (x, s) \in F(U) \\ &\iff s = xy \ for\ some\ y \in U_x \\ &\iff s \in xU_x \end{aligned}$$

so $(F(U))_x = xU_x$. Now,

$$\begin{aligned} (m_G \times m_G)(U) &= \int_G (U_x)\ m_G(dx) = \int_G m_G(xU_x)\ m_G(dx) \\ &= \int_G m_G(F(U))_x\ m_G(dx) = (m_G \times m_G)(F(U)) \end{aligned}$$

As $m_G$ is regular so as $m_G \times m_G$ moreover as $F$ is homeomorphism this gives that $(m_G \times m_G)F$ is regular. This implies

$$\begin{aligned} (m_G \times m_G)(A) &= inf\{(m_G \times m_G)(U) :\ U\ is\ open\ and\ A \subset U\} \\ &= inf\{(m_G \times m_G)(F(U)) :\ U\ is\ open\ and\ A \subset U\} \\ &= (m_G \times m_G)F((A)). \end{aligned}$$

Hence proved. □

Now we are heading towards defining the convolution operation between two elements of $L^1(G)$. For that we need following proposition.

**Proposition 2.3.3.** *Let $G$ be locally compact group. Let $f, g$ in $L^1(G)$ and define mapping $\phi : G \times G \to \mathbb{C}$ by $\phi(x, y) = f(x)g(x^{-1}y)$ then the function $x \mapsto \phi(x, y)$ is integrable for $m_G$ almost every $y$ in $G$.*

*Proof.* Define $\phi' : G \times G \to \mathbb{C}$ by

$$\phi'(x, y) = f(x)g(y).$$

It is clear that $\phi'$ is composition of maps $(x, y) \mapsto (f(x), g(y)$ and $(x, y) \mapsto xy$, from which it is evident that $\phi'$ is measurable. As $f$ and $g$ are in $L^1(G)$ so from lemma 2.3.1 we have compact sets $K_n$ and $K'_n$ such that $f$ *and* $g$ vanishes outside $A = \bigcup_{n\in\mathbb{N}} K_n$

and $B = \bigcup_{n\in\mathbb{N}} K'_n$ respectively. Which also implies that $\phi'(x,y) = f(x)g(y) = 0$ when $(x,y) \notin \bigcup_{n,m=1}^{\infty}(K_n \times K'_m)$. Now using Fubini's theorem we have,

$$\int_G \int_G |\phi'(x,y)|\ m_G(dx)m_G(dy) = \int_G \int_G |f(x)g(y)|\ m_G(dx)m_G(dy) = ||f||_1\ ||g||_1 < \infty.$$

This gives that $\phi'$ belongs to $L^1(G\times G)$. Now define $F : G\times G \to G\times G$ as $F(x,y) = (x,xy)$. Then $\phi = \phi' \circ F^{-1}$. From proposition 2.3.2 we have that $F$ is measure preserving homeomorphism so,

$$\begin{aligned}\int_{G\times G} &= \int_{G\times G} |\phi' \circ F^{-1}|\ d(m_G \times m_G)\\ &= \int_{G\times G} |\phi'|\ d((m_G\times m_G)F)\\ &= \int_{G\times G} |\phi'|\ d(m_G \times m_G) < \infty.\end{aligned}$$

So we get that $\phi \in L^1(G\times G)$. Let $A_{n,m} = K_n \times K'_m$, $A_{n,m}$ is compact and so by regularity of measure, it is of finite measure. Again it is easy to see that $\phi(x,y) = 0$ whenever $(x,y) \notin \bigcup_{n,m=1}^{\infty} F(A_{n,m})$. Since $m_G(F(A_{n,m})) = m_G(A_{n,m})$ for all $n,m$ in $\mathbb{N}$. It follows that $\phi$ vanishes outside a $\sigma-$compact set. So by Fubini's theorem the map $x \mapsto \phi(x,y)$ belongs to $L^1(G)$ for almost every $y$ in $G$. □

**Definition 2.3.4.** *Let $f$ and $g$ in $L^1(G)$, define convolution operation of two elements as*

$$x \mapsto \int_G f(y)g(y^{-1}x)\ m_G(dy)$$

*denoted as $(f * g)(x)$.*

From proposition 2.3.3 it is integrable and hence belongs to $L^1(G)$. Actually we have the following,

**Proposition 2.3.5.** *$L^1(G)$ is a Banach algebra with convolution as algebra multiplication.*

*Proof.* Let $f, f', g$ and $g'$ in Banach space $L^1(G)$ such that $f = f'$ and $g = g'$ almost everywhere, then

$$||f*g - f'*g'||_1 \le ||f*(g-g')||_1 + ||(f-f')*g'||_1 \le ||f||_1||g-g'||_1 + ||f-f'||_1||g'||_1 = 0.$$

This implies that $f*g = f'*g'$ almost everywhere, hence it is clear that convolution is well defined. From Proposition 2.3.3, it is clear that convolution of two elements of $L^1(G)$ is again an element of $L^1(G)$. That is $L^1(G)$ is closed under convolution operation. Moreover

from linearity of integration it is evident that $f * g$ is bilinear map.
Furthermore, by Fubini's theorem and left invariance property of $m_G$, we have

$$\begin{aligned}
||f * g||_1 &= \int_G |(f * g)(x)| \ m_G(dx) \\
&= \int_G | \int_G f(y)g(y^{-1}x) \ m_G(dx)| m_G(dy) \\
&\leq \int_G |f(x)| \left( \int_G g(y) m_G(dy) \right) m_G(dx) \\
&= ||f||_1 ||g||_1
\end{aligned}$$

That is $||f * g||_1 \leq ||f||_1 ||g||_1$. Now the only thing which is left to prove is the associativity of convolution. Let $f, g, h \in C_c(G)$ with $K = supp\ f$ and $L = supp\ g$ then for each $x$ in $G$, the function $\psi : G \times G \to \mathbb{C}$, defined as

$$\psi(s,t) = f(s)g(s^{-1}t)h(t^{-1}x),$$

is supported by the set $K \times KL$, hence $\psi \in C_c(G \times G)$. Now from Fubini's theorem for continuous compactly supported functions, we have;

$$\begin{aligned}
f * (g * h) &= \int_G f(s)(g * h)(s^{-1}x) \ m_G(ds) \\
&= \int_G \int_G f(s)g(t)h(t^{-1}s^{-1}x) \ m_G(dt) m_G(ds) \\
&= \int_G \int_G f(s)g(t)h((st)^{-1}x) \ m_G(dt) m_G(ds) \\
&= \int_g \int_G f(s)g(s^{-1}t)h(t^{-1}x) \ m_G(dt) m_G(ds) \\
&= \int_g \int_G f(s)g(s^{-1}t)h(t^{-1}x) \ m_G(ds) m_G(dt) \\
&= \int_G (f * g)(t) h(t^{-1}x) \ m_G(dt) \\
&= (f * g) * h
\end{aligned}$$

As we know that $C_c(G)$ is dense in $L^1(G)$. Then for $f, g$ and $h$ in $L^1(G)$ there exists sequence $(f_n)$, $(g_n)$ and $(h_n)$ respectively in $C_c(G)$ such that $f_n \to f$, $g_n \to g$ and $h_n \to h$. so,

$$\begin{aligned}
||(f * g) * h - (f_n * g_n) * h_n||_1 &\leq ||(f * g) * h - (f * g) * h_n||_1 + ||(f * g) * h_n - (f * g_n) * h_n||_1 \\
&\quad + ||(f * g_n) * h_n - (f_n * g_n) * h_n||_1 \\
&\leq ||f * g||_1 ||h - h_n||_1 + ||f||_1 ||g - g_n||_1 ||h_n||_1 \\
&\quad + ||f - f_n||_1 ||g_n||_1 ||h_n||_1 \\
&\to 0 \ \ as\ n \to \infty.
\end{aligned}$$

Similarly, we have that $||f*(g*h)-f_n*(g_n*h_n)||_1 \to 0$ as $n \to \infty$. So by using associativity of convolution for $C_c(G)$, we get that

$$||f*(g*h)-(f*g)*h||_1 \leq ||f*(g*h)-f_n(g_n*h_n)||_1 + ||(f_n*g_n)*h_n-(f*g)*h||_1$$
$$\to 0.$$

Hence $(f*g)*h = f*(g*h)$. We are done with showing that $L^1(G)$ is a Banach algebra. □

**Remark 2.3.6.** *Now we will look for some identities that we will use in our further discussions. By using left invariance of $m_G$ and some propositions we get following;*

1.
$$\begin{aligned}(f*g)(x) &= \int_G f(y)g(y^{-1}x)\ m_G(dy)\\ &= \int_G f(xy)g(y^{-1})\ m_G(dy)\\ &= \int_G f(y^{-1})g(yx)\Delta(y^{-1})m_G(dy)\\ &= \int_G f(xy^{-1})g(y)\Delta(y^{-1})m_G(dy).\end{aligned}$$

*Where the first equality is obtained by replacing $y$ by $xy$ and left invariance of Haar measure $m_G$. Second and third inequality is obtained by replacing $y$ by $x^{-1}y^{-1}$ and by using proposition 2.2.16. While to get third equality we are replacing $y$ by $yx^{-1}$.*

2.
$$\begin{aligned}(f*L_xg)(y) &= \int_G f(h)g(xh^{-1}y)\ m_G(dh)\\ &= \Delta(x^{-1})^{-1}\int_G R_x(f(h)g(xh^{-1}y))\ m_G(dh)\\ &= \Delta(x)\int_G f(hx)g(h^{-1}y)\ m_G(dh)\\ &= \Delta(x)(R_xf*g)(y)\end{aligned}$$

*To get the second equality we are using proposition 2.2.13 and next equality by replacing $h$ by $hx$.*

3. $L_x(f*g) = (L_xf*g)$
4. $R_x(f*g) = f*(R_xg)$

5. *Now we will go to next level, we will convolve $f$ from $L^1(G)$ and $\phi$ from $L^\infty(G)$. We observe that $f * \phi$ is an element of $L^\infty(G)$. Indeed $||f * \phi||_\infty \leq ||f||_1||\phi||_\infty$. Moreover, by using proposition 2.2.18 we can actually prove that $f * \phi$ is continuous on $G$. Let $(x_\alpha)_\alpha$ be a net in $G$ converging to $x \in G$, then*

$$\begin{aligned}|f * \phi(x_\alpha) - f * \phi(x)| &\leq |\int_G f(y)\phi(y^{-1}x_\alpha)\ m_G(dy) - \int_G f(y)\phi(y^{-1}x)m_G(dy)| \\ &\leq \int_G |(f(x_\alpha y) - f(xy))\phi(y^{-1})\ m_G(dy)| \\ &\leq ||\phi||_\infty ||L_{x_\alpha} f - L_x f||_1 \\ &\to 0.\end{aligned}$$

We discussed some elementary properties of convolution and also observed that $L^1(G)$ is actually the group algebra forms a Banach algebra with convolution as algebra multiplication. Now we turn to another Banach algebra that is measure algebra $M(G)$ which consist of **all finite complex valued regular Borel measure** on $G$. Again to prove that $M(G)$ is an algebra we will define convolution of measures.

**Lemma 2.3.7.** *Let $G$ be a locally compact group. If $\mu$ and $\nu$ are finite positive regular Borel measures on $G$ and if $\mu \times \nu$ is the regular Borel product of $\mu$ and $\nu$, then*

$$(\mu * \nu)(A) = (\mu \times \nu)\{(x, y) \in G \times G |\ xy \in A\}$$

*defines a regular Borel measure on $G$. Furthermore*

$$(\mu * \nu)(A) = \int_A \nu(x^{-1}A)\mu(dx) = \int_G \mu(Ay^{-1})\nu(dy)$$

*holds for each $A$ in $\mathcal{B}(G)$.*

*Proof.* For notational convenience let denote group operation multiplication as $m$. Then by definition

$$(\mu * \nu)(A) = (\mu \times \nu)(m^{-1}(A))$$

so by proposition B.2.8, second identity makes sense. It is easy to check that $(\mu * \nu)$ is actually a measure on $G$. Now to show the regularity of $(\mu * \nu)$ consider $A \in \mathcal{B}(G)$, as $A$ is $\sigma-$finite so we will prove that

$$(\mu * \nu)(A) = sup\{(\mu * \nu)(K)|\ K\ is\ compact\ and\ K \subset A\}$$

Let $\epsilon > 0$ then there exists some compact set $K_0$ such that

$$(\mu \times \nu)(K_0) > (\mu \times \nu)(m^{-1}(A)) - \epsilon.$$

Let $K = m(K_0)$ which implies that $K$ is subset of $G$ and $m^{-1}(K) \supset K_0$. Thus

$$(\mu * \nu)(K) > (\mu * \nu)(A) - \epsilon.$$

As $\epsilon$ is arbitrary it gives that $(\mu * \nu)$ is inner regular.
Now to show outer regularity of $(\mu * \nu)$, let $A^c \in \mathcal{B}(G)$ this implies for given $\epsilon > 0$ there exists compact set $K'$ contained in $A^c$ such that

$$(\mu * \nu)(K') > (\mu * \nu)(A^c) - \epsilon.$$

Furthermore

$$\begin{aligned}(\mu * \nu)(K')^c &= (\mu * \nu)(G) - (\mu * \nu)(K') \\ &< (\mu * \nu)(G) - (\mu * \nu)(A^c) + \epsilon \\ &= (\mu * \nu)(A) + \epsilon\end{aligned}$$

As $(K')^c$ is open and contains $A$, it gives that $(\mu * \nu)$ is outer regular.

□

**Note 2.3.8.** *For any general $\mu$ and $\nu$ from $M(G)$, define convolution as,*

$$\mu * \nu(A) = \int \mu(Ay^{-1})\nu(dy) = \int \nu(x^{-1}A)\mu(dx).$$

*From above lemma and Jordan decomposition theorem it is easy to see that there integral exists and are equal moreover $(\mu * \nu)$ is regular and so $\mu * \nu \in M(G)$.*

**Remark 2.3.9.** *Let $\mu$ and $\nu$ belongs to $M(G)$ and $f$ be a bounded Borel measurable function. Then by using Dominated convergence theorem and linearity of integrals, we observe that*

$$\int f(x, y)\ d(\mu * \nu) = \int\int f(xy)\ \mu(dx)\nu(dy) = \int\int f(xy)\ \nu(dy)\mu(dx).$$

**Remark 2.3.10.** *Let $x \in G$ and define measure, known as Dirac measure as,*

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

*Now we will see how this measure is actually helpful for us while discussing about measure algebra. Dirac measure induced by identity $e$ of group is unity in $M(G)$. Let us discuss some properties of Dirac measure. Let $x, y \in G,\ A \in \mathcal{B}(G)$ we have*

$$\delta_x * \delta_y(A) = \int_G \delta_y(t^{-1}A)\delta_x(dt) = \chi_{x^{-1}A}(y) = \chi_A(xy) = \delta_{xy}(A).$$

*Hence* $\delta_x * \delta_y = \delta_{xy}$. *Let* $\mu \in M(G)$ *we have*

$$\begin{aligned}
\delta_e * \mu(A) &= \int_G \mu(x^{-1}A)\delta_e(dx) \\
&= \mu(e^{-1}A) \\
&= \mu(A) \\
&= \mu(Ae^{-1}) \\
&= \int_G \mu(Ay^{-1})\delta_e(dy) \\
&= \mu * \delta_e(A)
\end{aligned}$$

*As we have that a function* $f \in L^1(G)$ *can be considered as measure. So observe the following relation* $\delta_x * f = L_{x^{-1}}f$ *and* $f * \delta_x = \Delta(x^{-1})R_{x^{-1}}f$. *Indeed we have*

$$\begin{aligned}
\delta_x * f(A) &= \int_G \delta_x(Ay^{-1})f(y)\ m_G(dy) \\
&= \int_G \chi_A(xy)f(y)\ m_G(dy) \\
&= \int_G \chi_A(y)f(x^{-1}y)\ m_G(dy) \\
&= L_{x^{-1}}f(A) \quad (x \in G, A \in \mathcal{B}(G)).
\end{aligned}$$

*Similarly by using proposition 2.2.13, we have*

$$\begin{aligned}
f * \delta_x(A) &= \int_G \delta_x(Ay^{-1})f(y)\ m_G(dy) \\
&= \int_G \chi_A(yx)f(y)\ m_G(dy) \\
&= \Delta(x^{-1})\int_G \chi_A(y)f(yx^{-1})\ m_G(dy) \\
&= \Delta(x^{-1})\int_A R_{x^{-1}}f(y)\ m_G(dy) \\
&= \Delta(x^{-1})R_{x^{-1}}f(A) \quad (x \in G, A \in \mathcal{B}(G)).
\end{aligned}$$

**Note 2.3.11.** *Recall that for a complex measure* $\mu$ *on* $G$ *we define the variation* $|\mu|$ *of* $\mu$ *as*

$$|\mu|(E) := sup\{\sum_i |\mu(E_i)| : E_i \textit{ is a finite boerl measurable partition of } E\},$$

*variation of a measure is itself a measure on $G$ and it is finite also. We define the total variation as*

$$||\mu|| = |\mu|(G).$$

*The interesting part is $(M(G), ||.||)$ is a Banach space (for justification refer [Cohn] proposition 4.1.18).*

**Proposition 2.3.12.** *$M(G)$ is unital Banach algebra with convolution of measures as algebra multiplication.*

*Proof.* Let $\mu_1, \mu_2$ and $\mu_3$ belongs to $M(G)$ then,

$$\begin{aligned}
(\mu_1 * \mu_2) * \mu_3(A) &= \int \mu_3(z^{-1}A)(\mu_1 * \mu_2)(dz) \\
&= \int \int \mu_3((xy^{-1})A)\mu_2(dy)\mu_1(dz) \\
&= \int \mu_3(y^{-1}x^{-1}A)\mu_2(dy)\mu_1(dx) \\
&= \int (\mu_2 * \mu_3)(x^{-1}A)\mu_3(dx) \\
&= \mu_1 * (\mu_2 * \mu_3).
\end{aligned}$$

The bilinearity of convolution follows from linearithy of integral. Now for given $\mu$ and $\nu$ in $M(G)$, we will try to show that $||\mu * \nu|| \leq ||\mu||||\nu||$. For that let $\{A_i\}$ be borel measurable partition of $G$. Then $\{Ay^{-1}\}$ is also a finite partition of $G$ for each $y$ in $G$. So

$$\begin{aligned}
\sum |(\mu * \nu)(A_i)| &= \sum_i |\int \mu(A_i y^{-1})\nu(dy)| \\
&\leq \int \sum |\mu(Ay^{-1})||\nu|(dy) \\
&\leq \int ||\mu|| \ d|\nu| \\
&= ||\mu||||\nu||
\end{aligned}$$

Hence, $||\mu * \nu|| \leq ||\mu||||\nu||$ We have already observes that $\delta_e$ is unity in $M(G)$. Thus we get that $M(G)$ is unital Banach algebra. □

We discussed two Banach algebra one group algebra and other is measure algebra. Now we try to seek into the relationship between these two algebras. Let us consider the relationship between the convolution of functions and the convolution of measures. Corollary 2.2.17 implies that an element of $M(G)$ is absolutely continuous with respect to the left Haar measures on $G$ if and only if it is absolutely continuous with respect to the right Haar measures on $G$. Thus we can define $M_a(G)$ to be the collection of elements of

$M(G)$ that are absolutely continuous with respect to some (and hence every) Haar measure on $G$. Recall that an ideal in an algebra $A$ is a linear subspace $I$ of $A$ such that $\mu * \nu$ and $\nu * \mu$ belong to $I$ whenever $\mu$ belongs to $I$ and $\nu$ belongs to $A$.

**Proposition 2.3.13.** *Let $G$ be locally compact group. Then*

1. *$M_a(G)$ is an ideal of $M(G)$.*
2. *if $\mu$ is left Haar measure on $G$, then the mao $f \mapsto \nu_f$ (where $\nu_f(A) = \int_A f\ d\mu$) induces a norm preserving algebra homomorphism of $L^1(G, \mathcal{B}(G), \mu)$ into $M(G)$,*
3. *the image of $L^1(G)$ under this homomorphism is $M_a(G)$.*

*Proof.* $M_a(G)$ is a linear subspace of $M(G)$. Let $\mu$ is left Haar measure on $G$. Let $\nu_1 \in M(G)$ and $\nu_2 \in M_a(G)$. Let $A$ be a Borel subset of $G$ that satisfies $\mu(A) = 0$, this also implies that $\mu(x^{-1}A) = 0$ for all $x$ in $G$. Since $\nu_2 << \mu$ so $\nu_2(x^{-1}A) = 0$ for all $x$ in $G$. So by definition $(\nu_1 * \nu_2)(A) = 0$. Now as $\hat{\mu}(A^{-1}) = \mu(A)$, it gives that $\hat{\mu}(Ay^{-1}) = \mu(yA^{-1})$. Since $\mu(A) = 0$ implies that $\hat{\mu}(A^{-1}) = 0$. We get that $\hat{\mu}(Ay^{-1}) = 0 = \mu(ya^{-1})$. As $\nu_2 << \hat{m_G}$, we have

$$(\nu_2 * \nu_1)(A) = \int \nu_2(Ay^{-1})\nu_1(dy) = 0.$$

Hence we get that $M_a(G)$ is ideal in $M(G)$.
As $f \mapsto \nu_f$ is norm preserving linear map, moreover $\nu_f << m_G$. The image is this map is $M_a(G)$.

$$\begin{aligned}
\nu_{f*g}(A) &= \int_A (f * g)\ d(\mu) \\
&= \int_G \chi_A(t) \int_G f(s)g(s^{-1}t)\mu(ds)\mu(dt) \\
&= \int_G \int_G \chi_A(st)f(s)g(t)\mu(ds)\mu(dt) \\
&= \int_G f(s) \left( \int_G \chi_A(st)g(t)\mu(dt) \right) \mu(ds) \\
&= \int_G \int_G (\chi_A(st)\nu_g(dt))\nu_f(ds) \\
&= \int_G \nu_g(s^{-1}A)\nu_f(ds) \\
&= (\nu_f * \nu_g)(A) \quad (f, g \in L^1(G)).
\end{aligned}$$

Hence we get desired results.

□

**Note 2.3.14.** *It is evident from above discussion that $M_a(G)$ is an ideal in $M(G)$ and $L^1(G)$ can be identify as $M_a(G)$. This actually imply that $L^1(G)$ can be viewed as an ideal in $M(G)$.*

**Remark 2.3.15.** *For two Banach algebras $A$ and $B$ where $A$ is closed ideal of $B$ we define the strict topology on $B$ with respect to $A$ to be the locally convex topology induced by the seminorms*

$$p_a(b) := ||ab|| + ||ba|| \quad (a \in A, b \in B).$$

*Furthermore a net $(x_\alpha))\alpha$ in $B$ converges to some $x$ in $b$ iff $p_a(x_\alpha - x) \to 0$ for every $a$ in $A$.*

**Proposition 2.3.16.** *Let $G$ be locally compact group. Then the map*

$$G \to M(G), \qquad x \mapsto \delta_x$$

*is continuous with respect to $w^*-$topology and the strict topology on $M(G)$ induced by $L^1(G)$.*

*Proof.* Let $(x_\alpha)_\alpha$ be a net in $G$ converging to some $x \in G$. Then for all $f \in C_0(G)$,

$$\delta_{x_\alpha}(f) = f(x_\alpha) \to f(x) = \delta_x(f),$$

since $f$ is continuous on $G$. Therefore $x \mapsto \delta_x$ is continuous with respect to $w^*-$topology on $M(G)$.
Let $f \in L^1(g)$ then by remark 2.3.10 and proposition 2.2.18, we get the following,

$$||(\delta_{x_\alpha} - \delta_x) * f||_1 = ||L_{x_{\alpha^{-1}}} f - L_{x^{-1}} f||_1 \to 0$$

and

$$\begin{aligned} ||f * (\delta_{x_\alpha} - \delta_x)||_1 &= ||\Delta(x_\alpha^{-1}) R_{x_\alpha^{-1}} f - \Delta(x^{-1}) R_{x^{-1}} f||_1 \\ &\leq ||\Delta(x_\alpha^{-1}) R_{x_\alpha^{-1}} f - \Delta(x_\alpha^{-1}) R_{x^{-1}} f||_1 + ||\Delta(x_\alpha^{-1}) R_{x^{-1}} f - \Delta(x^{-1}) R_{x^{-1}} f||_1 \\ &\leq |\Delta(x_\alpha^{-1})| \; ||R_{x_\alpha^{-1}} f - R_{x^{-1}} f||_1 + |\Delta(x_\alpha^{-1}) - \Delta(x^{-1})| \; ||R_{x^{-1}} f||_1 \to 0. \end{aligned}$$

Since $x_\alpha^{-1} \to x^{-1}$. Hence proved. □

# Chapter 3

# Amenable and Contractible Algebras

We have seen that the group algebra and measure algebra are actually Banach algebras. Now in this chapter we will introduce the concept of amenable and contractible algebras and in the further section we will characterize them. We will see that there is very close connection between contractible and semi simple algebra which will lead us to have that every contractible algebra is finite dimensional. Further as we have seen that every algebra can be embeded in a unital algebra. But what is next best for an algebra which is not unital. In this chapter we will introduce the concept of approximate identity and will observe some characterisations of algebras having approximate identity.

## 3.1 Banach-bimodules and Hoschchild Cohomology group

**Definition 3.1.1** (Banach Bi module)**.** *Let $A$ be an algebra. A left $A$-module is a vector space $X$ together with a map*

$$A \times X \to X, \qquad (a, x) \mapsto ax$$

*satisfying following property,*

1. $(a + b)x = ax + bx$
2. $a(x + y) = ax + ay$
3. $(\alpha a)x = a(\alpha x) = \alpha(ax)$
4. $(ab)x = a(bx)$

*for all $a,\ b \in A\ x,\ y \in X$ and $\alpha \in \mathbb{F}$.*

Similarly for right $A$ module. A space $X$ is called a $A$ bi-module if it is both a left $A$ module as well as right $A$ module satisfying extra condition

$$a(xb) = (ax)b \qquad (a,\ b \in A,\ x \in X).$$

If $X$ is Banach Space and $A$ is a Banach algebra and if there exists $M \geq 0$ satisfying

$$||ax||_X \leq M||a||_A||x||_X \quad \textit{and} \quad ||xa||_X \leq M||x||_X||a||_A$$

Then $X$ is called a Banach $A$ bi-module.

Now we will introduce the concept of amenability. Here we fix our notation. $A$ will denote Banach Algebra with unit, $X$ a Banach $A$ bi-module and $X'$ the dual Banach $A$ bi-module.

**Note 3.1.2.** *Here it can be checked that if $X$ is a Banach $A-$bi module then $X^*$ is also a Banach $A-$bi module under the following actions,*

$$(a.f)(x) = f(xa) \quad \textit{and} \quad (f.a)(x) = f(ax)$$

*It should be observed that if $X$ is left $A$ module then $X^*$ is right $A$ module and similarly if $X$ is right $A$ module then $X^*$ is left $A$ module.*

**Examples 3.1.3.**

1. *Define left and right action of $A$ on $X$ as trivial action that is*

$$a.x = 0 \quad \textit{and} \quad x.a = 0.$$

2. *Let $X = A$ then $A$ is $A-$bi module with action defined as follows,*

$$a.x = ax \quad \textit{and} \quad x.a = xa$$

   *such action is so called canonical action.*

3. *Let $A = L^1(G)$ and $X = L^\infty(G)$ and for all $f \in L^1(G)$ and $\phi \in L^\infty(G)$ define action as follows,*

$$f.\phi = f * \phi \quad \textit{and} \quad \phi.f = \left(\int_G f\ d(m_G)\right)\phi$$

**Definition 3.1.4.** *A bounded $X$-derivation is a bounded linear mapping $D$ of $A$ into $X$ such that*

$$D(ab) = (Da)b + a(Db) \qquad (a, b \in A).$$

*The set of all bounded $X$ derivation is denoted by $Z^1(A, X)$*

Given $x \in X$, let $\delta_x$ be the mapping of $A$ into $X$ given by

$$\delta_x(a) = ax - xa \qquad (a \in A).$$

Then it is easy to verify that $\delta_x \in Z^1(A, X)$. We call $\delta_x$ an inner $X$ derivation. We will Denote by $B^1(A, X)$, the set of all inner $X$ derivations. $B^1(A, X)$ is a linear subspace of $Z^1(A, X)$. Now define $H^1(A, X)$ as $Z^1(A, X)$ modulo $B^1(A, X)$, that is

$$H^1(A, X) = Z^1(A, X)/B^1(A, X).$$

Then $H^1(A, X)$ is called the first cohomology group of $A$ with coefficient in $X$. Now we have all the technicalities to define the amenable and contractible algebras.

**Definition 3.1.5.** *A Banach algebra $A$ is said to be contractible if $H^1(A, X) = \{0\}$ for every Banach $A$-bi module $X$, i.e. if every bounded $X$ derivation is inner.*

**Proposition 3.1.6.** *Every contractible Banach algebra is unital*

*Proof.* Let $X = A$ and define left and right action as follows

$$a.x = ax \quad \textit{and} \quad x.a = 0,$$

for all $a$ in $A$ and $x$ in $X$. Then observe that identity map $I$ is a derivation. Since $A$ is contractible there exists some $x$ in $X$ such that for all $a$ in $A$

$$a = I(a) = a.x - x.a = ax,$$

hence $x$ acts as right unit for $A$.
Similarly if we define left and right actions as

$$a.x = 0 \quad \textit{and} \quad x.a = xa$$

we get left unit $y$ for $A$. Define $1 := x + y - xy$ This is unit for $A$. □

We see that the definition of contractible algebra is very restrictive so we have the definition of amenable algebra as follows which gives very interesting theory.

**Definition 3.1.7.** *Let $A$ be an Banach algebra. Then $A$ is called amenable if*

$$H^1(A, X^*) = \{0\}$$

*for every Banach $A-$bi module $X$. That is if every bounded $X^*$ derivation is inner.*

As here we defined first cohomology group similarly we can define $n-$cohomology group $H^n(A, X)$. There we have very interesting reduction formulas and how the amenability of algebra is related with higher order cohomology group. Refer [Runde] (section 2.4) for this and following theorem.

**Theorem 3.1.8.** *For a Banach algebra following are equivalent,*

1. *$A$ is amenable (respectively contractible)*
2. *$H^n(A, X^*) = \{0\}$ (respectively $H^n(A, X) = \{0\}$) for all Banach $A$-bi module $X$ and for all $n \in \mathbb{N}$.*

## 3.2 Characterisation of Contractible Banach Algebra

As we promised, now we will discuss some characterisation of contractible Banach algebra. First recall some concepts from section 1.3.2, as definition of projective tensor norm and completion of space with respect to this norm.

**Definition 3.2.1.** *Let $A$ be a Banach algebra. The diagonal operator $\pi$ on $A$ is the bounded linear operator defined by the linear extension of*

$$\pi : A\otimes_p A \to A \quad x \otimes y = xy$$

$$\sum_{i=1}^{\infty} x_i \otimes y_i \mapsto \sum_{i=1}^{\infty} x_i y_i$$

**Remark 3.2.2.** *Let $A$ be Banach algebra and $X = A \otimes_p A$ then $X$ is Banach $A$-bi module under the following actions,*

$$a.(x \otimes y) = ax \otimes y \quad \textit{and} \quad (x \otimes y).a = x \otimes ya$$

*where $a, x$ and $y \in A$*

**Definition 3.2.3.** *An element $u$ in $A \otimes_p A$ is called projective diagonal for $A$ if for all $a$ in $A$*

$$a\pi(u) = a \quad \textit{and} \quad a.u - u.a = 0.$$

**Remark 3.2.4.**

1. *If $A$ has a projective diagonal then $A$ is unital. Indeed $\pi(u) = e_A$, as*

$$\pi(u)a = \pi(ua) = \pi(au) = a\pi(u) = a$$

2. *Projective diagonal $u$ for $A$ is an element of $A \otimes_p A$ such that there exists bounded sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ in $A$ and $u = \sum_i x_i \otimes y_i$. Moreover $\sum_i ||x_i||||y_i|| < \infty$, $\sum_i x_i y_i = e_A$ and $\sum_i ax_i \otimes y_i = \sum_i x_i \otimes y_i a$ for all $a$ in $A$.*

3. *$\pi$ is bounded and so continuous, we observe that diagonal operator is a bounded module homomorphism with respect to canonical $A-$bi module structure on $A \otimes_p A$.*

4. *$\ker \pi$ is closed submodule of $A \otimes_p A$ and hence Banach $A-$bi module.*

We already observe a nice property of contractible Banach algebra that it has unity, now the following proposition tells that every contractible Banach algebra possesses projective diagonal.

**Proposition 3.2.5.** *Let $A$ be unital Banach algebra with unit $1$, and $\pi$ the diagonal operator on $A$. If $H * 1(A,\ ker\ \pi) = \{0\}$, then $A$ has a projective diagonal.*

*Proof.* As we have that $H*1(A,\ ker\ \pi)=\{0\}$, so we will first try to find out a derivation on $A$ to $\ker\pi$ that will be inner so from there we will get some $x$ in $\ker\pi$. With the help of that $x$ we will try to extract our result.
Let $D: A\to ker\ \pi$, defined as $D(a)=a\otimes 1-1\otimes a$. It is evident that $D$ is linear for boundedness observe that

$$||D(a)||=||a\otimes 1-1\otimes a||\leq ||a||||1||+||1||||a||=2||a||||1||.$$

Let $a,b\in A$ then we have

$$D(ab)=ab\otimes 1-1\otimes ab=ab\otimes 1-a\otimes b+a\otimes b-1\otimes ab$$

$= a.(b\otimes 1-1\otimes b)+(a\otimes 1-1\otimes a).b = a.D(b)+D(a).b$ Hence $D\in Z^1(A,ker\ \pi)$, but as we have $H^1(A,ker\ \pi)=\{0\}$ this gives that there exists some $x$ in $\ker\pi$ such that for all $a$ in $A$,

$$a\otimes 1-1\otimes a=a.x-x.a$$

which gives that for all $a\in A$

$$a.(1\otimes 1-x)-(1\otimes 1-x).a=0$$

Furthermore by using linearity of $\pi$ we have

$$\pi(1\otimes 1-x)=\pi(1\otimes 1)-\pi(x)=1^2-0=1.$$

So we can conclude that $1\otimes 1-x$ is the projective diagonal for $A$. □

**Remark 3.2.6.** *Let $A$ be a Banach algebra and $X$ a Banach $A-bi$ module. If $S$ is a bounded linear map from $A$ to $X$, we can define bounded left module homomorphism $T\in\mathcal{L}(A\otimes_p A,X)$ as the linear extension of*

$$T(a\otimes b)=a.S(b)\quad (a,b\in A).$$

*Now the following are some observation,*

1. *$T$ is well defined as,*

$$T(a\otimes\lambda b)=a.S(\lambda b)=a.\lambda S(b)=\lambda a.S(b)=T(\lambda a\otimes b).$$

2. *$T$ is bounded as*

$$||T(a\otimes b)||=||a.S(b)||\leq k||a||\ ||S(b)||\leq k||S||\ ||a||\ ||b||.$$

3. *For all $a \in A$ and $u \in A \otimes_p A$ we also have*

$$a.T(u) = T(a.u)$$

Here comes the conclusive part of this section. We observed that every contractible Banach algebra has unity and also possesses projective diagonal, is the reveres true. The following theorem answers it in yes.

**Theorem 3.2.7.** *Let $A$ be a Banach algebra. Then the following are equivalent,*

1. *$A$ is contractible*
2. *$A$ is unital and possesses projective diagonal.*

*Proof.* We have already proved that (1) $\Rightarrow$ (2). See proposition 3.1.6 and proposition 3.2.5. For other side, Let $X$ be Banach $A-$bimodule and let $u = \sum_{i=1}^{\infty}$ be a projective diagonal for $A$. Let $D \in Z^1(A, X)$. Since

$$D(a) = D(1.a) = 1.D(a) - D(1).a$$

So now we will show that $a \mapsto 1.D(a) \quad and \quad a \mapsto D(1).a$ both are inner derivation and as $B^1(A, X)$ is linear space ti will imply that $D$ is inner derivation.
Define $T$ as the linear extension of $T(a \otimes b) = a.D(b)$, since $D$ is bounded linear map so from remark 3.2.6 it follows that $T$ is also bounded linear map. Now as for any $a$ and $b$ in $A$ $b.D(a) = D(ba) - D(b).a$, so by linearity and continuity of $\pi$ we get,

$$\begin{aligned}
1.D(a) &= \pi(u).D(a) \\
&= \lim_n \sum_{i=1}^{n} a_i b_i.D(a) \\
&= \lim_n \sum_{i=1}^{n} a_i(D(b_i a) - D(b_i).a) \\
&= \lim_n (T(\sum_{i=1}^{n} a_i \otimes b_i a) - T(\sum_{i=1}^{n} a_i \otimes b_i).a) \\
&= T(u.a) - T(u).a \\
&= T(a.u) - T(u).a \\
&= a.T(u) - T(u).a
\end{aligned}$$

This gives that $a \mapsto 1.D(a)$ is inner derivation.
Again since $D$ is a derivation,

$$1.D(1).a = 1.D(1.a) - 1.D(a) = 1.D(a) - 1.D(a) = 0$$

$$\begin{aligned} D(1).a &= a.D(1) - a.D(1) - 1.D(1).a + D(1).a \\ &= a.D(1.D(1) - D(1)) - (1.D(1) - D(1)).a \end{aligned}$$

Hence we get that both of our maps are inner. Hence proved that $A$ is contractible. □

## 3.3 Characterisation of Contractible Algebra

In previous section we characterise the contractible Banach algebra, now if we have algebra which is not so much structured i.e. let say not Banach so what are the characterisations. We will try to look closer into this that how a contractible algebra and semi-simple algebra are related, actually we will prove with the help of Wedderburn structure theorem that contractible algebra are semi-simple.
In this section we work with algebras without any topology on it. All the definition will be same except for diagonal operator which is now defined from $A \otimes A$ to $A$ instead of $A \otimes_p A$ to $A$,

$$\pi : A \otimes A \to A$$

as the linear extension of $a \otimes b = ab$. Theorem 3.2.7 holds true in the case so contractble algebras also. We can prove it without limiting process.
Now we will define semi-simple algebras and then after looking at a basic property of semi-simple algebra, we will look into the center piece of this section that tells that every contractible algebra is finite dimensional and semi-simple.

**Definition 3.3.1.** *Let $A$ be a unital algebra with unit $1$. A proper left ideal $I$ of $A$ is called left modular if there exists $a \in A$ such that $A(1 - a) \subset I$. The intersection of all maximal left modular ideals in $A$, denoted by rad $A$, is also a left ideal and is called the radical of $A$. If rad $A = \{0\}$ then $A$ is called semi-simple.*

From properness of $I$ it follows that $a$ does not belong to $I$. The following lemma says that there is no non trivial idempotent element in semi simple algebra.

**Lemma 3.3.2.** *Let $A$ be a unital algebra and $a \in rad\ A$. If $a^2 = a$ then $a = 0$.*

*Proof.* Let $a \in A$ such that $a^2 = a$. Let $I = A(1 - a)$ suppose $a \notin I$. then $I \neq A$ and thus $I$ is left modular ideal. Let $M$ be a maximal left modular ideal such that $I \subset M \quad and \quad a /$ $]inM$(Such a maximal ideal exists by Zorn's lemma). This implies that $a \notin rad\ A$ which is contradiction. Hence $a \in I$ and which further implies that there exists $b \in A$ such that $a = b(1 - a)$. So

$$a = a^2 = b(1 - a)a = b(a - a^2) = b.0 = 0.$$

□

Now we will state theorem due to J. Wedderburn. For a proof see theorem 1.5.9 of [Dales].

**Theorem 3.3.3.** *Let $A$ be a non empty semi simple unital and finite dimensional algebra. Then there exists $n_1, ..., n_k \in \mathbb{N}$ such that*

$$A = M_{n_1} \oplus M_{n_2} \oplus \oplus, ..., \oplus M_{n_k}$$

Now let conclude this section by following theorem.

**Theorem 3.3.4.** *Let $A$ be an algebra. Then the following are equivalent,*

1. *$A$ is contractible.*
2. *$A$ is unital and has a diagonal.*
3. *$A$ is semi simple and finite dimensional.*

*Proof.* The equivalence of (i) and (ii) has already proven in proposition 3.2.7.
$(ii) \implies (iii)$: Let $m = \sum_{i=1}^{n} a_i \otimes b_i$ be a diagonal for $A$, and define

$$B := span\{a_i b_j : 1 \leq i, j \leq n\}.$$

**Claim-1:** $A = B$ and thus $A$ is finite dimensional.
Let $\phi$ be linear projection of $A$ onto $span\{b_1, ..., b_n\} \subset A$. Now define $T : A \otimes A \to A$ as the linear extension of

$$T(a \otimes b) = a.\phi(b).$$

Now for $a \in A$

$$\begin{aligned} a = a\pi(m) = a \sum a_i b_i &= a \sum a_i \phi(b_i) \\ &= \sum a a_i \phi(b_i) = \sum T(a a_i \otimes b_i) \\ &= T(a \sum a_i \otimes b_i) = T(a.m) \\ &= T(m.a) = \sum T(a_i \otimes b_i a) \\ &= \sum a_i \phi(b_i a) \in B. \end{aligned}$$

This implies that actually $A \subset B$. Obviously $B \subset A$ giving that $A = B$. Hence $A$ is finite dimensional algebra.
**Claim-2:** $A$ is semi simple.
Let $psi$ be linear projection of $A$ onto rad $A$ and define $T$ be the linear extension of

$$T(a \otimes b) = a.\psi(b).$$

Now define $S : a \mapsto T(m.a) \quad (a \in A)$ which is again a linear projection of $A$ onto rad $A$. Since rad $A$ is left ideal we have the following,

$$S(a) = T(m.a) = \sum_{i=1}^{n} T(a_i \otimes b_i a) = \sum_{i=1}^{n} a_i \psi(b_i a) = \sum_{i=1}^{n} a_i b_i a = \pi(m)a = a.$$

Observe that for all $i \in \mathbb{N}$ we have

$$a_i\psi(b_i)(\sum_{j=1}^{n} a_j\psi(b_j))$$

indeed by using that $m = \sum_{i=1}^{n} a_i \otimes b_i$ is projective diagonal and rad $A$ is left ideal, we have that

$$\begin{aligned} a_j\psi(b_j)(\sum a_i\psi(b_i)) &= \sum a_j\psi(b_j)a_i\psi(b_i) \\ &= T(\sum a_j\psi(b_j)a_i \otimes b_i) = T(a_j\psi(b_j).m) \\ &= T(m.a_j\psi(b_j)) = \sum a_i\psi(b_ia_j\psi(b_j)) \\ &= \sum a_ib_ia_j\psi(b_j) = \pi(m)a_j\psi(b_j) \\ &= a_j\psi(b_j) \end{aligned}$$

Hence,

$$\begin{aligned} (T(m))^2 &= (\sum a_i\psi(b_i))^2 \\ &= a_1\psi(b_1)(\sum a_i\psi(b_i)) + ... + a_n\psi(b_n)(\sum a_i\psi(b_i)) \\ &= a_1\psi(b_1) + ... + a_n\psi(b_n) \\ &= T(m) \end{aligned}$$

Which implies that $T(m)$ is idempotent but from lemma 3.3.2 we have $T(m) = 0$. Consequently, we have

$$S(a) = T(m.a) = T(a.m) = a.T(m) = 0. \quad (a \in A)$$

but as $S(a) = a \implies a = 0$. This gives rad $A = \{0\}$. So $A$ is semi simple.
$(iii) \implies (ii)$ : By theorem 3.3.3 there exists $n_1, ..., n_k$ such that

$$A = M_{n_1} \oplus +... + M_{n_k}$$

Now take $\epsilon_{i,j} \in M_{n_r}$ such that

$$\epsilon_{i,j}(k,l) = \begin{cases} 1 & \text{if } (k,l) = (i,j) \\ 0 & \text{if } (k,l) \neq (i,j) \end{cases}$$

Then $m_r = \sum_{i,j=1}^{n} \epsilon_{i,1} \otimes \epsilon_{1,j}$ is diagonal for $M_{n_r}$. So for $A$ take $u = \oplus_{i=1}^{k} m_i$ is diagonal. $\square$

# Chapter 4

# Bounded Approximate Identity

## 4.1 Bounded Approximate identities for Banach Algebra

As we observed that every contractible Banach algebra is unital. But such a nice property can not be expect from arbitrary algebras, so what is next best is so called approximate identity that we can expect. It is defined as,

**Definition 4.1.1.** *Let $A$ be normed algebra. A left approximate identity for $A$ is a net $(e_\alpha)_\alpha$ in $A$ such that for each $a$ in $A$ $(ae_\alpha)$ converges to $a$. Similarly we define right approximate identity, a net $(e_\alpha)$ is called right approximate identity if for all $a$ in $A$ $(e_\alpha a)$ converges to $a$.*
*An approximate identity for an algebra $A$ is a net $(e_\alpha)\alpha \in \lambda$ which is both right and left approximate identity. An approximate identity is said to be bounded if there exists some positive real number $M$ such that $||e_\alpha|| \leq M$ for all $\alpha \in \lambda$*

**Remark 4.1.2.** *Let $X$ be a Banach $A$-bimodule and suppose that $A$ has a bounded left approximate identity $(e_\alpha)_{\alpha\in\lambda}$. If for all $x$ in $X$ we have that*

$$\lim_\alpha e_\alpha.x = x,$$

*then $(e_\alpha)$ is called bounded approximate identity for $X$. Similarly we define bounded approximate identity for $X$ and again an approximate identity which is both right and left is called approximate identity for $X$. We will denote the $m-$sphere as*

$$b_m(A := \{x \in A : ||x|| \leq m\})$$

**Example 4.1.3.** *1. For sequence space $l^1$, it is easy to check that the sequence $(x_n)$, where $x_n = (1, 1, 1, 1, ..., 1(n^{th}\ place), 0, 0, ...)$ is an approximate identity.*

Suppose if we have bounded left approximate identity an bounded right approximate identity so can we get approximate identity, the following remark answers it.

**Remark 4.1.4.** *Let $A$ be a Banach algebra. Suppose $(e_\alpha)_{\alpha \in I}$ is a bounded left approximate identity for $A$ and that $(f_\beta)_{\beta \in J}$ is a bonded right approximate identity for $A$ with bounds $M$ and $N$ respectively. Now define the relation on $I \times J$ as*

$$(\alpha, \beta) \leq (\alpha', \beta') \iff \alpha \leq \alpha' \text{ and } \beta \leq \beta'$$

*Then $I \times J$ is a directed set. Define*

$$g_{(\alpha.\beta)} = e_\alpha + f_\beta - f_\beta e_\alpha,$$

*now observe the following , that for all $a$ in $A$,*

$$||g_{(\alpha,\beta)a-a}|| \leq ||e_\alpha - a|| + ||f_\beta(a\ell_\alpha a)|| \leq ||e_\alpha a - a|| + N||a - e_\alpha a|| \to 0.$$

*Similarly for left approximate identity. Moreover it is bounded as,*

$$||g_{(\alpha,\beta)||\leq||e_\alpha||+||f_\beta||+||f_\beta e_\alpha||\leq M+N+MN},$$

*this implies that the net $(g_{(\alpha,\beta)})_{(\alpha,\beta)\in I\times J}$ is an approximate identity for $A$.*

**Definition 4.1.5.** *Let $A$ be a Banach algebra and let $m \geq 1$. Suppose that for each $a \in A$ and $\epsilon > 0$ there exists $u$ with $||u|| \leq m$ such that $||a - ua|| < \epsilon$. Then $A$ is said to have left approximate units of bound $m$. In a similar fashion we define right approximate units of bound $m$.*

Clearly if $A$ has a left (respectively right) approximate identity of bound $m$ then it has left (respectively right) approximate units of bound $m$. In fact, the converse is also true.

**Proposition 4.1.6.** *Let $A$ be a Banach algebra with left (respectively right) approximate units of bound $m \geq 1$. Then $A$ has a bounded left (resp. right) approximate identity of bound $m$.*

**Note 4.1.7.** *We defined the approximate identity where the convergence of net is norm convergence. If there exists a net $(e_\alpha)_{\alpha\in\lambda}$ in $A$ such that for all $a$ in $A$ the convergence of $(ae_\alpha)$ to $a$ is in weak sense, i.e. if for all $a \in A$*

$$f(ae_\alpha) \to f(a) \quad (\forall\ f \in A^*),$$

*then we say that $(e_\alpha)$ is weak left approximate identity. Similarly we define weak right approximate identity. A net which is both left and right weak approximate identity is called weak approximate identity.*

The following result bridge the gap between the bounded approximate identity and bounded weak approximate identity using Mazur's theorem A.0.25.

**Proposition 4.1.8.** *Let $A$ be a Banach algebra and suppose that $(e_\alpha)_{\alpha\in\lambda}$ is a weak left (resp. right) identity for $A$, then there exists a bounded left (resp. right) approximate identity for $A$.*

*Proof.* Let $(e_\alpha)$ be a bounded weak left approximate identity for $A$. Let $m = sup_\alpha ||e_\alpha||$ and $a \in A$. As each $e_\alpha \in b_m(A)$ we get that $a \in \overline{b_m(A).a}^w$, i.e. weak closure. It is easy to check by definition that $b_m(A).a$ is convex set so by Mazur's theorem A.0.25 we have that $a \in \overline{b_m(A).a}^{||.||}$ i.e. norm closure. By definition of closure only we get that there exists left approximate units of bounds $m$. So from proposition 4.1.6, $A$ has a bounded left approximate identity. Similarly for right approximate identity.

□

Contractible algebras are unital and amenable algebras possesses bounded approximate identity. This is proved in following theorem.

**Theorem 4.1.9.** *Let $A$ be amenable Banach algebra. Then $A$ has a bounded approximate identity.*

*Proof.* Let $X = A$ as a Banach space then for $a \in A$ and $x \in X$ define left and right module action as,

$$a.x = ax \quad and x.a = 0$$

So, we have $X^*$ and $X^{**}$ are Banach $A-$bimodule. Now the canonical map

$$J : A \to X^{**}$$

is a derivation. Indeed

$$\begin{aligned}(J(ab))(f) &= f(ab) = f(a.b)\\ &= (f.a)(b) = (J(b))(f.a)\\ &= (a.J(b))(f)\\ &= (a.J(b) + J(a).b)(f) \quad (as\ X^{**}\ has\ Zero\ right\ action)\end{aligned}$$

Since $A$ is amenable, there exists $\phi \in X^*{*}$ such that

$$J(a) = a.\phi - \phi.a = a.\phi$$

By Goldstein's theorem A.0.29 there exists a bounded net $(e_\alpha)$ in $X$ such that $J(e_\alpha) \xrightarrow{w^*} \phi$. This implies that for all $f \in X^*$

$$||(J(e_\alpha)f - \phi(f)|| \longrightarrow 0 \quad (\forall\ f \in X^*)$$

In particular for $a \in A$

$$||(J(e_\alpha))(fa) - \phi(fa)|| \longrightarrow 0.$$
$$||(J(a.e_\alpha))f - (a.\phi)f|| \longrightarrow 0.$$

This implies that for all $a \in A$, $J(a.e_\alpha) \xrightarrow{w^*} a.\phi$. So from above discussion actually it becomes $J(a.e_\alpha) \xrightarrow{w^*} J(a)$ which further give that $a.e_\alpha \xrightarrow{w} a$. Hence $e_\alpha$ is bounded weak right approximate identity for $A$. So from proposition 4.1.8 we get bounded approximate identity for $A$.
Similar observation with module action defined as

$$a.x = 0 \quad and \quad x.a = xa \quad (a \in A,\ x \in X)$$

we get bounded left approximate identity for $A$ and from remark 4.1.4 we get bounded approximate identity for $A$.

□

## 4.2 Bounded Approximate identity for Group Algebra

As we can see easily that for finite locally compact group $G$ has identity as

$$f(x) = \begin{cases} 1 & \text{if } x = e \\ 0 & \text{otherwise} \end{cases}$$

but we can not expect this for infinite group as it turns out to be that $f \sim 0$, hence for all $g \in L^1(G)$, $f * g = 0$. Therefore we will depend on approximate identity. Infact we will observe that for every locally compact group, $L^1(G)$ possesses approximate identity. The following proposition give us the desired result.

**Proposition 4.2.1.** *Let $G$ be locally compact group with left Haar measure $m_G$ and let $\mathcal{U}$ be a collection of neighborhood of $e$ directed by the inverse inclusion relation, that is,*

$$U \leq V \iff V \subset U.$$

*Denote $\mathcal{P}(G) := \{f \in L^1(G) : f \geq 0 \text{ and } ||f||_1 = 1\}$. Let $(e_U)_{U\in\mathcal{U}}$ be a net in $\mathcal{P}(G)$ such that for each $U \in \mathcal{U}$ the support of $e_U$ is compact and contained in $U$ also for all $x \in G$ $e_U(x) = e_U(x^{-1})$. Then $(e_U)_{U\in\mathcal{U}}$ is a bounded approximate identity for $L^1(G)$.*

*Proof.* Let $U \in \mathcal{U}$. Now as $e_U \in \mathcal{P}(G)$ it implies that $\int_G e_U \; d(m_G) = 1$, so by using this we have following,

$$\begin{aligned} f * e_U(x) - f(x) &= \int_G f(xy)e_U(y^{-1})m_G(dy) - f(x)\int_U e_U(y)\; m_G(dy) \\ &= \int_G f(xy)e_U(y)\; m_G(dy) - \int_G f(x)e_U(y)\; m_G(dy) \quad (as\ e_U(y^{-1}) = e_U(y) \\ &= \int_G (R_y f(x) - f(x))e_U(y)\; m_G(dy) \end{aligned}$$

Now by using Fubini's theorem,

$$\begin{aligned}
||f * e_U - f||_1 &= \int_G \left| \int_G (R_y f(x) - f(x)) e_U \; m_G(dy) \right| \; m_G(dx) \\
&\leq \int_G \int_G |R_y f(x) - f(x)| e_U(y) \; m_G(dy) \; m_G(dx) \\
&= \int_G ||R_y f - f||_1 e_U(y) \; m_G(dy) \\
&\leq \sup_{y \in U} ||R_y f - f||_1 \int_G e_U(y) \; m_G(dy) \\
&= \sup_{y \in U} ||R_y f - ff||_1
\end{aligned}$$

As from proposition 2.2.18, we get that for appropriate neighborhood of $e$

$$\sup_{y \in U} ||R_y f - f||_1 < \epsilon.$$

Hence we get that

$$||f * e_U - f||_1 < \epsilon$$

Thus $(e_U)$ is bounded right approximate identity. Similarly for left approximate identity,

$$e_U * f(x) - f(x) = \int (L_y f(x) - f(x)) e_U(y) \; m_G(dy)$$

and by similar argument as above we obtain

$$||e_U * f - f||_1 \leq \sup_{y \in U} ||L_y f - f||_1 < \epsilon$$

Hence $(e_U)_{U \in \mathcal{U}}$ is bounded approximate identity for $L^1(G)$. □

**Remark 4.2.2.** *By the properties of locally compact group (proposition 2.1.3)Then for $U \in \mathcal{U}$ we can define $e_U$ to be as following*

$$e_U = m_G(U)^{-1} \chi_U$$

*As $m_G$ is left Haar measure so $m_G(U) > 0$. Here $\mathcal{U}$ is collection of all compact and symmetric neighborhood of $e$.*

**Proposition 4.2.3.** *Let $I$ be a closed linear subspace of $L^1(G)$ such that for all $x$ in $G$, $L_x f \in I$. Then $I$ is left ideal in $L^1(G)$*

*Proof.* As we know that $L^\infty(G) \equiv (L^1(G))^*$ so for $\phi \in L^\infty(G)$ and $f \in L^1(G)$.

$$< f, \phi >= \int f(x)\phi(x)\ m_G(dx).$$

Let $\phi \in I^\perp$ which is contained in $L^\infty(G)$, then define $\check{\phi} \in L^\infty(G)$ as,

$$\check{\phi}(x) = \phi(x^{-1}) \quad (x \in G)$$

Observe the following

$$\begin{aligned}(f * \check{\phi})(x) &= \int f(y)\phi(y^{-1}x)\ d(m_G) = \int f(xy)\phi(y^{-1})d(m_G) \\ &= \int_G L_x f(y)\phi(y)\ m_G(dy) \\ &= 0.\end{aligned}$$

Last equality holds as $L_x f \in I$ and $\phi \in I^\perp$. Now

$$\begin{aligned}\int_G (g * f)(y)\phi(y)\ m_G(dy) &= \int_G \left(\int_G g(x)f(x^{-1}y)\ m_G(dx)\right)\check{\phi}(y^{-1})\ m_G(dy) \\ &= \int_G g(x)\left(\int_G f(x^{-1})\check{\phi}(y^{-1})\ m_G(dy)\right)\ m_G(dx) \\ &= \int_G g(x)(f * \check{\phi})(x^{-1})\ m_G(dx) = 0.\end{aligned}$$

This implies that for all $f \in I$ and $g \in L^1(G)$, $g * f \in^\perp (I^\perp)$ (refer definition A.0.5) Now as $I$ is closed linear subspace hence $^\perp(I \perp) = I$. Thus $I$ is closed ideal in $L^1(G)$.

□

**Corollary 4.2.4.** *Let $I$ be a closed linear subspace of $L^1(G)$ such that $L_x f \in I$ for all $x \in G$ and $f \in I$. Then $I$ is closed left ideal in $M(G)$.*

*Proof.* Let $(e_\alpha)_{\alpha\in\lambda}$ be a bounded approximate identity for $L^1(G)$. Now from proposition 4.2.3 we know that $I$ is closed left ideal in $L^1(G)$ and as $L^1(G)$ itself is an ideal in $M(G)$. So for all $\mu \in M(G)$, $f \in I$ and $\alpha \in \lambda$ we get that,

$$(\mu * e_\alpha) * f = \mu * (e_\alpha * f) \longrightarrow \mu * f.$$

hence $(\mu * e_\alpha) * f \in I$. Since $I$ is closed in $L^1(G)$ so also closed in $M(G)$ giving us our desired result. That is $\mu * f \in I$ fro all $f \in I$ *and* $\mu \in M(G)$. So we get that $I$ is closed left ideal in $M(G)$. □

We defined what does it means by strict topology (remark 2.3.15) and also what is Dirac measure (point measure). The interesting point is that the linear span of collection of point measure relative to a locally compact group are dense in measure algebra. For a proof see [Henrik].

**Theorem 4.2.5.** *The linear span of* $\{\delta_x : x \in G\}$ *is strictly dense in* $M(G)$.

Now we will state the Cohen-Hewitt factorisation theorem.

**Theorem 4.2.6.** *(Cohen-Hewitt) Let* $A$ *be a Banach algebra with a bounded left (respectively right) approximate identity* $(e_\alpha)_{\alpha\in\lambda}$ *with bound* $m \geq 1$*, and let* $X$ *be a Banach* $A$*-bimodule. Then* $A.X$ *(respectively* $X.A$*) is a closed submodule of* $X$*.*

**Corollary 4.2.7.** *(Cohen's factorization theorem) Let* $A$ *be a Banach algebra and* $X$ *a Banach* $A$*-bimodule and suppose that* $(e_\alpha)_{\alpha\in\lambda}$ *is a bounded left approximate identity for* $X$*. Then* $A.X = X$*. In particular* $A.A = A$

## 4.3 Pseudo-Unital Banach Algebra

We stated the cohen Hewitt factorisation theorem which gives us the factorisation for $A$. To get this we define following structure.

**Definition 4.3.1.** *Let* $A$ *be a Banach Algebra and* $X$ *be a Banach* $A$*-bimodule. If* $X = A.X.A$*, then we say that* $X$ *is pseudo-unital.*

**Remark 4.3.2.** *From the definition of pseudo unital, it is clear that if* $(e_\alpha)_{\alpha\in\lambda}$ *is approximate unit for* $A$ *then it is approximate unit for* $X$ *as well. The reason is, let* $x \in X$*, then there exists* $a, b \in A$ *and* $y \in Y$ *such that* $x = a.y.b$*. Now,* $\lim_\alpha e_\alpha x = \lim_\alpha e_\alpha a.y.b = (\lim_\alpha e_\alpha a).y.b = a.y.b = x$
*In particular, if* $1_A$ *is unity of* $A$*, then* $1_A.x = x = x.1_A$ *for all* $x \in X$*.*

**Proposition 4.3.3.** *Let* $A$ *and* $B$ *be Banach Algebra such that* $A$ *is closed ideal of* $B$ *and* $X$ *be pseudo-unital Banach* $A$*-bimodule. If* $A$ *has a bounded approximate identity then* $X$ *is a Banach* $B$*-bimodule in a canonical way.*

*Proof.* Let $(e_\alpha)_\alpha$ be a bounded approximate identity for $A$. Now, as $X$ is a Banach pseudo $A$-bimodule, for any $x \in X$, there will exist $a, b \in A$ and $y, z \in Y$ be such that $x = a.z.b$. As $X$ is module let $z.b = y$, which is in $X$. So, $x = a.y$. Now for $b \in B$, define $B \times X \to X$ by

$$b.x = ba.y$$

It is well-defined since,

$$ba'.y' = (\lim_\alpha be_\alpha).a'.y' = (\lim_\alpha be_\alpha).(a.y) = ba.y.$$

Since $A$ is an ideal, this will imply that $b.x \in A.X$ for all $b \in B$ and since, $X$ is a $A$-bimodule, it follows that $X$ is a left $B$-module. Similarly define $X \times B \to X$ by

$$x.b = y.a'b' \qquad \text{for } x = y.a'$$

This implies that $X$ is a Banach $B$-bimodule.

□

**Proposition 4.3.4.** *Let $A$ and $B$ be Banach Algebras such that $A$ has a bounded approximate identity and $A$ is closed ideal of $B$ and let $X$ be pseuso-unital Banach $A$-bimodule. Let $D \in Z^1(A, X^*)$. Then there is a unique extension $\tilde{D} \in Z^1(B, X^*)$ of $D$ such that*

1. $\tilde{D}|_A = D$
2. $\tilde{D}$ *is continuous with respect to strict topology on $B$ and with respect to $w^*$ topology on $X^*$.*

*Proof.* Let $(e_\alpha)$ be an approximate identity on $A$ and let $x \in X$ be arbitrary. Since $X$ is pseudo unital from remark 4.3.2 we get the following,

$$D(a)(x) = \lim_\alpha D(a)(e_\alpha.x) = \lim_\alpha (D(a).e_\alpha)(x) = \lim_\alpha (D(ae_\alpha) - a.D(e_\alpha))(x)$$

and thus we get that,

$$D(a) = w^* - \lim_\alpha (D(ae_\alpha) - a.D(e_\alpha)) \quad (a \in A) \tag{4.1}$$

Let $b \in B$. Since $X$ is pseudo unital, we can write $x \in X$ as $x = y.a$ where $a \in A$ and $y \in X$. So from continuity of bimodule action it follows that

$$\begin{aligned} (D(be_\alpha) - b.D(e_\alpha))(x) &= (D(be_\alpha) - b.D(e_\alpha))(y.a) = (a.D(be_\alpha) - ab.D(e_\alpha))(y) \\ &= (D(abe_\alpha) - D(a).be_\alpha - D(abe_\alpha) + D(ab).e_\alpha)(y) \\ &= D(ab(e_\alpha.y)) - D(a)(be_\alpha.y) \\ &\to D(ab)(y) - D(a)(b.y) \end{aligned} \tag{3.2}$$

Since the limit of a net is unique in $\mathbb{C}$, the calculations above do not depend on the factorisation of $x$. Hence $\tilde{D} : B \to X^*$ defined by

$$\tilde{D}(b)(x) = \lim_\alpha (D(be_\alpha)(x) - b.D(e_\alpha))(x) \quad (x \in X)$$

is well defines and by (3.2) if $x = y.a$ where $a \in A,\ y \in X$ we have

$$(\tilde{D}(b))(x) = (D(ab))(y) - (D(a))(b.y) \quad (b \in B)$$

We now show that $\tilde{D} \in Z^1(B, X^*)$. For this, let $b, c \in B$ and let $x = y.a$ be as above. Thus we have

$$\begin{aligned}
(b.\tilde{D}(c) + \tilde{D}(b).c)(x) &= (b.\tilde{D}(c) + \tilde{D}(b).c)(y.a) \\
&= \tilde{D}(c)(y.ab) + (\tilde{D}(b))(c.y.a) \\
&= D(abc)(y) - D(ab)(c.y) + D(ab)(c.y) - D(a)(bc.y) \\
&= D(abc)(y) - D(a)(bc.y) \\
&= \tilde{D}(bc)(x)
\end{aligned}$$

The linearity od $\tilde{D}$ follows easily from the linearity of $D$ and the module operation. Finally let $m \geq 1$ be a bound for $(e_\alpha)_{\alpha \in \lambda}$. Then for any $b \in B,\ x \in X$ we have

$$\begin{aligned}
|\tilde{D}(b(x))| &= \lim_\alpha |(D(be_\alpha) - b.D(e_\alpha))(x) \leq \lim_\alpha (|D(be_\alpha)(x)| + |b.D(e_\alpha)(x)|) \\
&\leq (k+1)m||x||\ ||D||\ ||b|| < \infty
\end{aligned}$$

and thus $\tilde{D}$ is bounded and we conclude that $\tilde{D} \in Z^1(B, X^*)$. Also, by (3.1), $\tilde{D}|_A = D$. To show the desired continuity, let $(b_\alpha)_\alpha$ be a net in $B$ such that $b_\alpha \to b$ for some $b \in B$ in the strict topology on $B$ with respect to $A$, that is,

$$||a(b_\alpha - b)|| + ||(b_\alpha - b)a|| \to 0$$

for all $a \in A$. Since $X$ is pseudo unital we get by writing any $y \in X$ as $y = a_1.y_1$, where $a_1 \in A$ and $y_1 \in X$, that

$$||(b_\alpha - b).y|| = ||(b_\alpha - b)a_1.y_1|| \leq k||(b_\alpha - b)a_1||\ ||y_1|| \to 0.$$

Hence by writing $x = y.a$ we obtain by (3.3) that

$$\begin{aligned}
|(\tilde{D}(b_\alpha) - \tilde{D}(b))(x)| &= |(D(ab_\alpha) - D(ab))(y) - D(a)((b_\alpha - b).y)| \\
&\leq |D(a(b_\alpha - b))(y)| + |D(a)((b_\alpha - b).y)| \\
&\leq ||y||\ ||D(a(b_\alpha - b))|| + ||(b_\alpha - b).y||\ ||D(a)|| \to 0.
\end{aligned}$$

by continuity of $D$. Thus $\tilde{D}$ is continuous with respect to the strict topology in $B$ and the $w^*-$topology on $X^*$.

□

**Proposition 4.3.5.** *Let $A$ be a Banach Algebra with bounded approximate identity and let $X$ be a Banach $A$-bimodule with a trivial left or right module action. Then $H^1(A, X^*) = 0$.*

*Proof.* Let $A.X = 0$, which implies that $X^*.A = 0$. Now, for $D \in Z^1(A, X^*)$, we have from module action $D(ab) = a.D(b)$. If $(e_\alpha)_\alpha$ be a bounded approximate identity for $A$, then $(De_\alpha)_\alpha$ is a bounded net in $X^*$. So, by Banach Alaoglu theorem A.0.28, there exists a $w^*$-accumulation point, $f$ of $(De_\alpha)_\alpha$.

$$w^* - \lim_\alpha(D(e_\alpha)) = f$$

Since any subnet of $(e_\alpha)_\alpha$ is also a bounded approximate identity for $A$. So, for every $a \in A$,

$$\begin{aligned} D(a)(x) &= \lim_\alpha [D(ae_\alpha)](x) = \lim_\alpha [a.D(e_\alpha)](x) \\ &= \lim_\alpha [D(e_\alpha)](x.a) = f(x.a) \\ &= (a.f)(x) = (a.f - f.a)(x) \end{aligned}$$

Hence, $D \in B^1(A, X^*)$ and therefore $H^1(A, X^*) = 0$. □

For proof of the following refer [Henrik].

**Proposition 4.3.6.** *Let $A$ be a Banach Algebra with bounded approximate identity, $(e_\alpha)_\alpha$. Suppose that $H^1(A, X^*) = 0$ for each pseudo unital Banach $A$-bimodule, $X$. Then $A$ is amenable.*

## 4.4 Characterization of Amenable Banach Algebra

**Definition 4.4.1.** *Let $A$ be a Banach Algebra and $\pi : A\hat{\otimes}A \to A$ be the diagonal operator on $A$. A bounded net $(m_\alpha)_\alpha$ in $A\hat{\otimes}A$ is called approximate diagonal for $A$ if for every $a \in A$,*

1. $\lim_\alpha a.m_\alpha - m_\alpha.a = 0$
2. $\lim_\alpha a\pi(m_\alpha) = a$

*An element $M \in (A\hat{\otimes}A)^{**}$ is a virtual diagonal for $A$ if for every $a \in A$, we have,*

1. $a.M - M.a = 0$
2. $a.\pi^{**}(M) = \hat{a}$, *where, $\hat{a} = J(a)$ and $J$ is canonical map from $A$ to $A^{**}$.*

**Remark 4.4.2.** *Consider, $\lim_\alpha a.\pi(m_\alpha) = a$. This implies $(\pi(m_\alpha))_\alpha$ is a right approximate identity. Moreover,*

$$\begin{aligned} ||\pi(m_\alpha).a - a|| &= ||a.\pi(m_\alpha) - \pi(m_\alpha).a|| + ||\pi(m_\alpha).a|| \\ &= ||\pi(a.m_\alpha - m_\alpha).a|| + ||a.\pi(m_\alpha) - a|| \\ &\longrightarrow 0 \end{aligned}$$

*Actually $(\pi(m_\alpha))_\alpha$ is an approximate identity.*

From the definition only the following theorem is expected.

**Theorem 4.4.3.** *Let $A$ be a Banach algebra. Then $A$ has an approximate diagonal if and only if $A$ has a virtual diagonal.*

*Proof.* (1) $\Rightarrow$ (2)
Suppose that $(m_\alpha)_\alpha$ is an approximate diagonal for $A$. Then $(\hat{m_\alpha})_\alpha$ is a bounded net in $(A\hat{\otimes}A)^{**}$ and thus by Banach Alaoglu theorem A.0.28 there exists a $w^*$-accumulation point, $M \in (A\hat{\otimes}A)^{**}$ of $(\hat{m_\alpha})$. Again since any subnet of $(m_\alpha)$ is also an approximate diagonal for $A$, we may assume that $w^*$-$\lim_\alpha \hat{m_\alpha} = M$. Then

$$a.M - M.a = w^* - \lim_\alpha a.\hat{m}_\alpha - \hat{m}_\alpha.a = w - \lim_\alpha a.m_\alpha - m_\alpha.a = 0 \quad (a \in A)$$

where the last equality follows from the definition of approximate diagonal. Furthermore

$$a.\pi^{**}(M) = w^* - \lim_\alpha a.\pi^{**}(\hat{m}_\alpha) = w - \lim_\alpha a.\pi(m_\alpha) = \hat{a}$$

(2)$\Rightarrow$ (1)
Suppose that $M$ is a virtual diagonal for $A$. By Goldstine's Theorem A.0.29 there exists a bounded net $(m_\alpha)_\alpha$ in $(A\hat{\otimes}A)$ such that $M = w^* - \lim_\alpha \hat{m}_\alpha$. Then,

$$w - \lim_\alpha a.m_\alpha - m_\alpha.a = w^* - \lim_\alpha a.\hat{m}_\alpha - \hat{m}_\alpha.a = a.M - M.a = 0 \qquad (a \in A)$$

and

$$w - \lim_\alpha a.\pi(m_\alpha) = w^* - \lim_\alpha a.\pi^{**}(\hat{m}_\alpha) = a.\pi^{**}(M) = a \qquad (a \in A)$$

Let $\mathcal{F} := \{F \subset A : |F| < \infty\}$. Then for any $F = \{a_1, a_2, ..., a_n\} \in \mathcal{F}$ and $\epsilon > 0$, the bounded net,

$$((a_1 m_\alpha - m_\alpha.a_1\ ,\ a_1\pi(m_\alpha) - a_1), ..., (a_n m_\alpha - m_\alpha.a_n\ ,\ a_n\pi(m_\alpha) - a_n)),$$

in the product space $((A\hat{\otimes}A) \times A)^n$ converges to 0 in the weak topology. Now let

$$H = Conv\{m_\alpha : \alpha \in I\} \quad (convexhull).$$

$$0 \in (a_i.\bar{H}^w - \bar{H}^w.a_i) \cap (a_i.\bar{H}^w - a_i) \qquad (i \in \mathbb{N}_n)$$

By Mazur's Theorem A.0.25, $\bar{H}^w = \bar{H}$ and hence there exists $u_{F,\epsilon} \in H$ such that

$$||a_i.u_{F,\epsilon} - u_{F,\epsilon}.a_i|| < \epsilon \text{ and } ||a_i.\pi(u_{F,\epsilon}) - a_i|| < \epsilon \qquad (i \in \mathbb{N}_n)$$

As $\mathcal{F} \times \mathbb{R}^+$ is a directed set with partial order defined by

$$(F_1, \epsilon_1) \leq (F_2, \epsilon_2) \iff F_1 \subset F_2 \text{ and } \epsilon_1 \geq \epsilon_2$$

and hence the net $(u_{F,\epsilon})$ is an approximate diagonal for $A$. $\square$

Moreover the following is true.

**Theorem 4.4.4.** *Let $A$ be a Banach Algebra. Then the following are equivalent:*

1. *$A$ is amenable.*
2. *$A$ has a bounded approximate diagonal and $H^1(A, X^*) = \{0\}$ for every pseudo unital Banach $A$-bimodule.*
3. *$A$ has a bounded approximate diagonal*
4. *$A$ has a virtual diagonal.*

# Appendix A

# Functional Analysis

In this section we will look at some basics of functional analysis and topology. For detailed proofs refer [Rob], [Kreyszig] and [Kehe-Zhu]. We will consider vector spaces over field $\mathbb{C}$ until and unless stated otherwise. Primary knowledge of vector spaces and linear algebra will be taken granted.

We will first look at some properties of Banach algebras.

In general, a Banach algebra need not have a unit. There is a canonical way to embed a non-unital Banach algebra into a unital one. Let $A$ is a non-unital Banach algebra, then consider the set $A' := \mathbb{C} \times A$ with pointwise addition and scalar multiplication and with algebra multiplication defined by,

$$(\alpha,\ a), (\beta,\ b) \ = \ (\alpha\beta,\ ab + \alpha b + \beta a).$$

Then $A'$ is unital Banach algebra with unity $1_A = (1, 0)$ and norm defined as,

$$||(\alpha, a)|| = |\alpha| + ||a|| \qquad (\alpha \in \mathbb{C},\ a \in A).$$

So, $||1_A|| = 1$. $A$ is embedded in $A'$ as $i : A \hookrightarrow A',\ \ i(a) = (0, a)$.

Consider a unital Banach algera $A$ with unit $1_A$, we have one sufficient condition that says if for $a \in A$, $||a|| < 1$ then $(1_A - a)$ is invertible. Moreover, $(1_A - a)^{-1} = \sum_{i=0}^{\infty} a^n$. As $||a|| < 1$ so underline power series is convergent and it is easy to see that the inverse of $(1_A - a)$ is this series only.

For a normed space $X$, we denote by $X^*$ the set of all continuous linear functionals on $X$ and set of all linear functional on $X$ by $X'$. We recall some central theorems of functional analysis. Before going there look at some basic terminologies.

**Definition A.0.5.** *Let $X$ be a normed space and let $A$ abd $B$ be subsets of $X$ and $X^*$ respectively. Define $A^{\perp}$ and ${}^{\perp}B$ as follows*

$$A^{\perp} := \{f \in X^* : f(x) = 0 \textit{ for each } x \textit{ in } A\}$$

$$^{\perp}B := \{x \in X : \ f(x) = 0 \textit{ for each } f \textit{ in } B\}$$

*Then* $A^{\perp}$ *is the annihilator of* $A$ *in* $X^*$*, while* $^{\perp}B$ *is the annihilator of* $B$ *in* $X$.

**Definition A.0.6.** ***(Sublinear functional)*** *Let* $X$ *be a vector space, then a sublinear functional* $p$ *on* $X$ *is a linear real valued functional which is sub-additive and positive homogeneous, i.e.;* $\forall \ x, y \in X$

$$p(x + y) \leq p(x) + p(y) \quad \textit{and}$$

$$p(\alpha x) = \alpha p(x) \qquad\qquad \forall \ \alpha \geq 0 \textit{ in } \mathbb{R} \textit{ and } x \textit{ in } X$$

**Definition A.0.7.** ***(partially ordered set, chain)*** *A partially ordered set (poset) is a set* $M$ *with an order relation* $\leq$*, which is reflexive, anti-symmetric and transitive.*
*A chain is a poset in which any two elements are comparable.*

An upper bound of a subset $W$ of a partially ordered set $M$ is an element $u \in M$ such that, for every $x \in W$

$$x \leq u.$$

A maximal element of $M$ is an $m \in M$ such that,

$$m \leq x \qquad \textit{implies} \quad m = x.$$

**Definition A.0.8.** ***(Zorn's Lemma)*** *Let* $M \neq \phi$ *be a partially ordered set. If every chain* $C \subset M$ *has an upper bound. Then* $M$ *has at least one maximal element.*

Now we will see the very important theorem in connection with bounded linear functionals. Hahn-Banach theorem is about the extension of linear functional from a subspace to whole space with same bound. First we discuss for vector space and then go for normed spaces.

**Theorem A.0.9.** ***(Hahn-Banach theorem for vector spaces)(HBT)*** *Let* $X$ *be a real or complex vector space and* $p$ *a real-valued functional on* $X$ *which is subadditive, and for every scalar* $\alpha$ *satisfies*

$$p(\alpha x) = |\alpha| x,$$

*Furthermore, let* $f$ *be a linear functional which is defined on a subspace* $Z$ *of* $X$ *and satisfies,*

$$|f(x)| \leq p(x) \qquad\qquad \forall x \in Z.$$

*Then* $f$ *has a linear extension* $\tilde{f}$ *from* $Z$ *to* $X$ *satisfying,*

$$|\tilde{f}(x)| \leq p(x) \qquad\qquad \forall x \in X$$

Idea of the proof is very simple first we extend the functional to a space of dimension one greater than $Z$ and containing $Z$. On repeating the process, Zorn's Lemma ensures the termination of this process and we get our extension map to whole $X$.

**Theorem A.0.10.** ***(Hahn-Banach for normed spaces)*** *Let $Y$ be linear subspace of normed space $X$. Then for each $f \in Y^*$ there exist $\tilde{f} \in X^*$ such that $\tilde{f}|_Y = f$ and $||\tilde{f}||_X = ||f||_Y$. where,*

$$||\tilde{f}||_X = \sup_{x \in X, ||x||=1} |f(x)|, \qquad ||f||_Y = \sup_{x \in Y, ||x||=1} |f(x)|.$$

To prove the above we consider $p(x) = ||f||_Y ||x||$, and the rest is achieved by HBT for vector spaces.
In the following corollary HBT ensures that the dual(topological) of a normed space is non-empty iff the space is itself non-empty. It also relates the norm of a vector with norm of functionals and characterizes the zero vector.

**Corollary A.0.11.** ***(Bounded Linear Functional)*** *Let $X$ be a normed space and let $x_0 \neq 0$ be any element of $X$. Then there exists a bounded linear functional $f$ on $X$ such that,*

$$||f|| = 1 \quad \text{and} \quad f(x_0) = ||x_0||.$$

**Corollary A.0.12.** ***(Zero Vector)*** *For every $x$ in normed space $X$, we have*

$$||x|| = \sup_{f \in X', ||f||=1} |f(x)|.$$

*Hence if $x_0$ is such that $f(x_0) = 0$ for all $f \in X'$, then $x_0 = 0$.*

The following are the three basic and most powerful theorems of Functional analysis.

**Theorem A.0.13.** *(Open Mapping Theorem) Every bounded linear operator from a Banach space onto a Banach space is an open mapping.*

**Theorem A.0.14.** *(Uniform Boundedness Principle) Let $\mathcal{J}$ be a nonempty family of bounded linear operators from a Banach space $X$ into a normed space $Y$. If sup$\{||Tx|| : T \in \mathcal{J}\}$ is finite for each $x$ in $X$, then sup$\{||T|| : T \in \mathcal{J}\}$ is finite.*

**Theorem A.0.15.** *(Closed Graph Theorem) Let $T$ be linear operator from a Banach space $X$ into a Banach space $Y$. Suppose that whenever a sequence $(x_n)$ in $X$ converges to some $x$ in $X$ and $(Tx_n)$ converges to some $y$ in $Y$, it follows that $y = Tx$. Then $T$ is bounded.*

Now we will go for some concepts and theorems from Banach Space theory. What are weak and weak* topologies? How it helps to have some good structures on spaces? Without going deep we will only prove and see some applications of important theorems. For details refer to [Rob] and [Kehe-Zhu].

**Definition A.0.16.** ***(Locally Convex Topological Vector Space)(LCTVS)*** *A vector space $(X, +, *)$ with a topology $\tau$, such that the vector addition $+$ and scalar multiplication $*$ are continuous, is called a topological vector space.*
*If the topology has a basis consisting of convex sets, then $(X, \tau)$ is a locally convex topological vector space.*

**Proposition A.0.17.** *Let $X$ be a set, $\mathcal{F}$ be a family of functions and let $\{(Y_f, \tau_f) : f \in \mathcal{F}\}$ a family of topological spaces such that each $f \in \mathcal{F}$ maps $X$ into the corresponding $Y_f$. Then there is a smallest topology for $X$ with respect to which each member of $\mathcal{F}$ is continuous, called the topology induced by topologizing family $\mathcal{F}$*
*That is, there is a unique topology $\tau_{\mathcal{F}}$ for $X$ such that, each $f$ in $\mathcal{F}$ is $\tau_{\mathcal{F}}-$continuous and if $\tau$ is any topology for $X$ such that each $f$ in $\mathcal{F}$ is $\tau-$continuous, then $\tau_{\mathcal{F}} \subset \tau$. The topology $\tau_{\mathcal{F}}$ has $\{f^{-1}(U) : f \in \mathcal{F} \text{ and } U \in \tau_{\mathcal{F}}\}$ as a subbasis.*

**Definition A.0.18.** ***(Weak Topology)*** *Let $X$ be a normed space. Then the topology for $X$ induced by the topologizing family $X^*$ is the weak topology of $X$ or $X^*$-topology of $X$ and denoted by $\sigma(X, X^*)$.*

**Definition A.0.19.** ***(Separating family)*** *Let $X$ be a set and $\mathcal{F} := \{f : X \to Y_f\}$. Then we say that the family $\mathcal{F}$ is separating if for each pair $x, y$ of distinct elements of $X$ there exists $f$ in $\mathcal{F}$ such that $f(x) \neq f(y)$.*

That is, the weak topology of a normed space is the smallest topology that every element of dual of normed space is continuous. Definition of weak topology simply gives that it is subset of norm topology. It is easy to check that as $X^*$ is separating family for $X$ and $\mathbb{F}$ is completely regular space, weak topology is itself completely regular and locally convex.

**Proposition A.0.20.** *Let $X$ be a set and $\mathcal{F} = \{f : X \to (Y_f, \tau_f)\}$ be a separating topologizing family for $X$. Then the map $x \mapsto (f(x)_{f \in \mathcal{F}})$ is a homeomorphism from $X$ with $\mathcal{F}$-topology onto a topological subspace of $\prod_{f \in \mathcal{F}} Y_f$ with the product topology.*

**Definition A.0.21.** *(Bounded Set) A subset of a topological vector space is bounded if, for each neighborhood $U$ of zero, there is a positive $S_U$ such that $A \subset tU$ whenever $t > S_U$.*

**Definition A.0.22.** *(weakly bounded set) A subset $A$ of normed space $X$ is said to be weakly bounded if $E := \{x^*x : x \in A\}$ is bounded for all $x^* \in X^*$.*

We have seen the basic definition and some theorems which will now help us to build our theory. As every open set in weak topology is open in norm topology, so what can be said for closed set, bounded set and etc?

By the use of Uniform boundedness principle we can see that a set is weakly bounded iff it is norm bounded.

Mazur theorem gives that a convex set is weakly closed iff it is norm closed. To prove it we will prove a general result and Mazur theorem is just a special case of this general result.

**Theorem A.0.23.** *Let $K$ and $C$ be disjoint nonempty convex subsets of an LCS $X$ such that $K$ is compact and $C$ is closed. Then there is a member $x^*$ of $X^*$ such that*

$$max\{x^*x : x \in K\} < \ inf\{x^*x : x \in C\}.$$

**Corollary A.0.24.** *Suppose that a vector space $X$ has two locally convex topologies $\tau_1$ and $\tau_2$ such that the dual spaces of $X$ under the two topologies are the same. Let $C$ be a convex subset of $X$. Then the $\tau_1-$closure of $C$ is the same as its $\tau_2-$closure. In particular, the set $C$ is $\tau_1$ closed if and only if it is $\tau_2$ closed.*

*Proof.* Let $C \neq \phi$. Let $X^*$ denote the dual of $X$ under each of the topologies. For each $x$ in $X \backslash \overline{C}^{\tau_1}$, from above theorem we get $x_x^*$ in $X^*$ such that

$$x_x^* = max\{x_x^*y : y \in x\} < inf\{x_x^*y : y \in \overline{C}^{\tau_1},\}$$

let $A_x = \{z : z \in X, x_x^*z \geq inf\{x_x^*y : y \in \overline{C}^{\tau_1}\}$. Then each $A_x$ is $\tau_2$ closed, and $\overline{C}^{\tau_1} = \cap\{A_x : x \in X\backslash\overline{C}^{\tau_1}\}$. This gives that $\overline{C}^{\tau_1}$ is $\tau_2$ closed. Similar argument shows that $\overline{C}^{\tau_2}$ is $\tau_1$ closed. So $\overline{C}^{\tau_1} = \overline{C}^{\tau_2}$ □

Now Mazur theorem is a special case of the above corollary as weak and norm dual are the same. Explicit statement of Mazur theorem is as follows,

**Theorem A.0.25.** *(Mazur Theorem) The closure and weak closure of a convex subset of normed space are the same. In particular, a convex subset of a normed space is closed iff it is weakly closed.*

We looked at the smallest topology on $X$ generated by $X^*$. Similarly we have weak topology on $X^*$ generated by $X^{**}$ and so on. Now here something special for $X^*$, We can see $X$ embedded in $X^{**}$, the topology generated by $X$ on $X^*$ such that some special kind of function are continuous, is called weak* topology. Define $J : X \to X^{**}$ by $x \mapsto J(x)$, where $J(x)(f) = f(x)$ for all $f \in X^*$. $J(x) \in X^{**}$ whenever $x \in X$, $J$ is isometric isomorphism from $X$ into $X^{**}$. The subspace $J(X)$ is closed in $X^{**}$ iff $X$ is a banach space. $J$ is called natural embedding or canonical mapping from $X$ into $X^{**}$.

**Definition A.0.26.** *(weak* topology) Let $X$ be a normed space. Then the topology for $X^*$ induced by topologizing family $J(X)$ is the weak* topology of $X^*$ and denoted by $\sigma(X^*, X)$.*

**Remark A.0.27.** *It is noteworthy here that $J(X)$ is the separating family for $X^*$. Due to similar argument as for weak topology, weak* topology of $X^*$ is also a completely regular locally convex topology and the dual of $X^*$ with respect ot weak* topology is $J(X)$. It can be proven that weak* topology is induced by norm if and only if the space $X$ is finite dimensional.*

As from Riesz' lemma for normed space, we have that the closed unit ball is compact if and only if the space is finite dimensional. Weak* topology provides us the compactness of closed unit ball of $X^*$. This strong result was given by Banach-Alaoglu theorem.

**Theorem A.0.28.** *(Banach-Alaoglu Theorem) Let $X$ be normed space.Then closed unit ball of $X^*$ is weak* compact.*

*Proof.* Natural embedding of $X$ into $X^{**}$ gives that $J(B_X)$ is a separating family of $B_{X^*}$ that induces the restriction on weak* topology of $X^*$ to $B_{X^*}$. Let $I := \{\alpha \in \mathbb{F} : |\alpha| \leq 1\}$. Let $I^{B_X} = \prod_{x \in B_X} I_x$ where $I_x = I$ for each $x$ in $X$. As each $I_x$ is compact by Hiene-Borel theorem, it gives that $I^{B_X}$ is compact by Tychnoff's Theorem. Define map $F$ from $B_{X^*}$ to $I^{B_X}$ as $x^* \mapsto (x^*x)_{x \in B_X}$. From proposition A.0.20 map $F$ is a homeomorphism from $B_{X^*}$ onto the topological subspace of $I^{B_X}$. Now we will show that image of $B_{X^*}$ is closed in $I^{B_X}$ and thus we are done because closed set of compact set is itself closed and inverse image of compact set under a homeomorphism is compact.
Let $(x^*_\beta)_{\beta \in \Lambda}$ be a net in $B_{X^*}$ such that $(F(x^*_\beta))$ converges to some $(\alpha_x)_{x \in B_X}$ in $I^{B_X}$. Now our claim is that there exist some $x^*$ in $B_{X^*}$ such that $F(x^*) = (\alpha_x)_{x \in B_X}$.
For each $x \neq 0$ in $X$, let $x^*x = ||x||\alpha_{(||x||^{-1}x)}$ and let $x^*(0) = 0$. Since

$$x^*x = ||x|| \lim_\beta x_{\beta^*}(||x||^{-1}x)$$

$$x^*x = \lim_\beta x^*_\beta(x) \qquad \forall x \in X$$

Thus $x^*$ is a linear functional, also as $|x^*x| = \lim_\beta |x^*_\beta x| \leq ||x||$ which implies $x^*$ belongs to $B_{X^*}$. Now for each $x$ in $B_X$,

$$\begin{aligned}(F(x^*)) &= (x^*x)_{x \in B_X} \\ &= (\lim_\beta x^*_\beta x)_{x \in B_X} \\ &= \lim_\beta (F(x^*_\beta))_{x \in B_X} \\ &= (\alpha_x)_{x \in B_X}.\end{aligned}$$

This implies that $(F(x^*_\beta))$ converges to $F(x^*) = (\alpha_x)_{x \in B_X}$. Hence we are done.

□

Now we will discuss another theorem of Banach space theory that tells about the closed unit ball of $X$. The image of $X$ under natural map is closed and weakly closed but it is weakly* dense in $X^{**}$. Goldstine's theorem says that the closed unit ball of $X$ is weakly* dense in $B_{X^{**}}$.

**Theorem A.0.29.** *(Goldstine's Theorem) Let $X$ be a normed space and let $J$ be the natural map from $X$ into $X^{**}$. Then $J(B_x)$ is weakly* dense in $B_{X^{**}}$.*

*Proof.* As it is clear that $\overline{J(B_X)}^{w^*} \subset B_{X^{**}}$. So it is enough to show other way containment. Let $x_0^{**} \in X^{**} \backslash \overline{J(B_X)}^{w^*}$. It is enough to show that $||x_0^{**}|| > 1$. Since $\overline{J(B_X)}^{w^*}$ is convex and weak* closed. So from A.0.23 there exists $x_0^* \in X^*$ as $(X^{**}, \sigma(X^{**}, X^*)) = X^*$ such that,

$$\begin{aligned} |x_0^{**}| &\geq x_0^{**} x_0^* \\ &> sup\{x^{**} x_0^* : x^{**} \in \overline{J(B_X)}^{w^*}\} \\ &\geq sup\{x_0^* x : x \in B_X\} \\ &= ||x_0^{**}|| \end{aligned}$$

which implies $||x_0^{**}|| > 1$. Hence we are done. □

Here we can say something more with the help of **??** as below corollary says,

**Corollary A.0.30.** *Let $X$ be a normed space and let $J$ be natural map from $X$ into $X^{**}$. Then $\overline{J(X)}^{w^*} = X^{**}$.*

Now here comes the conclusive discussion. We will define what is $C^*$-algebra and just state the very sound theorem named as Gelfand representation theorem. What is $C^*$ algebra, In first chapter we discussed about what is algebra so here comes one more operation on the defined algebra. We are doing all this to decorate our structure to make our life easy. The more stringent conditions, the better structure we have.

**Definition A.0.31.** *($C^* - algebra$) Let $(\mathcal{A}, +, \times, \circ, ||.||)$ be a Banach algebra together with a unitary map on $\mathcal{A}$, $x \mapsto x^*$ satisfying the following conditions,*

1. *$(x^*)^* = x$ for all $x \in \mathcal{A}$*
2. *$(ax + by)^* = \bar{a}x^* + \bar{b}y^*$ for all $x, y \in \mathcal{A}$ and $a, b \in \mathbb{F}$*
3. *$(xy)^* = y^* x^*$ for all $x, y \in \mathcal{A}$*
4. *$||x^* x|| = ||x||^2$ for all $x \in \mathcal{A}$*

**Remark A.0.32.** *An algebra satisfying 1,2 and 3 is called $^* - algebra$. The $^*$ map is called involution map on $\mathcal{A}$. It is to be noted here that $1^* = 1$.*

Now we will define some more terminologies, then we will go for the centerpiece of this discussion.

**Definition A.0.33.** *Suppose $\mathcal{A}$ and $\mathcal{B}$ are $C^*-$ algebras. A mapping $\phi$ from $\mathcal{A}$ to $\mathcal{B}$ is called $C^*-$ homomorphism if, it preserves the all the operations, i.e. for all $x, y$ in $\mathcal{A}$ and $a, b$ in $\mathbb{F}$*

1. *$\phi(ax + by) = a\phi(x) + b\phi(y)$,*
2. *$\phi(xy) = \phi(x)\phi(y)$*

*3.* $\phi(x^*) = (\phi(x)^*)$

*4.* $\phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$

*further, if $\phi$ is one-one, we say $\phi$ is a $C^*$ homomorphism. In other words $\mathcal{A}$ is isomorphic to its image under the map $\phi$.*

**Definition A.0.34.** *A linear functional $f$ on a Banach algebra $\mathcal{A}$ is said to be multiplicative if, $\phi$ is non-trivial and $\phi(xy) = \phi(x)\phi(y)$ for all $x$ and $y$ in $\mathcal{A}$.*

Noteworthy points regarding multiplicative functionals are $\phi(1) = 1$ and $||\phi|| = 1$.

**Remark A.0.35.** *For a Banach algebra $\mathcal{A}$ let $\mathcal{M}_{\mathcal{A}}$ denote the collection of all multiplicative functionals on $\mathcal{A}$. $\mathcal{M}_{\mathcal{A}}$ is called maximal ideal space of $\mathcal{A}$. $\mathcal{M}_{\mathcal{A}}$ is subset of $B_{X^*}$.*

**Proposition A.0.36.** *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{M}_{\mathcal{A}}$ is a compact Hausdorff space with the weak* topology inherited from the dual space of $\mathcal{A}$*

**Remark A.0.37.** *When a Banach algebra is commutative the set $\mathcal{M}_{\mathcal{A}}$ is in one-one correspondence with the set of all maximal ideals in $\mathcal{A}$.*

Now we are going to conclude this discussion by defining what are the Gelfand maps and what Gelfand representation says.

**Definition A.0.38.** *Let $\mathcal{A}$ be a Banach algebra. The Gelfand transformation is the mapping $\Gamma$ from $\mathcal{A}$ to $C(\mathcal{M}_{\mathcal{A}})$ defined as,*

$$\Gamma(x)(\phi) = \phi(x) \qquad x \in \mathcal{A},\ \phi \in \mathcal{M}_{\mathcal{A}}.$$

We can easily verify that $\Gamma$ is linear, multiplicative and bounded, moreover $\Gamma(1) = 1$.

**Theorem A.0.39.** *(Gelfand Representation Theorem) Every commutative $C^*$ algebra is $C^*$ isomorphic to $C(K)$ for some compact Hausdorff space $K$. Specifically for commutative $C^*$ algebra $\mathcal{A}$, the Gelfand representation is an isometric $C^*$ isomorphism from $\mathcal{A}$ onto $\mathcal{M}_{\mathcal{A}}$. If $\mathcal{A}$ is unital $\mathcal{M}_{\mathcal{A}}$ is compact.*

# Appendix B

# Measure Theory

After discussing functional analysis what else is left to discuss as preliminary is measure theory. In this section we will look into very important concepts like introduction to $L^P$-spaces, Radon-Nikodym theorem and at last but not the least Fubini's Theorem. Here we are not proving many theorem so for a proof and detailed discussion about the following, do refer [Cohn] and [Roy-Fitz]

## B.1 $L^P$ spaces

Let $(X, \Gamma, \mu)$ be a measure space, and $P \in [1, \infty)$. Suppose $\mathcal{L}^P(X, \Gamma, \mu)$ denote the set of $\Gamma$- measurable functions, $f : x \to \mathbb{C} \ \ni \ |f|^p$ is integrable with respect to $\mu$. Define, $||.||_p$ as,

$$||f||_p \ = \left( \int |f|^p d\mu \right)^p$$

It is easy to check that it is the seminorm on $\mathcal{L}^p(X, \Gamma, \mu)$. Now to get rid of this follow the classical way. Define the equivalence relation on $\mathcal{L}^p(X, \Gamma, \mu)$ as,

$$f \ \sim \ g \iff f \ = \ g \text{ a.e.}$$

This defines an equivalence relation on $\mathcal{L}^p(X, \Gamma, \mu)$. Now define,

$$L^p(X, \Gamma, \mu) \ := \ \{[f] \mid f \in \mathcal{L}^p(X, \Gamma, \mu)\},$$

where, $[f] \ = \ \{g \in \mathcal{L}^p(X, \Gamma, \mu) \ \mid f = g \text{ a.e}\}$. Addition and scalar multiplication is defined as

$$[af + bg] = a[f] + b[g] \quad (a, b \ \in \mathbb{C} \ , f, g \in \mathcal{L}^p)$$

We get that $(\mathcal{L}^p, ||.||_p)$ is a normed space. For a measurable function, $f$ define

$$\text{ess sup} f \ := \ \inf\{t > 0| \ |f| \leq t \text{ a.e.}\}$$

$\mathcal{L}^\infty$ contains all essentially bounded functions. That is,

$$L^\infty(X,\Gamma,\mu) := \{[f] \mid f \in \mathcal{L}^\infty(X,\Gamma,\mu), \text{f is essentially bounded}\}$$

$(L^{\infty)}, ||.||_\infty$, where $||f|| = \text{ess sup} f$ is a normed space.

**Young's Inequality** :- For $1 < p < \infty$ and $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and any $a > 0, b > 0$, the young's inequality is,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

**Theorem B.1.1.** *For $(X,\Gamma,\mu)$, $E \in \Gamma$, $1 < p < \infty$ and $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$ if $f \in L^p(E)$, $g \in L^q(E)$, then $f, g \in L'(E)$.*

**Holder's Inequality** :- For $(X,\Gamma,\mu)$, $E \in \Gamma$, $1 < p < \infty$ and $q$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ if $f \in L^p(E)$, $g \in L^q(E)$, $\int_E |f.g| \leq ||f||_p ||g||_q$ .

**Minkowski's Inquality** :- Suppose that $E \in \Gamma$, $1 \leq p \leq \infty$, and $f, g \in L^p(E)$, then $f + g \in L^p(E)$, and $||f + g|| \leq ||f||_p + ||g||_p$ .

**Cauchy-Schwartz Inequality** :- Suppose that $E \in \Gamma$, and $f$ and $g$ are measurable functions for which $f^2$ and $g^2$ are integrable over $\mathbb{C}$. Then their product $f.g$ will also be integrable over $E$ and

$$\int_E |fg| \leq \sqrt{\int_E f^2}.\sqrt{\int_E g^2}$$

**Corollary B.1.2.** *Suppose that $E \in \Gamma$ , $\mu(E) < \infty$ and $1 \leq p_1 < p_2 \leq \infty$. Then,*

$$L^{p_2} \subseteq L^{p_1}(E) \ .$$

**Theorem B.1.3.** *Riesz-Fisher Theorem Let $X$ be a measure space and $1 \leq p \leq \infty$, then the space $L^p(X)$ is a complete normed linear space.*

*Proof.* For $p = \infty$,
Let $(f_n) \subseteq L_\infty$ be a cauchy sequence and suppose that $M \subset X$, $\mu(M) = 0$ such that,

$$|f_n(x)| \leq ||f||_\infty \qquad x \notin M \quad n = 1, 2, ...$$

and also

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty \quad \forall x \notin M, \quad m, n = 1, 2, ...$$

This implies that $(f_n)$ is uniformally cauchy, which further implies that $(f_n)$ is uniformally convergent on $X \setminus M$.

Now let,

$$f(x) = \begin{cases} lim f_n(x) & \text{if } x \in M \\ 0 & \text{if } x \notin M \end{cases}$$

This implies that $f$ is measurable and also, as $||f_n - f||_\infty \longrightarrow 0$, hence this implies that $L^\infty$ is complete.

For $1 \leq p < \infty$ : Let $(f_n) \subset L^p$ be cauchy. That is $\forall\ \epsilon > 0,\ \exists\ M \in \mathbb{N}, \forall\ m, n \geq M$

$$\int |f_m - f_n|_p d\mu \ = \ ||f_n - f_m||_p^p \ < \ \epsilon^p$$

There exists a subsequence $(g_k)$ of $(f_n)$ such that

$$||g_{k+1} - g_k||_p < 2^{-k}, \qquad\qquad k \in \mathbb{N}$$

Define,

$$g = |g_1(x)| + \sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|$$

Then $g$ is a non-negative measurable function.
From Fatou's Lemma,

$$\int |g|^p d\mu \ \leq \ \lim_n inf \ \int \{|g| + \sum_{k=1}^{n} |g_{k+1} - g_k|\}^p d\mu$$

$$\begin{aligned} \left\{\int |g|^p d\mu\right\}^{1/p} &\leq \ \lim_n inf \ \{||g||_p + \sum_{k=1}^{n} ||g_{k+1} - g_k||_p\} \\ &\leq \ ||g_1||_p + 1 \end{aligned}$$

If $E \ = \ \{x \in X |\ g(x) < \infty\}$ , $E \in X$ , $\mu(X \setminus E) = 0 \Rightarrow$ Series in (1.1) is convergent a.e. and $g \in L^p$
Define,

$$f(x) = \begin{cases} g_1(x) + \sum_{k=1}^{\infty} |g_{k+1}(x) - g_k(x)|, & x \in E \\ \\ 0, & x \notin E \end{cases}$$

as $|g_k| \ \leq \ |g_1| + \sum_{j=1}^{k-1} |g_{j+1}(x) - g_j(x)| \ \leq \ |g|$
Since, $(g_k)$ converges a.e to $f$, from Dominated convergence Theorem (in $p^{th}$ norm), $f \in L^p$.

Also, since, $|f - g_k|^p \;\leq\; \left(\dfrac{1}{2^k}\right)^p \;\to\; 0$ as $k \;\to\; \infty$, thus from DCT, $\lim ||f - g_k||_p = 0$, that is $(g_k) \;\to\; f$ in $L^p$.
Now, for $m \geq M_\epsilon$ and for sufficiently large $k$ $(k \geq M_\epsilon)$

$$\int |f_m - g_k|^p \, d\mu \;<\; \epsilon^p$$

From Fatou's Lemma, $\int |f_m - f|^p \, d\mu \;\leq\; \lim \; inf\{\int |f_m - g_k|^p d\mu \;\leq\; \epsilon^p\}$. Hence, $(f_n)$ converges to $f$ in $L^p$.

□

We observed that $L^p$ spaces are not the collection of functions but actually they contain equivalence classes, but for our convenience we will work with $f$ as an element of $L^p(X, \Sigma, \mu)$ rather than using $[f]$.
Now we will define the concept of regular measure which will lead us to very interesting and vast theory of measures.

**Definition B.1.4.** *Let $X$ be a Hausdorff space. A Borel measure $\mu$ (measure defined on Borel $\sigma$-Algebra) on $X$ is regular if the following conditions are satisfied:*

1. *$\mu(K) < +\infty$, for all compact sets $K \subset X$,*
2. *$\mu(U) = sup\{\mu(K)|K is\ compact\ and K \subset U\}$ for all open $U \subset X$,*
3. *$\mu(A) = inf\{\mu(V)|V is\ open\ and A \subset V\}$ for all $A \in \mathbb{B}(X)$.*

*(1) and (3) are referred as inner and outer irregularity of $\mu$ respectively.*

If we have nice measure space in the sense that if measure is finite or $\sigma-$finite then we have the following. Also see proposition B.1.27

**Proposition B.1.5.** *Let $X$ be a Hausdorff space and let $\mu$ be a finite regular Borel measure on $X$. Then, $\mu(A) = sup\ \{\mu(K)|K$ is compact and $K \subset A\}$ for every $A \in \mathbb{B}(X)$.*

*Proof.* Let $\epsilon > 0$ be given and let $A \in \mathbb{B}(X)$. By the regularity of $\mu$, we have an open set $U \subset X$ and a compact set $K$ such that $A \subset U$, $K \subset U$ and

$$\mu(U) - \epsilon < \mu(K) \leq \mu(U) < \mu(A) + \epsilon.$$

We also have, $\mu(U \setminus A) < \epsilon$. So, again by regularity, we have an open set $V$ such that $U \setminus A \subset V$ and $\mu(V) < \epsilon$. Since $K \setminus V$ is closed and is contained in $K$ and so is compact. Also, $K \setminus V \subset A$ (as $K \subset A$). Consider,

$$\mu(K \setminus V) = \mu(K) - \mu(K \cap V) > \mu(U) - \epsilon - \epsilon \geq \mu(A) - 2\epsilon$$

which implies that

$$\mu(A) = sup\ \{\mu(K)|\ K \text{ is compact and } K \subset A\}.$$

Hence proved ☐

**Remark B.1.6.** *Let $(X, \Gamma, \mu)$ be a measurable space. A function, $f : X \to \mathbb{C}$ defined as,*

$$f = \sum_{i=1}^{n} a_i \chi_{A_i}, \quad (A_i \subset X,\ a_i \in \mathbb{C},\ n \in \mathbb{N})$$

*where the sets $A_i$ are pairwise disjoint, is called a simple function on $X$. Thus, $f$ is measurable iff $A_i$ is measurable for each $i = 1, 2, ..., n$.*

Why we are discussing simple function? Firstly it is clear from their structure itself that they are easy to handle moreover the interesting part is...

**Proposition B.1.7.** *Let $(X, \Gamma, \mu)$ be a measurable space and $1 \leq p \leq \infty$. Then the set of all simple functions is dense in $L^p(X, A, \mu)$ and thus makes its dense subspace.*

*Proof.* We will prove it for real valued functions. Corresponding results for complex valued functions can be proved by separating $f = f_1 + if_2$, where $f_1$ and $f_2$ are real valued functions.
For $1 \leq p \leq \infty$, $f \in L^p(X, A, \mu)$. We can choose non-decreasing sequences $(g_k)$ and $(h_k)$ of non-negative $A$-measurable functions such that $f = \lim_k h_k$.
Define, $f_k = g_k - h_k$. Observe that for all $k$, $f_k$ is $A$-measurable that satisfies $|f_k| \leq |f|$ (as $|f| = f^+ + f^-$). Thus we can say that $f_k \in L^p(X, A, \mu)$.
Since, $|f_k(x) - f(x)| \leq |f(x)|$, and, for every $x$ in $X$, $\lim(f_k(x) - f(x)) = 0$, therefore, from DCT, (applied to p$^th$ power of function, $f_k - f$),

$$\lim_k ||f_k - f|| = 0, \quad or, \quad \{f_k\} \to f\ .$$

For $p = \infty$, let $f$ be from $L^\infty(X, A, ]mu)$ and $\epsilon > 0$ be given. Choose real numbers, $a_0, a_1, ..., a_n$ such that $a_0 < a_1 < ... < a_n$ and such that the intervals, $(a_{i-1}, a_i]$ cover the interval, $[-||f||_\infty, ||f||_\infty]$ and have the length at-most $\epsilon$.
Let $A_i = f^{-1}((a_{i-1}, a_i])$ $i = 1, 2, ..., n$ and $f_\epsilon = \sum_{i=1}^{n} a_i \chi_{A_i}$. Then $f_\epsilon$ is a simple $A$-measurable function that satisfies $||f - f_\epsilon|| \leq \epsilon$. Since $f$ and $\epsilon$ are arbitrary hence the proof. ☐

**Proposition B.1.8.** *1. Suppose that $[a, b]$ is a closed and bounded interval and $1 \leq p \leq +\infty$. Then the subspace of continuous functions on $[a, b]$ is dense in $L^p[a, b]$.*
*While for general setup on locally compact Hausdorff space,*

*2. If $X$ is locally compact Hausdorff space, then* *(a)* $C_c(X)$ *is dense in* $C_0(X)$*, and*

*(b) If* $\mu$ *is a regular measure, then* $C_c(X)$ *is dense in* $L_p(X, \mathbb{B}(X), \mu)$*,* $\quad p \in [1, \infty)$*.*

**Definition B.1.9.** *A complex measure on sigma algebra* $\Gamma$ *is a complex valued measure,* $\mu : \Gamma \to \mathbb{C}$*, such that* $\mu(\phi) = 0$ *and* $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$*,* $A_i \cap A_j = \phi$ *,* $for\ i \neq j$*. Thus, complex measure has only complex value and so has no infinite values.* $M(X, \Gamma)$ *denote the set of all complex measures on* $(X, \Gamma)$*, which is a vector space.*

Now we have a vector space, can we have a norm on it and is it Banach space. These are the question that we will answer in our further discussion. So let us first try to define a norm.

**Definition B.1.10.** *The variation* $(\mu)$ *of a complex measure* $\mu$ *on* $(X, \Gamma)$ *is defined by*

$$|\mu|(E) := sup\ \left\{ \sum_i |\mu(E_i)| : \{E_i\}\ is\ finite\ \Gamma - partition\ of\ E \right\}$$

Obviously it is not a norm, but it will help us to define a norm. First observe the following property of variation of a complex measure.

**Proposition B.1.11.** *Let* $(X, \Gamma)$ *be a measurable space and let* $\mu$ *be a complex measure. Then the variation,* $|\mu|$ *of* $\mu$ *is a finite measure on* $(X, \Gamma)$*.*

*Proof.* First and the easy one, as $\mu$ is a measure so we have $|\mu|(\phi) = 0$.
We will check the finite additivity of $|\mu|$ by $|\mu|(B_1 \cup B_2) = |\mu|(B_1) + |\mu|(B_2)$. Let $\{A_j\}_{j=1}^{n}$ be a finite partition of $B_1 \cup B_2$ into $A$, then

$$\sum_{j=1}^{n} |\mu(A_j)| \ \leq \ \sum_{j=1}^{n} |\mu(A_j \cap B_1)| \ + \ \sum_{j=1}^{n} |\mu(A_j + B_2)|$$

, which is less than or equal to $|\mu|(B_1) + |\mu|(B_2)$. Thus, we have,

$$|\mu|(B_1 \cup B_2) \ \leq \ |\mu|(B_1) + |B_2|.$$

Similarly, based on partitioning of $B_1$ and $B_2$, we have,

$$|\mu|(B_1) + |\mu|(B_2) \ \leq \ |\mu|(B_1 \cup B_2)$$

and thus,

$$|\mu|(B_1 \cup B_2) \ = \ |\mu|(B_1) + |\mu|(B_2).$$

Hence, $\mu$ is finitely additive.
From Jordan Decomposition of $\mu$, $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$. This implies,

$$|\mu|(A) \ \leq \mu_1(A) + \mu_2(A) + \mu_3(A) + \mu_4(A), \quad \forall\ A \in \mathbb{A}$$

Since all $\mu_i,\ i = 1, ..., 4$ are finite, $|\mu|$ will also be finite.
Furthermore, if $\{A_n\}$ is a decreasing sequence of $\mathbb{A}$-measurable sets such that $\cap A_n = \phi$ then $\lim_n \mu_k(A_n) = 0,$ for $k = 1, ..., 4$. So from abovw inequality $\lim_n |\mu|(A_n) = 0$. Hence, $|\mu|$ is countably additive. □

Now define a norm on

**Definition B.1.12.** *The total variation, $||\mu||$, of the complex measure $\mu$ is defined as*

$$||\mu|| = |\mu|(X)$$

**Proposition B.1.13.** *Let $(X, \Gamma)$ be a measurable space. Then $M(X, \Gamma)$ is a Banach space under the total variation norm.*

**Remark B.1.14.** *Let $(X, \Gamma)$ be a measurable space and $\nu$ be a finite signed measure of $(X, \Gamma)$. We note that $|\nu|(A) \geq |\nu(A)|$ , $\forall A \in \Gamma$ and thus, $\nu_1 = \frac{1}{2}(|\nu| + \nu),\ \ \nu_2 = \frac{1}{2}(|\nu| - \nu)$ are finite measures and $\nu = \nu_1 - \nu_2$. And thus, every finite measure can be written as the difference of two finite measures. Moreover, every complex measure $\mu$ on $(X, \Gamma)$ can be written as, $\mu = \mu_1 + i\mu_2$, where, $\mu_1$ and $\mu_2$ are finite signed measures.*

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$$

*This representation of complex measure referred as Jordan Decomposition of $\mu$.*

**Remark B.1.15.** *Let $\mu$ be a complex measure and*

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$$

*be the Jordan Decomposition of $\mu$. Let $B(X, \Gamma) := \{f : X \to \mathbb{C} | f \text{ is } \Gamma\text{−measurable and bounded}\}$. Let $f \in B(X, \Gamma)$. Then,*

$$\int f d\mu = \int f d\mu_1 - \int f d\mu_2 + i \int f d\mu_3 - i \int f d\mu_4$$

*Thus, $|\int f d\mu| < \infty$ for any $f \in B(X, \Gamma)$ .*

**Remark B.1.16.** *A complex Borel measure $\mu$ is called regular if $|\mu|$ is regular or equivalently if each of the finite measures in the Jordan Decomposition of $\mu$ is regular. Set of complex Borel measures is denoted as*

$$M_r(X) := \{\mu : X \to \mathbb{C} | \mu \text{ is regular (Borel)}\}$$

Now the following proposition give us a relation between continuous functions and regular Borel measure. Which will be very useful in so many places like Radon-Nikodym theorem.

**Proposition B.1.17.** *Let $X$ be a locally compact Hausdorff space, $\mu$ be a regular borel measure on $X$ and $f : X \to (0, \infty)$ be a continuous function. Then, $\nu : \mathbb{B}(X) \to [0, +\infty]$ defined by*

$$\nu(A) = \int_A f d\mu$$

*is a regular Borel measure on $X$.*

*Proof.* Firstly, for pairwise disjoint Borel sets, $A_1, A_2, ...$, we have by Beppo Levi's theorem,

$$\begin{aligned}\nu(\cup_{n=1}^{\infty} A_n) &= \int_{\cup A_n} f d\mu = \int \sum_{n=1}^{\infty} \chi_{A_n} f d\mu \\ &= \sum_{n=1}^{\infty} \int \chi_{A_n} f d\mu = \sum_{n=1}^{\infty} \nu(A_n)\end{aligned}$$

Also, since $\nu(\phi) = \int \chi_{\phi} f d\mu = 0$. Thus, $\nu$ is a Borel measure.
Now we will show that $\nu$ is regular. Let $K \subset X$ be compact set and $M = \max_{x \in K} f(x)$. Then, since $\mu$ is a regular measure,

$$\nu(K) = \int_K f d\mu \leq M\mu(K) < +\infty$$

We claim that $\nu$ is inner regular. Define,

$$U_n := \{x \in X \mid 1/n < f(x) < n, \ n \in \mathbb{N}\}$$

Each $U_n$ is open by continuity of $f$. Let $U$ be open. Then, since, $U \cap U_1 \subset U \cap U_2 \subset ...U$ and $\cup_{n=1}^{\infty} (U \cap U_n) = U$. We have $\nu(U) = \lim_{n\to\infty} \nu(U \cap U_n)$. So it would be sufficient to show that,

$$\nu(U \cap U_n) = \sup\{\nu(K) | K \text{ is compact and } K \subset (U \cap U_n)\} \qquad \text{for each } n \in \mathbb{N}$$

Let $n \in \mathbb{N}$ $\epsilon > 0$ be given. Suppose that $\mu(U \cap U_n) < +\infty$. Then by the regularity of $\mu$, there exists a compact set $K \subset U \cap U_n$ , $\mu[(U \cap U_n) \setminus K] < \epsilon/n$. So,

$$\nu[(U \cap U_n) \setminus K] = \int_{(U \cap U_n) \setminus K} f d\mu \leq n\mu(U \cap U_n) \setminus K < \epsilon$$

Now suppose $\mu(U \cap U_n) = \infty$ and let $M > 0$. So again by regularity, there exists a compact $K \subset U \cap U_n$ such that $\mu(K) > nM$. So,

$$\nu(K) = \int_K f d\mu \geq 1/n\mu(K) > M$$

Thus, $\nu$ is inner regular.
Now we need to show that $\nu$ is outer regular.
Let $A$ be an arbitrary Borel set. If $\nu(A) = +\infty$, then $X$ is open such that $\nu(x) = +\infty$ and if $\nu(A) < +\infty$, then $\forall n \in \mathbb{N}$,

$$\mu(A \cap U_n) = n \int_{A \cap U_n} f d\mu = n\nu(A \cap U_n) < +\infty$$

For $\epsilon > 0$, from the outer regularity of $\mu$, we have an open subset $V_n$, such that $A \cap U_n \subset V_n$ and $\mu(V_n) < \mu(A \cap U_n) + \epsilon/2^n$. Consider an open set $V_n \cap U_n$. Suppose that $V_n \subset U_n$.

$$\begin{aligned}
\nu(V_n \setminus A) &= \nu(V_n \setminus (U_n \cap A)) \\
&= \int_{V_n \setminus A \cap V_n} f d\mu \\
&\leq n\mu(V_n \setminus A \cap U_n) \\
&= n(\mu(V_n) - \mu(U_n \cap A) \\
&< \epsilon/2^n
\end{aligned}$$

Take $V = \cup_{n=1}^{\infty} V_n$ open and $A \subset V$.

$$\begin{aligned}
\nu(V \setminus A) &= \nu(\cup_{n=1}^{\infty} V_n \setminus A) \\
&= \nu(\cup_{n=1}^{\infty} (V_n \setminus A) \\
&= \sum_{n=1}^{\infty} \nu(V_n \setminus A) \\
&= \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon
\end{aligned}$$

Hence, $\nu$ is outer regular. $\square$

**Definition B.1.18.** *Let $X$ and $Y$ be sets and $E \subset X \times Y$. Then the sections of $E$ are defined as,*

$$E_x := \{y \in Y | (x, y) \in E\}, \qquad E_y := \{x \in Y | (x, y)$$

*Similarly, we define the sections of a function $f$ on $X \times Y$ by*

$$f_x(y) = f(x, y), \qquad f_y(x) = f(x, y)$$

**Definition B.1.19.** *Let $(X, \mathcal{A})$ be a measurable space and let $\mu$ and $\nu$ be positive measures on $(X, \mathcal{A})$. Then $\nu$ is said to be absolutely continuous with respect to $\mu$ if for any $A \in \mathcal{A}$, $\mu(A) = 0$ implies that $\nu(A) = 0$ and is denoted by $\nu << \mu$.*

**Lemma B.1.20.** *Let $(X,\mathcal{A})$ be a measurable space, $\mu$ be a positive measure and $\nu$ be a finite positive measure on $(X,\mathcal{A})$. Then $\nu$ is absolutely continuous with respect to $\mu$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $\mathcal{A}$-measurable set, $A$, which satisfies $\mu(A) < \delta$ also satisfies $\nu(A) < \epsilon$.*

*Proof.* If $\mu(A) = 0$, then $\mu(A) < \delta$ for each $\delta$, which implies that $\nu(A) < \epsilon$ for all $\epsilon > 0$. This implies $\nu(A) = 0$. Hence, $\nu << \mu$.
Conversely, Let $\nu << \mu$. Suppose on contrary that there exists $\epsilon > 0$ for which there is no suitable $\delta$. Then for each $k \in \mathbb{N}$, we can choose an $\mathcal{A}$-measurable set $A_k$ that satisfies $\mu(A_k) < 1/2^k$ and $\nu(A_k) \geq \epsilon$.
Consider the inequalities,

$$\mu(\cup_{k=n}^{\infty} A_k) \;\leq\; \sum_{k=n}^{\infty} \mu(A_k) \;<\; 1/2^{n-1}$$

and

$$\nu(\cup_{k=n}^{\infty} A_k) \;\geq\; \nu(A_n) \;\geq\; \epsilon \quad \forall n$$

So, the set $A$ defined by $A = \cap_{n=1}^{\infty}(\cup_{k=n}^{\infty} A_k)$ satisfies $\mu(A) = 0$ and $\nu(A) \geq \epsilon$. Thus, $\mu(A) = 0$ but $\nu(A) \neq 0$, which is a contradiction. □

**Theorem B.1.21.** *(Radon-Nikodym Theorem)(for complex measure) Let $\mu$ be a $\sigma-$finite measure on $X$ and let $\nu$ be finite or signed measure on $X$. Let $\nu$ be absolutely continuous with respect to $\mu$. Then there exists unique $f$ in $L^1(X, \sum, \mu)$ such that*

$$\nu(A) \int_A f \; d(\mu).$$

*The function $f$ is unique upto $\mu$ almost everywhere equality.*

**Remark B.1.22.** *The function $f$ is called as the Radon-Nikodym derivative of $\nu$ with respect to $\mu$ denoted by $d\nu/d\mu$.*

**Theorem B.1.23** (Radon-Nikodym Theorem)**.** *Let $\mu$ and $\nu$ be finite measures on $(X,\mathcal{A})$ and $\nu$ be absolutely continuous with respect to $\mu$. Then there exists an $\mathcal{A}$-measurable function, $g : X \to [0, +\infty]$ such that $\nu(A) = \int_A g d\mu$ holds for each $A$ in $\mathcal{A}$. Moreover $g$ is unique upto $\mu$-almost everywhere equality.*

*Proof.* Firstly, let $\mu$ and $\nu$ both be $\sigma$-finite measures. Consider,

$$\mathcal{F} := \{f : X \to [0. +\infty] : \; f \; is \; \mathcal{A} - \text{measurable function and } \int_A f d\mu \leq \nu(A) \; \forall \; A \in \mathcal{A}\}$$

We claim that there exists $g \in \mathcal{F}$ such that $\int_A g d\mu = \sup\{\int_A f d\mu : f \in \mathcal{F}\}$ and $\nu(A) = \int_A g d\mu$ for all $A \in \mathcal{A}$. Let $f_1$ and $f_2 \in \mathcal{F}$ and $A \in \mathcal{A}$. Suppose,

$$A_1 = \{x \in A : f_1(x) > f_2(x)\} \text{ and } A_2 = \{x \in A : f_2(x) > f_1(x)\}$$

Then,

$$\begin{aligned}\int_A (f_1 \vee f_2)d\mu &= \int_{A_1} f_1 d\mu + \int_{A_2} f_2 d\mu \\ &\leq \nu(A_1) + \nu(A_2) \\ &= \nu(A)\end{aligned}$$

Hence, $f_1 \vee f_2 \in \mathcal{F}$. Also $\mathcal{F} \neq \phi$ as $0 \in \mathcal{F}$.
Now choose $\{f_n\} \subset \mathcal{F}$ for which

$$\lim_n \int_A f_n d\mu = \sup\ \{\int_A f d\mu : f \in \mathcal{F}\}$$

Replace $f_n$ with $f_1 \vee f_2 \vee ... \vee f_n$. $\{f_n\}$ is an increasing sequence. Let $g = \lim_n f_n$. Then by Monotone Convergence Theorem,

$$\int_A g d\mu = \lim_n \int_A f_n d\mu \leq \nu(A)\ .$$

The above inequalities hold for each $A \in \mathcal{A}$. Therefore, $g \in \mathcal{F}$.

$$\int_A g d\mu = \sup\ \{\int_A f d\mu : f \in \mathcal{F}\}$$

We will show that $\int_A g d\mu = \nu(A)$ for each $A \in \mathcal{A}$.
Since $g \in \mathcal{F}$, $\nu_0(A) = \nu(A) - \int_A g d\mu$ defines a positive measure on $\mathcal{A}$.
We now claim that $\nu_0 = 0$. Let $\nu_0 \neq 0$. As $\mu$ is finite, there is $\nu_0(X) > \epsilon\mu(X)$. Let $(P, N)$ be the Hahn-Decomposition for the signed measure $\nu_0 - \epsilon\mu$. Note that, for all $a \in \mathcal{A}$, we have $\nu_0(A \cap P) \geq \epsilon\mu(A \cap P)$. So we have,

$$\begin{aligned}\nu(A) = \int_A g d\mu + \nu_0(A) &\geq \int_A g d\mu + \nu_0(A \cap P) \\ &\geq \int_A g d\mu + \epsilon\mu(A \cap P) \\ &= \int_A (g + \epsilon\chi_p) d\mu\end{aligned}$$

Observe that $\mu(P) > 0$, as if $\mu(P) = 0$, then $\nu(P) = 0$, which implies that $\nu_0(P) = 0$. This will imply that $\nu_0(X) - \epsilon\mu(X) = \nu_0(N) - \epsilon\mu(N) \leq 0$ which contradicts that $\nu_0(X) > \epsilon\mu(X)$.
We have $g \in \mathcal{F}$. This implies $\int_A g d\mu \leq \nu(X) < +\infty$.
As we have $\int_A (g + \epsilon\chi_p) d\mu < \nu(A)$ for all $A \in \mathcal{A}$ and $\mu(P) > 0$. This implies $g + \epsilon\chi_p \in \mathcal{F}$ and $\int_A (g + \epsilon\chi_p) d\mu > \int_A g d\mu$, which contradicts that $\int_A g d\mu$ is supremum. Thus, $\nu_0 = 0$.
Hence, $\int_A g d\mu = \nu(A)$ for each $A \in \mathcal{A}$.

Secondly, let $\mu$ and $\nu$ both be $\sigma$-finite measures. Then there exists $\{B_n\} \subset \mathcal{A}$ such that $X = \cup_{n=1}^{\infty} B_n$ and $\mu(B_n) < +\infty$ and $\nu(B_n) < +\infty$ for every $n \in \mathbb{N}$.
For each $n$, we have, $g_n : B_n \to (0, +\infty)$ such that $\nu(A) = \int_A g_n d\mu$ for each $\mathcal{A}$-measurable subset $A$ of $B_n$. The function $g : X \to [0, +\infty]$ that agrees on each $B_n$ with $g_n$, is then the required function.
Now we will prove the uniqueness of $g$.
Let $g, h : X \to [0, +\infty)$ be $\mathcal{A}$-measurable functions that satisfy

$$\nu(A) = \int_A g d\mu = \int_A h d\mu \qquad\qquad \forall A \in \mathcal{A} \ .$$

If $\nu$ is finite, then $g - h$ is integrable and $\int_A (g-h) d\mu = 0 \quad \forall A \in \mathcal{A}$ . A may be the set where $g < h$ or $g > h$, then it follows that

$$\int_A (g-h)^+ d\mu = 0 \qquad\quad and \qquad\quad \int_A (g-h)^- d\mu = 0$$

This implies that $(g-h)^+$ and $(g-h)^+$ vanish $\mu$ almost everywhere and hence, $g = h$ $\mu$ almost everywhere.
If $\nu$ is $\sigma-$finite and $|B_n| \subset \mathcal{A}$ such that $\nu(B_n) \leq +\infty$ and $X = \cup_{n=1}^{\infty} B_n$, then $g$ and $h$ agree $\mu$ almost everywhere on each $B_n$ and thus on $X$.

□

*Proof.* (Proof of theorem B.1.21) If $\nu$ is a complex measure such that $\nu << \mu$ and $\nu = \nu_1 - \nu_2 + i\nu_3 - i\nu_4$, where $\nu_i$ are finite positive measures that are absolutely continuous with respect to $\mu$, then from the above theorem we have $g_j$, such that $\nu_j(A) = \int_A g_j d\mu \quad \forall A \in \mathcal{A}$ , $g = g_1 - g_2 + ig_3 - g_4$. (Simiarly, we can show it in case of finite signed measures.) □

**Remark B.1.24.** *We have $\mu << |\mu|$. So from Radon-Nikodym theorem,*

$$\mu(A) = \int_A \frac{d\mu}{d|\mu|} d|\mu| \qquad \forall A \in \mathcal{A}.$$

**Remark B.1.25.** *By moving from characteristic functions to simple functions and using Dominated Convergence Theorem, we have,*

$$\int f d\mu = \int_A f \frac{d\mu}{d|\mu|} d|\mu| \qquad \forall f \in B(X, \mu).$$

**Proposition B.1.26.** *Suppose that $(X, \mathcal{A}, \mu)$ is a measure space and that $f$ belongs to $\mathcal{L}^1(X, \mathcal{A}, \mu, \mathbb{R} \setminus \mathbb{C})$ and that $\nu$ is a finite or signed or complex measure defined by $\nu(A) = \int_A f d\mu$. Then,*

$$|\nu|(A) = \int_A |f| d\mu \quad (\forall A \in \mathcal{A}).$$

**Proposition B.1.27.** *Let $X$ be a Hausdorff space and $\mathcal{A}$ be a $\sigma$-algebra on $X$ that includes $\mathbb{B}(X)$. Also, let $\mu$ be a regular measure on $\mathcal{A}$. If $A \in \mathcal{A}$ and $A$ is $\sigma$-finite under $\mu$, then,*

$$\mu(A) = sup\{\mu(K)|K \subseteq A, K \text{ is compact}\}$$

**Remark B.1.28.** *Let $X$ be a locally compact Hausdorff space. For a regular Borel measure $\mu$ on $X$, we denote by $\mathcal{M}_a(X,\mu)$, the Banach space of all complex measures $\nu$ such that $\nu << \mu$.*
*For $f \in \mathcal{L}^1(X,\mu), \nu_f$, defined by $\nu_f(A) = \int_A f d\mu$, is a complex measure. Also see remark B.1.34*

**Proposition B.1.29.** *Let $X$ be a locally compact Hausdorff space and $\mu$ be a regular Borel measure on $(X, \mathbb{B}(X))$. Also let $f \in \mathcal{L}^1(X,\mu)$. Then $\nu_f$ defined by $\nu_f(A) = \int_A f d\mu$ be the finite signed or complex measure on $(X, \mathbb{B}(X))$. Moreover $\nu_f$ is regular.*

**Remark B.1.30.** *Clearly, $\nu_f << \mu$ and hence the map, $f \mapsto \nu_f$ is a linear isometric map from $f \in \mathcal{L}'(X,\mu)$ ,$\nu_f$ to $\mathcal{M}_a(X,\mu)$. By Radon-Nikodym theorem, the map is onto and hence an isometric isomorphism.*

**Proposition B.1.31.** *Let $X$ be a locally compact Hausdorff space and $\mu$ be a regular Borel measure on $X$. For a complex regular measure, $\nu$ on $X$, the following are equivalent:*

- *$\nu << \mu$,*
- *There is a function $f \in \mathcal{L}'(X,\mu)$ such that $\nu(A) = \int_A f d\mu$*

*Thus, we can identify $\mathcal{L}'(X,\mu)$ with $\mathcal{M}_a(X,\mu)$ as Banach space.*

**Definition B.1.32.** *Let $f$ be continuous real or complex valued function on a topological space $X$. The support of $f$, Supp$(f)$ is the closure of $A := \{x \in X | f(x) \neq 0\}$. That is Supp$(f) = \bar{A}$. If $X$ is a locally compact Hausdorff space, then the set of all continuous functions $f : X \to \mathbb{R} \setminus \mathbb{C}$ for which Supp$(f)$ is compact are denoted by $\mathcal{K}(X)$.*

$$\mathcal{K}(X) := \{f : X \to \mathbb{R} \setminus \mathbb{C} | f \text{ is continuous and } Supp(f) \text{ is compact}\}$$

**Definition B.1.33.** *Let $U$ be an open subset of locally compact hausdorff space, $X$ and if $0 \leq f \leq \chi_U$ and also supp$(f) \subseteq U$, then we denote it as $f \prec U$.*

**Theorem B.1.34** (Riesz Representation Theorem)**.** *Let $X$ be a locally compact Hausdorff space and let $I$ be a positive linear functional on $C_c(X)$ $[\mathcal{K}(X)]$ that is $I(f) \geq 0$ if $f \geq 0$. Then there is a unique regular Borel measure on $X$ such that*

$$I(f) = \int_X f d\mu$$

**Theorem B.1.35.** *Riesz Representation Theorem for Complex Measures Let $X$ be a locally compact Hausdorff space. For each complex regular Borel measure $\mu$ on $X$, define a linear functional $\phi : C_0(X) \to \mathbb{C}$ by*

$$\phi_\mu(f) = \int_X f d\mu$$

*Then the map, $\mu \mapsto \phi_\mu$ is an isometric isomorphism from $M_r(X)$ onto $(C_0(X))^*$.*

**Remark B.1.36.** *This theorem says that if we have a locally compact hausdorff space with some positive linear function of $C_c(X)$. Then there exists a unique regular Borel Measure on space, that is, if we have a positive linear functional on $X$, then there exists a regular Borel measure, $\mu$ on $X$ such that $I(f) = \int_X f d\mu$. Moreover, for every $I$ there exists a unique $\mu$ in correspondence.*

Now we will just give the outline of the proof where we will be using following lemma.

**Lemma B.1.37.** *Let $X$ be a locally compact Hausdorff space and $\mu$ be a regular Borel measure on $X$. If $U$ is an open subset of $X$, then*

$$\mu(U) = sup\{\int f d\mu | f \in C_c(X) \text{ and } 0 \leq f \leq \chi_U\}$$

*Proof.* (Outline of the proof)

1. Use the lemma B.1.37 to prove that, in Riesz Representation Theorem, if there exists such $\mu$, then it is unique.

2. Define a function $\mu^*$ on the open subsets of $X$ by

$$\mu^*(U) = sup\{\int f d\mu | f \in C_c(X) \text{ and } f \prec U\}$$

and then extend it to any subset of $X$ by

$$\mu^*(A) = inf\{\mu^*(U) : U \text{ is open and } A \subseteq U\}$$

3. Now show that $\mu^*$ is an outer measure on $X$ and every Borel subset of $X$ is $\mu^*$-measurable.

4. Let $A \subseteq X$ and $f \in C_c(X)$. Now show that

   - If $\chi_A \leq f$, then $\mu^*(A) \leq T(f)$.
   - $0 \leq f \leq \chi_A$ and $A$ is compact, then $I(f) \leq \mu^*(A)$.

5. Let $\mu$ be the restriction of $\mu^*$ on $\mathbb{B}(X)$ and let $\mu_1$ be the restriction of $\mu^*$ to the $\sigma$-algebra $\mathcal{M}_{\mu^*}$ of $\mu^*$-measurable sets. Then $\mu$ and $\mu_1$ are regular measures and $\int f d\mu = \int f d\mu_1 = I(f)$ for all $f \in C_c(X)$.
Hence the proof.

□

## B.2 Product Measures

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces and $X \times Y$ be their cartesian product. A subset of $X \times Y$ is a rectangle with measurable sides if it has the form $A \times B$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$. $\sigma$-algebra on $X \times Y$ generated by all rectangles with measurable sides is called the product of the $\sigma$-algebras $\mathcal{A}$ and $\mathcal{B}$ and is denoted by $\mathcal{A} \times \mathcal{B}$.

**Lemma B.2.1.** *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces.*

1. *If $E \subset X \times Y$ and $E \in \mathcal{A} \times \mathcal{B}$, then each section $E_x$ and $E^y$ are $\mathcal{B}$- measurable and $\mathcal{A}$- measurable respectively.*

2. *If $f$ is an extended real or complex valued $\mathcal{A} \times \mathcal{B}$-measurable function on $X \times Y$, then each section $f_x$ is $\mathcal{B}$- measurable and each section $f^y$ is $\mathcal{A}$- measurable.*

**Theorem B.2.2.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two $\sigma$-finite measurable spaces. Then there is a unique measure $\mu \times \nu$ on the $\sigma$-Algebra $\mathcal{A} \times \mathcal{B}$ such that $\mu \times \nu(A \times B) = \mu(A)\nu(B)$, holds for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*
*Furthermore, the measure under $\mu \times \nu$ of an arbitrary set $E$ in $\mathcal{A} \times \mathcal{B}$ is given by*

$$\mu \times \nu(E) = \int_X \nu(E_x)\mu(dx) = \int_Y \mu(E^y)\nu(dy)$$

*The measure $\mu \times \nu$ is called as product measure of $\mu$ and $\nu(E)$.*

**Remark B.2.3.** *In general $X$ and $Y$ are not $\sigma$-finite. We cannot expect such a good behaviour of product measure. $\mathbb{B}(X) \times \mathbb{B}(Y)$ do not necessarily contain all Borel subsets of $X \times Y$ and so measure defined on $\mathbb{B}(X) \times \mathbb{B}(Y)$ is not a Borel measure.*

**Proposition B.2.4.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be the $\sigma$-finite measure spaces. If $E$ belongs to the $\sigma$-algebra of $\mathcal{A} \times \mathcal{B}$, then the function $x \mapsto \nu(E_x)$ is $\mathcal{A}$-measurable and the function $y \mapsto \mu(E^y)$ is $\mathcal{B}$-measurable.*

**Theorem B.2.5** (Tonelli's Theorem)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be the $\sigma$-finite measure spaces and $f : X \times Y \to [0, +\infty]$ be $\mathcal{A} \times \mathcal{B}$-measurable, then*

1. *the function $x \mapsto \int_Y f_x d\nu$ is $\mathcal{A}$-measurable, and,*

2. *$f$ satisfies*

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X (\int_Y f_x d\nu)\mu(dx) = \int_Y (\int_X f^y d\mu)\nu(dy)$$

**Theorem B.2.6.** *Let $X$ and $Y$ be a locally compact Hausdorff space and $\mu$ and $\nu$ be a regular Borel measures on $X$ and $Y$ respectively. Then for $f \in C_c(X \times Y)$, the functions,*

$$x \mapsto \int_Y f_x(y)\nu(dy) \ \text{and} \ \ y \mapsto \int_X f^y(x)\mu(dx)$$

*belongs to* $C_c(X)$ *and* $C_c(Y)$ *respectively and*

$$\int_X \int_Y f(x,y)\nu(dy)\mu(dx) = \int_Y \int_X f(x,y)\mu(dx)\nu(dy)$$

**Remark B.2.7.** *Now we can define positive linear functional* $I$ *on* $C_c(X \times Y)$ *by* $I(f) = \int_X \int_Y f(x,y)\nu(dy)\mu(dx)$ *and then by Riesz Representation Theorem, there exists a unique regular Borel measure on* $X \times Y$*, which we will denote by* $\mu \times \nu$ *such that*

$$\int_X \int_Y f(x,y)\nu(dy)\mu(dx) = \int_{X\times Y} f(x,y)\nu(dy)\mu(dx)$$

*This gives* ***Product Regular Borel Measure****.*

**Theorem B.2.8.** *Let* $X$ *and* $Y$ *be a locally compact Hausdorff space and* $\mu$ *and* $\nu$ *be a regular Borel measures on* $X$ *and* $Y$ *respectively and* $E$ *be an open subset of* $X \times Y$ *or a Borel subset of* $X \times Y$ *such that there exists* $\sigma$*-finite Borel subsets of* $A \subset X$ *and* $B \subset Y$ *for which* $E \subset A \times B$*. Then the functions* $x \mapsto \nu(E_x)$ *and* $y \mapsto \mu(E^y)$ *are measurable and*

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\mu(dx) = \int_Y \mu(E^y)\nu(dy).$$

**Theorem B.2.9** (Fubini's Theorem (1))**.** *Let* $X$ *and* $Y$ *be a locally compact Hausdorff space and* $\mu$ *and* $\nu$ *be a regular Borel measures on* $X$ *and* $Y$ *respectively. Let* $f$ *be a non-negative real Borel measurable function on* $X \times Y$ *and suppose that there exist* $\sigma$*-finite Borel sets* $A \in \mathbb{B}(X)$ *and* $B \in \mathbb{B}(Y)$ *such that* $f(x,y) = 0$ *if* $(x,y) \notin A \times B$*. Then the functions,*

$$x \mapsto \int_Y f(x,y)\nu(dy) \text{ and } y \mapsto \int_X f(x,y)\mu(dx)$$

*are measurable and*

$$\int_{X\times Y} fd(\mu \times \nu) = \int_Y \int_X f(x,y)\mu(dx)\nu(dy) = \int_X \int_Y f(x,y)\nu(dy)\mu(dx)$$

*Proof.* Let $f = \chi_E,\ E \in \mathbb{B}(X,Y)$. By the assumption, there exists $A \times B,\ A \in \mathbb{B}(X),\ B \in \mathbb{B}(Y),\ E \subset A \times B$.
Note that, $\nu(E_x) = \int_Y \chi_E(x,y)\nu(dy)$ and $\mu(E^y) = \int_X \chi_E(x,y)\mu(dx)$. Hence by the previous proposition,

$$x \mapsto \int_Y \chi_E(x,y)\nu(dy) \text{ and } y \mapsto \int_X \chi_E(x,y)\mu(dx)$$

are measurable and

$$\int_{X\times Y} \chi_E(x,y)d(\mu\times\nu) = (\mu\times\nu)(E) = \int_Y \int_X \chi_E(x,y)\mu(dx)\nu(dy) = \int_X \int_Y \chi_E(x,y)\nu(dy)\mu(dx)$$

By linearity of integration and Monotone Convergence Theorem, it is true for any general function. $\square$

**Theorem B.2.10** (Fubini's Theorem (2)). *Let $X$ and $Y$ be a locally compact Hausdorff space and $\mu$ and $\nu$ be a regular Borel measures on $X$ and $Y$ respectively. Let $f \in L'(X,Y)$ and suppose that there exist $\sigma$-finite Borel sets $A \in \mathbb{B}(X)$ and $B \in \mathbb{B}(Y)$ such that $f(x,y) = 0$ if $(x,y) \notin A \times B$. Then,*

1. *$f_x$ is $\nu$-integrable for $\mu$ almost everywhere $x$ and $f^y$ is $\mu$-integrable for $\nu$ almost everywhere $y$.*
2. *The functions*
$$x \mapsto \int_Y f(x,y)\nu(dy) \ \ \textit{and} \ \ y \mapsto \int_X f(x,y)\mu(dx)$$
*are $\mu$ and $\nu$-integrable respectively.*
3. $$\int_{X\times Y} fd(\mu \times \nu) = \int_Y \int_X f(x,y)\mu(dx)\nu(dy) = \int_X \int_Y f(x,y)\nu(dy)\mu(dx)$$

# Bibliography


[Bonsall-Duncan] F.F. Bonsall and J. Duncan *Complete Normed algebras, Springer-Verlag, 1st edition-1973, 230-236*

[Cohn] Donald L. Cohn *Measure theory, Birkhausher, Second edition*

[Dales] H. G. Dales *Banach Algebras and Automatic Continuity. Oxford University Press, 2001.*

[Folland] Gerald B. Folland *A Course in Abstract Harmonic Analysis, CRC Press, 1995*

[Henrik] Henrik Wirzenius *Amenable Banach Algebras and Johnson's Theorem For the Group Algebra $L^1(G)$, Master's Thesis, University of Helsinki, 2012*

[Kehe-Zhu] Kehe-Zhu *An Introduction to operator Algebras, CRC Press 1993*

[Kreyszig] Erwin Kreyszig *Introductory to Functional Analysis with Applications, Wiley student edition 2013*

[Rob] Robert E. Megginson *An Introduction to Banach Space Theory, Springer-Verlag*

[Roy-Fitz] Halsey Royden and P.M. Fitzpatrick *Real analysis, Pearson Education, Fourth edition*

[Runde] Volker Runde *Lectures on Amenability. Lecture notes in mathematics, Springer Verlag, 2002.*

[Ryan] Raymond A. Ryan *introduction to Tensor Products of Banach Spaces, Springer-Verlag London Limited 2002*