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M.Sc. THESIS

Representation Theory on Locally Compact Groups

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Approval Sheet

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Declaration

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Contents

List of Notation

$\mathbb{T}$	Unit torus
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_0$	The set of natural numbers extended by zero
$\mathbb{Q}$	The set of rational numbers
$\mathbb{H}$	The upper half plane
$GL_n(\mathbb{C})$	The set of all $n \times n$ invirtible matrices in $\mathbb{C}$
$\mathbb{F}_N$	Free group generated by N elements
$\mathfrak{g}$	Lie algebra of the group G
$D(a)$	Derivation of a operator a

Abstract:

In this project we try to reach one of the famous problems, namely similarity problems. It can be stated as follows: Is every uniformly bounded continuous representation of locally compact group G is unitarizable? This problem is open for a while until, in 1955, Ehrenpreeis and Mautner gave counterexample for $G = SL_2(\mathbb{R})$. In the positive direction, the most general result seems to be Dixmiers theorem which says that if G is amenable then the above problem is affirmative. But the converse remains an open problem. It can be stated as, if all uniformly bounded continuous representations of a group G are uniterizable is G necessarily amenable?

Introduction:

The theory of representations of finite groups was initiated in the 1890s by people like Frobenius, Schur and Burnside. In the 1920s representations of arbitrary compact groups, and finite-dimensional (possibly nonunitary) representations of The classical matrix groups were investigated by Weyl and others. In the 1940s mathematicians such as Gelfand started to study (possibly in finite-dimensional) unitary representations of locally compact groups. Other important figures in representation theory include Harish-Chandra, Kirillov and Mackey. A (linear) representation of a group G is, to begin with, simply a homomorphism $f : G \to GL(E)$ where E is a vector space over some field K and $GL(E)$ is the group of invertible K-linear maps on E. Thus the idea of representation theory is to represent an algebraic object, such as a locally compact group or an algebra, as a more concrete group or algebra consisting of matrices or operators. In this way we can study an algebraic object as collection of symmetries of a vector space. Hence we can apply the methods of linear algebra and functional analysis to the study of groups and algebras. Representation theory also provides a generalization of Fourier analysis to groups.

Chapter 1

Finite Abelian Group

We study finite abelian group because it has a nice structure. Fundamental theorem says that any finite Abelian group is isomorphic to a direct sum of cyclic groups.

1.1 The Dual Group

Definition 1.1.1. *Let $\mathcal{A}$ be a finite abelian group. Character of $\mathcal{A}$ is a group homomorphism form $\mathcal{A}$ to the unit torus $\mathbb{T}$ ie if χ be character of $\mathcal{A}$ then $\chi(a.b)$=$\chi(a).\chi(b)$ $\forall$ a, $b \in \mathcal{A}$. Let $\hat{\mathcal{A}}$ be the set of all characters of $\mathcal{A}$.*

Among the characters is one that we denote by 1, with values $1(a) = 1 \quad \forall a \in \mathcal{A}$. It is not obivious whether other characters exist in general.

Lemma 1.1.2. *Two maps $\mathcal{P}$ and $\mathcal{I}$ makes $\hat{\mathcal{A}}$ an abelian group. Where $\mathcal{P} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is defined by $\mathcal{P}(\chi,\eta)$=$\chi.\eta$ and the map $\mathcal{I}$: $A \to \mathcal{A}$ is defined by $\mathcal{I}(\chi)$=χ^{-1} for any χ , $\eta \in \hat{\mathcal{A}}$. Where*

$$\chi.\eta(a) = \chi(a).\eta(a)$$
$$\chi^{-1}(a)=\chi(a^{-1})$$

for any $a \in \mathcal{A}$. We call $\hat{\mathcal{A}}$ the dual group or Pontryagin dual of $\mathcal{A}$.

Lemma 1.1.3. *Let $\mathcal{A}$ be a cyclic group of order $\mathcal{N}$ and τ is a generator of $\mathcal{A}$ then characters of the group $\mathcal{A}$ are given by*

$$\eta_\ell(\tau^k) = e^{\frac{2\pi i k\ell}{\mathcal{N}}}$$

for $\ell = 0, 1, 2, .., \mathcal{N} - 1$. The group $\hat{\mathcal{A}}$ is again a cyclic group of order $\mathcal{N}$. i.e for every finite cyclic group $\mathcal{A}$ its dual $\hat{\mathcal{A}}$ is also a cyclic group of same order. This then implies that those two groups must be isomorphic.

Now using the **Fundamental theorem of finite abelian group** can easily characterise any finite abelian group and also we can say that the dual group of a finite Abelian group $\mathcal{A}$ is isomorphic to $\mathcal{A}$.

Proof: 1. *Idea of this proof is 1st we will show that η_ℓ are the characters of $\mathcal{A}$ for each $\ell = 0, 1, \dots, \mathcal{N} - 1$. Then we show that if we take any character of $\mathcal{A}$ it is η_ℓ for some ℓ. To prove $\hat{\mathcal{A}}$ is cyclic we can show $\hat{\mathcal{A}}$ is generated by η_1 order of which is $\mathcal{N}$.*

Problem 1.1.4. *Show that for $\mathcal{A}$, $\mathcal{B}$ finite abelian groups we have $\widehat{\mathcal{A} \times \mathcal{B}}$ is isomorphic to $\hat{\mathcal{A}} \times \hat{\mathcal{B}}$. Then conclude that for every finite abelian group $\mathcal{A}$ we have $|\mathcal{A}| = |\hat{\mathcal{A}}|$.*

Solution 1. *Let us define a function $F : \widehat{\mathcal{A} \times \mathcal{B}} \to \hat{\mathcal{A}} \times \hat{\mathcal{B}}$ by $F(\chi) = (\chi_{\mathcal{A}}, \chi_B)$ where $\chi_{\mathcal{A}}(a) = \chi(a, 1)$ and $\chi_{\mathcal{B}}(b) = \chi(1, b)$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$. This function F is bijective as well as homomorphism and hence isomorphism.*

Problem 1.1.5. *Let $\mathcal{A}$ and $\mathcal{B}$ are two finite abelian groups and $\phi : \mathcal{A} \to \mathcal{B}$ be a group homomorphism. Show that the prescription $\phi^*(\chi) = \chi \circ \phi$ defines a group homomorphism $\phi^* : \hat{\mathcal{B}} \to \hat{\mathcal{A}}$.*

Solution 2. *for χ_1 , $\chi_2 \in \hat{\mathcal{B}}$ $\quad \phi^*(\chi_1.\chi_2) = (\chi_1.\chi 2) \circ \phi$
now for any arbitrary $a \in \mathcal{A}$*

$$\begin{aligned}
(\chi_1.\chi_2) \circ \phi(a) \qquad &= \chi_1 (\phi(a)) . \chi_2 (\phi(a)) \\
&= (\chi_1 \circ \phi)(a) . (\chi_2 \circ \phi)(a) \\
&= \phi^*(\chi_1).\phi^*(\chi_1)(a).
\end{aligned}$$

So ϕ^* is a homomorphism.

Lemma 1.1.6. *Let $\mathcal{A}$ be a finite abelian group and let $a \in \mathcal{A}$. Suppose that $\chi(a) = 1$ for every $\chi \in \hat{\mathcal{A}}$. Then $a = 1$.*

Theorem 1.1.7. *Let $\mathcal{A}$ be a finite abelian group. There is a canonical isomorphism to the bidual $\mathcal{A} \to \hat{\hat{\mathcal{A}}}$ given by $a \mapsto \delta_a$, where δ_a is the point evaluation at aa, i.e.,*

$$\begin{aligned}
\delta_a : \hat{\mathcal{A}} &\to \mathbb{T} \\
\chi &\mapsto \chi(a).
\end{aligned}$$

Proof: 2. *The idea of the proof is we show that the map $a \mapsto \delta_a$ is homomorphism and bijective. To prove that this function is bijective we use the Lemma 1.1.6.*

1.2 The Fourier Transform

Our main aim in this section is to show that the Hilbert spaces $\ell^2(\mathcal{A})$ and $\ell^2(\hat{A})$ isomorphic. Through out this section we consider $\mathcal{A}$ as a finite abelian group.

Lemma 1.2.1. *Let χ , η be two characters of $\mathcal{A}$; then*

$$\langle \chi, \eta \rangle = \begin{cases} |\mathcal{A}| & \text{if } \chi = \eta \ , \\ 0 & \text{otherwise} \end{cases}$$

Proof. Set of all characters of $\mathcal{A}$ forms a vector space over $\mathbb{C}$ so the standered inner product between two characters χ and η is

$$\langle \chi, \eta \rangle = \sum_{a \in \mathcal{A}} \chi(a).\overline{\eta(a)}.$$

Now if $\chi = \eta$ then
$\langle \chi, \eta \rangle = \sum_{a \in \mathcal{A}} |\chi(a)|^2 = \sum_{a \in \mathcal{A}} 1 = |\mathcal{A}|$ since $\chi(a) \in \mathbb{T}$
for $\chi \neq \eta$ let $\alpha = \chi\eta^{-1}$ since $\hat{\mathcal{A}}$ forms a group, $\alpha \in \hat{\mathcal{A}}$.
$\alpha \neq 1$ implies $\exists\ b \in \mathcal{A}$ such that $\alpha(b) \neq 1$ then

$$\begin{aligned} \alpha(b)\langle \chi, \eta \rangle \qquad & = \alpha(b) \sum_{a \in \mathcal{A}} \chi(a)\eta^{-1}(a) \\ & = \alpha(b) \sum_{a \in \mathcal{A}} \alpha(a) \\ & = \sum_{a \in \mathcal{A}} \alpha(ab) \\ & = \sum_{a \in \mathcal{A}} \alpha(a) \\ & = \langle \chi, \eta \rangle. \end{aligned}$$

So we get

$$(\alpha(b) - 1)\langle \chi, \eta \rangle = 0$$

and since $\alpha(b) \neq 1$ we will get $\langle \chi, \eta \rangle = 0$.

□

Definition 1.2.2. $\ell^2(\mathcal{A})$ *is the set of functions $f : \mathcal{A} \to \mathbb{C}$ satisfying*

$$\sum_{a \in \mathcal{A}} |f(a)|^2 < \infty.$$

In particular,the characters are the elements of $\ell^2(\mathcal{A})$.

Definition 1.2.3. *For $f \in \ell^2(\mathcal{A})$ we define its Fourier transform $\hat{f} : \hat{\mathcal{A}} \to \mathbb{C}$ by $\hat{f}(\chi) = \frac{1}{\sqrt{|\mathcal{A}|}}\langle f, \chi\rangle$.*

Obsevation 1. *If $\chi \in \hat{\mathcal{A}}$ then by Lemma 1.2.1 we can get that Fourier transform of χ is*

$$\hat{\chi}(\eta) = \begin{cases} \sqrt{|\mathcal{A}|} & \text{if } \eta = \chi \ , \\ 0 & \text{otherwise} \end{cases}$$

Obsevation 2. *If $f \in \ell^2(A)$ then $\hat{f} \in \ell^2(\hat{A})$.*

Theorem 1.2.4. *The map $f \mapsto \hat{f}$ is an isomorphism of the Hilbert spaces $\ell^2(\mathcal{A}) \to \ell^2(\hat{\mathcal{A}})$. This can also be applied to the group $\hat{\mathcal{A}}$, and the composition of the two Fourier transforms gives a map $f \mapsto \hat{\hat{f}}$. For the latter map we have $\hat{\hat{f}}(\delta_a) = f(a^{-1})$.*

Proof. For the first part idea of the proof is 1st we show that the map $F : \ell^2(a) \to \ell^2(\hat{A})$ where $F(f) = \hat{f}$ is a linear map then we show that for any $g \in \ell^2(\hat{\mathcal{A}})$ there is a function $f \in \ell^2(\mathcal{A})$ such that $g = F(f)$ i.e F is onto. Finally we show that $\langle \hat{f}, \hat{g}\rangle = \langle f, g\rangle$ for $f, g \in \ell^2(\mathcal{A})$.
For the next part we have

$$\begin{aligned}
\hat{\hat{f}}(\delta_a) &= \frac{1}{\sqrt{|\mathcal{A}|}}\langle \hat{f}, \delta_a\rangle \\
&= \frac{1}{\sqrt{|\mathcal{A}|}}\sum_{\chi\in\hat{\mathcal{A}}} \hat{f}(\chi)\overline{\delta_a(\chi)} \\
&= \frac{1}{\sqrt{|\mathcal{A}|}}\sum_{\chi\in\hat{\mathcal{A}}} \left(\frac{1}{\sqrt{|\mathcal{A}|}}\sum_{b\in\mathcal{A}} f(b)\overline{\chi(b)}\right)\overline{\delta_a(\chi)} \\
&= \frac{1}{|\mathcal{A}|}\sum_{\chi\in\hat{\mathcal{A}}}\sum_{b\in\mathcal{A}} f(b)\chi(b^{-1})\overline{\chi(a)} \\
&= \frac{1}{|\mathcal{A}|}\sum_{\chi\in\hat{\mathcal{A}}}\sum_{b\in\mathcal{A}} f(b^{-1})\chi(b)\overline{\chi(a)} \\
&= \frac{1}{|\mathcal{A}|}\sum_{b\in\mathcal{A}} f(b^{-1})\left(\sum_{\chi\in\hat{\mathcal{A}}} \chi(b)\overline{\chi(a)}\right) \\
&= \frac{1}{|\mathcal{A}|}\sum_{b\in\mathcal{A}} f(b^{-1})\left(\sum_{\chi\in\hat{\mathcal{A}}} \delta_a(\chi)\overline{\delta_a(\chi)}\right) \\
&= \frac{1}{|\mathcal{A}|}\sum_{b\in\mathcal{A}} f(b^{-1})\langle \delta_a, \delta_b\rangle \\
&= \frac{1}{|\mathcal{A}|} f(a^{-1})|\mathcal{A}| \\
&= f(a^{-1}).
\end{aligned}$$

□

1.3 Convolution

For functions on a finite abelian group there is a convolution product. Let f and g be in $\ell^2(\mathcal{A})$; we define their convolution product by

$$f * g(a) = \frac{1}{\sqrt{|\mathcal{A}|}} \sum_{b \in \mathcal{A}} f(b) g(b^{-1}a).$$

We can think convolution product as a multiplication in $\ell^2(\mathcal{A})$ space and with respect to this multiplication $\ell^2(\mathcal{A})$ forms a ring.

Theorem 1.3.1. *For $f, g \in \ell^2(\mathcal{A})$ we have $\widehat{f * g} = \hat{f}\hat{g}$.*

This theorem can be easily proved by using defination of Fourier transform and defination of convolution product.

Chapter 2

LCA Groups

Definition 2.0.2. *A metrizable abelian group is an abelian group $\mathcal{A}$ together with a class of metrics $[d]$ (or a topology that comes from a metric) such that the multiplication and inversion maps are continuous.*

Since here the group is metrizabe we can use sequentianal criteria for continuity. i.e when tow sequences (x_n) and (y_n) in $\mathcal{A}$ are converges to X and y respictively then the sequence $(x_n y_n)$ converge to xy and (x_n^{-1}) converge to x^{-1}.

Examples 2.0.3.

- *The groups $(\mathbb{R}, +)$ and $(\mathbb{R}*, .)$ with the standered topology in $\mathbb{R}$ are metrizable abelian groups.*

- *Any group with the discrete metric is a metrizable group. Since every functions are continuous if the domain has discrete metric.*

Definition 2.0.4. *A metrizable σ-locally compact abelian group is called an LCA group.*

A metrizable group $\mathcal{A}$ is called σ-compact if there is a increasing sequence (K_n) of compact subsets of $\mathcal{A}$ such that $\mathcal{A} = \cup_{n\in\mathbb{N}} K_n$, such a sequence is called compact exhaustion of $\mathcal{A}$. $\mathcal{A}$ is called locally compact if every point $a \in \mathcal{A}$ has a compact neighborhood.

Examples 2.0.5.

- *The groups $(\mathbb{R}, +)$ and $(\mathbb{R}*, .)$ with the standered topology in $\mathbb{R}$ are LCA groups. Since $\mathbb{R} = \cup_{n\in\mathbb{N}}[-n, n]$ and $[-n, n]$ are closed and bounded set in $\mathbb{R}$ so compact.*

- *$(GL_n(\mathbb{C}), .)$ is metizable but not a LCA group.*

2.1 Properties of a LCA Group

Lemma 2.1.1. *An LCA group contains a countable dense subset.*

Proof. Let (x_n) be a sequence in $\mathcal{A}$ converges to x. Then x_n is called dense sequence if for every point $a \in \mathcal{A}$ there is a subsequence of (x_n) that converges to x. Idea of this proof is using the compact exhaustion of the LCA group we can find a sequence which is dense in that group.

□

A compact exhaustion (K_n) of $\mathcal{A}$ is called absorbing if for every compact set $K \subset \mathcal{A}$ there is an index $n \in \mathbb{N}$ such that $K \subset K_n$; i.e., the exhaustion absorbs all compact sets.

Lemma 2.1.2. *Let $\mathcal{A}$ be an LCA group, then there exists an absorbing exhaustion.*

Proof. Let (L_n) be a compact exhaustion of $\mathcal{A}$. Since $\mathcal{A}$ locally compact, there is an open neighbourhood U of the unit such that $\overline{U}$ is compact, say $V = \overline{U}$. Now let $K_n = VL_n = \{vl : v \in V, l \in L_n\}$ is compact again, since it is the image of the compact set $V \times L_n$ under the multiplication map, which is continuous. and $\bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} VL_n = V \bigcap_{n\in\mathbb{N}} L_n = V\mathcal{A} = \mathcal{A}$. Since (L_n) is increasing (K_n) is also increasing sequence. Hence (K_n) is again a compact exhaustion of $\mathcal{A}$. Now if we can show that this (K_n) is absorbing we are done.
Now let us assume that (K_n) is not absorbing and K be a compact subset of $\mathcal{A}$. So we can construct a squence (x_n) such that for each n $x_n \in K$ that is not in K_n. Since K is compact it has a convergent subsequence say (x_{n_r}). Let (x_{n_r}) converges to $x \in \mathcal{A}$. Therefore $\exists$ a $n_0 \in \mathbb{N}$ such that $x \in L_{n_0}$.The set Ux is an open neighborhood of x, so there exists n_1 such that $x_n \in Ux$ for $n \geq n_1$. Now take $n_2 = max\{n_0, n_1\}$ then for $n \geq n_2$ we have

$$\begin{aligned} x_n \in Ux \quad & \subset UL_{n_0} \\ & \subset VL_{n_0} \\ & \subset VL_n \\ & = K_n. \end{aligned}$$

Which is a contradiction. Therefore our assumption is wrong. Hence (K_n) is absorbing exhaustion of $\mathcal{A}$. □

2.2 The structure of LCA- Groups

Proposition 2.2.1. *Let G be a topological group, and let U be a compact and open neighborhood of the identity e in G. Then U contains a compact and open subgroup K of G.*

Proof. We know that we can find an open neighbourhood V of e such that

$$VU = UV = U.$$

Now since $e \in U$, we have $V \subset U$ i.e $V^2 \subset VU \subset U$. By induction $V^n \subset U$ for every $n \in \mathbb{N}$. Now let $K = \cup_n \in \mathbb{N} V^n$, then K is both open and close subgroup of G contained in U.

□

Proposition 2.2.2. *Every totally disconnected locally compact space X has a base for its topology consisting of open and compact subsets of X.*

Using this two propositions we can say that every unit neighbourhood U in a locally compact group G contains an open and compact subgroup of G.

Theorem 2.2.3. *Let G be an LCA group. Then there exists $n \in \mathbb{N}_0$ and an LCA group H such that*
(a) *A is isomorphic to $\mathbb{R}^n \times H$.*
(b) *H contains an open compact subgroup K.*

This theorem is known as **First Structure Theorem**.

Theorem 2.2.4. *Let G be a compactly generated LCA group. Then there exist $n, m \in \mathbb{N}_0$ and a compact group K such that G is isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m \times K$.*

This theorem is known as **Second Structure Theorem**.

Chapter 3

The Dual Group

In this chapter we will discuss about the dual of a LCA group. We wil see that dual group of a LCA group is again forms a LCA group. This then paves the way for the famous Pontryagin duality theorm.

3.1 Characterization of the groups $\mathbb{Z}$, $\mathbb{R}/\mathbb{Z}$, $\mathbb{R}$

Definition 3.1.1. *A character of a metrizable abelian group $\mathcal{A}$ is a continuous group homomorphism $\chi : \mathcal{A} \to \mathbb{T}$. The set of all characters of $\mathcal{A}$ is denoted by $\hat{\mathcal{A}}$.*

Proposition 3.1.2.

(a) The characters of the group $\mathbb{Z}$ are given by $k \mapsto e^{2\pi ikx}$, where $x \in \mathbb{R}/\mathbb{Z}$.

(b) The characters of $\mathbb{R}/\mathbb{Z}$ are given by $x \mapsto e^{2\pi ikx}$, where $k \in \mathbb{Z}$.

(c) The characters of R are given by $x \mapsto e^{2\pi ixy}$, where $y \in \mathbb{R}$.

Proof. (a) 1st we prove that the map $\chi : \mathbb{Z} \to \mathbb{T}$ where $\chi(k) = e^{2ikx}$ is a character of $\mathbb{Z}$ Since exponential functions are always continuous it remains to show that χ is a group homomorphism. Now for $k_1, k_2 \in \mathbb{Z}$ we have,

$$\begin{aligned} \chi(k_1 + k_2) &= e^{2\pi i(k_1+k_2)x} \\ &= e^{2\pi ik_1x}e^{2\pi ik_2x} \\ &= \chi(k_1)\chi(k_2). \end{aligned}$$

Therefore the map χ is a character of $\mathbb{Z}$.

Next we have to show that if η be any character of $\mathbb{Z}$ then for any arbitrary $k \in \mathbb{Z}$

$$\eta(k) = e^{2\pi ikx}$$

for some $x \in \mathbb{R}/\mathbb{Z}$.

Now

$$\begin{aligned} \eta(k) &= \eta(1+1+\ldots+1) \\ &= \eta(1)^k. \end{aligned}$$

Now $\eta(1) \in \mathbb{T}$ implies

$$\begin{aligned} \eta(1) &= e^{2\pi i\alpha} && \text{for some } \alpha \in \mathbb{R} \\ &= e^{2\pi i\overline{\alpha}} && \text{for some } \overline{\alpha} \in \mathbb{R}/\mathbb{Z}. \end{aligned}$$

Hence we get $\eta(k) = e^{2\pi ikx}$ for some $x \in \mathbb{R}/\mathbb{Z}$.

(b) 1st part of this proof is same as previous one.

Now let $\eta \in \widehat{\mathbb{R}/\mathbb{Z}}$ then for $\overline{r} \in \mathbb{R}/\mathbb{Z}$ we have

$$\begin{aligned} \eta(\overline{r}) &= \eta(r.\overline{1}) \\ &= \eta(\overline{1}+\overline{1}+\ldots+\overline{1}) \\ &= \eta(\overline{1})^r. \end{aligned}$$

$\eta(1) = e^{2\pi i\overline{\alpha}}$ for some $\overline{\alpha} \in \mathbb{R}/\mathbb{Z}$ this implies

$$\begin{aligned} \eta(\overline{r}) &= e^{2\pi i\overline{\alpha}r} \\ &= e^{2\pi i(\overline{1}\alpha)r} \\ &= e^{2\pi i(\overline{1}r)\alpha} \\ &= e^{2\pi i\overline{r}\alpha}. \end{aligned}$$

Hence we get $\eta(\overline{r}) = e^{2\pi i\overline{r}\alpha}$ for some $\alpha \in \mathbb{Z}$.

(c) Here also 1st part remain same. Let $\chi \in \hat{\mathbb{R}}$ then for $s, t \in \mathbb{R}$ we have

$$\chi(s+t) = \chi(s) + \chi(t).$$

Since χ is continuous at 0 and $\chi(0) = 1$, we can say that there exist a $\delta > 0$ such that

$$\int_0^\delta \chi(s)ds \neq 0.$$

Then

$$\begin{aligned} \int_0^\delta \chi(s+t)ds &= \chi(t)\int_0^\delta \chi(s)ds \\ &= \chi(t).C && \text{for some constant } C. \end{aligned}$$

This implies

$$\int_t^{t+\delta} \chi(s)ds = \chi(t).C.$$

Then by Fundamental theorem of interal calculas we can say that χ is differentable. so,

$$\begin{aligned}
\chi'(t) &= \lim_{h\to 0} \frac{\chi(t+h)-\chi(t)}{h} \\
&= \lim_{h\to 0} \frac{\chi(t+h)-\chi(t+0)}{h} \\
&= \lim_{h\to 0} \frac{\chi(t)\left(\chi(h)-\chi(0)\right)}{h} \\
&= \chi(t)\lim_{h\to 0} \frac{\chi(h)-\chi(0)}{h} \\
&= \chi(t)\chi'(0).
\end{aligned}$$

Now let $\chi'(0) = k$ then solving $\chi'(t) = k\chi(t)$ with initial condition $\chi(0) = 1$ we get,

$$\chi(t) = e^{kt}.$$

Since $\chi(t) \in \mathbb{T}$ there exists a $y \in \mathbb{R}$ such that $k = 2\pi iy$. Hence we get

$$\chi(t) = e^{2\pi ity}$$

for every $t \in \mathbb{R}$ as claimed.

□

Obsevation 3.

- *The dual group of* $\mathbb{Z}$ *is isomorphic to* $\mathbb{R}/\mathbb{Z}$*.*
- *The dual group of* $\mathbb{R}/\mathbb{Z}$ *is isomorphic to* $\mathbb{Z}$*.*
- *The dual group of* $\mathbb{R}$ *is isomorphic to* $\mathbb{R}$*.*

3.2 The Dual as LCA Group

In this section we will prove that for a given LCA group $\mathcal{A}$ the dual $\hat{\mathcal{A}}$ is an LCA group again. For that we need to define a metric on $\mathcal{A}$ with the help of metric on $\mathcal{A}$.

Fix an absorbing compact exhaustion $\mathcal{A} = \cup_{n\in\mathbb{N}} K_n$. For $\chi, \eta \in \hat{\mathcal{A}}$ and $n \in \mathbb{N}$ let

$$\hat{d_n} = \sup_{x\in K_n} |\chi(x) - \eta(x)|$$

and

$$\hat{d} = \sum_{n=1}^{\infty} \frac{1}{2^n}\hat{d_n}(\chi, \eta).$$

Lemma 3.2.1. *The function $\hat{d}$ is a metric on the set $\mathcal{A}$.*

Proof. Proof is not much difficult. Non-negativity, identity of indiscernibles and symmetricity property are obivous. Only for triangle inequality let $\chi, \eta, \alpha \in \hat{\mathcal{A}}$ then

$$\begin{aligned}
\hat{d}_n(\chi, \eta) &= \sup_{x \in K_n} |\chi(x) - \eta(x)| \\
&= \sup_{x \in K_n} |\chi(x) - \alpha(x) + \alpha(x) - \eta(x)| \\
&\leq \sup_{x \in K_n} |\chi(x) - \alpha(x)| + \sup_{x \in K_n} |\alpha(x) - \eta(x)| \\
&= \hat{d}_n(\chi, \alpha) + \hat{d}_n(\alpha, \eta).
\end{aligned}$$

Now to prove $\quad \hat{d}(\chi, \eta) \leq \hat{d}(\chi, \alpha) + \hat{d}(\alpha, \eta).$

We need to show that the series $\sum_{n=1}^{\infty} \frac{1}{2^n} \hat{d}_n(\chi, \eta)$ is convergent. This is true by **Derichelet's test** because here $\sum_{n=1}^{\infty} \hat{d}_n(\chi, \eta)$ is bounded by 2 and $\lim_{n \to \infty} \frac{1}{2^n} = 0$. □

Theorem 3.2.2. *With the metric above, the group $\hat{\mathcal{A}}$ is a topological abelian group. A sequence (χ_n) converges in this metric if and only if it converges locally uniformly, so the metric class or topology does not depend on the exhaustion chosen. With this topology $\hat{\mathcal{A}}$ is an LCA group.*

Proof. To prove that $\hat{\mathcal{A}}$ is a topological abelian group we can easily show that the group operation on $\hat{\mathcal{A}}$ are continuous. For this we use sequential critera for continuity. Now, a sequence (χ_j) in $\hat{\mathcal{A}}$ converges if and only if it converges uniformly on each K_n. Since the exhaustion (K_n) was absorbing, this means that the sequence converges if and only if it converges uniformly on every compact subset of $\mathcal{A}$, which is equivalent to locally uniform convergence, since $\mathcal{A}$ is locally compact.

It remains to show that $\hat{\mathcal{A}}$ is locally compact and σ-compact. That is not very easy because for that need to constuct a compact exhaustion for $\hat{\mathcal{A}}$. Using the followig lemma 3.2.3 we can able to show that

$$L_n = \{\chi \in \hat{\mathcal{A}} : \chi(B_{\frac{1}{n}}) \subset \{Re(z) \geq 0\}\}$$

form a compact exhaustion for $\hat{\mathcal{A}}$. Then automatically $\hat{\mathcal{A}}$ becomes locally compact. □

Lemma 3.2.3. *Let $n \in \mathbb{N}$ For every $\epsilon > 0$ there is $\delta > 0$ such that for every $\chi \in L_n$,*

$$\chi(B_\delta) \subset \{z \in \mathbb{T} : |z - 1| < \epsilon\}.$$

Proposition 3.2.4. *The group isomorphism $\mathbb{R} \to \hat{\mathbb{R}}$ is homeomorphism homeomorphisms; i.e., they are continuous and so are their inverse maps. So in particular, we can say that $\hat{\mathbb{R}}$ is isomorphic to $\mathbb{R}$ as an LCA group.*

Proof. Let us define a map $\phi : \mathbb{R} \to \hat{\mathbb{R}}$ by

$$\phi(x) = \phi_x$$

where $\phi_x(y) = e^{2\pi i x y}$ for some $y \in \mathbb{R}$.
Then it can be easily proved that ϕ is a group isomorphism.
To show the continuity of ϕ and ϕ^{-1} here also we use the sequential defination.

Let (x_n) be a sequence in $\mathbb{R}$ which converge to x. Then for any arbitrary $y \in \mathbb{R}$ we have

$$\begin{aligned} |\phi_{x_n}(y) - \phi_x(y)| &= |e^{2\pi i x_n y} - e^{2\pi i x_n y}| \\ &= \int_x^{x_n} 2\pi i y . e^{2\pi i t y} dt \\ &\leq |2\pi i y| \int_x^{x_n} |e^{2\pi i t y}| dt \\ &= 2\pi |y| |x_n - x|. \end{aligned}$$

This implies that on every bounded interval the sequence of functions (ϕ_{x_n}) will converge uniformly to the function ϕ_x; hence we have that (ϕ_{x_n}) converges to ϕ_x locally uniformly on $\mathbb{R}$. We conclude that the map ϕ is continuous.

Next we prove that the inverse ϕ^{-1} is continuous. For that let (ϕ_{x_n}) be sequence in $\hat{\mathbb{R}}$ converges to ϕ_x. i.e $(\phi_{x_n})(y)$ converges uniformly to $\phi_x(y)$. This implies

$$(x_n - x)y = k_n + \epsilon_n$$

where $k_n \in \mathbb{Z}$ and ϵ_n tends to 0 in $\mathbb{R}$.
Then x_n must be a bounded sequence. by **Bolzano weierstrass theorem** (x_n) has a convergent subsequence say, (x_{n_k}). Let x' be its limit. Now using the 1st part of the proof we can say that $(\phi_{x_{n_k}})$ converges to $(\phi_{x'})$ and since limit is unique $x' = x$. Since this holds for every convergent subsequence, it follows that (x_n) converges to x. □

3.3 Pontryagin Duality

Proposition 3.3.1. *If A is compact, then $\hat{\mathcal{A}}$ is discrete. If $\mathcal{A}$ is discrete, then $\hat{\mathcal{A}}$ is compact*

Theorem 3.3.2. *Let $\mathcal{A}$ denote an LCA group. Then the map $\mathcal{A} \to \hat{\hat{\mathcal{A}}}$ where $a \mapsto \delta_a$ and $\delta_a(\chi) = \chi(a)$ is an isomorphism of LCA groups.*

Chapter 4

Matrix Group

A **matrix group** means a group of invertible matrices. You know from linear algebra that invertible matrices represent geometric motions (i.e., linear transformations) of vector spaces, so maybe its not so surprising that matrix groups are useful within geometry.

4.1 Martix group $GL_n(\mathbb{C})$ and $U(n)$

In this section we will discuss about the topological nature of $GL_n(\mathbb{C})$ and $U(n)$. We will see that $GL_n(\mathbb{C})$ is a LC group and $U(n)$ is a compact subgroup of $GL_n(\mathbb{C})$. Now first question that comes to our mind is what is the norm on space of matices?

Let n be a natural number. On the vector space of complex $n \times n$ matrices $M_n(\mathbb{C})$ we can define norm in many ways. For $A = (a_{i,j})$ let $||A||_1 = \sum_{i,j=1}^{n} |a_{i,j}|$ and $||A||_2 = \sqrt{\sum_{i,j=1}^{n} |a_{i,j}|^2}$, smilarly for any $p \in \mathbb{N}$ we can define $||A||_p$. Let d_i be the corresponding metric for norm $||.||_i$.

Lemma 4.1.1. *A sequence of matrices $A^{(k)} = (a_{i,j}^{(k)})$ converges in d_1 if and only if for each pair of indices (i, j), the sequence of entries $a_{i,j}^{(k)}$ converges in $\mathbb{C}$. The same holds for d_2, so the metrics d_1 and d_2 are equivalent.*

Proof. The proof is very simple only we have to use defination of $||.||_1$ and $||.||_2$ and defintion of a convergent sequence in any metric space. □

Proposition 4.1.2. *With the topology or metric class given above, the group of complex invertible matrices, $GL_n(\mathbb{C})$, is an LC group; i.e., it is a metrizable, σ-compact, locally compact group.*

Proof. We already know that $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ and $\mathbb{C}^n$ is locally compact. Being an open subset of the locally compact space $M_n(\mathbb{C})$ the topogical group $GL_n(\mathbb{C})$ is locally compact. So now it remain to show that it is σ- compact. For this our claim is

$$K_n = \{A \in GL_n(\mathbb{C}) : ||A||_1 \leq n, ||A||_1^{-1} \leq n\}$$

forms a compact exhaustion. □

Lemma 4.1.3. *$U(n) = \{g \in M_n(\mathbb{C}) : g^*g = I\}$ is a compact subgroup of $GL_n(\mathbb{C})$.*

Proof. First we show that $U(n)$ is a subgroup of $GL_n(\mathbb{C})$, then we show that it is close and bounded in $M_n(\mathbb{C})$ because it is sufficient to tell that it is compact.

For $a, b \in U(n)$

$$\begin{aligned}(ab)^*ab &= b^*a^*ab \\ &= b^*Ib \qquad \text{since } a \in U(n) \\ &= b^*b \\ &= I \qquad \text{since } b \in U(n).\end{aligned}$$

So $ab \in U(n)$.
Since $g^*g = I$ we get $g^{-1} = g^*$ and $(g^*)^*g^* = gg^* = I$ this implies $g^* \in U(n)$ i.e $g^{-1} \in U(n)$. Therefore $U(n)$ forms a group.

To show $U(n)$ closed, take $g \in \overline{U(n)}$ i.e $\exists$ a sequence (g_j) in $M_n(\mathbb{C})$ converging to g. Now since the map $g \mapsto g^*$ is continuous we get $g \in U(n)$. Now to prove $U(n)$ is bounded, for every $a \in U(n)$ we have

$$\begin{aligned}||a||_2^2 &= \sum_{i,j=1}^{n} |a_{i,j}|^2 \\ &= \sum_{i,j=1}^{n} a_{i,j}\overline{a_{i,j}} \\ &= \sum_{i,j=1}^{n} a_{i,j}(a_{j,i})^* \\ &= \sum_{k=1}^{n}(aa^*)_{k,k} \\ &= tr(a^*a).\end{aligned}$$

Hence we get $||a||_2 = \sqrt{tr(I)} = \sqrt{n}$. So $U(n)$ is bounded. □

4.2 The Lie Algebra of a Matrix Lie Group

Definition 4.2.1. *A matrix Lie group is any subgroup G of $Gl_n(\mathbb{C})$ with the following property: If (A_n) is any sequence of matrices in G, and (A_n) converges to some matrix A then either $A \in G$, or A is not invertible. It is equivalant to say that a matrix Lie group is a closed subgroup of $Gl_n(\mathbb{C})$.*

Examples 4.2.2.

- *$Gl_n(\mathbb{C})$ and $Gl_n(\mathbb{R})$ are matrix Lie group, since all the entries of (A_n) forms sequence in $\mathbb{C}$ and $\mathbb{R}$ and if it converges then limit must be in $\mathbb{C}$ and $\mathbb{R}$ if not then limit of (A_n) is not intvertable.*

- *$SL(n, \mathbb{C})$, the the group of $n \times n$ invertible matrices complex entrie having determinant one is a matrix Lie group, since determinant is a continuous function.*

- *The set of all $n \times n$ invertible matrices all of whose entries are real and rational is not a matrix Lie group because $\mathbb{Q}$ is dense over $\mathbb{R}$.*

Definition 4.2.3. *Let G be a matrix Lie group. The Lie algebra of G, denoted $\mathfrak{g}$, is the set of all matrices X such that e^{tX} is in G for all real number t.*

4.3 The Matrix Exponential

A series of matrices in $M_n(\mathbb{C})$ of the form $\sum_{\nu=0}^{\infty} A_\nu$ converges if the sequence of partial sums $s_k = \sum_{\nu=0}^{k} A_\nu$ converges.

Lemma 4.3.1. *For $A, b \in M_n(\mathbb{C})$ we have*

$$||AB||_1 \leq ||A||_1 ||B||_1.$$

In particular, for $j \in \mathbb{N}$, $||A^j||_1 \leq ||A||_1^j$.

Lemma 4.3.2. *If (A_ν) be a sequence in $M_n(\mathbb{C})$ and $\sum_{\nu=0}^{\infty} ||A_\nu||_1 < \infty$, then the series $\sum_{\nu=0}^{\infty} A_\nu$ converges in $M_n(\mathbb{C})$.*

Proposition 4.3.3. *For every $A \in M_n(\mathbb{C})$ we have*

$$det(exp(A)) = exp(trace(A)).$$

Proof. We know that every square matirx is conjugate to an upper triangular matrix i.e there always exist $S \in GL_n(\mathbb{C})$ such that $A = SUS^{-1}$ where U is upper triangular. Then,

$$det(exp(SUS^{-1})) = det(S\ exp(U)S^{-1}) = det(exp(U)).$$

Hence we get,

$$det(exp(A)) = det(exp(U)).$$

So it suffices to prove the proposition for an upper triangular matrix U.
Let

$$U = \begin{bmatrix} u_1 & & & & & * \\ & u_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & u_n \end{bmatrix}.$$

Then for any $\nu > 0$

$$U^\nu = \begin{bmatrix} u_1^\nu & & & & & * \\ & u_2^\nu & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & u_n^\nu \end{bmatrix}.$$

Now

$$exp(U) = \begin{bmatrix} e^{u_1} & & & & & * \\ & e^{u_2} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & e^{u_n} \end{bmatrix}.$$

This gives

$$\begin{aligned} det(exp(U)) \qquad &= e^{u_1} \dots e^{u_n} \\ &= e^{u_1 + \dots u_n} \\ &= e^{tr(U)}. \end{aligned}$$

□

The Lie algebra $\mathfrak{sl}_\mathfrak{n}(\mathbb{R})$ of $SL_n(\mathbb{R})$ is given by

$$\begin{aligned} \mathfrak{sl}_\mathfrak{n}(\mathbb{R}) \qquad &= \{X \in M_n(\mathbb{R}) : e^{tX} \in SL_n(\mathbb{R}) \quad \forall t \in \mathbb{R}\} \\ &= \{X \in M_n(\mathbb{R}) : det(e^{tX}) = 1 \quad \forall t \in \mathbb{R}\} \\ &= \{X \in M_n(\mathbb{R}) : exp(tr(tX)) = 1 \quad \forall t \in \mathbb{R}\} \\ &= \{X \in M_n(\mathbb{R}) : tr(tX) = 0 \quad \forall t \in \mathbb{R}\} \\ &= \{X \in M_n(\mathbb{R}) : tr(X) = 0\}. \end{aligned}$$

The Lie algebra $\mathfrak{U}(\mathfrak{n})$ of $U(n)$ is given by

$$\begin{aligned}
\mathfrak{U}(\mathfrak{n}) \quad &= \{X \in M_n(\mathbb{C}) : e^{tX} \in U(n) \quad \forall t \in \mathbb{R}\} \\
&= \{X \in M_n(C) : e^{tX}(e^{tX})^* = I \quad \forall t \in \mathbb{R}\} \\
&= \{X \in M_n(\mathbb{C}) : (e^{tX})^* = (e^{tX})^{-1} \quad \forall t \in \mathbb{R}\} \\
&= \{X \in M_n(\mathbb{C}) : e^{tX^*} = e^{-tX} \quad \forall t \in \mathbb{R}\} \\
&= \{X \in M_n(\mathbb{C}) : X^* = -X\}.
\end{aligned}$$

The Lie algebra $\mathfrak{su}(\mathfrak{n})$ of $SU(n)$ is given by

$$\begin{aligned}
\mathfrak{su}(\mathfrak{n}) \quad &= \{X \in M_n(\mathbb{C}) : e^{tX} \in SU(n) \quad \forall t \in \mathbb{R}\} \\
&= \{X \in M_n(\mathbb{C}) : e^{tX} \in SL_n(\mathbb{C}), e^{tX} \in U(n) \quad \forall t \in \mathbb{R}\} \\
&= \{X \in M_n(\mathbb{C}) : tr(X) = 0, X^* = -X\}.
\end{aligned}$$

The Lie algebra $\mathfrak{h}$ of 3×3 Heisenberg group H is given by

$$\mathfrak{h} \quad = \{X \in M_3(\mathbb{R}) : e^{tX} \in H \quad \forall t \in \mathbb{R}\}.$$

Now we know that,

$$X = \left[\frac{d}{dt}e^{tX}\right]_{t=0}.$$

This implies $\mathfrak{h}$ is the space of all 3×3 real matrices that are strictly upper triangular.

4.4 Properties of Lie Algebra

Proposition 4.4.1. *Let G be a matrix Lie group, with Lie algebra $\mathfrak{g}$. Let X be an element of $\mathfrak{g}$ and A an element of G. Then AXA^{-1} is in $\mathfrak{g}$.*

Proof. We know that,

$$e^{t(AXA^{-1})} = Ae^{tX}A^{-1}$$

and now $e^{tX} \in G \quad \forall t \in \mathbb{R}$ implies that

$$Ae^{tX}A^{-1} \in G \quad \forall t.$$

□

Theorem 4.4.2. *Let G be a matrix Lie group, with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ is a real subspace of the space $M_n(\mathbb{C})$ and is also closed under Lie bracket. More precisely if X and Y are elements of $\mathfrak{g}$ then $[X, Y] = XY - YX$ also lies in $\mathfrak{g}$.*

Proof. Let $X \in \mathfrak{g}$, then $e^{tX} \in G$ for all $t \in \mathbb{R}$. Which imples for any $s \in \mathbb{R} \quad e^{s(tX)} \in G$ i.e $sX \in \mathfrak{g}$.
Now we have to show that $X+Y \in \mathfrak{g}$ for $X, Y \in \mathfrak{g}$. To show this we will use the Lie product formula, which states that,

$$e^{t(X+Y)} = \lim_{m \to \infty} (e^{\frac{tX}{m}} e^{\frac{tY}{m}})^m \quad \forall t \in \mathbb{R}.$$

Now since $X, Y \in \mathfrak{g}$,
$e^{\frac{tX}{m}}, e^{\frac{tY}{m}} \in G$ and since G is a group this implies $(e^{\frac{tX}{m}} e^{\frac{tY}{m}})^m \in G$.
By our hypothesis G is matrix Lie group, so the limit of things in G must be again in G. Hence

$$e^{t(X+Y)} \in G \quad \forall t \in \mathbb{R}.$$

Implies $X + Y \in \mathfrak{g}$. Therefore the 1st part of this theorem is done.
For last part recall that,

$$\frac{d}{dt} e^{tX} = X e^{tX}.$$

So

$$\left[\frac{d}{dt} e^{tX} Y\right]_{t=0} = XY.$$

Hence by product rule,

$$\left[\frac{d}{dt} e^{tX} Y e^{-tX}\right]_{t=0} = XY - YX.$$

Now, by the proposition 4.4.1, $e^{tX} Y e^{-tX} \in \mathfrak{g}$. Furthermore we have established that $\mathfrak{g}$ is real subspace of $M_n(\mathbb{C})$, this means $\mathfrak{g}$ is closed subset of $M_n(\mathbb{C})$. Ti follows that,

$$[X, Y] = XY - YX = \left[\frac{d}{dt} e^{tX} Y e^{-tX}\right]_{t=0} = \lim_{h \to 0} \frac{e^{tX} Y e^{-tX} - Y}{t} \in \mathfrak{g}.$$

□

4.5 Matrix Lie Group $SL_2(\mathbb{R})$

$SL_2(\mathbb{R})$ is the special linear group of degree 2, i.e

$$SL_2(\mathbb{R}) = \{X \in M_2(\mathbb{R}) : det(X) = 1\}.$$

As we calculated before,

$$\mathfrak{sl}_2(\mathbb{R}) = \{X \in M_2(\mathbb{R}) : tr(X) = 0\}.$$

The elements,

$$X_0 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad X_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

form a basis of $\mathfrak{sl}_2(\mathbb{R})$.

Definition 4.5.1. *Let θ, t, ζ be arbitrary real numbers and put*

$$k_\theta = exp(\theta X_0) \quad a_t = exp(tX_1) \quad n_\zeta = exp(\zeta X_2).$$

Then the subgroups K, A, N of $SL_2(\mathbb{R})$ are defined by

$$K = \{k_\theta : \theta \in \mathbb{R}\} \quad A = \{a_t : t \in \mathbb{R}\} \quad N = \{n_\zeta : \zeta \in \mathbb{R}\}.$$

Then we have,

$$\begin{aligned} k_\theta &= exp(\theta X_0) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(\theta X_0)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\theta}{2}\right)^n (X_0')^n \quad , \qquad X_0' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\theta}{2}\right)^{2n} .1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\theta}{2}\right)^{2n+1} X_0' \\ &= \begin{bmatrix} cos\frac{\theta}{2} & sin\frac{\theta}{2} \\ -sin\frac{\theta}{2} & cos\frac{\theta}{2} \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} a_t &= \sum_{n=0}^{\infty} \frac{1}{n!}(tX_1)^n \\ &= \begin{bmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} n_\zeta &= \sum_{n=0}^{\infty} \frac{1}{n!}(\zeta X_2)^n \\ &= (\zeta X_2)^0 + (\zeta X_2) \\ &= I + (\zeta X_2) \\ &= \begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

4.5.1 Iwasawa Decomposition

Let $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$ and the group $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by linear functional, i.e for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ and for $z \in \mathbb{H}$ we define

$$gz = \frac{az + b}{cz + d}.$$

Then the stabilizer of the point $i \in \mathbb{H}$ is the rotation group K.

Theorem 4.5.2. *The group $SL_2(\mathbb{R})$ can be decomposed as*

$$SL_2(\mathbb{R}) = ANK.$$

More precisely, the map $\phi : A \times N \times K \to SL_2(\mathbb{R})$ where $\phi(a, n, k) = ank$ is a homeomorphism. This decomposition is called Iwasawa Decomposition.

Proof. Let $g \in SL_2(\mathbb{R})$, then $gi = x + iy$
implies

$$g = \begin{bmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix}.$$

Now let

$$a = \begin{bmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} \quad n = \begin{bmatrix} 1 & \frac{x}{y} \\ 0 & 1 \end{bmatrix}.$$

Then $a \in A$, $n \in N$ and $an = g$. So,

$$gi = ani \implies (g^{-1}an)i = i.$$

Since K is sebilizer of i, $g^{-1}an$ lies in K, which means there exist a $k \in K$ with $g = ank$. Now let

$$\phi' : SL_2(\mathbb{R}) \to A \times N \times K$$

be a map given by, $\phi'(g) = (\bar{a}(g), \bar{n}(g), \bar{k}(g)$ for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where,

$$\bar{a}(g) = \begin{bmatrix} \frac{1}{\sqrt{c^2+d^2}} & 0 \\ 0 & \sqrt{c^2 + d^2} \end{bmatrix}$$

$$\bar{n}(g) = \begin{bmatrix} 1 & ac + bd \\ 0 & 1 \end{bmatrix}$$

$$\bar{k}(g) = \frac{1}{\sqrt{c^2 + d^2}} \begin{bmatrix} d & -c \\ c & d \end{bmatrix}.$$

Then a straightforward computation shows that $\phi\phi' = I$ and $\phi'\phi = I$. □

Chapter 5

Representation Theory

5.1 Representations

Let G be a (metrizable) topological group and $(V, \langle ., . \rangle)$ be Hilbert space . A reprsentation of G on V is a group homomorphism $\Pi : G \to GL(v)$ such that the map $F : G \times V \to V$ is continuous, where $F(g, v) = \Pi(g)v$. The dimension of V is called the degree of the representation. One should think of a representation as a linear action of a group on a vector space.

The representation Π is called unitary if for every $x \in G$ the operator $\Pi(x)$ is unitary on V i.e

$$\langle \Pi(x)v, \Pi(x)w \rangle = \langle v, w \rangle$$

for all $v, w \in V$.

A closed subspace W of V is called invariant for Π if $\Pi(x)W \subset W$ for all $x \in G$. The representation is called irreducible if there is no proper invariant subspace.

Example.The identity map $\rho : U(n) \to GL(\mathbb{C}^n) = GL_n(\mathbb{C})$ is a unitary representation.

Theorem 5.1.1. *Let G be a finite group and $\rho : G \to GL(V)$ is representation on $(V, \langle ., . \rangle)$ Then there is another inner product $(., .)$ on V in which $\rho(g)$ is unitary for any $g \in G$.*

Proof. For $v, w \in V$ let

$$(v, w) = \frac{1}{O(G)} \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle.$$

It is easy to see that $(., .)$ is indeed an inner product. Moreover, the representation ρ is unitary with respect to $(.,.)$, since for $g \in G$ and $v, w \in V$ we have

$$\begin{aligned}
(\rho(g_1)v, \rho(g_1)w) &= \frac{1}{O(G)} \sum_{g \in G} \langle \rho(g_1)\rho(g)v, \rho(g_1)\rho(g)w \rangle \\
&= \frac{1}{O(G)} \sum_{g \in G} \langle \rho(g_1 g)v, \rho(g_1 g)w \rangle \\
&= \frac{1}{O(G)} \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle \\
&= (v, w).
\end{aligned}$$

□

Theorem 5.1.2. *Let G be a finite group and $\rho : G \to GL(V)$ is representation on $(V, \langle .,. \rangle)$ and W be an invariant for ρ, then $W^\perp = \{v \in V : \langle v, w \rangle = 0 \ \forall w \in W\}$ is also ρ invariant.*

Proof. Let $v \in W^\perp$ then for any $w \in W$ and $g \in G$, we have

$$\begin{aligned}
\langle \rho(g)v, w \rangle &= \langle v, \rho(g)^* w \rangle \\
&= \langle v, \rho(g^{-1})w \rangle \\
&= \langle v, w_1 \rangle \qquad \text{for some } w_1 \in W \\
&= 0.
\end{aligned}$$

Hence $\rho(g)v \in W^\perp$. So $W^\perp$ is ρ invariant. □

5.2 Decomposition of Representation

Lemma 5.2.1. *Let (V, V_π) be a finite dimensional unitary representation of the locally compact group G. Then π splits into a direct sum of irreducible representation.*

Proof. Let $\dim(V_\pi) = n$. We prove this by induction on dimension on V_π. If $\dim(V_\pi) = 1$, then clearly π is irreducible. Now suppose the claim is true for all spaces of dimension $< n$. Then either, in that case we are done or it has a proper subrepresentation W. then by the previous theorem says that $W^\perp$ is also a subrepresentation. Now since V_π is the direct sum of W and W^{perp}, which are both of smaller dimension than n, and hence decompose into irreducible representation. Hence V_π is direct sum of irreducible representations. □

5.3 Schur's Lemma

Let K be compact subgroup of $GL_n(\mathbb{C})$. Let τ and γ be two irreducible finite dimensional representation of K. Let H be the space of all linear maps from V_γ to V_τ. Let $Hom_K(V_\gamma, V_\tau)$

be the space of K-homomorphisms i.e., the space of all linear maps $T : V_\gamma \to V_\tau$ such that

$$T\gamma(k) = \tau(k)T$$

for every $k \in K$.

Lemma 5.3.1. *The space $Hom_K(V_\gamma, V_\tau)$ is at most one-dimensional.*

Proof. Let $T \in Hom_K(V_\gamma, V_\tau)$ and $T \neq 0$. Then the kernel $ker(T)$ is an invariant subspace of V_γ. Since if, $v \in ker(T)$ then

$$T(\gamma(k)) = \tau(k)T(v) = 0$$

for all $k \in K$. Therefore, $\gamma(k)v$ again lies in $ker(T)$, which is thus invariant. Since γ is irreducible and $T \neq 0$, it follows that $ker(T) = \{0\}$, implies T is injective. Likewise, the image of T is an invariant subspace, and since τ is irreducible, too, it follows that T is surjective, and hence is an isomorphism. Now let $T, S \in Hom_K(V_\gamma, V_\tau)$ and assume that both are non-zero. Since both are bijective map, both maps are invertible and $S^{-1}T \in Hom_K(V_\gamma, V_\gamma)$. Since V_γ is a finite dimensional space there always exist a eigenvalue of $S^{-1}T$. Let λ be an eigenvalue of $S^{-1}T$ and $Eig(\lambda)$ be the corresponding eigen space. Then $Eig(\lambda)$ is a invariant space. By irreducibility and $S^{-1}T \neq 0$ it follows that $S^{-1}T = V_\gamma$, which implies $\forall v \in V_\gamma$

$$(S^{-1}T)v = \lambda v.$$

Hence $T = \lambda S$, so dimension of $Hom_K(V_\gamma, V_\tau)$ is at most one.

□

Definition 5.3.2. *Two unitary representation γ and τ are called unitarily equivalant if there exist a unitary map $T : V_\gamma \to V_\tau$ such that*

$$T\gamma(k) = \tau(k)T$$

for every $k \in K$.

Lemma 5.3.3. *Two finite-dimensional irreducible representations γ, τ of K are unitarily equvalant if and only if $Hom_K(V_\gamma, V_\tau) \neq 0$, regardless of the inner products.*

Proof. If γ, τ of K are unitarily equvalant then by defination there exist a map in $Hom_K(V_\gamma, V_\tau)$, which says that the space $Hom_K(V_\gamma, V_\tau) \neq 0$.
Conversely let, $T \neq 0$ lies in $Hom_K(V_\gamma, V_\tau)$, then $T^*T \in Hom_K(V_\gamma, V_\gamma)$ and also identity map I lies in $Hom_K(V_\gamma, V_\gamma)$. By previous Lemma we can say that there exist a $c \in \mathbb{C}$ such that

$$T^*T = cI.$$

Since T^*T is positive self adjoint operator, there is a $\lambda \in \mathbb{C}$ such that $c = \frac{1}{|\lambda^2|}$. So we get,

$$T^*T = \frac{1}{|\lambda^2|} I$$

$$\implies (\lambda T)^*(\lambda T) = I.$$

Hence (λT) is a unitary operator in $Hom_K(V_\gamma, V_\tau)$.

□

Corollary 5.3.4. *If (γ, V_γ) and (τ, V_τ) be two irreducible finite dimensional representation of K and there is a linear map $T : V_\gamma \to V_\tau$ such that*

$$T\gamma(k) = \tau(k)T$$

*for every $k \in K$, then either $T \neq 0$ or γ and τ are unitarily equivalant. This is known as **Schur's Lemma**.*

5.4 Unitarizable Representation

Definition 5.4.1. *Let G be a metrizable group and V be a Hilbert space and $\pi : G \to GL(V)$ be a continuous representation of the group G. If there exist an invertible operator $S : H \to H$ such that the representation $\tilde{\pi}(t) = S^{-1}\pi(t)S$ is a unitary representation. Then we will say that π is unitarizable.*

Lemma 5.4.2. *Let K be a compact group and ρ be a representation on a finite-dimensional Hilbert space $(V, \langle ., . \rangle)$. Then there is $S \in GL(V)$ such that the representation $S\rho S^{-1}$ is unitary.*

Proof. Let us definie a inner product $(.,.)$ on V such that for v, $w \in V$

$$(v, w) = \int_K \langle \rho(k^{-1})v, \rho(k^{-1})w \rangle dk.$$

Then for any $k_0 \in K$ and v, $w \in V$ we have,

$$\begin{aligned}
(\rho(k_0)v, \rho(k_0)w) \quad &= \int_K \langle \rho(k^{-1})\rho(k_0)v, \rho(k^{-1})\rho(k_0)w \rangle dk \\
&= \int_K \langle \rho(k^{-1}k_0)v, \rho(k^{-1}k_0)w \rangle dk \\
&= \int_K \langle \rho(k_0^{-1}k)^{-1}v, \rho(k_0^1 k)^{-1}w \rangle dk \\
&= \int_K \langle \rho(k^{-1})v, \rho(k^{-1})w \rangle dk \\
&= (v, w).
\end{aligned}$$

Which implies that ρ is unitary with respect to $(.,.)$.

Now we know that every inner product on V is of the form $(v, w) = \langle Sv, Sw\rangle$ for some $S \in GL(V)$. From this it follows that $S^{-1}\rho S$ is unitary. □

Problem 5.4.3. *Let $B(H)$ be the space of all bounded operators on a complex Hilbert space H and $\pi : G \to B(H)$ be a continuous representation of a locally compact group G. Assume π is uniformly bounded, i.e. assume*

$$\sup_{t\in G} ||\pi(t)||_{B(H)} < \infty.$$

Then does this representation π is unitarizable?

This problem remained open for a while until, in 1955, Ehrenpreis and Mautner gave a counterexample for $G = SL_2(R)$. Later it was realized that rather simpler counterexamples can be described on the free groups with at least 2 generators. In the positive direction, the most general result seems to be Dixmier's theorem (1955) which says that if G is amenable then the answer to the above Problem is affirmative.

5.5 Amenable Group

Definition 5.5.1. *A locally compact group G is called amenable if there exists a left invariant mean on G, i.e. if there exists a positive linear functional $\phi : L_\infty(G) \to \mathbb{C}$ satisfying $||\phi|| = \phi(1) = 1$, and*

$$\forall f \in L_\infty(G) \quad \forall t \in G \quad \phi(\delta_t * f) = \phi(f).$$

Here $\delta_t * f(s) = f(t^{-1}s)$.

Examples 5.5.2.

- *Finite groups are amenable. The measure is counting measure.*
- *Compact groups are amenable. The Haar measure is an invariant mean*

The free group on two generators say a, b is denoted by $\mathbb{F}_2$. It is the typical example of a nonamenable group. To check this, assume there is an invariant mean ϕ on $\mathbb{F}_2$, then since this group is infinite, we have necessarily $\phi(1_t) = 0$ for all $t \in \mathbb{F}_2$. Let $F(x)$ is the set of (reduced) words which have x as their first letter, then

$$\mathbb{F}_2 = F(a) \cup F(a^{-1}) \cup F(b) \cup F(b^{-1}) \cup \{e\}.$$

Now if we apply ϕ both side in the above equation we will get

$$1 = \phi(1_{\mathbb{F}_2}) = \phi(1_{F(a)}) + \phi(1_{F(a^{-1})}) + \phi(1_{F(b)}) + \phi(1_{F(b^{-1})}).$$

On the other hand we have,

$$F(a) = a[\mathbb{F}_2 - F(a^{-1})]$$

$$F(a^{-1}) = a^{-1}[\mathbb{F}_2 - F(a)]$$

$$F(b) = b[\mathbb{F}_2 - F(b^{-1})]$$

$$F(b^{-1}) = b^{-1}[\mathbb{F}_2 - F(b)].$$

Hence,

$$\phi(1_{F(a)}) = 1 - \phi(1_{F(a^{-1})})$$

$$\phi(1_{F(a^{-1})}) = 1 - \phi(1_{F(a)})$$

$$\phi(1_{F(b)}) = 1 - \phi(1_{F(b^{-1})})$$

$$\phi(1_{F(b^{-1})}) = 1 - \phi(1_{F(b)}).$$

If we add the last four identities we obtain $1 = 4 - 1$ which is the desired contradiction.

Theorem 5.5.3. ***(Day, Dixmier 1950)***

Let G be a locally compact group. If G is amenable, then every uniformly bounded (u.b. in short) representation $\pi : G \to B(H)$ is unitarizable. More precisely, if we define,

$$|\pi| = \sup_{t \in G} ||\pi(t)||_{B(H)}.$$

then, if $|\pi| < \infty$, there exists $S : H \to H$ invertible with $||S||||S^{-1}|| \leq |\pi^2|$ such that $S^{-1}\pi(.)S$ is a unitary representation.

Proof. Let $|\pi| = c < \infty$ and for any x, y in H, we denote,

$$\forall t \in G \quad f_{xy}(t) = \langle \pi(t^{-1})x, \pi(t^{-1})y \rangle$$

then $f_{xy} \in L_\infty(G)$.

Let ϕ be an invariant mean on G. We define $|||x||| = (\phi(f_{xx}))^{\frac{1}{2}}$
i.e $|||x|||^2 = (\phi(f_{xx}))$. Then $|||.|||$ will be a norm on H, called Hilbertien norm. Now,

$$\begin{aligned} f_{xx}(t) \quad &= \langle \pi(t^{-1})x, \pi(t^{-1})x \rangle \\ &= ||\pi(t^{-1})x||^2 \\ &\leq c^2||x||^2 \\ \implies \phi(f_{xx}) \quad &\leq c^2||x||^2\phi(1) \\ &= c^2||x||^2. \end{aligned}$$

Hence we get,

$$\boxed{|||x||| \leq c||x||}$$

On the other hand,

$$\begin{aligned} ||x||^2 &= ||\pi(tt^{-1})x||^2 \\ &= ||\pi(t)\pi(t^{-1})x||^2 \\ &\leq ||\pi(t)||^2||\pi(t^{-1})x||^2 \\ &\leq c^2 f_{xx}(t) \\ \implies \phi(||x||^2) &\leq c^2\phi(f_{xx}(t)) \\ \implies ||x||^2 &\leq c^2\phi(f_{xx}). \end{aligned}$$

We get,

$$\boxed{||x|| \leq c|||x|||}.$$

Which says that $(H, |||.|||)$ is isomorphic to H, it is actually isometric to $(H, ||.||)$. We have clearly for all $s \in G$ and for $x \in H$ we have,

$$\begin{aligned} |||\pi(s)x|||^2 &= \pi(f_{\pi(s)x\pi(s)x}) \\ &= \pi(\delta_s * f_{xx}) \\ &= \pi(f_{xx}) \\ &= |||x|||^2. \end{aligned}$$

Which shows that $\pi(s)$ is a unitary map with respect to $|||.|||$. □

In his 1950 paper, Dixmier [5] asked two questions which can be rephrased as follows
Q1: Is every G unitarizable?
Q2: If not, is it true that conversely unitarizable implies amenable?

5.6 Representation of a Discrete Group

In thins section we consider G as a disctere group. Let $\lambda : G \to B(l_2(G))$ defined by,

$$\forall h \in l_2(G) \quad \lambda(t)h = \delta_t * h.$$

Where $\delta_t * h(s) = h(t^{-1}s)$ for $s \in G$. Let $\rho : G \to B(l_2(G))$ defined by,

$$\forall h \in l_2(G) \quad \rho(t)h = h * \delta_{t^{-1}}.$$

Where $h * \delta_{t^{-1}}(s) = h(st)$. We can show that both λ and ρ are representation of G. Let $t_1, t_2 \in G$,

$$\lambda(t_1)\lambda(t_2)h = \delta_{t_1} * (\lambda(t_2)h).$$

Now,

$$\begin{aligned}\delta_{t_1} * (\lambda(t_2)h(s) \quad &= (\lambda(t_2)h)(t_1^{-1}s) \\ &= (\delta_{t_2} * h)(t_1^{-1}s) \\ &= h(t_2^{-1}t_1^{-1}s) \\ &= h((t_1t_2)^{-1}s) \\ &= \delta_{t_1t_2} * h(s).\end{aligned}$$

Hence,

$$\lambda(t_1)\lambda(t_2)h = \delta_{t_1} * (\lambda(t_2)h) = \delta_{t_1t_2} * h = \lambda(t_1t_2)h.$$

Therefore λ is Representation of G. Similarly we can prove for ρ. This λ is called left regular representaton and ρ is called right regular representation. Observe that λ and ρ commute, i.e.

$$\forall s, t \in G \quad \lambda(t)\rho(s) = \rho(s)\lambda(t).$$

Let $B(G)$ is the space of all functions $f : G \to \mathbb{C}$ for which there is a Hilbert space H, a unitary representation $\pi : G \to B(H)$ and elements x, y in H such that,

$$f(t) = \langle \pi(t)x, y\rangle.$$

Now in this space if we difine a norm by,

$$||f||_{B(G)} = \inf\{||x||.||y||\}$$

Then $(B(G), ||.||_{B(G)})$ will be a Banach space.
Let G be a group and let $S \subset G$ be a semi-group included in G, i.e.

$$\forall s, t \in S \quad st \in S.$$

Now let $T_p(S)$ be a space of all the functions $f : S \to \mathbb{C}$ which admit the following decomposition: there are functions $f_1 : S \times S \to \mathbb{C}$ and $f_2 : S \times S \to \mathbb{C}$ such that,

$$f(st) = f_1(s,t) + f_2(s,t).$$

Where,

$$\sup_{s\in S}\sum_{t\in S}|f_1(s,t)|^p < \infty, \quad \sup_{t\in S}\sum_{s\in S}|f_2(s,t)|^p < \infty.$$

We equip this space with the norm,

$$||f||_{T_p} = \inf\{\sup_s(\sum_t |f_1(s,t)|^p)^{\frac{1}{p}} + \sup_t(\sum_s |f_2(s,t)|^p)^{\frac{1}{p}}\}.$$

Where the infimum runs over all such decompositions. Then $(T_p(G), ||.||_{T_p(G)})$ will form a Banach space.
Note that,

$$l_p(G) \subset B(G) \subset l_\infty(G)$$
$$l_p(S) \subset T_p(S) \subset l_\infty(S).$$

Theorem 5.6.1. *If every uniformly bounded representation on a discrete group G is unitarizable, then*

$$T_1(G) \subset B(G).$$

Proof. Let $f \in T_1(G)$, then we can find a decomposition of the form

$$f(s^{-1}t) = a_1(s,t) + a_2(s,t)$$

and for some constant C we have,

$$\sup_s \sum_t |a_1(s,t)| \leq C \quad \sup_t \sum_s |a_2(s,t)| \leq C.$$

Let us denote A_1 and A_2 as the representative matrices of a_1 and a_2 respictively. Let $\tilde{f}$ be the function defined by $\tilde{f}(x) = f(x^{-1})$. Observe that $f(s^{-1}t)$ is nothing but the matrix of the operator $\rho(f)$ defined by

$$\rho(f)g = g * \tilde{f}$$

, Hence we have

$$\rho(f) = A_1 + A_2$$

and

$$||A_1||_{B(l_\infty(G))} \leq C, \quad ||A_2||_{B(l_1(G))} \leq C$$

By an obvious calculation we have,

$$\rho(f)\lambda(t) = \lambda(t)\rho(f) \quad \forall t \in G.$$

Hence we have,

$$0 = [\rho(f), \lambda(t)] = [A_1 + A_2, \lambda(t)] = [A_1, \lambda(t)] + [A_2, \lambda(t)].$$

Therefore,

$$[A_1, \lambda(t)] = -[A_2, \lambda(t)].$$

For any linear operatopr a,let $D := [A_2, a] := A_2a - aA_2$. As it is well known that D is a derivation, we have

$$D(ab) = D(a)b + aD(b).$$

Now let $H = l_2(G) \oplus l_2(G)$ and $\pi(t) : H \to H$ defined by,

$$\pi(t) = \begin{bmatrix} \lambda(t) & D(\lambda(t)) \\ 0 & \lambda(t) \end{bmatrix}.$$

Then for $t,\, s \in G$ we have,

$$\begin{aligned}
\pi(t)\pi(s) \qquad &= \begin{bmatrix} \lambda(t) & D(\lambda(t)) \\ 0 & \lambda(t) \end{bmatrix} \begin{bmatrix} \lambda(s) & D(\lambda(s)) \\ 0 & \lambda(s) \end{bmatrix} \\
&= \begin{bmatrix} \lambda(t)\lambda(s) & D(\lambda(t)\lambda(s)) \\ 0 & \lambda(t)\lambda(s) \end{bmatrix} \\
&= \begin{bmatrix} \lambda(ts) & D(\lambda(ts)) \\ 0 & \lambda(ts) \end{bmatrix} \\
&= \pi(ts).
\end{aligned}$$

Hence π is a representation of G on the Hilbert space H. Our next aim is to show that this representation is uniformly bounded.
$[A_2, \lambda(t)] : l_1(G) \to l_1(G)$ is a bounded operator. Since $[A_2, \lambda(t)] = -[A_1, \lambda(t)]$, the operator $[A_2, \lambda(t)]$ bounded on $l_\infty(G)$ also. By the Riesz-Thorin classical interpolation theorem it follows that,

$$||[A_2, \lambda(t)]||_{B(l_2(G))} \leq 2C$$

and since $[A_2, \lambda(t)] = D(\lambda(t))$ we have,

$$||\pi(t)|| \leq 1 + 2C.$$

Now if π is unitarizable, then we have $\langle \pi(t)x, y\rangle = \langle \tilde{\pi}(t)Sx, S^{-1*}y\rangle$ for some $\tilde{\pi}$ unitary and some similarity S. Finally, let $x = (0, \delta_e) \quad y = (\delta_e, 0)$ then,

$$\begin{aligned}
\langle \pi(t)x, y\rangle \qquad &= \langle \begin{bmatrix} \lambda(t) & D(\lambda(t)) \\ 0 & \lambda(t) \end{bmatrix} \begin{bmatrix} 0 \\ \delta_e \end{bmatrix}, \begin{bmatrix} \delta_e \\ 0 \end{bmatrix} \rangle \\
&= \langle \begin{bmatrix} D(\lambda(t)).\delta_e \\ \lambda(t).\delta_e \end{bmatrix}, \begin{bmatrix} \delta_e \\ 0 \end{bmatrix} \rangle \\
&= \langle D(\lambda(t)).\delta_e, \delta_e\rangle + \langle \lambda(t)).\delta_e, 0\rangle \\
&= \langle D(\lambda(t)).\delta_e, \delta_e\rangle \\
&= \langle [A_2, \lambda(t)].\delta_e, \delta_e\rangle \\
&= \langle A_2\lambda(t)\delta_e, \delta_e\rangle - \langle \lambda(t)A_2\delta_e, \delta_e\rangle \\
&= a_2(e, t) - a_2(t^{-1}, e).
\end{aligned}$$

Now if we defini a function $F : G \to \mathbb{C}$ where $F(t) = a_2(e,t) - a_2(t^{-1}, e)$ then,

$$F(t) = \langle \pi(t)x, y \rangle = \langle \tilde{\pi}(t)Sx, S^{-1*}y \rangle.$$

Which implies that $F \in B(G)$. Now,

$$f(t) = a_1(e,t) + a_2(t^{-1}, e) + [a_2(e,t) - a_2(t^{-1}, e)].$$

Since the map $t \mapsto a_1(e,t) + a_2(t^{-1}, e) \in l_1(G) \subset B(G)$.
So we conclude that $f \in B(G)$. □

Lemma 5.6.2. *Let f be the indicator function of the set of words of length 1 in $\mathbb{F}_\infty$. Then $f \in T_1(\mathbb{F}_\infty)$ but $f \notin B(\mathbb{F}_\infty)$.*

Theorem 5.6.3. *Any discete group G containing $\mathbb{F}_2$ as subgroup is not uniterizable.*

Proof. Since $\mathbb{F}_2$ conatains $\mathbb{F}_\infty$, so G contains $\mathbb{F}_\infty$, then we have $T_1(G) \not\subset B(G)$ and, by previous theorem we can say that G is not uniterizable. □

So $\mathbb{F}_2$ is a non-amenable group which is not uniterizable. Which says that every group is not uniterizable.
Now if every non-amenable discrete group contain a copy of $\mathbb{F}_2$, then it follows that every non-amenable discrete groups are not uniterizable. Then the answer of Dixmier's 2nd question will be affermative for discrete case. But in 1980 A. Yu. Olshanski [8] established such discrete group which does not contain any free subgroup, by using the solution by AdianNovikov [1] of the famous Burnside problem, and also Grigorchuks cogrowth criterion [6]. Later, Adian [2] showed that the Burnside group $B(m,n)$ are all non-amenable when $m \geq 2$ and odd $n \geq 665$.

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