Stability of time-periodic wake flow

Meenu Agrawal

Department of Mechanical & Aerospace Engineering

June 2015

Declaration

I declare that this written submission represents my ideas in my own words, and where ideas or words of others have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be a cause for disciplinary action by the Institute and can also evoke penal action from the sources that have thus not been properly cited, or from whom proper permission has not been taken when needed.

Meenu Agrawal

(Signature)

(Meenu Agrawal)

(ME13M1036)

Approval Sheet

This Thesis entitled Stability of time-periodic wake flow by Meenu Agrawal is approved for the degree of Master of Technology from IIT Hyderabad

(Dr.Kirti Chandra Sahu) Examiner
Dept. of Chemical Engineering
IITH

(Dr. Chandrika Prakash Vyasarayani) Examiner
Dept. of Mechanical & Aerospace Engineering
IITH

(Dr. Harish N. Dixit) Adviser
Dept. of Mechanical & Aerospace Engineering
IITH

Acknowledgements

I would like to thank my guide for giving me an opportunity to work with him and for his constant support and timely suggestions during demanding times of the work. His extreme energy, creativity and commitment toward work have always been a constant source of motivation for me. He is one of the best mentors, I will always be thankful to him.

I would also like to thank Dr. Chandrika Prakash Vyasarayani for guiding me through the basics of the stability of time-periodic flow and constantly helping and clearing my doubts. A special note of thanks to all my classmates from Thermal Engineering for inspiring me to do my best always. I would also like to thank my junior and room-mate Priyamvada Sharma for making my stay at IIT Hyderabad memorable and enjoyable.

I would like to make a special mention of the excellent computational facilities provided by Dr.Harish N. Dixit. I would also like to thank Mr. Madhu Pandicheri for his timely support in the CAE lab.

I would like to dedicate this thesis to my amazingly loving and supportive parents who have always been with me, no matter where I am.

Abstract

Understanding the stability of open shear flows such as wakes and jets has important implications for flows in industry and nature. When a fluid flows past a stationary bluff body such as a cylinder, the well known Karman vortex shedding occurs where the shed vortices align themselves in a zig-zag anti-symmetric pattern. Recent experiments and numerical simulations show that if the inlet flow is oscillatory or if the cylinder oscillates about a fix position, the vortex shedding pattern can sometimes be symmetric rather than asymmetric. The goal of this thesis is to gain insight into the pattern formation process resulting in a symmetric shedding pattern. This can be achieved by carrying out a stability analysis on an oscillating wake profile.

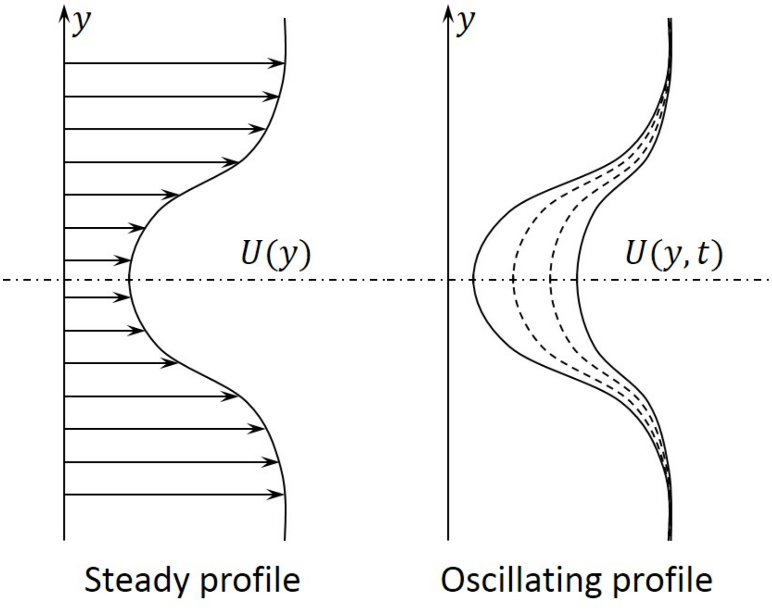
In this thesis, we carry out a linear stability analysis on an oscillating wake profile. To keep the analysis analytically tractable, we restrict the analysis to piece-wise continuous profiles. Since the base-flow is time-periodic, a generalized version of the Rayleigh stability equation is first derived. This is a partial differential equation with both time and spatial derivatives unlike the classical Rayleigh equation which is an ordinary differential equation. Generalization of kinematic and pressure jump conditions at vorticity interfaces are also derived. Two methods are used to analyse the stability of the oscillating base flow, a small frequency asymptotic analysis and a Floquet analysis. In all the cases, it was found that sinuous (asymmetric) modes were more unstable compared to varicose modes. A more extensive stability analysis, perhaps with smooth base-state profiles along with viscosity is needed to shed further light on why experiments and numerical simulations find a symmetric mode of vortex shedding.

Contents

Chapter 1

Introduction

Within the geophysical sciences, shear instability is known to be an important cause of turbulence and mixing in the atmosphere and oceans. The first step in the study of a shear flow is to perform a linear stability analysis to determine whether small perturbations applied to the flow will grow in time Fig. 1.1.



(a) (b)

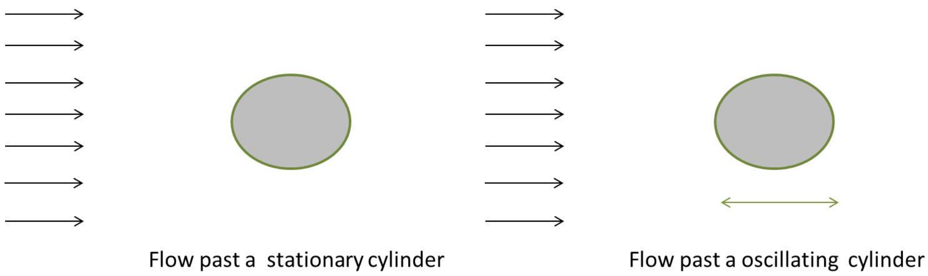
Figure 1.1: steady oscillating wake

The primary goal of this work is to study the instability process. The main focus is on understanding the stability properties of piecewise-linear profiles. However more realistic smooth profiles, as well as the application to geophysical flows will also be discussed.

1.1 Motivation of problem

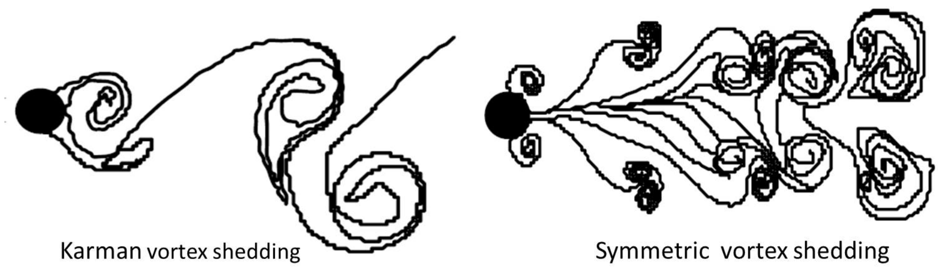
In fluid dynamics, a Karman vortex street Fig. 1.3 is a repeating pattern of swirling vortices caused by the unsteady separation of flow of a fluid around bluff bodies. It is named after the engineer and fluid dynamicist Theodore von Krmn and is responsible for such phenomena as the "singing" of suspended telephone or power lines, and the vibration of a car antenna at certain speeds.

The motivation for this work is to investigate the instabilities of the wake flow. The great advantage of this interpretation is that it gives one a physically based understanding of the often nonintuitive results from a stability analysis.



(a) (b)

Figure 1.2: flow past a stationry and oscillating cylinder



(a) (b) S-II mode

Figure 1.3: karman and symmetric vortex shedding

1.2 Literature review

[KELLY(1965)]:Discussion of stability for interface between two inviscid fluids for unsteady shear layers using a form of Mathieu equation to get the sub-harmonic motion, such that a wave which is neutrally stable in the absence of the oscillations and whose frequency is half of the frequency of the flow oscillations becomes unstable. The sub-harmonic response is very important in stability, because viscous effects would tend to eliminate the higher harmonic responses. [Srikanth T. & Govindarajan(2011)]The S-II mode of shedding is obtained computationally for the first time. A new symmetric mode, named here as S-III, is also found. At low oscillation amplitudes, the vortex shedding pattern transitions from anti-symmetric to symmetric smoothly via a regime of intermediate phase. Fig. 1.3 [S. J. X U & G(2006)] discovered a new mode of symmetric shedding, which named S-II. Two vortices of opposite sense were shed from each side (top and bottom) during each cycle. This mode was observed for high frequencies and amplitudes.There was considerable reverse flow during a part of the cycle, which aided in the formation of opposite signed vortices on a given side of the cylinder.Fig. 1.3 [Hultgren & Aggarwal(1987)] Agrawal is investigate the effects of viscosity on the absolute instability of wake flows using as a model the simple gaussian mean velocity profile. [BRIDGES & MORRIS(1984)] [T. B. Benjamin(1954)] [Suresh & Homsy(2004)] [Schmid & Henningson(2000)] [von Kerczek & Tozzi(1986)] [YOSHIKAWA & WESFREID(2011)] [Davis(1976)] [William E. Boyce(2009)]

1.3 Thesis structure

The thesis is divided into five chapters .The first chapter covers the introduction to stability of time periodic wake flow. Solvin governing equation using normal mode analysis and further simplification of the derived equation using different methods have been discussed in the second chapter. In the third chapter we discuss Steady state problem of kelvin helmholtz (KH) instability for piecewise continuous mixing layer profile.Chapter four deals with unsteady state problem of oscillatory kelvin helmholtz profile. In the last chapter is Conclusion.

Chapter 2

Governing Equation

Let us consider the system of governing differential equations for incompressible flow and 2D Euler equations, describing the conservation of mass and momentum:

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0 \\ \frac{\partial \tilde{u}}{\partial t} + \tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y} &= -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial x} \\ \frac{\partial \tilde{v}}{\partial t} + \tilde{u}\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial y} &= -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial y} \end{aligned} \tag{2.1}$$

Where $\tilde{u}, \tilde{v}, \tilde{p}$ are total quantities.($\tilde{u}, \tilde{v}$ are the velocity in x, y direction and $\tilde{p}$ is the pressure)

To derive the disturbance equations, each of the velocity vector components is divided into mean and disturbance parts, with the disturbance being of order ϵ. The functional form for the mean part only involves the axisymmetric parallel mean flow assumption.
So, it is clear from the above discussion that one cannot predict the behavior of the solutions of differential equation by just looking at the form of the equations.

2.1 Derivation of governing stability equations

The following steps define the procedure to identify the nature of a mathematical equation.

Step 1: Write the Navier Stokes (NS) equation for linear stability analysis. To get the special case of rotational incompressible flow, write

the vorticity function for the flow equation...

$$\tilde{u} = U(y,t) + u(x,y,t) \qquad u, v << U$$
$$\tilde{v} = v(x,y,t)$$
$$\tilde{p} = P(x,y,t) + p(x,y,t) \qquad p << P$$

$$\frac{\partial \tilde{\omega}}{\partial t} + \tilde{u}\frac{\partial \tilde{\omega}}{\partial x} + \tilde{v}\frac{\partial \tilde{\omega}}{\partial y} = \upsilon(\frac{\partial^2 \tilde{\omega}}{\partial x^2} + \frac{\partial^2 \tilde{\omega}}{\partial y^2}) \tag{2.2}$$

Where $\tilde{\omega} = \frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y}$ is the vorticity function

Equations are obtained after substituting the expressions of the velocity vector components in the Navier Stokes equations, considering contributions of first order in ϵ. Contributions of order one only satisfy Navier Stokes equations, where as ϵ^2 are ignored in the small disturbance limit.

Step 2: Consider the vorticity function and simplicity let $\upsilon = 0$ ***to get the PDEs***

$$\tilde{\omega} = \omega + \Omega = \frac{\partial v}{\partial x} - \frac{\partial (U+u)}{\partial y}$$

$$\omega + \Omega = -\frac{\partial U}{\partial x} + \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

$$\Omega = -\frac{\partial U}{\partial x} \quad , \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\frac{\partial \omega}{\partial t} + U\frac{\partial \omega}{\partial x} = vU'' \tag{2.3}$$

Substituting $\tilde{\omega}, \tilde{u}, \tilde{v}$ in equation (2.2) to get PDEs (2.3) and for simplicty inviscid flows to get $\upsilon = 0$

Step 3:By using the normal mode analysis to get PDEs

$$u = \bar{u}(y,t)e^{ikx} \qquad v = \bar{v}(y,t)e^{ikx} \qquad \omega = \bar{\omega}(y,t)e^{ikx}$$

$$\frac{\partial \bar{\omega}}{\partial t} + ik\bar{\omega}U - \bar{v}\frac{\partial^2 U}{\partial y^2} = 0 \tag{2.4}$$

Continuity Equation:

$$\bar{u} = -\frac{1}{ik}\frac{\partial \bar{v}}{\partial y}$$

Vorticity:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\bar{\omega} = -\frac{(D^2 - k^2)\bar{v}}{ik}$$

$$\omega = -\nabla^2 \psi \qquad \text{and } v = \frac{\partial \psi}{\partial x}$$

$$\boxed{\left(U + \frac{1}{ik}\frac{\partial}{\partial t}\right)(D^2 - k^2)\hat{v} - \hat{v}\frac{\partial^2 U}{\partial y^2} = 0} \tag{2.5}$$

Equation 2.9 is the main stability equation governing vertical velocity perturbation $\bar{v}$ for a time-dependent velocity field $U(t)$.

Step 4: Eigenvalue problem for steady state flow field

If U is not a function of time, we can assume the disturbance to be in the normal mode form, i.e. $\bar{v}(y,t) = \hat{v}(y)e^{-ikct}$. This form is used to obtain a simplified perturbation, especially in the continuous form. Equation 2.9 simplifies into an ordinary differential equation called the Rayleigh equation:

$$\boxed{(U - c)(\mathcal{D}^2 - k^2)\hat{v} - U''\hat{v} = 0.} \tag{2.6}$$

The Rayleigh equation can be written as an eigenvalue problem with eigenvalue c.

$$\left[U(D^2 - k^2) - U''\right]\hat{v} = c(D^2 - k^2)\hat{v}. \tag{2.7}$$

This is in the form

$$A\hat{v} = cB\hat{v} \tag{2.8}$$

Step 5: Matrix differential equation

Equation 2.9 can be rewritten in the form

$$\boxed{\frac{\partial}{\partial t}(D^2 - k^2)\hat{v} = ikU(D^2 - k^2)\hat{v} - ik\hat{v}\frac{\partial^2 U}{\partial y^2}} \tag{2.9}$$

The above equation can be solved using Floquet theory for time periodic differential equations. The details of this procedure will be discussed later. We briefly discuss below the procedure involved with Floquet theory. If $\mathbf{A(t)}$ is a time-periodic matrix governing the evolution of vector $\mathbf{X}(t)$ by the equation

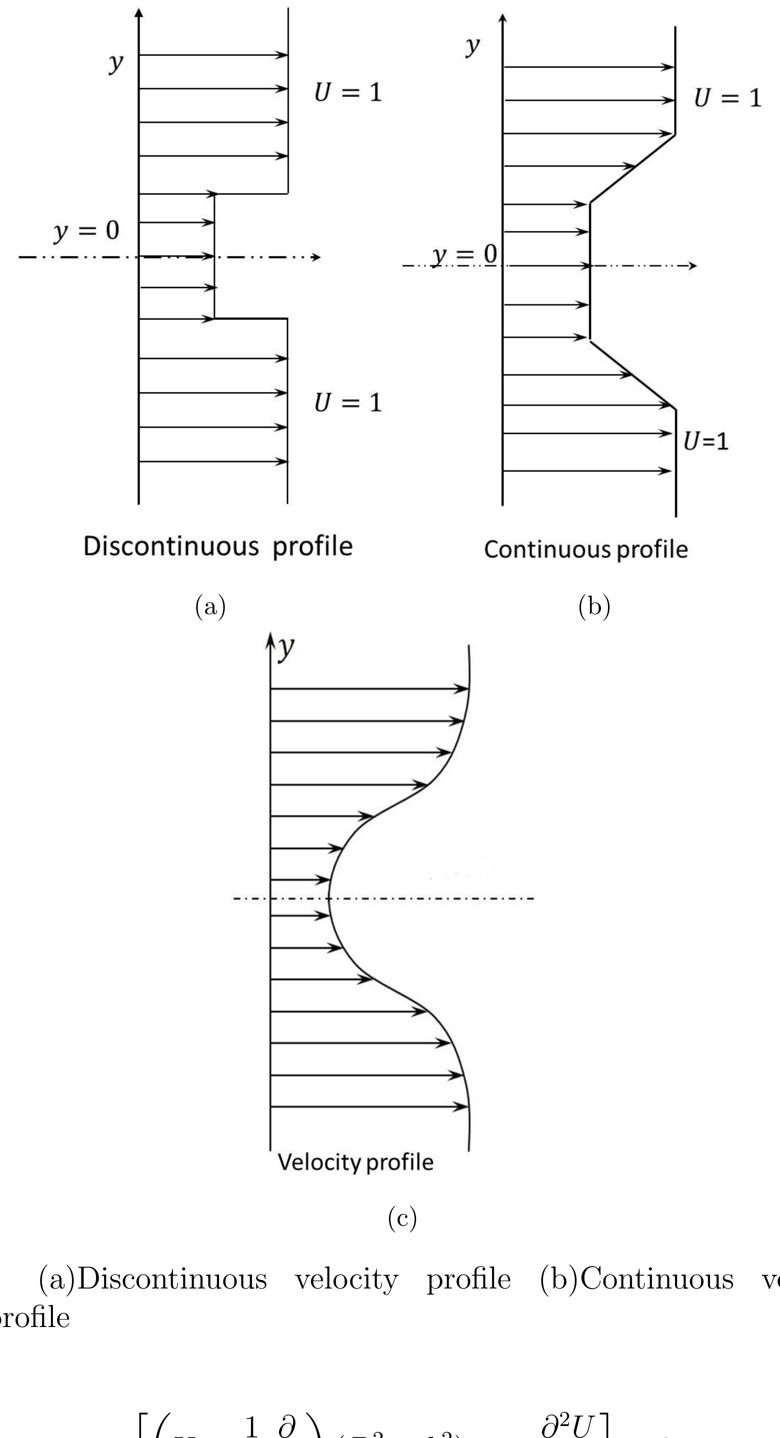
$$\frac{d}{dt}\mathbf{X} = \mathbf{A}(t)\mathbf{X}. \tag{2.10}$$

- Let $X = Df(c)$ be the matrix of first-order partial derivative of (Jacobian Matrix)evaluated at c.
- Every solution is stable if all eigenvalues of X has negative real parts.
- Every solution is unstable if at least one eigenvalue of X has positive real part.
- Floquet theory is very important for the study of dynamical systems.

$$\frac{d}{dt}\begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues of a real or complex $(n \times n)$ matrix X are the roots of its characteristic polynomial $det(X - \lambda I)$. Since the degree of the characteristic polynomial equals n, the dimension of X, it has n roots, so X has n eigenvalues. The eigenvalues may be real or complex, even if X is real. In case of complex roots, eigenvalues appear in pairs (complex conjugates).

2.2 Piecewise Discontinuous & Continuous velocity profile



(a) (b) (c)

Figure 2.1: (a)Discontinuous velocity profile (b)Continuous velocity profile (c)Velocity profile

$$\left[\left(U + \frac{1}{ik}\frac{\partial}{\partial t}\right)(D^2 - k^2)\bar{v} - \bar{v}\frac{\partial^2 U}{\partial y^2}\right] = 0$$

For piecewise profiles, $U'' = 0$, and the above equation becomes

$$(D^2 - k^2)\hat{v} = 0$$

2.3 Spatial & Temporal Stability analysis (c & k)

$$\omega(x, y, t) = \bar{\omega}(y)e^{ik(x-ct)}$$

The exponential structure allows the solution to oscillate and grow/decay in space and time, depending on the real and imaginary parts of k and c .In the temporal analysis, the solution grows/decay and oscillates with time, only in space: $k \in \mathbb{R}$ is given, and one obtains c from the dynamic equations. In the spatial analysis, one assume that the solution oscillates in time at a given spatial position, but is allowed to grow/decay and oscillate in space: $c \in \mathbb{R}$ is given, and k is obtained from the dynamics equations.

- If $c_i > 0$: Flow is unstable, e^{kc_it} increase with time.
- If $c_i < 0$: Flow is stable, e^{kc_it} decrease with time.
- If $c_i = 0$: Flow is neutrally Stable

Three types of analysis will be described. The analysis making the complex frequency ω the eigenvalue while fixing the axial wavenumber k is called a temporal stability analysis. On the other hand, if ω is fixed and k is the eigenvalue, a spatial stability analysis is performed. If the relationship between k and ω is restricted such that the combination of both values defines a point of vanishing group velocity, the resulting analysis determines the absolute stability of the flow field.

2.4 Sinuous and Varicose modes

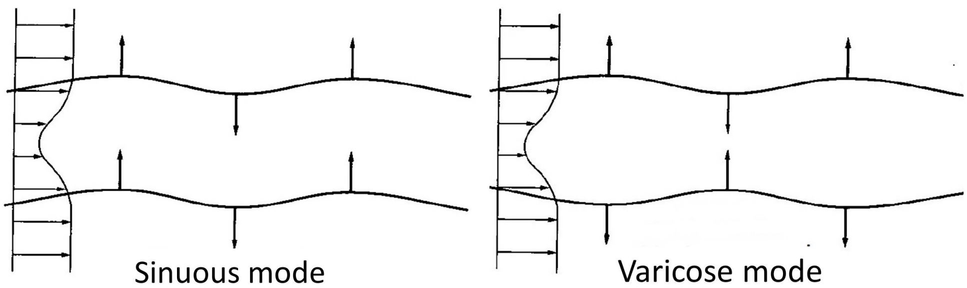
Introducing a second shear layer allows the interaction betwenn shear layers. These interactions take place in two different configurations; sinuous motions (I) which are anti-symmetric about the centreline and varicose motions (II) which are symmetric about the centreline. These two configurations are shown in Fig. 2.2

Since the governing equations are linear, any initial normal mode disturbance can be decomposed into a combination of varicose and sinuous modes, thus in turn it is sufficient to study the stability of each type of configuration individually.

- Sinuous mode, where the shear layers move parallel to each other.

- Varicose mode, where the shear layers move as mirror images of each other.

- Taking advantage of these symmetries, we may consider just one half of the domain for both kinds of mode.
- Sinuous Mode: At $y = 0$: $\mathcal{D}\bar{v} = 0$
- Varicose Mode: At $y = 0$: $\mathcal{D}^2\bar{v} = 0$



(a) (b)

Figure 2.2: Wake stability mode. (a)sinuous mode with $v(x, -y, t) = v(x, y, t)$ and (b) varicos mode with $v(x, -y, t) = -v(x, y, t)$

2.5 Mathieu Equation

Mathieu equation is a linear differential equation of second order.A pendulum has two critical points in its swing: the lowest position and the highest. The stability diagram for the Mathieu equation is shown in Fig. 2.3 .The lower position,one of the classic examples of an approximate harmonic oscillator, is completely stable.The upper position is unstable, falling with even the slightest change in the position or velocity of the pendulum

Its characteristics lead us to expect, for instance, that an inverted pendulum can be stabilized by suitably oscillating it in the vertical direction, thus causing the effective gravitational force to vary periodically with time.Normally stable pendulum can be made unstable by the vertical oscillations, especially if the frequency of vertical oscillation is exactly twice the frequency of the pendulums natural motion shown in Fig. 2.3)

- The Mathieu Equation is a second order linear ODE:

$$\begin{aligned} &\frac{\partial^2 Y}{\partial \tau^2} + [\delta + \varepsilon \cos \tau]Y = 0, \\ &Y(T+\tau) = Y(\tau). \end{aligned} \tag{2.11}$$

- T is the periodicity.

- If Mathieu Equation has the solution that grows exponentially with time then motion will be considered as unstable.

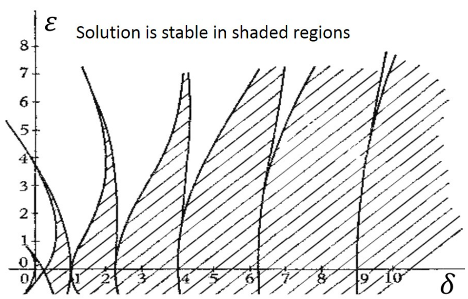


Figure 2.3: Stability boundary for Mathieu equation

Chapter 3

Steady Problem

"Every perfect geometrically sharp edge by which a fluid flows must tear it as under and establish a surface of separation however slowly the rest of the fluid may move" ***Helmholtz***

In this section we wish to consider the stability of two-dimensional flows. Flows of this type were first studied by Reynolds (1883),[Drazin(1981)]: who observed that instability could occur in quite different ways depending on the form of the basic velocity distribution. By comparing the flow of a viscous fluid with that of an inviscid fluid, both flows being assumed the same basic velocity distribution, he was led to formulate two fundamental hypotheses which can be stated as follows:

First hypothesis. The inviscid fluid may be unstable and the viscous fluid stable. The effect of viscosity is then purely stabilizing.

Second hypothesis. The inviscid fluid may be stable and the viscous fluid stable. In this case viscosity would be the cause of the instability.

The main focus is on understanding the stability properties of piecewise-linear profiles..[W. O. Criminale(2003)]

3.1 Piecewise continuous mixing layer profile

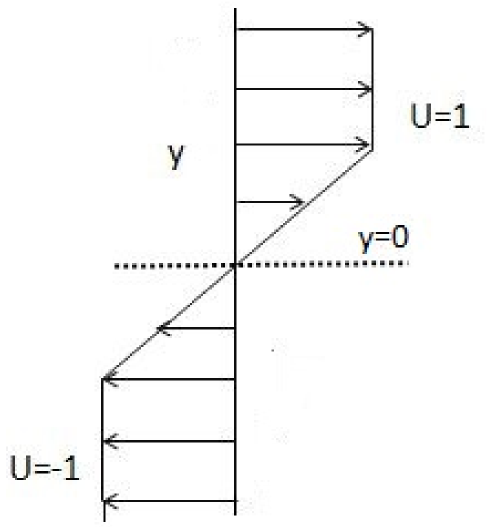


Figure 3.1: Piecewise continuous mixing layer profile

- Assuming viscous effect is neglected, 2D flow , steady state .
- We are using N . S. Equation to get the equation for Vorticity.

$$\frac{\partial \omega}{\partial t} + U\frac{\partial \omega}{\partial x} = vU''$$

We will analyze stability of the mean flow with respect to wave like velocity and pressure perturbations. We assume $v(x, y, t) = \hat{v}(y)e^{ik(x-ct)}$

- Then we get the Rayleigh equation:

$$[(U - c)(\mathcal{D}^2 - k^2)\hat{v} - U''\hat{v}] = 0 \tag{3.1}$$

(A)Continuity of pressure: $P|_+ = P|_-$

$$[(U - c)\mathcal{D}\hat{v} - U'\hat{v}] = 0$$

(B)Continuity of particle displacement: $\frac{D\eta}{Dt} = 0$

$$\left[\frac{\hat{v}}{U - c}\right] = 0$$

The equation (3.1) can be derived from the vorticity equation . Note that Rayleigh equation are unchanged when k is replaced by $-k$. Thus we shall always consider $k \geq 0$ and the criterion for instability then becomes that there exists a solution with $c_i > 0$ for some $k > 0$. Also if D is an eigen function with c for some k then so is D^* with eigen value c^* for the same k. Thus to each unstable mode, there exists a correspong stable mode.

The first jump condition is obtained from the fact that pressure be continuous at the material interface. Since the basic pressure is constant, we must have the perturbed pressure continuous across the interface. To derive the second jump condition we note that the normal velocity of the fluid must be continuous at the interface. Thus we must have from the definition of vertical velocity.

Unbounded shear layer $U'' = 0$ and thus Rayleigh's equation reduced to $(U - c)(\mathcal{D}^2 - k^2)\hat{v} = 0$. If we ignore the continuous spectrum then we have $(\mathcal{D}^2 - k^2)\hat{v} = 0$ which is equivalent to the vanishing of the y component of the vorticity. Since the perturbation vanish as $y \to \pm\infty$, the solution can be written as $v = C_2 e^{-ky} y \geq 0$ $v = C_5 e^{ky} y \leq 0$ If we assume $C_2 = 1$ then the solution can be written as

$$U(y) = \begin{cases} 1 & y \geq 1 \\ y & -1 \leq y \leq 1 \\ -1 & y \leq 1 \end{cases}$$

$$v(y) = \begin{cases} e^{-ky} & y \geq 1 \\ C_3 e^{ky} + C_4 e^{-ky} & -1 \leq y \leq 1 \\ C_5 e^{ky} & y \leq 1 \end{cases}$$

By using the jump condition you get equation in form of matrix.

$$\begin{bmatrix} 1 & e^{2k} & 0 \\ -k(1-c) - 1 & (k(1-c) - 1)e^{2k} & 0 \\ e^{2k} & 1 & -1 \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \\ C_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -k(1-c) \\ 0 \end{bmatrix}$$

3.1.1 Results for mixing layer profile

[Drazin(1981)]:

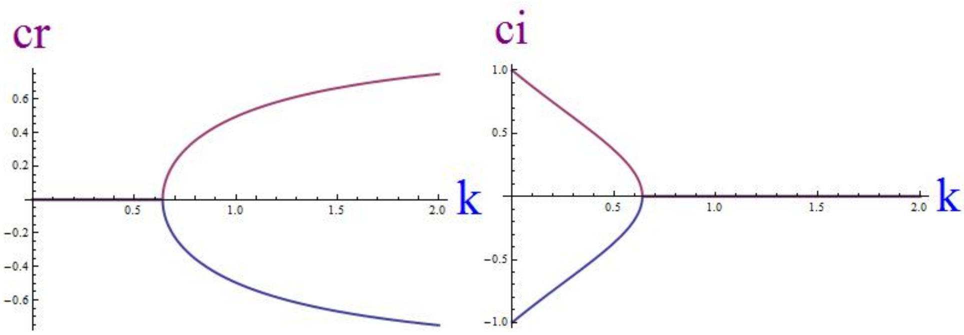


Figure 3.2: Eigenvalue for the piecewise linear mixing layer(a)Real part of the eigenvalue (phase speed) as a function of wave number(b) Imaginary part of the eigenvalue (phase speed) as a function of wave number

3.2 Stability of constant wake profile

$$\left[\left(U + \frac{1}{ik}\frac{\partial}{\partial t}\right)(D^2 - k^2)\bar{v} - \bar{v}\frac{\partial^2 U}{\partial y^2}\right] = 0$$

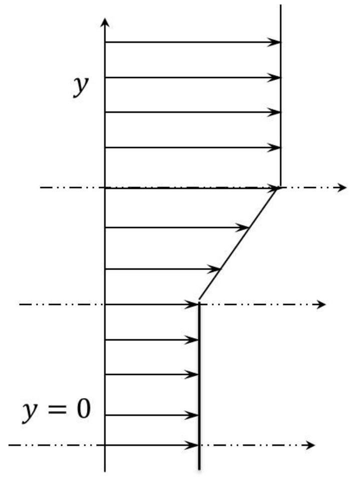
Assume normal mode: $v(y,t) = \hat{v}(y)e^{ik(x-ct)}$

$$\left[(U - c)(\mathcal{D}^2 - k^2)\hat{v} - U''\hat{v}\right] = 0 \tag{3.2}$$

Equation (3.2) is the well known Rayleigh Equation.

$$U(y) = \begin{cases} V(y) & 0 \le y \le b \\ \frac{1-V(y)}{1-b}y + \frac{V(y)-b}{1-b} & b \le y \le 1 \\ 1 & y \ge 1 \end{cases}$$

$$\hat{v}(y) = \begin{cases} c_1 e^{ky} + c_2 e^{-ky} & 0 \le y \le b \\ c_3 e^{ky} + c_4 e^{-ky} & b \le y \le 1 \\ c_5 e^{ky} + c_6 e^{-ky} & y \ge 1 \end{cases}$$



(a)

Figure 3.3: Piecewise continuous profile

3.2.1 Jump & Boundary Condition

Boundary conditions:

- As $y \to +\infty, \bar{v} \to 0 \implies c_5 = 0$
- At $y = 0, D^2\bar{v} = 0 \implies c_1 = -c_2$
  (A)Continuity of pressure: $P|_+ = P|_-$

$$[(U - c)\mathcal{D}\hat{v} - U'\hat{v}] = 0$$

  (B)Continuity of particle displacement: $\frac{D\eta}{Dt} = 0$

$$\left[\frac{\hat{v}}{U - c}\right] = 0$$

Jump conditions:

- At $y = b, \bar{v}|_{b^+} = \bar{v}|_{b^-} \quad \implies c_2 = \dfrac{c_3 e^{kb} + c_4 e^{-kb}}{e^{-kb} - e^{kb}}$
- At $y = 1, \bar{v}|_{1^+} = \bar{v}|_{1^-} \quad \implies c_6 = c_3 e^{2k} - c_4$
- At $y = b, \bar{P}|_{b^+} = \bar{P}|_{b^-}$

$$\left[(V(y) - c)k\{\frac{e^{kb} + c_4 e^{-kb}}{e^{-kb} - e^{kb}}\}(-e^{-kb} - e^{kb})\right] -$$
$$\left[\{(\frac{1 - V(y)}{1 - b}y + \frac{V(y) - b}{1 - b}) - c\}k(e^{kb} - c_4 e^{-kb}) - (e^{kb} + c_4 e^{-kb}) * (\frac{1 - V(y)}{1 - b})\right] = 0$$

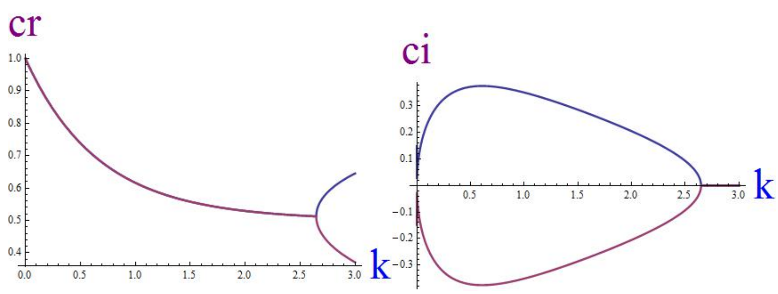
- At $y = 1, \bar{P}|_{1^+} = \bar{P}|_{1^-}$

$$\left[\{(\frac{1-V(y)}{1-b}y + \frac{V(y)-b}{1-b}) - c\}k(e^k - c_4 e^{-k}) - (e^k + c_4 e^{-k}) * (\frac{1-V(y)}{1-b})\right]$$
$$-\left[(1-c)k(-e^k + c_4 e^{-k})\right] = 0$$

Where

$$V(y) = 0.5 \qquad\qquad b = 0.5$$

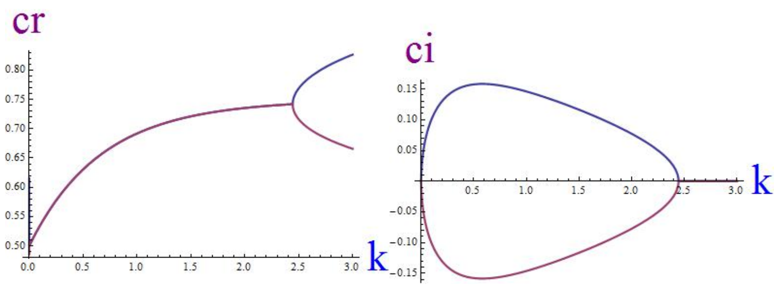
3.2.2 Result for Sinuous mode: Analytical solution



(a) (b)

Figure 3.4: Sinuous mode results

3.2.3 Result for Varicose mode: Analytical solution

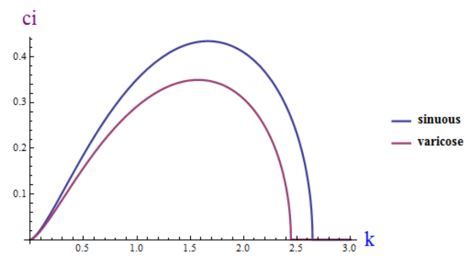


(a) (b)

Figure 3.5: Varicose mode result

3.2.4 Comparison of growth rate for constant profile: Sinuous vs Varicose

The varicos (II) mode is smaller than that of the sinuous (I) mode for the same flow parameters.



(a)

Figure 3.6: Comparison of growth rate: Sinuous vs Varicose

Chapter 4

Unsteady Problem

4.1 Stability of oscillatory Kelvin Helmholtz profile

The investigation concerns the stability of an interface between two inviscid fluids of same density which flow parallel to each other in an oscillatory manner. When the difference in the mean speeds is below the steady, critical speed for instability but is large compared to the amplitude of the fluctuations,parametric amplification of waves at the interface occurs, and the interface exhibits a resonance of a subharmonic nature.[KELLY(1965)]:

There would seem to be at least two reasons why the stability of time-dependent flows is of interest. First, one may be interested in how external effects which cause the basic flow to be unsteady but still laminar affect the stability of that flow.

A second reason for interest in such flows is that, prior to its final breakdown into turbulence, a flow may develop from its steady, laminar form through one or more stages of finite-amplitude oscillation. Non-linear analyses based upon perturbing the primary, steady flow have provided information on these states of oscillation, but not on the final breakdown into turbulence. Hence it would seem worth while to take the view that the instability has grown to such a degree that the basic flow must be taken to be time-dependent and to perform a linear stability analysis based upon a model of this unsteady flow.

$$-\delta p_j = \rho_j U_j \delta u_j + g\rho_j\eta + \rho_j \frac{\partial \delta\phi_j}{\partial t} \tag{4.1}$$

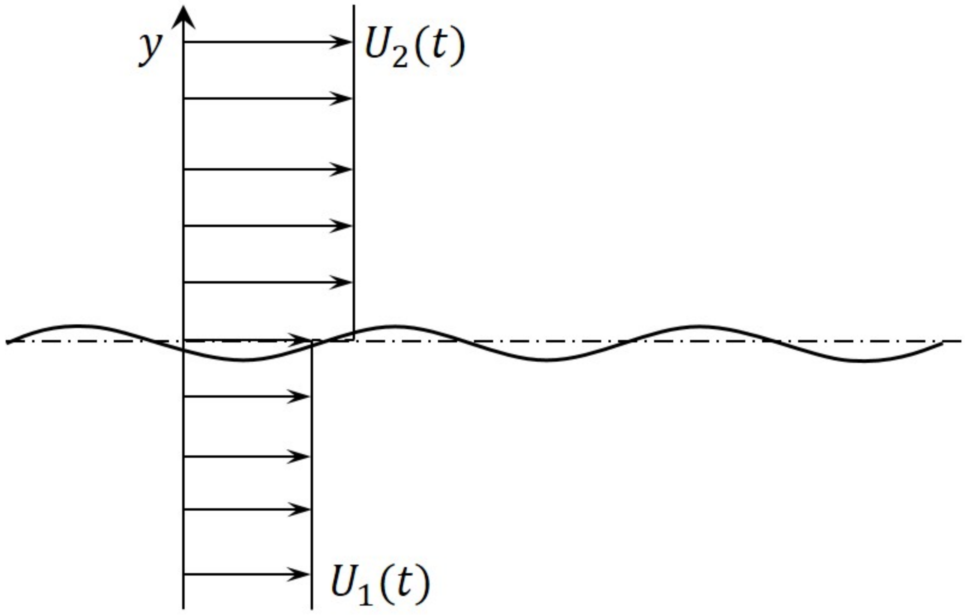


Figure 4.1: Oscillating velocity profile

We assume that the disturbance may be expressed in terms of normal modes, i.e. $f(x, y, t) = \bar{f}(y, t)e^{ikx}$.

Solving for the interface position, we get

$$\frac{\partial^2 \bar{\eta}}{\partial t^2} - \frac{1}{4}\left[k^2(U_1(t) - U_2(t))^2\right]\bar{\eta} = 0 \tag{4.2}$$

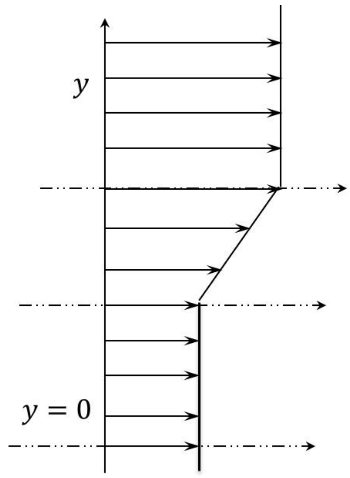
If $(U_2(t) - U_1(t)) = U_0 + \Delta U \cos ft$, the above equation reduces to the well known Mathieu equation in the limit of $\Delta U \ll U_0$.

4.2 Methods employed for analysis of governing equations:Small frequency asymptotic analysis

$$\left[\left(U + \frac{1}{ik}\frac{\partial}{\partial t}\right)(D^2 - k^2)\bar{v} - \bar{v}\frac{\partial^2 U}{\partial y^2}\right] = 0$$

$$U(y,t) = \begin{cases} 1 & y \geq 1 \\ \frac{1-V(t)}{1-b} y + \frac{V(t)-b}{1-b} & b \leq y \leq 1 \\ V(t) & 0 \leq y \leq b \end{cases}$$

$$\bar{\phi}(y,t) = \begin{cases} A_0(t)e^{ky} + A_1(t)e^{-ky} & y \geq 1 \\ A_2(t)e^{ky} + A_3(t)e^{-ky} & b \leq y \leq 1 \\ A_4(t)e^{ky} + A_5(t)e^{-ky} & 0 \leq y \leq b \end{cases}$$



(a)

Figure 4.2: Piecewise continuous profile

4.2.1 Jump & Boundary Condition

- As $y \to +\infty, \bar{v} \to 0 \implies A_0(t) = 0$
- At $y = 0, D^2\bar{v} = 0 \implies A_4(t) = A_5(t)$

  (A) Continuity of particle displacement: $\frac{DF}{Dt} = 0$

$F(x,y,t) = y - \eta(x,t) \qquad \eta = \bar{\eta}e^{ikx} \qquad \bar{\eta} = \tilde{\eta}e^{-i\lambda t} \qquad \bar{v} = \phi(y,t)e^{-i\lambda t}$

$$\left[\frac{\phi}{kU - \lambda}\right]_{-v}^{+v} = 0 \tag{4.3}$$

(B) Continuity of pressure: $P|_+ = P|_-$

$$\left[\frac{\partial^2 \phi}{\partial y \partial t} + ikU\frac{\partial \phi}{\partial y} - ik\phi\frac{\partial U}{\partial y}\right]_{-p}^{+p} = 0 \tag{4.4}$$

Jump conditions:

- At $y = b, \bar{v}|_{b^+} = \bar{v}|_{b^-} \implies A_2(t)e^{2kb} - A_3(t) = A_4(t)[e^{2kb} - 1]$
- At $y = 1, \bar{v}|_{1^+} = \bar{v}|_{1^-} \implies A_1(t) = A_3(t) - A_2(t)e^{2k}$
- At $y = b, \bar{P}|_{b^+} = \bar{P}|_{b^-}$

$$k\left[-2e^{kb}\frac{dA_2(t)}{dt} + 2e^{kb}A_3(t)\right] \quad -k^2V(t)\left[-2e^{kb}A_2(t) + 2e^{kb}A_3(t)\right]$$
$$+k(e^{2kb}-1)\left[\left(\frac{1-V(t)}{1-b}\right)(A_2(t)e^{kb} - A_3(t)e^{-kb}\right] = 0$$

- At $y = 1, \bar{P}|_{1^+} = \bar{P}|_{1^-}$

$$\frac{dA_2(t)}{dt} + \frac{i}{2}\left[2kU_1 - \left(\frac{1-V(t)}{1-b}\right)\right]A_2(t) + \frac{i}{2}\left(\frac{1-V(t)}{1-b}\right)A_3(t)e^{-2k} = 0$$

$$\frac{dA_3(t)}{dt} + \left[ik - ikV(t) - \frac{i}{2}\left(\frac{1-V(t)}{1-b}\right)e^{2kb}\right]A_2(t)$$
$$+\left[\frac{i}{2}\left(\frac{1-V(t)}{1-b}\right)e^{-2k} + ikV(t) + \frac{i}{2}\left(\frac{1-V(t)}{1-b}\right)(1-e^{-2kb})\right]A_3(t) = 0$$

We are solve for $A_2(t)$ and $A_3(t)$ using the form of matrix.

$$\frac{d}{dt}\begin{bmatrix} A_2(t \\ A_3(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix}\begin{bmatrix} A_2(t) \\ A_3(t) \end{bmatrix}$$

where

$$a_{11}(t) = -\frac{i}{2}\left[2kU_1 - \left(\frac{1-V(t)}{1-b}\right)\right]$$
$$a_{12}(t) = -\frac{i}{2}\left(\frac{1-V(t)}{1-b}\right)e^{-2k}$$
$$b_{21}(t) = \left[-ik + ikV(t) + \frac{i}{2}\left(\frac{1-V(t)}{1-b}\right)e^{2kb}\right]$$
$$b_{22}(t) = -\left[\frac{i}{2}\left(\frac{1-V(t)}{1-b}\right)e^{-2k} + ikV(t) + \frac{i}{2}\left(\frac{1-V(t)}{1-b}\right)(1-e^{-2kb})\right]$$

- $\vec{X}' = A\vec{X} + \vec{g}$ To find the particular solution.

- If $\vec{g}$ involves exponential term,try a particular solution of the form $\vec{X}_p = \vec{V}($ exponential term).

- If the exponential term is part of the homogeneous solution ,the try $\vec{X}_p = t\vec{V}($ exponential term)+$\vec{\eta}$(exponential term).

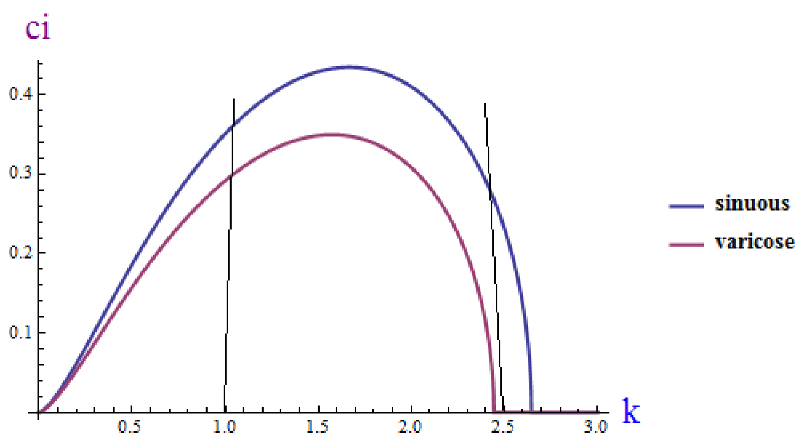
- If $\vec{g}$ has a 'cos' or 'sine' term ,solve the more general problem with the nonhomogeneous term replaced with $e^{i\omega t}$ then $\cos(\omega t) = Ree^{i\omega t}$ $\sin(\omega t) = Ime^{i\omega t}$

$$\frac{dX(t)}{dt} = A(tf)X(t) \qquad \tau f \frac{dX(\tau)}{d\tau} = A(\tau f)X(\tau)$$

$$X(t) = X^0(t) + (ft)X^1(t) + (ft)^2X^2(t) + ...$$

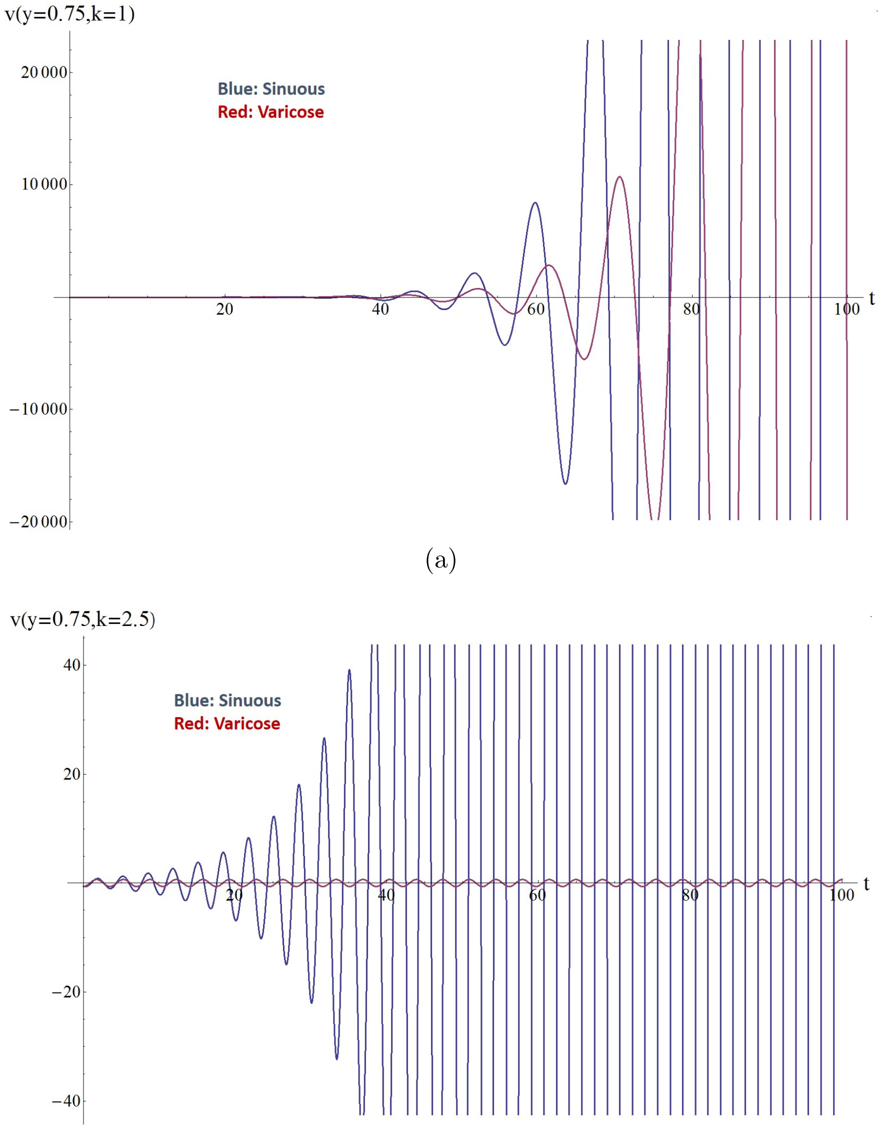
$$X^0(t) = A^0X^0 \qquad O(0)$$

$$X^1(t) = A^0X^1 + A^1(t)X^0 \qquad O(1)$$



(a)

Figure 4.3: Comparison of growth rate: Sinuous vs Varicose at different k value k=1 and k=2.5



(a)

(b)

Figure 4.4: Comparison of growth rate: Sinuous vs Varicose (a)for ft=0 and k=1 (b)for ft=0 and k=2.5

4.3 Methods employed for analysis of governing equations:Floquet analysis

$$\left[\left(U+\frac{1}{ik}\frac{\partial}{\partial t}\right)(D^2-k^2)\bar{v}-\bar{v}\frac{\partial^2 U}{\partial y^2}\right]=0$$

$$U(y,t)=\begin{cases} V(t) & 0\leq y\leq b \\ \frac{1-V(t)}{1-b}y+\frac{V(t)-b}{1-b} & b\leq y\leq 1 \\ 1 & y\geq 1 \end{cases}$$

$$\bar{v}(y,t)=e^{\sigma t}\sum_m e^{im\omega t}v_m(y)\begin{cases} c_1e^{ky}+c_2e^{-ky} & 0\leq y\leq b \\ c_3e^{ky}+c_4e^{-ky} & b\leq y\leq 1 \\ c_5e^{ky}+c_6e^{-ky} & y\geq 1 \end{cases}$$

4.3.1 Jump & Boundary Condition

Boundary conditions:

- As $y\to+\infty, \bar{v}\to 0 \implies c_5=0$
- At $y=0, D^2\bar{v}=0 \implies c_1=-c_2$

  (A) Continuity of particle displacement: $\frac{DF}{Dt}=0$

$$v(y,t)|_+=v(y,t)|_- \tag{4.5}$$

(B) Continuity of pressure: $P|_+=P|_-$

$$\left[\frac{\partial^2\bar{v}}{\partial y\partial t}+ikU\frac{\partial\bar{v}}{\partial y}-ik\bar{v}\frac{\partial U}{\partial y}\right]_{-p}^{+p}=0 \tag{4.6}$$

Jump conditions:

- At $y=b, \bar{v}|_{b^+}=\bar{v}|_{b^-} \implies c_2=\dfrac{c_3e^{kb}+c_4e^{-kb}}{e^{-kb}-e^{kb}}$
- At $y=1, \bar{v}|_{1^+}=\bar{v}|_{1^-} \implies c_6=c_3e^{2k}-c_4$
- At $y=1, \bar{P}|_{1^+}=\bar{P}|_{1^-}$

$$\left[\frac{-\beta}{i}(e^k-c_4e^{-k})-k(e^k-c_4e^{-k})+(e^k+c_4e^{-k})*(\frac{1-V(t)}{1-b})\right] - \left[\frac{\beta}{i}(e^k+c_4e^{-k})-(e^k+c_4e^{-k})\right]=0$$

- At $y = b, \bar{P}|_{b^+} = \bar{P}|_{b^-}$

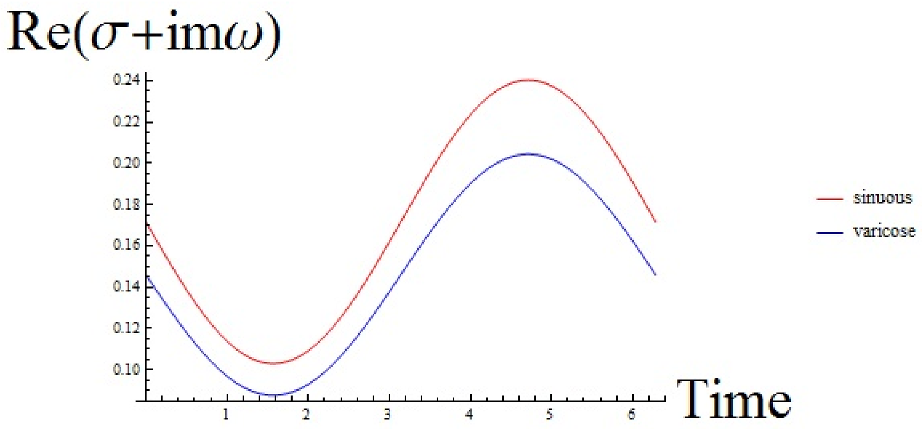
$$\left[\frac{-\beta}{i}\{\frac{e^{kb}+c_4e^{-kb}}{e^{-kb}-e^{kb}}\}(-e^{-kb}-e^{kb}) - v(t)k\{\frac{e^{kb}+c_4e^{-kb}}{e^{-kb}-e^{kb}}\}(-e^{-kb}-e^{kb})\right] + \left[\frac{\beta}{i}(e^{kb}-c_4e^{-kb}) + v(t)k(e^{kb}-c_4e^{-kb}) - (e^{kb}+c_4e^{-kb}) * (\frac{1-V(t)}{1-b})\right] = 0$$

We are trying to solve for c_4 in terms of β

Where

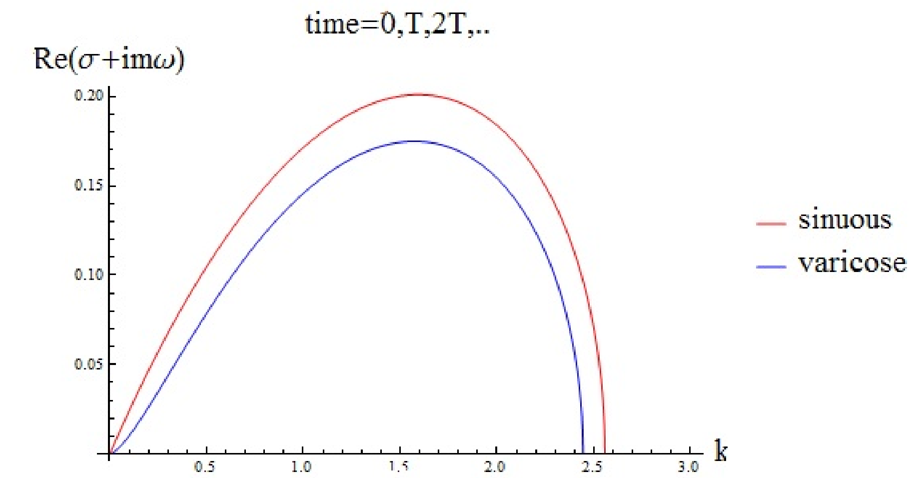
$$\beta = \sigma + im\omega \qquad V(t) = U_m + U_0 \sin \omega t$$
$$b = \frac{1}{2} \qquad Time = t = \frac{2\pi}{\omega}$$
$$U_m = 0.5 \qquad U_0 = 0.25$$

4.3.2 Result

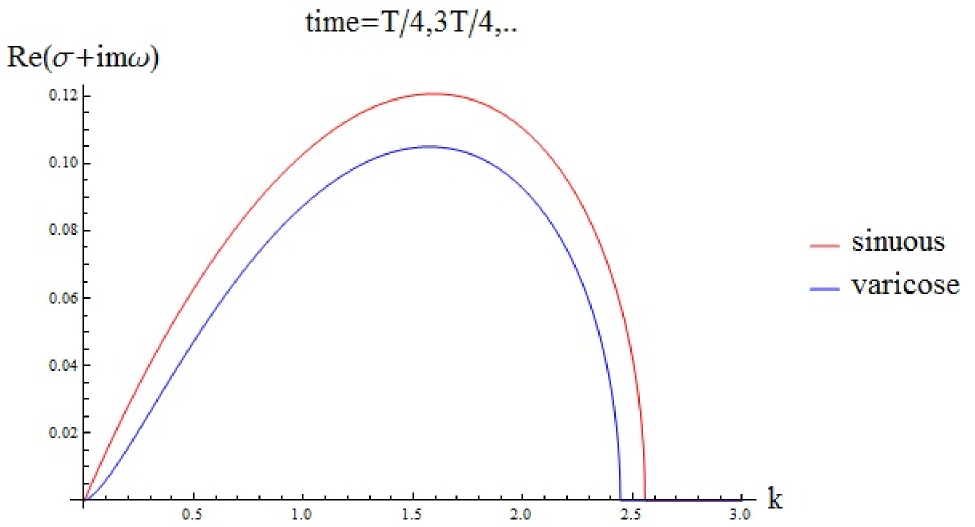


(a)

Figure 4.5: Comparison of growth rate: Sinuous vs Varicose in between $Re(\sigma + im\omega)$ Vs $Time$

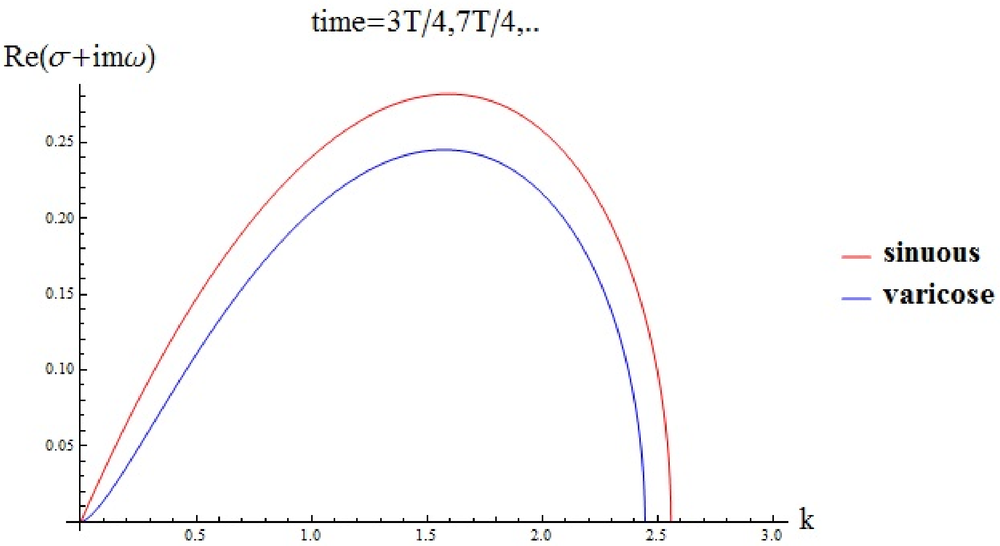


(a)

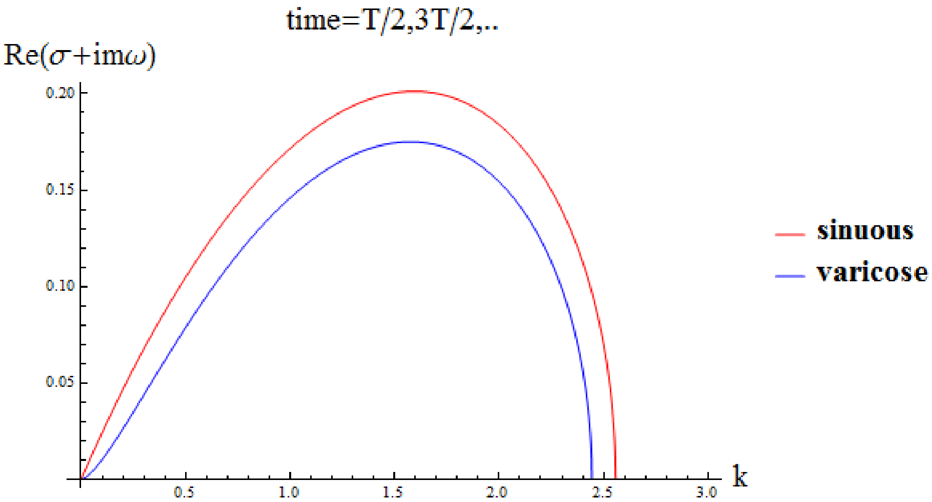


(b)

Figure 4.6: Comparison of growth rate: Sinuous vs Varicose (a)t=0,T,2T(b)t=T/4,3T/4



(a)



(b)

Figure 4.7: Comparison of growth rate: Sinuous vs Varicose (a)t=3T/4,7T/4(b)t=T/2,3T/4

Chapter 5

Conclusion

For a constant wake profile sinuous mode is more unstable than varicose mode.For an oscillating base flow using two methods, (1) Small frequency asymptotic analysis and (2) Floquet analysis the same result as of a cnstant wake profie is obtained i.e. sinuous mode is more unstable than varicose mode

Chapter 6

Appendix

```
Remove["Global`*"]
```

$$m = \begin{pmatrix} 1 & e^{2k} & 0 \\ -k(1-c)-1 & (k(1-c)-1)\,e^{2k} & 0 \\ e^{2k} & 1 & -1 \end{pmatrix};$$

$$b = \begin{pmatrix} 1 \\ -k(1-c) \\ 0 \end{pmatrix};$$

```
x = FullSimplify[LinearSolve[m, b]]
```

$$\left\{\left\{1 + \frac{1}{2(-1+c)k}\right\}, \left\{\frac{e^{-2k}}{2k - 2ck}\right\}, \left\{\frac{e^{-2k}(-1 + e^{4k}(1 + 2(-1+c)k))}{2(-1+c)k}\right\}\right\}$$

```
Remove["Global`*"]
```

$$c3 = 1 + \frac{1}{2(-1+c)k};$$

$$c4 = \frac{e^{-2k}}{2k - 2ck};$$

$$c5 = \frac{e^{-2k}(-1 + e^{4k}(1 + 2(-1+c)k))}{2(-1+c)k};$$

$$Eq = (1+c)\,k\,c5 - c3 * e^{2k}(1 - k(1+c)) - c4(k(1+c)+1);$$

```
Solve[Eq == 0, c]
```

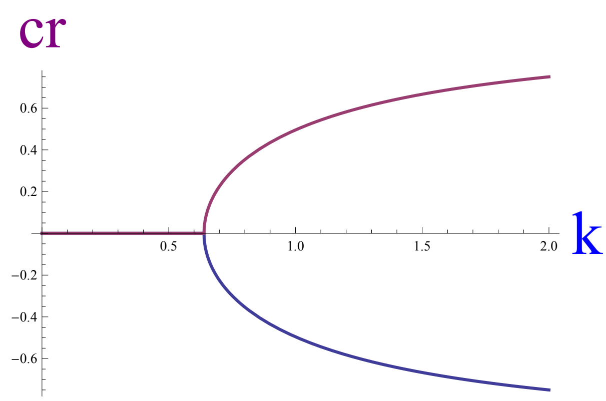
layer new1.pdf

In[57]:=
$$\left\{\left\{c \to -\frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k}\right\}, \left\{c \to \frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k}\right\}\right\}$$

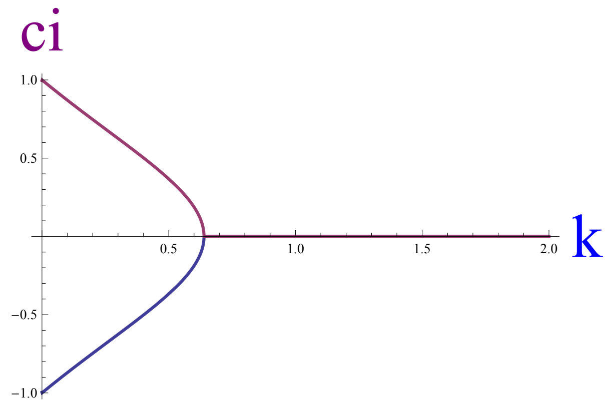
```
Plot[{Re[-Sqrt[1 - E^(-4 k) - 4 k + 4 k^2]/(2 k)], Re[Sqrt[1 - E^(-4 k) - 4 k + 4 k^2]/(2 k)]},
 {k, 0, 2}, PlotStyle → Thick,
 AxesLabel → {Style["k", Blue, FontSize → 40], Style[cr, Purple, FontSize → 40]}]
Plot[{Im[-Sqrt[1 - E^(-4 k) - 4 k + 4 k^2]/(2 k)], Im[Sqrt[1 - E^(-4 k) - 4 k + 4 k^2]/(2 k)]},
 {k, 0, 2}, PlotStyle → Thick,
 AxesLabel → {Style["k", Blue, FontSize → 40], Style[ci, Purple, FontSize → 40]}]
```

Out[57]=
$$\left\{\left\{c \to -\frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k}\right\}, \left\{c \to \frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k}\right\}\right\}$$

Out[58]=



Out[59]=



$$\left\{\left\{c \to -\frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k}\right\}, \left\{c \to \frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k}\right\}\right\}$$

$$\left\{\left\{c \to -\frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k}\right\}, \left\{c \to \frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k}\right\}\right\}$$

In[60]:= **Remove["Global`*"]**

$$c1 = -\frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k};$$

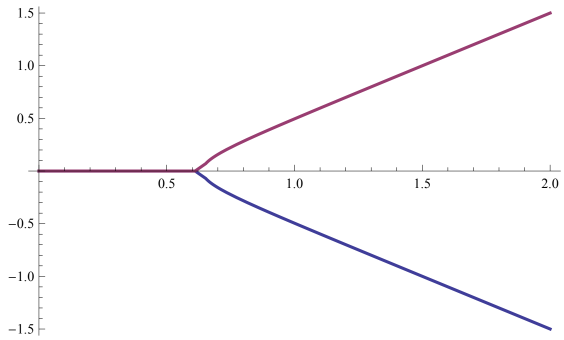
$$c2 = \frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k};$$

$$\omega 1 = k * -\frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k};$$

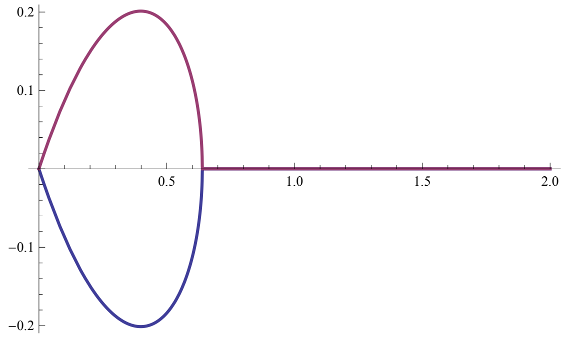
$$\omega 2 = k * \frac{\sqrt{1 - e^{-4k} - 4k + 4k^2}}{2k};$$

```
Plot[{Re[ω1], Re[ω2]}, {k, 0, 2}, PlotStyle → Thick]
Plot[{Im[ω1], Im[ω2]}, {k, 0, 2}, PlotStyle → Thick]
```

Out[65]=



Out[66]=



layer new3.pdf

Figure 6.1: mixing layer

```
Remove["Global`*"]
(* Stability analysis for a Piece-
 wise continuous time dependent wake profile *)
(* Velocity profile: U = 1 for y>1
                     U = (1-V[t])y/b + (V[t]-1+b)/b for 1-b<y<1
                     U = v[t] for 0≤y≤1-b *)
(* Larger the deficit, larger the growth rate *)

(* X' = A X where X = [c3 c4] and A = [a11 a12; a21 a22] *)
```

$$a11 = -\frac{1}{2}\left(2\,\mathrm{i}\,k - \mathrm{i}\left(\frac{1-V[t]}{b}\right)\right)$$

$$a12 = \frac{\mathrm{i}}{2}\,\mathrm{Exp}[-2\,k]\left(\frac{1-V[t]}{b}\right)$$

$$a21 = \left(-\mathrm{i}\,\frac{1}{2}\left(\frac{1-V[t]}{b}\right)\mathrm{Exp}[2\,k\,(1-b)] - \mathrm{i}\,k + \mathrm{i}\,k\,V[t]\right)$$

$$a22 = -\frac{1}{2}\left(-\mathrm{i}\,\mathrm{Exp}[-2\,k]\left(\frac{1-V[t]}{b}\right) + \mathrm{i}\left(\frac{1-V[t]}{b}\right)\left(\frac{\mathrm{Exp}[2\,k\,(1-b)]+1}{\mathrm{Exp}[2\,k\,(1-b)]}\right) + 2\,\mathrm{i}\,k\,V[t]\right)$$

```
Eigenvalues[{{a11, a12}, {a21, a22}}]
```

$$\frac{1}{2}\left(-2\,\mathrm{i}\,k + \frac{\mathrm{i}\,(1-V[t])}{b}\right)$$

$$\frac{\mathrm{i}\,e^{-2\,k}\,(1-V[t])}{2\,b}$$

$$-\mathrm{i}\,k - \frac{\mathrm{i}\,e^{2\,(1-b)\,k}\,(1-V[t])}{2\,b} + \mathrm{i}\,k\,V[t]$$

$$\frac{1}{2}\left(\frac{\mathrm{i}\,e^{-2\,k}\,(1-V[t])}{b} - \frac{\mathrm{i}\,e^{-2\,(1-b)\,k}\left(1+e^{2\,(1-b)\,k}\right)(1-V[t])}{b} - 2\,\mathrm{i}\,k\,V[t]\right)$$

$$
\begin{aligned}
\Big\{ & \frac{1}{4\,b}\, e^{-2\,k-2\,(1-b)\,k} \\
& \Big(-i\, e^{2\,k} + i\, e^{-2\,(-1+b)\,k} - 2\, i\, b\, e^{-2\,(-2+b)\,k}\, k + i\, e^{2\,k}\, V[t] - i\, e^{-2\,(-1+b)\,k}\, V[t] - 2\, i\, b\, e^{-2\,(-2+b)\,k}\, k \\
& \quad V[t] - \sqrt{\Big(\Big(i\, e^{2\,k} - i\, e^{-2\,(-1+b)\,k} + 2\, i\, b\, e^{-2\,(-2+b)\,k}\, k - i\, e^{2\,k}\, V[t] + i\, e^{-2\,(-1+b)\,k}\, V[t] +} \\
& \qquad 2\, i\, b\, e^{-2\,(-2+b)\,k}\, k\, V[t]\Big)^2 - 4\,\Big(e^{-2\,(-3+b)\,k} + e^{-4\,(-2+b)\,k} - e^{-2\,(-3+2\,b)\,k} - \\
& \qquad e^{-2\,(-4+3\,b)\,k} - 2\, b\, e^{-2\,(-3+b)\,k}\, k - 2\, b\, e^{-4\,(-2+b)\,k}\, k - 2\, e^{-2\,(-3+b)\,k}\, V[t] - \\
& \qquad 2\, e^{-4\,(-2+b)\,k}\, V[t] + 2\, e^{-2\,(-3+2\,b)\,k}\, V[t] + 2\, e^{-2\,(-4+3\,b)\,k}\, V[t] + \\
& \qquad 2\, b\, e^{-2\,(-3+b)\,k}\, k\, V[t] + 4\, b\, e^{-4\,(-2+b)\,k}\, k\, V[t] + 2\, b\, e^{-2\,(-3+2\,b)\,k}\, k\, V[t] - \\
& \qquad 4\, b^2\, e^{-4\,(-2+b)\,k}\, k^2\, V[t] + e^{-2\,(-3+b)\,k}\, V[t]^2 + e^{-4\,(-2+b)\,k}\, V[t]^2 - e^{-2\,(-3+2\,b)\,k}\, V[t]^2 - \\
& \qquad e^{-2\,(-4+3\,b)\,k}\, V[t]^2 - 2\, b\, e^{-4\,(-2+b)\,k}\, k\, V[t]^2 - 2\, b\, e^{-2\,(-3+2\,b)\,k}\, k\, V[t]^2\Big)\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{4\,b}\, e^{-2\,k-2\,(1-b)\,k}\, \Big(-i\, e^{2\,k} + i\, e^{-2\,(-1+b)\,k} - 2\, i\, b\, e^{-2\,(-2+b)\,k}\, k + i\, e^{2\,k}\, V[t] - \\
& \quad i\, e^{-2\,(-1+b)\,k}\, V[t] - \\
& \quad 2\, i\, b\, e^{-2\,(-2+b)\,k}\, k\, V[t] + \\
& \quad \sqrt{\Big(\Big(i\, e^{2\,k} - i\, e^{-2\,(-1+b)\,k} + 2\, i\, b\, e^{-2\,(-2+b)\,k}\, k - i\, e^{2\,k}\, V[t] +} \\
& \qquad i\, e^{-2\,(-1+b)\,k}\, V[t] + 2\, i\, b\, e^{-2\,(-2+b)\,k}\, k\, V[t]\Big)^2 - \\
& \qquad 4\,\Big(e^{-2\,(-3+b)\,k} + e^{-4\,(-2+b)\,k} - e^{-2\,(-3+2\,b)\,k} - e^{-2\,(-4+3\,b)\,k} - 2\, b\, e^{-2\,(-3+b)\,k}\, k - 2\, b\, e^{-4\,(-2+b)\,k} \\
& \qquad k - 2\, e^{-2\,(-3+b)\,k}\, V[t] - 2\, e^{-4\,(-2+b)\,k}\, V[t] + 2\, e^{-2\,(-3+2\,b)\,k}\, V[t] + 2\, e^{-2\,(-4+3\,b)\,k} \\
& \qquad V[t] + 2\, b\, e^{-2\,(-3+b)\,k}\, k\, V[t] + 4\, b\, e^{-4\,(-2+b)\,k}\, k\, V[t] + 2\, b\, e^{-2\,(-3+2\,b)\,k}\, k\, V[t] - \\
& \qquad 4\, b^2\, e^{-4\,(-2+b)\,k}\, k^2\, V[t] + e^{-2\,(-3+b)\,k}\, V[t]^2 + e^{-4\,(-2+b)\,k}\, V[t]^2 - e^{-2\,(-3+2\,b)\,k}\, V[t]^2 - \\
& \qquad e^{-2\,(-4+3\,b)\,k}\, V[t]^2 - 2\, b\, e^{-4\,(-2+b)\,k}\, k\, V[t]^2 - 2\, b\, e^{-2\,(-3+2\,b)\,k}\, k\, V[t]^2\Big)\Big)\Big)\Big\}
\end{aligned}
$$

In[28]:=
```
Remove["Global`*"]
V[t] = 0;
b = 1 / 2;
λs1 =
```

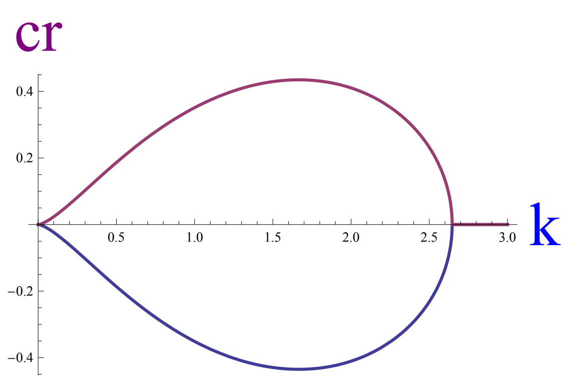
$$\frac{1}{4\,b} e^{-2k-2(1-b)k}\Big(-i e^{2k} + i e^{-2(-1+b)k} - 2 i b e^{-2(-2+b)k} k + i e^{2k} V[t] - i e^{-2(-1+b)k} V[t] - 2 i b e^{-2(-2+b)k} k V[t] - \sqrt{\Big(\big(i e^{2k} - i e^{-2(-1+b)k} + 2 i b e^{-2(-2+b)k} k - i e^{2k} V[t] + i e^{-2(-1+b)k} V[t] + 2 i b e^{-2(-2+b)k} k V[t]\big)^2 - 4\big(e^{-2(-3+b)k} + e^{-4(-2+b)k} - e^{-2(-3+2b)k} - e^{-2(-4+3b)k} - 2 b e^{-2(-3+b)k} k - 2 b e^{-4(-2+b)k} k - 2 e^{-2(-3+b)k} V[t] - 2 e^{-4(-2+b)k} V[t] + 2 e^{-2(-3+2b)k} V[t] + 2 e^{-2(-4+3b)k} V[t] + 2 b e^{-2(-3+b)k} k V[t] + 4 b e^{-4(-2+b)k} k V[t] + 2 b e^{-2(-3+2b)k} k V[t] - 4 b^2 e^{-4(-2+b)k} k^2 V[t] + e^{-2(-3+b)k} V[t]^2 + e^{-4(-2+b)k} V[t]^2 - e^{-2(-3+2b)k} V[t]^2 - e^{-2(-4+3b)k} V[t]^2 - 2 b e^{-4(-2+b)k} k V[t]^2 - 2 b e^{-2(-3+2b)k} k V[t]^2\big)\Big)}\Big);$$

$$\lambda s2 = \frac{1}{4\,b} e^{-2k-2(1-b)k}\Big(-i e^{2k} + i e^{-2(-1+b)k} - 2 i b e^{-2(-2+b)k} k + i e^{2k} V[t] - i e^{-2(-1+b)k} V[t] - 2 i b e^{-2(-2+b)k} k V[t] + \sqrt{\Big(\big(i e^{2k} - i e^{-2(-1+b)k} + 2 i b e^{-2(-2+b)k} k - i e^{2k} V[t] + i e^{-2(-1+b)k} V[t] + 2 i b e^{-2(-2+b)k} k V[t]\big)^2 - 4\big(e^{-2(-3+b)k} + e^{-4(-2+b)k} - e^{-2(-3+2b)k} - e^{-2(-4+3b)k} - 2 b e^{-2(-3+b)k} k - 2 b e^{-4(-2+b)k} k - 2 e^{-2(-3+b)k} V[t] - 2 e^{-4(-2+b)k} V[t] + 2 e^{-2(-3+2b)k} V[t] + 2 e^{-2(-4+3b)k} V[t] + 2 b e^{-2(-3+b)k} k V[t] + 4 b e^{-4(-2+b)k} k V[t] + 2 b e^{-2(-3+2b)k} k V[t] - 4 b^2 e^{-4(-2+b)k} k^2 V[t] + e^{-2(-3+b)k} V[t]^2 + e^{-4(-2+b)k} V[t]^2 - e^{-2(-3+2b)k} V[t]^2 - e^{-2(-4+3b)k} V[t]^2 - 2 b e^{-4(-2+b)k} k V[t]^2 - 2 b e^{-2(-3+2b)k} k V[t]^2\big)\Big)}\Big);$$

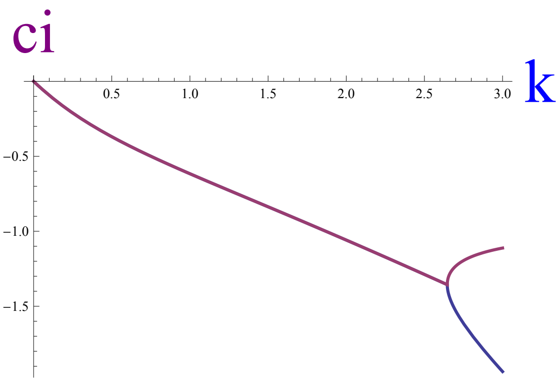
```
FullSimplify[λs1]
FullSimplify[λs2]
Plot[{Re[λs1], Re[λs2]}, {k, 0, 3} , PlotStyle → Thick,
 AxesLabel → {Style["k", Blue, FontSize → 40], Style[cr, Purple, FontSize → 40]} ]
Plot[{Im[λs1], Im[λs2]}, {k, 0, 3}, PlotStyle → Thick,
 AxesLabel → {Style["k", Blue, FontSize → 40], Style[ci, Purple, FontSize → 40]}]
```

Out[33]= $-\frac{1}{2}\, i\, e^{-3k}\left(e^{k}\left(-1+e^{k}+e^{2k}k\right) - i\sqrt{e^{2k}\left(-1+e^{k}\left(2+e^{k}\left(3-e^{2k}(-2+k)^2+2k+2e^{k}k\right)\right)\right)}\right)$

Out[34]= $\frac{1}{2}\, e^{-3k}\left(-i\, e^{k}\left(-1+e^{k}+e^{2k}k\right) + \sqrt{e^{2k}\left(-1+e^{k}\left(2+e^{k}\left(3-e^{2k}(-2+k)^2+2k+2e^{k}k\right)\right)\right)}\right)$

Out[35]=


Out[36]=



.pdf

Figure 6.2: constant sinouse

```
Remove["Global`*"]
(* Stability analysis for a Piece-
 wise continuous time dependent wake profile *)
(* Velocity profile: U = 1 for y>1
                     U = (1-V[t])y/b + (V[t]-1+b)/b for 1-b<y<1
                     U = v[t] for 0≤y≤1-b *)
(* Larger the deficit, larger the growth rate *)

(* X' =
 A X where X = [c3 c4] and A = [a11 a12; a21 a22] *)
```

$$a11 = -\frac{1}{2}\left(2\,i\,k - i\left(\frac{1-V[t]}{b}\right)\right);$$

$$a12 = \frac{i}{2}\,\text{Exp}[-2\,k]\left(\frac{1-V[t]}{b}\right);$$

$$a21 = \frac{1}{2}\left(2\,i\,k - i\left(\frac{1-V[t]}{b}\right) - 2\,i\,k\,V[t] - i\left(\frac{1-V[t]}{b}\right)(\text{Exp}[2\,k\,(1-b)]-1)\right);$$

$$a22 = -\frac{1}{2}\left(i\,\text{Exp}[-2\,k]\left(\frac{1-V[t]}{b}\right) + 2\,i\,k\,V[t] + i\left(\frac{1-V[t]}{b}\right)\frac{\text{Exp}[2\,k\,(1-b)]-1}{\text{Exp}[2\,k\,(1-b)]}\right);$$

```
FullSimplify[Eigenvalues[{{a11, a12}, {a21, a22}}]]
```

$$\Big\{\frac{1}{4\,b}\,e^{-2k}\Big(-i\,e^{2bk}(-1+V[t]) - e^{2(-1+b)k}\sqrt{\Big(-e^{(4-6b)k}\big(-4\,e^{4k}+e^{2bk}-2\,e^{4bk}+e^{6bk}+4\,e^{2k+4bk}(-1+bk)+4\,e^{2(2+b)k}(-1+bk)^2+4\,e^{2(1+b)k}(1+bk)\big)(-1+V[t])^2\Big)} + i\big(-1+V[t]-2\,b\,e^{2k}k\,(1+V[t])\big)\Big),\ \frac{1}{4\,b}\,e^{-2k}\Big(-i\,e^{2bk}(-1+V[t]) + e^{2(-1+b)k}\sqrt{\Big(-e^{(4-6b)k}\big(-4\,e^{4k}+e^{2bk}-2\,e^{4bk}+e^{6bk}+4\,e^{2k+4bk}(-1+bk)+4\,e^{2(2+b)k}(-1+bk)^2+4\,e^{2(1+b)k}(1+bk)\big)(-1+V[t])^2\Big)} + i\big(-1+V[t]-2\,b\,e^{2k}k\,(1+V[t])\big)\Big)\Big\}$$

In[37]:=
```
Remove["Global`*"]
V[t] = 1 / 2;
b = 1 / 2;
λv1 =
  1/(4 b) ⅇ^(-2 k-2 (1-b) k)
   (ⅈ ⅇ^(2 k) - ⅈ ⅇ^(-2 (-1+b) k) - 2 ⅈ b ⅇ^(-2 (-2+b) k) k - ⅈ ⅇ^(2 k) V[t] + ⅈ ⅇ^(-2 (-1+b) k) V[t] - 2 ⅈ b ⅇ^(-2 (-2+b) k) k
     V[t] - √((-ⅈ ⅇ^(2 k) + ⅈ ⅇ^(-2 (-1+b) k) + 2 ⅈ b ⅇ^(-2 (-2+b) k) k + ⅈ ⅇ^(2 k) V[t] - ⅈ ⅇ^(-2 (-1+b) k) V[t] +
          2 ⅈ b ⅇ^(-2 (-2+b) k) k V[t])^2 - 4 (-ⅇ^(-2 (-3+b) k) + ⅇ^(-4 (-2+b) k) + ⅇ^(-2 (-3+2 b) k) -
          ⅇ^(-2 (-4+3 b) k) + 2 b ⅇ^(-2 (-3+b) k) k - 2 b ⅇ^(-4 (-2+b) k) k + 2 ⅇ^(-2 (-3+b) k) V[t] -
          2 ⅇ^(-4 (-2+b) k) V[t] - 2 ⅇ^(-2 (-3+2 b) k) V[t] + 2 ⅇ^(-2 (-4+3 b) k) V[t] -
          2 b ⅇ^(-2 (-3+b) k) k V[t] + 4 b ⅇ^(-4 (-2+b) k) k V[t] - 2 b ⅇ^(-2 (-3+2 b) k) k V[t] -
          4 b^2 ⅇ^(-4 (-2+b) k) k^2 V[t] - ⅇ^(-2 (-3+b) k) V[t]^2 + ⅇ^(-4 (-2+b) k) V[t]^2 + ⅇ^(-2 (-3+2 b) k) V[t]^2 -
          ⅇ^(-2 (-4+3 b) k) V[t]^2 - 2 b ⅇ^(-4 (-2+b) k) k V[t]^2 + 2 b ⅇ^(-2 (-3+2 b) k) k V[t]^2)));
λv2 = 1/(4 b) ⅇ^(-2 k-2 (1-b) k) (ⅈ ⅇ^(2 k) - ⅈ ⅇ^(-2 (-1+b) k) - 2 ⅈ b ⅇ^(-2 (-2+b) k) k - ⅈ ⅇ^(2 k) V[t] +
     ⅈ ⅇ^(-2 (-1+b) k) V[t] - 2 ⅈ b ⅇ^(-2 (-2+b) k) k V[t] +
     √((-ⅈ ⅇ^(2 k) + ⅈ ⅇ^(-2 (-1+b) k) + 2 ⅈ b ⅇ^(-2 (-2+b) k) k + ⅈ ⅇ^(2 k) V[t] - ⅈ ⅇ^(-2 (-1+b) k) V[t] +
          2 ⅈ b ⅇ^(-2 (-2+b) k) k V[t])^2 - 4 (-ⅇ^(-2 (-3+b) k) + ⅇ^(-4 (-2+b) k) + ⅇ^(-2 (-3+2 b) k) -
          ⅇ^(-2 (-4+3 b) k) + 2 b ⅇ^(-2 (-3+b) k) k - 2 b ⅇ^(-4 (-2+b) k) k + 2 ⅇ^(-2 (-3+b) k) V[t] -
          2 ⅇ^(-4 (-2+b) k) V[t] - 2 ⅇ^(-2 (-3+2 b) k) V[t] + 2 ⅇ^(-2 (-4+3 b) k) V[t] -
          2 b ⅇ^(-2 (-3+b) k) k V[t] + 4 b ⅇ^(-4 (-2+b) k) k V[t] - 2 b ⅇ^(-2 (-3+2 b) k) k V[t] -
          4 b^2 ⅇ^(-4 (-2+b) k) k^2 V[t] - ⅇ^(-2 (-3+b) k) V[t]^2 + ⅇ^(-4 (-2+b) k) V[t]^2 + ⅇ^(-2 (-3+2 b) k) V[t]^2 -
          ⅇ^(-2 (-4+3 b) k) V[t]^2 - 2 b ⅇ^(-4 (-2+b) k) k V[t]^2 + 2 b ⅇ^(-2 (-3+2 b) k) k V[t]^2)));
FullSimplify[λv1]
FullSimplify[λv2]
Plot[{Re[λv1], Re[λv2]}, {k, 0, 3}, PlotStyle → Thick,
 AxesLabel → {Style["k", Blue, FontSize → 40], Style[cr, Purple, FontSize → 40]}]
Plot[{Im[λv1], Im[λv2]}, {k, 0, 3}, PlotStyle → Thick,
 AxesLabel → {Style["k", Blue, FontSize → 40], Style[ci, Purple, FontSize → 40]}]
```

Out[42]=

$$-\frac{1}{4}\, i\, e^{-3 k} \left(e^{k} - e^{2 k} + 3 e^{3 k} k - i \sqrt{-e^{2 k} \left(1 + e^{k} \left(-2 + e^{k} \left(5 + 2 e^{k} (-4 + k) + e^{2 k} (-2 + k)^2 + 2 k\right)\right)\right)}\right)$$

Out[43]=

$$\frac{1}{4}\, e^{-3 k} \left(\sqrt{\left(e^{2 k} \left(-1 + e^{k} \left(2 + e^{k} \left(-5 + e^{k} \left(8 - e^{k} (-2 + k)^2 - 2 k\right) - 2 k\right)\right)\right)\right)} - i\, e^{k} \left(1 + e^{k} \left(-1 + 3 e^{k} k\right)\right)\right)$$

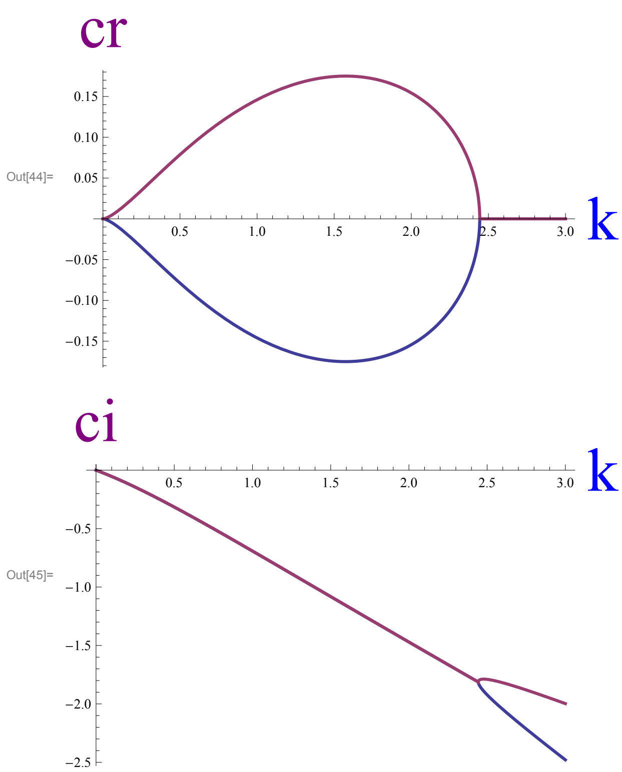


Figure 6.3: constant varicos

```
In[50]:= Remove["Global`*"]
λs =
 1/2 e^(-3 k) (-i e^k (-1 + e^k + e^(2 k) k) + Sqrt[e^(2 k) (-1 + e^k (2 + e^k (3 - e^(2 k) (-2 + k)^2 + 2 k + 2 e^k k)))])
(* comes from lambda_2 of sinuous *)
λv = 1/2 e^(-3 k)
  (Sqrt[e^(2 k) (-1 + e^k (2 + e^k (-5 + e^k (8 - e^k (-2 + k)^2 - 2 k) - 2 k)))] - i e^k (1 + e^k (-1 + e^k k)))
(* comes from lambda_2 of varicose *)
Plot[{Re[λs], Re[λv]}, {k, 0, 3}, PlotStyle → Thick,
 AxesLabel → {Style["k", Blue, FontSize → 40], Style[λ, Purple, FontSize → 40]}]
```

Out[51]= $\frac{1}{2} e^{-3 k} \left(-i\, e^{k} \left(-1 + e^{k} + e^{2 k} k\right) + \sqrt{e^{2 k} \left(-1 + e^{k} \left(2 + e^{k} \left(3 - e^{2 k} (-2 + k)^{2} + 2 k + 2 e^{k} k\right)\right)\right)}\right)$

Out[52]= $\frac{1}{2} e^{-3 k} \left(\sqrt{e^{2 k} \left(-1 + e^{k} \left(2 + e^{k} \left(-5 + e^{k} \left(8 - e^{k} (-2 + k)^{2} - 2 k\right) - 2 k\right)\right)\right)} - i\, e^{k} \left(1 + e^{k} \left(-1 + e^{k} k\right)\right)\right)$

Out[53]=

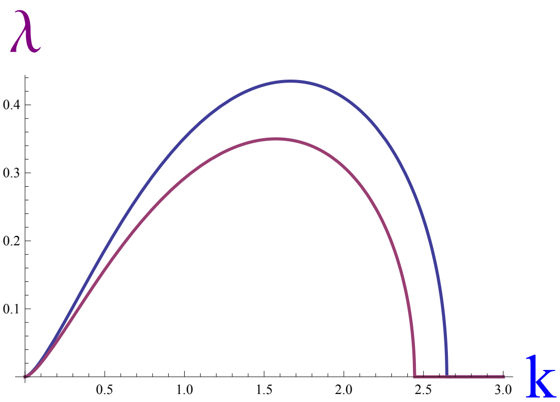


Figure 6.4: constant sinouse varicos

Bibliography

[BRIDGES & MORRIS(1984)] BRIDGES, T. J. & MORRIS, P. J. 1984 Differential eigenvalue problems in which the parameter appears nonlinearly. *J. OF COMPUTATIONAL PHYSICS* **55**, 437–460.

[Davis(1976)] DAVIS, STEPHEN H. 1976 The stability of time-periodic flows. *Ann. Rev. Fluid Mech.* **8**, 57–74.

[Drazin(1981)] DRAZIN, P. G. 1981 *Introduction to hydrodynamic stability*. Cambridge University Press.

[Hultgren & Aggarwal(1987)] HULTGREN, LENNART S. & AGGARWAL, ARUN K. 1987 Absolute instability of the gaussian wake profile. *Phys. Fluids* **30**, 3383–3387.

[KELLY(1965)] KELLY, R. E. 1965 The stability of an unsteady kelvin-helmholtz flow. *J . Fluid Mech.* **22**, 547–560.

[von Kerczek & Tozzi(1986)] VON KERCZEK, C. H. & TOZZI, J. T. 1986 The stability of oscillatory hagen-poiseuille flow. *J. Appl. Mech* **53**, 187–192.

[S. J. X U & G(2006)] S. J. X U, Y. Z H O U & G, M. H. W A N 2006 A symmetric binary-vortex street behind a longitudinally oscillating cylinder. *J. Fluid Mech.* **556**, 27–43.

[Schmid & Henningson(2000)] SCHMID, PETER J. & HENNINGSON, DAN S. 2000 *Stability and Transition in Shear flows*. springer.

[Srikanth T. & Govindarajan(2011)] SRIKANTH T., HARISH N. DIXIT, RAO TATAVARTI & GOVINDARAJAN, RAMA 2011 Vortex shedding patterns, their competition, and chaos in flow past inline oscillating rectangular cylinders. *Phys. Flows* **23**, 073603.

[Suresh & Homsy(2004)] SURESH, V. & HOMSY, G. M. 2004 Stability of time-modulated electroosmotic flow. *Phys. Fluids* **16**, 7.

[T. B. Benjamin(1954)] T. B. BENJAMIN, F. URSELL 1954 The stability of tlane free surface of a liquid in vertical periodic motion. *Royal Society* **225**, 505–515.

[W. O. Criminale(2003)] W. O. CRIMINALE, T. L. JACKSON, R. D. JOSLIN 2003 *Theory and Computation in Hydrodynamic Stability*. Cambridge University Press.

[William E. Boyce(2009)] WILLIAM E. BOYCE, RICHARD C. DIPRIMA 2009 *Elementary Differential Equations and Boundary value Problems*. Wileyplus.

[YOSHIKAWA & WESFREID(2011)] YOSHIKAWA, HARUNORI N. & WESFREID, JOSE EDUARDO 2011 Oscillatory kelvinhelmholtz instability. part 1.a viscous theory. *J . Fluid Mech.* **675**, 223–248.