A Note on Even Cycles and Quasi-Random Tournaments

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Abstract

A cycle $C = \{v_1, v_2, \dots, v_1\}$ in a tournament T is said to be even, if when walking along C, an even number of edges point in the wrong direction, that is, they are directed from v_{i+1} to v_i. In this short paper, we show that for every fixed even integer $k \geq 4$, if close to half of the k-cycles in a tournament T are even, then T must be quasi-random. This resolves an open question raised in 1991 by Chung and Graham [5].

1 Introduction

Quasi-random (or pseudo-random) objects are *deterministic* objects that possess the properties we expect truly *random* ones to have. One of the most surprising phenomena in this area is the fact that in many cases, if an object satisfies a single *deterministic* property then it must "behave" like a typical random object in many useful aspects. In this paper we will study one such phenomenon related to quasi-random tournaments. The notion of quasi-randomness has been widely studied for different combinatorial objects, like graphs, hypergraphs, groups and set systems [4, 6, 7, 9, 13, 14]. We refrain from giving a detailed discussion of this area in this short paper, and instead refer the reader to the surveys of Gowers [8] and Krivelevich and Sudakov [12] for more details and references.

A directed graph $D = (V, E)$ consists of a set of vertices and a set of directed edges $E \subseteq V \times V$. We use the ordered pair $(u, v) \in V \times V$ to denote directed edge from u to v. A tournament $T = (V, E)$ is a directed graph such that given any two distinct vertices $u, v \in V$, there exists exactly one of the two directed edges (u, v) or (v, u) in $E(T)$. One can also think of a tournament as an orientation of an underlying complete graph on V. We shall use n to denote $|V|$.

Consider a tournament $T = (V, E)$. For $Y \subseteq V$, and $v \in V$, let $d^+(v, Y)$ denote the number of directed edges going from v to Y and $d^-(v, Y)$ denote the number of directed edges going from

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Y to v. A purely random tournament is one where for each pair of distinct vertices u and v of V, the directed edge between them is chosen randomly to be either (u, v) or (v, u) with probability $1/2$. It is clear that in a random tournament T, we have $\sum_{v \in X} |d^+(v, Y) - d^-(v, Y)| = o(n^2)$ for all $X, Y \subseteq V(T)$. Let us define the corresponding property $\mathcal{Q}$ as follows:

Definition 1.1. *A tournament T on n vertices satisfies property $\mathcal{Q}$ if*

$$\sum_{v \in X} \left| d^+(v, Y) - d^-(v, Y) \right| = o(n^2) \quad \textit{for all } X, Y \subseteq V(T).$$

The notion of quasi-randomness in tournaments was introduced by Chung and Graham [5]. They defined several properties of tournaments, all of which are satisfied by purely random tournaments, including the property $\mathcal{Q}$ above. They also showed that all these properties are equivalent, namely, if a tournament satisfies one of these properties, then it must also satisfy all the other. They then defined a tournament to be quasi-random if it satisfies any (and therefore, all) of these properties. For the sake of brevity, we will focus on property $\mathcal{Q}$ (defined above) which will turn out to be the easiest one to work with in the context of the present paper.

Another property studied in [5] was related to even cycles in tournaments. A k-cycle is an ordered sequence of vertices $(v_1, v_2, \ldots, v_k, v_1)$ such that no vertex is repeated immediately in the sequence. That is, $v_i \neq v_{i+1}$ for all $i \leq k-1$ and $v_k \neq v_1$. We say that a k-cycle (for an integer $k \geq 2$) is even if as we traverse the cycle, we see an even number of directed edges opposite to the direction of the traversal. If a k-cycle is not even, we call it odd. Let $\mathsf{E}_k(T)$ denote the number of even k-cycles in a tournament T. Clearly, the number of k-cycles in an n-vertex tournament is $n^k - o(n^k)$. In fact, it is not hard to see that that the exact number is given by $(n-1)^k + (-1)^k(n-1)$ (see Section 3). In a random tournament, we expect about half of the k-cycles to be even. This motivated Chung and Graham [5] to define the following property.

Definition 1.2. *A tournament T on n vertices satisfies*[1] *property $\mathcal{P}(k)$ if $\mathsf{E}_k(T) = (1/2 \pm o(1))n^k$.*

Notice that when k is an odd integer, $\mathsf{E}_k(T)$ is *exactly* half the number of k-cycles in T, since an even cycle becomes odd upon traversal in the reverse direction. Hence, property $\mathcal{P}(k)$ cannot be equivalent to property $\mathcal{Q}$ when k is odd.

Chung and Graham [5] proved that $\mathcal{P}(4)$ is quasi-random. In other words, a tournament has (approximately) the correct number of even 4-cycles we expect to find in a random tournament, if and only if it satisfies property $\mathcal{Q}$. A question left open in [5] was whether $\mathcal{P}(k)$ is equivalent to $\mathcal{Q}$ for all even $k \geq 4$. Our main result answers this positively by proving the following.

Theorem 1. *The following holds for every fixed even integer $k \geq 4$: A tournament satisfies property $\mathcal{Q}$ if and only if it satisfies property $\mathcal{P}(k)$.*

[1] Observe that our definition of a k-cycle allows repeated vertices in the cycle. Note however, that forbidding repeated vertices (that is, requiring the k-cycles to be simple) would have resulted in the same property $\mathcal{P}(k)$ since the number of k-cycles with repeated vertices is $o(n^k)$. Allowing repeated vertices simplifies some of the notation.

As usual, when we say that property $\mathcal{Q}$ implies property $\mathcal{P}(k)$ we mean that for every ε there is a $\delta = \delta(\varepsilon)$, such that any large enough tournament satisfying $\sum_{v \in X} |d^+(v,Y) - d^-(v,Y)| \le \delta n^2$ for all X, Y has $(1/2 \pm \varepsilon)n^k$ even cycles. The meaning of $\mathcal{P}(k)$ implies $\mathcal{Q}$ is defined similarly.

2 Proof of Main Result

To prove Theorem 1, we shall go through a spectral characterization of quasi-randomness. We use the following adjacency matrix A to represent the tournament T. For every $u, v \in V$

$$A_{u,v} = \begin{cases} 1 & \text{if } (u,v) \in E(T) \\ -1 & \text{if } (v,u) \in E(T) \\ 0 & \text{if } u = v \end{cases}$$

A key observation that we will use is that the matrix A is skew-symmetric. Recall that a real skew symmetric matrix can be diagonalized and all its eigenvalues are purely imaginary. It follows that all the eigenvalues of A^2 are non-positive. This implies the following claim, which will be crucial in our proof.

Claim 2.1. *For $k \equiv 2 \pmod 4$, all the eigenvalues of A^k are non-positive. For $k \equiv 0 \pmod 4$, all the eigenvalues of A^k are non-negative.*

For a matrix M, we let $\mathrm{tr}(M) = \sum_{i=1}^n M_{i,i}$ denote the trace of the matrix M. Before we prove Lemmas 2.3 and 2.4, we make the following claim.

Claim 2.2. *Let A be the adjacency matrix of the tournament T. Then for an even integer $k \ge 4$, we have*

$$tr(A^k) = 2\mathsf{E}_k(T) - (n-1)^k - (n-1).$$

In particular, T satisfies the property $\mathcal{P}(k)$ if and only if $|tr(A^k)| = o(n^k)$.

Proof. Notice that the (u,u)-th entry of A^k is the number of even k-cycles starting and ending at u minus the number of odd k-cycles starting and ending at u. So the sum of all diagonal entries, $\mathrm{tr}(A^k)$, is the difference between all labeled even k-cycles and all labeled odd k-cycles. Recall that the total number of k-cycles is $(n-1)^k + (n-1)$ for even k. Thus we have that $\mathrm{tr}(A^k) = 2\mathsf{E}_k(T) - (n-1)^k - (n-1)$.

We have $\mathrm{tr}(A^k) = 2\mathsf{E}_k(T) - n^k + o(n^k)$. Notice that T satisfies property $\mathcal{P}(k)$ when $\mathsf{E}_k(T) = (1/2 \pm o(1))n^k$, which happens if and only if $|\mathrm{tr}(A^k)| = o(n^k)$. □

We are now ready to prove the first direction of Theorem 1.

Lemma 2.3. *Let $k \ge 4$ be an even integer. If a tournament satisfies $\mathcal{P}(k)$ then it satisfies $\mathcal{Q}$.*

Proof. Let $\lambda_1(A), \ldots, \lambda_n(A)$ be the eigenvalues of A sorted by their absolute value, so that $\lambda_1(A)$ has the largest absolute value. We first claim that $|\lambda_1(A)| = o(n)$. Assume first that $k \equiv 0 \pmod 4$. Then by Claim 2.1 all the eigenvalues of A^k are non-negative, implying that

$$\operatorname{tr}(A^k) = \sum_{i=1}^{n} \lambda_i(A^k) \geq \lambda_1(A^k) = \lambda_1(A)^k \ . \tag{1}$$

Now, since we assume that T satisfies $\mathcal{P}(k)$, we get from Claim 2.2 that $|\operatorname{tr}(A^k)| = o(n^k)$. Equation (1) now implies that $|\lambda_1(A)| = o(n)$. If $k \equiv 2 \pmod 4$, then since Claim 2.1 tells us that all eigenvalues are non-positive, we have

$$\operatorname{tr}(A^k) = \sum_{i=1}^{n} \lambda_i(A^k) \leq \lambda_1(A^k) = \lambda_1(A)^k \ . \tag{2}$$

As in (1), the fact that $|\operatorname{tr}(A^k)| = o(n^k)$ and that all the terms in (2) are non-positive, implies that $|\lambda_1(A)| = o(n)$.

We now claim that the fact that $|\lambda_1(A)| = o(n)$ implies that T satisfies $\mathcal{Q}$. Suppose it does not, and let $X, Y \subseteq V$ be two sets satisfying $\sum_{v \in X} |d^+(v,Y) - d^-(v,Y)| = cn^2$, for some $c > 0$. Let $\mathbf{y} \in \{0,1\}^n$ be the indicator vector for Y. We pick the vector $\mathbf{x}$ in the following way: if $v \notin X$, then set the corresponding coordinate $\mathbf{x}_v = 0$. For $v \in X$ such that $d^+(v,Y) - d^-(v,Y) \geq 0$, we set $\mathbf{x}_v = 1$. For all other $v \in X$, we set $\mathbf{x}_v = -1$. Now notice that for these vectors $\mathbf{x}$ and $\mathbf{y}$, we have $\mathbf{x}^T A \mathbf{y} = \sum_{v \in X} |d^+(v,Y) - d^-(v,Y)| = cn^2$. We can normalize $\mathbf{x}$ and $\mathbf{y}$ to get unit vectors $\tilde{\mathbf{x}} = \mathbf{x}/\sqrt{|X|}$ and $\tilde{\mathbf{y}} = \mathbf{y}/\sqrt{|Y|}$ satisfying

$$\tilde{\mathbf{x}}^T A \tilde{\mathbf{y}} = (\mathbf{x}^T A \mathbf{y})/\sqrt{|X||Y|} \geq cn^2/n = cn \ , \tag{3}$$

where the inequality follows since $|X|, |Y| \leq n$. We have thus found two unit vectors $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}$ such that $\tilde{\mathbf{x}}^T A \tilde{\mathbf{y}} \geq cn$.

We finish the proof by showing that (3) contradicts the fact that $|\lambda_1(A)| = o(n)$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the orthonormal eigenvectors corresponding to the eigenvalues of A. Let $\tilde{\mathbf{x}} = \sum_i \alpha_i \mathbf{v}_i$ and $\tilde{\mathbf{y}} = \sum_i \beta_i \mathbf{v}_i$ be the decomposition of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ along the eigenvectors (note that α_i and β_i might be complex numbers). We have

$$\tilde{\mathbf{x}}^T A \tilde{\mathbf{y}} = \left| \sum_i \alpha_i \lambda_i(A) \beta_i \right| \leq \sqrt{\sum_i |\overline{\alpha_i}|^2 \cdot \sum_i |\lambda_i(A)\beta_i|^2} = \sqrt{\sum_i |\lambda_i(A)|^2 |\beta_i|^2} \leq |\lambda_1(A)| \tag{4}$$

where the first inequality follows by using Cauchy-Schwarz ($\overline{\alpha}$ denotes the complex conjugate of α). We then use the fact that $\sum_i |\alpha_i|^2 = \sum_i |\beta_i|^2 = 1$ which follow from the fact that $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}$ are unit vectors. Finally, since we have that $|\lambda_1(A)| = o(n)$ and that $\tilde{\mathbf{x}}^T A \tilde{\mathbf{y}} \geq cn$ equation (4) gives a contradiction. So T must satisfy $\mathcal{Q}$. $\square$

We now turn to prove the second direction of Theorem 1.

Lemma 2.4. *Let $k \geq 4$ be an even integer. If a tournament satisfies $\mathcal{Q}$ then it satisfies $\mathcal{P}(k)$.*

Proof. Suppose T satisfies $\mathcal{Q}$. Then by the result of [5] mentioned earlier, T must also satisfy $\mathcal{P}(4)$. From Claim 2.2, we have that

$$|\mathrm{tr}(A^4)| = \left| \sum_{i=1}^{n} \lambda_i^4 \right| = o(n^4) \ , \tag{5}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. We will now apply induction to show that $|\mathrm{tr}(A^k)| = o(n^k)$ for all even integers $k \geq 4$. Claim 2.2 would then imply that $\mathcal{P}(k)$ is true for all even integers $k \geq 4$.

Now note the following for an even integer $k > 4$:

$$|\mathrm{tr}(A^k)| = \left| \sum_i \lambda_i^k \right| \leq \sqrt{\sum_i \lambda_i^4 \sum_i \lambda_i^{2k-4}} \leq \sqrt{\sum_i \lambda_i^4} \cdot \left| \sum_i \lambda_i^{k-2} \right| = o(n^k) \ .$$

The first inequality is Cauchy-Schwarz. For the second inequality, recall that by Claim 2.1 we have that λ_i^k are either all non-negative or non-positive. This means that $(\sum_{i=1}^n \lambda_i^{k-2})^2 \geq \sum_{i=1}^n \lambda_i^{2k-4}$ since we lose only non-negative terms. The last equality follows by applying the induction hypothesis and (5). □

3 Concluding Remarks

- The proof of Lemma 2.3 shows that if T satisfies the property $\mathcal{P}(4)$, then $|\lambda_1(A)| = o(n)$ which in turn implies that T satisfies $\mathcal{Q}$. Since we also know that $\mathcal{Q}$ implies $\mathcal{P}(4)$ we conclude that a tournament T is quasi-random if and only if $|\lambda_1(A)| = o(n)$. This is in line with other spectral characterizations of quasi-randomness for other combinatorial objects [1, 2, 3, 7, 11].

- Let $k \geq 4$ be an even integer. Now we make an observation about $\mathsf{E}_k(T)$ for an arbitrary tournament T (which is not necessarily quasi-random). The total number of distinct k-cycles of T is $\mathrm{tr}(B^k)$, where B is the adjacency matrix of the undirected complete graph on n vertices. Since the spectrum of B is $\{n-1, -1, \ldots, -1\}$ we get $\mathrm{tr}(B^k) = (n-1)^k + (n-1)$. For $k \equiv 0 \pmod 4$, by Claim 2.1, the eigenvalues of A^k are all non-negative and thus we have $\mathrm{tr}(A^k) \geq 0$. By Claim 2.2, we have that $\mathsf{E}_k(T) \geq ((n-1)^k + (n-1))/2$. For $k \equiv 2 \pmod 4$, we can conclude similarly using Claims 2.1 and 2.2 that $\mathsf{E}_k(T) \leq ((n-1)^k + (n-1))/2$.

- We note that we can use the ideas we used in this paper to prove similar results for general directed graphs as defined by Griffiths [10]. Since the ideas required to obtain this more general result do not deviate significantly from those we have used here, we defer them to the first author's Ph.D. thesis.

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