ON INTEGRATION IN BANACH SPACES

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Approval Sheet

This Thesis on **Integration in Banach spaces** by Vijay Yadav is approved for the degree of Master of Science from Indian Institute of Technology Hyderabad.

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Vijay Yadav,
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Dedicated to my parents

Chapter 0

Notations and Definitions

λ represents the Lebesgue measure on $\mathbb{R}$.

$\mathbb{N}$ represents set of natural numbers.

$\mathbb{Z}$ represents set of integers.

$\mathbb{R}$ represents set of real numbers.

X denotes real Banach space.

X^* represents dual of Banach space X.

$\mathcal{R}[a, b]$ is set of all Riemann integrable functions over $[a, b]$.

$(R)\int_a^b f dt$ or $\int_a^b f dt$ represent Riemann integral of f in $[a, b]$.

$(D) - \int_E f d\mu$ represents Dunford integral of f over E.

$(P) - \int_E f d\mu$ represents Pettis integral of f over E.

$\mathcal{R}(\alpha)[a, b]$ is set of all Riemann Stieltjes integrable functions with respect to α, where α is a monotonic increasing function on $[a, b]$.

Definition 0.0.1. (i) A partition P of a closed interval $[a, b]$ is a finite set of points $\{x_0, x_1, x_2 ... x_n\}$, where $a = x_0 < x_1 < x_2 < ... < x_n = b$ and $\Delta x_i = x_i - x_{i-1}$ $(i = 1, 2, 3, ..., n)$

(ii) A tagged partition is a partition $\{x_0, x_1, x_2, ... x_n\}$ together with a set of points $\{s_i : 1 \leq i \leq n\}$ satisfy $s_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$.

For example, $P = \{s_i, [x_{i-1}, x_i] : 1 \leq i \leq n\}$ where $s_i \in [x_{i-1}, x_i]$ is a tagged partition over $[a, b]$. Let $\mathcal{P}$ be the set of all tagged partitions over $[a, b]$.

(iii) A tagged partition is a partition $\{x_0, x_1, x_2, ...x_n\}$ together with a set of points $\{s_i : 1 \leq i \leq n\}$ satisfy $s_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$.

(iv) Norm of partition P is denoted by $||P||$, $||P|| = \max\{x_i - x_{i-1} : 1 \leq i \leq n\}$.

Definition 0.0.2 (Refinement of partition)**.** The tagged partition P_1 is a refinement of tagged partition P_2 if points of P_2 form a subset of the points of P_1.

Let α be a monotonic increasing and bounded function on [a,b]. For any partition P on $[a, b]$, $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$, $\Delta\alpha_i$ is non-negative for all i, as α is a monotonic increasing function.

Let f is a bounded real function on $[a, b]$ and P $= \{a = x_0 < x_1 < x_2... < x_n = b\}$ is partition of $[a, b]$, then define,

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i; \text{ where } M_i = \max\{f(x_i) : x_i \in [x_{i-1}, x_i]\}$$
$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i; \text{ where } m_i = \min\{f(x_i) : x_i \in [x_{i-1}, x_i]\}$$

the upper Riemann integral is defined as $\overline{\int_a^b f d\alpha} = \inf U(P, f, \alpha)$ and lower Riemann integral is $\underline{\int_a^b f d\alpha} = \sup L(P, f, \alpha)$, the infimum and superimum are taken over all partitions.

Definition 0.0.3. (i) f is said to be Riemann Stieltjes integral on $[a, b]$ if upper Riemann Stieltjes integral and lower Stieltjes Riemann integral are equal and denoted by $\int_a^b f d\alpha$. And we write $f \in \mathcal{R}(\alpha)[a, b]$.

Sometimes we use the notation $\mathcal{R}(\alpha)$ if the underlying interval is known.

(ii) If $f : [a, b] \to X$ and $P = \{s_i, [x_{i-1}, x_i] : 1 \leq i \leq n\}$ is a tagged partition of [a,b] then $f(P)$ will denote the Riemann sum, $f(P) = \sum_{i=1}^{n} f(s_i)\Delta x_i$.

Definition 0.0.4. (i) For a point $x \in [0, 1]$ define $O_f(x, \delta) = \sup\{|f(y) - f(z)| : y, z \in [x-\delta, x+\delta]\}$. We call $O_f(x)$ the oscillation of f at x, defined by $\lim_{\delta \downarrow 0} O_f(x, \delta)$. It is clear that f is continuous at x if and only if $O_f(x) = 0$.

(ii) A partially ordered set $(A, \preceq)$ is called directed if, for any α, $\beta \in A$, there is $\gamma \in A$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

Being a finite subset of $[0, 1]$, a partition of $[0, 1]$ is an element of the directed set generated by the finite subsets of $[0, 1]$. For a given $f : [0, 1] \to (X, \tau)$, one can define a net on the set of all tagged partition on $[0, 1]$ to X by defining $f(P) = f(s_i)\Delta x_i : P \mapsto X$.

Definition 0.0.5. (Ω, Σ, μ) be a measurable space and X is a Banach space.

(i) $f : \Omega \to X$ is said to be measurable if there exists a sequence of simple functions (s_n) such that $\|s_n(t) - f(t)\| \to 0$ a.e. $[\mu]$.

(ii) A μ-measurable function $f : \Omega \to X$ is said to be p-th Bochner integrable if there exists a sequence of simple function (s_n) such that, $\lim_n \int_\Omega ||f(t) - s_n(t)||^p d\mu(t) = 0$.

(iii) Define $L_p(\mu, X) = \{f : \Omega \to X$ and f is p-th Bochner integrable$\}$, for $f \in L_p(\mu, X)$, $||f||_p = \left(\int_\Omega ||f||^p d\mu\right)^{\frac{1}{p}}$ defines a norm on $L_p(\mu, X)$ which makes it a Banach space.

(iv) Define $\ell_p(\Gamma) = L_p(\Gamma, 2^\Gamma, \mu)$, where μ is counting measure.
For $f \in \ell_p$, $||f||_p = (\int_\Gamma |f(t)|^p d\mu(t))^{\frac{1}{p}}$.

Remark 0.0.6. It is clear that for $f \in \ell_p(\Gamma)$, $\|f\|_p = \sup_{F \in \mathcal{F}} S_F$, where $\mathcal{F}$ stands for the set of all finite subsets of Γ and for $F \in \mathcal{F}, S_F = \sum_{\gamma \in F} f(\gamma)$.

Definition 0.0.7. Let $P = \{t_i : 0 \leq i \leq N\}$ is a partition of $[a, b]$, then

$$\omega(f, P) = \sum_{i=1}^{N} \omega(f, [t_{i-1}, t_i])(t_i - t_{i-1})$$

Where $\omega(f, [t_{i-1}, t_i]) = \sup\{||f(v) - f(u)|| : u, v \in [t_{i-1}, t_i]\}$ is the oscillation of the function f on the interval $[t_{i-1}, t_i]$.

Chapter 1

Introduction

1.1 Origin of Riemann integration

Bernhard Riemann (1826-66) no doubt acquired his interest in problems connected with trigonometric series through contact with Dirichlet when he spent a year in Berlin. Riemann began with the question: *when is a function integrable*? By that he meant, when do the Cauchy sums converge? He assumed this to be the case if and only if

$$\lim_{\|P\|\to 0}(D_1\delta_1 + D_2\delta_2 + \ldots + D_n\delta_n) = 0 \tag{1.1}$$

Where P is a partition of $[a, b]$ with δ_i's are the lengths of the subintervals and D_i is the oscillation of f in the i-th interval.

$$D_i = |\sup f(x) - \inf f(x)|$$

For a given partition P and $\delta > 0$, define

$$S(P, \delta) = \sum_{D_i > \delta} \delta_i$$

Riemann proved that the following is a necessary and sufficient condition for integrability.

$$\forall \varepsilon, \sigma > 0 \exists d > 0 \text{ such that if } P \text{ is any partition with } \|P\| \leq d, \text{ then } S(P, \sigma) < \varepsilon. \tag{1.2}$$

These conditions (1.1) and (1.2) are germs of the idea of measurability of the set with nonzero oscillation. But the time was not yet ready for measure theory.

Thus, with (1.1) and (1.2) Riemann has integrability without explicit continuity conditions. Yet it can be proved that Riemann integrability implies $f(x)$ is continuous almost everywhere.

Riemann gives this example: Define $m(x)$ to be the integer that minimizes $|x - m(x)|$. Let

$$(x) = \begin{cases} x - m(x), & \text{if } x \neq n/2, \text{ n odd} \\ 0, & \text{if } x = n/2, \text{ n odd }. \end{cases}$$

(x) is discontinuous at $x = n/2$ when n is odd. Now define,

$$f(x) = (x) + \frac{(2x)}{2^2} + ... + \frac{(nx)}{n^2} + ...$$

This series converges and $f(x)$ is discontinuous at every point of the form $x = m/2n$, where $(m, n) = 1$. This is a dense set. At such points the left and right limiting values of this function are

$$f(x\pm) = f(x) \mp (\pi^2/16n^2)$$

This function satisfies (1.2) and thus f is R-integrable.

The Riemann integral lacks important properties for limits of sequences and series of functions. The basic theorem for the limit of integrals is,

Theorem 1.1.1. Let J be a closed interval $[a, b]$, and let $\{f_n(x)\}$ be a sequence of functions such that

$$\lim_{n\to\infty} (R) \int_a^b f_n(x)dx \text{ exists}$$

and such that $f_n(x)$ tends uniformily to f(x) in J as $n \to \infty$. Then,

$$\lim_{n\to\infty} (R) \int_a^b f_n(x)dx = (R) \int_a^b f(x)dx$$

That this is unsatisfactory is easily seen from an example. Consider the sequence of functions defined on $[0, 1]$ by $f_n(x) = x^n, n = 1, 2, 3, ...$ Clearly, as $n \to \infty, f_n(x) \to 0$ pointwise on $[0, 1)$, for all n. Because the convergence is not uniform, we cannot conclude from the above theorem that

$$\lim_{n\to\infty} (R) \int_a^b f_n(x)dx = 0,$$

which of course, it is.

What is needed is something stronger. Specifically if $|f_n(x)| \leq g(x)$ and $\{f_n\}$, g (nonnegative) are integrable and if $\lim_n f_n = f(x)$ then f may not be Riemann integrable. This is basic flaw that was finally resolved with Lebesgue integration.

1.2 A cue from Measure theory

A measure can be defined with the help of a distribution function,

Definition 1.2.1 (Measure)**.** $\mathcal{F}$ is a field of finite disjoint union of left open and right closed intervals over $[0,1]$ and $\alpha : [0,1] \to \mathbb{R}$ is a monotonic increasing function on $[0,1]$. μ is pre-measure defined on $\mathcal{F}$ such that,

$$\mu(\bigcup_{i=1}^{n}(a_i, b_i]) = \sum_{i=1}^{n} \alpha(b_i) - \alpha(a_i)$$

In the next Chapter we have shown that the integration of a function with respect to a measure is same as the Riemann-Stieltjes integral with respect to that monotone increasing function. This fact clearly lead to the fact that the space of continuous functions on a compact interval in $\mathbb{R}$ has Bounded Variation dual.

One of the major characterizations for a real valued function to be Riemann integrable is that the function is continuous except a *negligible* set (in the sense of Lebesgue) *Lebesgue property*. Similar conclusion can be drawn for the class of functions taking values in finite dimensional spaces. Unfortunately the result is not necessarily true for arbitrary vector valued functions. The main thrust of this part of our work is to explore the nonavailability of this property in the Banach spaces. This work also extends to define various notions of integrations in Banach spaces, the interplay between these integrals and the properties obtained by the Banach spaces under the assumption of convergence of these integrals. Basically integrability of a function taking values over a Banach space leads to a vector valued measure, which is sometime relevant to discuss the hidden properties of Banach spaces. In Chapter 3 we study on various notions of integrations for vector valued functions, several examples are given which lacks the *Lebesgue property*. Towards the end of this Chapter we have listed a family of Banach spaces which lacks the above property, though ℓ_1 has this property.

Chapter 2

Riemann Stieltjes Integral

2.1 Few basic facts

In this Section $\alpha : [a, b] \to \mathbb{R}$ represents a monotone increasing, continuous function. Let us recall the class of functions $\mathcal{R}(\alpha)$ from Chapter 0. Our first Theorem characterizes the elements of $\mathcal{R}(\alpha)$.

Theorem 2.1.1. $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

Proof. Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

As upper Riemann integral is infimum of upper sums and lower Riemann integral is supremum of lower sums, therefore there exist partition P_1 and P_2 such that,

$$U(P_1, f, \alpha) < \overline{\int_a^b f d\alpha} + \frac{1}{2}\varepsilon = \int_a^b f d\alpha + \frac{1}{2}\varepsilon$$

and

$$L(P_2, f, \alpha) > \underline{\int_a^b f d\alpha} - \frac{1}{2}\varepsilon = \int_a^b f d\alpha - \frac{1}{2}\varepsilon$$

Let partition $P = P_1 \cup P_2$, P is refinement of P_1 and P_2.

$$\text{Now } \; U(P, f, \alpha) \le U(P_1, f, \alpha) < \int_a^b f d\alpha + \frac{1}{2}\varepsilon < L(P_2, f, \alpha) + \varepsilon.$$

This implies $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

Conversely, assume that for $\varepsilon > 0$ there exists partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

Now $\overline{\int_a^b f d\alpha} - \underline{\int_a^b f d\alpha} \leq U(P, f, \alpha) - L(P, f, \alpha)$ and as $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ we have the result. □

As we have discussed the correspondence between the class of all Lebesgue Stieltjes measures on an interval $[a, b]$ with the class of all distribution functions modulo scalars, our next Theorem induces relationship between $\mathcal{R}(\alpha)$ and the set of functions for which the set of points with non zero oscillation is *negligible.*

Theorem 2.1.2. $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if f is continuous a.e $[\mu]$, where μ be the measure induced by α.

Proof. Let $f \in \mathcal{R}(\alpha)$.

Let $|f(x)| \leq M$ for $x \in [0, 1]$. Let $D = \{x \in [a, b] : f \text{ is discontinuous at } x\}$. Clearly $D = \cup_n D_n$, where $D_n = \{x \in [a, b] : O_f(x) \geq 1/n\}$.

Claim: $\mu(D_n) = 0$ for all n.

Let us fix $n \in \mathbb{N}$. Since f is Riemann Stieltjes Integrable, given $\varepsilon > 0$ there exists a partition P over $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon/n$. Hence $\sum_{i=1}^{n}(M_i - m_i)\Delta x_i < \varepsilon/n$. Let $\{I_k : 1 \leq k \leq N\}$ be the sub intervals of P, after a suitable relabellings, which contain at least one point of D_n. Then,

$$1/n \sum \Delta\alpha_k \leq \sum_{k=1}^{N}(M_k - m_k)\Delta\alpha_k \leq \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i < \varepsilon/n.$$

Clearly we have $D_n \subseteq \cup_k I_k$ and hence $\mu(D_n) < \varepsilon$. Since ε is arbitrary we have $\mu(D_n) = 0$ and finally $\mu(D) = 0$.

Now let f is is continuous a.e $[\mu]$.

Let D be the set defined above. We have that $\mu(D) = 0$, hence by regularity of Lebesgue measure get an open set U and then an open set F of disjoint union of finite intervals such that $F \supseteq D$ and $\mu(F) < \varepsilon/2M$.

Hence $[a, b] \setminus F$ is a finite pairwise disjoint union of closed intervals where f is continuous. Hence for $\varepsilon > 0$ get a $\delta > 0$ such that for $|f(y) - f(z)| < \varepsilon$ whenever $|y - z| < \delta, y, z \in [a, b] \setminus F$. Let P_1 be a union of partitions of the disjoint intervals of $[a, b] \setminus F$ each of which is less than δ. Extend it by adding the end points, if necessary, to a partition in $[a, b]$.

CLAIM: $U(P, f, \alpha) - L(P, f, \alpha) < 2\varepsilon$.

$$\begin{aligned}
\text{We have, } U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i \\
&= \sum_{\Gamma_1}(M_i - m_i)\Delta\alpha_i + \sum_{\Gamma_2}(M_i - m_i)\Delta\alpha_i \\
&< \varepsilon\sum_{\Gamma_1}\Delta\alpha_i + 2M\mu(F) \\
&< \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

Γ_1 : indices of P contained in $[0, 1] \setminus F$ and Γ_2 : indices of P not in Γ_1.

This completes the proof. □

It is now an easy consequence of Theorem 2.1.2 that a function is Riemann Integrable if and only if f is continuous almost everywhere, in the sense of Lebesgue.

2.2 The main characterization

Proposition 2.2.1. Let α be a monotone increasing differentiable function such that α' is bounded over $[a, b]$, then $\mu \ll \lambda$ where μ is the measure over $[a, b]$ induced by α.

Proof. It is enough to prove that given $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $E \subseteq [a, b]$ measurable with $\lambda(E) < \delta$ we have $\mu(E) < \varepsilon$. But it is enough to prove the result for a class of generating subsets viz $\{(x, y] : a \leq x < y \leq b\}$. Since $\mu((x, y]) = \alpha(y) - \alpha(x)$, the result now follows from Mean Value Theorem for α. □

The main result of this Section is stated in Theorem 2.2.3. An elementary proof can be found in any standard Calculus book but our proof is more elegant and fancy which is due to Johann Radon and Otto Nikodym.

Theorem 2.2.2. (Radon-Nikodym) Given a measurable space (Ω, Σ), if a σ-finite signed measure ν on (Ω, Σ) is absolutely continuous with respect to a σ-finite measure μ on (Ω, Σ), then there is a measurable function $f : X \to \mathbb{R}, f \in L_1(\mu)$, such that for any measurable subset $E \subset \Omega$, $\nu(A) = \int_A f d\mu$.

Theorem 2.2.3. Let α be a monotone increasing differentiable function such that α' is Riemann integrable over $[a, b]$, then $f \in \mathcal{R}(\alpha)$ if and only if $f.\alpha'$ is Riemann integrable over $[a, b]$ and $\int_a^b f d\alpha = \int_a^b f\alpha' dt$.

Proof. From Theorem 2.1.2 we have $\int_a^b f d\alpha = \int_{[a,b]} f d\mu$. Now $\int_{[a,b]} f d\mu = \int_{[a,b]} f \frac{d\mu}{d\lambda} d\lambda$, where $\frac{d\mu}{d\lambda}$ is the Radon-Nikodym derivative of μ with respect to λ. We now show that $\alpha' = \frac{d\mu}{d\lambda}$ a.e. $[\lambda]$. Which is enough to show that for any measurable $E, \mu(E) = \int_E \alpha' d\lambda$. Similar to Proposition 2.2.1 we show that the result is true for the sets of the form $(x, y]$. Since α' is Riemann integrable we have $\int_{[x,y]} \alpha' d\lambda = \int_x^y \alpha'(t) dt$. Now the last integral is the limit of a sum of the type $\sum_{i=1}^n \alpha'(t_i)\Delta_i$, the Riemann sum corresponding to the partition P of $[x, y]$. Let $J_i = [s_{i-1}, s_i]$ be the i-th subinterval of P then $\alpha(s_i) - \alpha(s_{i-1}) = \alpha'(\xi_i)\Delta_i$ for some $\xi_i \in J_i$. For these choice of $\xi_i \in J_i$ the corresponding Riemann sum for partition P is close to $\sum_{i=1}^n \alpha'(t_i)\Delta_i$ if $\|P\|$ is sufficiently small. Hence it is clear that $\mu((x, y]) = \int_x^y \alpha' dt$ and finally we have $\int_{[a,b]} f \frac{d\mu}{d\lambda} d\lambda = \int_{[a,b]} f\alpha' d\lambda$. Being a product of two Riemann integrable functions, $f\alpha'$ is Riemann integrable and we have $\int_{[a,b]} f\alpha' d\lambda = \int_a^b f\alpha' dt$. □

Chapter 3

Various Notions of Integrations in Banach spaces

3.1 Bochner integral

Definition 3.1.1. A μ-measurable function $f : \Omega \to X$ is called Bochner integrable if there exists a sequence of simple functions (f_n) such that $\lim_n \int_\Omega ||f_n - f||d\mu = 0$ In this case, $\int_E f d\mu$ is defined for each $E \in \sum$ by $\int_E f d\mu = \lim_n \int_E f_n$.

Theorem 3.1.2. Let $f : I \to X$ be a measurable function, f is Bochner integrable if and only if $\int_I ||f(t)||d\lambda(t) < \infty$

Proof. If f is measurable if and only if $(s_n) \to f$ uniformly, where (s_n) is a sequence of simple functions which takes countably many values.

Claim: $\int_I ||s_n(t)||d\lambda(t) < \infty$

We can write s_n as

$$s_n = \sum_{i=1}^{k_n} (\alpha_i)^{(n)} \chi_{E_i^{(n)}}$$

Now,

$$\int_I ||s_n(t)||d\lambda(t) = \sum_{i=1}^{n} ||\alpha_i^{(n)}||||\lambda(E_i)^n|| < \infty$$

Claim: f is Bochner integrable, then $\int_I ||f(t)||d\lambda(t) < \infty$.

As f is Measurable, then $||f(t) - s_n(t)|| < \frac{1}{n}$ for all $t \in I$. Now,

$$||f(t)|| \leq ||f(t) - s_n(t)|| + ||s_n(t)||$$

Then,

$$\begin{aligned}
\int_I ||f(t)||d\lambda(t) &\leq \int_I ||f(t) - s_n(t)||d\lambda(t) + \int_I ||s_n(t)||d\lambda(t) \\
\int_I ||f(t)||d\lambda(t) &< \frac{1}{n}\lambda(I) + \int_I ||s_n(t)||d\lambda(t) \\
\int_I ||f(t)||d\lambda(t) &< \infty
\end{aligned}$$

Claim: If $\int_I ||f(t)||d\lambda(t) < \infty$, then f is Bochner integrable.

$\int_I ||s_n(t)||d\lambda(t) < \infty$ is a absolutely convergent sequence. So, for each n there exists a natural number k_n such that,

$$\sum_{i=k_n+1}^{\infty} ||\alpha_i^{(n)}\lambda(E_i^{(n)})|| < \frac{1}{n}$$

Now take $p_n = \sum_{i=1}^{k_n} \alpha_i^{(n)}\chi_{E_i^{(n)}}$, then

$$\begin{aligned}
||f(t) - p_n(t)|| &\leq ||f(t) - s_n(t)|| + ||s_n(t) - p_n(t)|| \\
\int_I ||f(t) - p_n(t)||d\lambda(t) &\leq \int_I ||f(t) - s_n(t)||d\lambda(t) + \int_I ||s_n(t) - p_n(t)||d\lambda(t)
\end{aligned}$$

This implies,

$$\begin{aligned}
\int_I ||f(t) - p_n(t)||d\lambda(t) &< \frac{1}{n}\lambda(I) + \frac{1}{n}\lambda(I) \\
\int_I ||f(t) - p_n(t)||d\lambda(t) &< \frac{2}{n}\lambda(I)
\end{aligned}$$

As n tends to infinity $\int_I ||f(t) - p_n(t)||d\lambda(t)$ tends to zero, hence f is Bochner integrable.

□

Theorem 3.1.3 (Dominated Convergence Theorm)**.** Let (Ω, Σ, μ) be a finite measure space and (f_n) be a sequence of Bochner integrable X-valued function on Ω. If $\lim_n f_n$ in μ-measure,(*i.e.*, $\lim_n \mu\{\omega \in \Omega : ||f_n - f|| \geq \varepsilon\} = 0$ for every $\varepsilon > 0$) and if there exists a real valued Lebesgue integrable function g on Ω with $||f_n|| \leq g$ μ-almost everywhere, then f is Bochner integrable and $\lim_n \int_E f_n d\mu = \int_E f d\mu$ for each $E \in \Sigma$. In fact, $\lim_n \int_\Omega ||f - f_n||d\mu = 0$.

Theorem 3.1.4. If f is Bochner integrable function, then

(*a*) $|| \int_E f d\lambda|| \leq \int_E ||f||d\lambda$

(*b*) $\lim_{\lambda\to 0} \int_E f d\lambda = 0$

(c) if (E_n) is a sequence of pairwise disjoint members of Σ and $E = \bigcup_{i=1}^{\infty} E_n$, then

$$\int_E f d\mu = \sum_{n=1}^{\infty} \int_{E_n} f d\mu$$

(d) if $F(E) = \int_E f d\mu$, then F is of bounded variation and $|F|(E) = \int_E ||f|| d\mu$ for all $E \in \Sigma$

Corollary 3.1.5. If f and g are Bochner integrable and $\int_E f d\mu = \int_E g d\mu$ for each $E \in \Sigma$, then $f = g$ μ-almost everywhere.

Theorem 3.1.6. Let T be a bounded linear operator defined inside X and having value in Banach space Y. If f and Tf are Bochner integrable with respect to μ, then

$$T\Big(\int_E f d\mu\Big) = \int_E T f d\mu$$

for all $E \in \Sigma$.

Corollary 3.1.7. Let f be a Bochner integrable with respect to μ. Then for each $E \in \Sigma$ with $\mu(E) > 0$,

$$\frac{1}{\mu(E)} \int_E f d\mu \in \overline{co}(f(E))$$

Proof. We will prove it by contradiction, let there is a set $E \in \Sigma$ of positive μ-measure such that $\frac{1}{\mu(E)} \int_E f d\mu \notin \overline{co}(f(E))$

Now, by geometric version of Hahn-Banach theorm, choose a $x^* \in X^*$ and a real number β such that,

$$x^*\left(\frac{1}{\mu(E)} \int_E f d\mu\right) < \beta \leq x^* f(\alpha)$$

for all $\alpha \in E$. And ,

$$\frac{1}{\mu(E)} \int_E x^* f d\mu < \beta \leq x^* f(\alpha)$$

For all $\alpha \in E$. Now,

$$\int_E x^* f d\mu < \beta\mu(E) \leq x^* f(\alpha)\mu(E)$$

$$\int_E x^* f d\mu < \beta\mu(E) \leq x^* f(\alpha) \int_E d\mu$$

$$\int_E x^* f d\mu < \beta\mu(E) \leq \int_E x^* f(\alpha) d\mu$$

This holds for all $\alpha \in E$, and hence

$$\int_E x^* f d\mu < \beta\mu(E) \leq \int_E x^* f d\mu$$

Which is a contradiction. □

Theorem 3.1.8. Let f be a Bochner integrable on $[0,1]$ with respect on $[0,1]$ with respect to Lebesgue measure. Then

$$\lim_{h\to 0}\int_x^{x+h} ||f(t)-f(s)||d\lambda = 0 \qquad a.e.[\lambda]$$

Consequently, for almost $x \in [0,1]$,

$$\lim_{h\to 0}\int_x^{x+h} f(t) = f(s).$$

3.2 Dunford and Pettis integral

Definition 3.2.1. Let X and Y be two normed linear spaces and T is a linear operator then $T : X \to Y$ is said to be a closed linear operator if $x_n \to x$ in X and $Tx_n \to y$ in Y then $Tx = y$.

It is clear that every bounded linear operator is a closed linear operator but the converse is not true.

Example 3.2.2. Example of a closed linear operator which is not bounded linear operator.

Let us define $T : C^1[0,1] \to C[0,1]$ by $T(f) = f^{'}$.

Now, let $(f_n) \subseteq C^1[0,1]$, $||f_n - f||_\infty \to 0$ for some $f \in C^1[0,1]$ and $||f_n^{'} - g||_\infty \to 0$ for some $g \in C[0,1]$.

CLAIM: $f(t) = \int_0^t g(x)dx$ for all $t \in [0,1]$.

$$\begin{aligned}|f_n(t) - \int_0^t g(x)dx| &= |\int_0^t f_n^{'}(x)dx - \int_0^t g(x)dx| \\ &\leq \int_0^t |f_n^{'}(x) - g(x)|dx \\ &\leq ||f_n^{'} - g||_\infty\end{aligned}$$

As $||f_n^{'} - g||_\infty$ converging to zero, hence $\lim_{n\to\infty} |f_n(t) - \int_0^t g(x)dx| = 0$ for all $t \in [0,1]$ and $|\lim_{n\to\infty} f_n(t) - \int_0^t g(x)dx| = 0$, this implies $f(t) = \int_0^t g(x)dx$.
Hence T is closed.

Claim: T is not bounded. Now, let us consider a sequence $x_n \subseteq C^1[0,1]$ such that $x_n(t) = t^n$.

$$||T(x_n)|| = \sup |nt^{n-1}| = n \text{ and } ||x_n|| = 1$$

Now, for each positive number K, we can find a positive number n such that $n > K$. For such n, $||T(x_n)|| > K||x_n||$, so T is not bounded.

Lemma 3.2.3 (Dunford)**.** Suppose f is weakly μ-measurable function on the abstract measure space (Ω, Σ, μ) and $x^*f \in L_1(\mu)$ for each $x^* \in X^*$. Then for each $E \in \Sigma$ there exists $x_E^{**} \in X^{**}$ satisfying,

$$x_E^{**}(x^*) = \int_E x^*(f)d\mu$$

for all $x^* \in X^*$.

Proof. Let $E \in \Sigma$ and define a function $T : X^* \to L_1(\mu)$ by $Tx^* = x^*(f\chi_E)$ Claim: T is closed operator. If $x_n^* \to x_0$ and $x_n^*(f\chi_E) \to g$ in $L_1(\mu)$ and as μ is finite there exists a subsequence $x_{n_k}^*(f\chi_E(t)) \to g(t)$ almost everywhere in μ. Hence,

$$\begin{aligned}
\int_E |g(t) - x_0^*(f\chi_E)(t)| &\leq \int_E |g(t) - x_{n_k}^* f\chi_E(t)|d\mu(t) \\
&\quad + \int_E |x_{n_k}^* f\chi_E(t) - x_0^* f\chi_E|d\mu(t) \\
&= \int_E |g(t) - x_{n_k}^* f\chi_E(t)|d\mu(t) \\
&\quad + \int_E |(x_{n_k}^* - x_0^*)(f\chi_E(t))|d\mu(t) \\
&\leq \int_E |g(t) - x_{n_k}^* f\chi_E(t)|d\mu(t) \\
&\quad + \int_E ||x_{n_k}^* - x_0^*||||f\chi_E(t)||d\mu(t)
\end{aligned}$$

Since $x_{n_k}^*(f\chi_E)(t) \to g(t)$ a.e. in μ and $x_n^* \to x_0^*$ then,

$\int_E |g(t) - x_0^*(f\chi_E)(t)|$ can be made less than ε, which is arbitrary positive number. Hence $g(t) = x_0^*(f\chi_E)(t)$, implies T is closed.
Now, by closed graph theorem T is bounded and hence continuous.

Let us define $\psi : L_1(\mu) \to \mathbb{R}$ by $\psi(g) = \int_\Omega g d\mu$.

Claim: ψ is bounded and linear.

$$\begin{aligned}\psi(g_1+g_2) &= \int_\Omega (g_1+g_2)d\mu \\ &= \int_\Omega g_1 d\mu + \int_\Omega g_2 d\mu \\ &= \psi(g_1)+\psi(g_2)\end{aligned}$$

Hence, ψ is bounded. And,

$$\begin{aligned}|\phi(g)| &= (\int_\Omega g d\mu) \\ &\leq \int_\Omega |g| d\mu \\ \leq ||g||_1\end{aligned}$$

This implies ψ is bounded and linear.
Now, as T and ψ both bounded and linear implies $\psi \circ T : X^* \to \mathbb{R}$ is also bounded linear. And,

$$\begin{aligned}\psi \circ T(x^*) &= \psi(T(x^*)) \\ &= \psi(x^* f(\chi_E)) \\ \psi \circ T(x^*) &= \int_E x^* f(\chi_E) d\mu\end{aligned}$$

Hence, for each $E \in \Sigma$ there exists $\psi \circ T \in X^{**}$ satisfying $\psi \circ T(x^*) = \int_E x^* f(\chi_E) d\mu$.

□

Definition 3.2.4. If f is weakly μ-measurable function on Ω and $x^* f \in L_1(\mu)$ for each $x^* \in X^*$, then f is called Dunford integrable. The Dunford integral of f over $E \in \Sigma$ is called by the element x_E^{**} of X^{**} such that,

$$x_E^{**}(x^*) = \int_E x^* f d\mu$$

For all $x^* \in X^*$, and we write $x_E^{**} = (D) - \int_E f d\mu$.
In case that $x_E^{**} \in X$ for each $E \in \Sigma$, then f is called Pettis integrable.

Remark 3.2.5. Pettis integral function is Dunford integral but converse is not true.

Example 3.2.6. A Dunford integrable function that is not Pettis integrable.

Define $f : [0,1] \to c_0$ by,

$$f(t) = (\chi_{(0,1]}(t), 2\chi_{(0,1/2]}(t), 3\chi_{(0,1/3]}(t), ..., n\chi_{(0,1/n]}(t), ...)$$

If $x^* \in c_0^* = \ell_1$ and $x^* = \{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n, ...\}$ then $x^* f = \sum_{i=1}^{\infty} \alpha_n n \chi_{(0,i/n]}$, and $x^* f$ is Lebesgue integrable, and

$$\int_{(0,1]} \sum_{i=1}^{\infty} \alpha_n n \chi_{(0,i/n]} d\lambda = \sum_{i=1}^{\infty} \alpha_n$$

Now, we have to find $x^{**} \in X^{**}$ such that,

$$x^{**}(x^*) = \int_{(0,1]} x^* f d\lambda = \sum_{i=1}^{\infty} \alpha_n$$

As $x^{**} \in X^{**} = \ell_\infty$ then let $x^{**} = (x_1, x_2, x_3, ..., x_n, ...)$ and,

$$x^{**}(x^*) = \sum_{i=1}^{\infty} x_i \alpha_i = \sum_{i=1}^{\infty} \alpha_n$$

Hence, $x^{**} = (1, 1, 1, ..., 1, ...) \in \ell_\infty \setminus c_0$

$$(D) - \int_{(0,1]} f d\lambda = (1, 1, 1, ..., 1, ...)$$

So, f is Dunford integral but not a Pettis integral.

3.3 Riemann Integration on Banach space

Let $f : [a, b] \to X$.

Definition 3.3.1 (R_δ integrable)**.** f is R_δ integrable on [a,b] if there exists a vector z in X such that
for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$||f(P) - z|| < \epsilon \text{ where } P \text{ is a tagged partition of [a,b] with } ||P||_1 < \delta,$$

The z is called R_δ integral of f on $[a, b]$.

Definition 3.3.2 (R_Δ integrable)**.** The function f is R_Δ integrable on [a,b] if there exists z in X such that,

for each $\epsilon > 0$ there exists partition P_ϵ such that;

$$||f(P) - z|| < \epsilon \text{ where } P \text{ is a tagged partition of [a,b] that refine } P_\epsilon.$$

The z is called R_Δ integral of f on $[a,b]$.

Theorem 3.3.3. A function $f : [a,b] \to X$ is R_Δ integrable if and only if f is R_δ integrable.

Proof. Suppose f is R_Δ integrable. Let z is R_Δ integral of f on $[a,b]$. Let $\varepsilon > 0$ and choose a partition $P_\varepsilon = \{a = x_0 < x_1 < x_2 ... < x_n = b\}$ such that $||f(P) - z|| < \varepsilon$, where P is tagged partition on $[a,b]$ that refines P_ε.

Let $\delta = \frac{\varepsilon}{2MN}$, where M is superimum of f on $[a,b]$.

To prove that f is R_δ integrable on $[a,b]$ we have to show that, $||f(P) - z||$ whenever P is a tagged partition with $||P|| < \delta$

Let P_1 is a tagged partition on $[a,b]$ such that $P_1 = P_\varepsilon \cup P$, i.e points of P_1 are the points of both P_ε and P. And the tag of P_1 for the intervals that coincides with an interval P_ε is same as tag for P, and tags for remaining intervals are arbitrary. Let $\{[c_k, d_k] : 1 \leq k \leq N\}$ be the intervals of P that contains where $N < n-1$. In the interval $[c_k, d_k]$ let $\{c_k = u_0^k < u_1^k < u_2^k ... < u_k^n = d_k\}$ where the points $\{u_i^k : 1 \leq i \leq n_{k-1}\}$ are the points of P_ε in $[c_k, d_k]$. Let s_k be tag of P for $[c_k, d_k]$ and let v_i^k be the tag of P_1 for $[u_{i-1}^k, u_i^k]$. Then,

$$\begin{aligned}
||f(P) - f(P_1)|| =& || \sum_{k=1}^{N} \{ f(s_i)(d_k - c_k) - \sum_{i=1}^{n_1} f(v_i^k)(u_i^k - u_{i-1}^k) \} || \\
\leq& \sum_{k=1}^{N} \sum_{i=1}^{n_k} ||f(s_k) - f(v_i^k)|| (u_i^k - u_{i-1}^k) \\
\leq& 2Mn\delta < \varepsilon
\end{aligned}$$

Claim : If f is R_δ integrable than f is R_Δ integrable. Suppose f is R_δ integrable and let z is R_δ integral of f, then for any $\varepsilon > 0$ there exists a δ such that;

$$||f(P) - z|| < \epsilon \text{ where } P \text{ is a tagged partition of } [a,b] \text{ with } ||P|| < \delta.$$

Now for each $\varepsilon > 0$ we have a P such that $||f(P) - z|| < \epsilon$, take $P_\varepsilon = P$, for any tagged partition which refines P_ε have norm less than delta.

Hence, For each ε there exists P_ε such that $||f(P_1) - z|| < \varepsilon$ where P_1 is a tagged partition of $[a,b]$ that refines P_ε. That is f is R_Δ integrable. □

Definition 3.3.4. The function $f : [a,b] :\to X$ is Riemann integrable on [a,b] if f is either R_δ or R_Δ integrable on $[a,b]$.

Theorem 3.3.5. Let $f : [a,b] \to X$, then following are equivalent.

(a) The function f is Riemann integrable on $[a,b]$.

(b) For each $\varepsilon > 0$ there exists $\delta > 0$ such that $||f(P_1) - f(P_2)|| < \varepsilon$ for all tagged partition P_1 and P_2 of $[a,b]$ with norms less than delta.

(c) For each $\varepsilon > 0$ there exists partition P_ε on $[a,b]$ such that $||f(P_1) - f(P_2)|| < \varepsilon$ for all tagged partition P_1 and P_2 of $[a,b]$ that refines P_ε.

(d) For each $\varepsilon > 0$ there exists partition P_ε on $[a,b]$ such that $||f(P_1) - f(P_2)|| < \varepsilon$ for all tagged partition P_1 and P_2 of $[a,b]$ that have same points as P_ε.

Proof. Statement (1) and (2) are equivalent.

Now, suppose f is Riemann integrable on $[a,b]$, then f is R_δ integrable on $[a,b]$, by definition of R_δ there exists a vector z such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $||f(P_1) - z|| < \frac{\varepsilon}{2}$ where P_1 is any tagged partition with norm less than δ.

Now,

$$||f(P_1) - f(P_2)|| = ||f(P_1) - z + z - f(P_2)|| \leq ||f(P_1) - z|| + ||z - f(P_2)|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

implies,

$$||f(P_1) - f(P_2)|| < \varepsilon$$

Where P_1 and P_2 are any two tagged partition on $[a,b]$.

Now, suppose For each $\varepsilon > 0$ there exists $\delta > 0$ such that $||f(P_1) - f(P_2)|| < \varepsilon$ for all tagged partition P_1 and P_2 of $[a,b]$ with $\|P_1\|, \|P_2\| \leq \delta$.

Claim: - Set of partitions on $[a,b]$ forms a derived set.

As set of partition form a partial ordered set with $P_1 \preceq P_2$ if $P_1 \subseteq P_2$. And for any P_1 and P_2 we can find partition P which is intersection of P_1, P_2 such that $P \preceq P_1$ and $P \preceq P_2$, hence Set of partitions on $[a,b]$ forms a derived set. And $f((P))$ forms a net which is cauchy net in X where P is any tagged partition on $[a,b]$ as for each $\varepsilon > 0$ there exists $\delta > 0$ such that $||f(P_1) - f(P_2)|| < \varepsilon$ for all tagged partition P_1 and P_2 of $[a,b]$ with norms less than delta. Now , X is Banach space hence this cauchy net converge to some point z in X.

Now for each $\varepsilon > 0$ there exists $\delta > 0$ such that $||f(P_1) - z|| < \varepsilon$ for all tagged partition P_1 whose norm is less than delta. Hence f is Riemann integrable.

Claim : Statement (1) and (3) are equivalent.

f is R_Δ integrable as f is Riemann integrable, by definition of R_Δ there exists a vector z called R_Δ integral of f such that for all $\varepsilon > 0$ there exists a partition P_ε for which $||f(P_1) - z|| < \frac{\varepsilon}{2}$ where P_2 is a tagged partition that refines P_ε.

Now for any given $\varepsilon > 0$, we have P_ε and for any two tagged partition P_1 and P_2 those refine P_ε such that

$$||f(P_1) - f(P_2)|| = ||f(P_1) - z + z - f(P_2)|| \leq ||f(P_1) - z|| + ||z - f(P_2)|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

implies

$$||f(P_1) - f(P_2)|| < \varepsilon$$

.

Now suppose For each $\varepsilon > 0$ there exists partition P_ε on $[a, b]$ such that $||f(P_1) - f(P_2)|| < \varepsilon$ for all tagged partition P_1 and P_2 of $[a, b]$ that refines P_ε. And as defined above $f((P))$ forms a net which is cauchy net in X where P is any tagged partition on $[a, b]$ as for each $\varepsilon > 0$ there exists a partition P_ε such that $||f(P_1) - f(P_2)|| < \varepsilon$ for all tagged partition P_1 and P_2 of $[a, b]$ which refines P_ε. Now , X is Banach space hence this cauchy net converge to some point z in X.

Now for each $\varepsilon > 0$ there exists a partition P_ε such that $||f(P_1) - z|| < \varepsilon$ for all tagged partition P_1 which refines P_ε. Hence f is Riemann integrable.

Claim: Statement (3) and (4) are equivalent.

Suppose For each $\varepsilon > 0$ there exists partition P_ε on $[a, b]$ such that $||f(P_1) - f(P_2)|| < \varepsilon$ for all tagged partition P_1 and P_2 of $[a, b]$ that refines P_ε, hence $||f(P_1) - f(P_2)|| < \varepsilon$ where P_1 and P_2 are tagged partition of $[a, b]$ which has same points as P_ε because P_1 and P_2 are refines P_ε.

Now, suppose for each $\varepsilon > 0$ there exists partition P_ε on $[a, b]$ such that $||f(P_1) - f(P_2)|| < \varepsilon$ for all tagged partition P_1 and P_2 of $[a, b]$ that have same points as P_ε. Let $\varepsilon > 0$ and choose partition $P_\varepsilon = \{t_i : 0 \leq \leq N\}$ of $[a, b]$ such that $||f(P_1) - f(P_2)|| < \frac{\varepsilon}{2}$ for all tagged partition P_1 and P_2 of $[a, b]$ that have same points as P_ε. Let P_0 be the tagged partition $\{(t_i, [t_{i-1,t_i}]) : 1 \leq i \leq N\}$. For each i, let W_i be the set $\{(t_i - t_{i-1})f(t) : t \in [t_{i-1}, t_i]\}$ and let $W = \sum_{i=1}^{N} W_i$. Note that $||x|| < \frac{\varepsilon}{2}$ for all x in $Co(W - W)$ where CoA denotes the convex hull of A.

Let $P = \{(v_k, [u_{k-1}, u_k]) : 1 \leq k \leq m\}$ be a tagged partitions of $[a, b]$ that refines P_ε. For each i, let k_i be the index k for which $u_k = t_i$. Then

$$\begin{aligned} f(P_0) - f(P_1) &= \sum_{i=1}^{N} \{f(t_i)(t_i - t_{i-1}) - \sum_{k=k_{i-1}+1}^{k_i} f(v_k)(u_k - u_{k-1})\} \\ &= \sum_{i=1}^{N} \sum_{k=k_{i-1}+1}^{k_i} \frac{u_k - u_{k-1}}{t_i - t_{i-1}} \{(t_i - t_{i-1})f(t) - (t_i - t_{i-1})f(v_k)\} \\ &\in \sum_{i=1}^{N} Co(W_i - W_i) = Co(W - W) \end{aligned}$$

And it follows that $||f(P_0) - f(P)|| < \frac{\varepsilon}{2}$. Now let P_1 and P_2 be tagged partition of $[a, b]$ that refines P_ε and commute

$$||f(P_1) - f(P_2)|| \leq ||f(P_1) - f(P_2)|| + ||f(P_0) - f(P_2)|| < \varepsilon$$

This completes the proof. □

Theorem 3.3.6. If $f : [a, b] \to X$, where X is a finite dimensional Banach space then f is Riemann integral if and only if f is continuous almost everywhere.

Proof. Let f be Riemann integrable that is there exists $z \in X$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $||f(P) - z|| < \varepsilon$ for every partition P with $||P|| < \delta$. Since, X is finite dimensional it can be identified with $\mathbb{R}^n$ for some $n \in \mathbb{N}$ with $||.||_1$ norm (since any two norms in a finite dimensional normed linear space are equivalent). Then we can write f as $f = (f_1, f_2, ..., f_n)$, where each f_i for $1 \leq i \leq n$ is component function of f.

Let $P = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq m\}$ be a tagged partition on $[a, b]$ such that $||P|| < \delta$. Then,

$$\begin{aligned} & ||f(P) - z|| < \varepsilon \\ \Rightarrow \quad & ||\sum_{i=1}^{m} f(t_i)\Delta x_i - z|| < \varepsilon \\ \Rightarrow \quad & ||\sum_{i=1}^{m} (f_1(t_i), f_2(t_i), ..., f_n(t_i))\Delta x_i - (z_1, z_2, ..., z_n)|| < \varepsilon \\ \Rightarrow \quad & ||(\sum_{i=1}^{m} f_1(t_i)\Delta x_i - z_1, ..., \sum_{i=1}^{m} f_m(t_i)\Delta x_i - z_n)|| < \varepsilon \\ \Rightarrow \quad & |\sum_{i=1}^{m} f_j(t_i)\Delta x_i - z_i| < \varepsilon \qquad \text{for each j=1,2,...,n} \end{aligned}$$

And this holds for every partition P with $||P|| < \delta$, which implies that f_j is Riemann integrable for each $1 \leq j \leq n$. Therefore each f_j for $1 \leq j \leq n$ is continuous almost everywhere, i.e., each f_j is continuous on A_j^c where $A_j \subseteq [a, b]$ is a measure zero set for each $j = 1, 2, ..., n$. Since the finite of measure zero sets is also a measure zero set, it is clear that f is continuous on the complement of measure zero set , i.e., f is continuous almost everywhere.

Conversely, suppose f is continuous almost everywhere, i.e., f is continuous on the complement of a measure zero set say A. Therefore each f_j for $j = 1, 2, ..., n$ is continuous on A^c. i.e., each f_j for $j = 1, 2, .., n$ is continuous almost everywhere and hence Riemann integrable.

CLAIM: f is Riemann integrable.

Let $\varepsilon > 0$ be given. Then there exists unique $z_j \in \mathbb{R}$ $\delta_j > 0$ such that $|\sum_{i=1}^{m} f_j(P) - z_j| < \varepsilon/n$ for every tagged partition $P = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq m\}$ with $||P|| < \delta_j$ for each $j = 1, 2, ..., n$. Take $\delta = \min\{\delta_1, \delta_2, ..., \delta_n\}$. Clearly $\delta > 0$. Now let P be a tagged partition with $||P|| < \delta$. Now,

$$\begin{aligned} ||f(P) - (z_1, z_2, ..., z_n)|| &= ||(f_1(P) - z_1, f_2(P) - z_2, ..., f_n(P) - z_n)|| \\ &= |f_1(P) - z_1| + |f_2(P) - z_2| + ... + |f_n(P) - z_n| \\ &< \varepsilon/n + \varepsilon/n + ... + \varepsilon/n = \varepsilon \end{aligned}$$

Therefore, f is Riemann integrable. □

Definition 3.3.7. (i) Let $f : [a, b] \to X$ is vector valued function, then the f is scalary measurable if x^*f is measurable for each x^* in X^*.

(ii) Let $f : [a, b] \to X$ is vector valued function, then the function f is of weak bounded variation on $[a, b]$ if x^*f is of bounded variation on $[a, b]$ for each x^* in X^*.

(iii) Let $f : [a, b] \to X$ is vector valued function, then the function f is of outside bounded variation on $[a, b]$ if $\sup\{||\sum_{i=1}^{n}(f(d_i) - f(c_i))||\}$ is finite where the supremum is taken over all finite collection $\{[c_i, d_i]\}$ of non overlapping intervals in $[a, b]$.

(iv) Let $f : [a, b] \to X$ is vector valued function, then the function f is a Scalar derivative of $F[a, b] \to X$ on $[a, b]$ if for each x^* in X^* the function x^*F is differentiable almost everywhere on $[a, b]$ and $(x^*F) = x^*f$ almost everywhere on $[a, b]$.

Theorem 3.3.8. Let $f : [a, b] \to X$ be Riemann integrable on $[a, b]$. Then

(a) The function f is Riemann integrable on every subinterval of $[a, b]$.

(b) If M is bound for f, then $|| \int_a^b f|| \leq M(b - a)$.

(c) If $T : X \to Y$ is a continuous liner operator, then Tf is a Riemann integral on $[a, b]$ and $\int_a^b (Tf) = T(\int_a^b f)$.

(d) For each x^* in X^*, the function x^*f is Riemann integrable on $[a, b]$ and $\int_a^b (x^*f) = x^* \int_a^b f$. Hence, the function f is scalary measurable , and for each x^* in X^*, the function x^*f is continuous almost every where on $[a, b]$.

Proof. CLAIM 1: The function f is Riemann integrable on every subinterval of $[a, b]$.

As f is Riemann integrable on $[a, b]$ then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $||f(P_1) - f(P_2)|| < \varepsilon$ for all tagged partition P_1 and P_2 of $[a, b]$ with norms less than delta,

Let $[c, d]$ is any subinterval of $[a, b]$ then for each $\varepsilon > 0$, there exists $\delta > 0$ which is same for $[a, b]$. Now, let P_a and P_b are any tagged partitions on $[c, d]$ with norm less than δ. We extend these tagged partitions on $[a, b]$ by adding n points at equal distance $\frac{c-a}{n}$ from a to c where n is such that $\delta > \frac{c-a}{n}$ and m points at equal distance $\frac{b-d}{m}$ between d to b where m is $\delta > \frac{b-d}{m}$ and we add same tags for extended partitions and call the partitions P_{a_1}, and P_{b_2}.

Now for extended partition P_{a_1}, and P_{b_2}

$$\varepsilon > ||f(P_{a_1}) - f(P_{b_2})|| = ||f(P_a) - f(P_b)||$$

where P_a and P_b are any tagged partitions whose norm is less than delta.

CLAIM:-2 If M is bound for f, then $|| \int_a^b f|| \leq M(b - a)$.
For any tagged partition P,

$$||f(P)|| = ||\sum_{i=1}^{n} f(\xi_i)\Delta x_i|| \leq M \sum_{i=1}^{n} \Delta x_i \leq M(b - a)$$

As f is Riemann integrable on $[a, b]$, then there exists z such that for each $\varepsilon > 0$ there $\delta > 0$ such that

$$||f(P) - z|| < \epsilon \text{ where } P \text{ is a tagged partition of [a,b] with } ||P|| < \delta$$

Now,

$$||z - (z - \sum_{i=1}^{n} f(\xi_i)\Delta x_i)|| \leq M(b - a)$$

this implies

$$||z|| - ||z - \sum_{i=1}^{n} f(\xi_i)\Delta x_i|| \leq ||z - (z - \sum_{i=1}^{n} f(\xi_i)\Delta x_i)|| \leq M(b-a)$$

implies $||z|| - \varepsilon \leq M(b-a)$ that is $||z|| \leq M(b-a) + \varepsilon$ for arbitrary ε. Hence, $|| \int_a^b f|| \leq M(b-a)$.

Claim: 3 $T : X \to Y$ is a continuous liner operator, then Tf is a Riemann integral on $[a, b]$ and $\int_a^b (Tf) = T(\int_a^b f)$.
As f is Riemann integrable for each $\varepsilon > 0$ there exists $\delta > 0$ such that $||f(P_1) - f(P_2)|| < \frac{\varepsilon}{M}$, where M is bound of T as T is continuous, hence T is bounded and P_1, P_2 are tagged partition on $[a, b]$ whose norm is less than delta.Now,

$$\begin{aligned}||Tf(P_1) - Tf(P_2)|| &= ||\sum_{i=1}^{n} Tf(\xi_i)\Delta x_i - \sum_{i=1}^{m} Tf(\eta_i)\Delta x_i|| \\ &= ||\sum_{i=1}^{n} T(f(\xi_i))\Delta x_i - \sum_{i=1}^{m} T(f(\eta_i))\Delta x_i|| \\ &= ||T(\sum_{i=1}^{n} f(\xi_i)\Delta x_i - \sum_{i=1}^{m} f(\eta_i)\Delta x_i)||\end{aligned}$$

As T is bounded ,

$$\begin{aligned}||T(\sum_{i=1}^{n} f(\xi_i)\Delta x_i - \sum_{i=1}^{m} f(\eta_i)\Delta x_i)|| &\leq M||(\sum_{i=1}^{n} f(\xi_i)\Delta x_i - \sum_{i=1}^{m} f(\eta_i)\Delta x_i)|| \\ ||Tf(P_1) - Tf(P_2)|| &\leq M||(\sum_{i=1}^{n} f(\xi_i)\Delta x_i - \sum_{i=1}^{m} f(\eta_i)\Delta x_i)|| \\ ||Tf(P_1) - Tf(P_2)|| &< M(\frac{\varepsilon}{M})\end{aligned}$$

Where P_1 and P_2 are any tagged partition whose norm is less than delta.

Claim: $\int_a^b Tf = T(\int_a^b f)$. Now as f is Riemann integrable, then there exists z such that $\int_a^b f = z$.

For every $\varepsilon > 0$ there exists $\delta > 0$ such that $||f(P_1) - z|| < \varepsilon$, where P_1 is tagged partition on $[a, b]$ with norm less than delta. Now,

$||Tf(P_1) - T(\int_a^b f)|| = ||T(f(P_1) - z)|| < M(\frac{\varepsilon}{M})$ where P_1 is any tagged partition whose norm is less than delta. Hence, $\int_a^b Tf = T(\int_a^b f)$.

Claim:4 For each x^* in X^*, the function x^*f is Riemann integrable on $[a,b]$ and $\int_a^b(x^*f) = x^*\int_a^b f$. Hence, the function f is scalary measurable , and for each x^* in X^*, the function x^*f is continuous almost every where on $[a,b]$.

This is special case of pervious Claim if we take $Y = \mathbb{R}$. Hence x^*f is Riemann integrable on $[a,b]$ and$\int_a^b(x^*f) = x^*\int_a^b f$. Now $x^*f : [a, b \to \mathbb{R}]$ is Riemann integrable on $[a,b]$, hence it is lebesgue integrable on $[a,b]$ then it must be measurable. And as Riemann integrable function from a closed interval to $\mathbb{R}$ is continuous almost every where implies that x^*f is continuous almost every where on $[a,b]$. □

Theorem 3.3.9. Let $f : [a,b] \to X$ be a Riemann integrable continuous function on $[a,b]$ and let $F(t) = \int_a^t f$. Then F is absolutely continuous on $[a,b]$ and f is scalar derivative of F on $[a,b]$. Furthermore, at each point t of continuity of f the function F is differentiable and $F' = f(t)$.

Proof. Let $I = \{[c_i,d_i] : 1 \le N\}$ be disjoint subintervals of $[a,b]$, let $\varepsilon > 0$ is given,

$$\sum_{i=1}^{N} F(d_i) - F(c_i) < \varepsilon$$
$$\sum_{i=1}^{N} \int_a^{d_i} f - \int_a^{c_i} f < \varepsilon$$
$$\sum_{i=1}^{N} \int_{c_i}^{d_i} f < \varepsilon$$

As f is Riemann integrable, and let M is maximum value of f on $[a,b]$,

$$\sum_{i=1}^{N} \int_{c_i}^{d_i} f \le \sum_{i=1}^{N} M(d_i - c_i) < \varepsilon$$

And $\sum_{i=1}^{N}(d_i - c_i) < \delta = \frac{\varepsilon}{M}$. Hence, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sum_{i=1}^{N} F(d_i) - F(c_i) < \varepsilon$ whenever $\sum_{i=1}^{N}(d_i - c_i) < \delta$, where $[c_i,d_i]$ are disjoint for all $1 \le i \le N$. Hence F is absolutely continuous on $[a,b]$.

□

Theorem 3.3.10. If $f : [a,b] \to X$ is outside bounded variation on $[a,b]$ then f is Riemann integrable on $[a,b]$. Consequently, a function of weak bounded variation is Riemann integrable.

Proof. Let $\varepsilon > 0$ is given, Let M is outside bounded variation of f on $[a,b]$. Choose a positive integer N such that $\frac{b-a}{N} < \frac{\varepsilon}{M}$.

Let $P_\varepsilon = \{t_i : t_i = a + \frac{i(b-a)}{N}, 1 \leq i \leq N\}$ be the partition of $[a, b]$. Let $P_1 = \{(u_i, [t_{i-1,t_i}]) : 1 \leq i \leq N\}$ and $P_2 = \{(v_i, [t_{i-1,t_i}]) : 1 \leq i \leq N\}$ are tagged partition of $[a, b]$ which have same points as P_ε, and we have

$$\begin{aligned}
||f(P_1) - f(P_2)|| &= ||\sum_{i=1}^{n} f(u_i)(t_i - t_{i-1}) - \sum_{i=1}^{n} f(v_i)(t_i - t_{i-1})|| \\
&= ||\sum_{i=1}^{n}(f(u_i) - f(v_i))(t_i - t_{i-1})|| \\
&= \frac{b-a}{N} ||\sum_{i=1}^{n}(f(u_i) - f(v_i))|| \\
&\leq \frac{b-a}{N} M \\
||f(P_1) - f(P_2)|| &< \varepsilon
\end{aligned}$$

That is for given $\varepsilon > 0$ there exists partition P_ε on $[a, b]$ such that $||f(P_1) - f(P_2)|| < \varepsilon$ for all tagged partition P_1 and P_2 of $[a, b]$ that have same points as P_ε. Hence f is Riemann integrable.

Claim: A function of weak bounded variation is also outside bounded variation, hence Riemann integrable. As f is weak bounded variation then for any $x^* \in X^*$, and for any collection of disjoint subintervals there exists a positive number M such that $|| \sum_{i=1}^{N} (f(c_i) - f(d_i))|| < M$.

Now, $\sup_{||x^*|| \leq 1} || \sum_{i=1}^{N} x^* f(c_i) - x^* f(d_i)|| = \sup_{||x^*|| \leq 1} ||x^*(\sum_{i=1}^{N}(f(c_i) - f(d_i)))|| = || \sum_{i=1}^{N}(f(c_i) - f(d_i)))||$.This implies,

$M > \sup_{||x^*|| \leq 1} || \sum_{i=1}^{N} x^* f(c_i) - x^* f(d_i)|| = || \sum_{i=1}^{N}(f(c_i) - f(d_i)))||$

Hence, $|| \sum_{i=1}^{N}(f(c_i) - f(d_i)))|| < M$, that is f is outside bounded variation, so f is Riemann integrable.

□

Example 3.3.11. A measurable, Riemann integrable function that is not continuous almost everywhere.

Let $\{r_n\}$ be listing of rational number in $[0, 1]$. And define $f : [0, 1] \to c_0$ by,

$$f(t) = \begin{cases} 0 = (0, 0, 0...) & \text{if t is irrational} \\ e_n & \text{if } t \in \{r_n\} \end{cases}$$

Claim: f is outside bounded variation and Riemann integrable.

Let N be a positive integer, and $\{[c_i, d_i] : 1 \leq i \leq N\}$ are the non-overlapping subintervals of $[0, 1]$, then $|| \sum_{i=1}^{N}(f(d_i) - f(c_i))|| = 0$ if all the c_i and d_i are irrational otherwise $|| \sum_{i=1}^{N}(f(d_i) - f(c_i))|| = 1$. This implies,

$$\sup || \sum_{i=1}^{N}(f(d_i) - f(c_i))|| \leq 1$$

Hence f is outside bounded variation and every outside bounded variation is Riemann integrable, so f is Riemann integrable.

CLAIM: f is not continuous at any point in $[0, 1]$.

Let α is any irrational number in $[0, 1]$, if we take $\varepsilon = \frac{1}{2}$ then there does not exists $\delta > 0$ such that $||f(t) - f(\alpha)|| < \varepsilon$ whenever $|t - \alpha| < \delta$ because whatever δ we choose there is a rational number satisfying $|t - \alpha| < \delta$ and then $||f(t) - f(\alpha)|| = 1$. Hence for $\varepsilon = \frac{1}{2}$ then there does not exists $\delta > 0$ and f is not continuous on any irrational number.

Similarly, Let β is any rational number in $[0, 1]$, if we take $\varepsilon = \frac{1}{2}$ then there does not exists $\delta > 0$ such that $||f(t) - f(\beta)|| < \varepsilon$ whenever $|t - \beta| < \delta$ because whatever δ we choose there is a rational number satisfying $|t - \beta| < \delta$ and then $||f(t) - f(\beta)|| = 1$. Hence for $\varepsilon = \frac{1}{2}$ then there does not exists $\delta > 0$ and f is not continuous on any rational number.

Hence f is not continuous almost everywhere as it is discontinuous at each point of $[0, 1]$ and $\lambda([0, 1]) \neq 0$.

Example 3.3.12. A measurable Riemann integrable function that is not of outside bounded variation.

Let $\{r_n\}$ be listing of rational number in $[0, 1]$ and define $f : [0, 1] \to \ell_2$ by,

$$f(t) = \begin{cases} 0 = (0, 0, 0...) & \text{if t is irrational} \\ e_n & \text{if } t \in \{r_n\} \end{cases}$$

CLAIM: f is Riemann integrable on $[0, 1]$.

Let $\varepsilon > 0$ and let $\delta = \varepsilon^2$, let $P = \{(v_i, [t_{i-1}, t_i]) : 1 \leq i \leq N\}$ is any tagged partition with $||P|| < \delta$,

$$\begin{aligned}||f(P) - 0|| =& || \sum_{i=1}^{N} f(v_i)(t_i - t_{i-1})|| \\ \leq& \left\{ \sum_{i=1}^{N} (t_i - t_{i-1})^2 \right\}^{\frac{1}{2}} \\ \leq& ||P||^{\frac{1}{2}} \left\{ \sum_{i=1}^{N} (t_i - t_{i-1}) \right\}^{\frac{1}{2}} \\ <& \varepsilon\end{aligned}$$

Hence, the function f is Riemann integrable on $[0, 1]$ with integral 0. CLAIM: f is not outside bounded variation on $[0, 1]$.

Let N be a positive integer and for each positive integer i, let c_i be an irrational number in interval $(\frac{1}{(i+1)}, \frac{1}{i})$, then

$$|| \sum_{i=1}^{N} \left(f(\frac{1}{i}) - f(c_i) \right) || = \left(\sum_{i=1}^{N} 1 \right)^{\frac{1}{2}} = \sqrt{N}$$

This shows that f is not of bounded variation on $[0, 1]$.

Example 3.3.13. A Riemann integrable function that is not measurable and not weakly continuous almost every where.

Define $f : [0, 1] \to \ell_\infty[0, 1]$ by $f(t) = \chi_{[0,t]}$.

CLAIM: f is outside bounded variation on $[0, 1]$, and hence Riemann integrable.

Let N be a positive integer and $[c_i, d_i] : 1 \leq i \leq N$ are disjoint subintervals of $[0, 1]$, And $|| \sum_{i=1}^{N} (f(c_i) - f(d_i))|| = || \sum_{i=1}^{N} (\chi_{[0,c_i]} - \chi_{[0,d_i]})|| = || \sum_{i=1}^{N} (\chi_{[c_i,d_i]})|| = ||\chi_{[c_1,d_1]} + \chi_{[c_2,d_2]} + ... + \chi_{[c_N,d_N]}|| \leq ||\chi_{[0,1]}||$, where $[c_1, d_1], [c_2, d_2]...[c_N, d_N]$ are all disjoint subintervals of $[0, 1]$. This implies,

$$|| \sum_{i=1}^{N} (f(c_i) - f(d_i))|| \leq ||\chi_{[0,1]}|| = 1$$

Hence, f is outside bounded variation on $[0, 1]$, and Riemann integrable.

CLAIM: f is not measurable.

Let $S \subseteq [0, 1]$ not measurable and let $G = \bigcup_{s \in S} B(\chi_{[o,s]}, 1/2)$. G is a subset of $\ell_\infty[0, 1]$.

CLAIM: $f^{-1}(G) = S$.

Let $t \in f^{-1}(G)$, this implies $f(t) \in G$ and by definition $f(t) = \chi_{[0,t]} \in G$ implies $t \in S$, and hence $f^{-1}(G) \subseteq S$.

Conversely, let $s \in S$ then $\chi_{[0,s]} \in G$ and by definition $f(s) = \chi_{[0,s]} \in G$ this implies $s \in f^{-1}(G)$, so $S \subseteq f^{-1}(G)$.

Hence $f^{-1}(G) = S$.

Now, $G = \bigcup_{s\in S} B(\chi_{[0,s]}, 1/2)$ is open in $\ell_\infty[0,1]$ and $f^{-1}(G) = S$, but S is not measurable. Hence f is not measurable.

Claim: f is not weakly continuous almost every where on $[0,1]$.

For proving this we have to find $x^* \in X^*$ such that x^*f is not continuous on a set whose measure is non zero.

Define $\chi_{\{t\}} : \ell_\infty[0,1] \to \mathbb{R}$ by $\chi_{\{t\}}(f) = f(t)$, where f is a element of $\ell_\infty[0,1]$.

Claim: $\chi_{\{t\}} \in (\ell_\infty[0,1])^*$

Let $h, g \in \ell_\infty[0,1]$, then $\chi_{\{t\}}(h+g) = (h+g)(t) = h(t)+g(t) = \chi_{\{t\}}(h) + \chi_{\{t\}}(g)$, hence $\chi_{\{t\}}$ is linear.

Consequently, $|\chi_{\{t\}}(g)| = |g(t)| \leq \sup_{t\in[0,1]} |g(t)| = ||g||$. This implies $\chi_{\{t\}}$ is bounded.

Hence, $\chi_{\{t\}} \in (\ell_\infty[0,1])^*$

Theorem 3.3.14. Let $([a,b], \sigma(\mathcal{F}), \mu)$ be a measure space, where $\mathcal{F}$ is a field generated by finite disjoint union of left open right closed subintervals of $[a,b]$. Then clearly, the smallest σ- field generated by $\mathcal{F}$ is equal to $\sigma(\mathcal{F})$ Then given $F \in \sigma(\mathcal{F})$ and $\varepsilon > 0$ there exists $E \in \mathcal{F}$ such that $\mu(E\Delta F) < \varepsilon$, where $(E\Delta F) = E \setminus F \cup F \setminus E$.

Theorem 3.3.15. If $f : [a,b] \to X$ is Riemann integrable then f is Pettis Integral. In addition if f is measurable then f is Bochner integrable.

Proof. Let $f : [a,b] \to X$ be Riemann integrable.

Claim: f is bounded and let M be the bound of f.

For any $x^* \in X^*$, $x^*(\int_a^b f) = \int_a^b x^*f$ and hence ,

$$\int_{[a,b]} |x^*f| d\lambda = \int_a^b |x^*f(t)| dt$$

By Dunford lemma $D - \int_{[a,b]} f \in X^{**}$.
Now, $\int_I f \in X$ if I is a subinterval of $[a,b]$. $\mathcal{F}$ is a field generated by finite disjoint union

of left open right closed subintervals of $[a, b]$. Then clearly, the smallest σ- field generated by $\mathcal{F}$ is equal to $\mathcal{B}_{[a,b]}$, the Boreal σ-field over $[a, b]$. Then $([a, b], \mathcal{B}_{[a,b]}, \lambda)$ is measure space where λ is the Lebesgue measure on $[a, b]$.

Let $F \in \mathcal{B}_{[a,b]}$ and $\varepsilon < 0$ be any positive number then by the pervious theorem there exists $E \in \mathcal{F}$ such that $\mu(E\Delta F) < \varepsilon$. Then clearly $E = I_1 \cup I_2 \cup ... \cup I_n$, where each $I_j, 1 \leq j \leq n$ is a left open right closed interval.

Claim: $\int_F f \in X$. Now,

$$\begin{aligned}
||\int_F f - \int_E f|| &= \sup_{||x^*|| \leq 1} |x^*(\int_F f - \int_E f)| \\
&= \sup_{||x^*|| \leq 1} |\int_F x^* f - \int_E x^* f| \\
&= \sup_{||x^*|| \leq 1} |\int_{F \cap E} x^* f + \int_{F \setminus E} x^* f - \int_{F \cap E} x^* f - \int_{E \setminus F} x^* f| \\
&\leq \sup_{||x^*|| \leq 1} |\int_{F \setminus E} x^* f| + |\int_{E \setminus F} x^* f| \\
&\leq \sup_{||x^*|| \leq 1} \int_{F \setminus E} |x^* f| + \sup_{||x^*|| \leq 1} \int_{E \setminus F} |x^* f| \\
&\leq M(\lambda(F \setminus E) + \lambda(E \setminus F)) \\
&\leq M(\lambda(E\Delta F))
\end{aligned}$$

Therefore $|| \int_F f - \int_E f||$ can be made less than any arbitrary positive number ε, which shows that $\int_F f \in X$, since X is Banach space. Therefore f is Pettis integrable.

Now if f is measurable then $x^* f$ is measurable for all $x^* \in X^*$. And $\int_{[a,b]} \in X$ then,

$$\begin{aligned}
||\int_{[a,b]} f d|| &= \sup_{||x^*|| \leq 1} |x^*(\int_{[a,b]} f)| \\
&= \sup_{||x^*|| \leq 1} |\int_{[a,b]} x^* f| \\
&= \sup_{||x^*|| \leq 1} |\int_a^b x^* f| \\
&= \int_a^b \sup_{||x^*|| \leq 1} |(x^* f)(t)| dt \\
&= \int_a^b ||f|| dt \\
&= \int_{[a,b]} ||f|| < \infty
\end{aligned}$$

Since $\int_{[a,b]} f \in X$. Therefore it is Bochner integrable.

□

Remark 3.3.16. We have obtained that Riemann integrability implies Pettis integrability implies Dunford integrability. We also obtained Bochner integrability implies Pettis integrability but the converse is not true.

3.4 Derboux integral

Definition 3.4.1. Let $f : [a, b] \to X$, then the following two conditions are equivalent,

(a) For each $\varepsilon > 0$ there exists $\delta > 0$ such that $\omega(f, P) < \varepsilon$ whenever P is a partition of $[a, b]$ that satisfy $||P|| < \delta$.

(b) For each $\varepsilon > 0$ there exits a partition P_ε of $[a, b]$ such that $\omega(f, P) < \varepsilon$ whenever P is a partition of $[a, b]$ that refine P_ε.

Proof. CLAIM: First condition implies second condition.
Let f satisfy first condition that is for given $\varepsilon > 0$ there exists $\delta > 0$ such that $\omega(f, P) < \varepsilon$ whenever P is a partition of $[a, b]$ that satisfy $||P|| < \delta$, now choose a N such that $N > \frac{b-a}{\delta}$ and we will construct a partition $P_\varepsilon = \{a + i\frac{b-a}{N} : 1 \leq i \leq N\}$. And for $\varepsilon > 0$ we got partition P_ε such that $\omega(f, P) < \varepsilon$, where P is any partition of $[a, b]$ that refines P_ε.

CLAIM: Second condition implies first condition.
Let f satisfy second condition that is for given $\varepsilon > 0$ there exists a partition P_ε such that $\omega(f, P) < \varepsilon$ where P is a partition on $[a, b]$ that refines P_ε and $||P_\varepsilon|| = \delta$, now for given $\varepsilon > 0$, we have $\delta > 0$ such that $\omega(f, P) < \varepsilon$, where P is any partition of $[a, b]$ such that $||P|| < \delta$. □

Theorem 3.4.2. The function $f : [a, b] \to X$ is Darboux integrable on $[a, b]$ if it satisfy one of the above equivalent conditions.

Theorem 3.4.3. A function $F : [a, b] \to X$ is Darboux integrable on $[a, b]$ if and only if it is bounded and continuous almost everywhere on $[a, b]$.

Proof. Suppose that f is Darboux integrable on $[a, b]$.

CLAIM: f is bounded.

As f is Darboux integrable, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\omega(f, P) < \varepsilon$ whenever $||P|| < \delta$. Now if function is not bounded then there exists a interval of partition such that $\omega(f,[t_{i-1},t_i])\to\infty$, then $\omega(f,[t_{i-1},t_i])\delta\to\infty$, this implies $\omega(f,P)\to\infty$ which is contradiction. Hence f must be bounded.

CLAIM: f is continuous almost everywhere.

Let $E_n=\{t\in[a,b]:\omega(f,t)\geq\frac{1}{n}\}$ for each positive integer n and let $E=\bigcup_n E_n$. Since each E_n is closed, the set E is measurable and we must show that $\mu(E)=0$ and if $\mu(E)\neq 0$ then there exists $\eta>0$ and a positive integer N such that $\mu(E_N)=\eta$. Let P be any partition of $[a,b]$ and let P_1 be the collection of intervals of P that contains points of E_N in there interior. then,

$$\omega(f,P)\geq\sum_{I\in P_1}\omega(f,I)\mu(I)\geq\frac{1}{N}\mu(E_N)=\frac{\eta}{N},$$

a contradiction to Darboux integrability of f. Thus, the function f is continuous almost everywhere on $[a,b]$.

Now suppose that f is bounded and continuous almost everywhere on $[a,b]$, and let M is bound of f. We will show that f is D_Δ integrable on $[a,b]$.

Let $\varepsilon>0$ and choose a positive integer N such that $\frac{b-a}{N}<\frac{\varepsilon}{2}$. Let $E_N=\{t\in[a,b]:\omega(f,t)\leq\frac{1}{N}\}$. We will construct a partition P_ε of $[a,b]$ such that the sum of the length of intervals of P_ε that intersect E_N is less than $\frac{\varepsilon}{4M}$ and the oscillation of f on each interval of P_ε that does not intersect E_N is less than $\frac{1}{N}$. Denoted the intervals of P_ε that intersect by $P_\varepsilon^{'}$ and remaining intervals by $P_\varepsilon^{''}$. Since $\mu(E_N)=0$, there exists a sequence $\{(c_i,d_i)\}$ of disjoint open intervals such that $E_N\subset\bigcup_i(c_i,d_i)$ and $\sum_i(d_i-c_i)<\frac{\varepsilon}{4M}$. Since the set E_N is closed and bounded, it is compact and therefore a finite a finite number of intervals $\{(c_i,d_i)\}$ cover E_N. The closure of each interval in the finite subcover intersected with $[a,b]$ is an element of $P_\varepsilon^{'}$. Let $[\alpha,\beta]$ be an intervals in $[a,b]$ that is contiguous to the interval of $P_\varepsilon^{'}$. Since $[\alpha,\beta]\cap E_N=\phi$, for each $t\in[\alpha,\beta]$ there exists $\delta_t>0$ such that $\omega(f,t[t-\delta_t,t+\delta_t])<\frac{1}{N}$. The collection $\{(t-\delta_t,t+\delta_t):t\in[\alpha,\beta]\}$ is an open cover of $[\alpha,\beta]$ and thus there exists a finite subcover. The end points of the interval comprising the finite subcover that belong to (α,β) together with α and β from a partition of $[\alpha,\beta]$. Put the intervals of this partition into $P_\varepsilon^{''}$ and do this for all of the intervals in $[a,b]$ that are contiguous to $P_\varepsilon^{'}$. It is easy checked that the interval of $P_\varepsilon^{'}$ and $P_\varepsilon^{''}$ combine to form partition of P_ε of $[a,b]$ with the desired properties.

Let P be a partition of $[a,b]$ that refine P_ε. Let $P_\varepsilon^{'}$ and $P_\varepsilon^{''}$ be the interval of P that are entirely contained within intervals $P_\varepsilon^{'}$ and $P_\varepsilon^{''}$, respectively and

$$\begin{aligned}\omega(f,P) &= \sum_{I\in P'_\varepsilon}\omega(f,I)\mu(I) + \sum_{I\in P''_\varepsilon}\omega(f,I)\mu(I)\\ &\leq 2M.\frac{\varepsilon}{4M} + \frac{1}{N}(b-a)\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Thus f is Darboux integrable on $[a,b]$. □

Corollary 3.4.4. If $f:[a,b]\to X$ is Darboux integral on $[a,b]$, then f is measurable and $||f||$ is Riemann integrable on $[a,b]$. Consequently, the function f is Bochner integrable on $[a,b]$.

Proof. By the above theorem the function f is continuous almost everywhere on $[a,b]$ and hence measurable. The function $||f||$ is bounded and continuous almost everywhere on $[a,b]$ and hence Riemann integrable on $[a,b]$.

Consequently a measurable function that is Riemann integrable is Bochnar integrable. □

Definition 3.4.5. A Banach space is said to have Lebesgue property if for any $f:[a,b]\to X$ which is Riemann integrable is continuous a.e.$[\lambda]$.

Theorem 3.4.6. Let Y be a subspace of X,

(a) If X has the property of Lebesgue, then Y has the property of Lebesgue.

(b) If Y does not have the property of Lebesgue, then X does not have the property of Lebesgue.

Theorem 3.4.7. The following spaces does not have Lesbesgue property.

(a) The space $c_0, c, C[a,b], \ell_\infty[0,1]$ and $L_\infty[0,1]$.

(b) The space ℓ_p for $1<p\leq\infty$.

(c) The space $L_1[a,b]$.

Proof. Claim: The space ℓ_p for $1<p<\infty$ does not have Lebesgue property.

Let $\{r_n\}$ be listing of rational number in $[0,1]$ and define $f:[0,1]\to\ell_p$ by,

$$f(t)=\begin{cases}0=(0,0,0...) & \text{if t is irrational}\\ e_n & \text{if } t\in\{r_n\}\end{cases}$$

Claim: f is Riemann integrable on $[0,1]$.

Let $\varepsilon > 0$ and let $\delta = \varepsilon^p$, let $P = \{(v_i, [t_{i-1}, t_i]) : 1 \leq i \leq N\}$ is any tagged partition with $||P|| < \delta$,

$$\begin{aligned} ||f(P) - 0||_p =& || \sum_{i=1}^{N} f(v_i)(t_i - t_{i-1})||_p \\ \leq& \left\{ \sum_{i=1}^{N} (t_i - t_{i-1})^p \right\}^{\frac{1}{p}} \\ \leq& ||P||^{\frac{1}{2}} \left\{ \sum_{i=1}^{N} (t_i - t_{i-1}) \right\}^{\frac{1}{2}} \\ <& \varepsilon \end{aligned}$$

Hence, the function f is Riemann integrable on $[0, 1]$ with integral 0.

Claim: f is not continuous at any point in $[0, 1]$.

Let α is any irrational number in $[0, 1]$, if we take $\varepsilon = \frac{1}{2}$ then there does not exists $\delta > 0$ such that $||f(t) - f(\alpha)||_p < \varepsilon$ whenever $|t - \alpha| < \delta$ because whatever δ we choose there is a rational number satisfying $|t - \alpha| < \delta$ and then $||f(t) - f(\alpha)||_p = 1$. Hence for $\varepsilon = \frac{1}{2}$ then there does not exists $\delta > 0$ and f is not continuous on any irrational number.

Similarly, Let β is any rational number in $[0, 1]$, if we take $\varepsilon = \frac{1}{2}$ then there does not exists $\delta > 0$ such that $||f(t) - f(\beta)||_p < \varepsilon$ whenever $|t - \beta| < \delta$ because whatever δ we choose there is a rational number satisfying $|t - \beta| < \delta$ and then $||f(t) - f(\beta)||_p = 1$. Hence for $\varepsilon = \frac{1}{2}$ then there does not exists $\delta > 0$ and f is not continuous on any rational number.

Hence f is not continuous almost everywhere as it is discontinuous at each point of $[0, 1]$ and $\lambda([0, 1]) \neq 0$.

This proves that f is a Riemann integrable function and not continuous almost everywhere, hence the space ℓ_p for $1 < p < \infty$ does not have Lebesgue property.

Claim: ℓ_2 is imbeds in $L_1[0, 1]$. To prove this firstly we will use a function known as Rademacher's function defined by $r_n(t) = sign(\sin(2^n \pi t))$ for $t \in [0, 1]$ and $n \in \mathbb{N}$.

Define a map $T : \ell_2 \to L_1[0, 1]$ by $T((a_n)) = \sum_{n=1}^{\infty} a_n r_n$, where r_n is Rademacher's function. By Khintchine inequality it follows that T is an isomorphism from $\ell_2 \to L_1[0, 1]$. □

Theorem 3.4.8. Khintchine's Inquality[5] Let r_n be the Rademacher's function on $[0, 1]$. For each $p \in [0, \infty]$, there exists positive constant A_p and B_p such that for every

$a_1, a_2, ..., a_m$.

$$A_p\big(\sum_{n=1}^{m}|a_n|^2\big)^{\frac{1}{2}} \leq \big(\int_0^1 |\sum_{n=1}^{m} a_n r_n(t)|^p\big)^{\frac{1}{p}} \leq B_p\big(\sum_{n=1}^{m}|a_n^2|\big)^{\frac{1}{2}}$$

By A_p and B_p we denote the best possible constants in this inequality. They are called Khintchine's constants and their values are known. We observe that $A_2 = B_2 = 1$. By Holder inequality, it follows that if $p > r$, then $\big(\int_0^1 |f|^p)dt\big)^{\frac{1}{p}} \leq \big(\int_0^1 |f|^r)dt\big)^{\frac{1}{r}}$. Consequently, $A_r \leq A_p$ and $B_r \leq B_p$.

Definition 3.4.9. A banach space X is said to be uniformly convex if for any two sequences $(x_n), (y_n) \in$ such that $||\frac{x_n+y_n}{2}|| \to 1$ then $||x_n - y_n|| \to 0$.

The space $L_1[0,1]$ is a example of uniform convex Banach space and the space $L_\infty[0,1]$ is not a uniform convex Banach space.

Remark 3.4.10. If $\{x_n\}$ is a normalized basis of the uniformly convex space X, then there exists $M > 0$ and $r > 1$ such that,

$$||\sum_{n=1}^{\infty}\alpha_n x_n|| \leq M(\sum_{n=1}^{\infty}|a_n|^r)^{\frac{1}{r}}$$

For all finite non-zero sequence $\{\alpha_n\}$ of real numbers.

Theorem 3.4.11. An infinite dimensional uniformly convex Banach space does not have the Lebesgue property.

Proof. Let X be an infinite dimensional uniformly convex Banach space. Since X is infinite dimensional Banach space, it contains a basic sequence $\{x_n\}$ and we may assume that $||x_n|| = 1$ for every n. Let Y be a closed linear space generated by $\{x_n\}$. Then Y is uniformly convex and $\{x_n\}$ is normalized basis of Y. To complete the proof it is sufficient to prove that Y does not have Lebesgue property.

Let $\{r_n\}$ be listing of rational number in $[0,1]$ and define $f : [0,1] \to Y$ by,

$$f(t) = \begin{cases} e_n & \text{if } t \in \{r_n\} \\ \theta & \text{otherwise} \end{cases}$$

Now, we will show that f is Riemann integrable on $[0,1]$ with integral θ.

Choose M and r as in the above remark and let $\varepsilon > 0$ be given. Let $\delta = (\frac{\varepsilon}{M})^{\frac{r}{r-1}}$, and let $P = \{(s_k, [t_{k-1}, t_k]) : 1 \leq k \leq N\}$ be a tagged partition of $[0, 1]$ that satisfy $||P|| < \delta$.then

$$\begin{aligned}||f(P)|| =& ||\sum_{k=1}^{N} f(s_k)(t_k - t_{k-1})|| \\ \leq& M\left(\sum_{k=1}^{N}(t_k - t_{k-1})^r\right)^{\frac{1}{r}} \\ \leq& M\delta^{\frac{r-1}{r}}\left(\sum_{k=1}^{N}(t_k - t_{k-1})\right)^{\frac{1}{r}} \\ \leq& M.\frac{\varepsilon}{M} = \varepsilon\end{aligned}$$

Therefore, the function f is Riemann integrable on $[0, 1]$ and it is clear that f is not continuous almost everywhere on $[0, 1]$, hence the space Y does not have Lebesgue property. □

Corollary 3.4.12. The following spaces do not have the property of Lebesgue.

(a) Infinite dimensional spaces.

(b) The space $L_p[a, b]$ for $1 < p < \infty$.

Proof. These spaces are infinite dimensional and uniformly convex. □

Theorem 3.4.13. The space ℓ_1 has the property of lebesgue.

Proof. It is sufficient to prove that a bounded function $f : [0, 1] \to \ell_1$ that is not continuous almost everywhere is not Riemann integrable on $[0, 1]$. Let $f : [0, 1] \to \ell_1$ is bounded but not continuous almost everywhere. As f is not continuous almost everywhere than there exists positive numbers α and β such that $\mu(H) = \alpha$, where $H = \{x : osc_f(x) \geq \beta\}$. We will prove that for each $\delta > o$ there exists tagged partitions P_1 and P_2 such that $||f(P_1) - f(P_2)|| \geq \alpha\beta/4$, and proof will complete.

Let $\delta > 0$ given, choose a natural number N such that $1/N < \delta$ and let $P_N = \{k/N : 0 \leq k \leq N\}$. Let $\{[c_i, d_i] : 1 \leq i \leq p\}$ be all intervals of P_N for which $\mu(H \cap [c_i, d_i]) > 0$ and also $\mu(p/N) > \alpha$. For each positive integer j, let G_j be the set of discontinuities of $e_j f$. If $\mu(G_j) \neq 0$ then e_j and consequently f is not Riemann integrable on $[0, 1]$. Otherwise the set $G = \bigcup_j G_j$ is a measure zero set and f will continuous on $[0, 1] \setminus G$.

Let $\varepsilon = \alpha\beta/16$ and we will construct sets $\{u_i : u_i \in (H \setminus G) \cup (c_i, d_i) \text{ for } 1 \leq i \leq p\}$, $\{v_i : v_i \in [c_i, d_i] \text{ for } 1 \leq i \leq p\}$ and $\{n_i : 0 \leq i \leq p\}$ where each n_i is an integer and $\{0 = a_0 < a_1 < a_2... < a_p\}$ that have the following property.

Let $z_i = f(u_i) - f(v_i) = \{a^i_j\}$

Then

$$\begin{aligned} ||z_i|| \geq & \beta/2 && \text{for all } i \geq 1 \\ \sum_{j=n_i}^{\infty} |a^i_j| < & \varepsilon 2^{-i} && \text{for all } i \geq 1 \\ \sum_{j=1}^{n_{i-1}} |a^i_j| < & \varepsilon 2^{-i} && \text{for all } i \leq 2. \end{aligned}$$

Now we proceed as follows,

Let $n_0 = 0$ and choose $u_1 \in (H \setminus G) \cup (c_1, d_1)$. Since $osc_f(u_1) \geq \beta$ then there exists a point $v_1 \in (c_1, d_1)$ such that $||f(u_1) - f(v_1)|| \geq \beta/2$.

Let $z_1 = f(u_1) - f(v_1) = \{a^1_j\}$ and choose an integer $n_1 > n_0$ such that $\sum_{j=n_1}^{\infty} |a^1_j| < \varepsilon/2$. Now choose $u_2 \in (H \setminus G) \cap (c_2, d_2)$, since $osc_f(u_2) \geq \beta$ and since $e_j f$ is continuous at u_2 for each $1 \leq j \leq n_1$, there exists v_2 such that $||f|f(u_1) - f(v_1)|| \geq \beta/2$ and $\sum_{j=1}^{n_1} |e_j f(u_2) - e_j(v_2)| < \varepsilon/4$. Let $z_2 = f(u_2) - f(v_2) = \{a^2_j\}$; then $\sum_{j=1}^{n_1} |a^2_j| < \varepsilon/4$. Choose an integer $n_2 > n_1$ such that $\sum_{j=n_2}^{\infty} |a^2_j| < \varepsilon/4$. We continue this process for p steps and arrive at desired sets.

Let $y_i = \sum_{j=n_{i-1}+1}^{n_i - 1} a^i_j e_j$ for each $1 \leq i \leq p$. Then

$$||z_i - y_i|| = \sum_{j=1}^{n_i - 1} |a^i_j| + \sum_{j=n_i}^{\infty} |a^i_j| < 2.\varepsilon 2^{-i}$$

and

$$||y_i|| = ||z_i|| - ||z_i - y_i|| \geq \frac{1}{2}\beta - 2\varepsilon 2^{-i}$$

For all $1 \leq i \leq p$. Therefore,

$$\begin{aligned} ||\sum_{i=1}^{p} z_i|| \geq & ||\sum_{i=1}^{p} y_i|| - ||\sum_{i=1}^{p} (y_i - z_i)|| \\ \geq & \sum_{i=1}^{p} ||y_i|| - \sum_{i=1}^{p} ||(y_i - z_i)|| \\ \geq & \sum_{i=1}^{p} \frac{1}{2}\beta - 2\varepsilon 2^{-i} - 2.\varepsilon 2^{-i} \\ \geq & \frac{1}{2} p\beta - 4\beta \end{aligned}$$

Now let P_1 and P_2 be two tagged partitions of $[0,1]$ that have the same points as P_N. The tags of P_1 and P_2 are the same in the remaining intervals. Then $||P_1|| < \delta$ and $||P_2|| < \delta$ and we have,

$$||f(P_1) - f(P_2)|| = ||\sum_{i=1}^{p} \frac{1}{N} z_i|| \geq \frac{1}{2}\frac{p}{N}\beta - \frac{4}{N} \geq \frac{1}{2}\alpha\beta - \frac{1}{4}\alpha\beta = \frac{\alpha\beta}{4}.$$

This completes the proof.

□

Definition 3.4.14. $f : [a,b] \to X$. The function f is scalarly Riemann integrable on $[a,b]$ if x^*f is Riemann integrable on $[a,b]$ for all $x^* \in X^*$. And if for each interval $I \subset [a,b]$ there exists a vector x_I in X such that $x^*(x_I) = \int_I x^* f dx$ for all $x^* \in X^*$, then f is Riemann-Pettis integrable on $[a,b]$.

Remark 3.4.15. Every Riemann integrable function is Riemann-Pettis integrable and every scalary Riemann integrable function is Dounford integrable. And Riemann-Pettis integral is also Pettis integral.

Theorem 3.4.16. A measurable, scalarly Riemann integrable function is Riemann-Pettis integrable. Consequently, in a separable space every scalarly Riemann integrable function is Riemann-Pettis integrable.

Proof. Let $f : [a,b] \to X$ is measurable and scalarly Riemann integrable on $[a,b]$ then f is Bochner integrable and hence Pettis integrable on $[a,b]$ this implies f is Pettis integrable. □

Corollary 3.4.17. A bounded function that is weakly continuous almost everywhere is Riemann-Pettis integrable.

Proof. Let $f : [a,b] \to X$ be bounded and weakly continuous almost everywhere on $[a,b]$. Then f is measurable and scalarly Riemann integrable. Then f is measurable and scalarly Riemann integrable on $[a,b]$. Then the function is Riemann-Pettis integrable on $[a,b]$. □

Theorem 3.4.18. In a weekly sequentially complete space every scalarly Riemann integrable function is Riemann-Pettis integrable.

Proof. Let X be a weakly sequentially complete that is every weakly Cauchy sequence is weakly convergent in X and let $f : [a,b] \to X$ be a scalarly Riemann integrable on $[a,b]$. Let $[c,d] \subset [a,b]$. We have to show that there exists a vector $z \in X$ such that $x^*(z) = \int_c^d x^* f$ for all $x^* \in X^*$.

For each positive integer n, let P_n be a tagged partition of $[c, d]$ with points $\{c+(\frac{k}{n})(d-c) : 0 \leq k \leq n\}$. Since each x^*f is Riemann integrable on $[c, d]$, the sequence $\{f(P_n)\}$ is weak cauchy sequence and as X is weakly sequentially complete this sequence converge to a vector $z \in X$. For each $x^* \in X^*$ we have,

$$x^*(z) = \lim_{n\to\infty} f(P_n) = \int_c^d x^*f$$

This completes the proof. □

Theorem 3.4.19. A scalarly Riemann integrable function that has a relatively compact range is Riemann integrable and in fact Darboux integrable.

Proof. Let $f : [a, b] \to X$ be scalarly Riemann integrable on $[a, b]$ and suppose that the range of f is relatively compact and hence f is measurable consequently Riemann-Pettis integrable on $[a, b]$. Let z be the vector in X such that $x^*(z) = \int_a^b x^*f$ for all x^* in X. We firstly show that f is Riemann integrable on $[a, b]$.

Let $V = \{f(t) : t \in [a, b]\}$, let V_1 be a closed convex hull of the closure of V, and let $W = (b-a)V_1$. Then W is compact set and W contains all Riemann sums of f. Suppose that f is not Riemann integrable on $[a, b]$. Then there exists $\eta > 0$ such that for each $\delta > 0$ there exists a tagged partition P_δ of $[a, b]$ such that $||P_\delta||\delta$ and $||f(P_\delta) - z|| \geq \eta$. For each positive integer n, choose a tagged partition P_n of $[a, b]$ such that $||P_n|| < \frac{1}{n}$ and $||f(P_n) - z|| \geq \eta$. Since z is Riemann-Pettis integral of f on $[a, b]$, the sequence $\{f(P_n)\}$ converse weakly to z, and since W is compact the sequence $\{f(P_n)\}$ must converge in norm to z. This contradiction establishes the Riemann integrability of f on $[a, b]$.

Since f is bounded, to prove that f is Darboux integrable on $[a, b]$, it is sufficient to prove that f is continuous almost everywhere on $[a, b]$. Since V_1 is separable, there exists a sequence $\{x_n^*\}$ in X such that $||v|| = \sup_n |x_n^*|$ for all $v \in V_1$. For each n, let D_n be the set of discontinuities of x^*f on $[a, b]$ and let $D = \bigcup_n D_n$. Then $\mu(D) = 0$ and we will show that f is continuous on $[a, b] \setminus D$.

Let $t \in [a, b] \setminus D$ and let $\{t_k\}$ be a sequence in $[a, b]$ that converge to t. For each n, the sequence $\{x^*f(t_k)\}$ converge to $x^*f(t)$. Since x_n^* separates the point V_1 and as V_1 is compact, the sequence $\{f(t_k)\}$ converges in norm to $f(t)$. This shows that f is continuous at t, hence f is continuous this implies f is Darboux integrable. □

Definition 3.4.20. [4] A Banach space is said to be Schur space if any sequence (x_n) in the Banach space X converging weakly to x implies that $\lim_{n\to\infty} ||x_n - x|| = 0$.

In other words, the Banach space in which weak and strong topologies share the same convergent sequences is called Schur space.

The space ℓ_1 is Schur space but ℓ_2 does not have Schur property, hence it is not a Schur space.

Theorem 3.4.21. If X is Schur space, then every function $f : [a, b] \to X$ that is scalarly Riemann integrable on $[a, b]$ is Riemann integrable on $[a, b]$.

Proof. Since Schur space is weakly sequentially complete then the function f is Riemann-Pettis integrable by 3.4.18 theorem. Since every weakly convergent sequence in X converge in norm, the Riemann integrability of f on $[a, b]$ follows as in first part of the proof of theorem 3.4.19. □

Theorem 3.4.22. A Banach space X is a Sahur space and has the property of Lebesgue if and only if every scalarly Riemann integrable function $f : [a, b] \to X$ is Darboux integrable.

Proof. Suppose that X is a Schur space and has the property of Lebesgue, let $f : [a, b] \to X$ be a scalarly Riemann integrable on $[a, b]$. By pervious theorem, the function f is Riemann integrable on $[a, b]$ and as X has Lebesgue property implies that f is continuous almost everywhere and bounded on $[a, b]$, hence f is Darboux integrable on $[a, b]$.

Now, suppose that X is not a schur space. There exists a sequence $\{x_n\} \in X$ such that $||x_n|| \geq 1$ for all n and converge weakly to θ.

Let $\{r_n\}$ be listing of rational number in $[0, 1]$. And define $f : [0, 1] \to X$ by,

$$f(t) = \begin{cases} x_n & \text{if } t \in \{r_n\}, \\ \theta = (0, 0, 0...) & \text{otherwise .} \end{cases}$$

Since f is not continuous almost everywhere on $[0, 1]$, f is not Darboux integrable on $[0, 1]$. However, the function is bounded and weakly continuous almost everywhere on $[0, 1]$ and therefore, scalarly Riemann integrable on $[0, 1]$. This completes the proof since the case in which X does not have the property of Lebesgue is trivial. □

Example 3.4.23. A weak continuous function that is not Riemann integrable:-

Let H be a perfect, nowhere dense subset of $[0, 1]$ with $\mu(H) \geq \frac{3}{4}$, and let $(0, 1) \setminus H = \bigcup_k (a_k, b_k)$. For each pair of positive integer k and $n \geq 2$, let

$$E_k^n = \{a_k, a_k + \frac{(b_k - a_k)}{(2n)}, a_k + \frac{(b_k - a_k)}{(n)}, b_k - \frac{(b_k - a_k)}{(n)}, b_k - \frac{(b_k - a_k)}{(2n)}, b_k\},$$

and let ϕ_k^n be the function that equal 1 at $a_k + \frac{(b_k - a_k)}{(2n)}$ and $b_k - \frac{(b_k - a_k)}{(2n)}$, equal 0 at other point of E_k^n, and is linear on the interval contiguous to E_k^n. For each n, let $f_n(t) = \sum_{k=1}^{n} \phi_k^n(t)$.

Then the sequence $\{f_n\}$ converges pointwise to the zero function on $[0,1]$ and

$$\int_0^1 f_n = \sum_{k=1}^{n}\int_0^1 \phi_k^n(t) = \frac{1}{n}\sum_{k=1}^{n}(b_k - a_k).$$

Now, Define $f : [a,b] \to c_0$ by $f(t) = \{f_n(t)\}$. We will first prove that f is weakly continuous. Let $x^* = \{\alpha_k\} \in \ell_1$, then $x^*f = \sum_n \alpha_n f_n$. Since each $\alpha_n f_n$ is continuous on $[0,1]$ and $|\alpha_n f_n| < |\alpha_n|$ on $[0,1]$, the function x^*f is continuous on $[0,1]$ begin the uniform limit of continuous function. This shows that f is weakly continuous.

To prove that f is not Riemann integrable on $[0,1]$, it is sufficient to prove that for each $\delta > 0$ there exists a tagged partition P on $[0,1]$ and an integer j_0 such that $||P|| < \delta$ and $||f_{j_0}(P) - \int_0^1 f_{j_0}|| \geq \frac{1}{2}$. Let $\delta > 0$ be given. Since H is nowhere dense, there exists a partition $\{t_m : 0 \leq m \leq M\}$ of $[0,1]$ such that $t_m \notin H$ for $1 \leq m \leq M-1$ and $t_m - t_{m-1} < \delta$ for $1 \leq m \leq M$. Let $\{I_k : 1 \leq k \leq N\}$ be the intervals of this partition that contain points of H in there interiors and let $\{K_i : 1leqi \leq L\}$ be the remaining intervals, for each k, there exists n_k such that $(a_{n_k}, b_{n_k})) \subset I_k$. Let $j_0 = \max\{n_k : 1 \leq k \leq N\}$, and for each k choose $t_k \in (a_{n_k}, b_{n_k})$ such that $\phi_{n_k}^{j_0}(t_k) = 1$. Let $s_i \in K_i$ be arbitrary and let $P = \{(t_k, I_k) : 1 \leq k \leq N\} \cup \{(s_i, K_i) : 1 \leq i \leq L\}$. Then P is a tagged partition of $[0,1]$ with $||P|| < \delta$ and we have

$$\begin{aligned}
f_{j_0} - \int_0^1 f_{j_0} &= \sum_{k=1}^{N} f_{j_0}(t_k)\mu(I_k) + \sum_{i=1}^{L} f_{j_0}(s_i) - \int_0^1 f_{j_0} \\
&= \sum_{k=1}^{N} \mu(I_k) - \frac{1}{j_0}\sum_{k=1}^{j_0}(b_k - a_k) \\
&\geq \mu(H) - (1 - \mu(H)) \\
&\geq \frac{1}{2}
\end{aligned}$$

This shows that f is not Riemann integral.

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