Pseudo-Differential Operators

Prashant

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भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Guide: Dr. D. Venku Naidu
Department of Mathematics

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APPROVAL SHEET

This thesis entitled "PSEUDO DIFFERENTIAL OPERATORS" by PRASHANT is approved for the degree of MASTER OF SCIENCE.

(Signature of the supervisor)

Dr Venku Naidu

(Name of the supervisor)

Date: 17/05/15

Place: IIT Hyderabad

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Declaration

I hereby declare that the matter embodied in this report is the result of investigation carried out by me in the Department of Mathematics, Indian Institute of Technology Hyderabad under the supervision of Dr. Venku Naidu Dogga.

In keeping with general practice of reporting scientific observations, due acknowledgement has been made wherever the work described is based on the findings of other investigators.

Venku Naidu 17/04/2015

Signature of the Supervisor
(Dr. Venku Naidu Dogga)

Name of the student: Prashant
Roll No.: MA13M1007

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Chapter 1

Discrete Fourier Transform

1.1 Basic Concepts

In this section we are going to study about the notations and the basic definition of Finite Fourier transform. We will start with the notation of a signal, then we will define Fourier transform, Fourier matrix, inverse Fourier transform and at the end translation of a signal in this section.

1.1.1 Signal

Let $\mathbb{C}$ be the set of complex numbers. For $N \geq 2$, $\mathbb{C}^{\mathbb{N}}$ represents a $N-$dimensional inner product space.
Now define a set $\mathbb{Z}_N = \{1, 2, ...N-1\}$ and a function $z : \mathbb{Z}_N \to \mathbb{C}^N$ Hence the function z can be viewed as a vector in $\mathbb{C}^N$. In other words, we can think of function $z : \mathbb{Z}_N \to \mathbb{C}^N$ as a finite sequence. If we let $L^2(\mathbb{Z}_N)$ be the set of all finite sequences then we can get $L^2(\mathbb{Z}_N) = \mathbb{C}^N$. These finite sequences, i.e. functions on $\mathbb{Z}_N$ are called as digital signals in electrical engineering. The set $\{e_1, e_2, \ldots, e_{N-1}\}$ is defines as :

$$e_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \qquad m = 0, 1, \ldots, N-1$$

where e_m has 1 in the m^{th} position and zeros elsewhere.

Proposition 1.1.1. $\{e_0, e_1, ..., e_{N-1}\}$ *is an orthonormal basis for* $L^2(\mathbb{Z}_N)$.

Now let $\{f_0, f_1, \ldots f_{N-1}\} \in L^2(\mathbb{Z}_N)$ defined as

$$F_m = \begin{pmatrix} f_m(0) \\ f_m(1) \\ \vdots \\ f_m(N-1) \end{pmatrix} \qquad m = 0, 1, \ldots, N-1$$

where

$$f_m(n) = \frac{1}{\sqrt{N}} e^{2\pi i m n/N} \qquad n = 0, 1, ...N-1.$$

Proposition 1.1.2. *$\{f_0, f_1, \ldots, f_{N-1}\}$ is an orthonormal basis for $L^2(\mathbb{Z}_N)$.*

Note 1.1.3. *$\{f_0, f_1 \ldots f_{N-1}\}$ is known as orthonormal Fourier basis for $L^2(\mathbb{Z}_N)$.*

Based on the above orthonormal basis ,we have the following :

Proposition 1.1.4. *For $z, w \in L^2(\mathbb{Z}_N)$ we have*

1. $z \quad = \sum_{n=0}^{N-1} \langle z, f_n \rangle f_n$
2. $\langle z, w \rangle = \sum_{n=0}^{N-1} \langle z, f_n \rangle \overline{\langle w, f_n \rangle}$
3. $\|z\|^2 \;\; = \sum_{n=0}^{N-1} |\langle w, f_n \rangle|^2.$

Definition 1.1.5. *Let $z \in L^2(\mathbb{Z}_N)$ then for $m = 0, 1, \ldots, N-1$ we have*

$$\langle z, f_m \rangle = \frac{1}{\sqrt{\mathbb{N}}} \sum_{n=0}^{N-1} e^{-2\pi imn/N} z(n).$$

1.1.2 Fourier and Inverse Fourier Transform

Definition 1.1.6. *Let $z \in L^2(\mathbb{Z}_N)$ then we define $\hat{z} \in L^2(\mathbb{Z}_N)$ as*

$$\hat{z} = \begin{pmatrix} \hat{z}(0) \\ \hat{z}(1) \\ \vdots \\ \hat{z}(N-1) \end{pmatrix}$$

where

$$\hat{z}(m) = \sum_{n=0}^{N-1} e^{-2\pi imn/N} z(n) \qquad , m = 0, 1, \ldots, N-1$$

Theorem 1.1.7. *For $z, w \in L^2(\mathbb{Z}_N)$ we have*

1. ***The Fourier Inversion Formula***

$$z(m) = \frac{1}{N} \sum_{n=0}^{N-1} \hat{z}(n) e^{2\pi imn/N}$$

2. ***Parseval's Identity***

$$\langle z, w \rangle = \frac{1}{N} \sum_{n=0}^{N-1} \hat{z}(n) \overline{\hat{w}(n)} = \frac{1}{N} \langle \hat{z}, \hat{w} \rangle$$

3. ***Plancherel's Formula***

$$\|z\|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |\hat{z}(n)|^2.$$

To understand more about the Fourier inversion formula we have following :

Definition 1.1.8. *Let* $\{F_0, F_1, \dots F_{N-1}\} \in L^2(\mathbb{Z}_N)$ *defined as*

$$F_m = \begin{pmatrix} F_m(0) \\ F_m(1) \\ \vdots \\ F_m(N-1) \end{pmatrix} \qquad m = 0, 1, \dots, N-1$$

where

$$F_m = \frac{1}{N} e^{2\pi i m n/N} \quad n = 0, 1, ...N-1.$$

The set $\{F_0, F_1, \dots F_{N-1}\} \in L^2(\mathbb{Z}_N)$ is orthogonal with N signals in $L^2(\mathbb{Z}_N)$ and also known as Fourier basis for $L^2(\mathbb{Z}_N)$ Hence, the Fourier inversion formula will become

$$z = \sum_{n=0}^{N-1} \hat{z}(n) F_n$$

Proposition 1.1.9. *The matrix representation of the Fourier transform* $\mathcal{F}_{\mathbb{Z}_N} : L^2(\mathbb{Z}_N) \rightarrow L^2(\mathbb{Z}_N)$ *is given by*

$$F_{\mathbb{Z}_N} z(m) = \sum_{n=0}^{N-1} z(n) \omega_N^{mn}$$

where

$$\omega_N = e^{-2\pi i/N}.$$

So if we let Ω_N is the Fourier matrix then it is defined as $\Omega_N = \omega_N^{mn}$ where $0 \leq m, n \leq N-1$ i.e.

$$\Omega_N = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N & \omega_N^2 & \omega_N^3 & \dots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \omega_N^6 & \dots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega_N N-1 & \omega_N^{2(N-1)} & \omega_N^{3(N-1)} & \dots & \omega_N^{(N-1)(N-1)} \end{pmatrix}$$

and

$$\hat{z} = \Omega_N z.$$

Proposition 1.1.10. *Finite Fourier transform of a given signal* $z \in L^2(\mathbb{Z}_N)$ *is calculated by*

$$\hat{z} = \Omega_N z.$$

Examples 1.1.11. *Let* $N = 2$ *Then*

$$\Omega_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Definition 1.1.12. *Let* $z \in L^2(\mathbb{Z}_N)$ *given by*

$$z = \begin{pmatrix} 1 \\ 0 \\ 1 \\ i \end{pmatrix}$$

Then

$$\hat{z} = \Omega_4 z = \begin{pmatrix} 2+i \\ -1 \\ 2-i \\ 1 \end{pmatrix}$$

Definition 1.1.13. *Let $w \in L^2(\mathbb{Z}_N)$ then we define inverse Fourier transform $\breve{w} \in L^2(\mathbb{Z}_N)$ as*

$$\breve{w} = \begin{pmatrix} \breve{w}(0) \\ \breve{w}(1) \\ \vdots \\ \breve{w}(N-1) \end{pmatrix}$$

where

$$\breve{w}(m) = \frac{1}{N}\sum_{n=0}^{N-1} w(n)e^{2\pi imn/N} \qquad , m = 0, 1, \ldots, N-1.$$

Proposition 1.1.14. *Let $z \in L^2(\mathbb{Z}_N)$ the $\breve{\hat{z}} = z$.*

Proposition 1.1.15. *The matrix representation of the inverse Fourier transform $\mathcal{F}_{\mathbb{Z}_N}^{-1} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is given by*

$$F_{\mathbb{Z}_N}^{-1}w(m) = \frac{1}{N}\sum_{n=0}^{N-1} w(n)\overline{\omega_N^{mn}} = \frac{1}{N}(\overline{\Omega_N}w)(m),$$

where $\overline{\Omega_N}$ is the conjugate transpose of the Fourier matrix.

$$F_{\mathbb{Z}_N}^{-1}w = \breve{w} = \frac{1}{N}\overline{\Omega_N}w, \qquad z \in L^2(\mathbb{Z}_N)$$

So the matrix of $\mathcal{F}_{\mathbb{Z}_N}^{-1} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is equal to $\frac{1}{N}\overline{\Omega_N}$.

Examples 1.1.16. *Let $N = 2$ Then*

$$\Omega_2^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Definition 1.1.17. *let w is given by*

$$w = \begin{pmatrix} 2+i \\ -1 \\ 2-i \\ 1 \end{pmatrix}$$

Then

$$z = \Omega_4^{-1}w = \begin{pmatrix} 1 \\ 0 \\ 1 \\ i \end{pmatrix}$$

Remark 1.1.18. $\{F_0, F_1, \ldots F_{N-1}\} \in L^2(\mathbb{Z}_N)$ *defined as*

$$F_m = \begin{pmatrix} F_m(0) \\ F_m(1) \\ \vdots \\ F_m(N-1) \end{pmatrix} \qquad m = 0, 1, \ldots, N-1$$

where

$$F_m = \frac{1}{N}e^{2\pi imn/N} \quad n = 0, 1, \ldots N-1$$

is the Fourier basis for $L^2(\mathbb{Z}_N)$.

.We assume that N is very large and even. For simplification, neglect the $\frac{1}{N}$ part in F_m and we assume only the real part of $e^{2\pi imn/N}$ i.e. $\cos(^{(2\pi mn)}/_N)$ If m =0 we will get the value 1 for $\forall\ n \in (\mathbb{Z}_N)$. For m=1 we will look at the graph of

$$f(x) = \cos(^{(2\pi x)}/_N)$$

on $[0, N]$ .So the resulting graph of the $f(x)$ has N evenly spaced sample point on one cycle. Similar,argument will hold for all values of $m = 0, 1, \ldots, {}^N/_2$ with m cycles. For the remaining values of m we will get the reverse implication. Hence wave F_m increases from 0 to ${}^N/_2$ and decrease from ${}^N/_2$ to N .So the frequency of the wave F_m is high when it is near to mid point and low when it is near to end points.

Remark 1.1.19. *From the above Remark we conclude that $\hat{z}(m)$ measures the amount of wave F_m that is needed in composing the signal z. If $|\hat{z}(m)|$ is big at m near ${}^N/_2$ then the signal z has strong high frequency components and if $|\hat{z}(m)|$ is big at 0 or $N-1$ then the signal z has strong low frequency components .*

1.1.3 Translation

Definition 1.1.20. *The signal z is said to be periodic if*

$$z(n+N) = z(n), \quad n \in \mathbb{Z}.$$

Definition 1.1.21. *Let z be a periodic function on $\mathbb{Z}$ with the period N then for every integer k*

$$\sum_{n=k}^{k+N-1} z(n) = \sum_{n=0}^{N-1} z(n).$$

Remark 1.1.22. *Let $z \in L^2(\mathbb{Z}_N)$ then the Fourier transform $\hat{z}$ and the inverse Fourier transform $\check{z}$ are periodic with period N.*

Definition 1.1.23. *Let $z \in L^2(\mathbb{Z}_N)$ and $k \in \mathbb{Z}$ then we define function $R_k z$ on $\mathbb{Z}$ by*

$$(R_k z)(n) = z(n-k) \quad n \in \mathbb{Z}.$$

Definition 1.1.24. *Let $z \in L^2(\mathbb{Z}_N)$ and $k \in \mathbb{Z}$ then $\forall\ m \in \mathbb{Z}$ we have*

$$(R_k z)^\wedge(m) = e^{-2\pi imk/N}\hat{z}(m).$$

Definition 1.1.25. *Let $z \in L^2(\mathbb{Z}_N)$ then we define $\bar{z} \in L^2(\mathbb{Z}_N)$ as*

$$\bar{z} = \begin{pmatrix} \overline{z(0)} \\ \overline{z(1)} \\ \vdots \\ \overline{z(N-1)} \end{pmatrix}$$

Proposition 1.1.26. *Let $z \in L^2(\mathbb{Z}_N)$ then*

$$\hat{\bar{z}}(m) = \overline{\hat{z}(-m)}.$$

Definition 1.1.27. *Let $z \in L^2(\mathbb{Z}_N)$ then z is said to be real if $\bar{z} = z$.*

Theorem 1.1.28. *Let $z \in L^2(\mathbb{Z}_N)$ then z is real iff*

$$\hat{z}(m) = \overline{\hat{z}(-m)}, \quad m \in \mathbb{Z}.$$

1.2 Operators and Fourier Multiplier

As we know that linear transformation plays an important role in Mathematics, In this chapter we are going to study about some operators on $L^2(\mathbb{Z}_N)$ and will try to find out the diagonalizablity of these operators. Mainly we are going to study about translation-invariant operator,circulant operator, convolution operator and Fourier multiplier and try to find out the diagonalizablity of these operators.

1.2.1 Translation-Invariant Operator

Definition 1.2.1. *A map $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is linear if*

$$A(az+bw) = aA(z) + bA(w) \;\; \forall \;\; z, w \in L^2(\mathbb{Z}_N) \text{ and } \; a\;, b \in \mathbb{C}.$$

A is said to be translation-invariant if

$$AR_k = R_kA \quad \forall\, k \in \mathbb{Z}.$$

Remark 1.2.2. *A translation-invariant linear operator A is the mathematical analog of a filters that transmit the signal in electrical engineering . Its function is to transmit the input signal z into the output signal Az in $L^2(\mathbb{Z}_N)$. As, A is a linear operator, so if we delay or advance an input signal by a certain amount, then the output signal should be delayed or advanced by the same amount.*

The chief result which in this section is as follows :

Theorem 1.2.3. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a translation-invariant linear operator then for $m = 0, 1, 2, \ldots, N-1, F_m$ is an eigenfunction of A.*

Corollary 1.2.2. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a translation invariant operator $F = \{F_0, F_1, \ldots, F_{N-1}\}$ be the Fourier basis for $L^2(\mathbb{Z}_N)$ Then the matrix A_F for A with respect to F is diagonal.*

To understand the above theorem let's take $\beta = \{z_0, z_1, \ldots, z_{N-1}$ is a a basis for $L^2(\mathbb{Z}_N)$. Then for $z \in L^2(\mathbb{Z}_N)$,we have

$$z = \sum_{k=0}^{N-1} a_k z_k.$$

where $a_i \in \mathbb{C}\ \forall\, i = 0, 1, 2, \ldots N-1.$

Definition 1.2.4. *We define $(z)_\beta$ by*

$$(z)_\beta = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}$$

and call z_β are the coordinate of z with respect to β.

Let A be the matrix corresponding the basis β and let $(A)_\beta = (\alpha_{mn}), m, n = 0, 1, \ldots, N-1$ and $\alpha_{mn} \in \mathbb{C}$, then

$$(A)_\beta (z)_\beta = \begin{pmatrix} \sum\limits_{k=0}^{N-1} a_k \alpha_{0k} \\ \sum\limits_{k=0}^{N-1} a_k \alpha_{1k} \\ \vdots \\ \sum\limits_{k=0}^{N-1} a_k \alpha_{(N-1),k} \end{pmatrix}$$

And when we will calculate, $(Az)_\beta$ we will get

$$(Az)_\beta = \begin{pmatrix} \sum_{k=0}^{N-1} a_k \alpha_{0k} \\ \sum_{k=0}^{N-1} a_k \alpha_{1k} \\ \vdots \\ \sum_{k=0}^{N-1} a_k \alpha_{(N-1),k} \end{pmatrix}$$

Remark 1.2.5. $(Az)_\beta = A_\beta z_\beta$.

Theorem 1.2.6. *$A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a translation-invariant operator iff A commutes with R_1.*

1.2.3 Circulant Matrices

Let $\{a_{mn}\}_{0 \leq m,n \leq N-1}$ be an $N \times N$ matrix. Then we define $a_{mn}\ \forall\, m, n \in \mathbb{Z}$ by periodic extension to all of $\mathbb{Z}$ in each of the variable in m and n we expect that

$$a_{m+N,n} = a_{mn}$$

and

$$a_{m,n+N} = a_{mn}.$$

From now onwards we assume the $N \times N$ matrix is periodic.

Definition 1.2.7. *Let $C = \{a_{mn}\}_{0 \leq m,n \leq N-1}$ be an $N \times N$ periodic matrix, then we say C to be circulant if*

$$a_{m+1,n+1} = a_{mn}\ \forall\, m, n \in \mathbb{Z}.$$

Let $C = \{a_{mn}\}_{0 \leq m,n \leq N-1}$ $N \times N$ be a circulant matrix, then $(n+1)^{th}$ column of the matrix is given as

$$\begin{pmatrix} a_{0,n+1} \\ a_{1,n+1} \\ \vdots \\ a_{m+1,n+1} \\ \vdots \\ a_{N-1,n+1} \end{pmatrix} = R_1 \begin{pmatrix} a_{0,n} \\ a_{1,n} \\ \vdots \\ a_{m+1,n} \\ \vdots \\ a_{N-1,n} \end{pmatrix}$$

Examples 1.2.8.

$$\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}$$

Theorem 1.2.9. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a translation-invariant linear operator Then A_S the matrix of A with respect to the standard basis is S circulant.*

Theorem 1.2.10. *Let A and B are two ciculant matrices then we have the following :*

1. *Product of A and B is again a circulant matrix.*
2. *Adjoint of A is again circulant.*
3. *A is translation-invariant.*
4. *A is normal.*

1.2.4 Convolution operator

Definition 1.2.11. *Let* $z, w \in L^2(\mathbb{Z}_N)$ *Then we define the signal* $z * w \in L^2(\mathbb{Z}_N)$ *as*

$$(z * w)(m) = \sum_{n=0}^{N-1} z(m-n)w(n), \quad m \in \mathbb{Z}.$$

We call $z * w$ is the convolution of the signals z and w.

Definition 1.2.12. *Let* $b \in L^2(\mathbb{Z}_N)$ *Then we define the convolution operator* $C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ *with the kernel b as*

$$C_b z = b * z, \quad z \in L^2(\mathbb{Z}_N).$$

Remark 1.2.13. $C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ *is a linear operator.*

Definition 1.2.14. *Let* $C = \{a_{mn}\}_{0 \le m,n \le N-1}$ *be an* $N \times N$ *circulant matrix, then* $\forall\, z \in L^2(\mathbb{Z}_N)$

$$C_b z = Cz,$$

where b is the first column of C i.e.

$$b = \begin{pmatrix} a_{0,0} \\ a_{1,0} \\ \vdots \\ a_{N-1,0} \end{pmatrix}$$

Proposition 1.2.15. $C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ *be a convolution operator with kernel b where* $b \in L^2(\mathbb{Z}_N)$*.Then* C_b *is translation invariant operator.*

Remark 1.2.16. $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ *be a linear operator then following are equivalent:*

1. *A is translation-invariant.*
2. *The matrix* A_S *with respect to standard basis is circulant.*
3. *A is convolution operator.*

Definition 1.2.17. *A signal* $\delta \in L^2(\mathbb{Z}_N)$ *is of the form*

$$\delta = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

is called as unit impulse. δ *has the property that for any* $z \in L^2(\mathbb{Z}_N)$,

$$z * \delta = z.$$

Proposition 1.2.18. *Let* $z, w \in L^2(\mathbb{Z}_N)$*. Then*

$$\widehat{z * w}(m) = \hat{z}(m)\hat{w}(m) \quad m \in \mathbb{Z}.$$

1.2.5 Fourier Multiplier

Definition 1.2.19. *Let $z, w \in L^2(\mathbb{Z}_N)$ Then we define $zw \in L^2(\mathbb{Z}_N)$ as*

$$zw = \begin{pmatrix} z(0)w(0) \\ z(0)w(0) \\ \vdots \\ z(N-1)w(N-1) \end{pmatrix}$$

Definition 1.2.20. *Let $\sigma \in L^2(\mathbb{Z}_N)$ and let $T_\sigma : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a mapping defined as*

$$T_\sigma z = (\sigma \hat{z})^\vee, \quad z \in L^2(\mathbb{Z}_N).$$

*T_σ is known as **Fourier multiplier or pseudo-differential operator**.*

Remark 1.2.21.

$$(T_\sigma z)^\wedge(m) = \sigma(m)\hat{z}(m), \quad m \in \mathbb{Z}.$$

Proposition 1.2.22. *Let $C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a convolution operator where $b \in L^2(\mathbb{Z}_N)$ Then*

$$C_b = T_\sigma,$$

where $\sigma = \hat{b}$ and vice versa

Theorem 1.2.23. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a linear operator, then A is Fourier multiplier iff A_F, the matrix of A with respect to the Fourier basis F is a diagonal matrix. Moreover,its diagonal entries are the values of σ.*

Conclusion

From the above four section we have the following remarks :

Remark 1.2.24. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a linear operator, then following are equivalent :*

1. *A is convolution operator.*
2. *A is a Fourier multiplier.*
3. *The matrix of A with respect to Fourier basis F, A_F is diagonal.*

Remark 1.2.25. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a linear operator, then following are equivalent :*

1. *A is translation-invariant linear operator.*
2. *The matrix A_S with respect to standard basis is circulant.*
3. *A is convolution operator.*
4. *A is a Fourier multiplier.*
5. *The matrix of A with respect to Fourier basis F, A_F is diagonal.*

The results obtained above can be used to obtained the eigenvalues of the filters given by translation-invariant linear operator.

Theorem 1.2.26. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a translation-invariant operator. Then the eigenvalues of A can be given by*

$$\sigma(0), \sigma(1), \ldots, \sigma(N-1),$$

where σ can be calculated by using the above remark.

1.2.6 The Fast Fourier Transform

The Fourier inversion formula given in chapter 1 gives the evaluation of signal $z \in L^2(\mathbb{Z}_N)$ as

$$(z)_F = \hat{z} = \Omega_N z = \Omega_N (z)_S.$$

But this evaluation require N^2 complex multiplication as all entries of Ω_N are nonzero. But when N is too large then it is very difficult to compute the signal z even by any computer. To overcome this problem, Cooley and Tukey gave a technic by dividing the Fourier matrix Ω_N given as follow:
Assume that N is powers of 2 i.e. $N = 2^l$ for some positive integer l.

$$\Omega_N = \begin{pmatrix} I_{N/2} & D_{N/2} \\ I_{N/2} & D_{N/2} \end{pmatrix} \begin{pmatrix} \Omega_{N/2} & 0 \\ 0 & \Omega_{N/2} \end{pmatrix} P_N,$$

where $I_{N/2}$ is the identity matrix of order $N/2, \Omega_{N/2}$ is the Fourier matrix with order $N/2$, $D_{N/2}$is the diagonal matrix with order $N/2$ and entries given by $1, \omega_N, \omega_N^2, \ldots, \omega_N^{N-2/2}$ and P_N is the permutation matrix obtained by permuting even before odd. We know that

$$\Omega_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

Then

$$\Omega_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & i \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 1.2.27. *The number of complex multiplication using fast Fourier transform is at most*

$$\frac{1}{2}Nl = \frac{1}{2}N \log_2 N.$$

1.3 Time - Frequency Analysis and Wavelets

In this chapter our main focus will be on time-frequency localized basis.We will show that we can not always find one signal in $L^2(\mathbb{Z}_N)$ with which we can get an orthonormal time-frequency localized basis. Although, by using two signals we can do that. As we know wavelet is an very important part of analysis. By using the the concept of time-frequency localized basis we will define the wavelets which includes mainly Haar wavelet and Daubechies wavelet. At the end, we will define the trace of a Fourier multiplier. **For the complete Chapter assume that N = 2M**.

1.3.1 Time-Frequency Localization

Definition 1.3.1. *Let z be a signal in $L^2(\mathbb{Z}_N)$.Then z is said to be time-localized near n_0 if all component $z(n)$ are 0 or relatively small except for a few values of n near n_0.*

Definition 1.3.2. *An othonormal basis β of $L^2(\mathbb{Z}_N)$ is said to be time-localized basis if every signal z in the basis β is time-localized.*

Benefits 1.3.3. *Let $\beta = \{z_0, z_1, \ldots, z_{N-1}\}$ be a time localized basis for $L^2(\mathbb{Z}_N)$.Then for any $z \in L^2(\mathbb{Z}_N)$, we have $\{a_0, a_1, \ldots, a_{N-1}\} \in \mathbb{C} \ni$*

$$z = \sum_{n=0}^{N-1} a_n z_n.$$

1. *Now let's assume, we want to study the signal z near the point n_0 .So to do this we just need to concentrate on the basis elements which are time-localized near n_0 and will ignore the rest and hence our sum reduces to a lower sum.This process is known as **Signal compression**.*

2. *Let us consider n to be a space variable instead a time variable.Suppose that a coefficient in the given sum is very big. Then by using a space-localized orthonormal basis,we can locate and concentrate on this big coefficient. This is the idea underlying medical imaging.*

Definition 1.3.4. *Let z be a signal in $L^2(\mathbb{Z}_N)$.Then z is said to be frequency-localized near n_0 if all component $\hat{z}(m)$ are 0 or relatively small except for a few values of n near n_0.*

Definition 1.3.5. *An orthonormal basis β of $L^2(\mathbb{Z}_N)$ is said to be frequency-localized basis if every signal z in the basis β is frequency-localized.*

Benefits 1.3.6. *The main benefits of a frequency-localized basis are as follow:*

1. *Fast Fourier transform can be performed by using frequency-localized basis.*

2. *Suppose we want to remove a high frequency component of a given signal without affecting adversely and quality of the resulting signal. Then we need to know which frequency to remove and this information is provided by frequency-localized basis mainly Fourier basis.*

Definition 1.3.7. *Let $z \in L^2(\mathbb{Z}_N)$, then the involution $z^*$of the signal z is defined as :*

$$z^*(n) = \overline{z(-n)}.$$

Proposition 1.3.8. *Let $z \in L^2(\mathbb{Z}_N)$, then*

$$\hat{z^*}(n) = \overline{\hat{z}(n)}.$$

Lemma 1.3.9. *Let $z, w \in L^2(\mathbb{Z}_N)$, then*

$$z * w = w * z.$$

Proposition 1.3.10. *Let $z, w \in L^2(\mathbb{Z}_N)$, then*

$$(w^*)^* = w. \tag{1.1}$$

$$(z * w^*)(m) = \langle z, R_m w \rangle. \tag{1.2}$$

$$(z * w)(m) = \langle z, R_m w^* \rangle. \tag{1.3}$$

Remark 1.3.11. *Let $\beta = \{R_0 w, R_1 w, \ldots, R_{N-1} w\}$ be an orthonormal basis obtained by successive translation of the signal z. A simple calculation show that the change in the basis from standard to β the coefficients of z with respect to β i.e. $(z)_\beta$ is given by*

$$\begin{pmatrix} (z * w^*)(0) \\ (z * w^*)(1) \\ \vdots \\ (z * w^*)(N-1) \end{pmatrix} = (z * w^*).$$

Lemma 1.3.12. *Let $z, w \in L^2(\mathbb{Z}_N)$, then*

$$\langle R_j z, R_k w\rangle = \langle z, R_{j-k}\rangle \quad \forall\, j,k \in \mathbb{Z}_N, j \leq k.$$

Theorem 1.3.13. *Let $w \in L^2(\mathbb{Z}_N)$, then $\{R_0 w, R_1 w, \dots, R_{N-1} w\}$ is an orthonormal basis for $L^2(\mathbb{Z}_N)$ iff*

$$|\hat{w}(m)| = 1, \quad m \in \mathbb{Z}.$$

Proof. We know that

$$\hat{\delta}(m) = 1 \quad m \in \mathbb{Z}.$$

Also $\{R_0 w, R_1 w, \dots, R_{N-1} w\}$ is an orthonormal basis for $L^2(\mathbb{Z}_N)$. So, we will get

$$\langle w, R_k w\rangle = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases},$$

which implies

$$\langle w, R_k w\rangle = \langle w * w^*\rangle(k) \quad k \in \mathbb{Z},$$

and hence

$$1 = \hat{\delta}(m) = \hat{w}(m)\hat{w^*}(m) = \hat{w}(m)\overline{\hat{w}(m)} = |\hat{w}(m)|^2.$$

In the same way, we can get the converse part. □

Remark 1.3.14. *Let $w \in L^2(\mathbb{Z}_N)$ be such that w is time localize near some point n_0 Then the ortonormal basis $\beta = \{R_0 w, R_1 w, \dots, R_{N-1} w\}$ is a time localized basis and also we can perform fast Fourier transform on this basis. But above theorem tells us that this basis need not be a frequency-localized basis.*

1.3.2 Time-Frequency Localized Basis

Definition 1.3.15. *Let N=2M Suppose that $\exists\, \phi, \varphi \in L^2(\mathbb{Z}_N) \ni \beta = \{R_0\phi, R_2\phi, \dots, R_{2M-2}\phi\} \cup \{R_0\varphi, R_2\varphi, \dots, R_{2M-2}\varphi\}$ is an orthonormal basis of $L^2(\mathbb{Z}_N)$. Then we call β is a time-frequency localized basis. The signals ϕ and φ are called **mother wavelet** and **father wavelet** for the time-frequency localized basis β of $L^2(\mathbb{Z}_N)$ respectively. We also write this basis as $\{R_{2j}\phi\}_{j=0}^{M-1} \cup \{R_{2j}\varphi\}_{j=0}^{M-1}$.*

Definition 1.3.16. *Let N=2M and $z \in L^2(\mathbb{Z}_N)$, then we define $z^+ \in L^2(\mathbb{Z}_N)$ by*

$$z^+(n) = (-1)^n z(n) \quad n \in \mathbb{Z}.$$

Proposition 1.3.17. *Let $z \in L^2(\mathbb{Z}_N)$, then*

$$(z^+)^\wedge(m) = \hat{z}(m + M).$$

Remark 1.3.18. *Let $z \in L^2(\mathbb{Z}_N)$, then $\forall\, n \in \mathbb{Z}$ we have*

$$(z + z^+)(n) = 2z(n).$$

Lemma 1.3.19. *Let $\phi \in L^2(\mathbb{Z}_N)$, then $\{R_0\phi, R_2\phi, \dots, R_{2M-2}\phi\}$ is an orthonormal set with M distinct signals iff*

$$|\hat{\phi}(m)|^2 + |\hat{\phi}(m+M)|^2 = 2, \quad m = 0, 1, \dots, M-1.$$

Definition 1.3.20. *Let $\phi, \varphi \in L^2(\mathbb{Z}_N)$, then $\forall\, m \in \mathbb{Z}$, we define the 2×2 matrix $A_{\phi,\varphi}(m)$ as*

$$A_{\phi,\varphi}(m) = \frac{1}{\sqrt{N}} \begin{pmatrix} \hat{\phi}(m) & \hat{\varphi}(m) \\ \hat{\phi}(m+M) & \hat{\varphi}(m+M) \end{pmatrix}.$$

We call this $A_{\phi,\varphi}(m)$ matrix is the system matrix of ϕ and φ at the integer m.

Theorem 1.3.21. *Let $\phi, \varphi \in L^2(\mathbb{Z}_N)$, then the set $\{R_{2j}\phi\}_{j=0}^{M-1} \cup \{R_{2j}\varphi\}_{j=0}^{M-1}$ is a time frequency localized basis for $L^2(\mathbb{Z}_N)$ iff $A_{\phi,\varphi}(m)$ is a unitary matrix for m= 0,1,2,. . . ,M-1. Equivalently, β is a time-frequency localized basis of $L^2(\mathbb{Z}_N)$ iff*

$$\hat{\phi}(m)|^2 + |\hat{\phi}(m+M)|^2 = 2. \tag{1.4}$$

$$|\hat{\varphi}(m)|^2 + |\hat{\varphi}(m+M)|^2 = 2. \tag{1.5}$$

$$\hat{\phi}(m)\overline{\hat{\varphi}(m)} + \hat{\phi}(m+M)\overline{\hat{\varphi}(m+M)} = 0. \tag{1.6}$$

Proof. Suppose β is the orthonormal basis for $L^2(\mathbb{Z}_N)$ and hence $\{R_{2j}\phi\}_{j=0}^{M-1}$ and $\{R_{2j}\varphi\}_{j=0}^{M-1}$ are orthonormaal sets with M distinct signals in $L^2(\mathbb{Z}_N)$ and hence from last lemma equation first and second follows. Now to prove equation third, orthonormality of β gives,

$$\langle \phi, R_{2k} \rangle = 0 \quad k = 0, 1, \ldots, M-1,$$

and hence from proposition,

$$(\phi * \varphi^*) = \langle \phi, R_{2k}\varphi \rangle = 0 \quad k = 0, 1, \ldots, M-1,$$

and hence by using remark, we will get

$$(\phi * \varphi^*) + (\phi * \varphi^*)^+ = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and by taking the Fourier transform, we will get

$$\hat{\phi}(m)\overline{\hat{\varphi}(m)} + \hat{\phi}(m+M)\overline{\hat{\varphi}(m+M)} = 0.$$

Conversely, let us assume all three equations holds, then by lemma $\{R_{2j}\phi\}_{j=0}^{M-1}$ and $\{R_{2j}\varphi\}_{j=0}^{M-1}$ are orthonormaal sets with M distinct signals in $L^2(\mathbb{Z}_N)$. Now by using periodicity argument, we will get that

$$\hat{\phi}(m)\overline{\hat{\varphi}(m)} + \hat{\phi}(m+M)\overline{\hat{\varphi}(m+M)} = 0 \quad m \in \mathbb{Z},$$

and hence

$$\{(\phi * \varphi^*) + (\phi * \varphi^*)^+\}^\wedge = 0 \quad m \in \mathbb{Z}.$$

Then

$$(\phi * \varphi^*) + (\phi * \varphi^*)^+ = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and hence

$$\langle \phi, R_{2k}\varphi \rangle = (\phi * \varphi^*)(2k),$$

so we will get $j, k \in \{0, 1, \ldots, M-1\}$, $j \leq k$

$$\langle R_{2j}\phi, R_{2k}\varphi \rangle = \langle \phi, R_{2k-2j}\varphi \rangle = 0.$$

Thus $\beta = \{R_{2j}\phi\}_{j=0}^{M-1} \cup \{R_{2j}\varphi\}_{j=0}^{M-1}$ is an orthonormal set in $L^2(\mathbb{Z}_N)$ with N distinct signals and hence is an orthonormal basis for $L^2(\mathbb{Z}_N)$. □

Remark 1.3.22. *So the relation between mother and father wavelet can be obtained as follow :*

$$|\hat{\phi}(m_0)|^2 = 2,$$

and

$$|\hat{\phi}(m_0 + M)|^2 = 0.$$

For some integer $m_0 \in \mathbb{Z}$,we will have

$$|\hat{\varphi}(m_0 + M)|^2 = 2,$$

and hence

$$|\hat{\varphi}(m_0)|^2 = 0.$$

This means that for an integer m_0 the amount of the wave F_{m_0} in the father wavelet φ is 0, while the amount of the same wave F_{m_0} in the mother wavelet ϕ is full that is 2. So, we can construct the mother wavelet and the father wavelet in such a way that ϕ contains only low frequency wave and φ contains only high frequency wave.

1.3.3 Wavelet Transform and Filter Banks

Let $\beta = \{R_{2j}\phi\}_{j=0}^{M-1} \cup \{R_{2j}\varphi\}_{j=0}^{M-1}$ is a time frequency localized basis for $L^2(\mathbb{Z}_N)$, with ϕ as mother wavelet and φ as father wavelet. Let $z \in L^2(\mathbb{Z}_N)$ and β is an orthonormal basis for $L^2(\mathbb{Z}_N)$, then

$$z = \sum_{k=0}^{M-1} \langle z, R_{2k}\phi\rangle R_{2k}\phi + \sum_{k=0}^{M-1} \langle z, R_{2k}\varphi\rangle R_{2k}\varphi,$$

so

$$(z)_\beta = \begin{pmatrix} \langle z, R_0\phi\rangle \\ \langle z, R_2\phi\rangle \\ \vdots \\ \langle z, R_{2M-2}\phi\rangle \\ \langle z, R_0\varphi\rangle \\ \langle z, R_2\varphi\rangle \\ \vdots \\ \langle z, R_{2M-2}\varphi\rangle \end{pmatrix} = \begin{pmatrix} (z * \phi^*)(0) \\ (z * \phi^*)(2) \\ \vdots \\ (z * \phi^*)(2M-2) \\ (z * \varphi^*)(0) \\ (z * \varphi^*)(2) \\ \vdots \\ (z * \phi^*)(2M-2) \end{pmatrix} .(*)$$

Let $V_{\phi,\varphi}$ be the $N \times N$ matrix given by

$$V_{\phi,\varphi} = \{R_0\phi|R_2\phi|\dots|R_{2M-2}\phi|, |R_0\varphi||R_2\varphi|\dots|R_{2M-2}\varphi|\}.$$

Thus, we get

$$z = (z)_\beta = V_{\phi,\varphi}(z)_\beta,$$

or

$$(z)_\beta = V_{\phi,\varphi}^{-1} z \quad z \in L^2(\mathbb{Z}_N).$$

Definition 1.3.23. *Let $W_{\phi,\varphi} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be defined as*

$$W_{\phi,\varphi} = V_{\phi,\varphi}^{-1}.$$

Then we call $W_{\phi,\varphi}$ is the wavelet transform associate to the mother wavelet ϕ and the father wavelet φ.

Remark 1.3.24. *The wavelet Transform $W_{\phi,\varphi}$ is the change of the basis from standard basis to a time-frequency localized basis of $L^2(\mathbb{Z}_N)$ generated by ϕ and φ.*

Remark 1.3.25. *As $\beta = \{R_{2j}\phi\}_{j=0}^{M-1} \cup \{R_{2j}\varphi\}_{j=0}^{M-1}$ is an orthonormal time-frequency localized basis of $L^2(\mathbb{Z}_N)$, which implies the matrix $V_{\phi,\varphi}$ is a unitary matrix. It follows that*

$$W_{\phi,\varphi} = V_{\phi,\varphi}^{-1} = V_{\phi,\varphi}^{*}.$$

Hence, an explicit formula for the wavelet transform $W_{\phi,\varphi}$ is available.

$$(z)_\beta = W_{\phi,\varphi} z.$$

As we know from our earlier discussion the computation of z_β requires N^2 complex multiplication which is very difficult to do as N is very large. So we always try to use the formula given in $(*)$ for the computation of (z_β) as we can perform Fast Fourier Transform on $(*)$. Now we are going to study more about $(*)$.

Definition 1.3.26. ***Downsampling Operator*** *: A linear operator $D : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_M)$ is defined as*

$$(Dz)(n) = z(2n) \qquad n = 0, 1, , \dot{M} - 1 \;\; \forall z \in L^2(\mathbb{Z}_N).$$

Examples 1.3.27. *Let $z \in L^2(\mathbb{Z}_N$ is given by*

$$z = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \\ 6 \\ 5 \\ 8 \\ 7 \end{pmatrix}.$$

Then Dz is a signal in $L^2(\mathbb{Z}_M)$ given by

$$Dz = \begin{pmatrix} 2 \\ 3 \\ 6 \\ 8 \end{pmatrix}.$$

Thus the linear operator D discard the values at evaluated as the odd integers. In Engineering Language D is written as $\downarrow 2$. By using the the downsampling operator the computation of $(z)_\beta$, $z \in L^2(\mathbb{Z}_M)$ is given schematically by the following process.

$$z \mapsto \begin{cases} z * \phi^* \mapsto D(z * \phi^*) \\ z * \varphi^* \mapsto D(z * \varphi^*) \end{cases} \mapsto \begin{cases} D(z * \phi^*) \\ D(z * \varphi^*) \end{cases} = (z)_\beta.$$

The above process is known as filter bank in multi-rated signal analysis or subband coding. This filter bank is use to compute $W_{\phi,\varphi} z$ in $L^2(\mathbb{Z}_M)$ by electrical engineer people.

Theorem 1.3.28. *Let ϕ and φ are the mother and father wavelet of a time-frequency localized basis of $L^2(\mathbb{Z}_M)$ then*

$$W_{\phi,\varphi} z = \begin{cases} D(z * \phi^*) \\ D(z * \varphi^*) \end{cases}, \qquad z \in L^2(\mathbb{Z}_M).$$

Definition 1.3.29. ***Upsampling Operator*** *: A linear operator $U : L^2(\mathbb{Z}_M) \to L^2(\mathbb{Z}_N)$ is defined as*

$$(Uz)(n) = \begin{cases} z(\frac{n}{2}) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

The operator U double the size of z by inserting 0 after every entry of z. In engineering language denoted by $\uparrow 2$ .This is a filter bank to compute the inverse wavelet transform $W_{\phi,\varphi}^{-1}z$ for $z \in L^2(\mathbb{Z}_N)$.

Examples 1.3.30. *Lets look at the signal*

$$w = \begin{pmatrix} 2 \\ 3 \\ 6 \\ 8 \end{pmatrix},$$

which is obtained after the downsampling the signal

$$z = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \\ 6 \\ 5 \\ 8 \\ 7 \end{pmatrix}.$$

Now upsampling of w is given by

$$UDz = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 6 \\ 0 \\ 8 \\ 0 \end{pmatrix}.$$

As we can see $UDz \neq z$. In fact

$$UDz = \frac{1}{2}(z + z^+).$$

So now we can give a two phase filter bank in which, the first phase is consist of analysis of a signal $z \in L^2(\mathbb{Z}_N)$ using the wavelet transform $W_{\phi,\varphi}z$ and the second phase is the reconstruction of the signal by using the inverse wavelet transform $W_{\phi,\varphi}^{-1}z$.

Theorem 1.3.31. *Let $z \in L^2(\mathbb{Z}_N)$ then*

$$z \mapsto \begin{cases} z * \phi^* \mapsto D(z * \phi^*) \mapsto UD(z * \phi^*) \mapsto \phi * D(z * \phi^*) \\ z * \varphi^* \mapsto D(z * \varphi^*) \mapsto UD(z * \varphi^*) \mapsto \varphi * D(z * \varphi^*) \end{cases} + \mapsto z,$$

where the final step in the filter bank is given by

$$z = \{\phi * D(z * \phi^*)\} + \{\varphi * D(z * \varphi^*)\}.$$

Proof. Let

$$w = \begin{pmatrix} w(0) \\ w(1) \\ \vdots \\ w(N-1) \end{pmatrix}.$$

Then by using the definition of downsampling operator

$$Dw = \begin{pmatrix} w(0) \\ w(2) \\ \vdots \\ w(2M-2) \end{pmatrix}.$$

Now by using the definition of upsampling operator we will get

$$UDw = \begin{pmatrix} w(0) \\ 0 \\ w(2) \\ 0 \\ \vdots \\ w(2M-2) \\ 0 \end{pmatrix} = \frac{1}{2}(w + w^+),$$

thus

$$UD(z * \phi^*) = \frac{1}{2}\{(z * \phi^*) + (z * \phi^*)^+\} \quad m \in \mathbb{Z},$$

if we take finite Fourier transform on both side we will get

$$\{UD(z * \phi^*)\}^\wedge = \hat{\phi}(m)\frac{1}{2}\{\hat{z}(m)\overline{\hat{\phi}(m)} + \hat{z}(m+M)\overline{\hat{\phi}(m+M)}\} \quad m \in \mathbb{Z} \quad (1*).$$

Similarly,

$$\{UD(z * \varphi^*)\}^\wedge = \hat{\varphi}(m)\frac{1}{2}\{\hat{z}(m)\overline{\hat{\varphi}(m)} + \hat{z}(m+M)\overline{\hat{\varphi}(m+M)}\} \quad m \in \mathbb{Z} \quad (2*).$$

If we add the equations $(1*)$ and $(2*)$, and by using the fact that ϕ and φ are the mother and father wavelet of the the frequency localized basis i.e.

$$|\hat{\phi}(m)|^2 + |\hat{\varphi}(m)|^2 = 2,$$

and

$$\hat{\phi}(m)\hat{\phi}(m+M) + \hat{\varphi}(m)\hat{\varphi}(m+M) = 0,$$

we will get

$$\{\{UD(z * \phi^*)\} + \{UD(z * \varphi\}\}^\wedge = \hat{z}(m),$$

and hence by using the inverse Fourier formula we will get

$$\{\phi * UD(z * \phi^*)\} + \{\varphi * UD(z * \varphi\} = z(m) \quad \forall m \in \mathbb{Z}.$$

□

Hence from the Theorem above, we can compute inverse wavelet transform $W^{-1}_{\phi,\varphi}$ by

$$\begin{cases} UD(z * \phi^*) \mapsto \phi * D(z * \phi^*) \\ UD(z * \varphi^*) \mapsto \varphi * D(z * \varphi^*) \end{cases} + \mapsto z.$$

Remark 1.3.32. *The upsampling operator and the downsampling operators are conjugate to each other i.e.*

$$\langle Dz, w\rangle = \langle z, Uw\rangle \quad \forall z, w \in L^2(\mathbb{Z}_N).$$

1.3.4 Haar Wavelet

Let $N = 2^l$ where l is a positive integer and let $z \in L^2(\mathbb{Z}_N)$, defined as

$$z = \begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix}.$$

Then we have the following definitions :

Definition 1.3.33. *Let a be defined as*

$$a = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{M-1} \end{pmatrix},$$

where

$$a_n = \frac{z(2n) + z(2n+1)}{\sqrt{2}}, \quad n = 0, 1, , \dot{M} - 1.$$

*This type of signal is known as **trend** of the signal z.*

Definition 1.3.34. *Let a be defined as*

$$d = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{M-1} \end{pmatrix},$$

where

$$a_n = \frac{z(2n) - z(2n+1)}{\sqrt{2}}, \quad n = 0, 1, , \dot{M} - 1.$$

*This type of signal is known as **fluctuation** of the signal z.*

Definition 1.3.35. *Let $W : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a linear operator, defined as*

$$Wz = \begin{pmatrix} a \\ d \end{pmatrix} \quad , z \in L^2(\mathbb{Z}_N).$$

Where a is the trend of the signal and d is the fluctuation of the signal $z \in L^2(\mathbb{Z}_N)$. This linear operator is known as **Haar transform**. More precisely, it is known as first level Haar transform.

$$W^{-1} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{M-1} \\ d_0 \\ d_1 \\ \vdots \\ d_{M-1} \end{pmatrix} = \begin{pmatrix} (a_0 + d_0)/\sqrt{2} \\ (a_0 - d_0)/\sqrt{2} \\ (a_1 + d_1)/\sqrt{2} \\ (a_1 - d_1)/\sqrt{2} \\ (a_2 + d_2)/\sqrt{2}\vdots \\ (a_{M-1} + d_{M-1})/\sqrt{2} \\ (a_{M-1} - d_{M-1})/\sqrt{2} \end{pmatrix}, \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{M-1} \\ d_0 \\ d_1 \\ \vdots \\ d_{M-1} \end{pmatrix} \in L^2(\mathbb{Z}_N).$$

In order to understand the Haar transform we will study following proposition.

Proposition 1.3.36. ***Small fluctuation Property:*** *The fluctuation is very small in the seance that*

$$d = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Proposition 1.3.37. ***Similar trend Property :*** *The trend behaves like the original signal.*

Remark 1.3.38. *As we had seen Haar transform splits the given signal into 2 parts, out of which 1 is trend and another one is fluctuation, but as we had seen the behavior of the fluctuation signal is very small and trend behave like the original signal. So we just need to transmit only the trend signal and it will not affect the signal much. The benefit of this process is that we just need to transmit only the half of the bits, this process of transmitting the signal is known as* ***Compression of the signal****.*

Let $z \in L^2(\mathbb{Z}_N)$ then we have

$$z \mapsto \begin{pmatrix} a^{(1)} \\ d^{(1)} \end{pmatrix} \mapsto \begin{pmatrix} a^{(2)} \\ d^{(2)} \\ d^{(1)} \end{pmatrix} \mapsto \begin{pmatrix} a^{(3)} \\ d^{(2)} \\ d^{(2)} \\ d^{(1)} \end{pmatrix} \mapsto \dots.$$

To compute the second level trend $a^{(2)}$ and the second level fluctuation $d^{2()}$, we have the following

$$a^{(2)} = \begin{pmatrix} \frac{a_0^{(1)}+a_1^{(1)}}{\sqrt{2}} \\ \frac{a_2^{(1)}+a_3^{(1)}}{\sqrt{2}} \\ \vdots \\ \frac{a_{M-2}^{(1)}+a_{M-1}^{(1)}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{z(0)+z(1)+z(2)+z(3)}{2} \\ \frac{z(4)+z(5)+z(6)+z(7)}{2} \\ \vdots \\ \frac{z(M-4)+z(M-3)+z(M-2)+z(M-1)}{2} \end{pmatrix},$$

and

$$d^{(2)} = \begin{pmatrix} \frac{a_0^{(1)}-a_1^{(1)}}{\sqrt{2}} \\ \frac{a_2^{(1)}-a_3^{(1)}}{\sqrt{2}} \\ \vdots \\ \frac{a_{M-2}^{(1)}-a_{M-1}^{(1)}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{z(0)+z(1)-z(2)-z(3)}{2} \\ \frac{z(4)+z(5)-z(6)-z(7)}{2} \\ \vdots \\ \frac{z(M-4)+z(M-3)-z(M-2)-z(M-1)}{2} \end{pmatrix}.$$

To study more about trend and fluctuation at higher level let

$$\{V_0^{(1)}, V_0^{(1)}, \dots, V_{M-1}^{(1)}, W_0^{(1)}, W_1^{(1)}, \dots, W_{M-1}^{(1)}\},$$

is the set of N signals at first level and

$$\{V_0^{(2)}, V_0^{(2)}, \dots, V_{\frac{M}{2}-1}^{(2)}, W_0^{(2)}, W_1^{(2)}, \dots, W_{\frac{M}{2}-1}^{(2)}\},$$

be the set of $\frac{N}{2}$ signals at second level.

Definition 1.3.39. *Let*

$$V_0^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, V_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ldots, V_{M-1}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

We call $\{V_0^{(1)}, V_0^{(1)}, \ldots, V_{M-1}^{(1)}\}$ *is the first-level Haar scaling signal and we denote it as*

$$\{V_0^{(1)}, V_0^{(1)}, \ldots, V_{M-1}^{(1)}\} = \left\{R_{2k} V_0^{(1)}\right\}_{k=0}^{M-1}.$$

Definition 1.3.40. *Let*

$$W_0^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, W_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ldots, W_{M-1}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

We call $\{W_0^{(1)}, W_0^{(1)}, \ldots, W_{M-1}^{(1)}\}$, *is the first-level Haar wavelet and we denote it as*

$$\{W_0^{(1)}, W_0^{(1)}, \ldots, W_{M-1}^{(1)}\} = \left\{R_{2k} W_0^{(1)}\right\}_{k=0}^{M-1}.$$

Remark 1.3.41.

$$\left\{R_{2k} V_0^{(1)}\right\}_{k=0}^{M-1} \cup \left\{R_{2k} W_0^{(1)}\right\}_{k=0}^{M-1},$$

is a time frequency localized basis with father wavelet $W_0^{(1)}$ *and the mother wavelet* $V_0^{(1)}$.

Similarly, we can find the second level Haar wavelet and Haar scaling signals given as :

Definition 1.3.42. *Let*

$$V_0^{(2)} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ `0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, V_1^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ `0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \ldots, V_{\frac{M}{2}-1}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

We call $\{V_0^{(2)}, V_0^{(2)}, \ldots, V_{\frac{M}{2}-1}^{(1)}\}$, *is the second-level Haar scaling signal.*

Definition 1.3.43. *Let*

$$W_0^{(2)} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \\ `0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, W_1^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \dots, W_{\frac{M}{2}-1}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}.$$

We call $\{W_0^{(2)}, W_0^{(2)}, \dots, W_{\frac{M}{2}-1}^{(1)}\}$*, is the second-level Haar wavelets.*

Remark 1.3.44. *The first and second level Haar scaling numbers and the wavelet signals can be written as the linear combination of standard ordered basis as*

$$V_m^{(1)} = \alpha_1 e_{2m} + \alpha_2 e_{2m+1} \quad m = 0, 1, 2, \dots, M-1,$$

and hence

$$V_m^{(2)} = \alpha_1 V_{2m}^{(1)} + \alpha_2 V_{2m+1}^{(1)} \quad m = 0, 1, 2, \dots, \frac{M}{2} - 1,$$

where $\alpha_1 = \alpha_2 = \frac{1}{\sqrt{2}}$.

$$W_m^{(1)} = \beta_1 e_{2m} + \beta_2 e_{2m+1} \quad m = 0, 1, 2, \dots, M-1,$$

and hence

$$W_m^{(2)} = \beta_1 V_{2m}^{(1)} + \beta_2 V_{2m+1}^{(1)} \quad m = 0, 1, 2, \dots, \frac{M}{2} - 1,$$

where $\beta_1 = \frac{1}{\sqrt{2}}, \beta_2 = -\frac{1}{\sqrt{2}}$*. Then we call* α_1, α_2 *as scaling numbers and* β_1, β_2 *as wavelets numbers.*

Proposition 1.3.45. *Let* $z \in L^2(\mathbb{Z}_N)$*. Then*

$$a^{(1)} = \begin{pmatrix} \langle z, V_0^{(1)} \rangle \\ \langle z, V_1^{(1)} \rangle \\ \vdots \\ \langle z, V_{\frac{N}{2}-1}^{(1)} \rangle \end{pmatrix}, d^{(1)} = \begin{pmatrix} \langle z, W_0^{(1)} \rangle \\ \langle z, W_1^{(1)} \rangle \\ \vdots \\ \langle z, W_{\frac{N}{2}-1}^{(1)} \rangle \end{pmatrix}.$$

And similarly, we can get the second level trend and fluctuation as follow :

Proposition 1.3.46. *Let* $z \in L^2(\mathbb{Z}_N)$*. Then*

$$a^{(2)} = \begin{pmatrix} \langle z, V_0^{(2)} \rangle \\ \langle z, V_1^{(2)} \rangle \\ \vdots \\ \langle z, V_{\frac{M}{2}-1}^{(2)} \rangle \end{pmatrix}, d^{(2)} = \begin{pmatrix} \langle z, W_0^{(2)} \rangle \\ \langle z, W_1^{(2)} \rangle \\ \vdots \\ \langle z, W_{\frac{M}{2}-1}^{(2)} \rangle \end{pmatrix}.$$

1.3.5 Multiresolution Analysis

We know that the inverse wavelet transform is given as, if $z \in L^2(\mathbb{Z}_N)$, then

$$\begin{pmatrix} (a_0+d_0)/\sqrt{2} \\ (a_0-d_0)/\sqrt{2} \\ (a_1+d_1)/\sqrt{2} \\ (a_1-d_1)/\sqrt{2} \\ (a_2+d_2)/\sqrt{2}\vdots \\ \vdots \\ (a_{M-1}+d_{M-1})/\sqrt{2} \\ (a_{M-1}-d_{M-1})/\sqrt{2} \end{pmatrix}.$$

Then it can be written as

$$z = \sum_{n=0}^{M-1} \langle z, V_n^1 \rangle V_n^1 \; + \sum_{n=0}^{M-1} \langle z, W_n^1 \rangle W_n^1,$$

$\forall z \in L^2(\mathbb{Z}_N)$, if we define $A^{(1)}$ and $D^{(1)}$ as

$$A^{(1)} = \sum_{n=0}^{M-1} \langle z, V_n^1 \rangle V_n^1,$$

and

$$D^{(1)} = \sum_{n=0}^{M-1} \langle z, W_n^1 \rangle W_n^1.$$

Then

$$z = A^{(1)} + D^{(1)}.$$

$A^{(1)}$ is called as the first-level **average signal** and $D^{(1)}$ is called as first-level **detail signal**. With the similar iteration we can find the higher order average and detail signals.

Remark 1.3.47. *Iteration of the average and detail signal will give us*

$$z = A^j + \sum_{l=1}^{j} D^l \quad z \in L^2(\mathbb{Z}_N).$$

Proposition 1.3.48.

$$\{W_0^{(1)}, W_1^{(1)}, \ldots, W_{\frac{N}{2}-1}^{(1)}\}, \{W_0^{(2)}, W_1^{(2)}, \ldots, W_{\frac{M}{2}-1}^{(2)}\}, \ldots \{W_0^{(l-1)}, W_1^{(l-1)}\}, \{W_0^{(l)}\}, \{V_0^{(l)}\}$$

forms an orthonormal basis for $L^2(\mathbb{Z}_N)$.

Remark 1.3.49. *Thee average signal and the detail signal at level j is given by*

$$A^j = \sum_{n=0}^{\frac{N}{2^j}-1} \langle z, V_n^j \rangle V_n^j,$$

and

$$D^j = \sum_{n=0}^{\frac{N}{2^j}-1} \langle z, W_n^j \rangle W_n^j.$$

Remark 1.3.50. *The first level average signal is a linear combination of Haar scaling signal* $\{W_0^{(1)}, W_1^{(1)}, \ldots, W_{\frac{N}{2}-1}^{(1)}\}$ *where each Haar scaling signal is a short lived signal which moves only 2 units on the time axis and hence live only for two time units. So the Haar scaling signals measure short-lived trend in the given signal. So similar argument can be given for the detail signal. A similar, analogy can be given for higher order average signals and detail signal.*

Remark 1.3.51. *The energy of a signal* $z \in L^2(\mathbb{Z}_N)$, $E(z)$ *is given by*

$$E(z) = \|z\|^2.$$

Lemma 1.3.52. *Let* $z, w \in L^2(\mathbb{Z}_N)$ *be the two orthogonal signal, then*

$$\|z + w\|^2 = \|z\|^2 + \|w\|^2.$$

Proposition 1.3.53. *Let* $z \in L^2(\mathbb{Z}_N)$ *and let* $N = 2^l$ *for some positive integer* l. *Then*

$$E(z) \doteqdot E(A^j) \quad , j = 1, 2, \ldots, l.$$

Examples 1.3.54. *Let* $z \in L^2(\mathbb{Z}_N)$ *is given by*

$$z = \begin{pmatrix} 4 \\ 6 \\ 10 \\ 12 \\ 8 \\ 6 \\ 5 \\ 5 \end{pmatrix}.$$

The first level trend and fluctuation is given by

$$a^{(1)} = \begin{pmatrix} 5\sqrt{2} \\ 11\sqrt{2} \\ 7\sqrt{2} \\ 5\sqrt{2} \end{pmatrix} d^{(1)} = \begin{pmatrix} -\sqrt{2} \\ -\sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}.$$

The second level average and detail signals are given as

$$a^{(2)} = \begin{pmatrix} 16 \\ 12 \end{pmatrix} d^{(2)} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}.$$

And hence

$$E(A^{(1)}) = 440.$$

Since,

$$E(z) = 446.$$

It follows that 98.7% of the energy of the signal z *is in the first level average.Also*

$$E(A^{(2)}) = 400,$$

which is 89.7% of the energy of z

1.3.6 Daubechies Wavelet

Let $z \in L^2(\mathbb{Z}_N)$ defined as

$$z = \begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix}.$$

Suppose that z is obtained by sampling a analog signal g and we assume that g has a second order continuous derivative. So,

$$z(n) = g(t_n), \qquad 0, 1, 2, \ldots, N-1,$$

where h is the step length at time t_n, is given by

$$h = t_{n+1} - t_n.$$

Now let us compute the fluctuation $\langle z, W_0^{(1)} \rangle$ where $W_0^{(1)}$ is the first Haar wavelet at the first level i.e.

$$W_0^{(1)} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

After using Taylor series we will get

$$\langle z, W_0^{(1)} \rangle = O(h),$$

and also we know that $\beta_1 = \frac{1}{\sqrt{2}}, \beta_2 = -\frac{1}{\sqrt{2}}$. Now, we will go one step further, we are looking for a new wavelet and hence we need to find the numbers $\beta_1, \beta_2, \beta_3, \beta_4$. So for finding these numbers, we know that at first level

$$W_0^{(1)} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So again by using Taylor's theorem we will get

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0,$$

and

$$\beta_2 + 2\beta_3 + 3\beta_4 = 0.$$

Definition 1.3.55.

$$W_0^{(1)} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, W_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, W_{\frac{N}{2}-1}^{(1)} = \begin{pmatrix} \beta_3 \\ \beta_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta_1 \\ \beta_2 \end{pmatrix}.$$

From the above definition of $\{W_0^{(1)}, W_0^{(1)}, \ldots, W_{M-1}^{(1)}\}$, we want to constrict a time frequency localized basis with wavelet signals as $\{W_0^{(1)}, W_0^{(1)}, \ldots, W_{M-1}^{(1)}\}$ and scaling signals as $\{V_0^{(1)}, V_0^{(1)}, \ldots, V_{M-1}^{(1)}\}$. So assume that this is an orthonormal basis so we will get following equations :

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0 \qquad (A).$$
$$\beta_2 + 2\beta_3 + 3\beta_4 = 0 \qquad (B).$$
$$\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 = 1 \qquad (C).$$
$$\beta_1\beta_3 + \beta_2\beta_4 = 0 \qquad (D).$$

So after solving these 4 equations, we will get

$$\beta_1 = \frac{1-\sqrt{3}}{4\sqrt{2}}, \beta_2 = \frac{\sqrt{3}-3}{4\sqrt{2}}, \beta_3 = \frac{3+\sqrt{3}}{4\sqrt{2}}, \beta_4 = \frac{-1-\sqrt{3}}{4\sqrt{2}},$$

or

$$\beta_1 = -\frac{1-\sqrt{3}}{4\sqrt{2}}, \beta_2 = -\frac{\sqrt{3}-3}{4\sqrt{2}}, \beta_3 = -\frac{3+\sqrt{3}}{4\sqrt{2}}, \beta_4 = -\frac{-1-\sqrt{3}}{4\sqrt{2}}.$$

Lemma 1.3.56. *Let $\varphi \in L^2(\mathbb{Z}_N)$ be such that $\{R_{2j}\varphi\}_{j=0}^{M-1}$ is an othonormal set with M distinct signals, if we define $\phi \in L^2(\mathbb{Z}_N)$ by*

$$\phi(n) = (-1)^{n-1}\overline{\varphi(1-n)},$$

then $\{R_{2j}\phi\}_{j=0}^{M-1} \cup \{R_{2j}\varphi\}_{j=0}^{M-1}$ is a time frequency localized basis for $L^2(\mathbb{Z}_N)$.

Hence by using the above lemma we will get

$$V_0^{(1)} = \begin{pmatrix} -\beta_4 \\ \beta_3 \\ -\beta_2 \\ \beta_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

and hence the scaling numbers are given by

$$\alpha_1 = -\beta_4, \alpha_2 = \beta_3, \alpha_3 = -\beta_2, \alpha_4 = \beta_1.$$

Remark 1.3.57. *Whatever analysis we had done for Haar wavelet also holds for Daubechies Wavelet.*

Benefits 1.3.58. *The main benefit to study Daubechies wavelet is the concentration of energy of average signal is more as compare to Haar wavelet and we can see this by an example which we had done for Haar wavelet also. Let $z \in L^2(\mathbb{Z}_N)$ is given by*

$$z = \begin{pmatrix} 4 \\ 6 \\ 10 \\ 12 \\ 8 \\ 6 \\ 5 \\ 5 \end{pmatrix}.$$

The first level trend and fluctuation is given by

$$a^{(1)} = \begin{pmatrix} \frac{16-3\sqrt{3}}{\sqrt{2}} \\ \frac{19+2\sqrt{3}}{\sqrt{2}} \\ \frac{11.5+\sqrt{3}}{\sqrt{2}} \\ \frac{9.5}{\sqrt{2}} \end{pmatrix}.$$

And hence,

$$E(A^{(1)}) = 443.5.$$

Since,

$$E(z) = 446,$$

it follows that the energy level in the average signal is 99.4% *while in Haar wavelet it was* 98.7%. *Thus w had improved the energy level of average signal while using Daubechies wavelet in spite of Haar wavelet.*

1.3.7 The Trace

Definition 1.3.59. *Let $\sigma, \varphi \in L^2(\mathbb{Z}_N)$, then we define $T_{\sigma,\varphi} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ as*

$$T_{\sigma,\varphi} z = \sum_{k=0}^{N-1} \sigma(k) \langle z, \pi_k \varphi \rangle \pi_k \varphi,$$

where

$$\pi_k \varphi = \sqrt{N} (R_k \phi)^{\vee} \quad , k = 0, 1, \ldots N-1.$$

Proposition 1.3.60. *Let $\sigma \in L^2(\mathbb{Z}_N)$ be any signal. Then*

$$T_{\sigma,\delta} = T_{\sigma}.$$

Remark 1.3.61. *Let* $\varphi \in L^2(\mathbb{Z}_N) \ni$

$$|\hat{\varphi}(m)| = 1, \quad m \in \mathbb{Z}.$$

Then the set $\{R_0\varphi, R_1\varphi, \ldots, R_{N-1}\varphi\}$ *is an orthonormal basis for* $L^2(\mathbb{Z}_N)$ *and hence by using Parseval's equality and Fourier inversion formula, we will get* $\{\pi_0\varphi, \pi_1\varphi, \ldots, \pi_{N-1}\varphi\}$ *is an orthonormal basis for* $L^2(\mathbb{Z}_N)$.

Proposition 1.3.62. *Let* $\sigma \in L^2(\mathbb{Z}_N)$ *be any signal and* $\varphi \in L^2(\mathbb{Z}_N)$ *be such that*

$$|\hat{\varphi}(m)| = 1, \quad m \in \mathbb{Z},$$

then the eigenvalues of the operator $T_{\sigma,\varphi}$ *are given by*

$$\sigma(0), \sigma(1), \sigma(2), \ldots, \sigma(N-1).$$

Definition 1.3.63. *The trace of a* $N \times N$ *matrix* A *with complex entries is the sum of eigenvalues.*

Proposition 1.3.64. *Let* A *be a* $N \times N$ *matrix with complex entries, then*

$$tr(A) = \sum_{n=0}^{N-1} a_{jj},$$

where $a_{jj}, j = 0, 1, \ldots, N-1$ *are the diagonal entries of the matrix.*

Proposition 1.3.65. *Let* A *be a* $N \times N$ *matrix with complex entries, then*

$$tr(A) = \sum_{n=0}^{N-1} \langle A\varphi_j, \varphi_j \rangle,$$

where $\{\varphi_0, \varphi_1, \ldots, \varphi_{N-1}\}$ *is an orthonormal basis for* $L^2(\mathbb{Z}_N)$.

Theorem 1.3.66. *Let* $\sigma, \varphi \in L^2(\mathbb{Z}_N)$, *then* $tr(T_{\sigma,\varphi})$ *of the linear operator* $T_{\sigma,\varphi}$ *associated to the signals* σ *and* φ *is given by*

$$tr(T_{\sigma,\varphi}) = \|\varphi\|^2 \sum_{k=0}^{N-1} \sigma(k).$$

Chapter 2

Pseudo-Differential Operator on S^1

2.1 Hilbert Space

Functional analysis is a very important branch of mathematics. It is also broadly used in many applications of science and technology. In signal analysis Hilbert space plays an important part. Before moving to continuous case of Fourier transform, we will study briefly about Hilbert spaces and the operators defined on it which we are going to use in the later part.In this chapter, we will mainly concentrate on brief study of inner product space, Hilbert spaces, self-adjoint operator,compact operator and spectral theorem.

2.1.1 Definition and basic results

Definition 2.1.1. *Let X be a complex vector space. An inner product $\langle , \rangle$ is a mapping from $X \times X$ into $\mathbb{C}$ such that*

1. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle \quad \forall\ x, y, z \in X$ *and* $a, b \in \mathbb{C}$.
2. $\langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0 \Leftrightarrow x = 0 \quad \forall\, x \in X$.
3. $\langle x, y\rangle = \overline{\langle y, x\rangle} \quad \forall\ x, y \in X$.

Remark 2.1.2. *Inner product is conjugate linear in second component i.e.* $\langle x, ay\rangle = \overline{a}\langle x, y\rangle \forall x \in X$ *and* $a \in \mathbb{C}$.

Proposition 2.1.3. *Let X be a complex inner product space and $x, y \in X$. Then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

This inequality is known as Cauchy-Schwartz inequality. In this inequality, equality holds iff x and y are linearly dependent.

Proposition 2.1.4. *Let X be a complex inner product space and $x, y \in X$. Then*

$$\|x + y\| \leq \|x\| + \|y\|.$$

This inequality is known as triangular inequality. In this inequality, equality holds iff x and y are linearly dependent.

Note 2.1.5. $\langle , \rangle$ *is a continuous function.*

Definition 2.1.6. *Let* $\{x_j\}_{j=1}^{\infty}$ *be a sequence in X, then we say that* $\{x_j\}_{j=1}^{\infty}$ *converges to* x *if* $\|x_j - x\| \to 0$ *as* $j \to \infty$.
Clearly, if $\{x_j\}_{j=1}^{\infty}$ *converges to* x *in X and we know that* $\langle , \rangle$ *is a continuous map and hence*

$$\langle x_j, y \rangle \to \langle x, y \rangle \quad \forall\ y \in X.$$

Definition 2.1.7. *Let* $\{x_j\}_{j=1}^{\infty}$ *be a sequence in X, then we say that* $\{x_j\}_{j=1}^{\infty}$ *is a Cauchy sequence if*

$$\|x_j - x_k\| \to 0,$$

as $j, k \to \infty$.

Definition 2.1.8. *The induced norm by an inner product for* $x \in X$ *is given by*

$$\|x\|^2 = \langle x, x \rangle.$$

Definition 2.1.9. *A complex vector space with* $\|.\|$ *is said to be normed linear space, where* $\|.\| : X \to [0, \infty)$ *satisfying following properties.*

1. $\|x\| \geq 0$.
2. $\|x\| = 0 \Leftrightarrow x = 0$.
3. $\|ax\| = |a|\|x\|$.
4. $\|x + y\| \leq \|x\| + \|y\| \quad \forall\ x, y \in X$ *and* $a \in \mathbb{C}$.

Definition 2.1.10. *A normed linear space Y is said to be a Banach space if every Cauchy sequence in Y converges in Y.*

Definition 2.1.11. *A infinite dimensional complex inner product space is said to be a Hilbert space if every Cauchy sequence in X converges in X .*

Remark 2.1.12. *Hilbert space is a particular case of Banach space as* $\|.\|$ *is induced in a Hilbert space by* $\langle , \rangle$.

Examples 2.1.13. *Some trivial examples of Hilbert spaces are:*

1. $\mathbb{R}^n$ *under* $\|.\|_2$ *norm.*
2. ℓ^2 *and* $\mathbb{L}^2$.

Definition 2.1.14. *A sequence* $\{x_j\}_{j=1}^{\infty}$ *in a Hilbert space X is said to be orthogonal sequence if*

$$\langle x_j, x_k \rangle = 0$$

$\forall$ *positive integer* j *and* k *with* $j \neq k$.

Definition 2.1.15. *An orthogonal sequence* $\{x_j\}_{j=1}^{\infty}$ *in a Hilbert space is said to be orthonormal sequence if*

$$\|x_j\| = 1$$

$\forall$ *j=1,2,3, . . .*

Definition 2.1.16. *An orthonormal sequence* $\{x_j\}_{j=1}^{\infty}$ *in a Hilbert space X is said to be complete if every element* x *in X with the property that*

$$\langle x, x_j \rangle = 0, \quad \forall j = 1, 2, 3...$$

is the zero element in X.

Definition 2.1.17. *$\{x_j\}_{j=1}^{\infty}$ is said to be an orthonormal basis for the Hilbert space X if it is a complete orthonormal sequence.*

We assume that every Hilbert space X has an orthonormal basis $\{w_j\}_{j=1}^{\infty}$.

Definition 2.1.18. *Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in a Hilbert space X. For every positive integer n, define s_n as*

$$s_n = \sum_{m=1}^{n} x_m.$$

If $\{s_j\}_{j=1}^{\infty}$ converges to s in X, then we say that $\sum_{m=1}^{\infty} x_m$ converges to s in X and we write

$$\sum_{m=1}^{\infty} x_m = s.$$

Definition 2.1.19. *The series $\sum_{m=1}^{\infty} x_m$ is said to be absolutely convergent if*

$$\sum_{m=1}^{\infty} \|x_m\| < \infty.$$

Proposition 2.1.20. *Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in Hilbert space X such that the series $\sum_{m=1}^{\infty} x_m$ is absolutely convergent. Then $\sum_{m=1}^{\infty} x_m$ converges in X.*

Proposition 2.1.21. ***Bessel inequality*** *: Let $\{z_j\}_{j=1}^{\infty}$ be an orthonormal sequence in a Hilbert space X. Then $\forall\, x \in X$ we have*

$$\sum_{m=1}^{\infty} |\langle x, z_j \rangle|^2 \leq \|x\|^2.$$

Proposition 2.1.22. *Let $\{z_j\}_{j=1}^{\infty}$ be an orthonormal sequence in a Hilbert space X and $\{a_j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{C}$. Then $\sum_{m=1}^{\infty} z_m.a_m$ converges in X iff*

$$\sum_{m=1}^{\infty} |a_m|^2 < \infty.$$

Proposition 2.1.23. ***Pythagoras' Theorem*** *: Let $\{z_j\}_{j=1}^{\infty}$ be an orthonormal sequence in a Hilbert space X and let $\{a_j\}_{j=1}^{\infty}$ be a sequence in $\mathbb{C}$ such that $\sum_{m=1}^{\infty} z_m.a_m$ converges in X . Then*

$$\|\sum_{m=1}^{\infty} z_m.a_m\|^2 = \sum_{m=1}^{\infty} |a_m|^2.$$

Proposition 2.1.24. *Let $\{x_j\}_{j=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ be a sequence in a Hilbert space X such that $\sum_{m=1}^{\infty} x_m$ and $\sum_{m=1}^{\infty} y_m$ both converges in X. Then*

$$\langle \sum_{m=1}^{\infty} x_m, \sum_{m=1}^{\infty} y_m \ \rangle = \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \langle x_j, y_j \rangle.$$

Theorem 2.1.25. *Let $\{w_j\}_{j=1}^{\infty}$ be an orthonormal basis for a Hilbert space X. Then $\forall\, x, y \in X$, we have the following conclusions:*

1. ***The Fourier Inversion Formula***

$$x = \sum_{m=1}^{\infty} \langle x, w_j \rangle w_j.$$

2. ***Parseval's Identity***

$$\langle x, y \rangle = \sum_{m=1}^{\infty} \langle x, w_j \rangle \langle w_j, y \rangle.$$

3. ***Plancherel's Theorem***

$$\|x\|^2 = \sum_{m=1}^{\infty} |\langle x, w_j \rangle|^2.$$

Proposition 2.1.26. ***Parallelogram Law :*** *Let X be a complex vector space in which the $\|.\|$ is induced by $\langle , \rangle$. Then $\forall\, x, y \in X$ we have*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

2.1.2 Bounded Linear Operator

Definition 2.1.27. *A linear functional is a linear transformation from a complex normed linear space X into $\mathbb{C}$.*

Definition 2.1.28. *A linear functional T on X is said to be a bounded linear functional on X if $\exists$ a positive constant K such that*

$$\|T(x)\| \leq K\|x\| \quad , x \in X.$$

Theorem 2.1.29. *Let M be a closed subspace of X . Let $x \in M^{\complement}$ and let d be the distance between x and M defined by*

$$d = \inf_{z \in M} \|x - z\|.$$

Then $\exists\, z \in M$ such that

$$\|x - z\| = d.$$

Definition 2.1.30. *Let M be a closed subspace of a Hilbert space X. Then the orthogonal complement $M^{\perp}$ of M is define as*

$$M^{\perp} = \{x \in X : \langle x, y \rangle = 0,\ y \in M\}.$$

Remark 2.1.31. *Null space of a bounded linear functional T on X is a closed subspace of X.*

Theorem 2.1.32. *Let M be a closed subspace of a Hilbert space X. Then $\forall\, x \in X$, we can find unique elements v and w such that $v \in M, w \in M^{\perp}$ and*

$$x = v + w.$$

Theorem 2.1.33. ***The Riesz Representation Theorem :***
Let T be a bounded linear functional on a Hilbert space X. Then $\exists$ a unique y in X such that

$$T(x) = \langle x, y \rangle \quad , x \in X.$$

Definition 2.1.34. *A linear operator A on X is said to be a bounded linear operator on X if* $\exists$ *a positive constant K such that*

$$\|A(x)\| \leq K\|x\| \quad \forall\, x \in X.$$

Definition 2.1.35. $\|.\|$ *on a bounded linear operator is defined as :*

$$\|A\|_* = \sup_{x\neq 0} \frac{\|A(x)\|}{\|x\|} = \sup_{\|x\|=1} \|A(x)\|$$

and from here it follows that

$$\|Ax\| \leq \|A\|_*\|x\|, \quad x \in X.$$

Definition 2.1.36. *A linear operator A on a Hilbert space X is said to be continuous at a point* $x \in X$ *if for every sequence* $\{x_j\}_{j=1}^{\infty}$ *converging to x in X, we have*

$$Ax_j \to Ax$$

in X as $j \to \infty$.

Theorem 2.1.37. *A linear operator A is bounded on a Hilbert space X iff A is continuous at a point in X.*

Note 2.1.38. *B(X) represents the set of all bounded linear operators on a Hilbert space X.*

Theorem 2.1.39. *B(X) is a Banach space with respect to the* $\|.\|_*$.

Definition 2.1.40. *B(*x_0, r*), the open ball with center* x_0 *and radius r in X is given by*

$$B(x_0, r) = \{x \in X : \|x - x_0\| < r\}.$$

Definition 2.1.41. *A subset W of X is said to be nowhere dense in X if the closure* $\overline{W}$ *of W contains no open balls.*

Theorem 2.1.42. ***Baire's category theorem :***
A complete metric space can not be expressed as countable union of nowhere dense sets.

Theorem 2.1.43. ***Uniform Boundedness Theorem :***
Let X be a Banach space and Y be norm linear space. Let B(X,Y) is the set of all bounded linear operators from X to Y. Let $W \subset X$ *be such that* $\forall\, x \in X$*,*

$$\sup_{A\in W} \|Ax\| < \infty.$$

Then

$$\sup_{A\in W} \|A\| < \infty.$$

This theorem implies the point wise boundedness of bounded linear operators on X to uniform boundedness on X.

Definition 2.1.44. *For a bounded linear operator T, the set*

$$\Sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \;\; \textit{is not invertible in } X\}$$

is known as spectrum of T.

Definition 2.1.45. *For a bounded linear operator T the set*

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \;\; \textit{is invertible in } X\}$$

where I is the identity operator, is known as resolvent set of T.

Definition 2.1.46. *Let T be bounded linear operator on a Hilbert space X. Let* $\lambda \in \mathbb{C}$ *is said to be the eigenvalue of T if* $\exists\, x \neq 0$ *such that*

$$Tx = \lambda x.$$

Remark 2.1.47. *The set of all eigenvalues is a proper subset of* $\Sigma(T)$ *for a bounded linear operator T.*

2.1.3 Self-Adjoint Operators

Definition 2.1.48. *Let T be a bounded linear operator on a Hilbert space X. A linear operator T^* is said to be adjoint of T if*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \ , \ x,y \ \in X.$$

Remark 2.1.49. *A linear operator on a Hilbert space has at most one adjoint.*

Theorem 2.1.50. *Every bounded linear operator on a Hilbert space X has an adjoint, which is also an bounded linear operator on X.*

Remark 2.1.51. *B(X) is closed under '$*$' operation.*

Theorem 2.1.52. *Let B(X) be the set of bounded linear operators on the Hilbert space X and let us define a map $T \to T^*$ on B(X). Then we have the following:*

1. *It is conjugate linear.*
2. *It is isometric i.e.* $\|T\|_* = \|T^*\|_* \ \ \forall \ T$.
3. *It is surjective.*
4. $T^{**} = T \ \ \forall \, T \in B(X)$.
5. $(TS)^* = S^*T^* \ \ \forall \, T, S \in B(X)$.
6. $I^* = I$.
7. *If T is invertible then so is T^* and* $(T^*)^{-1} = (T^{-1})^*$.
8. *The given map is continuous in the usual topology.*

Definition 2.1.53. *A bounded linear operator on a Hilbert space X is said to be self-adjoint if it is equal to its adjoint i.e.*

$$T = T^*.$$

Theorem 2.1.54. *Let T be a self-adjoint operator on a Hilbert space X. Then*

$$\|T\|_* = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

Theorem 2.1.55. *Let T be self-adjoint operator on a Hilbert space X. Then all eigenvalues of T are real. Moreover, if x,y are two eigenvectors of T corresponding to distinct eigenvalues, then*

$$\langle x, y\rangle = 0.$$

Theorem 2.1.56. *A bounded linear operator T is self-adjoint on a Hilbert space X iff*

$$\langle Tx, x\rangle \in \mathbb{R} \ \ \forall \ x \in X.$$

2.1.4 Compact Operator

Definition 2.1.57. *A sequence $\{x_j\}_{j=1}^{\infty}$ in a Hilbert space X is said to be bounded if* $\exists\, C > 0 \ni$

$$\|x_j\| \leq C, \ \ j = 1, 2, \ldots$$

Definition 2.1.58. *A bounded linear operator T on a Hilbert space X is said to be compact if for every bounded sequence $\{x_j\}_{j=1}^{\infty}$ in X, the sequence $\{Tx_j\}_{j=1}^{\infty}$ has a convergent subsequence in X.*

Definition 2.1.59. *A bounded linear operator T on a Hilbert space X is said to be an operator of finite rank if the range space of T given by*

$$R(T) = \{Tx : x \in X\}$$

is finite dimensional.

Proposition 2.1.60. *A finite rank operator on a Hilbert space is compact.*

Theorem 2.1.61. *Let $\{T_j\}_{j=1}^{\infty}$ be a sequence of compact operators on a Hilbert space X such that*

$$\|T_j - T\|_* \to 0$$

as $j \to \infty$, where T is a bounded linear operator on X. Then T is a compact operator on X.

Definition 2.1.62. *A sequence $\{x_j\}_{j=1}^{\infty}$ in a Hilbert space X is said to converge weakly to x in X if*

$$\langle x_j, y\rangle \to \langle x, y\rangle \ \ \forall y \in X \ as\, j \to \infty.$$

Proposition 2.1.63. *Let $\{x_j\}_{j=1}^{\infty}$ be a weakly convergent sequence. Then we have the following:*

1. *Limit of $\{x_j\}_{j=1}^{\infty}$ is unique.*
2. *If $\{x_j\}_{j=1}^{\infty}$ converge in X, then it also converges weakly in X.*
3. *$\{x_j\}_{j=1}^{\infty}$ is bounded.*
4. *If $x_j \to x$ in X weakly and $\|x_j\| \to \|x\|$ as $j \to \infty$, then $x_j \to x$ in X as $j \to \infty$.*

Theorem 2.1.64. *Let T be a compact operator on a Hilbert space X. Then T maps weakly convergent sequences into convergent sequences.*

Theorem 2.1.65. *Let T be a self-adjoint and compact operator on a Hilbert space X. Then $\|T\|_*\, or - \|T\|_*$ is an eigenvalue of T.*

2.1.5 The Spectral Theorem

Definition 2.1.66. *An operator T is said to be positive if*

$$\langle Tx, x\rangle \geq 0 \quad , x \in X.$$

Definition 2.1.67. *Let T be a positive compact operator on a Hilbert space and can be written as*

$$Tx = \sum_{j=1}^{\infty} \lambda_j \langle x, w_j\rangle w_j \quad , x \in X$$

where $\{w_j\}_{j=1}^{\infty}$ is an orthonormal basis for X consisting of eigenvectors of T and λ_j is eigenvalue of T corresponding to w_j. Then the square root of $T^{1/2}$ is defined as

$$T^{1/2}x = \sum_{j=1}^{\infty} \lambda_j^{1/2} \langle x, w_j\rangle w_j \quad , x \in X.$$

Theorem 2.1.68. ***The Spectral Theorem :*** *Let T be a self-adjoint and compact operator on a Hilbert space X. Then $\exists$ an orthonormal basis $\{w_j\}_{j=1}^{\infty}$ for X consisting of eigenvectors of T. Moreover, $\forall\ x \in X$*

$$Tx = \sum_{j=1}^{\infty} \lambda_j \langle x, w_j \rangle w_j,$$

where λ_j is the eigenvalue of T corresponding to eigenvector w_j.

2.2 Pseudo-Differential Operator on S^1

2.2.1 Fourier Series

In this section, we are going to give some basic concepts of Fourier series which we are going to use in the later sections. We will need basic measure theory and corresponding theory of integration, which we are going to assume as prerequisites. We are also assuming the basic concepts of Hilbert space specially L^p spaces.

Let $f \in L^1[-\pi, \pi]$. Then we define the Fourier transform $\hat{f}$ on the set $\mathbb{Z}$ of all integers as

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\iota n\theta} f(\theta) d\theta, \quad n \in \mathbb{Z}.$$

We call $\hat{f}(n)$ the Fourier coefficient of function f at frequency n. The formal series $\sum\limits_{n=-\infty}^{\infty} \hat{f}(n)e^{-\iota n\theta}$ is the Fourier series of f at $[-\pi, \pi]$.

So the common problem that arise here is whether the Fourier series $\sum\limits_{n=-\infty}^{\infty} \hat{f}(n)e^{-\iota n\theta}$ converges pointwise to $f(\theta)\ \forall\, \theta \in [-\pi, \pi]$ or not. To answer this problem let us assume $\{s_N\}_{N=0}^{\infty}$ denotes the sequence of partial sums of the Fourier series defined by :

$$s_N(\theta) = \sum_{n=-N}^{N} \hat{f}(n)e^{\iota n\theta}, \quad \theta \in [-\pi, \pi].$$

Then

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{\iota n\theta} = f(\theta)$$

means that the sequence of partial sums of the Fourier series converges to $f(\theta)\ \forall\ \theta \in [-\pi, \pi]$. So, we need to find a sufficient condition for pointwise convergence of the Fourier series.

We know that the interval $[-\pi, \pi]$ can be identified with the unit circle S^1. For $n \in \mathbb{Z}$, we define the function e_n on $[-\pi, \pi]$ by

$$e_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{\iota n\theta}, \quad \theta \in [-\pi, \pi].$$

We have the following lemma which is a very small result but very useful. We are omitting the proof as we can prove it just by using the inner product definition of $L^2(S^1)$.

Lemma 2.2.1. *$\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal set in $L^2(S^1)$.*

We are assuming the proof of the next lemma from measure theory.

Lemma 2.2.2. *$C_0^{\infty}(S^1)$ is dense in $L^p(S^1)$, $1 \le p < \infty$.*

Lemma 2.2.3. ***(Riemann-Lebesgue Lemma) :*** *Let* $f \in L^1(S^1)$. *Then* $\lim_{|n|\to\infty} \hat{f}(n) = 0$

Proof. First of all, let $f \in L^2(S^1)$. Then by using the orthonormality of the sequence $\{e_n\}_{n=-\infty}^{\infty}$ and the Bessel inequality, we will get

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq \frac{1}{2\pi}||f||_2^2.$$

So,

$$\hat{f} \to 0 \ \ as \ \ |n| \to \infty.$$

Now, let $f \in L^1(S^1)$ and let ϵ be a given positive number, then by the previous lemma $\exists\ \phi \in C_0^\infty(S^1)$ such that

$$||f - \phi||_1 < \pi\epsilon$$

Then by triangular inequality we will be having

$$|\hat{f}(n)| < |\hat{\phi}(n)| + \frac{1}{2\pi}||f - \phi||_1 < \frac{\epsilon}{2} + |\hat{\phi}(n)|.$$

Since $\phi \in L^2(S^1)$, it follows that

$$\hat{\phi}(n) \to 0,$$

as $|n| \to 0$. So, there exists a positive integer N such that

$$|n| \geq N \Rightarrow |\hat{\phi}(n)| < \frac{\epsilon}{2}.$$

Therefore,

$$|n| \geq N \Rightarrow |\hat{f}(n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and the proof is complete. □

Lemma 2.2.4. *Let* $f \in L^1(S^1)$. *Then for all* $N = 0, 1, 2, \ldots,$

$$s_N(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta - \phi)f(\phi)d\phi, \quad \phi \in [-\pi, \pi]$$

where

$$D_N(\theta) = \frac{sin(N + \frac{1}{2})\theta}{sin\frac{1}{2}\theta}.$$

The proof of the above lemma can be manupulated easily. D_N is known as Dirichlet kernel.

Corollary 2.2.2. *For* $N = 0, 1, 2, \ldots D_N$ *is an even function such that*

$$\int_{-\pi}^{\pi} D_N(\theta)d\theta = 2\pi$$

Theorem 2.2.5. ***(Dini's Condition):*** *Let* $f \in L^1(S^1)$. *If* $\theta \in [-\pi, \pi]$ *is such that*

$$\int_{-\pi}^{\pi} \frac{|f(\theta + \phi) - f(\theta)|}{|\phi|}d\phi < \infty,$$

then

$$s_N(\theta) \to f(\theta)$$

as $N \to \infty$.

Proof. By using the previous lemma and corollary, we get

$$s_N\theta - f(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\phi)(f(\theta-\phi) - f(\theta))d\phi.$$

Now by using the expression for $D_N(\theta)$, we will get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\theta+\phi) - f(\theta)}{sin\frac{1}{2}\phi} sin(N + \frac{1}{2}\phi)d\phi.$$

Without loss of generality, we can assume that f is a real valued function. Since,

$$\frac{f(\theta+\phi) - f(\theta)}{sin\frac{1}{2}\phi} \in L^1(S^1)$$

as a function of ϕ, and hence result follows from the Riemann-Lebesgue lemma. □

Definition 2.2.6. *A funtion $f \in L^1(S^1)$ is Lipschitz continuous function at a point $\theta \in [-\pi, \pi]$, if there exist positive constants M and α such that*

$$|f(\theta) - f(\phi)| \leq M|\theta - \phi|^{\alpha}, \quad \phi \in [-\pi, \pi].$$

The number α is said to be the order of Lipshitz continuity of the function f at θ.

Definition 2.2.7. *A function $f \in L^1(S^1)$ is Lipschitz continuous function of order α, if it is Lipshitz continuous of order α at all points in $[\pi, \pi]$.*

Lemma 2.2.8. *Let f be a continuous function on $[-\pi, \pi]$, such that*

$$f(-\pi) = f(\pi).$$

Let $f^{'}$ exists at all but possibly a finite number of points in $[-\pi, \pi]$ and

$$\int_{-\pi}^{\pi} |f^{'}(\theta)|^2 d\theta < \infty.$$

Then f is Lipshitz continuous of order $\frac{1}{2}$ on $[-\pi, \pi]$.

Detail of the proof is left as it can be easily done by using Schwarz inequality.

Theorem 2.2.9. *Let f satisfies all conditions given in the above lemma, then the Fourier series of f converges to f absolutely and uniformly on $[-\pi, \pi]$.*

Proof. Let $g = f^{'}$, then for all $n \in \mathbb{Z}$ we have

$$\hat{g}(n) = \iota n \hat{f}(n).$$

Now for all M and N with $M < N$, by Schwarz inequality and Bessel inequality, we get

$$\begin{aligned}
|s_N(\theta) - s_M(\theta)| &= \left|\sum_{M<|n|\leq N} \hat{f}(n)\right| \leq \sum_{M<|n|\leq N} |\hat{f}(n)| \\
&\leq \left(\sum_{M<|n|\leq N} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{M<|n|\leq N} n^2|\hat{f}(n)|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{M<|n|\leq N} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{M<|n|\leq N} |\hat{g}(n)|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{M<|n|\leq N} \frac{1}{n^2}\right)^{\frac{1}{2}} (2\pi)^{-1/2}||f^{'}||.
\end{aligned}$$

Hence for every ϵ, there exists a positive integer K such that

$$N > M \geq K \Rightarrow |s_N(\theta) - s_M(\theta)| < \epsilon.$$

So, there exists a continuous function h on $[-\pi, \pi]$ such that

$$s_N \to h,$$

absolutely and uniformly as $N \to \infty$. From the previous lemma, f is Lipschitz continuous of order $\frac{1}{2}$ on $[-\pi, \pi]$. So by the last theorem we have

$$s_N(\theta) \to f(\theta)$$

for all $\theta \in [-\pi, \pi]$. Thus,

$$f(\theta) = h(\theta), \quad \theta \in [-\pi, \pi]$$

and so

$$s_N \to f$$

absolutely and uniformly on $[-\pi, \pi]$ as $N \to \infty$. □

Theorem 2.2.10. $\{e_n\}_{n=-\infty}^{\infty}$ *is an orthonormal basis for* $L^2(S^1)$.

Proof. Let $f \in L^2(S^1)$. Then for every positive ϵ there exits a $\phi \in C_0^\infty(S^1)$ such that

$$||f - \phi||_2 < \epsilon.$$

Then by last theorem

$$\sum_{n=-N}^{N} \langle \phi, e_n \rangle_2 e_n(\theta) = \sum_{n=-N}^{N} \hat{g}(n) e^{\iota n\theta} \to \phi(\theta)$$

uniformly with respect to θ in $[-\pi, \pi]$ as $N \to \infty$. So,

$$\left\| \sum_{n=-N}^{N} \langle \phi, e_n \rangle_2 e_n - \phi \right\|_2 \to 0,$$

as $N \to \infty$. Then by Pythagoras theorem and Bessel inequality, we have

$$\begin{aligned}
&\left\| \sum_{n=-N}^{N} \langle f, e_n \rangle_2 e_n - \sum_{n=-N}^{N} \langle \phi, e_n \rangle_2 e_n \right\| \\
&= \left(\sum_{n=-N}^{N} |\langle f, e_n \rangle_2 - \langle \phi, e_n \rangle_2|^2 \right)^{1/2} \\
&\leq \left(\sum_{n=-\infty}^{\infty} |\langle f, e_n \rangle_2 - \langle \phi, e_n \rangle_2|^2 \right)^{1/2} \\
&\leq \|f - \phi\| < \frac{\epsilon}{3}.
\end{aligned}$$

Therefore, after using triangular inequality, we will get

$$\|f - s_N\| < \epsilon$$

in $L^2(S^1)$ *as* $N \to \infty$. Then, it is easy to see that if $f \in L^2(S^1)$ is such that

$$\hat{f}(n) = 0, \quad n \in \mathbb{Z},$$

then $f = 0$. Therefore, $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for $L^2(S^1)$. □

The L^2 *theory* of Fourier series can now be easily obtained as the corollaries of last theorem.

Theorem 2.2.11. *Let $f, g \in L^2(S^1)$. Then,*

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\bar{\hat{g}}(n) = \frac{1}{2\pi}\langle f, g\rangle.$$

We can prove this theorem by using Parseval's identity.

Theorem 2.2.12. *Let $f \in L^2(S^1)$. Then*

$$s_N \to f,$$

in $L^2(S^1)$ as $N \to \infty$.

Remark 2.2.13. *From last theorem we conclude that*

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{\iota n\theta}, \ \ \theta \in [-\pi, \pi],$$

*where the convergence of the Fourier series is in L^2 convergence. This is known as **Fourier inversion formula** for the Fourier series.*

2.2.3 Fourier Multiplier on S^1

In this section, we are going to define Fourier multiplier (also known as filters) on the unit circle. The Plancherel theorem and Fourier inversion formula are the basic ingredient for study of filters on the unit circle S^1. As we know,

$$f = \sum_{n=-\infty}^{\infty} \langle f, e_n\rangle e_n,$$

where the convergence is in L^2. The above equation can also be written as:

$$I = \sum_{n=-\infty}^{\infty} \langle ., e_n\rangle e_n,$$

where I is the identity operator. So, we can write identity operator as an infinite sum of one dimensional projections. Hence, we can find more interesting and more applicable operators by substituting a suitable function σ on $\mathbb{Z}$ in the Fourier inversion formula.

Theorem 2.2.14. *(Plancherel's Theorem) The linear operator $\mathcal{F}_{S^1} : L^2(S^1) \to L^2(\mathbb{Z})$, defined by*

$$(\mathcal{F}_{S^1} f)(n) = \hat{f}(n), \quad n \in \mathbb{Z}$$

is a bijection such that

$$\langle \mathcal{F}_{S^1} f, \mathcal{F}_{S^1} g\rangle = \frac{1}{2\pi}\langle f, g\rangle,$$

for all $f, g \in L^2(S^1)$. The linear operator defined here is known as Fourier transform on S^1.

Proof. We are just going to give the outlines of the proof. Clearly, from the above equality the linear operator is injective. So, we are left to prove that the map is surjective. Let $(a_n) \in L^2(\mathbb{Z})$. Then for $N = 0, 1, 2, \ldots define$ s_N by

$$s_N(\theta) = \sum_{n=-N}^{N} a_n \sqrt{2\pi} e_n(\theta).$$

Now we can easily show that s_N is a cauchy sequence. Since $L^2(S^1)$ is complete. Hence

$$s_N \to f$$

for some $f \in L^2(S^1)$ as $N \to \infty$. Now for a fixed $m \in \mathbb{Z}$, we have

$$\widehat{s_N}(m) = a_m,$$

for sufficiently large N. Moreover by using Schwartz inequality we can show that

$$|s_N^{\wedge}(m) - \hat{f}(m)| \to 0 \ as \ N \to \infty \forall m \in \mathbb{Z}.$$

And hence,

$$\hat{f}(m) = a_m \Rightarrow \mathcal{F}_{S^1} f = a.$$

□

Corollary 2.2.4. *$\forall\, a \in L^2(\mathbb{Z}), \mathcal{F}_{S^1}^{-1} f$ is a function defined on $L^2(S^1)$ by*

$$(\mathcal{F}_{S^1}^{-1} f)(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{\iota n\theta}, \quad \theta \in [-\pi, \pi].$$

Definition 2.2.15. *Let $\sigma \in L^\infty(S^1)$. Then for all $f \in L^2(S^1)$, we define the function $T_\sigma f$ on S^1 by*

$$(T_\sigma f)(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\sigma(n) e^{\iota n\theta}, \quad \theta \in [-\pi, \pi].$$

*The above defined operator is known as **Fourier multiplier** or filter.*

Theorem 2.2.16. *Let σ be a measurable function on $\mathbb{Z}$, then T_σ is the bounded linear operator from $L^2(S^1)$ to $L^2(S^1) \Leftrightarrow \sigma \in L^\infty(\mathbb{Z})$. Moreover, if $\sigma \in L^\infty(\mathbb{Z})$, then*

$$\|T_\sigma\| = \|\sigma\|.$$

Proof. Let $\sigma \in L^\infty(\mathbb{Z})$. Then for all $f \in L^2(S^1)$,

$$\sigma \hat{f} \in L^2(\mathbb{Z}).$$

and from last corollary,

$$T_\sigma f \in L^2(S^1),$$

and hence we will get

$$\|T_\sigma f\| \leq \|\sigma\| \|f\|$$

which implies T_σ is the bounded linear operator such that

$$\|T_\sigma\| \leq \|\sigma\|.$$

Now we are going to prove the converse part of this theorem. Let $\sigma \notin L^\infty(S^1)$. Let T_σ is the bounded linear operator. Then there exits a positive constant C such that

$$\|T_\sigma f\| \leq C\|f\|.$$

Now as $\sigma \notin L^\infty(S^1)$, for $N = 1, 2, 3, \ldots$, there exits a positive n_N such that

$$|\sigma(n_N)| > N.$$

Without loss of generality, assume that the sequence $|n_1| < |n_2| < \ldots$. Now for N=1,2,3, ..., let f_{n_N} be the function on S^1 defined by

$$f_{n_N} = e^{\iota n_N}\theta, \;\; \theta \in [-\pi, \pi].$$

Then for $N = 0, 1, 2, \ldots,$ we have

$$\widehat{f_{n_N}}(n) = \begin{cases} 1 & n = n_N, \\ 0 & n \neq n_N. \end{cases}$$

and hence by applying the definition of Fourier multiplier, we will get

$$(T_\sigma f_{n_N})(\theta) = e^{\iota n_N \theta}\sigma(n_N),$$

which gives

$$\|T_\sigma f_{n_N}\| > \sqrt{2\pi}N,$$

and hence we will get

$$\sqrt{2\pi}N < \sqrt{2\pi}, C$$

which will give us a contradiction on boundedness of T_σ. Hence $\sigma \in L^\infty$. Now we need to show the equality of norms. Let

$$\|T_\sigma\| < |\sigma\|.$$

Hence there exists an integer m such that

$$|\sigma(m)| > \|T_\sigma\|.$$

So for all non zero functions $f \in L^2(S^1)$, we have

$$\|T_\sigma f\|^2 < |\sigma(m)|^2\|f\|^2.$$

Also we have

$$\sum_{n=-\infty}^{\infty} |\sigma(n)|^2|\hat{f}(n)|^2 < |\sigma(m)|^2 \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

Now let $f \in Ł^2(S^1)$ such that

$$\widehat{f}(n) = \begin{cases} 1 & n = m, \\ 0 & n \neq m. \end{cases}$$

Thus, we have

$$|\sigma(m)|^2 < |\sigma(m)|^2,$$

which is obviously a contradiction. □

Theorem 2.2.17. *Let $\sigma \in L^\infty$. Then for all $n \in \mathbb{Z}$, $\sigma(n)$ is an eigenvalue of the operator T_σ and e_n is the corresponding eigenfunction. Moreover, the spectrum of T_σ is precisely given by*

$$\Sigma(T_\sigma) = \overline{\{\sigma(n) : n \in \mathbb{Z}\}}.$$

Proof. Let $m \in \mathbb{Z}$. Then

$$T_\sigma e_m = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \sigma(n) \widehat{e_m}(n) e_n,$$

where the convergence is in L^2. But

$$\widehat{e}_m(n) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & n = m, \\ 0, & n \neq m. \end{cases}$$

Therefore,

$$T_\sigma e_m = \sigma(m) e_m,$$

i.e. $\sigma(m)$ is an eigenvalue of T_σ and e_m is the corresponding eigenvector. Now, we are left with the second proof of the theorem. Let $\lambda \notin \overline{\{\sigma(n) : n \in \mathbb{Z}\}}$. Then there exits a positive constant C such that

$$|\sigma(n) - \lambda| \geq C, \ \ \forall \ n \in \mathbb{Z}.$$

Now, we can easily show that the operator $T_\sigma - \lambda I$ is a bijection and hence we are done. □

Now we are going to give a characterization of the compact Fourier multiplier. Although, we are not giving the proof of this theorem as we can prove it very easily just by using the definition of compact operator and the previous theorem i.e. for the Fourier multiplier $T_\sigma, e_m' s$ are the eigenfunctions corresponding to eigenvalues $\sigma(m)$.

Theorem 2.2.18. *Let $\sigma \in L^\infty$. Then T_σ is compact if and only if*

$$\lim_{|n| \to \infty} \sigma(n) = 0.$$

Lemma 2.2.19. *Let $\sigma \in L^\infty(\mathbf{Z})$. If we let*

$$|T_\sigma| = (T_\sigma^* T_\sigma)^{1/2},$$

then

$$|T_\sigma| = T_{|\sigma|}$$

Proof. As we know

$$T_\sigma^* T_\sigma = T_{\sigma\overline{\sigma}} = T_{|\sigma|^2} = T_{|\sigma|}^2.$$

□

Now by using the above lemma, we are going to prove the L^2 boundedness of the Fourier multiplier.

Theorem 2.2.20. *Let $1 \leq p < \infty$, the Fourier multiplier T_σ is in Schatten-von Newmann class S_p if and only if $\sigma \in L^p(\mathbb{Z})$. Moreover, if $\sigma \in L^p(\mathbb{Z})$, then*

$$\|T_\sigma\| = \|\sigma\|.$$

Proof. Since, $\sigma \in L^p(\mathbb{Z})$, hence

$$\lim_{|n| \to \infty} \sigma(n) = 0.$$

Hence T_σ is a compact operator. So the singular values of T_σ are given by $|\sigma(n)|, n \in \mathbb{Z}$. Therefore,

$$T_\sigma \in S_p \Leftrightarrow \sum_{n=-\infty}^{\infty} |\sigma(n)|^p < \infty \Leftrightarrow \sigma \in L^p(\mathbb{Z}).$$

Clearly, we can see that

$$\|T_\sigma\| = \|\sigma\|.$$

□

2.2.5 Pseudo-Differential Operator

In this section we will be concentrating on time-varying FM-filters. In mathematical language, a time-frequency FM-filter is known as pseudo-differential operator on the unit circle with center at origin. Mainly, we will be concentrating on the boundedness and compactness of the pseudo-differential operator defined on S^1

For $1 \leq p < \infty$,the set of measurable functions is denoted by $L^p(S^1 \times \mathbb{Z})$ such that

$$\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |\sigma(\theta, n)|^p d\theta < \infty.$$

In the similar fashion, we will define norms in L^p and Hilbert space L^2.

Definition 2.2.21. *Let $\sigma \in S^1 \times \mathbb{Z}$ be a measurable function. Then for all $f \in L^2(S^1)$, we define the function $T_\sigma f$ on S^1 formally by,*

$$(T_\sigma f)(\theta) = \sum_{n=-\infty}^{\infty} e^{\iota n\theta} \sigma(\theta, n) \hat{f}(n), \quad \theta \in [-\pi, \pi].$$

*We call T_σ the **pseudo-differential operator** corresponding to symbol σ.*

Here the natural question to ask is why we are calling T_σ to be differential operator? To answer this question, consider the linear differential operator $P(\theta, D)$ on S^1 defined by

$$P(\theta, D) = \sum_{j=0}^{m} a_j(\theta) D^j,$$

where $D = -\iota \frac{d}{d\theta}$, and $a_0, a_1, \ldots, a_m$ are measurable function on S^1. Let $f \in C^\infty(S^1)$, then

$$(\mathcal{F}_{S^1} D^j f)(n) = n^j \hat{f}(n), \quad n \in \mathbb{Z}.$$

Then by using the unitary of Fourier transform, we will get,

$$\begin{aligned} (P(\theta, D)f)(\theta) &= \sum_{j=0}^{m} a_j(\theta)(D^j f)(\theta) \\ &= \sum_{n=-\infty}^{\infty} e^{\iota n\theta} \left(\sum_{j=0}^{m} a_j(\theta) n^j \right) \hat{f}(n), \end{aligned}$$

for all $\theta \in [-\pi, \pi]$. The message of the integral representation for $(P(\theta, D))f$ is that a pseudo-differential operator T_σ corresponding to a polynomial

$$\sigma(\theta, n) = \sum_{j=0}^{m} a_j(\theta) n^j,$$

is a linear differential operator. These kind of filters are very important in signal analysis as here σ depends on both time and frequency. In next theorem, we are going to give an upper bound for T_σ when $\sigma \in L^2(S^1 \times \mathbb{Z})$. We can prove this theorem easily by using Minkowski's inequality. So we are omitting this proof.

Theorem 2.2.22. *Let $\sigma \in L^2(S^1 \times \mathbb{Z})$, then T_σ is a bounded linear operator. Moreover,*

$$\|T_\sigma\| \leq (2\pi)^{-1/2} \|\sigma\|.$$

Now we are going to give the L^p boundedness of T_σ with respect to e_n.

Theorem 2.2.23. *Let $\sigma \in L^2(S^1 \times \mathbb{Z}), 1 \leq p < \infty$. Then*

$$\sum_{n=-\infty}^{\infty} \|T_\sigma e_n\|^p = (2\pi)^{-p/2}\|\sigma\|^p.$$

Proof. Let $j \in \mathbb{Z}$. Then

$$(T_\sigma e_j)(\theta) = \sum_{n=-\infty}^{\infty} \sigma(\theta, n)\widehat{e_j}(n)e^{\iota n\theta}, \quad \theta \in [-\pi, \pi].$$

$$\widehat{e_j}(m) = \begin{cases} \frac{1}{\sqrt{2\pi}} & j = n \\ 0 & j \neq n \end{cases}.$$

So we will get

$$(T_\sigma e_j)(\theta) = \sigma(\theta, j)e_j(\theta).$$

Hence,

$$\sum_{n=-\infty}^{\infty} \|T_\sigma e_n\|^p = (2\pi)^{-p/2}\|\sigma\|^p.$$

□

Theorem 2.2.24. *The pseudo-differential operator $T_\sigma : L^2(S^1) \to L^2(S^1)$ is a Hilbert Schmidt operator if and only if $\sigma \in L^2(S^1 \times \mathbb{Z})$. Moreover, if T_σ is a Hilbert Schmidt operator then*

$$\|T_\sigma\| = (2\pi)^{-1/2}\|\sigma\|.$$

The proof of the previous theorem gives rises a necessary condition on a measurable function σ on $S^1 \times \mathbb{Z}$ for T_σ to be bounded. Therefor, a necessary condition for T_σ to be bounded is

$$\sup_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} |\sigma(\theta, n)|^2 d\theta < \infty.$$

To show that this condition is not sufficient, we can use the following example.

Examples 2.2.25. *Let σ is a measurable function on $S^1 \times \mathbb{Z}$, defined as*

$$\sigma(\theta, n) = e^{-\iota n\theta}, \ \ \theta \in [-\pi, \pi], \ \ n \in \mathbb{Z}.$$

We can easily verify that the above example will give a contradiction on the sufficient condition for the boundedness of L^2. Before giving a sufficient condition for the boundedness of T_σ, as a corollary to that theorem, we are going to prove **Young's inequality**.

Lemma 2.2.26. *Let $a_n \in L^1(\mathbb{Z})$, $b_n \in L^p(\mathbb{Z})$, where $1 \leq p \leq \infty$. Then the function $a*b$ defined by*

$$(a * b)_n = \sum_{k=-\infty}^{\infty} a_{n-k}b_k, \quad n \in \mathbb{Z},$$

is in $L^p(\mathbb{Z})$ and

$$\|a * b\| \leq \|a\|\|b\|.$$

We can prove this lemma just by using the definition of convolution and integral Minkowski's inequality. Hence we are leaving this proof. Now, we are ready to give a sufficient condition for the pseudo-differential operator to be bounded.

Theorem 2.2.27. *Let σ is a measurable function on $S^1 \times \mathbb{Z}$. Suppose that, we can find a positive constant C and a function w on $L^1(\mathbb{Z})$ such that*

$$|\hat{\sigma}(m,n)| \leq C|w(n)|,$$

where

$$\hat{\sigma}(m,n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\iota m\theta}\sigma(\theta,n)d\theta.$$

Then, T_σ is a bounded linear operator and

$$\|T_\sigma\| \leq C\|w\|.$$

Proof. Let $f \in C^\infty(S^1)$. Then, we have

$$\|T_\sigma f\|^2 = \int_{-\pi}^{\pi}\left|\sum_{k=-\infty}^{\infty} e^{\iota k\theta}\left(\sum_{n=-\infty}^{\infty}\hat{\sigma}(k-n,n)\hat{f}(n)\right)\right|^2 d\theta.$$

Using the orthogonality of $\{e_n\}_{n=-\infty}^{\infty}$ in $L^2(S^1)$, we will get

$$\|T_\sigma f\|^2 \leq \sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}|\hat{\sigma}(k-n,n)||\hat{f}(n)|\right)^2.$$

Now using the condition given to us, we will get

$$\|T_\sigma f\|^2 \leq 2\pi C^2 \sum_{k=-\infty}^{\infty}|(|w| * |\hat{f}|)(k)|^2.$$

Now, by using the Young's inequality we will get the result. □

Now, we are going to give a sufficient condition to get the boundedness of T_σ for a general measurable function σ and the compactness of the pseudo-differential operator. Like in Fourier multiplier

$$\int_{-\pi}^{\pi}|\sigma(\theta,n)|^2 d\theta \to 0,$$

is not sufficient for a pseudo-differential operator to be bounded. To see this consider, the following example.

Examples 2.2.28. *Consider*

$$\sigma(\theta,n) = \begin{cases} \frac{1}{\log n}e^{-\iota n\theta} & n > 1 \\ 0 & n \leq 1 \end{cases},$$

and

$$f(\theta) = \sum_{n=1}^{\infty}\frac{1}{n}e^{\iota n\theta}.$$

Clearly, $T_\sigma f \notin L^2(S^1)$. Hence this condition will not be enough for the boundedness of T_σ.

Theorem 2.2.29. *Let σ is a measurable function on $S^1 \times \mathbb{Z}$. Suppose that, we can find a function C on $\mathbb{Z}$ and a function w on $L^1(\mathbb{Z})$ such that*

$$\lim_{|n|\to\infty} C(n) = 0,$$

and

$$|\hat{\sigma}(m,n)| \leq C(n)|w(m)|, \quad m,n \in \mathbb{Z}.$$

Then T_σ is compact.

Proof. For all positive N, we define the function σ_N on $S^1 \times \mathbb{Z}$ such that

$$\sigma_N(\theta, n) = \begin{cases} \sigma(\theta, n) & |n| \le N \\ 0 & |n| > N \end{cases},$$

for all $\theta \in [-\pi, \pi]$ and $n \in \mathbb{Z}$. So for $N = 1, 2, \ldots$, we have

$$\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |\sigma_N(\theta, n)|^2 d\theta = \sum_{n=-N}^{N} \int_{-\pi}^{\pi} |\sigma(\theta, n)|^2 d\theta.$$

And hence T_σ is a Hilbert Schmidt operator. Now let $\tau_N = \sigma - \sigma_N$. Then by definition of σ_N, we will get

$$\tau_N(\theta, n) = \begin{cases} 0 & |n| \le N \\ \sigma(\theta, n) & |n| > N \end{cases}.$$

Let ϵ be a positive number. Then there exits a positive integer N_0 such that

$$|C(n)| < \epsilon,$$

whenever $|n| > N_0$. So for $N > N_0$, we have

$$\|T_{\sigma_N} - T_\sigma\|^2 = \int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} e^{\iota n\theta} \sigma_N(\theta, n) \hat{f}(n) \right|^2 d\theta.$$

Now, we are going to use the same proof as we had done it previous proof and the condition given in the theorem statement, we will get

$$\|(T_{\sigma_N} - T_\sigma)f\|^2 \le 2\pi\epsilon^2 \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |w(k-n)||\hat{f}(n)| \right)^2,$$

and hence using the same argument used in the proof of last theorem, we will get

$$\|T_{\sigma_N} - T_\sigma\| \le \sqrt{B}\epsilon.$$

In other words, T_σ is the limit of T_{σ_N} which are compact and hence T_σ is a compact operator. □

Now we are going to give the condition for L^p boundedness of the pseudo-differential operator. We are leaving the proof as this can be prove by using the same technique used in the earlier proof and a non trivial result which we will assume. First of all, we will give a sufficient condition for σ on $\mathbb{Z}$ for which the Fourier multiplier T_σ to be bounded where $T_\sigma : L^p(S^1) \to L^p(S^1), 1 < p < \infty$.

Lemma 2.2.30. *Let σ is a measurable function $\mathbb{Z}$ and let k be the smallest integer greater then $\frac{1}{2}$. Suppose there exists a positive number C such that*

$$|(\partial^j \sigma)(n)| \le C\langle n \rangle^{-j}, \quad n \in \mathbb{Z},$$

for $0 \le j \le k$, where ∂^j is the differential operator given by

$$(\partial^j \sigma)(n) = \sum_{l=0}^{j} (-1)^{j-1} \binom{j}{l} \sigma(n+1), \quad n \in \mathbb{Z}.$$

and

$$\langle n \rangle = (1 + |n|^2)^{1/2}.$$

Then T_σ is a bounded linear operator on L^p and there exists a positive constant B, depending upon p such that

$$\|T_\sigma f\| \le BC\|f\|.$$

So now we are ready to give a sufficient condition for the L^p boundedness of the pseudo-differential operator T_σ.

Theorem 2.2.31. *Let σ is a measurable function on $S^1 \times \mathbb{Z}$ and let k be the smallest integer greater than $\frac{1}{2}$. Suppose, we can find a positive number C and a function w on $L^1(S^1)$ such that*

$$|(\partial^j \hat{\sigma})(m,n)| \leq C|w(n)|\langle n\rangle^{-j}, \quad n, m \in \mathbb{Z},$$

for $0 \leq j \leq k$, where $\hat{\sigma}(m,n)$ is defined as earlier and ∂^j is the differential operator given by

$$(\partial^j \sigma)(n) = \sum_{l=0}^{j} (-1)^{j-1} \binom{j}{l} \sigma(n+1), \quad n \in \mathbb{Z}.$$

and

$$\langle n\rangle = (1 + |n|^2)^{1/2}.$$

Then T_σ is a bounded linear operator on L^p and there exists a positive constant B, depending upon p such that

$$\|T_\sigma\| \leq BC\|w\|.$$

To prove this theorem we will use Fubini's theorem, previous lemma and the techniques of the previous theorem which is easy to prove.

2.2.6 Pseudo-Differential Operator on $\mathbb{Z}$

This section is the "dual" of the previous section i.e. in the similar way, we can define the pseudo-differential operator on $\mathbb{Z}$ also and a similar kind of results can also be obtained.

Let $a \in \mathbb{Z}$. Then the Fourier transform $\mathcal{F}_{\mathbb{Z}} a$ on a is the function on the unit circle S^1 centered at origin which is defined as

$$(\mathcal{F}_{\mathbb{Z}} a)(n) = \sum_{n=-\infty}^{\infty} a_n e^{\iota n\theta}, \quad \theta[-\pi, \pi].$$

So, we can see that

$$\mathcal{F}_{\mathbb{Z}} = \mathcal{F}_{\mathbb{S}^1}^{-1}.$$

So, the Plancherel formula for Fourier series gives

$$\sum_{n=-\infty}^{\infty} |a(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |(\mathcal{F}_{\mathbb{Z}} a)(\theta)| d\theta.$$

The Fourier inversion formula is given by

$$a(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\iota n\theta} (\mathcal{F}_{\mathbb{Z}} a)(\theta) d\theta, \quad n \in \mathbb{Z}.$$

Definition 2.2.32. *Let $\sigma : \mathbb{Z} \times S^1 \to \mathbb{C}$ be a measurable function. Then for every sequence $a \in L^2(\mathbb{Z})$, we define the sequence $T_\sigma a$ formally by*

$$(T_\sigma a)(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma(n,\theta) e^{-\iota n\theta} (\mathcal{F}_{\mathbb{Z}} a)(\theta) d\theta, \quad n \in \mathbb{Z}.$$

T_σ is called as pseudo-differential operator on $\mathbb{Z}$ corresponding to symbol σ whenever the integral exists for all $n \in \mathbb{Z}$.

As same in last section, we are going to see the conditions on pseudo-differential operators to be Hilbert Schmidt, bounded on L^p and all.

Theorem 2.2.33. *The pseudo-differential operator $T_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is Hilbert Schmidt if and only if $\sigma \in L^2(\mathbb{Z} \times S^1)$. Moreover, if T_σ is a Hilbert Schmidt operator then*

$$\|T_\sigma\| = (2\pi)^{-1/2}\|\sigma\|.$$

Proof. We will start the proof by considering e_k,

$$e_k(n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}.$$

For $k \in \mathbb{Z}$, we get

$$(\mathcal{F}_{\mathbb{Z}} e_k)(\theta) = e^{\iota k\theta}, \quad \theta \in [-\pi, \pi],$$

and hence

$$(T_\sigma e_k)(n) = (\mathcal{F}_{S^1}\sigma)(n, n-k),$$

for all $n \in \mathbb{Z}$, where $(\mathcal{F}_{S^1}\sigma)(n,')$ is the Fourier transform used in the last section. Hence, by using the Fubini's theorem and Plancherel formula for Fourier transform, we will get

$$\|T_\sigma\|^2 = \frac{1}{2\pi}\|\sigma\|^2,$$

and hence the result follows. □

Before going to L^p boundedness and compactness of the pseudo-differential operator, we will give the simplest result about L^2 boundedness of the pseudo-differential operator.

Theorem 2.2.34. *Let σ be a measurable function on $\mathbb{Z} \times S^1$, such that we can find a positive constant C and a function w on $L^2(\mathbb{Z})$ for which*

$$|\sigma(n, \theta)| \leq C|w(n)|,$$

for all $n \in \mathbb{Z}$ and almost all $\theta \in [-\pi, \pi]$. Then $T_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is a bounded linear operator. Moreover,

$$\|T_\sigma\| \leq C\|w\|.$$

We are leaving the proof of this particular result as we can easily prove this result by using Schwarz inequality and Plancherel's theorem.

Now we are ready to give a condition on σ for the L^p boundedness of the pseudo-differential operator on $L^p, 1 \leq p < \infty$.

Theorem 2.2.35. *Let σ be a measurable function on $\mathbb{Z} \times S^1$, such that we can find a positive constant C and a function w on $L^1(\mathbb{Z})$ for which*

$$|(\mathcal{F}_{S^1}\sigma)(n, m)| \leq C|w(m)|,$$

for all $n, m \in \mathbb{Z}$. Then $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is a bounded linear operator. Moreover,

$$\|T_\sigma\| \leq C\|w\|.$$

Proof. Let $a \in L^1(\mathbb{Z})$. Then for all $n \in \mathbb{Z}$, we get

$$(T_\sigma a)(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\iota n\theta} \sigma(n,\theta)(\mathcal{F}_{\mathbb{Z}} a)(\theta) d\theta.$$

Now, by using the definition of Fourier transform on $\mathbb{Z}$ and the definition of convolution, we will get

$$(T_\sigma a)(n) = ((\mathcal{F}_{S^1}\sigma)(n,') * a)(n).$$

So

$$\|T_\sigma a\|^p \leq C^p \sum_{n=-\infty}^{\infty} |(|w| * |a|)(n)|^p.$$

The above result, we will get by using the hypothesis given in the statement of the theorem. Now by using the Young's inequality, we will get

$$\|T_\sigma\| \leq C\|w\|.$$

Since L^1 is dense in L^p hence the result is true for $1 \leq p < \infty$. $\square$

Theorem 2.2.36. *Let σ be a measurable function on $\mathbb{Z} \times S^1$, such that we can find a function C on $\mathbb{Z}$ and a function w on $L^1(\mathbb{Z})$ for which*

$$|(\mathcal{F}_{S^1}\sigma)(n,m)| \leq C(n)|w(m)|,$$

for all $n, m \in \mathbb{Z}$ and

$$\lim_{|n|\to\infty} C(n) = 0.$$

Then $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is a compact linear operator for $1 \leq p < \infty$.

The proof of the theorem is very easy. Using the same technique, used to prove the compactness of the Fourier multiplier and then proceed in the similar fashion as in the proof of L^p boundedness.

At the end of the chapter, we are going to give a application of pseudo-differential operator in numerical analysis.

Remark 2.2.37. *Since $w \in L^i(\mathbb{Z})$, roughly it follows that*

$$w(n) = \mathcal{O}(|m|^{-(1+\alpha)}),$$

as $|m| \to \infty$ where α is a positive number.

Theorem 2.2.38. *Let σ be a symbol satisfying the hypothesis of the L^p boundedness of the pseudo-differential operator. Then for $1 \leq p < \infty$, the matrix A_σ of the pseudo-differential operator defined on L^p is given by*

$$A_\sigma = (\sigma_{nk})_{n,k\in\mathbb{Z}},$$

where

$$\sigma_{nk} = (\mathcal{F}_{S^1}\sigma)(n, n-k).$$

Proof of the theorem is left as it is trivial. Just use the definition of pseudo-differential operator and the proof of L^p boundedness.

Now from the remark, we will have

$$\sigma_{nk} = \mathcal{O}(|n-k|^{-(1+\alpha)}),$$

as $|n-k| \to \infty$. In other words, the off diagonal entries in A_σ, are small and the matrix A_σ can be seen as almost diagonal.

Examples 2.2.39.

$$\sigma(n,\theta) = \left(n+\frac{1}{2}\right)^{-2} \sum_{k=-\infty}^{\infty} e^{\iota k\theta} \left(k+\frac{1}{2}\right)^{-2}, \ \ n \in \mathbb{Z}, \ \ \theta \in S^1.$$

Then

$$\sigma_{nk} = \left(n+\frac{1}{2}\right)^{-2} \left(n-k+\frac{1}{2}\right)^{-2}, \quad n,k \in \mathbb{Z}.$$

Chapter 3

Pseudo-Differential operator on $\mathbb{R}^n$

3.1 Fourier Transform on $\mathbb{R}^n$

Fourier transform will be used in defining the pseudo-differential operator in the coming chapter. In Fourier theory mainly for $L^2(\mathbb{R}^n)$, we are having two very important results namely Fourier inversion formula for $\mathcal{S}$ and the Plancherel's theorem.

3.1.1 Notations and Preliminaries

In this subsection, we are going to talk about the basic notations which we are going to use in the later sections. We are also going to introduce multi-index notations which is essential in defining the Fourier transform on $\mathbb{R}^n$.

Here $\mathbb{R}^n$ is the usual euclidean space. We denote points in $\mathbb{R}^n$ by x, y, ξ, η etc. We also define the usual inner product which is defined on $\mathbb{R}^n$ and the norm also.

On $\mathbb{R}^n$, the simplest differential operators are $\frac{\partial}{\partial x_j}, j = 1, 2, \ldots n$. We sometimes denote $\frac{\partial}{\partial x_j}$ by ∂_j. But we always take the differential operator D_j given by $D_j = -\iota\partial_j, \iota^2 = -1$, which is better in explaining some formulas. The most general partial differential operator of order m on $\mathbb{R}^n$ is defined by

$$\sum_{a_1+a_2+\ldots+a_n\leq m} a_{\alpha_1,\alpha_2,\ldots,\alpha_n}(x) D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the non negative constant and $a_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ is a complex valued function which is infinite times differentiable function on $\mathbb{R}^n$. To simplify the expression, we let

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n),$$

$$|a| = \sum_{j=1}^{n} \alpha_j$$

and

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}.$$

Hence, the expression will become

$$\sum_{|n|\leq m} a_\alpha(x) D^\alpha.$$

For each fixed x in $\mathbb{R}^n$, we can see the above expression as the polynomial in the derivatives D_j. If we replace ξ in the above expression in place of D, we will get

$$\sum_{|n|\leq m} a_\alpha(x) \xi^\alpha.$$

Hence, we got a polynomial in ξ over $\mathbb{R}^n$ which is denoted by $P(x,\xi)$. We call $P(x,\xi)$ the **symbol** corresponding to the operator D.

In this chapter, we are going to study about the partial differential operator and its generalized form known as pseudo-differential operator.

3.1.2 Remarks and Formulas

Before beginning of the main content let us give some remarks and formulas which we are going to use through out the next sections.

1. We denote set of all complex numbers by $\mathbb{C}$ and set of all real numbers by $\mathbb{R}$.
2. All vector spaces are assumed over the field of complex numbers. All functions are assumed to be complex valued unless specified.
3. Although the differential operator $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots, D_n^{\alpha_n}$ is more useful but at many places, we will use the differential operator $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots,\partial_n^{\alpha_n}$. In case, we are differentiating with respect to x variable we write ∂_x^α.
4. The set of all infinite time differentiable functions is denoted by $C^\infty(\mathbb{R}^n)$.
5. The L^p norm on $\mathbb{R}^n$ is given is the usual p norm only and is denoted by $\|f\|_p$.

   Let α and β are two multi-indices.
6. $\beta \le \alpha$ means that $\beta_j \le \alpha_j$ for $j = 1, 2, \dots, n$.
7. $\alpha - \beta$ is the multi-index given by $(\alpha_1 - \beta_1, \alpha_2 - \beta_2 \dots, \alpha_n - \beta_n)$.
8. $\alpha! = \alpha_1!\alpha_2!\dots\alpha_n!$.
9. $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\dots\binom{\alpha_n}{\beta_n}$.
10. $D^\alpha(fg) = \sum_{\beta\le\alpha}(D^\beta(f))(D^{\alpha-\beta}(g))$. This formula is known as Leibnitz formula.

3.1.3 The Convolution

In this subsection, we mainly introduce two main subsets of $C^\infty(\mathbb{R}^n)$ mainly denoted by $C_0^\infty(\mathbb{R}^n)$ and $\mathcal{S}$ i.e. the set of all infinitely times differential functions with compact support and the Schwartz class. The main aim of this subsection is to show that these subsets are dense in $L^p(\mathbb{R}^n), 1 \le p < \infty$.

We are starting this subsection with the Young's inequality, which we are not going to prove as this can be easily proved by using the Fubini's theorem.

Theorem 3.1.1. ***(Young's inequality)*** *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n), 1 \le p < \infty$. Then the integral*

$$\int_{\mathbb{R}^n} f(x-y)g(y)dy,$$

exists for almost all $x \in \mathbb{R}^n$, then $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\| \le \|f\|\|g\|.$$

We call $f * g$ to be the convolution.

As we know that $C_0(\mathbb{R}^n)$ denotes the set of all continuous functions with compact support. Now from the fact of measure theory, we assume the following result.

Proposition 3.1.2. *$C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n), 1 \leq p < \infty$.*

Now based on the above result we are going to prove the following result.

Proposition 3.1.3. *(L^p-**Continuity of Translation**) Let $f \in L^p(\mathbb{R}^n), 1 < p < \infty$. Then*

$$\lim_{|s| \to \infty} \|f_x - f\|_p = 0,$$

where the function f_x is defined as

$$f_x(y) = f(x + y), \quad y \in \mathbb{R}^n.$$

Proof. Let $\delta > 0$ and $f \in L^p(\mathbb{R}^n)$ then there is a function $g \in C_0(\mathbb{R}^n)$ such that

$$\|f - g\| < \frac{\delta}{3}.$$

Now using the triangular inequality, we get

$$\|f_x - f\| < \delta.$$

If $|x|$ is very small. □

Theorem 3.1.4. *$\phi \in L^1(\mathbb{R}^n)$ be such that*

$$\int_{\mathbb{R}^n} \phi(x)dx = a.$$

For $\epsilon > 0$, define the function ϕ_ϵ by

$$\phi_\epsilon(x) = \epsilon^{-n}\phi\left(\frac{x}{\epsilon}\right), \quad x \in \mathbb{R}^n.$$

Then for any function $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$, we have $f * \phi_\epsilon \to af$ in $L^p(\mathbb{R}^n)$ as $\epsilon \to 0$.

Proof.

$$\int_{\mathbb{R}^n} \phi_\epsilon(x)dx = a$$

for all $\epsilon > 0$, by using Minkowski's inequality in integral form, we get

$$\begin{aligned}
\|f * \phi_\epsilon - af\|_p &= \left(\int_{\mathbb{R}^n} |f * \phi_\epsilon(x) - af(x)|^p dx\right)^{\frac{1}{p}}. \\
&= \left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} (f(x - y) - f(x))\phi_\epsilon(y)dy\right|^p dx\right)^{\frac{1}{p}}. \\
&= \left(\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} (f(x - \epsilon y) - f(x))\phi(y)dy\right|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Now by using Fubini's theorem and solving we will get

$$\|f * \phi_\epsilon - af\|_p \leq \int_{\mathbb{R}^n} |\phi(y)| \|f_{-\epsilon y} - f\|_p dy.$$

Now by the previous proposition, we will get $\|f_{-\epsilon y} - f\|_p \to 0$ as $\epsilon \to 0$. Also by using the triangular inequality, we will get $\|f_{-\epsilon y} - f\|_p \leq 2\|f\|$. Hence by using the dominant Convergence theorem of Lebsegue integration, we will get

$$\|f * \phi_\epsilon - af\|_p \to 0,$$

as $\epsilon \to 0$. □

Definition 3.1.5. C_0^∞ *represents the set of all infinitely differentiable functions with compact support.*

Definition 3.1.6. $\mathcal{S}$ *represents the set of all differentiable functions* $\phi \in \mathbb{R}^n$ *such that for all multi-index* α, β, *we have*

$$\sup_{x\in\mathbb{R}^n} |x^\alpha (D^\beta)(x)| < \infty.$$

Remark 3.1.7. *Clearly, from above two definitions, we can see that* C_0^∞ *is contained in* $\mathcal{S}$. *But the converse of this is not true as we can see from the following example.*

Examples 3.1.8. *Consider*

$$\phi(x) = e^{-|x|^2}.$$

Clearly we can see that $\phi \in \mathcal{S}$ *but* $\phi \notin C_0^\infty$, *as it does not have any compact support.*

Now, we are going to give some small results which we are not going to prove. And at the end of this section, we will show that, the two subsets we have taken are dense in $L^p(\mathbb{R}^n), 1 \leq p < \infty$.

Proposition 3.1.9. *Let* $\phi \in \mathcal{S}$ *and* $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$. *Then* $f * \phi \in C^\infty(\mathbb{R}^n)$ *and*

$$\partial^\alpha (f * \phi) = f * \partial^\alpha \phi.$$

We can prove the above proposition by using the Holder's inequality.

Proposition 3.1.10. *Let* f *and* g *are two functions on* $\mathbb{R}^n$ *with the compact support, then the convolution* $f * g$ *also have the compact support. In fact,*

$$supp(f * g) \sqsubseteq supp(f) + supp(g).$$

To prove this just use the definition of f and g and definition of set addition.

Theorem 3.1.11. $C_0^\infty(\mathbb{R}^n)$ *is dense in* $L^p(\mathbb{R}^n), 1 \leq p < \infty$.

Proof. Let $\phi \in \mathcal{S}$ be a non negative function such that

$$\int_{\mathbb{R}^n} \phi(x)dx = 1.$$

For $\epsilon > 0$, define the function ϕ_ϵ by

$$\phi_\epsilon(x) = \epsilon^{-n}\phi\left(\frac{x}{\epsilon}\right), \quad x \in \mathbb{R}^n.$$

Then for all functions $g \in C_0(\mathbb{R}^n)$, we have $g * \phi_\epsilon \in C_0^\infty(\mathbb{R}^n)$ and

$$g * \phi_\epsilon \to g,$$

as $\epsilon \to 0$. Let $\delta > 0$ and $f \in L^p(\mathbb{R}^n)$. Then we will get a function $g \in C_0(\mathbb{R}^n)$ such that

$$\|f - g\| < \frac{\delta}{2}.$$

We can find a function $\varphi \in C_0^\infty(\mathbb{R}^n)$, such that

$$\|g - \varphi\| < \frac{\delta}{2}.$$

Hence by using the triangular inequality, we will get

$$\|f - \varphi\| < \delta.$$

□

Remark 3.1.12. *Clearly, from the above theorem we get* $\mathcal{S}$, *is dense in* $L^p(\mathbb{R}^n), 1 \leq p < \infty$.

3.1.4 Fourier Transformation

We are starting this subsection with the definition of Fourier transform.

Definition 3.1.13. *Let $f \in L^1(\mathbb{R}^n)$, we define $\hat{f}$ as*

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\iota x\xi} f(x)dx, \quad \xi \in \mathbb{R}^n.$$

The function $\hat{f}$ is called as the Fourier transform denoted by $\mathcal{F}f$.

Now we are going to give a basic result which is the Fourier transform of convolution. Fourier transform of the convolution of two function is the pointwise multiplication of indivisible Fourier transform of the functions.

Theorem 3.1.14. *Let $f, g \in L^1(\mathbb{R}^n)$. Then*

$$\widehat{(f * g)} = (2\pi)^{n/2} \hat{f}\hat{g}.$$

We are not giving proof of the above theorem as it is a very simple application of Fubini's theorem. Next, we are going to give some basic results about the Fourier transform and the differential operator which we are not going to prove as we can prove them very easily by the means of change of variable.

Proposition 3.1.15. *Let $\phi \in \mathcal{S}$. Then*

1. $\left(\widehat{D^\alpha \phi}\right)(\xi) = \xi^\alpha \hat{\phi}(\xi)$ *for every multi-index α.*
2. $\left(D^\beta \hat{\phi}\right)(\xi) = ((-x)^\beta \phi)^\wedge(\xi)$ *for every multi-index β.*
3. $\hat{\phi} \in \mathcal{S}$.

Proposition 3.1.16. ***(Riemann-Lebesgue Lemma)*** *: Let $f \in L^1\mathcal{R}$, then*

1. *$\hat{f}$ is continuous on $\mathbb{R}^n$.*
2. $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.
3. *$f_j \to f$ in $L^1(\mathbb{R}^n)$ implies that $\hat{f}_j \to \hat{f}$ uniformly.*

Proof. Let $f_j \to f$ in $L^1(\mathbb{R}^n)$, then

$$|\hat{f}_j - \hat{f}| \leq (2\pi)^{-n/2} \|f_j - f\|.$$

Hence $\hat{f}_j \to \hat{f}$ uniformly on $\mathbb{R}^n$. This proves the part (3). Then for $f \in \mathcal{S}$ we will get the result for part (1), (2) and hence by using the fact that $\mathcal{S}$ is dense in L^p and by using part (3) we will get the complete result. □

Now, we are defining some new operators for a measurable function on $\mathbb{R}^n$.

Definition 3.1.17. *Let f be a measurable function on $\mathbb{R}^n$. For a fixed $y \in \mathbb{R}^n$, we define $T_y f$ and $M_y f$ by*

$$(T_y f)(x) = f(x + y) \quad x \in \mathbb{R}^n,$$

and

$$(M_y f)(x) = e^{\iota x.y} f(x), \quad x \in \mathbb{R}^n.$$

Let α is a nonzero number then we define the function $D_\alpha f$ by

$$(D_\alpha f)(x) = f(\alpha x), \quad x \in \mathbb{R}^n.$$

We call these operators as Translation, Modulation and Dilation respectively.

Now, we are going to give the Fourier transform of the operators defined above.

Proposition 3.1.18. *Let us assume $f \in \mathbb{R}^n$. Then the Fourier transform of the operators defined above is given by:*

1. $\widehat{(T_y f)}(x) = (M_y \hat{f})(x) \quad x \in \mathbb{R}^n$.
2. $\widehat{(M_y f)}(x) = (T_{-y} \hat{f})(x) \quad x \in \mathbb{R}^n$.
3. $\widehat{(D_\alpha f)}(x) = |\alpha|^{-n}(D_{\frac{1}{\alpha}} \hat{f})(x) \quad x \in \mathbb{R}^n$.

We can prove above result by means of change of variables.

Proposition 3.1.19. *Let $\phi(x) = e^{-\frac{|x|^2}{2}}$. Then $\hat{\phi}(\xi) = e^{-\frac{|\xi|^2}{2}}$.*

Proof. First of all, we will compute

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\iota x.\xi - |x|^2} dx.$$

Note that

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\iota x.\xi - |x|^2} dx = \prod_{j=1}^{n} (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-\iota x_j.\xi_j - x_j^2} dx.$$

Hence it is sufficient to compute

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-\iota t.\eta - t^2} dt, \quad \eta \in (-\infty, \infty).$$

But

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-\iota t.\eta - t^2} dt = e^{\frac{-t^2}{4}} \int_L e^{-x^2} dx.$$

Where L is the contour $Imgz = \frac{\eta}{2}$ in the z plane. Using the Cauchy's integral formula and the fact that the integral goes to zero very fast, we get

$$\int_L e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$

Hence, we will get

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\iota x.\xi - |x|^2} dx = 2^{\frac{-n}{2}} e^{\frac{-|\xi|^2}{4}}.$$

So by using the result of Fourier transform of Dilation, we will get the required Fourier transform. □

Proposition 3.1.20. ***(Adjoint Formula)**: Let f and g be functions in $L^1(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} \hat{f}(x) g(x) dx = \int_{\mathbb{R}^n} f(x) \hat{g}(x) dx.$$

Again proof of the above proposition is a application of Fubini's theorem.

Theorem 3.1.21. ***(Fourier Inversion Formula)**: $(\hat{f})^\vee = f$ for all functons $f \in \mathcal{S}$. Here the operation is defined as*

$$g^\vee(x) = \int_{\mathbb{R}^n} e^{\iota x.\xi} g(\xi) d\xi dx.$$

The defined function is known as Inverse Fourier transform.

Proof. We have

$$(\hat{f})^{\vee}(x) = \int_{\mathbb{R}^n} e^{\iota x.\xi} \hat{f}(\xi) d\xi.$$

Let $\epsilon > 0$. Define

$$I_\epsilon(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{\iota x.\xi - \frac{\epsilon^2|\xi|^2}{2}} \hat{f}(\xi) d\xi.$$

Let

$$g(\xi) = e^{\iota x.\xi - \frac{\epsilon^2|\xi|^2}{2}} = (M_x D_\epsilon \phi)(\xi),$$

where

$$\phi(\xi) = e^{\frac{-|\xi|^2}{2}}.$$

Then by the formula for Fourier transform of Modulation and Dilation, we get

$$\hat{g}(\eta) = \epsilon^{-n} e^{\frac{-|\eta - x|^2}{2\epsilon^2}}.$$

Now use adjoint formula to get

$$I_\epsilon(x) = (2\pi)^{-1/2} (f * \phi_\epsilon)(x),$$

where

$$\phi_\epsilon(x) = \epsilon^{-n} \phi(\frac{x}{\epsilon}).$$

And hence, we will get

$$I_\epsilon \to (2\pi)^{-1/2} \left(\int_{\mathbb{R}^n} e^{\frac{-|x|^2}{2}} dx \right) f = f.$$

Hence after using Dominant convergence theorem we will get

$$I_\epsilon \to (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{\iota x.\xi} \hat{f}(\xi) d\xi,$$

for every $x \in \mathbb{R}^n$ as $\epsilon \to 0$. Hence,

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{\iota x.\xi} \hat{f}(\xi) d\xi = f(x).$$

□

Remark 3.1.22. *The main consequence of the Fourier inversion formula is that the Fourier transform is one to one on $\mathcal{S}$. If we define $\tilde{f}$ by*

$$\tilde{f}(x) = f(-x), \quad x \in \mathbb{R}^n.$$

Then the Fourier inversion formula will be equivalent to

$$(\hat{f})^{\vee} = \tilde{f}.$$

Proposition 3.1.23. *Let $f, g \in \mathcal{S}$ then $f * g \in \mathcal{S}$.*

Proof can be done easily.

Theorem 3.1.24. ***(Plancherel's Theorem)**: The mapping $f \to \hat{f}$ defined on $\mathcal{S}$ can be extended uniquely to a unitary operator on $L^2(\mathbb{R}^n)$.*

Proof of this theorem is also left. There is some other form of Fourier inversion formula.

Theorem 3.1.25. *Let $f \in \mathbb{R}^n$ and $\hat{f} \in \mathbb{R}^n$, then $(\hat{f})^{\vee} = f$.*

Proposition 3.1.26. *We have $|x^\alpha| = |x|^{|\alpha|}$.*

3.1.5 Distribution

In this subsection, we are just going to see about the distribution and tempered distribution which we are going to use in the later chapters. If we will need some other concepts, we will define at that point only.

Definition 3.1.27. *A sequence of functions ϕ_n in the Schwartz class is said to converge to zero in Schwartz class if for all multi-indecies α and β, we have*

$$\sup_{x\in\mathbb{R}^n} |x^\alpha (D^\beta \phi_n)(x)| \to 0,\ n \to \infty.$$

Definition 3.1.28. *A linear functional T on $\mathcal{S}$ is said to be tempered distribution if for any sequence $(x_j)_{j=1}^\infty$ of functions converging to zero in $\mathcal{S}$, we have*

$$T(\phi_j) \to 0, j \to \infty.$$

Definition 3.1.29. *Let f be a measurable function defined on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|^N)} dx < \infty,$$

for some positive integer then we call f to be tempered function.

Proposition 3.1.30. *Let f be a tempered function defined on $\mathbb{R}^n$. Then the linear functional T_f on $\mathcal{S}$ defined by*

$$T_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx, \quad \phi \in \mathcal{S},$$

is a tempered distribution.

To prove the above proposition just use the definition of convergence of sequence in Schwartz class and tempered function.

Proposition 3.1.31. *Any function $f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty$, is a tempered function.*

Proposition 3.1.32. *Let $f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty$. Then the linear functional T_f on $\mathcal{S}$ defined by*

$$T_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx, \quad \phi \in \mathcal{S},$$

is a tempered distribution.

Definition 3.1.33. *Let T be a tempered distribution. Then the Fourier transform of T is defined to be linear functional on $\mathcal{S}$ given by*

$$\hat{T}(\phi) = T(\hat{\phi}), \quad \phi \in \mathcal{S}.$$

Proposition 3.1.34. *$\hat{T}$ is also a tempered distribution.*

We can easily prove this statement just by using the definition of Schwartz class, dilation and its Fourier transform.

Theorem 3.1.35. ***(Fourier Inversion Formula)***: *Let T be a tempered distribution. Then*

$$\hat{T} = \tilde{T}$$

where $\tilde{T}$ is defined by

$$\tilde{T}(\phi) = T(\tilde{\phi}) \quad \phi \in \mathcal{S}.$$

3.2 Pseudo-Differential Operator

In this section, we are going to define the pseudo-differential operator on $\mathbb{R}^n$ by using some particular kind of functions known as symbol. We are also going to define asymptotic expansion and at the end, we will see some basic properties and L^P boundedness of pseudo-differential operators.

3.2.1 Definition And Asymptotic Expansion

We will start this chapter by recalling the basic linear partial differential operator $P(x, D)$ on $\mathbb{R}^n$ is given by

$$P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

where the $a'_\alpha s$ are functions defined on $\mathbb{R}^n$. If we replace D^α by the monomial ξ^α on $\mathbb{R}^n$, then we obtain the so-called symbol

$$P(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha.$$

In order to get another representation of the differential operator $P(x, D)$, let us take any function $\phi \in \mathcal{S}$. Then by the above two representation and Fourier inversion formula for Schwartz function we will get

$$(P(x, D)\phi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{\iota x.\xi} P(x, \xi) \hat{\phi}(\xi) d\xi.$$

So we represented partial differential operator $P(x, D)$ in terms of symbol by means of Fourier transformation. This representation suggest that we can obtained the more general operators by using a suitable symbol. The new operator obtained are know as pseudo-differential operator. We can impose different conditions on the symbol to obtain different kind of pseudo differential operator.

Definition 3.2.1. *Let $m \in (-\infty, \infty)$. Then we define S^m to be the set of all functions $\sigma(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any two multi-indices α and β, there is a positive constant $C_{\alpha,\beta}$ depending on α and β only, such that*

$$\left| \left(D_x^\alpha D_\xi^\beta \sigma(x, \xi) \right) \right| \le C_{\alpha,\beta} (1 + |\xi|)^{m - |\beta|}, \quad x, \xi \in C_{\alpha,\beta}.$$

We call any function $\sigma \in \cup_{m \in \mathbb{R}} S^m$ is a symbol.

Definition 3.2.2. *Let σ is a symbol. Then the pseudo-differential operator T_σ associated to σ is defined by*

$$(T_\sigma \phi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{\iota x.\xi} \sigma(x, \xi) \hat{\phi}(\xi) d\xi, \quad \phi \in \mathcal{S}.$$

We will give some examples of pseudo-differential operator which can be prove easily.

Examples 3.2.3. *Let*

$$P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

is the linear partial differential operator on $\mathbb{R}^n$. If the coefficients $a_\alpha \in C^\infty$ and have bounded derivative then the polynomial

$$P(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha,$$

is in S^m and hence $P(x, D)$ is the pseudo-differential operator.

Examples 3.2.4. *Let* $\sigma(\xi) = (1+|\xi|^2)^{m/2}, -\infty < m < \infty$. *Then* $\sigma \in S^m$ *and* T_σ *is a pseudo-differential operator. We can prove it just by using the definition of pseudo-differential operator.*

Now we will give a relation between the equality of symbols and the pseudo-differential operator but before that we need a result.

Lemma 3.2.5. *Let* f *be a continuous tempered function such that*

$$T_f(\phi) = 0, \quad \phi \in \mathcal{S}.$$

Then f *is identically zero on* $\mathbb{R}^n$.

Proposition 3.2.6. *Let* σ *and* τ *be two symbol such that*

$$T_\sigma = T_\tau.$$

Then $\sigma = \tau$.

To prove the proposition just use the definition of pseudo-differential operator and the above lemma.

Proposition 3.2.7. *Let* σ *be a symbol. Then* T_σ *maps Schwartz class to Schwartz class elements.*

Definition 3.2.8. *Let* $\sigma \in S^m$. *Suppose we can find* $\sigma_j \in S^{m_j}$ *where*

$$m = m_0 > m_1 > m_2 > \ldots > m_j > \to -\infty, j \to \infty,$$

such that

$$\sigma - \sum_{j=0}^{N-1} \sigma_j \in S^{m_N},$$

for every positive integer N. *Then we call* $\sum_{j=0}^{\infty} \sigma_j$ *is asympototic expansion of* σ *and we write*

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_j.$$

Based on the above definition we have an important theorem but before that we will give a lemma.

Lemma 3.2.9. *There exists a function* $\varphi \in C^\infty(\mathbb{R}^n)$ *such that*

1. $0 \leq \varphi(\xi) \leq 1, \quad \xi \in \mathbb{R}^n$.
2. $\varphi(\xi) = 0, \quad |\xi| \leq 1$.
3. $\varphi(\xi) = 1, \quad |\xi| \geq 2$.

Theorem 3.2.10. *Let* $m = m_0 > m_1 > m_2 > \ldots > m_j > \to -\infty, j \to \infty$. *Suppose* $\sigma_j \in S^{m_j}$. *Then there exists a symbol* $\sigma \in S^{m_0}$ *such that*

$$\sigma \sim \sum_{j=0}^{\infty} \sigma_j$$

. *Moreover, if* τ *be another symbol with the same property then* $\sigma - \tau \in \cap_{m \in \mathbb{R}} S^m$.

Proof. Let $\Psi \in C^\infty(\mathbb{R}^\ltimes)$ be a function satisfying above properties in the lemma. Let (ϵ_j) be a sequence of positive numbers such that $1 > \epsilon_0 > \epsilon_1 > \epsilon_2 > \ldots > \epsilon_j \to 0$ as $j \to \infty$. Define a function $\sigma \in \mathbb{R}^n \times \mathbb{R}^n$ by

$$\sigma(x,\xi) = \sum_{j=0}^{\infty} \psi(\epsilon_j \xi)\sigma_j(x,\xi), \quad x, \xi \in \mathbb{R}^n.$$

Note that for each $(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists a neighbouhood U of (x_0,ξ_0) and a positive integer N such that $\psi(\epsilon_j\xi)\sigma_j(x,\xi) = 0$ for all $(x,\xi) \in U$ and $j > N$. Hence, $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Furthermore, for any multi-index α and by using the properties of ψ, we will get $\partial_\xi^\alpha \{\psi(\epsilon\xi)\} = 0$ if $|\xi| \leq \frac{1}{\epsilon}$ or $|\xi| \geq \frac{2}{\epsilon}$ and $|\partial_\xi^\alpha \{\psi(\epsilon\xi)\}| \leq C_\alpha \epsilon^{|\alpha|}\ \forall\ \xi \in \mathbb{R}^n$, where $C_\alpha = \sup_{\xi\in\mathbb{R}^n} |\{\partial^\alpha\psi\}(\xi)|$. If $\frac{1}{\epsilon} \leq |\xi| \leq \frac{2}{\epsilon}$, then $\epsilon \leq \frac{2}{|\xi|} \leq \frac{4}{1+|\xi|}$. Hence,

$$|\partial_\xi^\alpha \{\psi(\epsilon\xi)\}| \leq C'_\alpha(1+|\xi|)^{-|\alpha|}, \quad \xi \in \mathbb{R}^n,$$

where $C'_\alpha = C_\alpha 4^\alpha$. Now, by using Leibnitz' formula and the fact that $\sigma_j \in S^{m_j}$, we can find the constants $C_{\alpha,\gamma}$ and $C_{j,\beta,\gamma}$ such that

$$|D_\xi^\alpha D_\alpha^\beta \psi(\epsilon_j\xi)\sigma_j(x,\xi)| \leq \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} C_{\alpha,\gamma} C_{j,\beta,\gamma}(1+|\xi|)^{m_j-|\alpha|}.$$

Let $C_{j,\alpha,\beta} = \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} C_{\alpha,\gamma} C_{j,\beta,\gamma}$. Hence, we will get

$$|D_\xi^\alpha D_\alpha^\beta \psi(\epsilon_j\xi)\sigma_j(x,\xi)| \leq C_{j,\alpha,\beta}(1+|\xi|)^{-1}(1+|\xi|)^{m_j-|\alpha|+1}.$$

Now, we choose ϵ_j such that

$$C_{j,\alpha,\beta}\epsilon_j \leq 2^{-j},$$

for all the multi-indices α and β such that $|\alpha+\beta| \leq j$. By definition of ψ, we have

$$\psi(\epsilon_j\xi) = 0,$$

whenever $1+|\xi| \leq \epsilon_j^{-1}$. Hence,

$$|D_\xi^\alpha D_\alpha^\beta \psi(\epsilon_j\xi)\sigma_j(x,\xi)| \leq 2^{-j}(1+|\xi|)^{m_j-|\alpha|+1},$$

whenever $x, \xi \in \mathbb{R}^n$ and $|\alpha+\beta| \leq j$. Now, for any multi-indices α_0 and β_0, we take j_0 so large that $j_0 \geq |\alpha_0+\beta_0|$ and $m_{j_0}+1 \leq m_0$. Write

$$\sigma(x,\xi) = \sum_{j=0}^{j_0-1} \psi(\epsilon_j\xi)\sigma_j(x,\xi) + \sum_{j=j_0}^{\infty} \psi(\epsilon_j\xi)\sigma_j(x,\xi).$$

Clearly, first part of the sum is a finite quantity and hence, it belongs to S^{m_0}. Now,

$$\begin{aligned}|D_\xi^\alpha D_\alpha^\beta \sum_{j=j_0}^{\infty} \psi(\epsilon_j\xi)\sigma_j(x,\xi)| &\leq \sum_{j=j_0}^{\infty} |D_\xi^\alpha D_\alpha^\beta \{\psi(\epsilon_j\xi)\sigma_j(x,\xi)\}|. \\ &\leq 2^{-j_0+1}(1+|\xi|)^{m_0-|\alpha_0|}.\end{aligned}$$

Hence, second part of the sum is also belongs to S^{m_0}, so $\sigma \in S^{m_0}$. Remaining part of the proof can be done easily. □

3.2.2 Partition of Unity

In this subsection, we are going to give three important theorems which we are going to use in next two subsections. First of all, we will construct the partition of unity. Then we use this partition of unity to decompose a signal $\sigma(x,\xi)$ into family $\{\sigma_k(x,\xi)\}$ of symbols with the compact support in the ξ variable. At the end, we will give multi-dimentional version of Taylor's formula with the integral remainder. Out of these three results, we will give proof only for the second result. So, let us start the subsection by the construction of partition of unity.

Theorem 3.2.11. *There is a sequence $\{\varphi_k\}_{k=0}^{\infty}$ of functions in $C_0^{\infty}(\mathbb{R}^n)$ such that*

1. $0 \leq \varphi_k(\xi) \leq 1, \quad \xi \in \mathbb{R}^n, \ k = 0, 1, 2, 3 \ldots,$
2. $\sum_{k=0}^{\infty} \varphi_k(\xi) = 1, \quad \xi \in \mathbb{R}^n,$
3. *for each $\xi \in \mathbb{R}^n$, at least one or at most three of the φ_k's are nonzero,*
4. *supp*$(\varphi_0) \sqsubseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$,
5. *supp*$(\varphi_k) \sqsubseteq \left\{\xi \in \mathbb{R}^n : 2^{k-2} \leq |\xi| \leq 2^{k+1}\right\}, \quad k = 0, 1, 2, 3, \ldots,$
6. *for each multi-index α, there is a multi-index $A_\alpha > 0$ such that*

$$\sup_{\xi \in \mathbb{R}^n} |(\partial^{\alpha}\varphi_k)(\xi)| \leq A_\alpha 2^{-k|\alpha|}.$$

Remark 3.2.12. *Let $\sigma \in S^m$. For $k = 0, 1, 2, \ldots,$ we write*

$$\sigma_k(x,\xi) = \sigma(x,\xi)\varphi_k(\xi)$$

for all $x, \xi \in \mathbb{R}^n$ and

$$K_k(x,z) = (2\pi)^{\frac{-n}{2}} \int\limits_{\mathbb{R}^n} e^{\iota z.\xi}\sigma_k(x,\xi)d\xi$$

for all $x, z \in \mathbb{R}^n$, where $\{\varphi_k\}_{k=0}^{\infty}$ is the partition of unity.

Lemma 3.2.13.

$$|x|^{2N} \leq n^N \sum_{|\gamma|=N} |x^{\gamma}|^2, \quad x \in \mathbb{R}^n.$$

Theorem 3.2.14. *For all non-negative integers N, and multi-indices $\alpha 0$ and β, there exists a constant A, depending on m, n, N, α and β only, such that*

$$\int\limits_{\mathbb{R}^n} |z|^N |(\partial_x^{\beta}\partial_z^{\alpha}K_k)(x,z)|dz \leq A 2^{(m+|\alpha|-N)k}$$

for all $k = 0, 1, 2, 3, \ldots.$

Proof. Let γ be any multi-index, then by using above remark,Plancherel's theorem, Leibnitz' formula and theorem 1, we will get

$$\int\limits_{\mathbb{R}^n} |z^{\gamma}(\partial_x^{\beta}\partial_z^{\alpha}K_k)(x,z)|^2 dz = \int\limits_{W_k} \left| \sum_{\gamma' \leq \gamma} \binom{\gamma}{\gamma'} \partial_{\xi}^{\gamma'} \left\{\xi^{\alpha}(\partial_x^{\beta}\sigma)(x,\xi)\right\} \left(\partial_{\xi}^{\gamma-\gamma'}\varphi_k\right)(\xi) \right|^2 d\xi,$$

where

$$W_0 = \{\xi \in \mathbb{R}^n : |\xi| \le 2\}$$

and

$$W_k = \left\{\xi \in \mathbb{R}^n : 2^{k-2} \le |\xi| \le 2^{k+1}\right\}, \quad k = 0, 1, 2, 3, \ldots.$$

Hence, by theorem 1 and the fact that $\xi^\alpha(\partial_x^\beta \sigma)$ is a symbol in $S^{m+|\alpha|}$, we will get positive constants $C_{\alpha,\beta,\gamma'}$ and $C_{\gamma,\gamma'}$ such that

$$\int\limits_{\mathbb{R}^n} |z^\gamma(\partial_x^\beta \partial_z^\alpha K_k)(x,z)|^2 dz \le 2^{k(n+2m+2|\alpha|-2|\gamma|)} C_n 2^n \left\{ \sum_{\gamma' \le \gamma} \binom{\gamma}{\gamma'} C_{\alpha,\beta,\gamma'} C_{\gamma,\gamma'} 2^{2(m+|\alpha|-|\gamma'|)} \right\}^2$$

where C_n is the constant with the property that the volume of the ball in $\mathbb{R}^n$ with radius r is equal to $C_n r^n$. Let

$$A_{\alpha,\beta,\gamma,m,n} = C_n 2^n \left\{ \sum_{\gamma' \le \gamma} \binom{\gamma}{\gamma'} C_{\alpha,\beta,\gamma'} C_{\gamma,\gamma'} 2^{2(m+|\alpha|-|\gamma'|)} \right\}^2.$$

Thus

$$\int\limits_{\mathbb{R}^n} |z^\gamma(\partial_x^\beta \partial_z^\alpha K_k)(x,z)|^2 dz \le A_{\alpha,\beta,\gamma,m,n} 2^{k(n+2m+2|\alpha|-2|\gamma|)}.$$

Let N be a nonnegative integer, then by lemma, we will get

$$\int\limits_{\mathbb{R}^n} |z|^{2N} |(\partial_x^\beta \partial_z^\alpha K_k)(x,z)|^2 dz \le A_1 2^{k(n+2m+2|\alpha|-2N)}$$

for all $k = 0, 1, 2, 3, \ldots$, where

$$A_1 = n^N \sum_{|\gamma|=N} A_{\alpha,\beta,\gamma,m,n}.$$

By taking square root on both sides, we will get

$$\left\{ \int\limits_{\mathbb{R}^n} |z|^{2N} |(\partial_x^\beta \partial_z^\alpha K_k)(x,z)|^2 dz \right\}^{\frac{1}{2}} \le 2^{k(\frac{n}{2}+m+|\alpha|-N)} A_2$$

for $k = 0, 1, 2, 3, \ldots$, where $A_2 = A_1^{\frac{1}{2}}$. Now, write

$$\int\limits_{\mathbb{R}^n} |z|^N |(\partial_x^\beta \partial_z^\alpha K_k)(x,z)| dz = \int\limits_{|z| \le 2^{-k}} + \int\limits_{|z| > 2^{-k}}.$$

By cauchy Schwarz inequality, there exists a constant $A_3 > 0$ depending on m, n, N, α and β only such that

$$\int\limits_{|z| \le 2^{-k}} \le A_3 2^{(m+|\alpha|-N)k}$$

for $k = 0, 1, 2, 3 \ldots$. Again, by cauchy Schwarz inequality, there is another constant $A_4 > 0$ depending on m, n, N, α and β only such that

$$\int\limits_{|z| > 2^{-k}} \le 2^{k(\frac{n}{2}+m+|\alpha|-N-n)} A_4 \left\{ \int\limits_{S^{n-1}} \int\limits_{2^{-k}}^{\infty} r^{-2n} r^{n-1} dr d\sigma \right\}^{\frac{1}{2}}$$

where $d\sigma$ is the surface measure on the unit sphere S^{n-1}. Hence

$$\int\limits_{|z|>2^{-k}} \leq 2^{k(\frac{n}{2}+m+|\alpha|-N-n)} A_4 |S^{n-1}|^{\frac{1}{2}} 2^{\frac{nk}{2}}.$$

Now, let $A_5 = A_4|S^{n-1}|^{\frac{1}{2}}$. Hence for $k = 0, 1, 2, 3, \ldots$, we have

$$\int\limits_{|z|>2^{-k}} \leq A_5 2^{(m+|\alpha|-N)k}.$$

Hence, we are done with the proof. □

Theorem 3.2.15. *Let $f \in C^\infty(\mathbb{R}^n)$. Then for all positive integers N,*

$$f(\xi+\eta) = \sum_{|\alpha|<N} \frac{(\partial^\alpha f)(\xi)}{\alpha!} \eta^\alpha + N \sum_{|\gamma|=N} \frac{\eta^\gamma}{\gamma!} \int\limits_0^1 (1-\theta)^{N-1} (\partial^\gamma f)(\xi+\eta\theta) d\theta$$

for all $\xi, \eta \in \mathbb{R}^n$.

3.2.3 Product of Pseudo-Differential Operators

In this subsection, we are going to define the product of two pseudo-differential operators and will show that the product is again a pseudo-differential operator. We also give the asymptotic expansion for the symbol for the product.

Before defining the product of two pseudo-differential operators, let us look at some motivation for it.

Let $\phi \in \mathcal{S}$. Then, we have

$$(T_{\sigma_k}\phi)(x) = (2\pi)^{\frac{-n}{2}} \int\limits_{\mathbb{R}^n} e^{\iota x.\xi} \sigma_k(x,\xi) \hat{\phi}(\xi) d\xi$$

for all $x \in \mathbb{R}^n$, where $\sigma_k(x,\xi) = \sigma(x,\xi)\phi_k(\xi)$ and $\{\phi_k\}$ is the partition of the unity. Hence,

$$\begin{aligned}\sum_{k=0}^{\infty} (T_{\sigma_k}\phi)(x) &= (2\pi)^{\frac{-n}{2}} \int\limits_{\mathbb{R}^n} e^{\iota x.\xi} \left\{ \sum_{k=0}^{\infty} \sigma_k(x,\xi) \right\} \hat{\phi}(\xi) d\xi. \\ &= (2\pi)^{\frac{-n}{2}} \int\limits_{\mathbb{R}^n} e^{\iota x.\xi} \sigma(x,\xi) \hat{\phi}(\xi) d\xi.\end{aligned}$$

Then by the definition of pseudo-differential operator, Fourier transform, Fubini's theorem and the definition of $K_k(x,\xi)$, we will get

$$(T_{\sigma_k} T_\sigma \phi)(x) = (2\pi)^{\frac{-n}{2}} \int\limits_{\mathbb{R}^n} K_k(x, x-y)(T_\sigma\phi)(y) dy$$

for all $x \in \mathbb{R}^n$. Hence, by the definition of pseudo-differential operator and Fubini's theorem, we will get

$$(T_{\sigma_k} T_\sigma \phi)(x) = (2\pi)^{\frac{-n}{2}} \int\limits_{\mathbb{R}^n} e^{\iota x.\eta} \lambda_k(x,\eta) \hat{\phi}(\eta) d\eta$$

for all $x \in \mathbb{R}^n$, where

$$\lambda_k(x,\eta) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{-\iota(x-y).\eta} K_k(x, x-y)\tau(y,\eta)dy.$$

Hence,by a simple change of variable, we will get

$$(T_\sigma T_\tau \phi)(x) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{\iota x.\eta} \lambda(x,\eta)\hat{\phi}(\eta)d\eta$$

for all $x \in \mathbb{R}^n$, where

$$\lambda(x,\eta) = \sum_{k=0}^{\infty} \lambda_k(x,\eta)$$

for all $\eta, x \in \mathbb{R}^n$.

Lemma 3.2.16. *For all positive integers M, there exists a positive constant C_{α,β,M,N_1} such that*

$$\left| \left(D_x^\alpha D_\xi^\beta T_{N_1}^{(k)} \right)(x,\xi) \right| \leq C_{\alpha,\beta,M,N_1}(1+|\xi|^2)^{-M}(1+|\xi|)^{m_2} 2^{(m_1+2M-N_1)k}$$

for all $x, \xi \in \mathbb{R}^n$ and $k = 0, 1, 2, 3, \ldots$.

Theorem 3.2.17. *Let $\sigma \in S^{m_1}$ and $\tau \in S^{m_2}$. Then the product $T_\sigma T_\tau$ of the pseudo-differential operators T_σ and T_τ is again a pseudo-differential operator T_λ, where $\lambda \in S^{m_1+m_2}$ and has the following asymptotic expansion :*

$$\lambda \sim \sum_{\mu} \frac{(-\iota)^{|\mu|}}{\mu!} \left(\partial_\xi^\mu \sigma\right)(\partial_x^\mu \tau).$$

This means

$$\lambda - \sum_{|\mu|<N} \frac{(-\iota)^{|\mu|}}{\mu!} \left(\partial_\xi^\mu \sigma\right)(\partial_x^\mu \tau)$$

is a symbol in $S^{m_1+m_2-N}$ for every positive integer N.

Proof.

$$\lambda_k(x,\xi) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{-\iota x.\xi} K_k(x,z)\tau(x-z,\xi)dz$$

for all $\xi, x \in \mathbb{R}^n$. Now, by Taylor's formula, we get

$$\tau(x-z,\xi) = \sum_{|\mu|<N_1} \frac{(-\iota)^\mu}{\mu!}(\partial_x^\mu \tau)(x,\xi) + R_{N_1}(x,z,\xi),$$

where

$$R_{N_1}(x,z,\xi) = N_1 \sum_{|\mu|=N_1} \frac{(-z)^\mu}{\mu!} \int_0^1 (1-\theta)^{N_1-1}(\partial_x^\mu \tau)(x-\theta z,\xi)d\theta$$

for all $z, \xi, x \in \mathbb{R}^n$. Now, by using above three equations, we get

$$\lambda_k(x,\xi) = \sum_{|\mu|<N_1} \frac{(-\iota)^{|\mu|}}{\mu!} \left(\partial_\xi^\mu \sigma_k\right)(x,\xi)(\partial_x^\mu \tau)(x,\xi) + T_{N_1}^{(k)}(x,\xi),$$

where

$$T_{N_1}^{(k)}(x,\xi) = (2\pi)^{\frac{-n}{2}} \int\limits_{\mathbb{R}^n} e^{-\iota x.\xi} K_k(x,z) R_{N_1}(x,z,\xi) dz$$

for all $z, \xi, x \in \mathbb{R}^n$. For any positive integer N, the function λ satisfies

$$\lambda - \sum_{|\mu|<N} \frac{(-\iota)^{|\mu|}}{\mu!} \left(\partial_\xi^\mu \sigma\right) \left(\partial_x^\mu \tau\right) = \lambda - \sum_{|\mu|<N_1} \frac{(-\iota)^{|\mu|}}{\mu!} \left(\partial_\xi^\mu \sigma\right) \left(\partial_x^\mu \tau\right) + \sum_{N\le|\mu|<N_1} \frac{(-\iota)^{|\mu|}}{\mu!} \left(\partial_\xi^\mu \sigma\right) \left(\partial_x^\mu \tau\right),$$

where N_1 is any positive integer larger then N. Obviously,

$$\sum_{N\le|\mu|<N_1} \frac{(-\iota)^{|\mu|}}{\mu!} \left(\partial_\xi^\mu \sigma\right) \left(\partial_x^\mu \tau\right) \in S^{m_1+m_2-N}.$$

Hence, if we can prove that for all multi-indices α and β, there exists a constant $C_{\alpha,\beta}$ such that

$$\left| \left\{ D_x^\alpha D_\xi^\beta \left[\lambda - \sum_{|\mu|<N_1} \frac{(-\iota)^{|\mu|}}{\mu!} \left(\partial_\xi^\mu \sigma\right) \left(\partial_x^\mu \tau\right) \right] \right\} (x,\xi) \right| \le C_{\alpha,\beta}(1+|\xi|)^{m_1+m_2+N-|\beta|}$$

for all $x, \xi \in \mathbb{R}^n$, then we can conclude that $\lambda \in S^{m_1+m_2}$. But as we know

$$\lambda - \sum_{|\mu|<N_1} \frac{(-\iota)^{|\mu|}}{\mu!} \left(\partial_\xi^\mu \sigma\right) \left(\partial_x^\mu \tau\right) = \sum_{k=0}^{\infty} T_{N_1}^{(k)}$$

and hence by lemma, we are done with the proof. □

Lemma 3.2.18. *Let*

$$R_{N_1}(x,z,\xi) = N_1 \sum_{|\mu|=N_1} \frac{(-z)^\mu}{\mu!} \int\limits_0^1 (1-\theta)^{N_1-1} (\partial_x^\mu \tau)(x-\theta z, \xi) d\theta.$$

Then, for all multi-indices α, β and γ, there exists a constant $C_{\alpha,\beta,\gamma} > 0$ such that

$$\left| \left(\partial_z^\gamma \partial_x^\alpha \partial_\xi^\beta R_{N_1}\right)(x,z,\xi) \right| \le C_{\alpha,\beta,\gamma} \left[\sum_{\gamma' \le \gamma} |z|^{N_1-|\gamma'|} \right] (1+|\xi|)^{m_2-|\beta|}$$

for all $z, x, \xi \in \mathbb{R}^n$.

Adjoint Of A Pseudo-Differential Operator

Definition 3.2.19. *Let σ be a symbol in S^m and T_σ its associated pseudo-differential operator. Suppose there exists a linear operator $T_\sigma^* : \mathcal{S} \to \mathcal{S}$ such that*

$$\langle T_\sigma \phi, \psi \rangle = \langle \phi, T_\sigma^* \psi \rangle \ , \ \phi, \psi \in \mathcal{S}.$$

Then, we call T_σ^* is a formal adjoint of the pseudo-differential operator T_σ.

We can clearly see that a pseudo-differential operator has at most one formal adjoint. Here, following three problems arises:

1. Does a formal adjoint exist?

2. If it exists, is it a pseudo-differential operator?

3. If it is a pseudo-differential operator, can we find an asymptotic expansion for its symbol?

o answer the above questions, we have the following result. We are omitting the proof.

Theorem 3.2.20. *Let σ be a symbol in S^m. Then the formal adjoint of the pseudo-differential operator T_σ is a again a pseudo-differential operator T_τ, where τ is a symbol in S^m and*

$$\tau(x,\xi) \backsim \sum_{\mu} \frac{(-\iota)^{|\mu|}}{\mu!} \left(\partial_x^\mu \partial_\xi^\mu \bar{\sigma}\right)(x,\xi).$$

Which means

$$\tau(x,\xi) - \sum_{|\mu|<N} \frac{(-\iota)^{|\mu|}}{\mu!} \left(\partial_x^\mu \partial_\xi^\mu \bar{\sigma}\right)(x,\xi),$$

is a symbol in S^{m-N} for every positive integer N.

The Parametrix Of An Elliptic Pseudo-Differential Operator

Among all pseudo-differential operator there exists a class of operators which come up frequently in application and are particularly easy to work with. They are called elliptic operator. They are nice because they have approximate inverse (or parametrices) which are again pseudo-differential operator. In this part, we will categories all these pseudo-differential operator.

Definition 3.2.21. *A symbol $\sigma \in S^m$ is said to be elliptic if there exits positive constants C and R such that*

$$|\sigma(x,\xi)| \geq C(1+|\xi|)^m, \quad |\xi| \geq R.$$

Definition 3.2.22. *A pseudo-differential operator T_σ is said to be elliptic if its symbol is elliptic.*

Theorem 3.2.23. *Let σ be an elliptic symbol in S^m. Then there exists a symbol τ in S^{-m} such that*

$$T_\tau T_\sigma = I + R$$

and

$$T_\sigma T_\tau = I + S,$$

where R and S are pseudo-differential operator in $\cap_{k\in\mathbb{R}} S^k$, and I is identity operator.

Remark 3.2.24. *In other words, above theorem says that if T_σ is an elliptic pseudo-differential operator, then it can be inverted modulo some error terms R and S with symbols in $\cap_{k\in\mathbb{R}} S^k$.*

3.2.4 L^p-Boundedness Of Pseudo-Differential Operators

At the end of this thesis, we are going to discus about the L^p boundedness of the pseudo-differential operators. We start this subsection with a lemma.

Lemma 3.2.25. *If $\varphi_k \to 0$ in $\mathcal{S}$ as $k \to \infty$, then $\varphi_k \to 0$ in $L^p(\mathbb{R}^n$ as $k \to \infty$, for $1 \leq p \leq \infty$.*

This is an easy application of dominant convergence theorem.

Lemma 3.2.26. *The Fourier transform $\mathcal{F}$ maps $\mathcal{S}$ continuously into $\mathcal{S}$. More precisely If $\varphi_k \to 0$ in $\mathcal{S}$ as $k \to \infty$, then $\hat{\varphi}_k \to 0$ in $\mathcal{S}$ as $k \to \infty$.*

Based on above two lemmas, we have the following result.

Proposition 3.2.27. *T_σ maps $\mathcal{S}$ continuously into $\mathcal{S}$. More precisely, if $\varphi_k \to 0$, then $T_\sigma\varphi_k \to 0$ in $\mathcal{S}$ as $k \to \infty$.*

Remark 3.2.28. *The pseudo-differential operator T_σ, initially defined on the Schwartz space $\mathcal{S}$, can be extended to a linear mapping defined on the space $\mathcal{S}'$ of tempered distributions. To wit, take a distribution $u \in \mathcal{S}'$ and define $T_\sigma u$ by*

$$(T_\sigma u)(\varphi) = u\overline{(T_\sigma^* \hat{\varphi})}, \quad \varphi \in \mathcal{S}.$$

Proposition 3.2.29. *T_σ is a linear mapping from $\mathcal{S}'$ into $\mathcal{S}'$.*

Definition 3.2.30. *A sequence of distributions $\{u_k\}$ in $\mathcal{S}'$ is said to converge to zero in $\mathcal{S}'$ if $u_k(\varphi) \to 0$ as $k \to \infty$ for all $\varphi \in \mathcal{S}$.*

Proposition 3.2.31. *T_σ maps $\mathcal{S}'$ continuously into $\mathcal{S}'$. More precisely, if $u_k \to 0$ in $\mathcal{S}'$ as $k \to \infty$, then $T_\sigma u_k \to 0$ in $\mathcal{S}'$ as $k \to \infty$.*

Definition 3.2.32. *Let u be a tempered distribution. Then for any multi-index α, we define $\partial^\alpha u$ by*

$$(\partial^\alpha u)(\varphi) = (-1)^{|\alpha|} u(\partial^\alpha \varphi), \quad \varphi \in \mathcal{S}.$$

Clearly, $\partial^\alpha u$ is also a tempered distribution.

Proposition 3.2.33. *Let u be a tempered distribution. Then for any multi-index α,*

$$(D^\alpha u)^\wedge = x^\alpha \hat{u},$$

where $x^\alpha \hat{u}$ is the tempered distribution given by

$$(x^\alpha \hat{u})(\varphi) = \hat{u}(x^\alpha \varphi), \quad \varphi \in \mathcal{S}.$$

Theorem 3.2.34. *Let $k \in C^k(\mathbb{R}^n - \{0\}), k > \frac{n}{2}$, be such that there is a positive constant B for which*

$$|(D^\alpha m)(\xi)| \leq B|\xi|^{-\|\alpha|}, \quad \xi \neq 0,$$

for all multi-indices α $|\alpha| \leq k$. Then, for $1 < p < \infty$, there is a positive constant C, depending on p and n only, such that

$$\|T\varphi\|_p \leq CB\|\varphi\|, \quad \varphi \in \mathcal{S},$$

where

$$(T\varphi)((x) = (2\pi)^{\frac{-n}{2}} \int\limits_{\mathbb{R}^n} e^{\iota x.\xi} m(\xi)\hat{\varphi}(\xi) d\xi, \quad x \in \mathbb{R}^n.$$

Lemma 3.2.35. *For all multi-indices α and positive integer N, there is a positive constant $C_{\alpha,N}$, depending on α and N only, such that*

$$\left|\left(D_\xi^\alpha \hat{\sigma}_m\right)(\lambda, \xi)\right| \leq C_{\alpha,N}(1+|\xi|)^{-|\alpha|}(1+|\lambda|)^{-N}, \quad \lambda, \xi \in \mathbb{R}^n.$$

Lemma 3.2.36. *Let $K(x,z) = (2\pi)^{\frac{-n}{2}} \int\limits_{\mathbb{R}^n} e^{\iota x.\xi}\sigma(x,\xi)d\xi$ in the distribution sense. Then*

1. *for each fixed $x \in \mathbb{R}^n, K(x,.)$ is a function defined on $\mathbb{R}^n - \{0\}$,*

2. *for each sufficiently large positive integer N, there is a positive constant C_N such that*

$$|K(x,z)| \leq C_N|z^{-N}|, \quad z \neq 0,$$

3. for each fixed $x \in \mathbb{R}^n$ and $\varphi \in \mathcal{S}$ vanishing in a neighbourhood of x,

$$(T_\sigma\varphi)(x) = (2\pi)^{\frac{-n}{2}} \int\limits_{\mathbb{R}^n} K(x, x-z)\varphi(x)dz.$$

Now based on the above results, we will prove the important result of this chapter.

Theorem 3.2.37. *Let σ be a symbol in S^0. Then $1 < p < \infty, T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator.*

Proof. Let $\mathbb{Z}^n$ be the set of n-tuples in $\mathbb{R}^n$ with integer coefficients. We write $\mathbb{R}^n$ as a union of cubes with disjoint interiors, i.e., $\mathbb{R}^n = \cup_{m\in\mathbb{Z}^n} Q_m$, where Q_m is the cube with center at m, edges of length one and parallel to the coordinate axes. Let Q_0 be the cube with center at the origin. Let η be any function in $C_0^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1\ \forall\ x \in Q_0$. For $m \in \mathbb{Z}^n$, define σ_m by

$$\sigma_m(x,\xi) = \eta(x-m)\sigma(x,\xi), \quad x, \xi \in \mathbb{R}^n.$$

Obviously, $T_{\sigma_m} = \eta(x-m)T_\sigma$, and

$$\int\limits_{Q_m} |(T_\sigma\varphi)(x)|^p dx \le \int\limits_{\mathbb{R}^n} |(T_{\sigma_m}\varphi)(x)|^p dx, \quad \varphi \in \mathcal{S}.$$

Since $\sigma_m(x,\xi)$ has compact support in x, it follows from Fourier inversion formula and Fubini's theorem that

$$\begin{aligned}(T_{\sigma_m}\varphi)(x) &= (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{\iota x.\xi}\sigma_m(x,\xi)\hat{\varphi}(\xi)d\xi.\\ &= (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{\iota x.\xi} \left\{ \int\limits_{\mathbb{R}^n} e^{\iota x.\lambda}\hat{\sigma}_m(\lambda,\xi)d\lambda \right\} \hat{\varphi}(\xi)d\xi.\\ &= (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{\iota x.\lambda} \left\{ \int\limits_{\mathbb{R}^n} e^{\iota x.\xi}\hat{\sigma}_m(\lambda,\xi)\hat{\varphi}(\xi)d\xi \right\} d\lambda.\end{aligned}$$

Where $\hat{\sigma}_m(\lambda,\xi)$ is the Fourier transform with respect to the variable x. Now, by using lemma and theorem above, we will get the map $\varphi \to T_\lambda\varphi$ defined on $\mathcal{S}$ by

$$(T_\lambda\varphi)(x) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{\iota x.\xi}\hat{\sigma}_m(\lambda,\xi)\hat{\varphi}(\xi)d\xi$$

can be extended to a bounded linear operator on $L^p\mathbb{R}^n$. Moreover, for any positive integer N, there is a positive constant C_N such that

$$\|T_\lambda\varphi\|_p \le C_N(1+|\lambda|)^{-N}\|\varphi\|_p, \quad \varphi \in \mathcal{S}.$$

Now, by using Minkowski's inequality in the integral form, choosing sufficiently large N and putting the value of $(T_\lambda\varphi)$, we will get

$$\|T_{\sigma_m}\varphi\| \le C_N\|\varphi\|_p \ , \ \varphi \in \mathcal{S}.$$

Hence,

$$\int\limits_{Q_m} |(T_\sigma\varphi)(x)|^p dx \le C_N^p\|\varphi\|_p^p.$$

Now, we will use last lemma. Let Q_m^{**} be double of Q_m. Now, let Q_m^* be another cube concentric with Q_m and Q_m^{**} such that $Q_m \subset Q_m^* \subset Q_m^{**}$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \psi(x) \leq 1$ for all $x \in \mathbb{R}^n$, $supp(\psi) \subseteq Q_m^{**}$ and $psi(x) = 1$ in a neighbourhood of Q_m^*. Write $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 = \psi\varphi$ and $\varphi_2 = (1-\psi)\varphi$. Then $T_\sigma\varphi = T_\sigma\varphi_1 + T_\sigma\varphi_2$. Write

$$I_m = \int\limits_{Q_m} |(T_\sigma\varphi)(x)|^p dx$$

and

$$J_m = \int\limits_{Q_m} |(T_\sigma\varphi_2)(x)|^p dx.$$

Then for sufficiently large positive integer N, from last inequality, thee exists a positive constant C_N such that

$$I_m \leq 2^p \|\varphi_1\|_p^p C_N^p + 2^p J_m.$$

By lemma,there is a positive constant C_{2N} such that for all $x \in Q_m$,

$$\begin{aligned}
|(T_{\sigma\varphi_2})(x)| &= \left| \int\limits_{\mathbb{R}^n} K(x, x-z)\varphi_2(z)dz \right| \\
&= \left| \int\limits_{\mathbb{R}^n - Q_m^*} K(x, x-z)\varphi_2(z)dz \right| \\
&\leq C_{2N} \int\limits_{\mathbb{R}^n - Q_m^*} |x-z|^{-2N} |\varphi_2(z)| dz.
\end{aligned}$$

Let $\lambda \geq \sqrt{n} + 1$. Then there exists a positive constant $C_{\lambda,N}$, depending on λ and N only, such that

$$\frac{|x-z|^{-2N}}{(\lambda + |x-z|)^{2N}} = \frac{(\lambda + |x-z|)^{2N}}{|x-z|^{2N}} \leq C_{\lambda,N}$$

for all $x \in Q_m$ and $z \in \mathbb{R}^n - Q_m^*$. Hence,

$$|(T_{\sigma\varphi_2})(x)| \leq C_{2N} C_{\lambda,N} \int\limits_{\mathbb{R}^n - Q_m^*} (\lambda + |x-z|^{-2N}) |\varphi_2(z)| dz, \quad x \in Q_m.$$

Next, we note that, for all $x \in Q_m$ and $z \in \mathbb{R}^n - Q_m^*$,

$$\begin{aligned}
\lambda + |x-z| &= \lambda + |x - z + m - m|. \\
& geq \lambda + |m-z| - |m-x|. \\
&\geq \left(\lambda - \frac{\sqrt{n}}{2}\right) + |m-z|. \\
&\geq \mu + |m-z|,
\end{aligned}$$

where $\mu = \frac{\sqrt{n}}{2} + 1$.

$$|(T_{\sigma\varphi_2})(x)| \leq C_{2N} C_{\lambda,N} \int\limits_{\mathbb{R}^n - Q_m^*} \frac{(\mu + |x-z|)^{-N} |\varphi_2(z)|}{(\mu + |x-z|)^N} dz, \quad x \in Q_m.$$

Now, by using Minkowski's inequality in integral form and after a simple calculation, we will get

$$J_m \leq C_{\lambda,N,p} \int\limits_{\mathbb{R}^n - Q_m^*} \frac{|\varphi_2(z)|^p}{(\mu + |m - z|)^{\frac{Np}{2}}} dz.$$

And hence summing over all $m \in \mathbb{Z}^n$, we get a positive constant C, depending only on n, p, λ and N such that

$$\int\limits_{\mathbb{R}^n} |(T_\sigma \varphi)(x)|^p dx \leq \left\{ C + 2^p C_{\lambda,N,p} \sum_{m \in \mathbb{Z}^n} \sum_{l \neq m} \frac{1}{(1 + |m - l|)^{\frac{Np}{2}}} \right\} \int\limits_{\mathbb{R}^n} |\varphi(x)|^p dx.$$

Since $\mathcal{S}$ is dense in $L^p(\mathbb{R}^n)$. It follows that T_σ can be extended to a bounded linear operator on $L^p(\mathbb{R}^n)$. $\square$

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