Bijective Transformations of Fuzzy Implications - An Algebraic Perspective

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Abstract

Bijective transformations play an important role in generating fuzzy implications from fuzzy implications. In [Representations through a Monoid on the set of Fuzzy Implications, Fuzzy Sets and Systems, 247, 51-67], Vemuri and Jayaram proposed a monoid structure on the set of fuzzy implications, which is denoted by $\mathbb{I}$, and using the largest subgroup $\mathbb{S}$ of this monoid discussed some group actions on the set $\mathbb{I}$. In this context, they obtained a bijective transformation which ultimately led to hitherto unknown representations of the Yager's families of fuzzy implications, viz., f-, g-implications. This motivates us to consider whether the bijective transformations proposed by Baczyński & Drewniak and Jayaram & Mesiar, in different but purely analytic contexts, also possess any algebraic connotations. In this work, we show that these two bijective transformations can also be seen as being obtained from some group actions of $\mathbb{S}$ on $\mathbb{I}$. Further, we consider the most general bijective transformation that generates fuzzy implications from fuzzy implications and show that it can also be obtained as a composition of group actions of $\mathbb{S}$ on $\mathbb{I}$. Thus this work tries to position such bijective transformations from an algebraic perspective.

Keywords: Bijective transformation, fuzzy implications, group action, equivalence relation, conjugacy classes, special property.
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1. Introduction

Fuzzy implications are one of the important operators in fuzzy logic, both for their theoretical and applicational values. They are binary operations on the unit interval $[0,1]$ defined as follows:

Definition 1.1 ([3], Definition 1.1.1). A function $I\colon [0,1]^2 \longrightarrow [0,1]$ is called a *fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$, the following conditions:

$$\text{if } x_1 \leq x_2, \text{ then } I(x_1, y) \geq I(x_2, y)\ , \text{ i.e., } I(\,\cdot\,, y) \text{ is decreasing}\ , \tag{I1}$$

$$\text{if } y_1 \leq y_2, \text{ then } I(x, y_1) \leq I(x, y_2)\ , \text{ i.e., } I(x,\,\cdot\,) \text{ is increasing}\ , \tag{I2}$$

$$I(0,0) = 1\ , I(1,1) = 1\ , I(1,0) = 0\ . \tag{I3}$$

The set of fuzzy implications will be denoted by $\mathbb{I}$.

Fuzzy implications have many applications in the areas like, *fuzzy control, approximate reasoning, decision making, fuzzy image processing* and *data mining* etc. Hence there is always a need to generate newer fuzzy implications with various properties. Among the many methods, *bijective transformations* are one of the earliest methods of generating fuzzy implications from fuzzy implications. Let Φ denote the set of all increasing bijections on $[0,1]$.

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1.1. Bijective transformations of fuzzy implications

The first such bijective transformation was proposed by Baczyński and Drewniak in [1]. For a given $I \in \mathbb{I}$ and $\varphi \in \Phi$, the φ-conjugate $I_\varphi \in \mathbb{I}$ is defined as follows:

$$I_\varphi(x,y) = \varphi^{-1}\left(I(\varphi(x),\varphi(y))\right), \qquad x,y \in [0,1]. \tag{1}$$

They further showed that I_φ preserves many of the desirable properties of the fuzzy implication I.

Following this, in [6], Jayaram and Mesiar while studying a class of fuzzy implications, namely, special fuzzy implications (see Definition 4.1), proposed the following transformation of an $I \in \mathbb{I}$:

$$\varphi(I)(x,y) = \varphi(I(x,y)), \qquad x,y \in [0,1]\ , \tag{2}$$

where once again, $\varphi \in \Phi$. Let us term this as the $\varphi(I)$-conjugate of $I \in \mathbb{I}$.

Recently, Vemuri and Jayaram, in [9], proposed the following transformation and obtained hitherto unknown representations of two families of fuzzy implications, viz., the Yager's f- and g-implications.

For a given $I \in \mathbb{I}$ and $\varphi \in \Phi$, the φ-pseudo conjugate $I^\varphi \in \mathbb{I}$ is defined as follows:

$$I^\varphi(x,y) = \varphi(I(x,\varphi^{-1}(y))), \qquad x,y \in [0,1]\ . \tag{3}$$

1.2. φ-pseudo conjugates and group actions on $\mathbb{I}$

Interestingly, the last of the above transformations, given by (3), not only gives rise to fuzzy implications but also has an algebraic connotation. In fact, (3) can be said to have had its beginnings in a purely algebraic context. In [9], the authors had proposed a monoid structure on the set of all fuzzy implications $(\mathbb{I}, \circledast)$- see Definition 2.1 - and determined the largest subgroup $\mathbb{S}$ of this monoid. The transformation (3) arose naturally while studying the action of the group $(\mathbb{S}, \circledast)$ on the set $\mathbb{I}$.

Motivated by the fact that any group action on a set partitions the set, in [9], they have defined a relation $\sim_{\mathcal{V}}$ on $\mathbb{I}$ based on the φ-pseudo conjugates in the following manner. Given $I, J \in \mathbb{I}$,

$$I \sim_{\mathcal{V}} J \Longleftrightarrow J = I^\varphi \text{ for some } \varphi \in \Phi\ . \tag{4}$$

Further they have shown that $\sim_{\mathcal{V}}$ is an equivalence relation on $\mathbb{I}$. In fact, as shown in [9], Remark 4.6, the $\sim_{\mathcal{V}}$ equivalence classes form precisely the partition obtained from the particular group action of $\mathbb{S}$ on $\mathbb{I}$.

1.3. Motivation of this work

Let us now define the following relations on $\mathbb{I}$ based on the φ-conjugates and the transformation (2). Given $I, J \in \mathbb{I}$ we define

$$I \sim_{\mathcal{B}} J \Longleftrightarrow J = I_\varphi \text{ for some } \varphi \in \Phi\ , \tag{5}$$

$$I \sim_{\mathcal{J}} J \Longleftrightarrow J = \varphi(I) \text{ for some } \varphi \in \Phi\ . \tag{6}$$

It can be easily seen that both $\sim_{\mathcal{B}}, \sim_{\mathcal{J}}$ are equivalence relations and hence partition the set $\mathbb{I}$ into equivalence classes. This leads one to investigate the following:

Question 1: Do the bijective transformations (1) and (2) have any algebraic connotations?

Further, given an $I \in \mathbb{I}$ and $\varphi, \psi, \mu \in \Phi$, let us define the most general bijective transformation of I as follows:

$$J_{\varphi,\psi,\mu}(x,y) = \varphi\left(I\left(\psi(x),\mu(y)\right)\right), \qquad x,y \in [0,1]\ . \tag{7}$$

Note that all of the above bijective transformations are special cases of the following general bijective transformation of an $I \in \mathbb{I}$.

Given $I, J \in \mathbb{I}$ if we define

$$I \sim_{\varphi,\psi,\mu} J \Longleftrightarrow J = I_{\varphi,\psi,\mu} \text{ for some } \varphi,\psi,\mu \in \Phi\ , \tag{8}$$

it can be easily seen that $\sim_{\varphi,\psi,\mu}$ is an equivalence relation and hence partitions $\mathbb{I}$ into equivalence classes. Thus the following question arises quite naturally:

Question 2: Does the general bijective transformation (7) have any algebraic connotation?

The above two posers form the basic motivation behind this study.

1.4. Main contributions of this work

If $([0,1],*)$ and $([0,1],\diamond)$ are two ordered groupoids, then $\varphi(x*y)=\varphi(x)\diamond\varphi(y)$ is a groupoid homomorphism for any $\varphi\in\Phi$. Conversely, given a binary groupoid operation, one could obtain new groupoid operations from the above as follows: $x*y=\varphi^{-1}\left(\varphi(x)\diamond\varphi(y)\right)$. Thus, viewing a fuzzy implication as a groupoid on $[0,1]$, the φ-conjugates of an $I\in\mathbb{I}$ proposed by Baczyński and Drewniak [1] can be seen as an automorphism on the groupoid $([0,1],I)$. However, note that an $I\in\mathbb{I}$ does not impose any regular algebraic structure on the unit interval $[0,1]$ and hence many of the algebraic tools, which would have otherwise been available for analysis, are not applicable in this setting. To the best of the authors' knowledge, this is the first work that attempts to assign an algebraic connotation to transformations that were proposed in a purely analytical context.

We answer **Question 1** in the affirmative by showing them to be obtained from some particular actions of the group $\mathbb{S}$ on $\mathbb{I}$. This further highlights the power of the monoid structure $(\mathbb{I},\circledast)$ proposed in [9]. Note that the monoid structure $(\mathbb{I},\circledast)$ is the richest algebraic structure available so far on the set $\mathbb{I}$ of fuzzy implications.

Further, the monoid $(\mathbb{I},\circledast)$ allows one to construct group actions such that any general bijective transformation of an $I\in\mathbb{I}$, as given in (7), can be seen as a composition of three particular actions of $\mathbb{S}$ on $\mathbb{I}$, thus also answering **Question 2**.

1.5. Outline of the paper

In Section 2, we recall the φ-pseudo conjugate transformations of fuzzy implications proposed by Vemuri and Jayaram in [9] and the concept of group action on any set. We recall also the representation results obtained through these φ-pseudo conjugate transformations. We deal with the φ-conjugate transformation proposed by Baczyński and Drewniak in Section 3 and show that the equivalence classes obtained from them are exactly the equivalence classes obtained from a group action of $\mathbb{S}$ on the set $\mathbb{I}$. In Section 4, we do an analogous study on the Jayaram and Mesiar's $\varphi(I)$-bijective transformations and obtain the group action of $\mathbb{S}$ on $\mathbb{I}$ whose equivalence classes turn out to be the equivalence classes of $\varphi(I)$-bijective transformations. Most general bijective transformations of the form (7) are considered in Section 5 and the equivalence classes of these bijective transformations are shown to be the equivalence classes of a relation defined in terms of some known actions of $\mathbb{S}$ on the set $\mathbb{I}$. Some special classes of (7) are also discussed.

2. Group actions and the φ-pseudo conjugate transformation

In this section, we begin by recalling the monoid structure $(\mathbb{I},\circledast)$ on the set of fuzzy implications proposed by Vemuri and Jayaram in [8]. Following this, we present relevant definitions and results relating the φ-pseudo conjugacy classes and a particular group action of the subgroup $\mathbb{S}$ of the monoid on the set $\mathbb{I}$. We clearly highlight the impact of the algebraic study of the bijective transformation (3) by presenting some of the representation results, especially that of the Yager's families of fuzzy implications, that were obtained with their help.

2.1. Monoid of fuzzy implications

As mentioned earlier, from a given pair (I,J) of fuzzy implications, Vemuri and Jayaram in [8] proposed the $\circledast$-composition $I\circledast J$ of I,J and showed that it is again a fuzzy implication. Further they have investigated the algebraic structure on $\mathbb{I}$ imposed by this generating method.

Definition 2.1 ([8], Definition 7). For any two fuzzy implications I,J, we define $I\circledast J$ as

$$(I\circledast J)(x,y)=I(x,J(x,y)),\qquad x,y\in[0,1]\,. \tag{9}$$

Theorem 2.2 ([8], Theorem 10). *The function $I\circledast J$ is a fuzzy implication, i.e., $I\circledast J\in\mathbb{I}$.*

As stated earlier, the $\circledast$-composition, when looked at as a binary operation on the set $\mathbb{I}$, makes it a monoid as the following result illustrates.

Theorem 2.3 ([8], Theorem 11)**.** $(\mathbb{I}, \circledast)$ *forms a monoid, whose identity element is given by*

$$I_{\mathbf{D}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \ , \\ y, & \text{if } x > 0 \ . \end{cases}$$

Further, observing that $(\mathbb{I}, \circledast)$ is only a monoid and can not be made a group, the authors, in [9], have characterized the set of invertible elements of $(\mathbb{I}, \circledast)$, which forms the largest subgroup of the monoid and obtained their representations as follows:

Theorem 2.4 ([9], Theorem 3.5)**.** *An $I \in \mathbb{I}$ is invertible w.r.t. $\circledast$ if and only if*

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \ , \\ \varphi(y), & \text{if } x > 0 \ , \end{cases} \tag{10}$$

where the function $\varphi \colon [0,1] \longrightarrow [0,1]$ is an increasing bijection, i.e., $\varphi \in \Phi$.

Let $\mathbb{S}$ be the set of invertible elements of $(\mathbb{I}, \circledast)$, i.e., $\mathbb{S}$ is the set of fuzzy implications of the form (10) for some $\varphi \in \Phi$. In fact, $(\mathbb{S}, \circledast)$ is the largest subgroup contained in $(\mathbb{I}, \circledast)$.

Let $\circ$ denote the usual composition of functions. Then it is well known that $(\Phi, \circ)$ is a group. Interestingly, the subgroup $(\mathbb{S}, \circledast)$ is isomorphic to $(\Phi, \circ)$, as the following result illustrates.

Theorem 2.5 ([9], Theorem 3.9)**.** *The groups $(\Phi, \circ)$, $(\mathbb{S}, \circledast)$ are isomorphic to each other.*

2.2. Action of $(\mathbb{S}, \circledast)$ *on the monoid* $\mathbb{I}$

In [9], the authors have defined a group action of $\mathbb{S}$ on $\mathbb{I}$ and obtained equivalence classes from it. Let us, firstly, recall the definition of an action of a group on a non-empty set.

Definition 2.6 ([7], Pg. 488). Let $(G, *)$ be a group and H be a non empty set. A function $\bullet \colon G \times H \longrightarrow H$ is called a left group action if, for all $g_1, g_2 \in G$ and $h \in H$, $\bullet$ satisfies the following two conditions:

(i) $g_1 \bullet (g_2 \bullet h) = (g_1 * g_2) \bullet h$.

(ii) $e \bullet h = h$, where e is the identity of G.

A right group action can also be defined analogously.

With the help of the subgroup $\mathbb{S}$, the authors in [9] have defined the following group action of $\mathbb{S}$ on the set $\mathbb{I}$.

Definition 2.7 ([9], Definition 4.2 & Lemma 4.3). Let $\bullet \colon \mathbb{S} \times \mathbb{I} \to \mathbb{I}$ be a map defined by

$$(K, I) \mapsto K \bullet I = K \circledast I \circledast K^{-1}.$$

Then the function $\bullet$ is both a left and right group action of $\mathbb{S}$ on $\mathbb{I}$.

Since any group action on a set partitions the set into equivalence classes, with the help of the above group action $\bullet$, the authors in [9] have defined an equivalence relation on the set $\mathbb{I}$, see [9], Definition 4.4 and Lemma 4.5, as follows:

Let $I, J \in \mathbb{I}$. Define $I \sim_{\bullet} J \iff J = K \bullet I$ for some $K \in \mathbb{S}$. In other words, $I \sim_{\bullet} J \iff J = K \circledast I \circledast K^{-1}$ for some $K \in \mathbb{S}$. The relation $\sim_{\bullet}$ is an equivalence relation and it partitions the set $\mathbb{I}$.

Further, the equivalence class containing $I \in \mathbb{I}$ will be of the form

$$[I]_{\sim_{\bullet}} = \{J \in \mathbb{I} \mid J = K \circledast I \circledast K^{-1} for\ some\ K \in \mathbb{S}\}.$$

Since any $K \in \mathbb{S}$ is of the form

$$K(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(y), & \text{if } x > 0, \end{cases}$$

for some $\varphi \in \Phi$, we have that, if $J \in [I]_{\sim_\bullet}$, then for all $x, y \in [0,1]$, J is given by $J(x,y) = \varphi(I(x, \varphi^{-1}(y)))$, for some $\varphi \in \Phi$.

Clearly, J is a φ-pseudo conjugate of I, for some $\varphi \in \Phi$ and the equivalence classes obtained from $\sim_\bullet$ and $\sim_\mathcal{Y}$ of (4) are identical.

Further, the investigation of the φ-pseudo conjugates of the Yager's families of fuzzy implications have led to some hitherto unknown representations of the same in terms of the three basic fuzzy implications. We present only the main results and for more details, refer the readers to [9]. For the definition and properties of the Yager's families of fuzzy implications, please see, [3] or [2].

Let us denote the families of f- and g-implications by $\mathbb{I}_\mathbb{F}, \mathbb{I}_\mathbb{G}$, respectively.

Theorem 2.8 ([9], Corollary 5.9). *An $I \in \mathbb{I}_\mathbb{F}$ if and only if for some $\varphi \in \Phi$,*

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ \varphi\left(\left[\varphi^{-1}(y)\right]^x\right), & \text{if } x > 0 \text{ or } y > 0 \ , \end{cases}$$

$$\text{or,} \qquad I(x,y) = \varphi\left(1 - x + x\varphi^{-1}(y)\right) \ .$$

Theorem 2.9 ([9], Corollary 5.17). *An $I \in \mathbb{I}_\mathbb{G}$ if and only if for some $\varphi \in \Phi$,*

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \ , \\ \varphi\left(\left[\varphi^{-1}(y)\right]^x\right), & \text{if } x > 0 \text{ or } y > 0 \ , \end{cases}$$

$$\text{or,} \qquad I(x,y) = \begin{cases} 1, & \text{if } \varphi(x) \le y, \\ \varphi\left(\dfrac{\varphi^{-1}(y)}{x}\right), & \text{if } \varphi(x) > y. \end{cases}$$

3. The φ-conjugate transformation as a group action on $\mathbb{I}$

Let us recall the bijective transformation of fuzzy implications proposed by Baczyński and Drewniak in [1]. For a given $I \in \mathbb{I}$ and $\varphi \in \Phi$, the φ-conjugate $I_\varphi \in \mathbb{I}$ is defined as follows:

$$I_\varphi(x,y) = \varphi^{-1}\left(I(\varphi(x), \varphi(y))\right), \qquad x, y \in [0,1]. \qquad (1)$$

Note also that the equivalence classes of $I \in \mathbb{I}$ w.r.t. the equivalence relation $\sim_\mathcal{B}$, see (5), are given by

$$[I]_{\sim_\mathcal{B}} = \{J(x,y) = \varphi^{-1}\left(I(\varphi(x), \varphi(y))\right) \text{for some } \varphi \in \Phi\} \ . \qquad (11)$$

The above equivalence relation preserves some of the most desirable basic properties of fuzzy implications as well as is closed w.r.t. some families of fuzzy implications, see for instance, Proposition 1.3.6, Theorem 2.4.5, Proposition 2.5.10 and Theorem 2.6.11 in [3]. However, so far, there have been no known attempts to position the φ-conjugates, or the conjugacy classes obtained from them, algebraically. In this section we attempt the same and show that these conjugacy classes are exactly the equivalence classes of fuzzy implications obtained from a group action of $\mathbb{S}$ on the set $\mathbb{I}$.

3.1. A semigroup structure on $\mathbb{I}$

Towards this end, we first propose yet another new generating method of fuzzy implications from fuzzy implications and show that this method imposes a semigroup structure on the set $\mathbb{I}$.

Definition 3.1. Let $I, J \in \mathbb{I}$. Define $I\Delta J \colon [0,1]^2 \longrightarrow [0,1]$ as follows:

$$(I\Delta J)(x,y) = I(J(1,x), J(x,y)), \qquad x, y \in [0,1]. \qquad (12)$$

Observe that Definitions 2.1 and 3.1 are not identically same on $\mathbb{I}$. To ensure this, please see Remark 3.2.

Remark 3.2. Let $I(x,y) = I_{\mathbf{RC}}(x,y) = 1 - x + xy$ and $J(x,y) = \max(1-x, y^2)$. Then it follows that

$$(I \circledast J)(x,y) = \max(1-x^2, 1-x+xy^2) \ ,$$

$$\text{while, } (I\Delta J)(x,y) = \max(1-x^3, 1-x^2+x^2y^2) \ ,$$

for all $x, y \in [0,1]$. When $x = 0.5, y = 0$, we see that $(I \circledast J)(x,y) = 0.75 \neq 0.87 = (I\Delta J)(x,y)$.

Now, in the following, we show that the Δ-composition of fuzzy implications is again a fuzzy implication.

Theorem 3.3. *The function $I\Delta J$ is a fuzzy implication. i.e., $I\Delta J \in \mathbb{I}$.*

Proof. Let $I, J \in \mathbb{I}$ and $x_1, x_2, y \in [0,1]$.

(i) Let $x_1 \leq x_2$. Then $J(x_1,y) \geq J(x_2,y)$ and $J(1,x_1) \leq J(1,x_2)$.

$$\begin{aligned}(I\Delta J)(x_1,y) &= I(J(1,x_1), J(x_1,y)) \geq I(J(1,x_1), J(x_2,y)) \\ &\geq I(J(1,x_2), J(x_2,y)) \geq (I\Delta J)(x_2,y).\end{aligned}$$

Thus Δ is decreasing in the first variable. Similarly, one can show that Δ is increasing in the second variable.

(ii) $(I\Delta J)(0,0) = I(J(1,0), J(0,0)) = I(0,1) = 1$.
$(I\Delta J)(1,1) = I(J(1,1), J(1,1)) = I(1,1) = 1$.
$(I\Delta J)(1,0) = I(J(1,1), J(1,0)) = I(1,0) = 0$.

Thus $I\Delta J$ is a fuzzy implication. □

From Theorem 3.3, it follows that $I\Delta J \in \mathbb{I}$ for all $I, J \in \mathbb{I}$. Algebraically speaking, Δ becomes a binary operation on the set $\mathbb{I}$. In fact, in the following, we show that Δ is associative in $\mathbb{I}$, thus making $(\mathbb{I}, \Delta)$ a semigroup.

Theorem 3.4. $(\mathbb{I}, \Delta)$ *is a semigroup.*

Proof. From Theorem 3.3, it is enough to show that Δ is associative in $\mathbb{I}$. To show this, let $I, J, K \in \mathbb{I}$ and $x, y \in [0,1]$. Then

$$\begin{aligned}(I\Delta(J\Delta K))(x,y) &= I((J\Delta K)(1,x), (J\Delta K)(x,y)) \\ &= I(J(K(1,1), K(1,x)), J(K(1,x), K(x,y))) \\ &= I(J(1, K(1,x)), J(K(1,x), K(x,y))), \\ \text{and, } ((I\Delta J)\Delta K)(x,y) &= (I\Delta J)(K(1,x), K(x,y)) \\ &= I(J(1, K(1,x)), J(K(1,x), K(x,y))).\end{aligned}$$

Thus Δ is associative in $\mathbb{I}$ and $(\mathbb{I}, \Delta)$ forms a semigroup. □

Remark 3.5. From Theorem 3.4, it follows that $(\mathbb{I}, \Delta)$ is a semigroup. However, unlike the monoid $(\mathbb{I}, \circledast)$, the semigroup $(\mathbb{I}, \Delta)$ is not a monoid. However, the fuzzy implication $I_{\mathbf{D}}$ is a right identity as the following illustrates. Let $I \in \mathbb{I}$ and $x, y \in [0,1]$. Then

$$\begin{aligned}(I\Delta I_{\mathbf{D}})(x,y) &= I(I_{\mathbf{D}}(1,x), I_{\mathbf{D}}(x,y)) \\ &= \begin{cases} 1, & \text{if } x = 0, \\ I(x,y), & \text{if } x > 0, \end{cases} = I(x,y) \ .\end{aligned}$$

From Remark 3.2, it is clear that the binary operations Δ and $\circledast$ are different on the set $\mathbb{I}$. However, in the following, we show that these binary operations are identically the same on $\mathbb{S}$, the set of invertible elements of $(\mathbb{I}, \circledast)$.

Lemma 3.6. *Let $I, J \in \mathbb{S}$. Then $I \circledast J = I \Delta J$.*

Proof. Let $I, J \in \mathbb{S}$, i.e., for some $\varphi, \psi \in \Phi$,

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0\ , \\ \varphi(y), & \text{if } x > 0\ , \end{cases} \text{ and } J(x,y) = \begin{cases} 1, & \text{if } x = 0\ , \\ \psi(y), & \text{if } x > 0\ . \end{cases}$$

$$\begin{aligned} \text{Now, } (I\Delta J)(x,y) &= I(J(1,x), J(x,y)) \\ &= I(\psi(x), J(x,y)) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(\psi(y)), & \text{if } x > 0, \end{cases} \\ \text{and } (I \circledast J)(x,y) = I(x, J(x,y)) &= \begin{cases} 1, & \text{if } x = 0, \\ \varphi(\psi(y)), & \text{if } x > 0. \end{cases} \end{aligned}$$

Thus $\circledast$ and Δ are equal on $\mathbb{S}$. □

From Lemma 3.6, the following remark is straightforward.

Lemma 3.7. *For all $I \in \mathbb{I}, K \in \mathbb{S}$, $K \circledast (I\Delta K^{-1}) = (K \circledast I)\Delta K^{-1}$.*

Proof. Let $I \in \mathbb{I}, K \in \mathbb{S}$. Then from Theorem 2.4, K is of the form (10), i.e.,

$$K(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi(y), & \text{if } x > 0, \end{cases}$$

for some $\varphi \in \Phi$. Also K^{-1} will be given by

$$K^{-1}(x,y) = \begin{cases} 1, & \text{if } x = 0, \\ \varphi^{-1}(y), & \text{if } x > 0. \end{cases}$$

Case (i): Let $x = 0$. Then $(K \circledast (I\Delta K^{-1}))(0,y) = 1 = ((K \circledast I)\Delta K^{-1})(0,y)$.
Case (ii): Let $x > 0$. Then

$$\begin{aligned} (K \circledast (I\Delta K^{-1}))(x,y) &= K(x, (I\Delta K^{-1})(x,y)) \\ &= K(x, I(K^{-1}(1,x), K^{-1}(x,y))) \\ &= \varphi(I(\varphi^{-1}(x), \varphi^{-1}(y))) \\ \text{and, } ((K \circledast I)\Delta K^{-1})(x,y) &= (K \circledast I)(K^{-1}(1,x), K^{-1}(x,y)) \\ &= K(K^{-1}(1,x), I(K^{-1}(1,x), K^{-1}(x,y))) \\ &= K(\varphi(x), I(\varphi^{-1}(x), \varphi^{-1}(y))) \\ &= \varphi(I(\varphi^{-1}(x), \varphi^{-1}(y))). \end{aligned}$$

Thus in all cases, we have proved that $(K \circledast (I\Delta K^{-1}))(x,y) = ((K \circledast I)\Delta K^{-1})(x,y)$, for all $x, y \in [0,1]$ and hence $K \circledast (I\Delta K^{-1}) = (K \circledast I)\Delta K^{-1}$. □

3.2. The φ-conjugate transformation as a group action on $\mathbb{I}$

We, now, define yet another group action of $\mathbb{S}$ on $\mathbb{I}$ and study the equivalence classes obtained from it.

Lemma 3.8. *Let $\sqcap \colon \mathbb{S} \times \mathbb{I} \longrightarrow \mathbb{I}$ be defined by*

$$K \sqcap I = K \circledast I\Delta K^{-1}, \qquad K \in \mathbb{S},\ I \in \mathbb{I}. \tag{13}$$

The operation $\sqcap$ is a group action of $\mathbb{S}$ on $\mathbb{I}$.

Proof. (i) Let $K_1, K_2 \in \mathbb{S}$ and $I \in \mathbb{I}$. Then

$$\begin{aligned} K_1 \sqcap (K_2 \sqcap I) &= K_1 \circledast (K_2 \sqcap I)\Delta K_1^{-1} \\ &= K_1 \circledast (K_2 \circledast I\Delta K_2^{-1})\Delta K_1^{-1} \\ &= K_1 \circledast K_2 \circledast I\Delta(K_1\Delta K_2)^{-1}, \qquad [\because \text{ by Lemma 3.7.} \\ &= K_1 \circledast K_2 \circledast I\Delta(K_1 \circledast K_2)^{-1}, \qquad [\because \text{ by Lemma 3.6.} \\ &= (K_1 \circledast K_2) \sqcap I. \end{aligned}$$

(ii) $I_{\mathbf{D}} \sqcap I = I_{\mathbf{D}} \circledast I\Delta I_{\mathbf{D}}^{-1} = I$ for all $I \in \mathbb{I}$.

Thus $\sqcap$ is a group action of $\mathbb{S}$ on $\mathbb{I}$. □

Definition 3.9. Define $\sim_\sqcap$ on $\mathbb{I}$ by $I \sim_\sqcap J \iff J = K \sqcap I = K \circledast I\Delta K^{-1}$ for some $K \in \mathbb{S}$.

It is easy to verify that $\sim_\sqcap$ is an equivalence relation (Lemma 3.7 is useful).

Theorem 3.10. *The conjugacy classes of fuzzy implications proposed by Baczyński* et.al.*, viz.,* (11)*, are the equivalence classes of fuzzy implications w.r.t. the equivalence relation $\sim_\sqcap$, i.e., for any $I \in \mathbb{I}$, we have that $[I]_{\sim_{\mathcal{B}}} = [I]_{\sim_\sqcap}$.*

Proof. Let $I \in \mathbb{I}$. Then

$$\begin{aligned} [I]_{\sim_\sqcap} &= \{J \in \mathbb{I} | I \sim_\sqcap J\} \\ &= \{J \in \mathbb{I} | J = K \sqcap I \text{ for some } K \in \mathbb{S}\} \\ &= \{J \in \mathbb{I} | J = K \circledast I\Delta K^{-1} \text{ for some } K \in \mathbb{S}\} \\ &= \{J \in \mathbb{I} | J(x,y) = (K \circledast I\Delta K^{-1})(x,y)\} \\ &= \{J \in \mathbb{I} | J(x,y) = K(x, (I\Delta K^{-1})(x,y))\ \} \\ &= \{J \in \mathbb{I} | J(x,y) = K(x, I(K^{-1}(1,x), K^{-1}(x,y)))\} \\ &= \{J \in \mathbb{I} | J(x,y) = K(x, I(\varphi^{-1}(x), K^{-1}(x,y)))\} \\ &= \{J \in \mathbb{I} | J(x,y) = \varphi(I(\varphi^{-1}(x), \varphi^{-1}(y)))\} \\ &= \{J \in \mathbb{I} | J = I_{\varphi^{-1}} \text{ for some } \ \varphi^{-1} \in \Phi\} = [I]_{\sim_{\mathcal{B}}}. \end{aligned}$$

□

From Theorem 3.10, we see that the conjugacy classes proposed by Baczyński and Drewniak can also be obtained by a group action of $\mathbb{S}$ on $\mathbb{I}$.

4. The bijective transformation $\varphi(I)$ as a group action

In Section 3, we have shown that the conjugacy classes proposed by Baczyński and Drewniak have algebraic connotations. In this section we investigate the algebraic connotations of conjugacy classes proposed by Jayaram and Mesiar in [6].

4.1. The bijective transformation $\varphi(I)$ of Jayaram and Mesiar

In [6], Jayaram and Mesiar studied a new class of fuzzy implications, namely, special fuzzy implications, which are defined as follows.

Definition 4.1 (cf. [4], [6], Definition 1.1). A fuzzy implication I is said to be special, if for any $\epsilon > 0$ and for all $x, y \in [0,1]$ such that $x + \epsilon, y + \epsilon \in [0,1]$ the following condition is fulfilled:

$$I(x,y) \le I(x+\epsilon, y+\epsilon), \tag{SP}$$

In the context of generating special fuzzy implications from special fuzzy implications, Jayaram and Mesiar proposed the following transformation.

Definition 4.2 ([6], Definition 9.7). Let $\varphi \in \Phi$ and $I \in \mathbb{I}$. The $\varphi(I)$-transformation of I is defined as follows

$$\varphi(I)(x,y) = \varphi(I(x,y)), \qquad x,y \in [0,1]. \tag{2}$$

and is shown that it is a fuzzy implication.

Further, Jayaram and Mesiar [6] showed that the $\varphi(I)$-transformation preserves many properties of fuzzy implications including (SP) (see, Proposition 9.8 in [6]). Note also that the equivalence classes of $I \in \mathbb{I}$ w.r.t. the equivalence relation $\sim_{\mathcal{J}}$, see (6), are given by

$$[I]_{\sim_{\mathcal{J}}} = \{J(x,y) = \varphi(I(x,y)) \text{ for some } \varphi \in \Phi\}\ . \tag{14}$$

Once again, it is not known whether any algebraic perspective can be assigned to the above equivalence classes. In our investigations below we show that these conjugacy or equivalence classes are exactly the equivalence classes obtained via a particular group action of $\mathbb{S}$ on $\mathbb{I}$.

4.2. The transformation $\varphi(I)$ as a group action on $\mathbb{I}$

We now propose yet another group action of $\mathbb{S}$ on $\mathbb{I}$ and show that the equivalence classes obtained through them are exactly the conjugacy classes proposed by Jayaram and Mesiar [6], viz., (14), in the context of special fuzzy implications.

Towards this end, in the following we define a function and show that it is, in fact, a left group action.

Lemma 4.3. *Let $\sqcup \colon \mathbb{S} \times \mathbb{I} \longrightarrow \mathbb{I}$ be defined by*

$$K \sqcup I = K \circledast I, \qquad K \in \mathbb{S},\ I \in \mathbb{I}. \tag{15}$$

The function $\sqcup$ is a left group action of $\mathbb{S}$ on $\mathbb{I}$.

Proof. Proof follows similar to that of Lemma 3.8. □

Definition 4.4. Define $\sim_{\sqcup}$ on $\mathbb{I}$ by $I \sim_{\sqcup} J \Longleftrightarrow J = K \circledast I$ for some $K \in \mathbb{S}$.

It is easy to verify that $\sim_{\sqcup}$ is an equivalence relation.

Theorem 4.5. *The equivalence classes of fuzzy implications as given in Definition 4.4 are exactly the conjugacy classes proposed by Jayaram and Mesiar, viz., (14), i.e., for any $I \in \mathbb{I}$, we have that $[I]_{\sim_{\mathcal{J}}} = [I]_{\sim_{\sqcup}}$.*

Proof. Proof follows along the lines of Theorem 3.10. □

In Sections 3 and 4, we have recalled the bijective transformations proposed by both Baczyński and Drewniak and Jayaram and Mesiar and shown that they are exactly the equivalence classes of fuzzy implications obtained as some particular group actions of $\mathbb{S}$ on $\mathbb{I}$. This settles **Question 1** completely.

However, in general, it is still unknown whether the general bijective transformations of the form (7) can also be seen in an algebraic perspective. In the following section, we take up this task and solve **Question 2** completely.

5. General bijective transformations of fuzzy implications

In the following we recall the most general bijective transformations of the form (7).

Definition 5.1. Let $I\colon [0,1]^2 \longrightarrow [0,1]$ be a function and $\varphi, \psi, \mu \in \Phi$. We define the general bijective transformation $J_{\varphi,\psi,\mu}\colon [0,1]^2 \longrightarrow [0,1]$ of I as follows:

$$J_{\varphi,\psi,\mu}(x,y) = \varphi(I(\psi(x), \mu(y))), \qquad x, y \in [0,1] \ . \tag{7}$$

The following result shows that any general bijective transformation of the form (7) can also generate fuzzy implications from fuzzy implications.

Lemma 5.2. *Let $I\colon [0,1]^2 \longrightarrow [0,1]$ be a function and $\varphi, \psi, \mu \in \Phi$. Let $J_{\varphi,\psi,\mu}$ be defined as in* (7)*. Then the following statements are equivalent:*

(i) *I is a fuzzy implication.*
(ii) *$J_{\varphi,\psi,\mu}$ is a fuzzy implication.*

Proof. A straight-forward verification. □

From Theorem 5.2, it follows that for every $I \in \mathbb{I}$ the function of the form (7) is always a fuzzy implication. In other words, one can always obtain fuzzy implications from given a fuzzy implication using (7).

As noted earlier in Section 1.3, the relation (8) defined based on the general bijective transformation (7) is an equivalence relation. For an $I \in \mathbb{I}$, if $[I]_{\sim_{\varphi,\psi,\mu}}$ denotes the equivalence class of fuzzy implications containing I w.r.t. (8), then

$$\begin{aligned} [I]_{\sim_{\varphi,\psi,\mu}} &= \{J \in \mathbb{I} | J \sim_{\varphi,\psi,\mu} I\} \\ &= \{J \in \mathbb{I} | J(x,y) = \varphi(I(\psi(x), \mu(y))) \text{ for some } \varphi, \psi, \mu \in \Phi\} \\ &= \{\varphi(I(\psi(x), \mu(y))) \mid \varphi, \psi, \mu \in \Phi\} \ . \end{aligned} \tag{16}$$

Now, we are ready to position the general bijective transformations (7) in an algebraic setting.

5.1. The bijective transformation $\varphi(I(\psi(x), \mu(y)))$ and actions of $\mathbb{S}$ on $\mathbb{I}$

In our quest towards answering **Question 2**, we propose two functions from $\mathbb{S} \times \mathbb{I} \longrightarrow \mathbb{I}$. One of these turns out to be a group action of $\mathbb{S}$ on $\mathbb{I}$, while the other is an anti-group action. The important role played by them will become apparent presently.

Definition 5.3. Let $\sqsubset\colon \mathbb{I} \times \mathbb{S} \longrightarrow \mathbb{I}$ be defined by

$$I \sqsubset K = I \circledast K, \qquad K \in \mathbb{S}, I \in \mathbb{I} \ . \tag{17}$$

Lemma 5.4. *$\sqsubset$ is a right group action of $\mathbb{S}$ on $\mathbb{I}$.*

Proof. We show that $\sqsubset$ is a right group action of $\mathbb{S}$ on $\mathbb{I}$. For this purpose, let $I \in \mathbb{I}$ and $K_1, K_2 \in \mathbb{S}$

(i) Now, $(I \sqsubset K_1) \sqsubset K_2 = (I \circledast K_1) \sqsubset K_2 = I \circledast K_1 \circledast K_2 = I \circledast (K_1 \circledast K_2) = I \sqsubset (K_1 \circledast K_2)$.
(ii) $I \sqsubset I_{\mathbf{D}} = I \circledast I_{\mathbf{D}} = I$, for all $I \in \mathbb{I}$.

Thus $\sqsubset$ is a right group action. □

Definition 5.5. Define $\sim_{\sqsubset}$ on $\mathbb{I}$ by $I \sim_{\sqsubset} J \Longleftrightarrow J = I \sqsubset K = I \circledast K$ for some $K \in \mathbb{S}$.

It is easy to verify that $\sim_{\sqsubset}$ is an equivalence relation.

Remark 5.6. Let $I \in \mathbb{I}$. If $[I]_{\sqsubset}$ denotes the equivalence class containing I, then

$$\begin{aligned} [I]_{\sqsubset} &= \{J \in \mathbb{I} | J \sim_{\sqsubset} I\} \\ &= \{J \in \mathbb{I} | J = I \circledast K \text{ for some } \ K \in \mathbb{S}\} \\ &= \{J \in \mathbb{I} | J(x,y) = I(x, K(x,y)) \text{ for some } \ K \in \mathbb{S}\} \\ &= \{J \in \mathbb{I} | J(x,y) = I(x, \varphi(y)) \text{ for some } \ \varphi \in \Phi\} \\ &= \{I(x, \varphi(y)) | \varphi \in \Phi\} \ . \end{aligned}$$

5.1.1. An anti-group action of $\mathbb{S}$ on $\mathbb{I}$

Similar to the concept of anti-homomorphism [5], we recall the definition of an anti-group action in the following.

Definition 5.7 ([5]). Let $(G, *)$ be a group with identity e and S a non-empty set. A map $\circ\colon G \times S \longrightarrow S$ is called anti-group action if for all $g_1, g_2 \in G, s \in S$ the map $\circ$ satisfies the following:

(i) $g_1 \circ (g_2 \circ s) = (g_2 * g_1) \circ s$.

(ii) $e \circ s = s$.

Firstly, note that, in the case G is abelian, then an anti-group action becomes a left-group action. Further, as the following result shows, anti-group actions can also give rise to equivalence classes and hence partition the set S on which they act.

Lemma 5.8. *Let $\circ$ be an anti-group action of $(G, *)$ on a non-empty set S. Let $s, t \in S$. Define $s \sim t$ if and only if $t = g \circ s$ for some $g \in G$. Then $\sim$ is an equivalence relation.*

Proof. We prove only the transitivity of $\sim$, since the proof of reflexivity and symmetry are easy to obtain. Let $s, t, u \in S$ and $s \sim t, t \sim u$. Then it follows that $t = g \circ s$ and $u = h \circ t$ for some $g, h \in G$. Now, $u = h \circ t = h \circ (g \circ s) = (g * h) \circ s = g' \circ s$ for some $g' \in G$. Thus $s \sim u$ and $\sim$ is an equivalence relation. $\square$

Now, in the following, we define a function $\sqsupset$ and show that it is an anti-group action of $\mathbb{S}$ on $\mathbb{I}$.

Lemma 5.9. *Let $\sqsupset\colon \mathbb{S} \times \mathbb{I} \longrightarrow \mathbb{I}$ defined by*

$$K \sqsupset I = (I \Delta K) \circledast K^{-1}, \qquad K \in \mathbb{S}, I \in \mathbb{I}\ , \tag{18}$$

is an anti-group action of $\mathbb{S}$ on $\mathbb{I}$.

Proof. (i) Let $I \in \mathbb{I}$ and $K_1, K_2 \in \mathbb{S}$. Then

$$\begin{aligned} K_1 \sqsupset (K_2 \sqsupset I) &= K_1 \sqsupset ((I \Delta K_2) \circledast K_2^{-1}) \\ &= ((I \Delta K_2) \circledast K_2^{-1} \Delta K_1) \circledast K_1^{-1} \end{aligned}$$

Since $K_1, K_2 \in \mathbb{S}$, from Theorem 2.4, it follows that, K_1, K_2 are of the following:

$$K_i(x, y) = \begin{cases} 1, & \text{if } x = 0\ , \\ \varphi_i(y), & \text{if } x > 0\ , \end{cases} \qquad \text{for } i = 1, 2,$$

where $\varphi_i \in \Phi$.

- Let $x = 0$. Then $(K_1 \sqsupset (K_2 \sqsupset I))(0, y) = 1 = ((K_2 \circledast K_1) \sqsupset I)(1, y)$, for all $y \in [0, 1]$.

- Let $x > 0$. Then

$$
\begin{aligned}
(K_1 \sqsupset (K_2 \sqsupset I))(x,y) &= (((I\Delta K_2) \circledast K_2^{-1}\Delta K_1) \circledast K_1^{-1})(x,y) \\
&= ((I\Delta K_2) \circledast K_2^{-1}\Delta K_1)(x, K_1^{-1}(x,y)) \\
&= ((I\Delta K_2) \circledast K_2^{-1}\Delta K_1)(x, \varphi_1^{-1}(y))) \\
&= ((I\Delta K_2) \circledast K_2^{-1})(K_1(1,x), K_1(x,\varphi_1^{-1}(y))) \\
&= ((I\Delta K_2) \circledast K_2^{-1})(\varphi_1(x), y) \\
&= ((I\Delta K_2)(\varphi_1(x), K_2^{-1}(\varphi_1(x), y)) \\
&= (I\Delta K_2)(\varphi_1(x), \varphi_2^{-1}(y)) \\
&= I(K_2(1,\varphi_1(x)), K_2(\varphi_1(x), \varphi_2^{-1}(y))) \\
&= I(\varphi_2(\varphi_1(x)), y) \ , \\
\text{while, } ((K_2 \circledast K_1) \sqsupset I)(x,y) &= (I\Delta(K_2 \circledast K_1) \circledast (K_2 \circledast K_1)^{-1})(x,y) \\
&= (I\Delta(K_2 \circledast K_1) \circledast K_1^{-1} \circledast K_2^{-1})(x,y) \\
&= (I\Delta(K_2 \circledast K_1) \circledast K_1^{-1})(x, \varphi_2^{-1}(y)) \\
&= (I\Delta(K_2 \circledast K_1)(x, \varphi_1^{-1}(\varphi_2^{-1}(y))) \\
&= I((K_2 \circledast K_1)(1,x), (K_2 \circledast K_1)(x, \varphi_1^{-1}(\varphi_2^{-1}(y)))) \\
&= I(\varphi_2(\varphi_1(x)), y) \ .
\end{aligned}
$$

Thus we have proved that $(K_1 \sqsupset (K_2 \sqsupset I))(x,y) = ((K_2 \circledast K_1) \sqsupset I)(x,y)$, for all $x > 0$.

Thus in all cases we have shown that $K_1 \sqsupset (K_2 \sqsupset I) = (K_2 \circledast K_1) \sqsupset I$, for all $K_1, K_2 \in \mathbb{S}$ and $I \in \mathbb{I}$.

(ii)Let $I \in \mathbb{I}$. Then $I_{\mathbf{D}} \sqsupset I = (I\Delta I_{\mathbf{D}}) \circledast I_{\mathbf{D}}^{-1} = I\Delta I_{\mathbf{D}} = I$, follows from Remark 3.5, and hence $\sqsupset$ is an anti-group action. □

Definition 5.10. Let $I, J \in \mathbb{I}$. Then the relation defined as follows is an equivalence relation:

$$I \sim_{\square} J \text{ if and only if } J = K_1 \sqcup ((K_3 \sqsupset I) \sqsubset K_2) \text{ for some } K_1, K_2, K_3 \in \mathbb{S}. \tag{19}$$

In fact, by expanding the above J as follows:

$$
\begin{aligned}
J &= K_1 \sqcup ((K_3 \sqsupset I) \sqsubset K_2) = K_1 \circledast ((K_3 \sqsupset I) \sqsubset K_2) \\
&= K_1 \circledast ((K_3 \sqsupset I) \circledast K_2) = K_1 \circledast (I\Delta K_3) \circledast K_3^{-1} \circledast K_2 \ ,
\end{aligned}
$$

(19) becomes

$$I \sim_{\square} J \text{ if and only if } J = K_1 \circledast (I\Delta K_3) \circledast K_3^{-1} \circledast K_2, \text{ for some } K_1, K_2, K_3 \in \mathbb{S}. \tag{20}$$

Theorem 5.11. *The equivalence classes of fuzzy implications as given in* (16) *are exactly the equivalence classes obtained from the relation* $\sim_{\square}$ *in* (20)*, i.e., for any* $I \in \mathbb{I}$, $[I]_{\sim_{\varphi,\psi,\mu}} = [I]_{\sim_{\square}}$.

Proof. Let $I \in \mathbb{I}$. Then

$$\begin{aligned}
[I]_{\sim_\square} &= \{J \in \mathbb{I} | J \sim_\square I\} \\
&= \{J \in \mathbb{I} | J = K_1 \circledast (I \Delta K_3) \circledast K_3^{-1} \circledast K_2 \text{ for some } K_1, K_2, K_3 \in \mathbb{S}\} \\
&= \{J \in \mathbb{I} | J(x,y) = (K_1 \circledast (I \Delta K_3) \circledast K_3^{-1} \circledast K_2)(x,y) \text{ for all } x, y \in [0,1]\} \\
&= \{J \in \mathbb{I} | J(x,y) = K_1(x, ((I \Delta K_3) \circledast K_3^{-1} \circledast K_2)(x,y)) \text{ for all } x, y \in [0,1]\} \\
&= \{J \in \mathbb{I} | J(x,y) = K_1(x, ((I \Delta K_3)(x, (K_3^{-1} \circledast K_2)(x,y)))) \text{ for all } x, y \in [0,1]\} \\
&= \{J \in \mathbb{I} | J(x,y) = K_1(x, I(K_3(1,x), K_3(x, (K_3^{-1} \circledast K_2)(x,y)))) \text{ for all } x, y \in [0,1]\} \\
&= \{J \in \mathbb{I} | J(x,y) = K_1(x, (I(K_3(1,x), K_2(x,y)))) \text{ for all } x, y \in [0,1]\} \\
&= \{J \in \mathbb{I} | J(x,y) = \begin{cases} 1, & \text{if } x = 0 \ , \\ \varphi(I(\psi(x), \mu(y))), & \text{if } x > 0 \ , \end{cases} \} \\
&= \{J \in \mathbb{I} | J(x,y) = \varphi(I(\psi(x), \mu(y))) \text{ for some } \varphi, \psi, \mu \in \Phi\} = [I]_{\sim_{\varphi,\psi,\mu}} \ . \qquad \square
\end{aligned}$$

In other words, Theorem 5.11 shows that any general bijective transformation of the form (7) can be represented by a composition of group actions and an anti-group action of $\mathbb{S}$ on $\mathbb{I}$ and provides an answer to **Question 2**.

5.2. General bijective transformations - Some special cases and group actions

From the above discussion, we see that, in general, a general bijective transformation of the form (7) may not be representable as a group action, at least not in the setting of the monoid $(\mathbb{I}, \circledast)$ considered here, even though each of them gives rise to partitions on the set $\mathbb{I}$.

Note that every bijective transformation of the form (7) involves three functions $\varphi, \psi, \mu \in \Phi$, which in general are different. However, if there is some relationship between the functions $\varphi, \psi, \mu \in \Phi$ then one can investigate to see if these special cases relate to some group actions of $\mathbb{S}$ on $\mathbb{I}$.

For instance, let $\lambda \in \Phi$ and let **id** denote the identity bijection on $[0,1]$. Now, if we let

- $\varphi = \lambda^{-1}, \psi = \mu = \lambda$, we obtain the φ-conjugate transformation of Baczyński and Drewniak [1],
- $\varphi = \lambda, \psi = \mu = \mathbf{id}$, we obtain the $\varphi(I)$-transformation of Jayaram and Mesiar [6],
- $\varphi = \lambda^{-1}, \psi = \mathbf{id}, \mu = \lambda$, we obtain the φ-pseudo conjugate transformation of [9].

In Table 1, we list all the distinct possibilities when $\varphi, \psi, \mu \in \{\mathbf{id}, \lambda, \lambda^{-1}\}$, the corresponding functions $f : \mathbb{S} \times \mathbb{I} \to \mathbb{I}$ that give these bijective transformations and also indicate if they are a group action or not.

6. Concluding remarks

In this paper, we have recalled the three different bijective transformations on fuzzy implications that are defined to generate fuzzy implications in different contexts. These transformations could be seen as automorphisms on the groupoids $([0,1], I)$, where $I \in \mathbb{I}$, the set of fuzzy implications. However, unlike a triangular norm T, which imposes an integral monoid structure on the unit interval $[0,1]$, an $I \in \mathbb{I}$ does not impose any further regular algebraic structure on $[0,1]$ and hence the above transformations lack any algebraic connotation and preclude the possibility of gleaning some new insights on fuzzy implications.

Interestingly, the bijective transformations proposed by Vemuri and Jayaram [9] do have a clear algebraic connotation as being a group action on the set $\mathbb{I}$. Hence, we have investigated whether the bijective transformations that were proposed by Baczyński and Drewniak in [1] and Jayaram and Mesiar in [6] can be seen in an algebraic perspective. Towards answering this question, we conducted this study in the setting of the monoid $(\mathbb{I}, \circledast)$ and its subgroup $\mathbb{S}$ proposed in [9].

Our study has shown that the equivalence classes obtained from these bijective transformations are exactly the equivalence classes of fuzzy implications obtained from some group actions of $\mathbb{S}$, the subgroup

(φ, ψ, μ)	$J(x,y) = \varphi(I(\psi(x), \mu(y)))$	$f : \mathbb{S} \times \mathbb{I} \to \mathbb{I}$	Group Action
$(\mathbf{id}, \mathbf{id}, \mathbf{id})$	$I(x,y)$	$\mathbf{id}$	✓
$(\mathbf{id}, \lambda, \mathbf{id})$	$I(\lambda(x), y)$	$\sqsupset$	×
$(\mathbf{id}, \mathbf{id}, \lambda)$	$I(x, \lambda(y))$	$\sqsubset$	✓
$(\mathbf{id}, \lambda, \lambda)$	$I(\lambda(x), \lambda(x))$	$I\Delta K$	×
$(\mathbf{id}, \lambda^{-1}, \lambda)$	$I(\lambda^{-1}(x), \lambda(x))$	$(I\Delta K^{-1}) \circledast K \circledast K$	×
$(\lambda, \mathbf{id}, \mathbf{id})$	$\lambda(I(x,y)$	$\sqcup$	✓
$(\lambda, \mathbf{id}, \lambda)$	$\lambda(I(x, \lambda(y)))$	$K \circledast I \circledast K$	×
$(\lambda, \mathbf{id}, \lambda^{-1})$	$\lambda(I(x, \lambda^{-1}(y)))$	$K \circledast I \circledast K^{-1}$	✓
$(\lambda, \lambda, \mathbf{id})$	$\lambda(I(\lambda(x), y))$	$K \circledast (I\Delta K) \circledast K^{-1}$	×
$(\lambda, \lambda, \lambda)$	$\lambda(I(\lambda(x), \lambda(y)))$	$K \circledast (I\Delta K)$	×
$(\lambda, \lambda, \lambda^{-1})$	$\lambda(I(\lambda(x), \lambda^{-1}(y)))$	$K \circledast (I\Delta K) \circledast K^{-1} \circledast K^{-1}$	×
$(\lambda, \lambda^{-1}, \mathbf{id})$	$\lambda(I(\lambda^{-1}(x), y))$	$K \circledast (I\Delta K^{-1}) \circledast K$	×
$(\lambda, \lambda^{-1}, \lambda)$	$\lambda(I(\lambda^{-1}(x), \lambda(y)))$	$K \circledast (I\Delta K^{-1}) \circledast K \circledast K$	×
$(\lambda, \lambda^{-1}, \lambda^{-1})$	$\lambda(I(\lambda^{-1}(x), \lambda^{-1}(y)))$	$\sqcap$	✓

Table 1: General Bijective Transformations - Some special cases and whether they form group actions of $\mathbb{S}$ on $\mathbb{I}$

of the monoid $(\mathbb{I}, \circledast)$, on $\mathbb{I}$ as discussed in [9]. In other words, the bijective transformations proposed in [1] and [6] can be seen as particular group actions of $\mathbb{S}$ on $\mathbb{I}$, thus answering the **Question 1** completely.

Further, we have considered the most general bijective transformations of fuzzy implications as given in (7). Unfortunately, we have observed that not all bijective transformations of the form (7) lead to equivalence classes that are also obtained from some group actions of $\mathbb{S}$ on $\mathbb{I}$. However, we have shown that these bijective transformations can be seen as a composition of group actions $\sqsubset, \sqcup$ and $\sqsupset$, thus answering **Question 2**.

The results in this work not only show that any general bijective transformation can be seen in an algebraic perspective, but also highlight the important role played by the monoid structure on $\mathbb{I}$ proposed in [9, 10]. In [9], the group actions obtained while exploring the monoid structure on $\mathbb{I}$ led to some hitherto unknown representations of some families of fuzzy implications and the study of some homomorphisms on this monoid in [11] led to some new solutions to an iterative functional equation involving fuzzy implications. All these, perhaps, point to a greater need to study the set $\mathbb{I}$ from an algebraic point of view.

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