On the Linear Combination of Gamma Conditionally Gaussian Distributions with Application to Decode and Forward Cooperation

Sachin Kumar, A. Rathnakar and G.V.V.Sharma
EE Department, IIT Hyderabad, India 502205
Email: {ee10b033,ee12m1005,gadepall}@iith.ac.in

***Abstract*—In this paper, exact statistics for the linear combination of gamma conditionally Gaussian random variables (CGRVs) are obtained. In particular, the probability density function (PDF), the cumulative distribution function (CDF) and the moment generating function (MGF) are derived. Closed form expressions are obtained for both integer and noninteger parameters, using the Mellin-Barnes integral representation of the extended Fox-$\hat{H}$ function. The significance of these results is then explained by obtaining performance metrics for decode and forward (DF) cooperation in Nakagami-m fading.**

***Index Terms*—Cooperative diversity, Residue theorem, DF.**

I. Introduction

Conditionally Gaussian random variables (CGRVs) appear in the decision variables for decode and forward (DF) cooperation [1]–[3]. While they also figured in the derivation of performance metrics for various fading channels in additive white Gaussian noise (AWGN), they did not attract enough attention, since knowledge of their statistics was not really necessary. However, the difficulty in the performance analysis for cooperative systems led to interest in the statistics of this new distribution.

Conditionally Gaussian distributions (CGDs) were first defined in [1] and their statistics subsequently derived for obtaining the bit error rate (BER) for maximum-likelihood decode and forward (ML-DF) cooperation in Rayleigh fading. Similar results were obtained for Gamma-CGDs in [3]. Related work can also be found in [4] and [5].

In this paper, we obtain statistics for a linear combination of Gamma-CGDs that are independent, with arbitrary parameters (integer as well as noninteger). This is done by using a Mellin-Barnes integral representation of special funtions [6]–[8]. The usefulness of these results is then demonstrated by obtaining the BER for λ-MRC (Maximal Ratio Combining) DF cooperation in Nakagami-m fading channels.

II. Problem Definition

Definition II.1. *X is gamma CG with parameters $a, b > 0$ if $X \mid A \sim \mathcal{N}(aA, bA), A \sim \mathcal{G}(c, m)$ being Gamma distributed [9] with scale parameter $c > 0$ and order $m > 0$ with PDF [9]*

$$p_A(y) = \frac{c^m}{\Gamma(m)} x^{m-1} e^{-cy}, y, c > 0. \tag{1}$$

We wish to obtain the statistics of

$$Y = \sum_{i=1}^{N} \lambda_i X_i, \tag{2}$$

where $X_i \mid A_i \sim \mathcal{N}(a_i A_i, b_i A_i), A_i \sim \mathcal{G}(c_i, m_i)$ are independent and $\lambda_i \in \mathbf{R}$. We begin by listing the complete statistics of X, which are partially available in the literature [3], [5].

III. Statistics of Gamma CGD

Lemma III.1. *The PDF and MGF of X are given by*

$$p_X(x) = \frac{\alpha^m \beta^m e^{-\frac{(\beta-\alpha)x}{2}}}{\Gamma(m)\sqrt{\pi}} \left(\frac{|x|}{\alpha+\beta}\right)^{m-\frac{1}{2}} K_{m-\frac{1}{2}}\left(\frac{(\alpha+\beta)|x|}{2}\right) \tag{3}$$

$$M_X(s) = E\left[e^{-sX}\right] = \frac{(\alpha\beta)^m}{(\alpha-s)^m(\beta+s)^m} \text{ where} \tag{4}$$

$$\beta, \alpha = \frac{\sqrt{a^2+2bc} \pm a}{b} \tag{5}$$

where $K(\cdot)$ is the modified bessel function of the second kind.

Proof. See Appendix A. Note that the above expressions are valid for arbitrary m and the proof is straightforward. The following Lemmas gives the expressions for the CDF for integer and noninteger m separately.

Lemma III.2. *(noninteger m) The CDF of X is given by*

$$F_X(x) = \begin{cases} 1 + \left(\frac{2c}{b}\right)^m \hat{H}^{1,2}_{3,3}\left[e^x \middle| \begin{smallmatrix}\Upsilon^1 \\ \Upsilon^2\end{smallmatrix}\right] & x \geq 0 \\ \left(\frac{2c}{b}\right)^m \hat{H}^{1,2}_{3,3}\left[e^{-x} \middle| \begin{smallmatrix}\Upsilon^1 \\ \Upsilon^2\end{smallmatrix}\right] & x < 0, \end{cases} \tag{6}$$

where

$$\Upsilon^1 = \begin{cases} \{(1,1,1),(1-\beta,1,m),(1+\alpha,1,m)\} & x \geq 0 \\ \{(1,1,1),(1-\alpha,1,m),(1+\beta,1,m)\} & x < 0 \end{cases} \tag{7}$$

$$\Upsilon^2 = \begin{cases} \{(0,1,1),(\alpha,1,m),(-\beta,1,m)\} & x \geq 0 \\ \{(0,1,1),(\beta,1,m),(-\alpha,1,m)\} & x < 0 \end{cases} \tag{8}$$

and $\hat{H}$ is the Fox-$\hat{H}$ function, [7, (T.I.1)], [8],

Proof. See Appendix B.

Lemma III.3. *(integer m) The CDF of X in this case is*

$$F_X(x) =$$
$$\begin{cases} 1 + \frac{\beta^m e^{-\beta x}}{(m-1)!} \sum_{i=0}^{m-1} \binom{m-1}{i} x^{m-1-i} \left[\frac{1}{\beta^i} + \sum_{k=1}^{m} \frac{\alpha^{k-1}(k+i)!}{k!(\alpha+\beta)^{k+i}} \right] & x \geq 0 \\ \frac{\alpha^m e^{\alpha x}}{(m-1)!} \sum_{i=0}^{m-1} \binom{m-1}{i} (-x)^{m-1-i} \left[\frac{1}{\alpha^i} + \sum_{k=1}^{m} \frac{\beta^{k-1}(k+i)!}{k!(\alpha+\beta)^{k+i}} \right] & x < 0 \end{cases} \tag{9}$$

Proof. See Appendix C

IV. Statistics of the linear combination of Gamma CGD

Lemma IV.1. *The MGF of $Y = \sum_{i=1}^{N} \lambda_i X_i$, defined in (2) is given by*

$$\begin{aligned} M_Y(s) &= \prod_{i=1}^{N} \frac{(\alpha_i \beta_i)^{m_i}}{(\alpha_i - s)^{m_i} (\beta_i + s)^{m_i}} \\ \text{where } \beta_i, \alpha_i &= \frac{\sqrt{a_i^2 + 2 b_i c_i} \pm a_i}{\lambda_i b_i} \end{aligned} \tag{10}$$

Proof. (10) is trivially obtained from (5) by noting that X_i are independent.

Corollary IV.1. *The sum of i.i.d Gamma CGD variables is Gamma CGD*

In the following, we obtain the expressions for the CDF and PDF of Y for integer and noninteger m. Note that the MGF is the same for both cases.

A. Integer m

Theorem IV.2. *(integer m) The CDF of Y is*

$$F_Y(x) = \begin{cases} 1 - \sum_{i=1}^{N} \frac{1}{(m_i-1)!} \left[\frac{e^{-\beta_i x}}{\beta_i} \prod_{j=1}^{N} \frac{(\alpha_j \beta_j)^{m_j}}{(\alpha_j+\beta_i)^{m_j}} \right. \\ \left. \times \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{(\beta_j - \beta_i)^{m_j}} \right] \sum_{k=1}^{m_i-1} B_{m_i-1,k} \left(G^{(1)}(-\beta_i), \right. & x \geq 0 \\ \left. G^{(2)}(-\beta_i), \ldots, G^{(m_i-k)}(-\beta_i) \right) \\ \hline \sum_{i=1}^{N} \frac{1}{(m_i-1)!} \left[\frac{e^{\alpha_i x}}{\alpha_i} \prod_{i=1}^{N} \frac{(\alpha_i \beta_i)^{m_i}}{(\beta_i+\alpha_i)^{m_i}} \right. \\ \left. \times \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{(\alpha_j - \alpha_i)^{m_j}} \right] \sum_{k=1}^{m_i-1} B_{m_i-1,k} \left(H^{(1)}(-\alpha_i), \right. & x < 0 \\ \left. H^{(2)}(-\alpha_i), \ldots, H^{(m_i-k)}(-\alpha_i) \right) \end{cases} \tag{11}$$

where

$$G(s) = \ln \left[\frac{e^{sx}}{s} \prod_{j=1}^{N} \frac{\left(\alpha_j \beta_j\right)^{m_j}}{(\alpha_j - s)^{m_j}} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{(\beta_j + s)^{m_j}} \right] \tag{12}$$

$$H(s) = \ln \left[\frac{e^{-sx}}{s} \prod_{j=1}^{N} \frac{\left(\alpha_j \beta_j\right)^{m_j}}{(\beta_j - s)^{m_j}} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{(\alpha_j + s)^{m_j}} \right], \tag{13}$$

$G^{(k)}, H^{(k)}$ are their k th derivaties and $B_{(.,.)}(\cdot)$ is the Bell polynomial [6].

Corollary IV.3. *(integer m) The PDF of Y is*

$$f_Y(x) = \begin{cases} \sum_{i=1}^{N} \frac{1}{(m_i-1)!} \left[e^{-\beta_i x} \prod_{j=1}^{N} \frac{(\alpha_j \beta_j)^{m_j}}{(\alpha_j+\beta_i)^{m_j}} \right. \\ \left. \times \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{(\beta_j - \beta_i)^{m_j}} \right] \sum_{k=1}^{m_i-1} B_{m_i-1,k} \left(\tilde{G}^{(1)}(-\beta_i), \right. & x \geq 0 \\ \left. \tilde{G}^{(2)}(-\beta_i), \ldots, \tilde{G}^{(m_i-k)}(-\beta_i) \right) \\ \hline - \sum_{i=1}^{N} \frac{1}{(m_i-1)!} \left[e^{\alpha_i x} \prod_{i=1}^{N} \frac{(\alpha_i \beta_i)^{m_i}}{(\beta_i+\alpha_i)^{m_i}} \right. \\ \left. \times \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{(\alpha_j - \alpha_i)^{m_j}} \right] \sum_{k=1}^{m_i-1} B_{m_i-1,k} \left(\tilde{H}^{(1)}(-\alpha_i), \right. & x < 0 \\ \left. \tilde{H}^{(2)}(-\alpha_i), \ldots, \tilde{H}^{(m_i-k)}(-\alpha_i) \right) \end{cases} \tag{14}$$

where

$$\tilde{G}(s) = \ln \left[e^{sx} \prod_{j=1}^{N} \frac{\left(\alpha_j \beta_j\right)^{m_j}}{(\alpha_j - s)^{m_j}} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{(\beta_j + s)^{m_j}} \right] \tag{15}$$

$$\tilde{H}(s) = \ln \left[e^{-sx} \prod_{j=1}^{N} \frac{\left(\alpha_j \beta_j\right)^{m_j}}{(\beta_j - s)^{m_j}} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{(\alpha_j + s)^{m_j}} \right], \tag{16}$$

Proof. See Appendix D

B. Non Integer m

Theorem IV.4. *(noninteger m) The CDF of Y is*

$$F_Y(y) = \begin{cases} 1 + \prod_{i=1}^{N} \left(\frac{2c_i}{b_i \lambda_i^2} \right)^{m_i} \hat{H}_{2N+3,2N+3}^{N+1,N+2} \left[e^{y} \middle| \begin{matrix} \Upsilon^1 \\ \Upsilon^2 \end{matrix} \right] & y \geq 0 \\ \prod_{i=1}^{N} \left(\frac{2c_i}{b_i \lambda_i^2} \right)^{m_i} \hat{H}_{2N+3,2N+3}^{N+1,N+2} \left[e^{-y} \middle| \begin{matrix} \Upsilon^1 \\ \Upsilon^2 \end{matrix} \right] & y < 0, \end{cases} \tag{17}$$

$$\Upsilon^1 = \begin{cases} \{(1,1,1), (1-\beta_1, 1, m_1), \\ (1+\alpha_1, 1, m_1), (1-\beta_2, 1, m_2), (1+\alpha_2, 1, m_2) & y \geq 0 \\ \ldots (1-\beta_N, 1, m_N), (1+\alpha_N, 1, m_N)\} \\ \hline \{(1,1,1), (1-\alpha_1, 1, m), \\ (1+\beta_1, 1, m), (1-\alpha_2, 1, m_2), (1+\beta_2, 1, m_2) & y < 0 \\ \ldots (1-\alpha_N, 1, m_N), (1+\beta_N, 1, m_N)\} \end{cases} \tag{18}$$

$$\Upsilon^2 = \begin{cases} (0,1,1), (\alpha_1, 1, m_1), \} \\ (-\beta_1, 1, m_1), (\alpha_2, 1, m_2), (-\beta_2, 1, m_2) \\ \ldots (\alpha_N, 1, m_N), (-\beta_N, 1, m_N)\} & y \geq 0 \\ \hline \{(0,1,1), (\beta_1, 1, m_1), \\ (-\alpha_1, 1, m_1), (\beta_2, 1, m_1), (-\alpha_2, 1, m_2) \\ \ldots, (\beta_N, 1, m_N), (-\alpha_N, 1, m_N)\} & y < 0 \end{cases} \tag{19}$$

Proof. Using the approach in Appendix B, the above expression can be easily obtained after some algebra.

V. BER Analysis: λ-MRC with N Relays

Consider a λ-MRC cooperative system [10] wih N relay nodes between the source and destination. For BPSK modulation, the expression for the BER is given by

$$P_e = \sum_{\mathbf{x}} \prod_{r=1}^{N} \left(\varepsilon_r^{\frac{1-x_r}{2}} (1-\varepsilon_r)^{\frac{1+x_i}{2}} \right) \Pr\left(\sum_{i=0}^{N} \lambda_i X_i < 0 | x_0 = 1, \mathbf{x} \right) \tag{20}$$

where $X_i|h_i \sim \mathcal{N}\left(a_i h_i^2, b_i h_i^2\right)$ account for the parameters on the source (S) - destination (D) link and the relay (R) - destination link [3]. The 0 index is used to represent the source parameters with $\lambda_0 = 1$. Without loss of generality, the different variables involved in (20) are listed in Table I. From Theorems IV.2

h	Nakagmi-m fading coefficient
E	Transmit power at a node
a	$\frac{4Ex}{N_0}$
b	$\frac{8E}{N_0}$
c, m	Nakagami fading figure
x	Transmitted symbol at a node
ε	BER on the S-R link
N	Number of relays
N_0	Noise variance
γ	Average signal to noise ratio (SNR)
$\mathbf{x}$	Set of all N tuples of the relay symbols x

TABLE I
DESCRIPTION OF PARAMETERS IN (20)

and IV.4, (20) can be expressed in closed form for integer and noninteger fading parameters respectively. This is explained in the following.

A. Single Relay Performance

For $N = 1$, we use the subscripts 0 and 1 for the source and relay parameters. The expression in (20) can then be expressed as

$$\begin{aligned} P_e = \varepsilon_1 \Pr(Y < 0|x_0 = 1, x_1 = 1) \\ + (1 - \varepsilon_1)\Pr(Y < 0|x_0 = 1, x_1 = -1) \end{aligned} \tag{21}$$

$$\text{where } Y = X_0 + \lambda_1 X_1 \quad (\because \lambda_0 = 1) \tag{22}$$

(21) consists of two error probabilities obtained with a) correct ($x_1 = 1$) and b) incorrect ($x_1 = -1$) decision at the relay which can be more conveniently written as

$$P_e = \varepsilon_1 F_{Y|x_0=1,x_1=-1}(0) + (1 - \varepsilon_1) F_{Y|x_0=1,x_1=1}(0) \tag{23}$$

1) Correct Decision at the Relay: Noting that X_0 and X_1 are gamma CGD and using Lemma IV.4,

$$F_{Y|x_0=1,x_1=1}(0) = \left(\frac{m_0}{4\gamma_0}\right)^{m_0} \left(\frac{m_1}{4\gamma_1\lambda_1^2}\right)^{m_1} \hat{H}^{2,3}_{5,5}\left[1 \middle| \begin{matrix} \Upsilon^1 \\ \Upsilon^2 \end{matrix}\right] \tag{24}$$

where Υ^1, Υ^2 consist of tuples involving

$$\beta_0, \alpha_0 = \frac{1}{2}\left(\sqrt{1 + \frac{m_0}{\gamma_0}} \pm 1\right) \tag{25}$$

$$\beta_1, \alpha_1 = \frac{1}{2\lambda_1}\left(\sqrt{1 + \frac{m_1}{\gamma_1}} \pm 1\right) \tag{26}$$

2) Incorrect Decision at the Relay: Similarly,

$$F_{Y|x_0=1,x_1=-1}(0) = \left(\frac{m_0}{4\gamma_0}\right)^{m_0} \left(\frac{m_1}{4\gamma_1\lambda_1^2}\right)^{m_1} \hat{H}^{2,3}_{5,5}\left[1 \middle| \begin{matrix} \Upsilon^1 \\ \Upsilon^2 \end{matrix}\right] \tag{27}$$

where

$$\beta_0, \alpha_0 = \frac{1}{2}\left(\sqrt{1 + \frac{m_0}{\gamma_0}} \pm 1\right) \tag{28}$$

$$\beta_1, \alpha_1 = \frac{1}{2\lambda_1}\left(\sqrt{1 + \frac{m_1}{\gamma_1}} \mp 1\right) \tag{29}$$

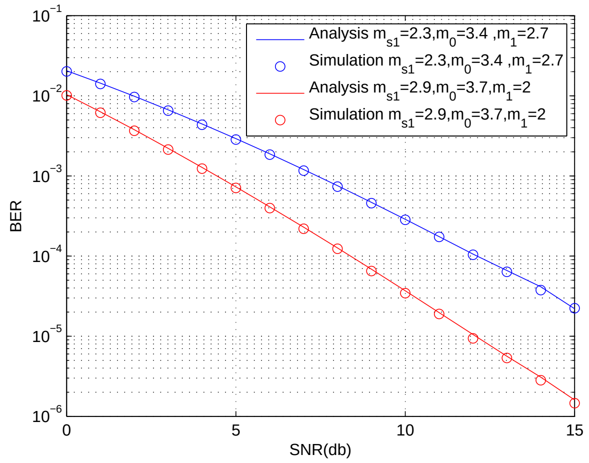

Fig. 1. Analysis and Simulation for Single Relay for $\lambda = 1$

Note that $x_1 = -1$ results in different values for β_1, α_1 above and distinguishes (27) from (24). Substituting (24) and (27) in (21) we obtain the final expression for BER. In Figure 1 the simulation and analytical results for various combinations of m across different links are provided for a single relay. m_{s1} is the Nakagami fading figure on the S-R link. The expressions in (24), (27) are used in (21) to evalute the exact BER. We have assumed $E_s = E_r$ for generating the results. The simulations perfectly follow the analysis, validating the expressions obtained for single relay.

VI. CONCLUSIONS

Exact statistics for the linear combination of Gamma CGDs have been obtained in this paper. A relatively new approach, using the Mellin-Barnes integral representation of the extended Fox-$\hat{H}$ function, was employed for this. The usefulness of the results was then demonstrated by obtaining the exact expression for the BER for a single relay λ-MRC cooperative system. While general closed form expressions for arbitrary values of m were obtained, computationally efficient expressions for integer m were also obtained separately. Exploiting the expressions for choosing appropriate λ values for cooperative diversity gain is a likely topic for further research.

REFERENCES

[1] G. V. V. Sharma, V. Ganwani, U. B. Desai, and S. N. Merchant, “Performance Analysis of Maximum Likelihood Decode and Forward Cooperative Systems in Rayleigh Fading,” *Proc. IEEE International Conference on Communications (ICC)*, June 2009.

[2] ——, “Performance Analysis of Maximum Likelihood Detection for Decode and Forward MIMO Relay Channels in Rayleigh Fading,” *IEEE Trans. Wireless Commun.*, vol. 9, no. 9, p. 28802889, September 2010.

[3] G. V. V. Sharma, U. B. Desai, and S. N. Merchant, “Conditionally Gaussian Distributions and their Application in the Performance of Maximum Likelihood Decode and Forward Cooperative Systems,” *IEEE Trans. Wireless Commun.*, vol. 10, no. 1, p. 2732, january 2011.

[4] R. K. Mallik, “Distribution of Inner Product of Two Complex Gaussian Vectors and its Application to MPSK Performance,” *Proc. IEEE International Conference on Communications (ICC)*, May 2008.

[5] Y. G. Kim and N. C. Beaulieu, "Exact Closed-Form Solutions for the BEP of Decode-and-Forward Cooperative Systems in Nakagami-m Fading Channels," *Communications, IEEE Transactions on*, vol. 59, no. 9, p. 23552361, september 2011.
[6] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, 1st ed. Cambridge University Press, 1999.
[7] I. S. Ansari, F. Yilmaz, M.-S. Alouini, and O. Kucur, "New Results on the Sum of Gamma Random Variates With Application to the Performance of Wireless Communication Systems over Nakagami-*m* Fading Channels." [Online]. Available: http://arxiv.org/abs/1202.2576
[8] A. K. Rathie, "A New Generalization of Generalized Hypergeometric Functions," *LE MATEMATICHE*, vol. LII, no. Fasc. II, pp. 297–310, 1997.
[9] W. Feller, *An introduction to Probability Theory and Its Applications, Vol II*, 2nd ed. John Wiley and Sons, 1970.
[10] A. Sendonaris, E. Erkip, and B. Aazhang, "User Cooperation Diversity, Part II: Implementation Aspects and Performance Analysis," *IEEE Trans. Commun.*, vol. 51, no. 11, p. 19391948, November 2003.
[11] G. V. V. Sharma, "Exact Error Analysis for Decode and Forward Cooperation with Maximal Ratio Combining," in *National Conference on Communications (NCC)*, January 2011.
[12] J. E. Marsden and M. J. Hoffman, *Basic Complex Analysis*, 3rd ed. W H Freeman and Company, 1999.
[13] C. F. D. Bruno, *Thorie des Formes Binaries*. Librairie Brero, Succ^r de P. Marietti, Turin, Italy, 1876.

Appendix A

The MGF of $X|A$ is [9]

$$M_{X|A}(s) = e^{asA+\frac{bs^2}{2}A} = e^{\left(-as+\frac{bs^2}{2}\right)A} \tag{30}$$

$$\text{and } M_A(s) = E\left[e^{-sA}\right] = \left(1+\frac{s}{c}\right)^{-m}. \tag{31}$$

Averaging (30) over A yields

$$M_Z(s) = \left(1+\frac{as-\frac{bs^2}{2}}{c}\right)^{-m} = \left(1+\frac{as}{c}-\frac{bs^2}{c}\right)^{-m} \tag{32}$$

which can be expressed as (5). The pdf of X is [5]

$$\begin{aligned}
p_X(x) &= \int_{-\infty}^{\infty} p_{X|A}\, p_A(z)\, dz \\
&= \frac{1}{\sqrt{2\pi b}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{x}}e^{-\frac{(x-az)^2}{2bx}}P_A(z)\, dz \\
&= \frac{c^m}{\Gamma(m)\sqrt{2\pi b}}\int_0^{\infty} z^{m-\frac{3}{2}}\exp\left\{-\frac{(x-az)^2}{2bz}-cz\right\}dz \\
&= \frac{c^m e^{\frac{ax}{b}}}{\Gamma(m)\sqrt{2\pi b}}\int_0^{\infty} z^{m-\frac{3}{2}}exp\left\{-\frac{|x|^2}{2bz}-\left(\frac{a^2}{2b}\right)z\right\}dz \\
&= \frac{2c^m e^{\frac{ax}{b}}}{\Gamma(m)\sqrt{2\pi b}}\left(\frac{|x|}{\sqrt{a^2+2bc}}\right)^{m-\frac{1}{2}} K_{m-\frac{1}{2}}\left(\frac{|x|}{b}\sqrt{a^2+2bc}\right)
\end{aligned}$$

where we have substituted for $p_A(z)$ from (1). After simplifying the above, we obtain (3).

Appendix B

The CDF of X can be expressed as[1]

$$\begin{aligned}
F_X(x) &= \mathscr{L}^{-1}\left\{\frac{M_X(s)}{s}\right\} \\
&= \begin{cases} 1+\frac{1}{2\pi j}\oint_C \frac{M_X(s)}{s}e^{sx}dy & x \ge 0 \\ -\frac{1}{2\pi j}\oint_C \frac{M_X(-s)}{s}e^{-sx}dy & x < 0 \end{cases}
\end{aligned} \tag{33}$$

[1] see [12] for mathematical details

where C is a suitable contour encompassing all poles of the above integrand(s) in the left half complex plane. Since

$$M_X(s) = \frac{(\alpha\beta)^m}{(\alpha-s)^m(\beta+s)^m}, \tag{34}$$

$$\frac{M_X(s)}{s} = (\alpha\beta)^m\frac{\Gamma(s)}{\Gamma(1+s)}\frac{\Gamma^m(\alpha-s)\,\Gamma^m(\beta+s)}{\Gamma^m(1+\alpha-s)\,\Gamma^m(1+\beta+s)} \tag{35}$$

$$\text{where } \beta,\alpha = \frac{\sqrt{a^2+2bc}\pm a}{b}. \tag{36}$$

Thus, the integrals in (33) fit into the Mellin-Barnes integral expression for the Fox $\hat{H}$ function, [7, (T.I.1)], [8]. Using [7, (T.I.1)], [8] in (33), we obtain (6).

Appendix C

From (33), the CDF of X for integer m can be expressed as

$$F_X(x) = \begin{cases} 1+\mathrm{Res}_{s=-\beta}\frac{M_X(s)}{s}e^{sx} & x \ge 0 \\ -\mathrm{Res}_{s=-\alpha}\frac{M_X(-s)}{s}e^{-sx} & x < 0 \end{cases} \tag{37}$$

$$= \begin{cases} 1+(\alpha\beta)^m\frac{1}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left[\frac{e^{sx}}{s(\alpha-s)^m}\right]_{s=-\beta} & x \ge 0 \\ (\alpha\beta)^m\frac{1}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left[\frac{e^{-sx}}{s(\beta-s)^m}\right]_{s=-\alpha} & x < 0 \end{cases} \tag{38}$$

Using Leibniz rule for differentiation,

$$\begin{aligned}
&\frac{d^{m-1}}{ds^{m-1}}\left[\frac{e^{sx}}{s(\alpha-s)^m}\right] \\
&\qquad= \sum_{i=0}^{m-1}\binom{m-1}{i}\frac{d^{m-1-i}}{ds^{m-1-i}}\left[e^{sx}\right]\frac{d^i}{ds^i}\left[\frac{1}{s(\alpha-s)^m}\right]
\end{aligned} \tag{39}$$

$$\text{Since, } \frac{1}{s(\alpha-s)^m} = \frac{1}{\alpha^m}\left[\frac{1}{s}+\sum_{k=1}^{m}\frac{\alpha^{k-1}}{(\alpha-s)^k}\right], \tag{40}$$

$$\frac{d^i}{ds^i}\left[\frac{1}{s(\alpha-s)^m}\right] = \frac{1}{\alpha^m}\left[\frac{(-1)^i}{s^i}+\sum_{k=1}^{m}\frac{\alpha^{k-1}(k+i)!}{k!(\alpha-s)^{k+i}}\right] \tag{41}$$

Substituting the above in (39),

$$\begin{aligned}
&\frac{d^{m-1}}{ds^{m-1}}\left[\frac{e^{sx}}{s(\alpha-s)^m}\right] \\
&\quad= \frac{e^{sx}}{\alpha^m}\sum_{i=0}^{m-1}\binom{m-1}{i}x^{m-1-i}\left[\frac{(-1)^i}{s^i}+\sum_{k=1}^{m}\frac{\alpha^{k-1}(k+i)!}{k!(\alpha-s)^{k+i}}\right]
\end{aligned} \tag{42}$$

Similarly,

$$\begin{aligned}
&\frac{d^{m-1}}{ds^{m-1}}\left[\frac{e^{-sx}}{s(\beta-s)^m}\right] \\
&= \frac{e^{-sx}}{\beta^m}\sum_{i=0}^{m-1}\binom{m-1}{i}(-x)^{m-1-i}\left[\frac{(-1)^i}{s^i}+\sum_{k=1}^{m}\frac{\beta^{k-1}(k+i)!}{k!(\beta-s)^{k+i}}\right]
\end{aligned} \tag{43}$$

Substituting from (42) and (43) in (38), we obtain (9) after simplification.

Appendix D

From (33), the CDF of Y can be expressed as,

$$F_Y(x) = \begin{cases} 1 + \sum_{i=1}^{N} \mathrm{Res}_{s=-\beta_i} \frac{M_Y(s)}{s} e^{sx} & x \geq 0 \\ -\sum_{i=1}^{N} \mathrm{Res}_{s=-\alpha_i} \frac{M_Y(-s)}{s} e^{-sx} & x < 0 \end{cases} \tag{44}$$

$$= \begin{cases} 1 + \sum_{i=1}^{N} \frac{1}{(m_i-1)!} \frac{d^{m_i-1}}{ds^{m_i-1}} \times \left[\frac{e^{sx}}{s} \prod_{j=1}^{N} \frac{\left(\alpha_j\beta_j\right)^{m_j}}{(\alpha_j-s)^{m_j}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{(\beta_j+s)^{m_j}} \right]_{s=-\beta_i} & x \geq 0 \\ -\sum_{i=1}^{N} \frac{1}{(m_i-1)!} \frac{d^{m_i-1}}{ds^{m_i-1}} \times \left[\frac{e^{-sx}}{s} \prod_{i=1}^{N} \frac{(\alpha_i\beta_i)^{m_i}}{(\beta_i-s)^{m_i}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{(\alpha_j+s)^{m_j}} \right]_{s=-\alpha_i} & x < 0 \end{cases}$$

$$\text{Letting } G(s) = \ln \left[\frac{e^{sx}}{s} \prod_{j=1}^{N} \frac{\left(\alpha_j\beta_j\right)^{m_j}}{(\alpha_j - s)^{m_j}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{(\beta_j + s)^{m_j}} \right] \tag{45}$$

$$= sx - \ln s + \sum_{j=1}^{N} m_j \left[\ln\left(\alpha_j\beta_j\right) - (\alpha_j - s) \right]$$

$$- \sum_{\substack{j=1 \\ j\neq i}}^{N} m_j \ln(\beta_j + s), \tag{46}$$

upon successive differentiation, we obtain

$$G^{(1)}(s) = x - \frac{1}{s} - \sum_{j=1}^{N} \frac{m_j}{(\alpha_j - s)} - \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{(\beta_j + s)} \tag{47}$$

$$G^{(2)}(s) = \frac{1}{s^2} - \sum_{j=1}^{N} \frac{m_j}{(\alpha_j - s)^2} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{(\beta_j + s)^2} \tag{48}$$

$$G^{(k)}(s) = (k-1)! \left[\frac{(-1)^k}{s^k} - \sum_{j=1}^{N} \frac{m_j}{(\alpha_j - s)^k} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(-1)^k m_j}{(\beta_j + s)^k} \right]$$

Similarly, letting

$$H(s) = \ln \left[\frac{e^{-sx}}{s} \prod_{j=1}^{N} \frac{\left(\alpha_j\beta_j\right)^{m_j}}{(\beta_j - s)^{m_j}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{(\alpha_j + s)^{m_j}} \right] \tag{49}$$

$$= -sx - \ln s + \sum_{j=1}^{N} m_j \left[\ln\left(\alpha_j\beta_j\right) - (\beta_j - s) \right]$$

$$- \sum_{\substack{j=1 \\ j\neq i}}^{N} m_j \ln(\alpha_j + s), \tag{50}$$

upon successive differentiation, we obtain

$$H^{(1)}(s) = -x - \frac{1}{s} - \sum_{j=1}^{N} \frac{m_j}{(\beta_j - s)} - \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{(\alpha_j + s)} \tag{51}$$

$$H^{(2)}(s) = \frac{1}{s^2} - \sum_{j=1}^{N} \frac{m_j}{(\beta_j - s)^2} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{(\alpha_j + s)^2} \tag{52}$$

$$H^{(k)}(s) = (k-1)! \left[\frac{(-1)^k}{s^k} - \sum_{j=1}^{N} \frac{m_j}{(\beta_j - s)^k} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(-1)^k m_j}{(\alpha_j + s)^k} \right]$$

Using the Fàa Di Bruno formula [13] in (44) and the above derivatives, we obtain (11). The PDF of Y can be expressed in terms of the MGF as

$$f_X(x) = \begin{cases} \frac{1}{2\pi j} \oint_C M_Y(s) e^{sx} dx & x \geq 0 \\ -\frac{1}{2\pi j} \oint_C M_Y(-s) e^{-sx} dx & x < 0 \end{cases} \tag{53}$$

$$= \begin{cases} \sum_{i=1}^{N} \frac{1}{(m_i-1)!} \frac{d^{m_i-1}}{ds^{m_i-1}} \times \left[e^{sx} \prod_{j=1}^{N} \frac{\left(\alpha_j\beta_j\right)^{m_j}}{(\alpha_j-s)^{m_j}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{(\beta_j+s)^{m_j}} \right]_{s=-\beta_i} & x \geq 0 \\ -\sum_{i=1}^{N} \frac{1}{(m_i-1)!} \frac{d^{m_i-1}}{ds^{m_i-1}} \times \left[e^{-sx} \prod_{i=1}^{N} \frac{(\alpha_i\beta_i)^{m_i}}{(\beta_i-s)^{m_i}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{(\alpha_j+s)^{m_j}} \right]_{s=-\alpha_i} & x < 0 \end{cases}$$

Letting

$$\tilde{G}(s) = \ln \left[e^{sx} \prod_{j=1}^{N} \frac{\left(\alpha_j\beta_j\right)^{m_j}}{(\alpha_j - s)^{m_j}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{(\beta_j + s)^{m_j}} \right] \tag{54}$$

$$= sx + \sum_{j=1}^{N} m_j \left[\ln\left(\alpha_j\beta_j\right) - (\alpha_j - s) \right] - \sum_{\substack{j=1 \\ j\neq i}}^{N} m_j \ln(\beta_j + s),$$

upon successive differentiation, we obtain

$$\tilde{G}^{(1)}(s) = x - \sum_{j=1}^{N} \frac{m_j}{(\alpha_j - s)} - \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{(\beta_j + s)} \tag{55}$$

$$\tilde{G}^{(2)}(s) = -\sum_{j=1}^{N} \frac{m_j}{(\alpha_j - s)^2} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{(\beta_j + s)^2} \tag{56}$$

$$\tilde{G}^{(k)}(s) = (k-1)! \left[-\sum_{j=1}^{N} \frac{m_j}{(\alpha_j - s)^k} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(-1)^k m_j}{(\beta_j + s)^k} \right] \tag{57}$$

Similarly, letting

$$\tilde{H}(s) = \ln \left[e^{-sx} \prod_{j=1}^{N} \frac{\left(\alpha_j\beta_j\right)^{m_j}}{(\beta_j - s)^{m_j}} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{(\alpha_j + s)^{m_j}} \right] \tag{58}$$

$$= -sx + \sum_{j=1}^{N} m_j \left[\ln\left(\alpha_j\beta_j\right) - (\beta_j - s) \right] - \sum_{\substack{j=1 \\ j\neq i}}^{N} m_j \ln(\alpha_j + s),$$

upon successive differentiation, we obtain

$$\tilde{H}^{(1)}(s) = -x - \sum_{j=1}^{N} \frac{m_j}{(\beta_j - s)} - \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{(\alpha_j + s)} \tag{59}$$

$$\tilde{H}^{(2)}(s) = -\sum_{j=1}^{N} \frac{m_j}{(\beta_j - s)^2} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{m_j}{(\alpha_j + s)^2} \tag{60}$$

$$\tilde{H}^{(k)}(s) = (k-1)! \left[-\sum_{j=1}^{N} \frac{m_j}{(\beta_j - s)^k} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{(-1)^k m_j}{(\alpha_j + s)^k} \right] \tag{61}$$

Using the Fàa Di Bruno formula [13] in (53), we obtain (14).