On the existence of equivalence class of RIP-compliant matrices

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***Abstract*—In Compressed Sensing (CS), the matrices that satisfy the Restricted Isometry Property (RIP) play an important role. But it is known that the RIP properties of a matrix Φ and its 'weighted matrix' $G\Phi$ (G being a non-singular matrix) vary drastically in terms of RIP constant. In this paper, we consider the opposite question: Given a matrix Φ, can we find a non-singular matrix G such that $G\Phi$ has compliance with RIP? We show that, under some conditions, a class of non-singular matrices (G) exists such that $G\Phi$ has RIP-compliance with better RIP constant. We also provide a relationship between the Unique Representation Property (URP) and Restricted Isometry Property (RIP), and a direct relationship between RIP and sparsest solution of a linear system of equations.**

Key words: Compressed Sensing, RIP, l_1-minimization, non-RIP.

I. Introduction

Recent developments at the intersection of algebra and optimization theory, by the name of Compressed Sensing (CS), aim at providing sparse descriptions to linear systems. These developments are found to have tremendous potential for several applications [2],[14]. Sparse representations of a function are a powerful analytic tool in many application areas such as image/signal processing and numerical computation [15] , to name a few. The need for the sparse representation arises from the fact that several real life applications demand the representation of data in terms of as few basis (frame) type elements as possible. The elements or the columns of the associated matrix Φ are called atoms and the matrix so generated by them is called the dictionary. The developments of CS Theory depend typically on sparsity and incoherence [15]. Sparsity expresses the idea that the information rate of a continuous time data may be much smaller than suggested by its bandwidth, or that a discrete-time data depends on a number of degrees of freedom which is comparably much smaller than its (finite) length. On the other hand, incoherence extends the duality between the time and frequency contents of data.

The matrices that satisfy the Restricted Isometry Property (RIP) provide sparse representation to certain classes of data. Though a sufficient condition, the compliance of RIP has become important concept in the field of CS Theory. Surprisingly, it has been shown [3] that an RIP compliant matrix Φ results in $G\Phi$ (for some invertible matrix G) in such a way that RIP constant of $G\Phi$ is bad compared to that of Φ. Motivated by this observation, in the present work, we attempt to provide existence of a class of non-singular matrices G such that $G\Phi$ has better RIP constant.

The main contributions of the present work are summarized below:

- characterization of the class of RIP matrices
- establishment of relationship between Unique Representation Property (URP) and Restricted Isometry property (RIP)
- improvement of RIP constant

The paper is organized in several sections. In section 2, we provide basics of CS theory. In sections 3 and 4, we provide motivation and equivalence class of RIP matrices respectively. In section 5, we discuss on the improvement of RIP constant. While in sections 6 and 7, we establish the relationship between URP and RIP, and a comparison with the result based on coherence respectively. In the last two sections, we discuss our future work and conclusion respectively.

II. Basics of Compressed Sensing

The objective of Compressed Sensing (CS) is to recover $x \in \mathbb{R}^M$ from a few of its linear measurements $y \in \mathbb{R}^m$ through a stable and efficient reconstruction process via the concept of sparsity. From the measurement vector y and the sensing mechanism, one gets a system $y = \Phi x$, where Φ is an $m \times M$ $(m < M)$ matrix. Now given the pair (y, Φ), the problem of recovering x can be formulated as finding the sparsest solution (solution containing most number of zero entries) of linear system of equations $y = \Phi x$. Sparsity is measured by $\|\cdot\|_0$ norm and $\|x\|_0 := |\{j \in \{1, 2, \dots, M\} : x_j \neq 0\}|$. Now finding the sparsest solution can be formulated as the following minimization problem, generally denoted as a P_0 problem:

$$P_0 : \min_x \|x\|_0 \text{ subject to } \quad \Phi x = y.$$

This P_0 problem is a combinatorial minimization problem and is known to be NP-hard [2]. One may use greedy methods and convex relaxation of P_0 problem to recover $k-$sparse signal x ($\|x\|_0 \leq k$). The convex relaxation of P_0 problem can be posed as P_1 problem [4] [6], which is defined as follows:

$$P_1 : \min_x \|x\|_1 \text{ subject to } \quad \Phi x = y.$$

The Orthogonal Matching Pursuit (OMP) algorithm and and l_1-minimization (also called basis pursuit) are two widely studied CS reconstruction algorithms [7].

Candes et. al. [5] have introduced the following isometry condition on matrices Φ and have established its important role in CS. An $m \times M$ matrix Φ is said to satisfy the Restricted Isometry Property (RIP) of order k with constant δ_k $(0 < \delta_k < 1)$ if for all $k-$sparse vectors $x \in \mathbb{R}^M$, we have

$$(1-\delta_k)\left\|x\right\|_2^2 \leq \left\|\Phi x\right\|_2^2 \leq (1+\delta_k)\left\|x\right\|_2^2. \qquad (1)$$

Equivalently, for all $k-$sparse vectors $x \in \mathbb{R}^M$ with $\left\|x\right\|_2 = 1$, one may rewrite (1) as

$$(1-\delta_k) \leq \left\|\Phi x\right\|_2^2 \leq (1+\delta_k).$$

The smaller the value of δ_k is, better the RIP is for that k value. Roughly speaking, RIP measures the "overall conditioning" of the set of $m \times k$ submatrices of Φ. The following theorem [6], establishes the equivalence between P_0 and P_1 problems:

Theorem 2.1: Suppose an $m \times M$ matrix Φ has the $(2k, \delta)$ RIP for some $\delta < \sqrt{2}-1$, then P_0 and P_1 have same $k-$sparse solution if P_0 has $k-$sparse solution.

The mutual-coherence $\mu(\Phi)$ of a given matrix Φ is the largest absolute normalized inner product between different columns of Φ, that is, $\mu(\Phi) = \max_{1\leq\, i,j \leq\, M,\, i\neq j} \frac{|\,\Phi_i^T \phi_j|}{\|\Phi_i\|_2 \|\Phi_j\|_2}$. Here, Φ_k stands for the k-th column in Φ. The following proposition [2] relates the RIP constant δ_k and μ.

Proposition 2.2: Suppose that $\Phi_1, \ldots, \Phi_M$ are the unit norm columns of the matrix Φ and have coherence μ. Then Φ satisfies RIP of order k with constant $\delta_k = (k-1)\mu$.

One of the important problems in CS theory deals with constructing CS matrices that satisfy the RIP for the largest possible range of k. It is known that the widest possible range of k is of the order $\frac{m}{\log(\frac{M}{m})}$ [1], [10], [11]. However the only known matrices that satisfy the RIP for this range are based on random constructions[1].

There are some attempts towards constructing the deterministic CS matrices [10], [12], [13], [14]. For all these methods, $k = O(\sqrt{m})$. The only known explicit matrix which surpasses this square root bottleneck bound has been constructed by Bourgain et. al. in [2].

III. Motivation for present work

The Null Space Property (NSP) [16] is another property that ensures the unique sparse recovery from P_1 problem.

Definition 3.1: A matrix $\Phi \in \mathbb{R}^{m\times M}$ is said to satisfy the Null Space Property (NSP) relative to set $S \subset \{1, 2, \ldots M\}$ if

$$\|v_S\|_1 < \|v_{S^C}\|_1, \;\; \forall\, 0 \neq v \in \text{Null}(\Phi). \qquad (2)$$

It is said to satisfy NSP of order s if it satisfies the NSP relative to any subset $S \subset \{1, 2, \ldots M\}$ with card$(S) \leq s$. The following theorem [16] states the importance of NSP.

Theorem 3.2: Given a matrix $\Phi \in \mathbb{R}^{m\times M}$, every $s-$sparse vector $x \in \mathbb{R}^M$ is the unique solution of P_1 problem with $y = \Phi x$ if and only if Φ satisfies the null space property of order s.

We know that the linear systems of equations $\Phi x = y$ and $G\Phi x = Gy$, for a non-singular matrix G, have the same set of solutions. Now as pre-multiplication with a non-singular matrix G leaves the null space unchanged, the NSP of Φ and $G\Phi$ remains same. But the RIP properties of Φ and $G\Phi$ may vastly differ from each other. In [3], Yin Zhang has demonstrated this fact showing that one can easily choose G to make the RIP of $G\Phi$ arbitrarily bad, no matter how good the RIP of Φ is. To see this, let us define,

$$\gamma_k(\Phi) = \frac{\lambda^k_{\text{max}}}{\lambda^k_{\text{min}}}, \lambda^k_{\text{max}} = \max\frac{\|\Phi x\|^2}{\|x\|^2}, \lambda^k_{\text{min}} = \min\frac{\|\Phi x\|^2}{\|x\|^2}. \qquad (3)$$

In (4), max and min are taken over all $k-$sparse signals.

Now equating $(1+\delta) = \lambda^k_{\text{max}}$ and $(1-\delta) = \lambda^k_{\text{min}}$, we obtain an explicit expression for the $k-$th RIP parameter of Φ as

$$\delta_k(\Phi) = \frac{\gamma_k(\Phi) - 1}{\gamma_k(\Phi) + 1}. \qquad (4)$$

Without loss of generality, let $A = [A_1 \;\; A_2]$ where $A_1 \in \mathbb{R}^{m\times m}$ is non-singular. Set $G = BA_1^{-1}$ for some non-singular $B \in \mathbb{R}^{m\times m}$. Then $GA = [GA_1 \;\; GA_2] = [B \;\; GA_2]$. Let $B_1 \in \mathbb{R}^{m\times k}$ consist of the first k columns of B and $\kappa(B_1^T B_1)$ be the condition number of $B_1^T B_1$ that is, $\kappa(B_1^T B_1) = \frac{\lambda_1}{\lambda_k}$ where λ_1 and λ_k are respectively the maximum and minimum eigenvalues of $B_1^T B_1$. Then $\kappa(B_1^T B_1) \leq \gamma_k(G\Phi)$ and

$$\delta_k(G\Phi) \geq \frac{\kappa(B_1^T B_1) - 1}{\kappa(B_1^T B_1) + 1}. \qquad (5)$$

Clearly, we can choose B_1 so that $\kappa(B_1^T B_1)$ is arbitrarily large and $\delta_k(G\Phi)$ is arbitrarily close to one.

For example, for any $k \leq \frac{m}{2}$, if we choose, $\kappa(B_1^T B_1) \geq 3$ for $B_1 \in \mathbb{R}^{m\times 2k}$, then it follows from (4) that $\delta_{2k}(G\Phi) \geq 0.5 > \sqrt{2} - 1$.

Our interest, however, is the converse of this point: Starting with a matrix Φ, can we find the existence of non-singular G such that $G\Phi$ satisfies RIP with better RIP constant?

IV. Equivalence class of RIP matrices:

The spark of a matrix Φ is the smallest number k' such that there exists a set of k' columns in Φ which are linearly dependent [8]. If spark of Φ is k', then any $(k'-1)$ number of columns of Φ are linearly independent. For a given matrix Φ, let us define the equivalence class of Φ as $\{G\Phi : \text{G is non-singular matrix}\}$, which is denoted as E_Φ.

Let $\Phi_{m\times M}$ be a matrix such that any k $(1 \leq k < k')$ columns of Φ are linearly independent. So there exist $\binom{M}{k}$ submatrices of size $m \times k$, consisting of k number of columns of Φ. Let us denote this collection of sub-matrices by L_k. Let us define C_k to be the minimum among the smallest singular values of the sub-matrices in L_k and D_k to be maximum among the largest singular values of the sub-matrices in L_k.

Now let us consider S_k to be $\{x \in \mathbb{R}^M : \|x\|_0 = k, \|x\|_2 = 1\}$. From the definitions of C_k and D_k, the following inequality holds:

$$C_k^2 \leq \|\Phi x\|^2 \leq D_k^2 \;\; \forall x \in S_k. \qquad (6)$$

For a non-singular matrix G of size $m \times m$, the following inequality holds:

$$\sigma_{min}^2(G)C_k^2 \leq \|G\Phi x\|^2 \leq \sigma_{max}^2(G)D_k^2 \ \ \forall x \in S_k, \quad (7)$$

where $\sigma_{min}(G)$ and $\sigma_{max}(G)$ are the smallest and largest eigen values of G respectively. In order for $G\Phi$ to satisfy RIP, we require the following to hold true:

$$\begin{aligned} &[\sigma_{min}(G)C_k]^2 \geq 1 - \delta_k \text{ and} \\ &[\sigma_{max}(G)D_k]^2 \leq 1 + \delta_k, \text{ for some } 0 < \delta_k < 1, \end{aligned} \quad (8)$$

which may be rewritten as

$$\frac{1-\delta_k}{C_k^2} \leq \sigma_{min}^2(G) < \sigma_{max}^2(G) \leq \frac{1+\delta_k}{D_k^2}. \quad (9)$$

From the fore-going inequalities, it is clear that if C_k and D_k are known and if (9) holds, then $G\Phi$ satisfies RIP of order k with RIP constant δ_k, where $\delta_k \in \left(\frac{D_k^2 - C_k^2}{D_k^2 + C_k^2}, 1\right)$.

In fact the non-singular matrix G may be defined as follows:

$$G = U \ . \ diag\left[\frac{\sqrt{(1-\delta_k)}}{C_k}, \lambda_1, \ldots, \lambda_{m-2}, \frac{\sqrt{(1+\delta_k)}}{D_k}\right] \ . \ V^T, \quad (10)$$

where $diag\left[\frac{\sqrt{(1-\delta_k)}}{C_k}, \lambda_1, \ldots, \lambda_{m-2}, \frac{\sqrt{(1+\delta_k)}}{D_k}\right]$ is a diagonal matrices of size $m \times m$ with diagonal entries $\frac{\sqrt{(1-\delta_k)}}{C_k}, \lambda_1, \ldots, \lambda_{m-2}, \frac{\sqrt{(1+\delta_k)}}{D_k}$ with the relation $\frac{\sqrt{(1-\delta_k)}}{C_k} \leq \lambda_1 \leq \ldots \leq \lambda_{m-2} \leq \frac{\sqrt{(1+\delta_k)}}{D_k}$, and U, V are any unitary matrices of size $m \times m$. The following theorem summarizes the ideas discussed in the preceding paragraphs.

Theorem 4.1: Suppose Φ is an $m \times M$ matrix such that all of its k-column subsets are linearly independent. Let $0 < C_k \leq D_k < \infty$ be such that $C_k^2 \leq \|\Phi x\|^2 \leq D_k^2$, $\forall \ x \in S_k$, then there exists a class of non-singular matrices (G) of the form as in (10) such that the matrix $G\Phi$ satisfies RIP of order k with RIP constant δ_k, where $\delta_k \in \left(\frac{D_k^2 - C_k^2}{D_k^2 + C_k^2}, 1\right)$.

Remark 4.2: In view of (6), if one considers rescaling Φ by a factor $\frac{1}{D_k^2}$, then the RIP constant becomes $\left(\frac{D_k^2 - C_k^2}{D_k^2}\right)$, which is greater than $\left(\frac{D_k^2 - C_k^2}{D_k^2 + C_k^2}\right)$.

Remark 4.3: An example of Φ, wherein all k-column subsets are linearly independent, is Vandermonde matrix. Consequently, a class of RIP compliant matrices can be generated from the Vandermonde matrix.

V. Improvement on RIP Constant (δ_k):

In this section, we discuss the issue of improvement upon the RIP constant (δ_k). Let us assume that the following inequality holds true on C_k and D_k for some $\delta'_k \in (0, 1)$

$$(1 - \delta'_k) \leq C_k^2 \leq \|\Phi x\|^2 \leq D_k^2 \leq (1 + \delta'_k) \ \forall x \in S_k, \quad (11)$$

which means that Φ satisfies RIP of order k with RIP constant δ'_k. It is also true that

$$1 - C_k^2 \leq \delta'_k, \delta'_k \geq D_k^2 - 1 \quad (12)$$

That means $\delta'_k \geq \max\{1 - C_k^2, D_k^2 - 1\}$. Now question is by pre-multiplying non-singular matrix G (the structure of G is given in (11)) with Φ can we improve on RIP constant? If we take G to be unitary matrix then it is clear that there is no change in terms of RIP constant. If we wish to show improvement of RIP constant then we have to show that $\delta_k < \delta'_k$ (δ_k is obtained from (10)). Let us investigate under what conditions the RIP constant can be improved. Two cases can arise.

Case-1: $1 - C_k^2 < D_k^2 - 1$, that is, $D_k^2 + C_k^2 > 2$, Now it is enough to show under the condition $D_k^2 + C_k^2 > 2$ that the following inequality holds:

$$\left(\frac{D_k^2 - C_k^2}{D_k^2 + C_k^2}\right) < (D_k^2 - 1). \quad (13)$$

It can be checked that the above inequality follows easily.

Case-2: $1 - C_k^2 > D_k^2 - 1$, that is, $D_k^2 + C_k^2 < 2$, Now it is enough to show under the condition $D_k^2 + C_k^2 < 2$ that the following inequality holds true:

$$\left(\frac{D_k^2 - C_k^2}{D_k^2 + C_k^2}\right) < (1 - C_k^2). \quad (14)$$

This inequality can also be verified easily. The above argument shows that RIP constant $\delta_k (< \delta'_k)$ is improved after pre-multiplication of G with Φ.

The theorem summarizes the above arguments

Theorem 5.1: Suppose there exist numbers $\delta'_k \in (0, 1)$, C_k and D_k such that $\Phi \in \mathbb{R}^{m \times M}$ satisfies the following inequality

$$(1 - \delta'_k) \leq C_k^2 \leq \|\Phi x\|^2 \leq D_k^2 \leq (1 + \delta'_k) \ \forall \ x \in S_k, \quad (15)$$

that is, Φ satisfies RIP of order k with RIP constant δ'_k. Then there exists a non-singular G such that $G\Phi$ satisfies RIP of order k with RIP constant $\delta_k (< \delta'_k)$.

A. Analysis of equivalence classes

In this section, we analyze further the class of matrices that can be used to generate RIP compliant matrices. Define two sets of matrices H_1 and H_2 as follows:

- $H_1 = \{\Phi \in \mathbb{R}^{m \times M} :$ $G\Phi$ is RIP of order k for some non-singular $G\}$
- $H_2 = \{\Phi \in \mathbb{R}^{m \times M} :$ k-column subsets of Φ are linearly independent$\}$.

Let $\Phi \in H_2$, then by theorem 4.1, there exists a non-singular matrix G such that $G\Phi$ satisfies RIP of order k. Consequently, $H_2 \subseteq H_1$. On the other-hand, for $\Phi \in H_1$, there exists a non-singular matrix G such that $G\Phi$ satisfies RIP of order k, which implies that any k number of columns of $G\Phi$ are linearly independent. Since G is non-singular, any k number of columns of Φ are linearly independent. Hence, $\Phi \in H_2$. Therefore $H_1 \subseteq H_2$, and we conclude that $H_1 = H_2$.

Remark 5.2: According to the Theorem 2.1 we ideally want $\delta_{2k} < \sqrt{2} - 1$, which implies that C_k and D_k have to satisfy $\frac{(D_{2k}^2 - C_{2k}^2)}{(D_{2k}^2 + C_{2k}^2)} < \sqrt{2} - 1$, or equivalently $(\sqrt{2} - 1)D_{2k}^2 < C_{2k}^2$.

The following theorem reiterates the equivalence between P_0 and P_1 problems in terms of C_k and D_k.

Theorem 5.3: Suppose Φ is an $m \times M$ matrix, whose any $2k$ column subset is linearly independent, such that $C_{2k} > 0$ and $D_{2k} > 0$ with the relation $(\sqrt{2}-1)D_{2k}^2 < C_{2k}^2$ and $C_{2k}^2 \leq \|\Phi x\|^2 \leq D_{2k}^2$, $\forall\ x \in S_{2k}$, then there exists a non-singular matrix G such that $G\Phi$ satisfies RIP of order $2k$ with RIP constant $\delta_{2k}(< \sqrt{2}-1)$.

VI. Relationship between URP and RIP:

A property termed the unique representation property (URP) [8], [9] of Φ is the key to understanding the behavior of the solution obtained from (P_0) problem.

Definition 6.1: A full rank matrix Φ is said to have URP of order k if any k columns of Φ are linearly independent.

It has been shown in [8], [9] that the URP of order $2k$ is both a necessary and a sufficient condition for uniquely recovering the sparest solution from P_0 problem, which is summarized as follows:

Theorem 6.2: An arbitrary k-sparse signal x can be uniquely recovered from $y = \Phi x$ as a solution to P_0 problem if and only if Φ satisfies the URP of order $2k$.

Let Φ satisfy URP of order k, that means any k column subsets of Φ are linearly independent. Now $\Phi \in H_2$ implies that $\Phi \in H_1$, which means there exists a non-singular matrix G such that $G\Phi \in E_\Phi$ satisfies RIP of order k. This interesting fact gives rise to following theorem:

Theorem 6.3: A full rank matrix satisfies URP of order k if and only if there exists a non-singular matrix G such that $G\Phi \in E_\Phi$ satisfies RIP of order k.

Above theorem gives a relation between URP and RIP of a matrix. For a given matrix Φ there exists $\Psi \in E_\Phi$ such that Ψ satisfies RIP up to order equal to spark(Φ). The next theorem establishes that RIP is necessary and sufficient condition for uniquely recovering the sparest solution from P_0 problem in the following sense.

Theorem 6.4: An arbitrary k-sparse signal x can be uniquely recovered from $y = \Phi x$ as a solution to P_0 problem if and only if there exists a non-singular matrix G such that $G\Phi \in E_\Phi$ satisfies RIP of order $2k$.

Proof: From theorem 6.2, it is clear that Φ satisfies URP of order $2k$. Then by theorem 6.3 the proof follows.

The above theorem provides us with the direct relation between P_0 problem and RIP of Φ.

VII. Comparison with the result based on coherence:

By proposition 2.2, we obtain that the coherence based argument provides us with RIP of order $k < 1 + \mu_\Phi^{-1}$, which can not cross the square root bottle-neck bound, namely in the order of $\sqrt{m}$ bound. But by the argument presented in section 6, for a given Φ, we are able to provide an existence of Ψ in its equivalence class such that Ψ satisfies RIP up to the order equal to spark(Φ) = spark(Ψ), where spark(Φ) $\geq (1 + \mu_\Phi^{-1})$. Full spark matrix can provide existence of RIP matrices of order m.

VIII. Future work:

In future we would like to investigate the following questions:

- when Φ is pre-multiplied with a non-singular matrix G, what is the relation between the coherence of Φ and the coherence $G\Phi$.
- can C_k be estimated numerically?

IX. Concluding Remarks

We have shown that for a matrix Φ whose any k column subsets are linearly independent, there exists a non-singular matrix G such that in its equivalence class $G\Phi$ becomes an RIP matrix. We hope that the class of matrices for which C_k and D_k are known explicitly helps in constructing deterministic matrices. Our future work will address this issue.

X. Acknowledgment:

The first author is thankful for the support (Ref No. 19-06/2011(i)EU-IV) that he receives from UGC, Govt of India. The second author (CSS) is thankful to CSIR, Govt. of India for its financial support (Ref: 25(219)/13/EMR-II).

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