NONNEGATIVE MOORE-PENROSE INVERSES OF UNBOUNDED GRAM OPERATORS

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Abstract. In this paper we derive necessary and sufficient conditions for the nonnegativity of Moore-Penrose inverses of unbounded Gram operators between real Hilbert spaces. These conditions include statements on acuteness of certain closed convex cones. The main result generalizes the existing result for bounded operators [11, Theorem 3.6].

1. Introduction

Monotonicity of Gram matrices and Gram operators has received a lot of attention in recent years. This has been primarily motivated by applications in convex optimization problems.

A real square matrix T is called monotone if $x \geq 0$, whenever $Tx \geq 0$. Here $x = (x_i) \geq 0$ means that $x_i \geq 0$ for all i. Collatz [4] has shown that a matrix is monotone if and only if it is invertible and the inverse is nonnegative. Gil gave sufficient conditions on the entries of an infinite matrix T in order for T^{-1} to be nonnegative [5]. An extension of the notion of monotonicity to characterize nonnegativity of generalized inverses in the finite dimensional case seems to have been first accomplished by Mangasarian [13]. Berman and Plemmons [2] made extensive contributions to nonnegative generalized inverses by proposing various notions of monotonicity. The book by Berman and Plemmons [2] contain numerous examples of applications of nonnegative generalized inverses that include Numerical Analysis and linear economic models.

The question of monotonicity and their relationships to nonnegativity of generalized inverses in the infinite dimensional setting, have been first taken up by Sivakumar ([14] and [15]). Three other types of operator monotonicity were studied later by Kulkarni and Sivakumar [10]. For applications of nonnegative Moore-Penrose inverses of operators to the solution of linear systems of equations defined by operators between infinite dimensional spaces, we refer to Kammerer and Plemmons ([6, Section 6]).

There is a well known result by Cegielski that characterizes nonnegative invertibility of Gram matrices in terms of obtuseness (or acuteness) of certain polyhedral cones. (See for instance [3, Lemma 1.6]). The results of Cegielski were generalized by Kurmayya and Sivakumar [11] in two directions; from finite dimensional real Euclidean spaces to infinite dimensional real Hilbert spaces and from classical inverses to Moore-Penrose inverses.

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In this paper we consider linear operators (not necessarily bounded) between real Hilbert spaces and obtain necessary and sufficient conditions for the nonnegativity of Moore-Penrose inverses of Gram operators in terms of acuteness of certain closed convex cones. This can be achieved by taking cones in the domain of the Gram operator. Because of this slight modification, we observe that there is a slight change in some of the existing results (see Lemmas 3.1 and 3.2, and the condition (2) in Theorem 3.4). Our results generalizes the existing results due to Kurmayya and Sivakumar [11] and the related results (See for instance Lemma 1.6, [3]) in the literature.

The paper is organized as follows. In section 2 we introduce some basic notations, definitions and results. In section 3, we present some preliminary results and prove the main theorem. In section 4, we illustrate the main theorem with some examples.

2. Notations and Preliminary results

Throughout the article we consider infinite dimensional real Hilbert spaces which will be denoted by H, H_1, H_2 etc . The inner product and the induced norm are denoted by $\langle , \rangle$ and $||.||$ respectively.

A subset K of a Hilbert space H is called cone if, (i) $x, y \in K \Rightarrow x + y \in K$ and (ii) $x \in K$, $\alpha \in \mathbb{R}$, $\alpha \geq 0 \Rightarrow \alpha x \in K$. For a subset K of a Hilbert space H, the dual of K denoted K^* is defined as $K^* = \{x \in H \colon \langle x, t\rangle \geq 0, \text{for all } t \in K\}$ and $K^{**} = (K^*)^*$. Note that in general, $K^{**} = \overline{K}$, where the bar denotes the closure of K. If $H = \ell^2$, the Hilbert space of all square summable real sequences and $K = \ell^2_+ = \{x \in \ell^2 \colon x_i \geq 0, \forall i\}$, then $K^* = \ell^2_+$ and hence $K^{**} = \ell^2_+$. A cone C is said to be acute if $\langle x, y\rangle \geq 0$, for all $x, y \in C$.

Let T be a linear operator with domain $D(T)$, a subspace of H_1 and taking values in H_2, then the graph $G(T)$ of T is defined by $G(T) := \{(x, Tx) : x \in D(T)\} \subseteq H_1 \times H_2$. If $G(T)$ is closed, then T is called a closed operator. If $D(T)$ is dense in H_1, then T is called a densely defined operator. For a densely defined operator there exists a unique linear operator $T^* : D(T^*) \to H_1$, where

$$D(T^*) := \{y \in H_2 : \text{the functional } x \to \langle Tx, y\rangle \text{ for all } x \in D(T) \text{ is continuous}\}$$

and $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in D(T)$ and $y \in D(T^*)$. This operator is called the adjoint of T. Note that T^* is always closed whether or not T is closed.

The set of all closed operators between H_1 and H_2 is denoted by $\mathcal{C}(H_1, H_2)$ and $\mathcal{C}(H) := \mathcal{C}(H, H)$. By the closed graph Theorem [18], an everywhere defined closed operator is bounded. Hence the domain of an unbounded closed operator is a proper subspace of a Hilbert space. For, $T \in \mathcal{C}(H_1, H_2)$, the null space and the range space of T are denoted by $N(T)$ and $R(T)$ respectively and the space $C(T) := D(T) \cap N(T)^{\perp}$ is called the carrier of T. In fact, $D(T) = N(T) \oplus^{\perp} C(T)$ [1, page 340]. For a closed subspace M of H, we denote the orthogonal projection on H with range M by P_M.

If $T \in \mathcal{C}(H_1, H_2)$ and $S \in \mathcal{C}(H_2, H_3)$, then $D(ST) = \{x \in D(T) : Tx \in D(S)\}$ and $(ST)(x) = S(Tx)$ for all $x \in D(ST)$.

If S and T are closed operators with the property that $D(S) \subseteq D(T)$ and $Sx = Tx$ for all $x \in D(S)$, then S is called the restriction of T and T is called an extension of S. For the details we refer to [16, 7, 18].

Next, we recall some of the definitions and important results that we use throughout the article.

Definition 2.1. *For a linear map $T : H_1 \longrightarrow H_2$, the operator T^*T is said to be the Gram operator of T.*

Definition 2.2. *(Moore-Penrose Inverse)*[1, definition 2, page 339] *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then there exists a unique densely defined operator $T^\dagger \in \mathcal{C}(H_2, H_1)$ with domain $D(T^\dagger) = R(T) \oplus^\perp R(T)^\perp$ and has the following properties:*

1. *$TT^\dagger y = P_{\overline{R(T)}}\, y$, for all $y \in D(T^\dagger)$*
2. *$T^\dagger T x = P_{N(T)^\perp}\, x$, for all $x \in D(T)$*
3. *$N(T^\dagger) = R(T)^\perp$.*

This unique operator $T^\dagger$ is called the Moore-Penrose inverse of T.
The following property of $T^\dagger$ is also well known. For every $y \in D(T^\dagger)$, let

$$L(y) := \Big\{x \in D(T) : ||Tx - y|| \leq ||Tu - y|| \quad \text{for all} \quad u \in D(T)\Big\}.$$

Here any $u \in L(y)$ is called a least square solution of the operator equation $Tx = y$. The vector $x = T^\dagger y \in L(y)$, $||T^\dagger y|| \leq ||u||$ for all $u \in L(y)$ and it is called the least square solution of minimal norm. A different treatment of $T^\dagger$ is given in [1], *where it is called "the Maximal Tseng generalized Inverse".*

We have the following equivalent definition:

Definition 2.3. *Let $P := P_{\overline{R(T)}}$. If $y \in R(T) \oplus^\perp R(T)^\perp$, the equation*

$$Tx = Py \tag{2.1}$$

always has a solution. This solution is called a ***least square solution****. If $x \in D(T)$ is a least square solution, then*

$$||Tx - y||^2 = ||Py - y||^2 = \min_{z \in D(T)} ||Tz - y||^2.$$

The unique vector with the minimal norm among all least square solutions, is called the least square solution of minimal norm of the Equation 2.1 and is given $x = T^\dagger y$.

Here we list the properties of the Moore-Penrose inverse, which we need to prove our main results.

Theorem 2.4. [1, theorem 2, page 341] *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then*

1. *$D(T^\dagger) = R(T) \oplus^\perp R(T)^\perp$, $\quad N(T^\dagger) = R(T)^\perp = N(T^*)$*
2. *$R(T^\dagger) = C(T)$*
3. *$T^\dagger$ is densely defined and $T^\dagger \in \mathcal{C}(H_2, H_1)$*
4. *$T^\dagger$ is continuous if and only $R(T)$ is closed.*
5. *$T^{\dagger\dagger} = T$*
6. *$T^{*\dagger} = T^{\dagger *}$*
7. *$N(T^{*\dagger}) = N(T)$*
8. *$(T^*T)^\dagger = T^\dagger T^{*\dagger}$*
9. *$(TT^*)^\dagger = T^{*\dagger} T^\dagger$.*

Proposition 2.5. [1] *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Then*

1. *$N(T) = R(T^*)^\perp$*
2. *$N(T^*) = R(T)^\perp$*
3. *$N(T^*T) = N(T)$ and*

(4) $\overline{R(T^*T)} = \overline{R(T^*)}$.

Proposition 2.6. [1, 7] *For a densely defined $T \in \mathcal{C}(H_1, H_2)$, the following statements are equivalent:*

(1) *$R(T)$ is closed*
(2) *$R(T^*)$is closed*
(3) *$R(T^*T)$ is closed. In this case, $R(T^*T) = R(T^*)$*
(4) *$R(TT^*)$ is closed. In this case, $R(TT^*) = R(T)$.*

Theorem 2.7. [12, Theorem 4.1] *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined. Assume that $R(T)$ is closed. Then*

$$(T^*T)^\dagger T^* \subset T^*(TT^*)^\dagger = T^\dagger.$$

For more information on generalized inverses we refer to [19, 20, 17].

3. Main results

For proving the main theorem (Theorem 3.4) we consider the following results.

Let H_1 and H_2 be real Hilbert spaces, $T \in \mathcal{C}(H_1, H_2)$ be densely defined with closed range. Let K be a closed convex cone in $D(T^*T)$ such that $K^* \subset D(T^*T)$. Let $C = TK$ and $D = (T^\dagger)^* K^*$.

Lemma 3.1. $u \in C^* \cap D(T^*) \Longrightarrow T^*u \in K^*$.

Proof. Let $u \in C^* \cap D(T^*)$ and $r \in K$. Then $0 \leq \langle u, Tr\rangle = \langle T^*u, r\rangle$. □

Lemma 3.2. *The following are equivalent :*

(1) *$C^* \cap D(T^*) \cap R(T)$ is acute.*
(2) *For all $x, y \in D(T^*T)$ with $T^*Tx \in K^*, T^*Ty \in K^*$, the inequality $\langle T^*Tx, y\rangle \geq 0$ holds.*

Proof. (1) $\Longrightarrow$ (2): Let $x, y \in D(T^*T)$ satisfy $T^*Tx \in K^*$ and $T^*Ty \in K^*$. For $r \in K$, we have $Tr \in C$ and hence

$$\langle Tx, Tr\rangle = \langle T^*Tx, r\rangle \geq 0.$$

So, $Tx \in C^*$. Similarly, we can show that $Ty \in C^*$. Since $C^* \cap D(T^*) \cap R(T)$ is acute, we have $0 \leq \langle Tx, Ty\rangle = \langle T^*Tx, y\rangle$.

(2) $\Longrightarrow$ (1): Let $u, v \in C^* \cap D(T^*) \cap R(T)$. Let $u = Tx$ for some $x \in D(T)$. Since $u \in D(T^*)$, T^*u is defined. That is $x \in D(T^*T)$. Similarly, $v = Ty$ for some $y \in D(T^*T)$.

Next we show that $\langle u, v\rangle \geq 0$. Since $u \in C^*$, for $r \in K$ we have

$$0 \leq \langle Tx, Tr\rangle = \langle T^*Tx, r\rangle.$$

Thus $T^*Tx \in K^*$. With a similar argument, we can conclude that $T^*Ty \in K^*$. By assumption,

$$\langle u, v\rangle = \langle Tx, Ty\rangle = \langle T^*Tx, y\rangle \geq 0.$$

Hence $C^* \cap D(T^*) \cap R(T)$ is acute. □

Lemma 3.3. *D is acute if and only if $\langle r, (T^*T)^\dagger s\rangle \geq 0$, for every $r, s \in K^*$.*

Proof. Let $x, y \in D$. Then $x = (T^\dagger)^* r,\ y = (T^\dagger)^* s$ for some $r, s \in K^*$. Then D is acute if and only if

$$0 \leq \langle x, y\rangle = \langle (T^\dagger)^* r, (T^\dagger)^* s\rangle = \langle r, T^\dagger (T^\dagger)^* s\rangle = \langle r, (T^*T)^\dagger s\rangle,$$

by (8) of Theorem 2.4. □

We are now in a position to prove the main result of this paper.

Theorem 3.4. *Let $T \in \mathcal{C}(H_1, H_2)$ be densely defined with closed range. Let K be a closed convex cone in $D(T^*T)$ with $T^\dagger T K \subseteq K$. Let $C = TK$ and $D = (T^\dagger)^* K^*$. Then the following conditions are equivalent:*

(1) $(T^*T)^\dagger (K^*) \subseteq K$
(2) $C^* \cap D(T^*) \cap R(T) \subseteq C$
(3) D *is acute*
(4) $C^* \cap D(T^*) \cap R(T)$ *is acute*
(5) $T^*Tx \in P_{R(T^*)}(K^*) \Longrightarrow x \in K$
(6) $T^*Tx \in K^* \Longrightarrow x \in K$.

Proof. (1)$\Longrightarrow$ (2): Let $u \in C^* \cap D(T^*) \cap R(T)$. Then $u = Tp$ for some $p \in C(T)$. Then $T^\dagger u = T^\dagger T p = P_{N(T)^\perp} p = p$. Since $u \in D(T^*)$, by Theorem 2.7, $T^\dagger u = (T^*T)^\dagger T^* u$. Set $z = T^\dagger u$. Then $Tz = TT^\dagger u = P_{R(T)} u = u$. Also $T^*u \in K^*$, by Lemma 3.1. So by the assumption, $z = (T^*T)^\dagger T^* u \in K$. Thus $u \in C$.

(2)$\Longrightarrow$ (3): Let $x = (T^\dagger)^* u$ and $y = (T^\dagger)^* v$ with $u, v \in K^*$. Since

$$\begin{aligned} R((T^\dagger)^*) = R((T^*)^\dagger) = C(T^*) &= D(T^*) \cap N(T^*)^\perp \\ &= D(T^*) \cap \overline{R(T)} \\ &= D(T^*) \cap R(T), \end{aligned}$$

$x, y \in D(T^*) \cap R(T)$. Let $r \in K$. We have $r' = T^\dagger T r \in K$ (as $T^\dagger T K \subseteq K$). Then

$$\langle x, Tr \rangle = \langle (T^\dagger)^* u, Tr \rangle = \langle u, T^\dagger T r \rangle = \langle u, r' \rangle \geq 0.$$

Thus $x \in C^*$. Since $C^* \cap D(T^*) \cap R(T) \subseteq C$, we have $x \in C$. Thus $x = Tp$ for some $p \in K$.

Finally, with $p' = T^\dagger T p \in K$, we have,

$$\langle x, y \rangle = \langle Tp, (T^\dagger)^* v \rangle = \langle T^\dagger T p, v \rangle = \langle p', v \rangle \geq 0.$$

Hence D is acute.

(3)$\Longrightarrow$ (4): Let x, y be such that $r = T^*Tx \in K^*$ and $s = T^*Ty \in K^*$. Since D is acute, by Lemma 3.3,

$$\begin{aligned} 0 \leq \langle r, (T^*T)^\dagger s \rangle &= \langle T^*Tx, (T^*T)^\dagger T^*Ty \rangle \\ &= \langle x, (T^*T)(T^*T)^\dagger (T^*T) y \rangle \\ &= \langle x, (T^*T) y \rangle \\ &= \langle T^*Tx, y \rangle. \end{aligned}$$

By Lemma 3.2, $C^* \cap D(T^*) \cap R(T)$ is acute.

(4)$\Longrightarrow$ (5): Let $T^*Tx = P_{R(T^*)} w$ for some $w \in K^*$. Since, $R(T^*T) = R(T^*)$, we have $T^*Tx = P_{R(T^*T)} w$. Hence $x = (T^*T)^\dagger w$ (By Definition 2.3).

Let $r \in K^*$. Then

$$\langle x, r \rangle = \langle (T^*T)^\dagger w, r \rangle = \langle T^\dagger (T^\dagger)^* w, r \rangle = \langle (T^\dagger)^* w, (T^\dagger)^* r \rangle.$$

Set $u = (T^\dagger)^* w,\ v = (T^\dagger)^* r$. Then, as was shown earlier, $u, v \in R(T) \cap D(T^*)$. For $t \in K$, with $t' = T^\dagger T t \in K$, we have

$$\langle u, Tt \rangle = \langle (T^\dagger)^* w, Tt \rangle = \langle w, T^\dagger T t \rangle = \langle w, t' \rangle \geq 0.$$

So $u \in C^*$. Along similar lines it can be shown that $v \in C^*$. Thus for all $r \in K^*$, $\langle x, r \rangle = \langle u, v \rangle \geq 0$. So $x \in (K^*)^* = K$.

(5)$\Longrightarrow$ (6): Choose x such that $T^*Tx \in K^*$. We have

$$T^*Tx = P_{R(T^*T)}(T^*Tx) = P_{R(T^*)}(T^*Tx) \in P_{R(T^*)}(K^*).$$

Hence by (5), $x \in K$.

(6)$\Longrightarrow$ (1): Let $u = (T^*T)^\dagger v$ with $v \in K^*$.

Then $T^*Tu = T^*T(T^*T)^\dagger v = P_{R(T^*)}v = T^\dagger Tv$. Then for $r \in K$ with $r' = T^\dagger Tr \in K$, we have

$$\langle T^*Tu, r\rangle = \langle T^\dagger Tv, r\rangle = \langle v, T^\dagger Tr\rangle = \langle v, r'\rangle \geq 0.$$

Thus $T^*Tu \in K^*$. As (6) holds, $u \in K$. Thus $(T^*T)^\dagger(K^*) \subseteq K$.

This completes the proof of the theorem. □

Remark 3.5.

(1) In [11], conditions (5) and (6) were shown to be equivalent to each other and also equivalent to the nonnegativity of $(T^*T)^\dagger$ under an assumption that $x \in R(T^*)$ by Kurmayya and Sivakumar. Here, we make a remark that the above mentioned assumption is redundant to prove equivalence of those conditions. If $T^*Tx \in P_{R(T^*)}(K^*)$ then it can be shown that $x \in R(T^*)$.

(2) If $K \subseteq C(T^*T)$, then the condition $T^\dagger TK \subseteq K$ is satisfied automatically.

(3) If T is one-to-one, then $T^\dagger T = I$ and hence in this case $T^\dagger TK \subseteq K$ holds for any cone in $D(T)$.

4. Examples

In this section, we illustrate Theorem 3.4 with examples.

Example 4.1. Let $H = \ell^2$ and $D(T) = \Big\{(x_1, x_2, \dots) \in H\colon \sum_{j=1}^{\infty} |jx_j|^2 < \infty\Big\}$. Define $T : D(T) \to H$ by

$$T(x_1, x_2, x_3, \dots, x_n, \dots) = (x_1, 2x_2, 3x_3, \dots, nx_n, \dots) \quad \text{for all } (x_1, x_2, \dots) \in D(T).$$

Since $D(T)$ contains c_{00}, the space of all sequences having at most finitely many nonzero terms, we have $\overline{D(T)} = H$. Clearly T is unbounded and closed since $T^* = T$. By [8, example 5.1], $R(T)$ is closed. In fact, T^{-1} exists and

$$T^{-1}(y_1, y_2, y_3, \dots, y_n, \dots) = (y_1, \frac{y_2}{2}, \frac{y_3}{3}, \dots, \frac{y_n}{n}, \dots), \text{ for all } (y_n) \in H.$$

Note that $D(T^*T) = \{(x_n) \in H : \sum_{n=1}^{\infty} n^4|x_n|^4 < \infty\}$. Let

$$K = \{(x_n) \in D(T^2) : x_n \geq 0 \text{ for all } n \in \mathbb{N}\}.$$

Clearly, $K^* = K$ and $T^\dagger TK = K$. Hence K satisfy the Hypothesis of Theorem 3.4. In this case, $D = T^{\dagger^*}(K^*) = T^{-1}(K)$. Let $x, y \in D$. Then $x = T^{-1}u,\ y = T^{-1}v$ for some $u, v \in H$. Then Let $u = \sum_{n=1}^{\infty}\langle u, e_n\rangle e_n$ and $v = \sum_{n=1}^{\infty}\langle v, e_n\rangle e_n$ (Here $\{e_n : n \in \mathbb{N}\}$

is the standard orthonormal basis for H). Then

$$\begin{aligned}\langle x,y\rangle &= \langle T^{-2}u,v\rangle \geq 0\\ &= \sum_{n=1}^{\infty}\frac{1}{n^2}\langle u,e_n\rangle\,\langle v,e_n\rangle\\ &\geq 0 \quad (\text{since } \langle u,e_n\rangle, \langle v,e_n\rangle \geq 0).\end{aligned}$$

Therefore D is acute. Hence by Theorem 3.4, $(T^*T)^\dagger$ is nonnegative with respect to the cone K. This can be easily verified independently by using the definition.

Example 4.2. Let $H=\ell^2$ and $D(T)=\big\{(x_1,x_2,\dots,x_n,\dots)\colon \sum_{j=2}^{\infty}|jx_j|^2<\infty\big\}$. Define $T:D(T)\to H$ by

$$T(x_1,x_2,\dots,x_n,\dots)=\big(0,2x_2,3x_3,4x_4,\dots\big) \quad \text{for all} \quad (x_1,x_2,\dots)\in H.$$

Observe that T is densely defined, $T=T^*$ and $N(T)=\{(x_1,0,0,\dots)\colon x_1\in\mathbb{C}\}$. Hence $C(T)=\Big\{(0,x_2,x_3,\dots)\colon \sum_{j=2}^{\infty}|jx_j|^2<\infty\Big\}$. We can show that $R(T)$ is closed (see [8, example 5.2] for details) and

$$T^\dagger(y_1,y_2,y_3,\dots,)=\big(0,\frac{y_2}{2},\frac{y_3}{3},\dots\big),\ (y_n)\in\ell^2.$$

It can be seen that $T=T^*$ and $D(T^2)=\{(x_n)\in H:\sum_{n=2}^{\infty}n^4|x_n|^4<\infty\}$. Take

$$K=\{(x_n)\in D(T^2):x_n\geq 0 \text{ for all } n=2,3,\dots\}.$$

It is easy to verify that $K^*=K$ and $T^\dagger TK\subseteq K$. Also $D=T^{\dagger^*}(K^*)=T^\dagger(K)$. Let $x,y\in D$. Then $x=T^\dagger u,\ y=T^\dagger v$ for some $u,v\in H$. Then Let $u=\sum_{n=1}^{\infty}\langle u,e_n\rangle e_n$ and $v=\sum_{n=1}^{\infty}\langle v,e_n\rangle e_n$. Then

$$\begin{aligned}\langle x,y\rangle &= \langle T^\dagger u,T^\dagger v\rangle \geq 0\\ &= \sum_{n=2}^{\infty}\frac{1}{n^2}\langle u,e_n\rangle\,\langle v,e_n\rangle\\ &\geq 0 \quad (\text{since } \langle u,e_n\rangle, \langle v,e_n\rangle \geq 0).\end{aligned}$$

Therefore D is acute. Hence by Theorem 3.4, $(T^*T)^\dagger$ is positive with respect to the cone K.

Example 4.3. Let $\mathcal{AC}[0,\pi]$ denote the space of all absolutely continuous functions on $[0,\pi]$. Let

$$\begin{aligned}H &:= \text{The real space } L^2[0,\pi] \text{ of real valued functions}\\ H' &:= \Big\{\phi\in\mathcal{AC}[0,\pi]:\phi'\in H\Big\},\\ H'' &:= \{\phi\in H':\phi'\in H'\}.\end{aligned}$$

Let $L:=\dfrac{d}{dt}$ with $D(L)=\{x\in H':\phi(0)=\phi(\pi)=0\}$.

It can be shown using the fundamental theorem of integral calculus that $L \in \mathcal{C}(H)$. Let $\phi_n = \sin(nt),\ n \in \mathbb{N}$. Then $\{\phi_n : n \in \mathbb{N}\}$ is an orthonormal basis for H and is contained in $D(L)$, hence L is densely defined. Also $C(L) = D(L)$. i.e., L is one-to-one. It can be shown that $R(L) = \{y \in H : \int\limits_0^\pi y(t)\,dt = 0\} = \text{span}\,\{1\}^\perp$. Hence in this case $D(L^\dagger) = H$. Let $\psi_n = \sqrt{\frac{2}{\pi}}\,\cos(nt),\ t \in [0,\pi],\ n \in \mathbb{N}$. Then $\{\psi_n : n \in \mathbb{N}\}$ is an orthonormal basis for $R(L)$.

We have, $L^*L = -\frac{d^2}{dt^2}$ with $D(L^*L) = \{\phi \in H'' : \phi(0) = 0 = \phi(\pi)\}$ [1, page 349]. By using the projection method (see [9, example 3.5]), we can show that

$$L^\dagger(y) = \sum_{n=1}^\infty \frac{1}{n}\langle y, \psi_n\rangle \phi_n. \tag{4.1}$$

Let $K = \{\phi \in D(L^*L) : \langle \phi, \phi_n\rangle \geq 0, \text{ for all } n \in \mathbb{N}\}$. Then K is a cone and $K^* = K$. We verify condition 1 of Theorem 3.4. First note that, by Equation 4.1, we have

$$L^{\dagger *}\phi = \sum_{n=1}^\infty \frac{1}{n}\langle \phi, \phi_n\rangle \psi_n, \text{ for all } \phi \in H. \tag{4.2}$$

Now, let $f \in K$. Then

$$\begin{aligned}(L^*L)^\dagger(f) &= L^\dagger(L^\dagger)^*(f)\\ &= \sum_{n=1}^\infty \frac{1}{n^2}\langle f, \phi_n\rangle \phi_n.\end{aligned}$$

Since $f \in K$, we have $\langle f, \phi_n\rangle \geq 0$ for all $n \in \mathbb{N}$ and so $\frac{1}{n^2}\langle f, \phi_n\rangle \geq 0$ for all $n \in \mathbb{N}$. This concludes that $(L^*L)^\dagger(K^*) \subseteq K$.

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