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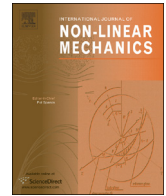
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# Surface and non-local effects for non-linear analysis of Timoshenko beams



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## ABSTRACT

In this paper, we present a non-local non-linear finite element formulation for the Timoshenko beam theory. The proposed formulation also takes into consideration the surface stress effects. Eringen's non-local differential model has been used to rewrite the non-local stress resultants in terms of non-local displacements. Geometric non-linearities are taken into account by using the Green–Lagrange strain tensor. A  $C^0$  beam element with three degrees of freedom has been developed. Numerical solutions are obtained by performing a non-linear analysis for bending and free vibration cases. Simply supported and clamped boundary conditions have been considered in the numerical examples. A parametric study has been performed to understand the effect of non-local parameter and surface stresses on deflection and vibration characteristics of the beam. The solutions are compared with the analytical solutions available in the literature. It has been shown that non-local effect does not exist in the nano-cantilever beam (Euler–Bernoulli beam) subjected to concentrated load at the end. However, there is a significant effect of non-local parameter on deflections for other load cases such as uniformly distributed load and sinusoidally distributed load (Cheng et al. (2015) [10]). In this work it has been shown that for a cantilever beam with concentrated load at free end, there is definitely a dependency on non-local parameter when Timoshenko beam theory is used. Also the effect of local and non-local boundary conditions has been demonstrated in this example. The example has also been worked out for other loading cases such as uniformly distributed force and sinusoidally varying force. The effect of the local or non-local boundary conditions on the end deflection in all these cases has also been brought out.

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## 1. Introduction

The classical theory of hyperelasticity is used to solve a large number of problems in engineering, wherein the stress at a given point uniquely depends on the current values and possibly also the previous history of deformation and temperature at that point only. Deformation in this case is characterized by the deformation gradient or by an appropriate strain tensor, that is, it is fully determined by the first gradient of the displacement field. In modeling micro/nano structures where the size effect becomes prominent, for example, study of elastic waves when dispersion effect is taken to account and the determination of stress at the crack tip when the singularity of the solution is of concern, the classical theory cannot model the material behavior accurately.

The inhomogeneities present in any material at the microscopic scale influence its properties at the macroscopic scale: materials such as suspensions, blood flows, liquid crystals, porous media, polymeric

substances, solids with microcracks, dislocations, turbulent fluids with vortices, and composites point to the need for incorporating micro-motions in continuum mechanical formulations [13]. There has been considerable focus towards the development of generalized continuum theories [19] that account for the inherent microstructure in such natural and engineering materials (see [36,15]). The notion of generalized continua unifies several extended continuum theories that account for such a size dependence due to the underlying microstructure of the material. A systematic overview and detailed discussion of generalized continuum theories has been given by Bazant and Jirasek [8]. These theories can be categorized as gradient continuum theories (see works by Mindilin et al. [45–47], Toupin [68], Steinmann et al. [13,34,66,37], and Casterzene et al. [52], Fleck et al. [20,63], Askes et al. [3–5]), microcontinuum theories (see works by Eringen [18,16,19]), Steinmann et al. [35,28], and non-local continuum theories (see works by Eringen [17], Jirasek [33], Reddy [54], and others [7,12,51,10]). Recently, the higher order gradient theory for finite deformation has been elaborated (for instance see [21,38,39,52]), within classical continuum mechanics in the context of homogenization approaches. A comparison of various higher order gradient theories can be found in [20]. A more detailed formulation of

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gradient approach in spatial and material setting has been presented in [35].

Classical continuum mechanics takes exclusively the bulk into account, nevertheless, neglecting possible contributions from the surface of the deformable body. However, surface effects play a crucial role in the material behavior, the most prominent example being surface tension. A mathematical framework was first developed by Gurtin [23] to study the mechanical behavior of material surfaces. The effect of surface stress on wave propagation in solids has also been studied by Gurtin [24]. The tensorial nature of surface stress was established using the force and moment balance laws. Bodies whose boundaries are material surfaces are discussed and the relation between surface and body stress examined in a recent work by Steinmann [64] and by Hamilton [25]. The surface effects have been applied to modeling two [31] and three-dimensional continua in the frame work of finite element method (see [32,14]). Similar studies on static analysis of nanobeams using non-local finite element models have been done by Mahmoud [43].

The focus of this work is on non-local non-linear formulation together with surface effects for static and free vibration analysis of Timoshenko beams. The non-local formulations can be of integral-type formulations with weighted spatial averaging or by implicit gradient models which are categorized as strongly non-local, while weakly non-local theories include for instance explicit gradient models [8]. Herein we consider a strongly non-local problem. The Timoshenko beam can be considered as a specific onedimensional version of a Cosserat continuum. Recently various beam theories such as Euler–Bernoulli, Timoshenko, Reddy, and Levinson beam theories were reformulated using Eringen's non-local differential constitutive model by Reddy [54]. The analytical solutions for bending, buckling and free vibrations were also presented in [54]. Various shear deformation beam theories were also reformulated in recent works by Reddy [55] using non-local differential constitutive relations. Similar works have been done to study bending, buckling and free vibration of nanobeams by Aydogdu [7], Civalek [12].

Eringen's non-local elasticity theory has also been applied to study bending, buckling and vibration of nanobeams using Timoshenko beam theory (see [40,60,72,48]). Numerical solutions were obtained by a meshless method. Two different collocation techniques, global (RDF) and local (RDF-FD), were used with multi-quadrics radial basis functions by Roque et al. [58]. Static deformation of micro- and nano-structures was studied using non-local Euler–Bernoulli and Timoshenko beam theory and explicit solutions have been derived for deformations for standard boundary conditions by Wang et al. (see [71,70]). Analytical solutions for beam bending problems for different boundary conditions were derived using non-local elasticity theory and Timoshenko beam theory by Wang et al. [69]. Iterative non-local elasticity for Kirchhoff plates has been presented in [62]. Thai et al. [67] developed a non-local shear deformation beam theory with a higher order displacement field that does not require shear correction factors. Some explicit solutions involving trigonometric expansions are also presented recently for non-local analysis of beams [74]. A finite element framework for non-local analysis of beams has also been made in a recent work by Sciarra et al. [61]. Size effects on elastic moduli of plate like nanomaterials have been studied in [65].

Non-local elastic rod models have been developed to investigate the small-scale effect on axial vibrations of the nanorods by Aydogdu [6] and Adhikari et al. [1]. Free vibration analysis of microtubules based on non-local theory and Euler–Bernoulli beam theory was done by Civalek et al. [12]. Free vibration analysis of functionally graded carbon nanotube with various thickness based on Timoshenko beam theory has been investigated to obtain numerical solutions using the Differential Quadrature Method (DQM) by Janghorban et al. [30] and others (see [11,27,2]). Studies to understand thermal vibration of single wall carbon nanotube embedded in an elastic medium using DQM have also

been reported in [49]. The recent studies have been towards the application of non-local non-linear formulations for the vibration analysis of functionally graded beams [53]. Analytical study on the non-linear free vibration of functionally graded nanobeams incorporating surface effects has been presented in [26,59,42]. The effect of non-local parameter, surface elasticity modulus and residual surface stress on the vibrational frequencies of Timoshenko beam has been studied in [73,41]. The coupling between non-local effect and surface stress effect for the non-linear free vibration case of nanobeams has been studied in [29]. The effect of surface stresses on bending properties of metal nanowires is presented in [75]. There has been some works on transforming non-local approaches to gradient type formulations [9]. Semi-analytical approach for large amplitude free vibration and buckling of non-local functionally graded beams has been reported in [50].

In this paper, we present a non-local non-linear finite element formulation for the Timoshenko beam theory. The proposed formulation takes into consideration the surface stress effects. Eringen's non-local differential model has been used to write the non-local stress resultants. Geometric non-linearities are taken into account by using Green–Lagrange strain tensor. Numerical solutions are obtained by performing a non-linear analysis for bending and free vibration cases. Simply supported and clamped boundary conditions have been considered in the numerical examples. A parametric study has been performed to understand the effect of non-locality and surface stresses on deflection and vibration characteristics of the beam. The solutions are compared with the analytical solutions available in the literature. The following Section 2 gives a background on Eringen's non-local theory. Section 3 gives the mathematical formulation for the non-local Timoshenko beam theory. The finite element formulation for the Timoshenko beam theory is explained in Section 4. In Section 5 numerical examples are presented together with parametric studies to demonstrate the effect of non-local and surface stresses on the bending and vibration characteristics of the beam.

## 2. Non-local theories

In classical elasticity, stress at a point is a function of strain at that point. Whereas in non-local elasticity, stress at a point is a function of strains at all points in the continuum. In non-local theories, forces between the atoms and internal length scale are considered in the constitutive equation. Non-local theory was first introduced by Eringen [19]. According to Eringen, the stress field at a point  $x$  in an elastic continuum not only depends on the strain field at that point but also on the strains at all other points of the body. Eringen attributed this fact to the atomic theory of lattice dynamics and experimental observation on phonon dispersion. The non-local stress tensor  $\sigma$  at a point  $x$  in the continuum is expressed as

$$\sigma = \int K(|\mathbf{x}' - \mathbf{x}|, \tau) \mathbf{t}(\mathbf{x}') d\mathbf{x}' \quad (1)$$

where  $\mathbf{t}(\mathbf{x})$  is the classical macroscopic stress tensor at point  $\mathbf{x}$  and the kernel function  $K(|\mathbf{x}' - \mathbf{x}|, \tau)$  represents the non-local modulus,  $|\mathbf{x}' - \mathbf{x}|$  is the distance and  $\tau$  is the material constant that depends on internal and external characteristic lengths.

Stress and strain at a point are related to each other by Hooke's law as

$$\mathbf{t}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) \quad (2)$$

where  $t$  is the macroscopic stress tensor,  $\boldsymbol{\varepsilon}$  is the strain tensor,  $\mathbf{C}$  is the fourth-order elasticity tensor and  $':'$  denotes double dot product. Eqs. (1) and (2) together form the non-local constitutive equations of Hookean solid. Constitutive equations can also be

expressed in equivalent differential form as

$$(1 - \tau^2 \nabla^2) \boldsymbol{\sigma} = \mathbf{t} \tag{3}$$

$$\tau = \frac{e_0 a}{l} \tag{4}$$

where  $e_0$  is a material constant, and  $a$  and  $l$  are the internal and external characteristic lengths respectively.

### 3. Mathematical formulations

#### 3.1. Classical Timoshenko beam theory

In Timoshenko beam theory, the effects of shear deformation are also considered. A linear distribution of transverse shear stress is assumed. The displacement field is given by

$$u = u(x, t) + z\varphi_x \tag{5a}$$

$$w = w(x, t) \tag{5b}$$

where  $\varphi_x$  is the measure of rotation of the beam cross-section. The non-zero components of Lagrangian strain tensor can be written as

$$\begin{aligned} \epsilon_{xx} &= \left[ \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right] + z \left[ \frac{d\varphi_x}{dx} \right] \\ &= [\epsilon_{xx}^{(0)}] + z [\epsilon_{xx}^{(1)}] \end{aligned} \tag{6a}$$

$$\epsilon_{xz} = \frac{1}{2} \left( \varphi_x + \frac{dw}{dx} \right) \tag{6b}$$

$$\epsilon_{zz} = \frac{1}{2} \varphi_x^2 \tag{6c}$$

Since  $\epsilon_{zz}$  is positive and non-zero, two-dimensional Hooke's law should be used. But it makes the beam flexible by 38% [56]. So, its contribution is through certain material length scale parameters (see [56,57]). Therefore, for an isotropic material, following stress-strain relationship is used:

$$\sigma_{xx} = E\epsilon_{xx} + \alpha\epsilon_{zz} \tag{7a}$$

$$\sigma_{xz} = GK_s\gamma_{xz} \tag{7b}$$

$$\sigma_{zz} = \alpha\epsilon_{xx} + \beta\epsilon_{zz} \tag{7c}$$

$$\sigma^s = \tau^0 + E^s\epsilon_{xx} \tag{7d}$$

where  $\alpha$  and  $\beta$  are certain material length scale parameters (for example, in the case of a soft material embedded with hard inclusions they may describe the geometry and distribution of the hard inclusions), and  $E$ ,  $G$  and  $K_s (= \frac{5}{6})$  are Young's moduli, shear moduli, and shear correction factor, respectively.  $\sigma^s$  is the surface stress in the direction of the length of the beam (see [23,22]). The stress resultants can be written as

$$N_{xx} = \int_A \sigma_{xx} dA + \oint_{\Gamma} \sigma^s ds \tag{8a}$$

$$M_{xx} = \int_A \sigma_{xx} z dA + \oint_{\Gamma} \sigma^s z ds \tag{8b}$$

$$N_{zz} = \int_A \sigma_{zz} dA \tag{8c}$$

$$N_{xz} = \int_A \sigma_{xz} dA \tag{8d}$$

Using Eqs. (6) and (7), the stress resultants in Eq. (8) can be written as

$$N_{xx} = \tilde{A}\epsilon_{xx}^{(0)} + \tilde{C}\epsilon_{zz} + 2\tau^0(b+h) \tag{9a}$$

$$M_{xx} = \tilde{D}\epsilon_{xx}^{(1)} \tag{9b}$$

$$N_{zz} = \tilde{C}\epsilon_{xx}^{(0)} + \tilde{F}\epsilon_{zz} \tag{9c}$$

$$N_{xz} = \tilde{G}\gamma_{xz} \tag{9d}$$

where

$$\tilde{A} = EA + 2E^s(b+h) \tag{10a}$$

$$\tilde{C} = \alpha A \tag{10b}$$

$$\tilde{D} = EI + E^s \left[ \frac{h^3}{6} + \frac{bh^2}{2} \right] \tag{10c}$$

$$\tilde{F} = \beta A \tag{10d}$$

$$\tilde{G} = K_s GA \tag{10e}$$

#### 3.2. Non-local Timoshenko beam theory

Using Eq. (3), the non-local stress resultants in terms of strains can be written as

$$N_{xx}^{nl} = \mu \frac{d^2 N_{xx}^{nl}}{dx^2} + \tilde{A}\epsilon_{xx}^{(0)} + \tilde{C}\epsilon_{zz} + 2\tau^0(b+h) \tag{11}$$

$$M_{xx}^{nl} = \mu \frac{d^2 M_{xx}^{nl}}{dx^2} + \tilde{D}\epsilon_{xx}^{(1)} \tag{12}$$

$$N_{zz}^{nl} = \tilde{C}\epsilon_{xx}^{(0)} + \tilde{F}\epsilon_{zz} \tag{13}$$

$$N_{xz}^{nl} = \mu \frac{d^2 N_{xz}^{nl}}{dx^2} + \tilde{G}\gamma_{xz} \tag{14}$$

#### 3.3. Equations of motion

By using the principle of virtual work, the equations of motion for the classical Timoshenko beam can be obtained as

$$\frac{dN_{xx}}{dx} + f_x = m_0 \frac{d^2 u}{dt^2} \tag{15}$$

$$\frac{d}{dx} \left[ N_{xz} + N_{xx} \frac{dw}{dx} \right] + f_z = m_0 \frac{d^2 w}{dt^2} \tag{16}$$

$$\frac{dM_{xx}}{dx} - (N_{xz} + N_{zz}\varphi_x) = m_1 \frac{d^2 \varphi}{dt^2} \tag{17}$$

where

$$m_i = \int_A \rho z^i dA$$

and  $f_x$  and  $f_z$  are the axially and transversely distributed forces, respectively. The boundary conditions are

Geometric :  $u, w, \varphi_x$

$$\text{Force : } N_{xx}, N_{xz} + N_{xx} \frac{dw}{dx}, M_{xx}$$

Considering the stress resultants in Eqs. (15)–(17) to be non-local and using the relations from (11) to (17), the following equations for non-local stress resultants are obtained:

$$N_{xx}^{nl} = \tilde{A}\epsilon_{xx}^{(0)} + \tilde{C}\epsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \tag{18a}$$

$$M_{xx}^{nl} = \tilde{D}\varepsilon_{xx}^{(1)} + \mu m_1 \frac{d^3 \varphi}{dx dt^2} + \mu m_0 \frac{d^2 w}{dt^2} - \mu f_z - \mu \frac{d}{dx} \left[ \left( \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{C}\varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{dw}{dx} \right] + \mu \frac{d}{dx} \left[ \left( \tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz} \right) \varphi_x \right] \quad (18b)$$

$$N_{zz}^{nl} = \tilde{C}\varepsilon_{zz}^{(0)} + \tilde{F}\varepsilon_{zz} \quad (18c)$$

$$N_{xz}^{nl} = \tilde{G}\gamma_{xz} - \mu \frac{df_z}{dx} - \mu \frac{d^2}{dx^2} \left[ \left( \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{C}\varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{dw}{dx} \right] \quad (18d)$$

It is clearly seen in the above equation that the non-locality reduces the resistance offered by the beam to the external forces. By substituting the expressions for non-local stress resultants (18) back in the equations of motion (15) to (17), we obtain the equilibrium equation for non-local Timoshenko beam theory including surface stress effects as

$$m_0 \frac{d^2 u}{dt^2} - \mu m_0 \frac{d^4 u}{dx^2 dt^2} = \frac{d}{dx} \left( \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{C}\varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} \right) + f_x \quad (19)$$

$$m_0 \frac{d^2 w}{dt^2} - \mu m_0 \frac{d^4 w}{dx^2 dt^2} = \frac{d}{dx} \left( \tilde{G}\gamma_{xz} \right) - \mu \frac{d^2 f_z}{dx^2} + f_z + \frac{d}{dx} \left[ \left( \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{C}\varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{dw}{dx} \right] - \mu \frac{d^3}{dx^3} \left[ \left( \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{C}\varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{dw}{dx} \right] \quad (20)$$

$$m_1 \frac{d^2 \varphi_x}{dt^2} - \mu m_1 \frac{d^4 \varphi_x}{dx^2 dt^2} = \frac{d}{dx} \left( \tilde{D}\varepsilon_{xx}^{(1)} \right) - \tilde{G}\gamma_{xz} - \varphi_x \left( \tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz} \right) + \mu \frac{d^2}{dx^2} \left[ \left( \tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz} \right) \varphi_x \right] \quad (21)$$

#### 4. Finite element formulation

The principle of virtual work for the Timoshenko beam has the form

$$0 = \int_0^l \left[ N_{xx}^{nl} \delta\varepsilon_{xx}^{(0)} + M_{xx}^{nl} \delta\varepsilon_{xx}^{(1)} + N_{xz}^{nl} \delta\gamma_{xz} + N_{zz}^{nl} \delta\varepsilon_{zz} - f_x \delta u - f_z \delta w + m_0 \ddot{u} \delta u + m_0 \ddot{w} \delta w + m_1 \ddot{\varphi} \delta \varphi \right] dx - Q_1 \delta u(0) - Q_4 \delta u(l) - Q_2 \delta w(0) - Q_5 \delta w(l) - Q_3 \delta \varphi(0) - Q_6 \delta \varphi(0) \quad (22)$$

After substituting the expressions for stress resultants from Eq. (18) into Eq. (22), we obtain

$$0 = \int_0^l \int_0^l \left( \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{C}\varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \delta\varepsilon_{xx}^{(0)} + \left( \tilde{D}\varepsilon_{xx}^{(1)} + \mu m_1 \frac{d^3 \varphi}{dx dt^2} + \mu m_0 \frac{d^2 w}{dt^2} - \mu f_z \right) \delta\varepsilon_{xx}^{(1)} - \mu \frac{d}{dx} \left[ \left( \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{C}\varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{dw}{dx} \right] \varepsilon_{xx}^{(1)} + \mu \frac{d}{dx} \left[ \left( \tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz} \right) \varphi_x \right] \delta\varepsilon_{xx}^{(1)} + \left( \tilde{G}\gamma_{xz} + \mu m_0 \frac{d^3 w}{dx dt^2} - \mu \frac{df_z}{dx} \right) \delta\gamma_{xz}$$

$$- \mu \frac{d^2}{dx^2} \left[ \left( \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{C}\varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{dw}{dx} \right] \delta\gamma_{xz} + \left( \tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz} \right) \delta\varepsilon_{zz} - f_x \delta u - f_z \delta w + m_0 \ddot{u} \delta u + m_1 \ddot{w} \delta w + m_1 \ddot{\varphi}_x \delta \varphi_x - [Q_1 \delta u(x_a) + Q_4 \delta u(x_b) + Q_2 \delta w(x_a) + Q_5 \delta w(x_b) + Q_3 \delta \varphi(x_a) + Q_6 \delta \varphi(x_b)] dT \quad (23)$$

The underlined expressions in the above equation do not allow us to construct a quadratic functional. So after omitting the underlined expressions in Eq. (23), it can be equivalently written into the following three equations:

$$\int_0^T \int_0^l \left[ \left( \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{C}\varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{d\delta u}{dx} - f_x \delta u + m_0 \ddot{u} \delta u \right] dx - [Q_1 \delta u(x_a) + Q_4 \delta u(x_b)] dT = 0 \quad (24)$$

$$\int_0^T \int_0^l \left( \tilde{A}\varepsilon_{xx}^{(0)} + \tilde{C}\varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{dw}{dx} \frac{d\delta w}{dx} + \left( \tilde{G}\gamma_{xz} + \mu m_0 \frac{d^3 w}{dx dt^2} - \mu \frac{df_z}{dx} \right) \frac{d\delta w}{dx} - f_z \delta w + m_0 \ddot{w} \delta w - [Q_2 \delta w(x_a) + Q_5 \delta w(x_b)] dT = 0 \quad (25)$$

$$\int_0^T \int_0^l \left( \tilde{G}\gamma_{xz} + \mu m_0 \frac{d^3 w}{dx dt^2} - \mu \frac{df_z}{dx} \right) \delta \varphi_x + \left( \tilde{D}\varepsilon_{xx}^{(1)} + \mu m_1 \frac{d^3 \varphi_x}{dx dt^2} + \mu m_0 \frac{d^2 w}{dt^2} - \mu f_z \right) \frac{d\delta \varphi_x}{dx} + \left( \tilde{C}\varepsilon_{xx}^{(0)} + \tilde{F}\varepsilon_{zz} \right) \varphi_x \delta \varphi_x + m_1 \ddot{\varphi}_x \delta \varphi_x - [Q_3 \delta \varphi_x(x_a) + Q_6 \delta \varphi_x(x_b)] dT = 0 \quad (26)$$

The generalized displacements  $(\bar{u}, \bar{w}, \bar{\varphi}_x)$  are approximated using the Lagrange interpolation functions

$$\bar{u}(x) = \sum_{j=1}^m \Delta_j^1 \psi_j^{(1)}(x) \quad (27a)$$

$$\bar{w}(x) = \sum_{j=1}^n \Delta_j^2 \psi_j^{(2)}(x) \quad (27b)$$

$$\bar{\varphi}_x(x) = \sum_{j=1}^p \Delta_j^3 \psi_j^{(3)}(x) \quad (27c)$$

By substituting Eq. (27) for  $\bar{u}$ ,  $\bar{w}$  and  $\bar{\varphi}_x$ , and putting  $\delta \bar{u} = \psi_i^1$ ,  $\delta \bar{w} = \psi_i^2$ ,  $\delta \bar{\varphi}_x = \psi_i^3$  into the weak form statements (24)–(26), the finite element model of the Timoshenko beam can be expressed as

$$\begin{bmatrix} K^{11} & K^{12} & K^{13} \\ K^{21} & K^{22} & K^{23} \\ K^{31} & K^{32} & K^{33} \end{bmatrix} \begin{Bmatrix} \Delta^1 \\ \Delta^2 \\ \Delta^3 \end{Bmatrix} + \begin{bmatrix} M^{11} & M^{12} & M^{13} \\ M^{21} & M^{22} & M^{23} \\ M^{31} & M^{32} & M^{33} \end{bmatrix} \left\{ \bar{\Delta}^1 \bar{\Delta}^2 \bar{\Delta}^3 \right\} = \begin{Bmatrix} F^1 \\ F^2 \\ F^3 \end{Bmatrix} \quad (28)$$

where the stiffness coefficients  $K_{ij}^{\alpha\beta}$ , mass matrix coefficients  $M_{ij}^{\alpha\beta}$  and force coefficients  $F_i^\alpha$  ( $\alpha, \beta = 1, 2, 3$ ) are defined as follows:

$$K_{ij}^{11} = \int_0^l \tilde{A} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx} dx$$

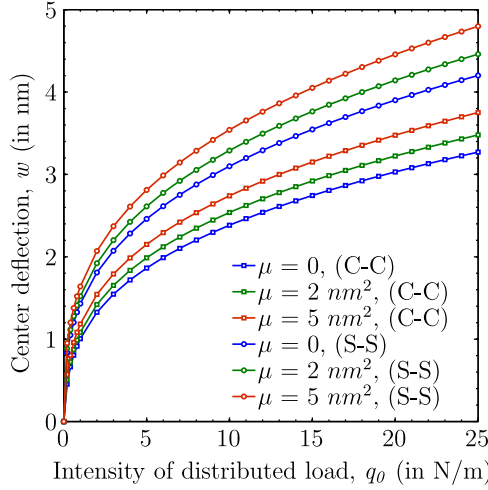
$$K_{ij}^{12} = \int_0^l \frac{1}{2} \tilde{A} \frac{dw}{dx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx$$

$$K_{ij}^{13} = \int_0^l \frac{1}{2} \tilde{C} \varphi_x \frac{d\psi_i^{(1)}}{dx} \psi_j^{(3)} dx$$

$$K_{ij}^{21} = \int_0^l \tilde{A} \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(1)}}{dx} dx$$

**Table 1**  
Variation of non-dimensionalized central deflection  $\bar{w}$  for various  $L/H$  ratios and non-local parameter  $\mu$  (for  $L/H = 100$ ).

$\mu$	Reddy [54]	Thai [67]	$\bar{w}$ (present)
0	1.3134	1.3024	1.3024
1	1.4492	1.4274	1.4274
2	1.5849	1.5525	1.5524
3	1.7207	1.6775	1.6774
4	1.8565	1.8025	1.8024



**Fig. 1.** Plot of load versus transverse center deflection for C–C beam and S–S beam under sinusoidally distributed load.

$$M_{ij}^{33} = \mu m_1 \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} + m_1 \psi_i^{(3)} \psi_j^{(3)}$$

$$M_{ij}^{12} = 0, \quad M_{ij}^{13} = 0, \quad M_{ij}^{23} = 0, \quad M_{ij}^{31} = 0$$

$$F_i^1 = \int_0^l \left\{ f_x \psi_i^{(1)} + \mu \frac{df_x}{dx} \frac{d\psi_i^{(1)}}{dx} - 2\tau^0 (b+h) \frac{d\psi_i^{(1)}}{dx} \right\} dx$$

$$+ Q_1 \psi_i^{(1)}(0) + Q_4 \psi_i^{(1)}(l)$$

$$F_i^2 = \int_0^l \left\{ f_z \psi_i^{(2)} + \mu \left[ \frac{df_z}{dx} + \frac{df_x}{dx} \frac{dw}{dx} \right] \frac{d\psi_i^{(2)}}{dx} - 2\tau^0 (b+h) \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \right\} dx$$

$$+ Q_2 \psi_i^{(2)}(0) + Q_5 \psi_i^{(2)}(l)$$

$$F_i^3 = \int_0^l \mu \left[ f_z \frac{d\psi_i^{(3)}}{dx} + \frac{df_z}{dx} \psi_i^{(3)} \right] dx + Q_3 \psi_i^{(3)}(0) + Q_6 \psi_i^{(3)}(l) \quad (31)$$

**5. Numerical results**

In this section we will present numerical examples to demonstrate the application of the above discussed non-local non-linear formulation. The first example deals with non-local non-linear bending analysis of beams and the second example deals with non-local non-linear free vibration analysis of beams. Various boundary conditions, such as both ends simply supported (S–S) and both ends clamped (C–C), are considered. Two cases of load distribution (uniformly distributed load and sinusoidally distributed load) with load intensity  $q_0$  are considered. Numerical implementation is made after developing a MATLAB code for the Timoshenko beam finite element as discussed in the previous section.

Based on the non-local non-linear analysis of the beam, a parametric study has been performed for all the boundary conditions considered. For the static bending analysis the following cases are considered for the parametric study, namely (a) the effect of non-local parameter  $\mu$ , (b) the effect of surface modulus  $E_s$  and (c) the effect of surface tension parameter  $\tau$  on the non-linear behavior of the beam. For the dynamic analysis the effect of non-local parameter  $\mu$  on (a) the variation of fundamental frequency ratio with aspect ratio (b) surface frequency ratio versus amplitude ratio (c) the effect of surface modulus  $E_s$  on the variation of fundamental frequency with aspect ratio of the beam are considered for the parametric study.

**5.1. Example 1: static bending analysis**

In this example a beam with aspect ratio  $L/H$  varying from 10 to 100 is considered for the non-local non-linear bending analysis. The material properties of the beam are taken as elastic modulus,  $E = 17.73 \times 10^{10}$  N/m<sup>2</sup> and Poisson’s ratio  $\nu = 0.27$ . The breadth  $B$  and height  $H$  of the beam are taken as 1 nm. Simply supported (S–S) and clamped (C–C) boundary conditions are considered. Taking the symmetry of the beam (about  $x = L/2$ ) in both cases, one half of the beam is only considered for analysis. The boundary conditions for the two cases read as

S–S beam :  $w_{(x=0)} = 0, \quad u_{(x=L/2)} = 0, \quad \phi_{(x=L/2)} = 0$   
 C–C beam :  $u_{(x=0)} = 0, \quad w_{(x=0)} = 0, \quad \phi_{(x=0)} = 0$

For verification of the non-local non-linear analysis results obtained from the present study as a specific case the linear analysis

$$K_{ij}^{22} = \int_0^l \tilde{G} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx$$

$$+ \frac{1}{2} \left\{ \tilde{A} \left( \frac{dw}{dx} \right)^2 + \tilde{C} (\varphi_x)^2 \right\} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} dx \quad (29)$$

$$K_{ij}^{23} = \int_0^l \tilde{G} \frac{d\psi_i^{(2)}}{dx} \psi_j^{(3)} dx$$

$$K_{ij}^{31} = \int_0^l \tilde{C} \varphi_x \psi_i^{(3)} \frac{d\psi_j^{(1)}}{dx} dx$$

$$K_{ij}^{32} = \int_0^l \tilde{G} \psi_i^{(3)} \frac{d\psi_j^{(2)}}{dx} dx$$

$$K_{ij}^{33} = \int_0^l \tilde{D} \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} dx + \tilde{G} \psi_i^{(3)} \psi_j^{(3)}$$

$$+ \frac{1}{2} \left\{ \tilde{C} \left( \frac{dw}{dx} \right)^2 + \tilde{F} (\varphi_x)^2 \right\} \psi_i^{(3)} \psi_j^{(3)}$$

$$M_{ij}^{11} = m_0 \psi_i^{(1)} \psi_j^{(1)} + \mu m_0 \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx}$$

$$M_{ij}^{21} = \mu m_0 \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(1)}}{dx}$$

$$M_{ij}^{22} = m_0 \psi_i^{(2)} \psi_j^{(2)} + \mu m_0 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} \quad (30)$$

$$M_{ij}^{32} = \mu m_0 \psi_i^{(3)} \frac{d\psi_j^{(2)}}{dx} + \mu m_0 \frac{d\psi_i^{(3)}}{dx} \psi_j^{(2)}$$

results are compared with the analytical solutions given by Reddy [54] and with other methods available in the literature [67]. Table 1 gives the values of dimensionless static deflection  $\bar{w} = \frac{100 \times w_{max} EI}{q_0 l^4}$  for various values of non-local parameter  $\mu$  ranging from 0 to 4. It is observed that with increase in the non-local parameter there is an increase in the dimensionless central deflection  $\bar{w}$ . This clearly shows a decrease in the stiffness with increase in the non-local parameter, and the trends and values obtained from present analysis are very close to the analytical results.

To study the effect of non-local parameter  $\mu$  on the non-linear behavior of the beam, the non-local parameter  $\mu$  is varied from 0 to 5 nm<sup>2</sup>. S-S and C-C boundary conditions are considered. A non-linear non-local analysis is performed. The plot of intensity of sinusoidally distributed load versus the central deflection  $w$  is shown in Fig. 1. It is observed that for both S-S and C-C boundary conditions with increase in the non-local parameter  $\mu$  there is an increase in the non-linear behavior and the values of the central deflection  $w$ . As expected the deflections in simply supported case are higher than those with clamped boundary conditions.

To study the effect of the surface modulus  $E_s$  on the non-local non-linear behavior, the surface modulus values of ( $E_s$ ) 0 N/m, 13 N/m and -3 N/m are taken. The plot of intensity of sinusoidally distributed load versus the central deflection  $w$  for various surface modulus is shown in Fig. 2(a) and (b). It is observed that for both S-S and C-C boundary conditions the effect of considering the surface effects results in an increased value of stiffness and hence a

reduction in the deflection of the beam. Positive value of  $E_s$  tends to increase the stiffness of the beam and hence results in reduced deflection. On the contrary, negative value of  $E_s$  reduces the stiffness of the beam and results in increased deflection. As expected, the deflections in simply supported case are higher than those with clamped boundary conditions.

To study the effect of surface tension  $\tau$  on the non-local non-linear bending behavior, a surface tension value of  $\tau = 1.7$  N/m is taken for the analysis. It is observed that non-local effect tends to relax the stiffness of the beam resulting in increased deflection as seen from Fig. 1. On the contrary the surface tension ( $\tau$ ) stiffens the beam and reduces the deflections. A plot of load versus transverse center deflection for SS and CC beams for different values of  $\tau$  is presented in Fig. 3(a) and (b).

### 5.2. Example 2: dynamic analysis

In this example a beam with aspect ratio  $L/H$  varying from 10 to 100 is considered for the non-local non-linear free vibration analysis. The material properties of the beam are taken as elastic modulus,  $E = 17.73 \times 10^{10}$  N/m<sup>2</sup> and Poisson's ratio  $\nu = 0.27$ . The width  $B$  and height  $H$  of the beam both are taken as 1 nm. Simply supported (S-S) and clamped (C-C) boundary conditions are considered. Taking the symmetry of the beam about  $x = L/2$  in both cases, one half of the beam is modeled.

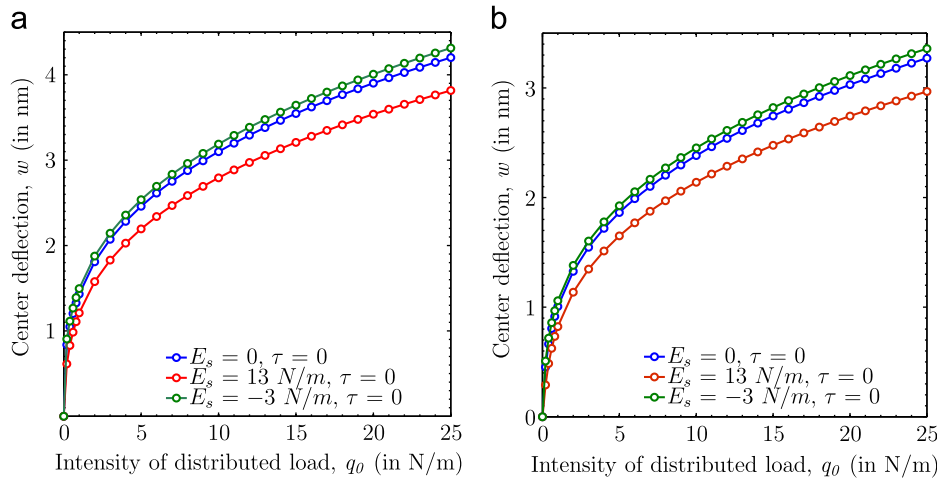


Fig. 2. Plot of load versus transverse center deflection for various values of surface parameter ( $E_s$ ) for (a) S-S beam and (b) C-C beam under sinusoidally distributed load.

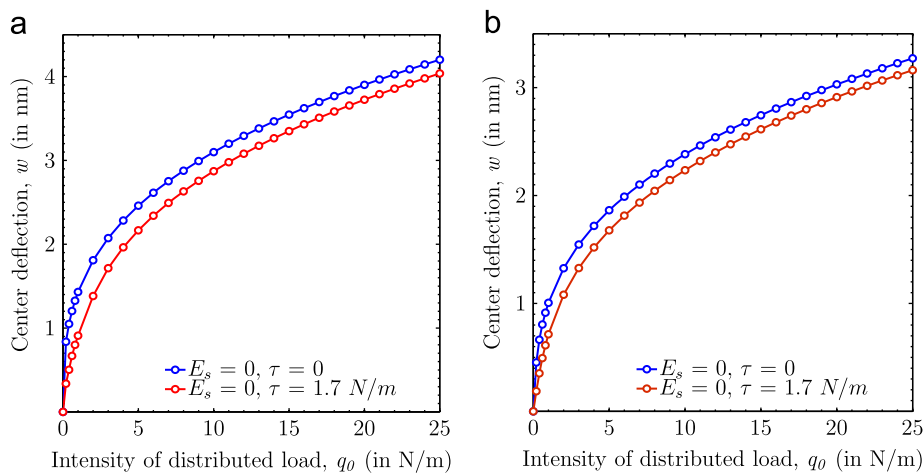


Fig. 3. Load versus transverse center deflection plot for different values of surface parameter ( $\tau$ ) for (a) SS beam and (b) CC beam under sinusoidally distributed load.

For a verification of the non-local non-linear free vibration analysis results obtained from the present study as a specific case the linear and the non-linear analysis results are compared with the analytical solutions given by Reddy [54] and with other methods available in the literature [67]. Table 2 gives the values of non-dimensionalized natural frequency  $\bar{\omega} = \omega \times l^2 \sqrt{\frac{m_p}{EI}}$  for various values of non-local parameter  $\mu$  ranging from 0 to 4. It is observed that with increase in the non-local parameter there is a decrease in the dimensionless natural frequency  $\bar{\omega}$ . This clearly reflects a decrease in the stiffness with increase in the non-local parameter and the trends and values obtained from present analysis are very close to the analytical results.

Table 3 contains the values of frequency parameter (ratio of non-linear natural frequency to linear natural frequency) for different amplitude ratio obtained from a non-linear analysis. The results are compared to those obtained from the DQM [44] and from the Ritz method [42]. There is an increase in frequency parameter with increase in amplitude ratio.

To study the effect of non-local parameter on the variation of frequency ratio with the aspect ratio of the beam, various non-local parameters  $\mu$  from 0 to 5 are considered. The frequency ratio

is defined as

$$\text{Frequency ratio} = \frac{\omega_{nl}(\text{with non - local effect})}{\omega_{nl}(\text{without non - local effect})}$$

The plot of frequency ratio versus aspect ratio  $L/H$  for S–S and C–C beams is presented in Fig. 4. It is observed that with the inclusion of non-local parameter there is a stiffening effect on the beam, thereby resulting in lower values of natural frequencies. For the values of  $L/H$  less than 25, the difference in frequencies for different values of  $\mu$  is very prominent. The trend is same for both S–S and C–C beams.

To study the effect of non-local parameter on the variation of surface frequency ratio with the aspect ratio of the beam, various non-local parameters  $\mu$  from 0 to 5 are considered. The surface frequency ratio is defined as

$$\text{Surface frequency ratio} = \frac{\omega_{nl}(\text{with surface effect})}{\omega_{nl}(\text{without surface effect})}$$

The plot of surface frequency ratio versus amplitude ratio plots for both beam cases is presented in Fig. 5 for various values of  $\mu$ . On the contrary to non-local effect, surface effect stiffens the beam which results in higher frequencies.

The effect of non-local parameter on non-linear natural frequency variation with aspect ratio is studied. The plots for both S–S and C–C beam cases are presented in Fig. 6. Positive values of  $E_s$  stiffen the beam and thus resulting in higher frequencies. Negative values of  $E_s$  have the opposite effect and decrease the frequencies. Surface tension  $\tau$  has no effect on the vibration characteristics of the beam.

**Table 2**  
Variation of the dimensionless natural frequency ratio obtained from linear analysis for various non-local parameter  $\mu$  (for  $L/H = 100$ ).

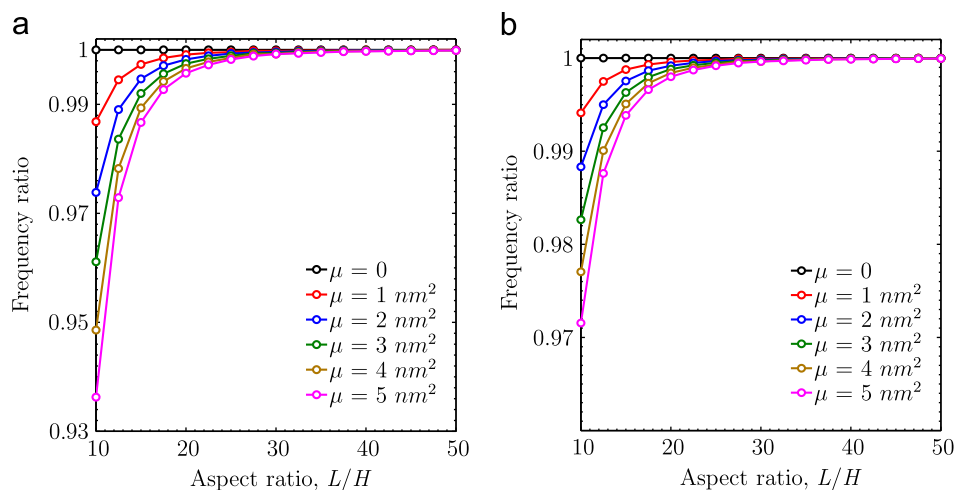
$\mu$	Reddy [54]	Thai [67]	$\bar{\omega}$ (present)
0	9.8683	9.8679	9.8677
1	9.4147	9.4143	9.4141
2	9.0183	9.0180	9.0179
3	8.6682	8.6678	8.6676
4	8.3558	8.3555	8.3553

**Table 3**  
Variation of frequency parameter ( $\omega_{non-linear}/\omega_{linear}$ ) with various amplitude ratios for a non-linear analysis.

$W_m/r$	DQM [44]	RM [42]	Present
1	1.0630	1.0486	1.0465
2	1.2505	1.1794	1.1751
3	1.5987	1.3635	1.3626

5.3. Example 3: cantilever beam example

In this example a cantilever beam with aspect ratio  $L/H$  varying from 10 to 50 is considered for the non-local non-linear bending analysis. The material properties of the beam are taken as elastic modulus,  $E = 17.73 \times 10^{10} \text{ N/m}^2$  and Poisson's ratio  $\nu = 0.27$ . The breadth  $B$  and height  $H$  of the beam are taken as 1 nm. The non-local parameter is varied from 0 to 5. In an earlier work [10] it has been shown that non-local effect does not exist in the nano-cantilever beam (Euler–Bernoulli beam) subjected to concentrated load at the end. However, it is shown in [10] that there is a significant effect of non-local parameter on deflections for other load cases such as uniformly distributed load and sinusoidally distributed load. In this study example, we mainly concentrate on studying the behavior of



**Fig. 4.** Frequency ratio versus aspect ratio  $L/H$  for (a) S–S beam and (b) C–C beam for different values of non-local parameter ( $\mu$ ).



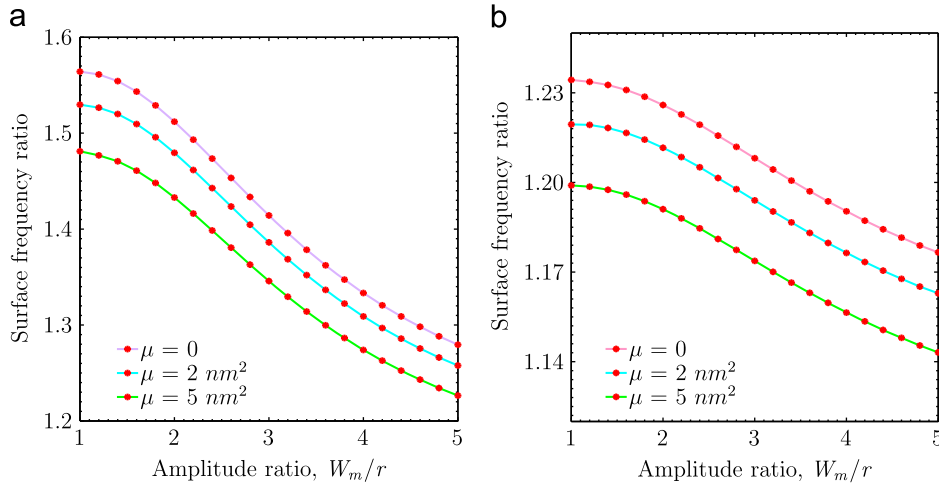


Fig. 5. Surface frequency ratio versus amplitude ratio ( $W_m/r$ ) for (a) S-S beam and (b) C-C beam for different values of non-local parameter ( $\mu$ ).

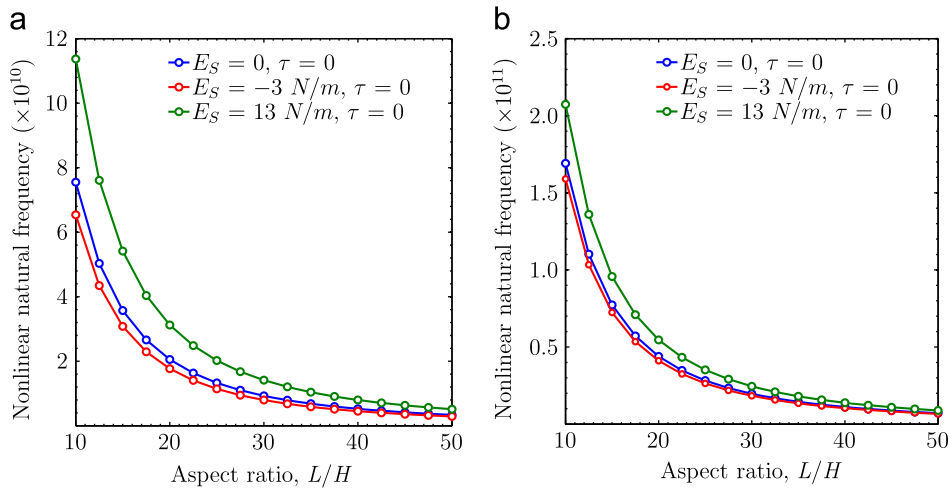


Fig. 6. Non-linear natural frequency versus aspect ratio  $L/H$  for (a) S-S beam and (b) C-C beam for different values of surface parameter ( $E_S$ ).

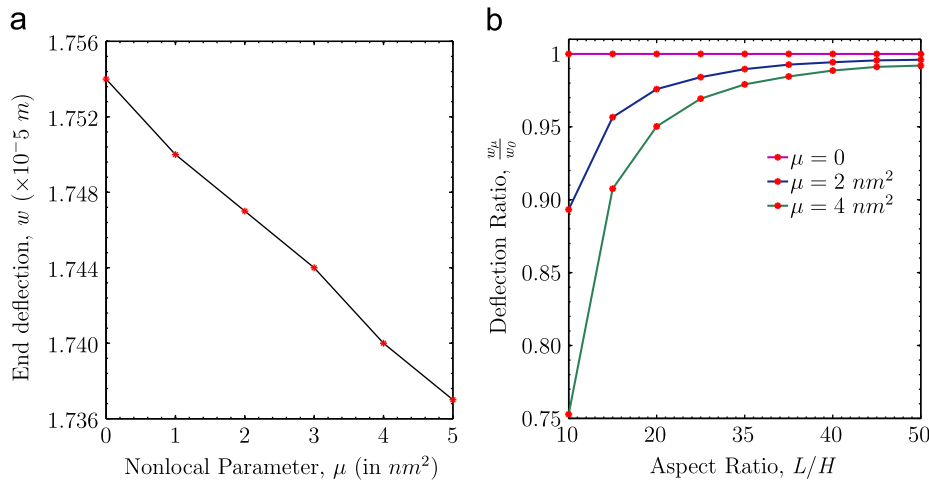


Fig. 7. (a) Non-local parameter versus end deflection of the cantilever beam subjected to point load ( $Q_0 = 10 \text{ N}$ ) at the end (for  $L/H = 50$ ). (b) Aspect ratio versus ratio of end deflection of the cantilever beam for different values of non-local parameter when subjected to the point load at the end ( $Q_0 = 10 \text{ N}$ ).

cantilever beams subjected to different load conditions namely a point load at free end, uniformly distributed force and sinusoidally varying force. The beam has been modeled using the Timoshenko beam formulation. The effect of the local or non-local boundary conditions on the end deflection has also been brought out.

Let us consider the transversely applied point load case on the cantilever beam as dirac delta function given as

$$P = Q_0 \delta(x - x_p) \tag{32}$$

where  $Q_0$  is the point load applied at the point  $x_p$  on the beam.

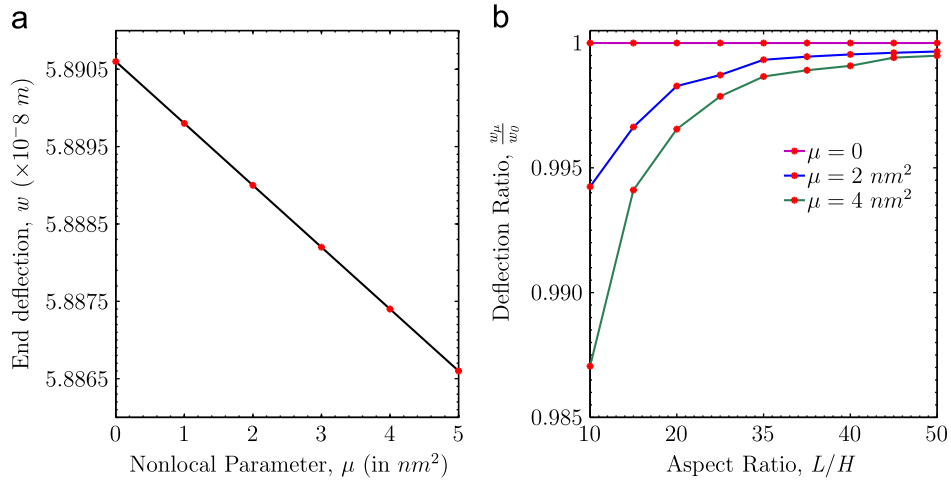


Fig. 8. (a) Non-local parameter versus end deflection of the cantilever beam subjected to sinusoidally varying load ( $q_0 = 10 N, L/H = 50$ ). (b) Aspect ratio versus ratio of end deflection of the cantilever beam for different values of non-local parameter when subjected to sinusoidally varying load ( $q_0 = 10 N$ ).

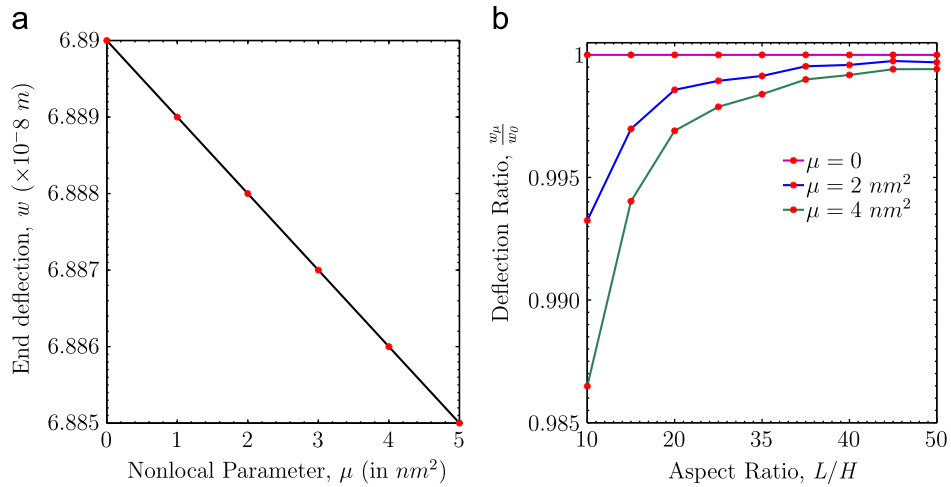


Fig. 9. (a) Non-local parameter versus end deflection of the cantilever beam subjected to uniformly varying load ( $q_0 = 10 N, L/H = 50$ ). (b) Aspect ratio versus ratio of end deflection of the cantilever beam for different values of non-local parameter when subjected to uniformly varying load ( $q_0 = 10 N$ ).

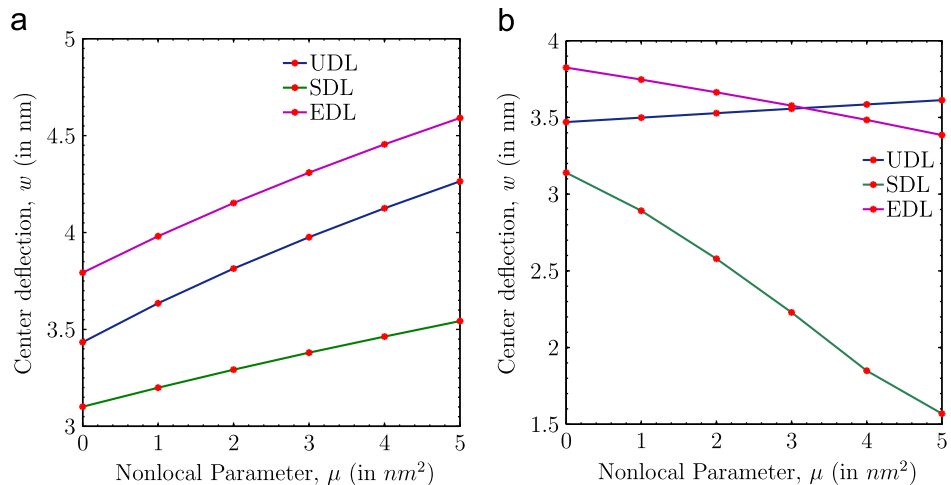
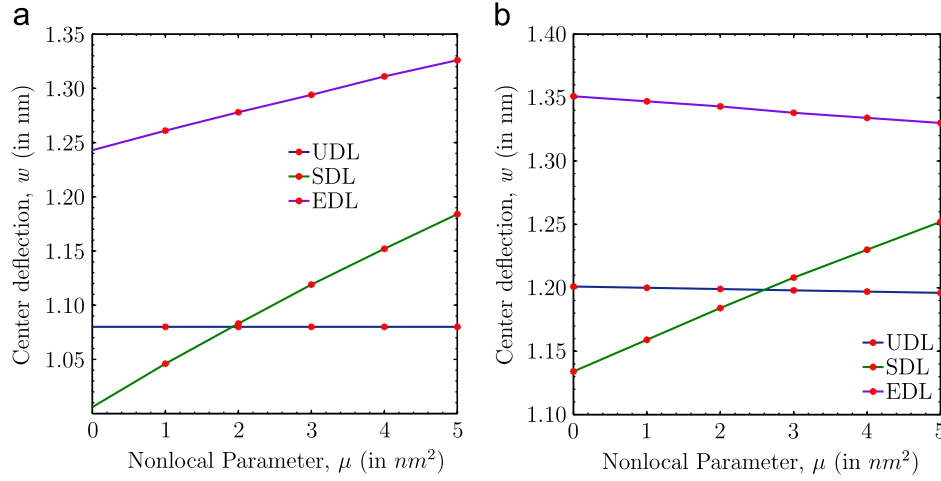


Fig. 10. Non-local parameter versus center deflection of the simply supported beam subjected to various load conditions (with  $q_0 = 10 N, L/H = 10$ ) with (a) local boundary conditions and (b) non-local boundary conditions.



**Fig. 11.** Non-local parameter versus center deflection of the clamped-clamped beam subjected to various load conditions (with  $q_0 = 1$  N,  $L/H = 10$ ) with (a) local boundary conditions and (b) non-local boundary conditions.

By using the principle of virtual work, the equations of equilibrium for the Timoshenko beam can be obtained as

$$\frac{dN_{xx}}{dx} + f_x = 0 \tag{33}$$

$$\frac{d}{dx} \left[ N_{xz} + N_{xx} \frac{dw}{dx} \right] + f_z + P = 0 \tag{34}$$

$$\frac{dM_{xx}}{dx} - (N_{xz} + N_{zz} \varphi_x) = 0 \tag{35}$$

and,  $f_x$  and  $f_z$  are the axially and transversely distributed forces, respectively. The boundary conditions are

Geometric :  $u, w, \varphi_x$

$$\text{Force : } N_{xx}, N_{xz} + N_{xx} \frac{dw}{dx}, M_{xx}$$

Manipulating Eqs. (33)–(35) and using Eringen's non-local differential model, the following relations are obtained:

$$N_{xx}^{nl} = \tilde{A} \varepsilon_{xx}^{(0)} + \tilde{C} \varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} \tag{36a}$$

$$M_{xx}^{nl} = \tilde{D} \varepsilon_{xx}^{(1)} - \mu f_z - \mu P - \mu \frac{d}{dx} \left[ \left( \tilde{A} \varepsilon_{xx}^{(0)} + \tilde{C} \varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{dw}{dx} \right] + \mu \frac{d}{dx} \left[ \left( \tilde{C} \varepsilon_{xx}^{(0)} + \tilde{F} \varepsilon_{zz} \right) \varphi_x \right] \tag{36b}$$

$$N_{zz}^{nl} = \tilde{C} \varepsilon_{xx}^{(0)} + \tilde{F} \varepsilon_{zz} \tag{36c}$$

$$N_{xz}^{nl} = \tilde{G} \gamma_{xz} - \mu \frac{df_z}{dx} - \mu \frac{d^2}{dx^2} \left[ \left( \tilde{A} \varepsilon_{xx}^{(0)} + \tilde{C} \varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{dw}{dx} \right] \tag{36d}$$

By substituting the expressions for non-local stress resultants (36) back in the equations of motion (33)–(35), we obtain the equilibrium equation for the non-local Timoshenko beam theory including surface stress effects as

$$0 = \frac{d}{dx} \left( \tilde{A} \varepsilon_{xx}^{(0)} + \tilde{C} \varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} \right) + f_x \tag{37}$$

$$0 = \frac{d}{dx} \left( \tilde{G} \gamma_{xz} \right) - \mu \frac{d^2 f_z}{dx^2} + f_z + P + \frac{d}{dx} \left[ \left( \tilde{A} \varepsilon_{xx}^{(0)} + \tilde{C} \varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{dw}{dx} \right] - \mu \frac{d^3}{dx^3} \left[ \left( \tilde{A} \varepsilon_{xx}^{(0)} + \tilde{C} \varepsilon_{zz} + 2\tau^0(b+h) - \mu \frac{df_x}{dx} + \mu m_0 \frac{d^3 u}{dx dt^2} \right) \frac{dw}{dx} \right] \tag{38}$$

$$0 = \frac{d}{dx} \left( \tilde{D} \varepsilon_{xx}^{(1)} \right) - \tilde{G} \gamma_{xz} - \varphi_x \left( \tilde{C} \varepsilon_{xx}^{(0)} + \tilde{F} \varepsilon_{zz} \right) + \mu \frac{d^2}{dx^2} \left[ \left( \tilde{C} \varepsilon_{xx}^{(0)} + \tilde{F} \varepsilon_{zz} \right) \varphi_x \right] \tag{39}$$

*Finite element formulation for Timoshenko beam with point load:* The principle of virtual work for the Timoshenko beam has the form

$$0 = \int_0^l \left[ N_{xx}^{nl} \delta \varepsilon_{xx}^{(0)} + M_{xx}^{nl} \delta \varepsilon_{xx}^{(1)} + N_{xz}^{nl} \delta \gamma_{xz} + N_{zz}^{nl} \delta \varepsilon_{zz} - f_x \delta u - f_z \delta w - P \delta w \right] dx - Q_1 \delta u(0) - Q_4 \delta u(l) - Q_2 \delta w(0) - Q_5 \delta w(l) - Q_3 \delta \varphi(0) - Q_6 \delta \varphi(l) \tag{40}$$

After substituting the expressions for stress resultants from Eq. (36) into Eq. (40) and by substituting the corresponding shape functions for virtual displacements, we obtain the finite element model as before. The stiffness coefficients  $K_{ij}^{\alpha\beta}$  and mass coefficients  $M_{ij}^{\alpha\beta}$  remain same. Force coefficients  $F_i^\alpha$  ( $\alpha, \beta = 1, 2, 3$ ) are given as

$$F_i^1 = \int_0^l \left\{ f_x \psi_i^{(1)} + \mu \frac{df_x}{dx} \frac{d\psi_i^{(1)}}{dx} - 2\tau^0(b+h) \frac{d\psi_i^{(1)}}{dx} \right\} dx + Q_1 \psi_i^{(1)}(0) + Q_4 \psi_i^{(1)}(l) \tag{41a}$$

$$F_i^2 = \int_0^l \left\{ f_z \psi_i^{(2)} + P \psi_i^{(2)} + \mu \left[ \frac{df_z}{dx} + \frac{df_x}{dx} \frac{dw}{dx} \right] \frac{d\psi_i^{(2)}}{dx} - 2\tau^0(b+h) \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \right\} dx + Q_2 \psi_i^{(2)}(0) + Q_5 \psi_i^{(2)}(l) \tag{41b}$$

$$F_i^3 = \int_0^l \mu \left[ f_z \frac{d\psi_i^{(3)}}{dx} + P \frac{d\psi_i^{(3)}}{dx} + \frac{df_z}{dx} \psi_i^{(3)} \right] dx + Q_3 \psi_i^{(3)}(0) + Q_6 \psi_i^{(3)}(l) \tag{41c}$$

As seen in Eq. (41c), the point load function ( $P$ ) is associated with the non-local parameter ( $\mu$ ). Therefore, point load subjected at any

point in the beam brings out non-local effect. If the point load is applied at the end, the terms highlighted will become the nodal quantities but will be still associated with non-local parameter.

**Numerical results:** In this example, cantilever beam case is studied under the following three load cases have been considered for analysis: (a) uniformly varying load (UVL), (b) sinusoidally varying load (SVL), and (c) point load acting at the end of the beam (PL). The material properties taken are same as those for examples in the paper. To study the effect of non-local parameter ( $\mu$ ) on the non-linear behavior of the beam, the non-local parameter  $\mu$  is varied from 0 to 5 nm<sup>2</sup>. Fig. 7(a) shows the non-local parameter versus end deflection of the cantilever beam subjected to point load  $Q_0 = 10$  N at the end (for  $L/H = 50$ ). It is observed that there is a decrease in the deflection with increase in non-local parameter for the cantilever beam. Fig. 7(b) shows the aspect ratio versus ratio of end deflection of the cantilever beam for different values of non-local parameter when subjected to the point load at the end ( $Q_0 = 10$  N). This clearly indicates that the non-local parameter has a significant effect on deflection characteristics of the cantilever beam.

Fig. 8(a) shows the non-local parameter versus end deflection of the cantilever beam subjected to sinusoidally varying load  $q_0 = 10$  N at the end (for  $L/H = 50$ ). It is observed that there is a decrease in the deflection with increase in non-local parameter for the cantilever beam. Fig. 8(b) shows the aspect ratio versus ratio of end deflection of the cantilever beam for different values of non-local parameter when subjected to the sinusoidally varying load ( $q_0 = 10$  N). This clearly indicates that the non-local parameter has a significant effect on deflection characteristics of the cantilever beam.

Fig. 9(a) shows the non-local parameter versus end deflection of the cantilever beam subjected to uniformly varying load ( $q_0 = 10$  N) at the end (for  $L/H = 50$ ). It is observed that there is a decrease in the deflection with increase in non-local parameter for the cantilever beam. Fig. 9(b) shows the aspect ratio versus ratio of end deflection of the cantilever beam for different values of non-local parameter when subjected to uniformly varying load  $q_0 = 10$  N. This clearly indicates that the non-local parameter has a significant effect on deflection characteristics of the cantilever beam.

#### 5.4. Example 4: cantilever, simply supported and clamped beam with varying boundary conditions

In [54], it is suggested that the boundary conditions for the non-local beam theory will remain same as that of local theories. In [10], it is suggested that bending moment and shear force in the boundary conditions should adopt the corresponding non-local expressions (i.e., the boundary conditions still keep non-local). To study the effect of local and non-local boundary conditions, the examples considered above were analyzed with both local and non-local boundary conditions. Results vary with the choice of boundary conditions (local or non-local). The choice of boundary condition did not affect the bending behavior of the cantilever beam. In the case of simply supported and clamped beams, the results had a dependency on the boundary conditions. Fig. 10 (a) and (b) shows the variation of deflection with the non-local parameter for both local and non-local boundary conditions for the simply supported case.

Fig. 11(a) and (b) shows the variation of deflection with the non-local parameter for both local and non-local boundary conditions for the clamped–clamped case. In both the cases it is observed that when local boundary conditions are applied the deflection is found to increase with non-local parameter and when non-local boundary conditions are applied the deflection is found to decrease with non-local parameter.

## 6. Summary and conclusions

Using Eringen's non-local differential model together with Gurtin and Murdoch surface elasticity theory, the effect of non-local parameter and surface stress on non-linear bending and vibration characteristics of beams is formulated. Green's strain tensor was used to model geometric non-linearity. The finite element method is used to solve the resulting non-linear equations. Parametric studies are carried out to investigate the influence of non-local parameter ( $\mu$ ) and surface parameters ( $E_s$  and  $\tau$ ) on bending and vibration characteristics of beams. It is found that the non-local parameter as well as the positive values of surface parameters relaxes the stiffness of the beam and results in larger deflections and lower frequencies. Negative values of  $E_s$  decrease the deflections and increase the frequencies. In the case of a cantilever beam with concentrated load, the non-local effects are found to significantly affect the results.

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