Homotopy Continuation for Characteristic Roots of Delay Differential Equation

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Dedication

This thesis is dedicated to my parents & siblings
for their love, endless support
and encouragement

Abstract

In this thesis we develop a homotopy continuation method to find the characteristic roots of scalar delay differential equations with multiple delays. We introduce a homotopy parameter $\mu \in [0, 1]$ in such a way that for $\mu = 0$, the characteristic equation contains only one delay term and for $\mu = 1$ the original characteristic equation is recovered. By selecting $\mu = 0$ allows us to express all the roots of the characteristic equation in closed form in terms of Lambert W function. A numerical continuation based scheme is then developed to trace the roots as μ is varied from 0 to 1. The roots of the characteristic equation for $\mu = 1$ correspond to the characteristic roots of the delay differential equation. We show several numerical examples to demonstrate the developed method.

Contents

Chapter 1

Introduction

1.1 Delay differential equations

The study of delay differential equations (DDEs), that is equations of the form

$$x'(t) = f(t, x(t), x(t-\tau_1), x(t-\tau_2), ..., x(t-\tau_N))) \tag{1.1}$$

was initially motivated mainly by problems in feedback control theory [1]. The delays, $\tau_i,\ i = 1, 2, .., N$ are measurable physical quantities and may be a constant, a function of t (the variable or time dependent case) or a function of t and x itself (the state dependent case). Examples of delays include the driver reaction time, time taken for a signal to travel to the controlled object, the time for the body to produce red blood cells and cell division, time in the dynamics of viral exhaustion or persistence. In the life sciences delays are often brought into account for hidden variables and processes which, although not well known, are known to cause a time lag (see [2–7] and the references therein).

Time delays are natural components of the dynamic processes of biology, economics, ecology, epidemiology, physiology and mechanics [8] and 'to ignore them is to ignore reality'.

Ordinary differential equations (ODEs) have been used as a fundamental tool of the mathematical modeler for long time. However, an ODE model formulation of a system ignores the presence of any delays. Formulation as a functional differential equation(FDE), which includes all DDEs, includes both the current and all previous values of a function and/or its derivatives when analyzing the future behavior of a system. This often leads to a better model of a process since 'an increase in the complexity of the mathematical models can lead to a better quantitative consistency with real data', but at a cost [2]. The size of the delay with respect to the underlying time-scales influences the modelers decision about the choice of model formulation [9]. Systems for which a model based on a delay differential equation is more appropriate than one based on an ODE can be referred to as "problems with memory". A delay differential equation model may also be used to approximate a high-dimensional model without delay by a lower dimensional model with delay, the analysis of which is more readily carried out. This approach has been used extensively in process control industry (see [10], p. 40-41).

There are many similarities between the theory of ODEs and that of DDEs and analytical methods for ODEs have been extended to DDEs when possible. However, their differences have necessitated new approaches. In Table 1.1 we highlight important differences between ODEs and DDEs, such as the need for an initial function and the infinite dimensionality of a DDE.

ODE Model	DDE Model
Assumes: effect of any changes to the system is instantaneous (A principle of causality, [1,11]).	**Assumes**: effect of any changes to the system is **not instantaneous** i.e., past history is taken into account.
Generates a system that is **finite** dimensional.	**Generates** a system that is **infinite** dimensional.
Needs an **initial value** (to determine a unique solution).	**Needs** an **initial function** (to determine a unique solution).
	Advantage: Enables a more accurate reflection of the system being modelled.
	Disadvantage: The analytical theory is less well developed.

Table 1.1: Important Differences between ODEs and DDEs

Changes in the qualitative behavior of the solution may be observed as a effect of a delay term. In biological models the presence of delays is a potent source of non-stationary phenomena such as periodic oscillations and instabilities [2–7]. The delay can act as a stabiliser or a destabiliser of ODE models [5,7,12].

1.2 Some applications

This generalization of the ODE is important, as it allows the mathematical treatment of models with delays. Indeed, many physical events do not occur instantaneously and can be modeled with delays. Here, some models with delays from engineering, physics and chemistry are mentioned.

The electronic signal of the control of a robot takes some time to go from the controller to the robot arm. Similarly, if the controllers of the wing-rudders of an airplane are located in the cockpit, the controllers can only control the rudders with a certain delay. When a human driver on a high-way observes that the next car is breaking, he will hit the breaks after a certain reaction time. In the modeling of a high-way congestion, this reaction time influences the length of the congestion. Chemical reactions do normally not occur instantaneously. Suppose a manufacturer wants to produce a material with a customer specified material property. An (unpolluted) construction of a material with the given properties is sometimes only possible with an accurate control of the chemical process.

Delays are also relevant in more critical applications. The accurate modeling and control of nuclear reactors is crucial. The temperature of the inner part of nuclear reactor may not be available for measurement. If the temperature in the inner part rises, after some time (delay), the temperature of the surface of the reactor will also rise. Hence, only old information about the state is available for measurement and can be used to control the process.

Since delay-differential equations appear in a large number of fields in science. It is not surprising that it has received different names in different fields. For instance, the following terms are used for slight variations of DDEs, time-delay systems, difference-differential equations, retarded systems, functional differential equations.

1.3 Literature review

The last 50 years have seen an increasing interest in the analysis of delayed systems. This interest is strongly motivated by the ubiquity of physical and biological models that include delays. The first delay models in engineering were introduced by von Schlippe and Dietrich [13] for modelling wheel shimmy, and by Minorsky [14] for ship stabilization. Delays are inherent to many physical systems such as machining processes and lasers [15, 16]. Delays can also enter the system's equation as a modeling decision to characterize complex processes that are known to take a certain amount of time, e.g. epilepsy seizure models [17]. In addition, delayed signals have been used in efficient control schemes even when the mathematical model of the system is too complicated or unavailable [18, 19].

Delays have important applications in science and engineering, but their infinite dimensional state space significantly complicates the analysis. Further, even low-order delay equations can exhibit very complicated dynamics [6]. Due to the difficulties associated with the analytical treatment of delay differential equations (DDEs), a significant number of works focus on solving DDEs numerically. The majority of these works have focused on either extending Runge-Kutta methods to DDEs or on using an implicit Radau method [5, 20, 21]

Some examples of systems with multiple delays include traffic stability models [22–25], laser systems [15, 26] and variable pitch mills [27, 28]. The existence of multiple delays in the system often leads to complicated stability structures in the delays parameter space. Specifically, increasing one of the delays may stabilize the system whereas increasing one of the other delays can destabilize the same system.

The difficulties associated with studying multiple time delay systems (MTDS) is evidenced by the limited literature on the topic. For example, time integration methods were developed based on either discretizing the solution operator [29–31] or discretizing the infinitesimal generator of the solution operator semi-group [32, 33]. Analytical methods have also been used to study the stability of MTDS. For instance, Hale and Huang studied the stability of autonomous first order MTDS [34] using D-subdivision. Stepan [35] and Niculescu [36] investigated autonomous second order MTDS using D-subdivision and a frequency domain approach, respectively, but without damping terms. Sipahi and Olgac used the Cluster Treatment of Characteristic Roots method to study the stability of autonomous second order MTDS including damping [37]. The stability of autonomous MTDS was also studied in Ref. [38] using the Continuous Time Approximation method (CTA). However, the CTA method often results in very large matrices which can lead to computational difficulties.

Insperger and Stepan investigated the stability of second order MTDS using the semi-discretization approach [39]. However, the semi-discretization approach often results in large matrices and long computational times. A collocation method for the stability of MTDS based on piecewise polynomials was analyzed in Refs. [40, 41]. However, as was shown in Refs. [42, 43], the spectral element method can have higher rates of convergence than collocation methods.

For any such mathematical model, it is important to know parameter ranges for which the system exhibits stability. DDEs have infinite dimensional state spaces and this complicates their analysis. Stability analysis of equilibrium of DDEs boils down to the problem of finding the roots of a quasi-polynomial (the characteristic equation) with infinitely many roots. If all the roots of this quasi-polynomial are in the left half complex plane then that equilibrium point is stable. Methods to determine stability without calculating the roots include Lyapunov functionals [44–46] and approaches based on linear matrix inequalities [47, 48]. These methods in general do not provide information about the location of the characteristic roots and their distance from the stability boundary.

Given an initial guess for the characteristic roots, the characteristic equation of the DDE which is a quasi-polynomial can be solved using any nonlinear root finding algorithms like Newton-Raphson or secant method. However finding the initial guess for the roots is very difficult. Therefore different methods are developed in the literature to find the roots approximately. These methods include semi-discretization [49], D-subdivision methods [50], finite difference methods [51, 52], time finite elements [53], and Galerkin approximations [54, 55]. The roots obtained from these approximate methods can be given as initial conditions for the nonlinear root finding algorithms to refine the approximate roots.

In this work we develop a homotopy algorithm [56] to determine the characteristic roots and stability of the DDEs. Homotopy methods are widely used in nonlinear root finding algorithms [57–59] particularly when appropriate initial guesses are unavailable [60]; in inverse kinematic problems [61–63], parameter identification problems [64] and in mechanism synthesis applications [65, 66]. We introduce a homotopy parameter $\mu \in [0, 1]$ into the characteristic equation in such a way that for $\mu = 0$, the characteristic equation has only one delay term and for $\mu = 1$ the original characteristic equation is recovered. By selecting $\mu = 0$ allows us to express all the roots of the characteristic equation in closed form in terms of Lambert W function. A numerical continuation based scheme is developed to trace the roots as μ is varied from 0 to 1. The roots of the characteristic equation for $\mu = 1$ correspond to the characteristic roots of the delay differential equation.

1.4 Background

1.4.1 Lambert W function

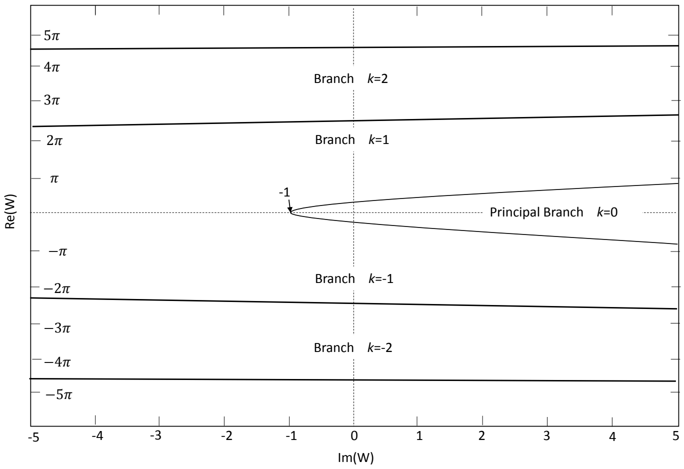
Introduced in the 1700's by Lambert and Euler [67], the Lambert W function is defined to be any function, $W(H)$, that satisfies

$$W(H)e^{W(H)} = H \tag{1.2}$$

The Lambert W function is complex valued, with a complex argument, H, and has an infinite number of branches, W_k, where $k = -\infty, ..., -1, 0, 1, ..., \infty$ [68]. Fig. 1.4.1 shows the range of each branch of the Lambert W function. For example, the real part of the principal branch, W_0, has a minimum value, -1. The principal and all other branches of the Lambert W function in Eq. (1.2) can be calculated analytically using a series expansion [67], or alternatively, using commands already embedded in the various commercial software packages, such as Matlab, Maple and Mathematica.

Many equations involving exponentials can be solved using the W function. The general strategy is to move all instances of the unknown to one side of the equation and make it look like $Y = Xe^X$ at which point the W function provides the value of the variable in X.
In other words

$$Y = Xe^X \iff X = W(Y)$$



(a)

Figure 1.1: (Ranges of each branch of the Lambert W function. Note that real part of the principal branch, W_0 , is equal to or larger than 1.

Example

For example consider an equation of the form

$$2^t = 5t \tag{1.3}$$

which can be written in Lambert W form by following steps

$$2^t = 5t \tag{1.4}$$

$$1 = \frac{5t}{2^t} \tag{1.5}$$

$$1 = 5te^{-}tln2 \tag{1.6}$$

$$\frac{1}{5} = te^{(-tln2)} \tag{1.7}$$

$$\frac{-ln2}{5} = (-tln2)e^{(-tln2)} \tag{1.8}$$

$$W(\frac{-ln2}{5}) = -tln2 \tag{1.9}$$

$$t = -\frac{W(\frac{-ln2}{5})}{ln2} \tag{1.10}$$

$$\tag{1.11}$$

More generally, the equation

$$p^{ax+b} = cx + d \tag{1.12}$$

where $p > 0$ and $c, a \neq 0$, can be transformed via the substitution

$$-t = ax + \frac{ad}{c} \tag{1.13}$$

into

$$tp^t = R = -\frac{a}{c}p^{b-\frac{ad}{c}} \tag{1.14}$$

giving

$$t = \frac{W(R\ln p)}{\ln p} \tag{1.15}$$

which yields the final solution

$$x = -\frac{W(-\frac{a\ln p}{c}p^{b-\frac{ad}{c}})}{a\ln p} - \frac{d}{c} \tag{1.16}$$

the above W equation can be solved by using inbuilt functions in software packages like Matlab, Maple or Mathematica.

1.5 Problem definition

Stability analysis of a DDE can be studied by using Homotopy method. In this work we consider a DDE system described by:

$$\dot{x} + ax + \sum_{i=1}^{N} b_i x(t-\tau_i) = 0, \tag{1.17}$$

where a and b_i, $i = 1, ..., N$ are non-zero real constants and $\tau_1, \tau_2, ..., \tau_N > 0$. On substituting $x = e^{st}$ in Eq. (1.17), we get the following characteristic equation

$$s + a + \sum_{i=1}^{N} b_i e^{-\tau_i s} = 0. \tag{1.18}$$

However, Eq. (1.18) is a quasi-polynomial having infinitely many roots and can be easily solved using numerical continuation methods, but the challenge lies in finding the initial guess for a given root. So, in order to solve this problem we use Homotopy method to find the characteristic roots of the DDE Eq. (1.17). The stability of the system can be easily found by analyzing the characteristic roots of the DDE. If all the roots of the characteristic equation are in the left half of the complex plane then the $x = 0$ solution of the DDE is stable, otherwise it is unstable.

1.6 Thesis structure

The entire thesis has been formatted in seven chapters. The first chapter covers the introduction; here we mentioned about the introduction to delay differential equations, some applications of DDEs, background to the Lambert W function and then we define the problem that we are going to focus on. In chapter 2 we discussed about some of the existing methods to solve the DDEs, like semi-discretization method, spectra-tau method etc., Chapter 3 discusses the solution procedure for single delay system using Lambert W function and some results for the case is shown. In chapter 4 the homotopy methodology for finding the roots of quasi-polynomials and also the numerical continuation method for tracing the roots for the quasi-polynomials are explained. In chapter 5 we showed the efficacy of our method in determining the characteristic roots and stability of DDE's using several examples. Chapter 6 discusses the cases when the method fails and finally we conclude the entire thesis in chapter 7.

Chapter 2

Existing methods to solve DDE

In last few decades many methods were developed by the researchers to solve DDEs. Among them Semi-discretization, D-subdivision, Finite difference methods, Galerkin approximations and spectral-tau are some of the most powerful methods. In this chapter some of these methods were explained briefly.

2.1 Semi-discretization method

Semi-Discretization is one of the most powerful method to solve the DDEs and main idea of this method is presented here for an undamped delayed oscillator

$$\ddot{x}(t) + a_0 x(t) = b_0 x(t-\tau) \tag{2.1}$$

The discrete time scale $t_i = ih$, $i \in Z$, is introduced with h being the discretization step, and consider the DDE

$$\ddot{y}(t) + a_0 y(t) = b_0 y(t_{i-r}), t \in [t_i, t_{i+1}), \tag{2.2}$$

with r being a positive integer. Here, the term $y(t_{i-r})$ refers to the state at $t_{i-r} = (i-r)h$, i.e., the delayed term is constant over the interval $[t_i, t_{i+1})$. Equation. (2.2) can also be written in the form

$$\ddot{y}(t) + a_0 y(t) = b_0 y(t - \rho(t)), \tag{2.3}$$

where

$$\rho(t) = rh - t_i + t, t \in [t_i, t_{i+1}), \tag{2.4}$$

is a sawtooth-like time-periodic time delay, i.e., if $r \to \infty$ and $h \to 0$ such that $(r + 1/2)h = \tau$, remains constant, then the time-periodic time delay $\rho(t)$ tends to the constant delay τ. In this sense, Eq. (2.2) gives an approximation of the original DDE Eq. (2.1) obtained such that the delayed term on the right-hand side of Eq. (2.1) is discretized, while all the other (non-delayed) terms on the left-hand side are left in their original form. This is the basic point of the semi-discretization method for delayed systems.

For given initial conditions $y_i := y(t_i), \dot{y}_i := \dot{y}(t_i)$ and for a given delayed state variable $y_{i-1} = y(t_{i-1})$, Eq. (2.3) can be solved over the discretization interval $[t_i, t_{i+1})$ as an ODE, and the state variable y and its derivative at $t = t_{i+1}$ can be given as

$$y_{i+1} = P_{11}y_i + P_{12}\dot{y}_i + R_1 y_{i-r}, \tag{2.5}$$

$$\dot{y}_{i+1} = P_{21}y_i + P_{22}\dot{y}_i + R_2 y_{i-r}, \tag{2.6}$$

where,

$$P_{11} = \frac{\theta_2}{\theta_2 - \theta_1}e^{\theta_1 h} - \frac{\theta_1}{\theta_2 - \theta_1}e^{\theta_2 h} \tag{2.7}$$

$$P_{12} = \frac{1}{\theta_2 - \theta_1}e^{\theta_2 h} - \frac{1}{\theta_2 - \theta_1}e^{\theta_1 h} \tag{2.8}$$

$$P_{21} = \frac{\theta_1\theta_2}{\theta_2 - \theta_1}e^{\theta_1 h} - \frac{\theta_1\theta_2}{\theta_2 - \theta_1}e^{\theta_2 h} \tag{2.9}$$

$$P_{22} = \frac{\theta_2}{\theta_2 - \theta_1}e^{\theta_2 h} - \frac{\theta_1}{\theta_2 - \theta_1}e^{\theta_1 h} \tag{2.10}$$

$$R_1 = (1 + \frac{\theta_1}{\theta_2 - \theta_1}e^{\theta_2 h} - \frac{\theta_2}{\theta_2 - \theta_1}e^{\theta_1 h})\frac{b_0}{a_0} \tag{2.11}$$

$$R_2 = (\frac{\theta_1\theta_2}{\theta_2 - \theta_1}e^{\theta_2 h} - \frac{\theta_1\theta_2}{\theta_2 - \theta_1}e^{\theta_1 h})\frac{b_0}{a_0} \tag{2.12}$$

and θ_1 and θ_2 are the roots of characteristic function

$$D(\theta) = \theta^2 + a_0 \tag{2.13}$$

of the homogeneous part of Eq. (2.2), i.e., $\theta_{1,2} = \pm\sqrt{-a_0} = i \pm \sqrt{a_0}$. Eq. (2.5) and Eq. (2.6) imply the discrete map

$$\begin{pmatrix} y_{i-1} \\ \dot{y}_{i+1} \\ y_{i+1} \\ y_i \\ . \\ . \\ . \\ y_{i-r+1} \end{pmatrix} = \underbrace{\begin{pmatrix} P_{11} & P_{12} & 0 & 0 & . & . & . & 0 & R_1 \\ P_{21} & P_{22} & 0 & 0 & . & . & . & 0 & R_2 \\ 1 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 & 0 \\ . & & & & . & & & & . \\ . & & & & & . & & & . \\ . & & & & & & . & & . \\ 0 & 0 & 0 & 0 & . & . & . & 1 & 0 \end{pmatrix}}_{:=\boldsymbol{G}} \begin{pmatrix} y_i \\ \dot{y}_i \\ y_{i-1} \\ y_{i-2} \\ . \\ . \\ . \\ y_{i-r} \end{pmatrix} \tag{2.14}$$

The trivial solution of this map, and consequently, the trivial solution of Eq. (2.2), are asymptotically stable if all the eigenvalues of the coefficient matrix $\boldsymbol{G}$ are of modulus less than one. According to the Floquet theory of time-periodic DDEs, matrix $\boldsymbol{G}$ is the $(r+2)$-dimensional matrix representation of the monodromy operator of Eq. (2.2), and its nonzero eigenvalues are the characteristic multipliers (μ_j, $j = 1, 2, ..., r+2$).

2.2 Spectral-tau method

Spectral-tau is an another powerful method through which the characteristic roots of the DDEs can be determined to find the stability of the system. In the following chapters the results obtained from the homotopy method are compared with the spectral-tau method to show the efficiency of homotopy method.

The algorithm of spectral-tau method is explained here by considering the DDE with m delays of the form

$$\dot{x}(t) + ax(t) + \sum_{q=1}^{m} b_q x(t - \tau_q) = 0, \tau_q > 0. \tag{2.15}$$

Here initial function is specified as

$$x(t) = \theta(t), -\tau \geq t \geq 0, \tag{2.16}$$

where, $\tau = max(\tau_1, \tau_2, .., \tau_m)$. By introducing the so-called shift of time $y(s,t) = x(t+s)$, $s \in [-\tau, 0)$, the initial value problem Eq. (2.15-2.16) is re-casted into the following initial-boundary value problem for the advection equation [69, 70]

$$\frac{\partial y(s,t)}{\partial t} = \frac{\partial y(s,t)}{\partial s} \in [-\tau, 0) \tag{2.17}$$

$$\frac{\partial y(s,t)}{\partial t}\Big|_{s=0} = -ay(0,t) - \sum_{q=1}^{m} b_q y(\tau_q, t), \tag{2.18}$$

$$y(s, 0) = \theta(s), s \in [-\tau, 0] \tag{2.19}$$

In this method the solution to the PDE Eq. (2.17) is assumed as of the following form:

$$y(s,t) = \sum_{i=1}^{\infty} \phi_i(s)\eta_i(t), \tag{2.20}$$

where, $\phi_i(s)$ are the basis functions and $\eta_i(t)$ are the time dependent coordinates. For practical reasons the sum is terminated at N terms, i.e.

$$y(s,t) = \phi(s)^T \eta(t) \tag{2.21}$$

where, $\phi(s) = [\phi_1(s), \phi_2(s), ..., \phi_N(s)]$ and $\eta(t) = [\eta_1(t), \eta_2(t), ..., \eta_N(t)]^T$. Substituting the series solution Eq. (2.21) in Eq. (2.17) we get (the symbol $'$ denotes derivative with respect to s)

$$\phi(s)^T \dot{\eta}(t) = \phi'(s)^T \eta(t), \tag{2.22}$$

On pre-multiplying Eq. (2.22) with $\phi(s)$ and integrating over the domain we get:

$$\int_{-\tau}^{0} \phi(s)\phi(s)^T ds\, \dot{\eta}(s) = \int_{-\tau}^{0} \phi(s)\phi'(s)^T ds\, \eta(s.) \tag{2.23}$$

In matrix form

$$A\dot{\eta}(t) = B\eta(t), \tag{2.24}$$

with

$$A = \int_{-\tau}^{0} \phi(s)\phi(s)^T ds, \tag{2.25}$$

$$B = \int_{-\tau}^{0} \phi(s)\phi'(s)^T ds, \tag{2.26}$$

Substituting Eq. (2.21) in Eq. (2.18) we get the scalar equation

$$\phi(0)^T \dot{\eta}(t) = \big[-a\phi(0)^T - \sum_{q=1}^{m} b_q \phi(-\tau_q)^T \big] \eta(t) \tag{2.27}$$

Note that Eq. (2.24), Eq. (2.27) provide $N+1$ independent equations. To arrive at a determinate system the Eq. (2.24) is truncated and augment it with Eq. (2.27) to form

$$M_{Tau} \dot{\eta}(t) = K_{Tau} \eta(t), \tag{2.28}$$

where,

$$M_{Tau} = \begin{bmatrix} \bar{A} \\ \phi(0)^T \end{bmatrix} \tag{2.29}$$

$$K_{Tau} = \begin{bmatrix} \bar{B} \\ -a\phi(0)^T - \sum_{q=1}^{m} b_q \phi(-\tau_q)^T \end{bmatrix} \tag{2.30}$$

and matrices $\bar{A}, \bar{B}$ are obtained by deleting the last row of matrix A and B, respectively. The initial conditions for Eq. (2.28) in $\eta(0) = M^{-1} \int_{-\tau}^{0} \phi(s)\theta(s) ds$ and the solution of the DDE can be obtained as $x(t) = y(0,t) - \phi(0)^T \eta(t)$. The finite dimensional system Eq. (2.28) represents an approximation for Eq. (2.15).

Chapter 3

Single delay system

First order DDE with a single delay can be solved directly by using Lambert W function by converting the characteristic equation of the DDE in to the Lambert W form.
Lets consider a single delay system of the form,

$$\dot{x} + ax + bx(t - \tau_1) = 0, \tag{3.1}$$

where a and b, are non-zero real constants and $\tau_1 > 0$. On substituting $x = e^{st}$ in Eq. (3.1), we get the following characteristic equation

$$s + a + b_1 e^{-\tau_1 s} = 0. \tag{3.2}$$

The stability of the DDE can be found by analyzing the characteristic roots. If all the roots of the characteristic equation are in the left half of the complex plane then the $x = 0$ solution of the DDE is stable, otherwise it is unstable. The roots of the above characteristic equation can be determined exactly by using the Lambert W function [68, 71, 72].
After some algebraic manipulation we get

$$(s + a)e^{\tau_1 s} = -b_1, \tag{3.3}$$

and multiplying Eq. (3.3) with $\tau_1 e^{(a\tau_1)}$ yields

$$\tau_1(s + a)e^{\tau_1(s+a)} = -b_1\tau_1 e^{\tau_1 a}. \tag{3.4}$$

The roots of the above Eq. (3.4) can be expressed in closed form using the Lambert W function and are given by

$$s_k = -a + \frac{W_k(-b_1\tau_1 e^{\tau_1 a})}{\tau_1}, \quad k = -\infty, ... - 1, 0, 1, ..., \infty. \tag{3.5}$$

Here, W_k corresponds to the k_{th} branch of the Lambert W function. If k is varied between $-r$ and $+r$, we get $2r + 1$ dominant characteristic roots. However, the rightmost root, which determines the stability, obtained when the principal branch, $k = 0$ is the dominant characteristic root [73].
The characteristic roots for the Eq. (3.2), s_k, are shown in the Fig. 3.1(a) and Fig. 3.1(c) when k is varied from -50 to $+50$ and the error in respective roots of the characteristic equation is

shown in Fig. 3.1(b) and Fig. 3.1(d). Here, Fig. 3.1(a), Fig. 3.1(b) are generated for the parameters $a = 0.8$, $b = 1.5, \tau = 1$ and Fig. 3.1(c), Fig. 3.1(d) are generated for $a = 1.2$, $b = 0.5$, $\tau = 0.75$. However, in general case of multiple time delays, the equation will be of the form

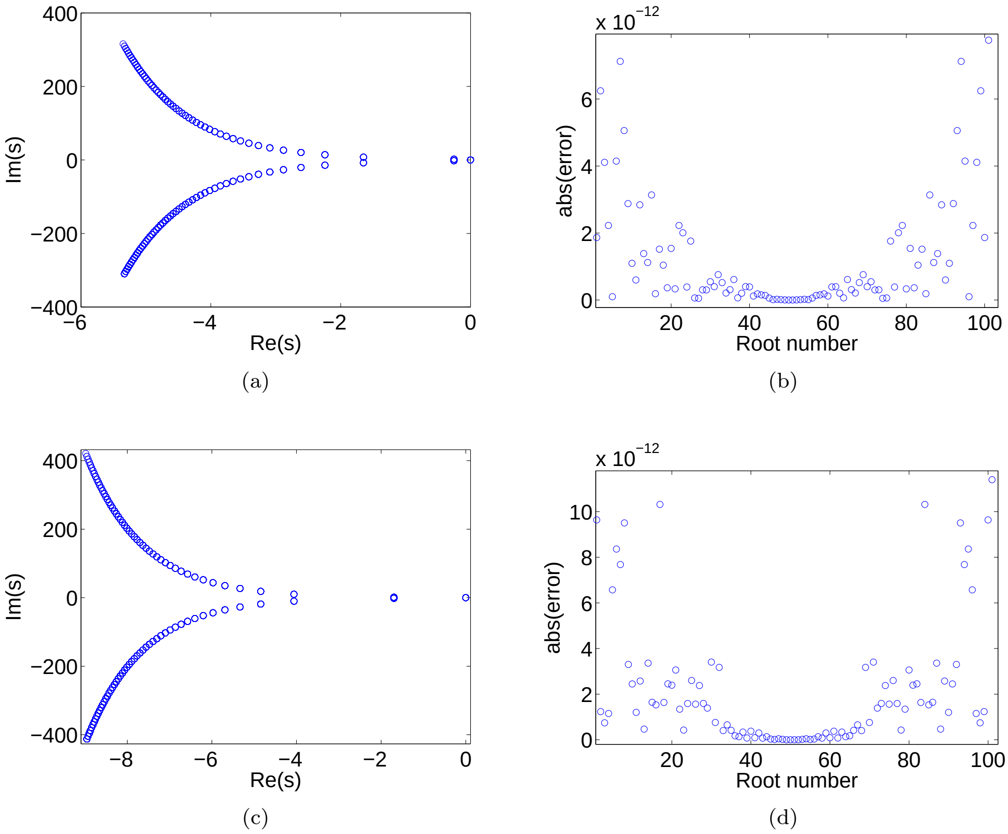


Figure 3.1: (a,b) Stability diagram for the first-order DDE with single delay.(d,e) Error for homotopy roots

$$\dot{x} + ax + \sum_{i=1}^{N} b_i x(t - \tau_i) = 0 \tag{3.6}$$

Here a and b_i, $i = 1, ..., N$ are non-zero real constants and $\tau_1, \tau_2, ..., \tau_N > 0$. with the corresponding characteristic equation

$$s + a + \sum_{i=1}^{N} b_i e^{-\tau_1 s} = 0 \tag{3.7}$$

Since the above characteristic equation contains more than one delay it cannot be solved directly using Lambert W function alone. The difficulty here is that Eq. (1.18) is a quasi-polynomial having infinitely many roots and can be only found by approximate methods. Here, in the subsequent chapters we developed a homotopy method to determine the characteristic roots for the multiple time delayed systems, Eq. (3.7) and further the stability.

Chapter 4

Homotopy method

The basic idea behind a homotopy method is that a known solution of a simple problem may continuously be "deformed" into a solution of a more difficult problem. Such deformation is called a homotopy. A simple example is a linear homotopy, a continuous "interpolation" $\mathcal{H}(x,\mu) = \mu f(x) + (1-\mu) g(x)$ between the two functions $f(x)$ and $g(x)$. The second argument of $\mathcal{H}(x,\mu)$ can thus be thought of as the deformation parameter such that for $\mathcal{H}(x,0) = f(x)$ is a solution of the simple problem and $\mathcal{H}(x,1) = g(x)$ is the hitherto unknown solution.
Here we seek the roots of a quasi-polynomial $\mathcal{H}(s)$ and split it into $\mathcal{H}(s) = P(s) + Q(s)$, where $P(s)$ is a quasi-polynomial with known roots. Then the following homotopy is constructed

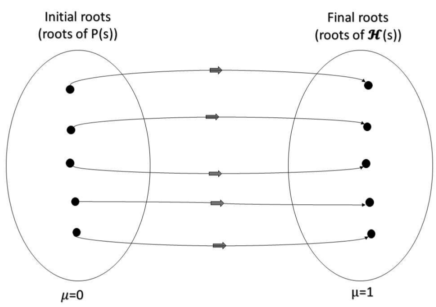
$$\mathcal{H}(s,\mu) = P(s) + \mu Q(s), \tag{4.1}$$

where $\mu \in [0,\,1]$ is the homotopy parameter. Now the idea is to increase μ by a small amount $\delta\mu << 1$ and solve for the roots of $\mathcal{H}(s,\delta\mu)$ using a nonlinear solver using the roots of $\mathcal{H}(s,0)$ as the initial guess. Once the roots of $H(s,\delta\mu)$ are found (within some tolerance), we increase μ by $\delta\mu$ again and use these roots as initial guess to solve $\mathcal{H}(s,2\delta\mu) = 0$. This procedure is then continued until $\mu = 1$ is reached. This kind of continuation where μ is varied in constant steps is known as natural parameter continuation and it is well known that it fails to trace the roots $s(\mu)$ at turning points. Therefore, we use pseudo-arclength continuation [74–76] to trace the roots as we vary μ from 0 to 1 as this method is capable of tracing the roots even at the turning points. Fig. 4.1 illustrates how the roots of $\mathcal{H}(s,\mu)$ are traced from $\mu = 0$ to $\mu = 1$. In the context of this work the homotopy is the following

$$\mathcal{H}(s,\mu) = \underbrace{s + a + b_1 e^{-\tau_1 s}}_{P(s)} + \underbrace{\mu \sum_{i=2}^{N} b_i e^{-\tau_i s}}_{Q(s)}. \tag{4.2}$$

It should be noted that $P(s)$ is constructed by taking the exponential containing the dominant time delay in Eq. (1.18) which will play a major role in the distribution of roots [77]. To emphasize the dependence of the roots on the homotopy parameter μ, we re-write Eq. (4.2) as

$$\mathcal{H}(s(\mu),\mu) = s(\mu) + a + b_1 e^{-\tau_1 s(\mu)} + \mu \sum_{i=2}^{N} b_i e^{-\tau_i s(\mu)} = 0. \tag{4.3}$$



(a)

Figure 4.1: Roots of $\mathcal{H}(s,\mu)$ as μ is varied from 0 to 1

For $\mu = 0$ (one time delay case), we have

$$\mathcal{H}(s(0), 0) = s(0) + a + b_1 e^{-\tau_1 s(0)} = 0, \tag{4.4}$$

the roots of which can be expressed in terms of the Lambert W function (see Eq. (3.5)). By using the closed form solutions of $\mathcal{H}(s(0), 0) = 0$ as initial guess (s_0, μ_0), we use pseudo-arclength continuation to trace roots of Eq. (4.2) as μ is varied from 0 to 1. The algorithm for continuing the roots is shown below:

Algorithm 1 Algorithm for Finding Characteristic Roots of DDE Using Homotopy Method

Input: Delay differential equation $\dot{x} + ax + \sum_{i=1}^{N} b_i x(t - \tau_i) = 0$.
Output: Characteristic roots of given DDE.
    Identify characteristic equation by substituting $x = e^{st}$ in DDE
    Introduce Homotopy parameter μ, and construct $H(s, \mu)$
    When $\mu = 0$ compute the roots $s_k(\mu = 0)$ for $H(s(\mu), \mu)$ using Lambert W function $\forall\ k \in [-r, r]$
        while $-r \leq k \leq r$ **do**
            while $\mu + \delta\mu \leq 1$ **do**
                Compute $s_k(\mu + \delta\mu)$ for $H(s_k(\mu + \delta\mu), \mu + \delta\mu) = 0$ with initial conditions $s_k(\mu)$
                $\mu \leftarrow \mu + \delta\mu$
            end while
        return $s_k(\mu = 1)$
        end while

4.1 Pseudo-arclength continuation method

In pseudo-arclength continuation method s and ζ are implicitly defined as a function of ζ as $s = s(\zeta)$ and $\mu = \mu(\zeta)$. Here ζ is the arc-length along a branch of solutions. On the path parametrized by the arc-length ζ, we seek s and μ such that

$$\mathcal{H}\left(s(\zeta), \mu(\zeta)\right) = 0 \tag{4.5}$$

Here the tangent predictor is used to determine the prediction (s^*_{i+1}, μ^*_{i+1}) at $\zeta_i + \delta\zeta$ along the tangent $\hat{\boldsymbol{t}}$. That is,

$$\mu^*_{i+1} = \mu_i + \mu'_i \delta\zeta \tag{4.6}$$

$$s^*_{i+1} = s_i + s'_i \delta\zeta \tag{4.7}$$

where i is the number of iteration. Now, Newton-Raphson method is used to correct the solution (s^*_{i+1}, μ^*_{i+1}) along the normal at (s^*_{i+1}, μ^*_{i+1}). This path is shown as a broken line in Fig. 4.2. On the solution path, (s_{i+1}, μ_{i+1}) satisfies

$$\mathcal{H}(s_{i+1}, \mu_{i+1}) = 0 \tag{4.8}$$

and since the vector $\hat{\boldsymbol{n}}$ is normal to the tangent vector $\hat{\boldsymbol{t}}$, we have

$$\hat{\boldsymbol{n}}^T \hat{\boldsymbol{t}} = 0 \tag{4.9}$$

here,

$$\hat{\boldsymbol{n}} = \left\{ \begin{array}{c} s_{i+1} - s^*_{i+1} \\ \mu_{i+1} - \mu^*_{i+1} \end{array} \right\} \tag{4.10}$$

and

$$\hat{\boldsymbol{t}} = \left\{ \begin{array}{c} s' \\ \mu' \end{array} \right\} \tag{4.11}$$

where s' and μ' are defined as $s' = \frac{ds}{d\zeta}$; $\mu' = \frac{d\mu}{d\zeta}$. On substituting Eq. (4.6) and Eq. (4.7) into Eq. (4.9) and using the definition of $\hat{\boldsymbol{t}}$, we obtain

$$(s_{i+1} - s_i)^T s'_i + (\mu_{i+1} - \mu_i)\mu'_i - [\mu'^2_i + (s'_i)^T s'_i]\delta\zeta = 0 \tag{4.12}$$

From Euclidean arc-length normalization we have

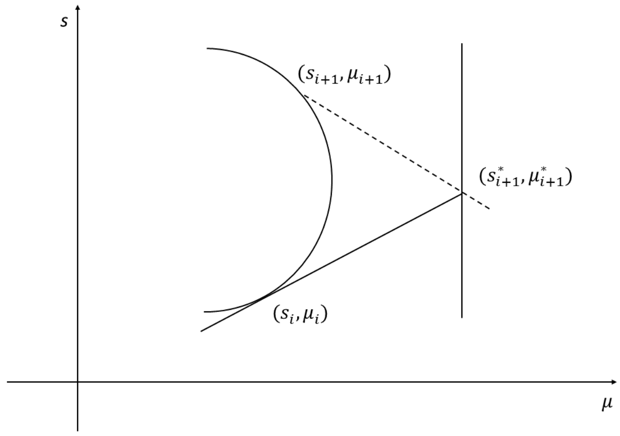
$$\mu'^2_i + (s'_i)^T s'_i = 1 \tag{4.13}$$

Therefore, Eq. (4.12) becomes

$$(s_{i+1} - s_i)^T s'_i + (\mu_{i+1} - \mu_i)\mu'_i - \delta\zeta = 0 \tag{4.14}$$

Equation. (4.8) and Eq. (4.14) constitute the pseudo-arclength continuation scheme. Equation. (4.8) and Eq. (4.14) must be solved for $(n + 1)$ unknowns in (s, μ).

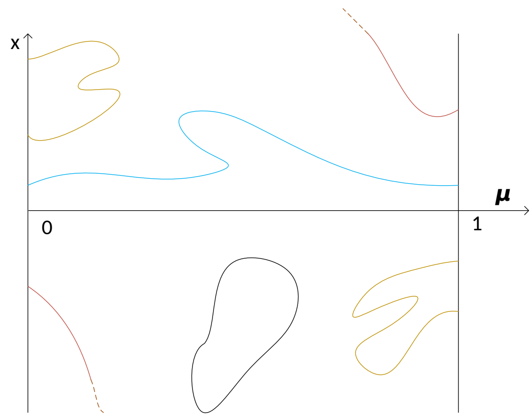
In homotopy method the path traced by the root is different with different initial guesses. The different paths that a root can take while λ reaches 1 is shown in Fig. 4.3. In Fig. 4.3 it is shown that some paths are actually reaching the actual root, while some others diverge to infinity and some turn back to the stage where $\lambda = 0$. Therefore, it is very important in selecting the $\mathcal{H}(s(0), 0) = P(s)$ polynomial which provides the initial guesses for solving the Eq. (1.18). From the extensive numerical



(a)

Figure 4.2: (a) Illustration of pseudo arc-length continuation method.

studies we have found that when we select $P(s)$ such that it contains the term with highest delay. All the roots in the homotopy continuation converged to the actual roots of $\mathcal{H}(s(1), 1) = 0$.



(a)

Figure 4.3: (a) possibilities of root convergence in homotopy method.

Chapter 5

Results and discussion

In this chapter stability charts are presented for different DDEs. To validate these, the stability charts obtained from the homotopy method are compared with the charts that obtained from direct numerical simulation of DDEs. In this work Matlab inbuilt DDE solver "dde23" is used for direct numerical simulation of the DDEs. The characteristic roots obtained by our method are also validated with those obtained from spectral-tau method [78]. Stability of the DDE can be ascertained by checking the real part of all the obtained roots. When real part of all the roots are negative then the DDE is stable, else it is unstable.

5.1 First order DDE case studies

5.1.1 First-order DDE with two delays

As a first example consider a DDE with two delays

$$\dot{x} + ax + b_1 x(t - \tau_1) + b_2 x(t - \tau_2) = 0 \tag{5.1}$$

with the corresponding characteristic equation

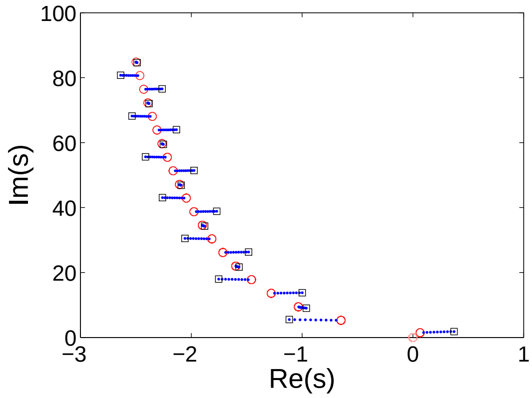
$$s + a + b_1 e^{-s\tau_1} + b_2 e^{-s\tau_2} = 0, \tag{5.2}$$

and homotopy

$$\mathcal{H}(s(\mu), \mu) = (s(\mu) + a + b_1 e^{-s(\mu)\tau_1}) + \mu b_2 e^{-s\tau_2} = 0. \tag{5.3}$$

The path traced by the roots of Eq. (5.3) when μ is varied from 0 to 1 in an interval of $\delta\mu = 0.1$ is shown in Fig. 5.1 (parameters are $\tau_1 = 1.5$, $\tau_2 = 1$, $a = 1$, $b_1 = 2$, and $b_2 = 2$.). Since the roots are complex conjugates, for clarity only upper half of the complex plane is shown in the Fig. 5.1. Circles show the roots of $H(s(0), 0)$ obtained using the Lambert W function, squares denote the roots of $H(s(1), 1) = R(s)$ and the dotted line indicate the intermediate roots of the homotopy equation as μ is varied between 0 and 1.

Fig. 5.2(a) shows the stability diagram for the DDE (Eq. (5.1)) for different time delays. For generating Fig. 5.2(a), τ_1 and τ_2 were varied from 0 to 5. Other parameters used for generating the figure are $a = 1$, $b_1 = 2$, $b_2 = 2$. In Fig. 5.2(a) red dots are stable points obtained from homotopy



(a)

Figure 5.1: Path traced by the roots of the Eq. (5.3) when μ is varied from from 0 to 1.

method and blue circles are the points obtained from direct numerical simulation of the DDE. In direct numerical simulation we integrate the DDE with unit initial history function for a long time and if the solution approaches zero we consider it as stable point. We can see from Fig. 5.2(a) that the stability diagram obtained from homotopy method agrees well with the one obtained from numerical simulation. Two random points from the stability diagram are picked in such a way that one is from the stable region and the other is from the unstable region and plotted the first 101 characteristic roots. The characteristic roots of the Eq. (5.1) for the stable point ($\tau_1 = 2$ and $\tau_2 = 0.5$) is shown in Fig. 5.2(b) and for unstable point ($\tau_1 = 2$ and $\tau_2 = 1$) is shown in Fig. 5.2(c). From the two figures it is clearly seen that for the stable point all the characteristic roots lie on the negative real axis where as for an unstable point some of the roots lie on the positive real axis which causes instability to the system. In Fig. 5.2(b) and Fig. 5.2(c) the red dots are roots obtained from homotopy method and blue circles are roots obtained from spectral-tau method. We can see that the roots obtained by both spectral-tau method and homotopy methods are in good agreement. Figure. 5.2(d) and Fig. 5.2(e) shows the error in the characteristic equation when the roots obtained from the homotopy method is substituted(see Fig. 5.2(b) and Fig. 5.2(c)) and we can see that error is very low and is less than 1×10^{-11}. For obtaining the roots with very high accuracy in spectral-tau method, one has to solve a large eigenvalue problem. However, in homotopy method we are using a nonlinear solver at every step of the continuation algorithm and we can set the tolerance on the admissible error for the roots in satisfying the characteristic equation and we can obtain very accurate characteristic roots.

5.1.2 First-order DDE with several delays

In several delay case, first order DDE with five delays ($N = 5$) is considered

$$\dot{x} + ax + \sum_{i=1}^{N} b_i x(t - \tau_i) = 0, \tag{5.4}$$

with the characteristic equation

$$s + a + \sum_{i=1}^{N} b_i e^{-\tau_i s} = 0. \tag{5.5}$$

The stability of the DDE (Eq. (5.4)) is explored by using the homotopy method. Figure 5.3(a) shows the correlation between the stability charts acquired from the homotopy method and the spectral-tau method. The stability chart is generated by varying τ_1 and τ_2 from 0 to 1 using the parameters $a = 1$, $b_1 = 3$, $b_2 = 2.8$, $b_3 = 0.6$, $b_4 = 0.8$, $b_5 = 1$, $\tau_3 = 1$, $\tau_4 = 1.5$ and $\tau_5 = 2$. In Fig. 5.3(a) red dots represent the stability boundary (region under which the system is stable) obtained from the homotopy method and blue dots are the stable points obtained from direct numerical simulation of the DDE. The first 101 roots obtained from the homotopy method for Eq. (5.5) for two different points from the stability chart Fig. 5.3 are plotted in the Fig. 5.3(b) and Fig. 5.3(c). Figure. 5.3(b) is generated for a stable point $\tau_1 = 0.2$, $\tau_2 = 0.5$ and Fig. 5.3(c) is generated for a unstable point $\tau_1 = 2.2$, $\tau_2 = 1.7$. The respective absolute errors in roots of the characteristic equation from the homotopy method are presented in the Fig. 5.3(d) and Fig. 5.3(e).

A DDE with ten delays having different coefficients is also considered to show the efficacy of the homotopy method. Figure. (5.4) is generated for the Eq. (5.4) for $N = 10$ with the coefficients $a = 1$, $b_1 = 3$, $b_2 = 2.8$, $b_3 = 0.6$, $b_4 = 0.8$, $b_5 = 1$, $b_6 = 2$, $b_7 = 0.1$, $b_8 = 0.5$, $b_9 = 0.9$ and $b_{10} = 1.2$ and delays $\tau_3 = 1$, $\tau_4 = 1.5$, $\tau_5 = 2$, $\tau_6 = 2.5$, $\tau_7 = 3$, $\tau_8 = 3.5$, $\tau_9 = 4$ and $\tau_{10} = 4.5$ by varying τ_1 and τ_2 between 0 and 1. Figure. 5.4(b) and Fig. 5.4(c) shows the characteristic roots of the Eq. (5.4) when $\tau_1 = 0.4$, $\tau_2 = 0.1$ and $\tau_1 = 0.4$, $\tau_2 = 0.3$ respectively and Fig. 5.4(d) and Fig. 5.4(e) shows the respective errors in the characteristic roots.

5.2 Higher-order DDE case studies

The homotopy approach can also be extended for higher order time delay systems. The effectiveness of the developed homotopy approach for higher order DDEs is demonstrated with some examples in the following sections.

5.2.1 Higher order DDE with single delay.

Higher order DDE having single delay cannot be solved directly by Lambert W function as in the case of first order DDE with single delay. In such case homotopy method can be applied to solve the problem. In order to show this, we took an example from [39], applied homotopy method and compared the results.
The second order DDE we considered is

$$\ddot{x} + ax - bx(t - \tau) = 0 \tag{5.6}$$

with the characteristic equation

$$s^2 + as - be^{-s\tau} = 0 \tag{5.7}$$

In order to solve the Eq. (5.7), the homotopy equation is considered by adding and subtracting s and introducing homotopy parameter μ to the Eq. (5.7).

$$\mathcal{H}(s,\mu) = s + a - be^{-s\tau} + \mu(s^2 - s) \tag{5.8}$$

The stability chart for the DDE Eq. (5.6) is plotted in Fig. 5.5(a) by varying the coefficients a between -1 and $+5$ and b between -1 and $+1$. In Fig. 5.5(a) the blue dots are the stable points obtained from the homotopy method and red line represents the stability boundary given in [39]. Figure. 5.5(b) and Fig. 5.5(c) represents the characteristic roots of the Eq. (5.7) for a stable point and as well as for a unstable point respectively. In Fig. 5.5(b) and Fig. 5.5(c) red dots represents the roots from homotopy method and blue circles represents the roots obtained from spectral-tau method. Errors in the respective roots are also shown in Fig. 5.5(d) and Fig. 5.5(e). Figure. 5.5 clearly shows that the results obtained from homotopy method exactly matches with the results published in [39].

5.2.2 Higher-order DDE with two delays

For higher order DDE we consider

$$\ddot{x} + a_1\dot{x} + a_2 x + b_1 x(t-\tau_1) + b_2 x(t-\tau_2) = 0 \tag{5.9}$$

with the following characteristic equation

$$\mathcal{H}(s) = s^2 + a_1 s + a_2 + b_1 e^{-\tau_1 s} + b_2 e^{-\tau_2 s} = 0 \tag{5.10}$$

On introducing homotopy parameter to the Eq. (5.10) we get,

$$\mathcal{H}(s(\mu),\mu) = a_1 s(\mu) + a_2 + b_1 e^{-\tau_1 s(\mu)} + \mu(s(\mu)^2 + b_2 e^{-\tau_2 s(\mu)}) \tag{5.11}$$

For $\mu = 0$, we have

$$\mathcal{H}(s(0),0) = a_1 s(0) + a_2 + b_1 e^{-s(0)\tau_1} \tag{5.12}$$

On performing simple algebraic manipulations, we get

$$\tau_1(s + \frac{a_2}{a_1})e^{\tau_1(s+\frac{a_2}{a_1})} = -\frac{b_1}{a_1}\tau_1 e^{\frac{a_2}{a_1}\tau_1} \tag{5.13}$$

The roots of the above Eq. (5.13) can be expressed in the closed form using Lambert W function and are given by

$$s_k(0) = \frac{W_k((-\frac{b_1}{a_1})\tau_1 e^{(\frac{a_2}{a_1})\tau_1})}{\tau_1} - (\frac{a_2}{a_1}) \tag{5.14}$$

Where W_k corresponds to the k_{th} branch of the Lambert W function. The exact roots obtained from Lambert W function are used as initial guess for numerical continuation process to solve for roots of the characteristic equation. Figure 5.6(a) shows the stability of the DDE (Eq. (5.9)) for the parameters $a_1 = 1$, $a_2 = 1$, $b_1 = 1$, $b_2 = 1$ and for different values of τ_1 and τ_2. In Fig. 5.6(a) red dots are the stable points obtained from the homotopy method and the blue circles are the stable points from the spectral-tau method. The 101 dominant roots for the characteristic equation

(Eq. (5.10)) for stable and unstable points are shown in Fig. 5.6(b) and Fig. 5.6(c). Where Fig. 5.6(b) is plotted for $\tau_1 = 1.7, \tau_2 = 2.8$ and Fig. 5.6(c) is plotted for $\tau_1 = 1, \tau_2 = 4$. Error in the roots of the quasi-polynomial (Eq. (5.10)) are shown in Fig. 5.6(d) and Fig. 5.6(e).

As a second example, second order DDE with two delays having different coefficients is considered and homotopy method is applied to Eq. (5.9) with the parameters $a = 3$, $b_1 = 2$ and $b_2 = 1$ and the stability diagram is plotted in Fig. 5.7(a) by varying τ_1 and τ_2 from 0 to 5. Figure. 5.7(b) and Fig. 5.7(c) represents the characteristic roots for the Eq. (5.9) when $\tau_1 = 2.7$, $\tau_2 = 0.3$ and $\tau_1 = 1.2$, $\tau_2 = 1.5$ respectively. Figure. 5.7(d) and Fig. 5.7(e) shows the error in the characteristic equation when the roots obtained from the homotopy method are substituted (see Fig. 5.7(b) and Fig. 5.7(c)).

5.2.3 Higher order DDE with several delays

In this case a second order DDE with three delays and along with a delay in the damping term is considered.

$$\ddot{x}(t) + a_1\dot{x}(t) + a_2x(t) + b_1\dot{x}(t-\tau_1) + b_2x(t-\tau_1) + b_3\dot{x}(t-\tau_2) + b_4x(t-\tau_1-\tau_2) = 0 \tag{5.15}$$

On applying homotopy method to the Eq. (5.15) and following the same derivation as shown in the previous example for higher order DDE with two delays. we get the following equations.

$$\mathcal{H}(s(\mu), \mu) = a_1s(\mu) + a_2 + b_4e^{-\tau_3 s} + \mu(s^2 + b_1se^{-\tau_1 s} + b_3se^{-\tau_2 s} + b_2e^{-\tau_1 s}) \tag{5.16}$$

$$s_k(0) = \frac{W_k(\frac{b_4}{a_2})\tau_3 e^{(\frac{a_2}{a_2})\tau_3}}{\tau_3} - (\frac{a_2}{a_1}) \tag{5.17}$$

After finding the roots of the Eq. (5.16) using homotopy technique, the stability of the system is shown in the Fig. 5.8(a). The plot is generated for the parameters $a_1 = 7.1$, $a_2 = 21.1425$, $b_1 = 6$, $b_2 = 14.8$, $b_3 = 2$, $b_4 = 8$ and $\tau_3 = \tau_1 + \tau_2$ for different values of τ_1 and τ_2. In Fig. 5.8(a) red dots represents the stability boundary obtained from the homotopy method and the blue dots shows the stable points obtained from spectral-tau method. Figure. 5.8(b), Fig. 5.8(c) represents characteristic roots for the equation (Eq. (5.15)) and Fig. 5.8(d), Fig. 5.8(e) represents errors in the respective roots. Here, Fig. 5.8(b) is generated for $\tau_1 = 1.6$, $\tau_2 = 3.1$ and Fig. 5.8(c) is generated for $\tau_1 = 1$, $\tau_2 = 0.5$

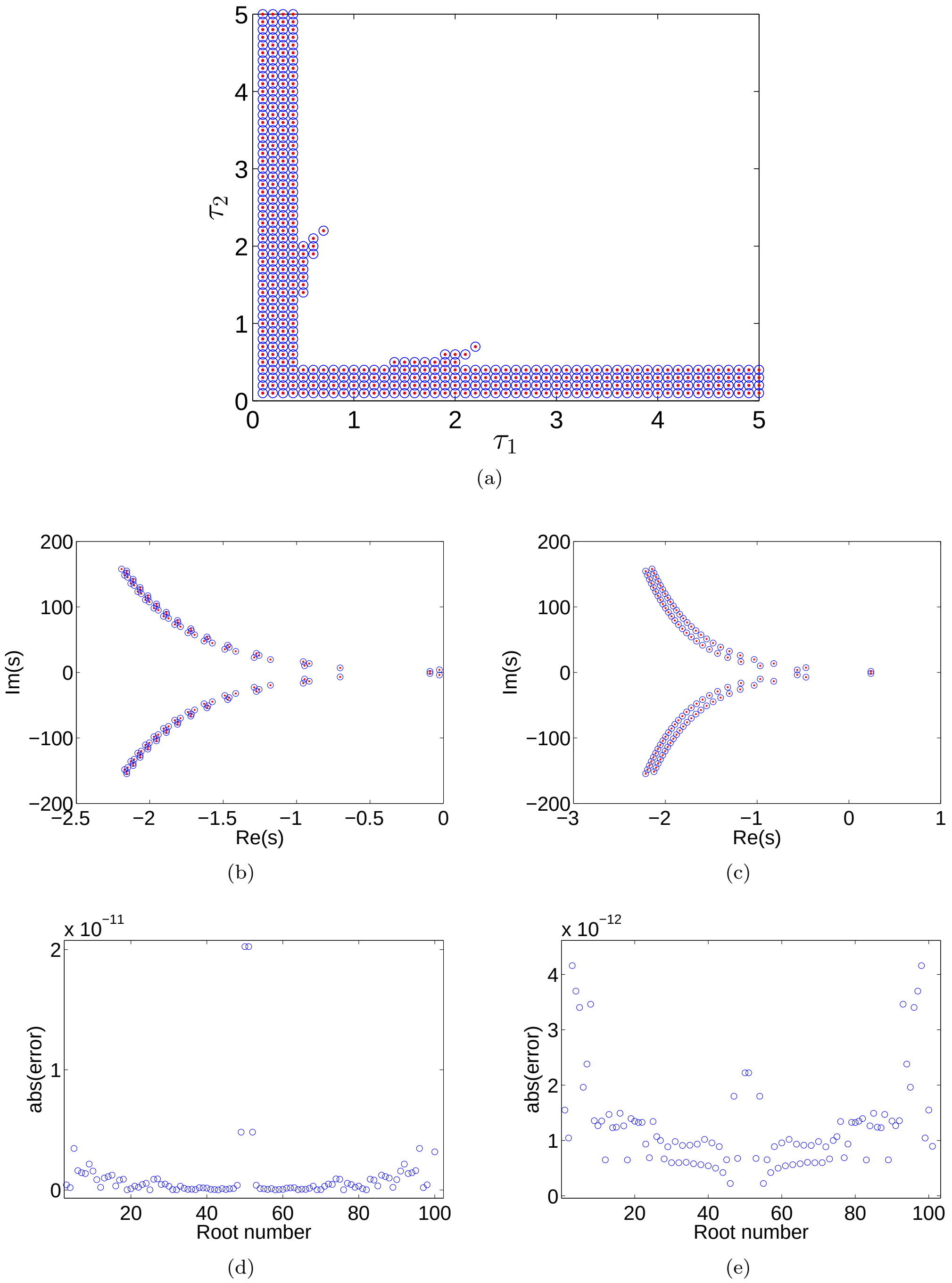


Figure 5.2: (a) Stability diagram for the first-order DDE with two delays. (b,c) Comparison between the roots obtained from homotopy and spectral-tau methods for stable and unstable points respectively. (d,e) Error for homotopy roots

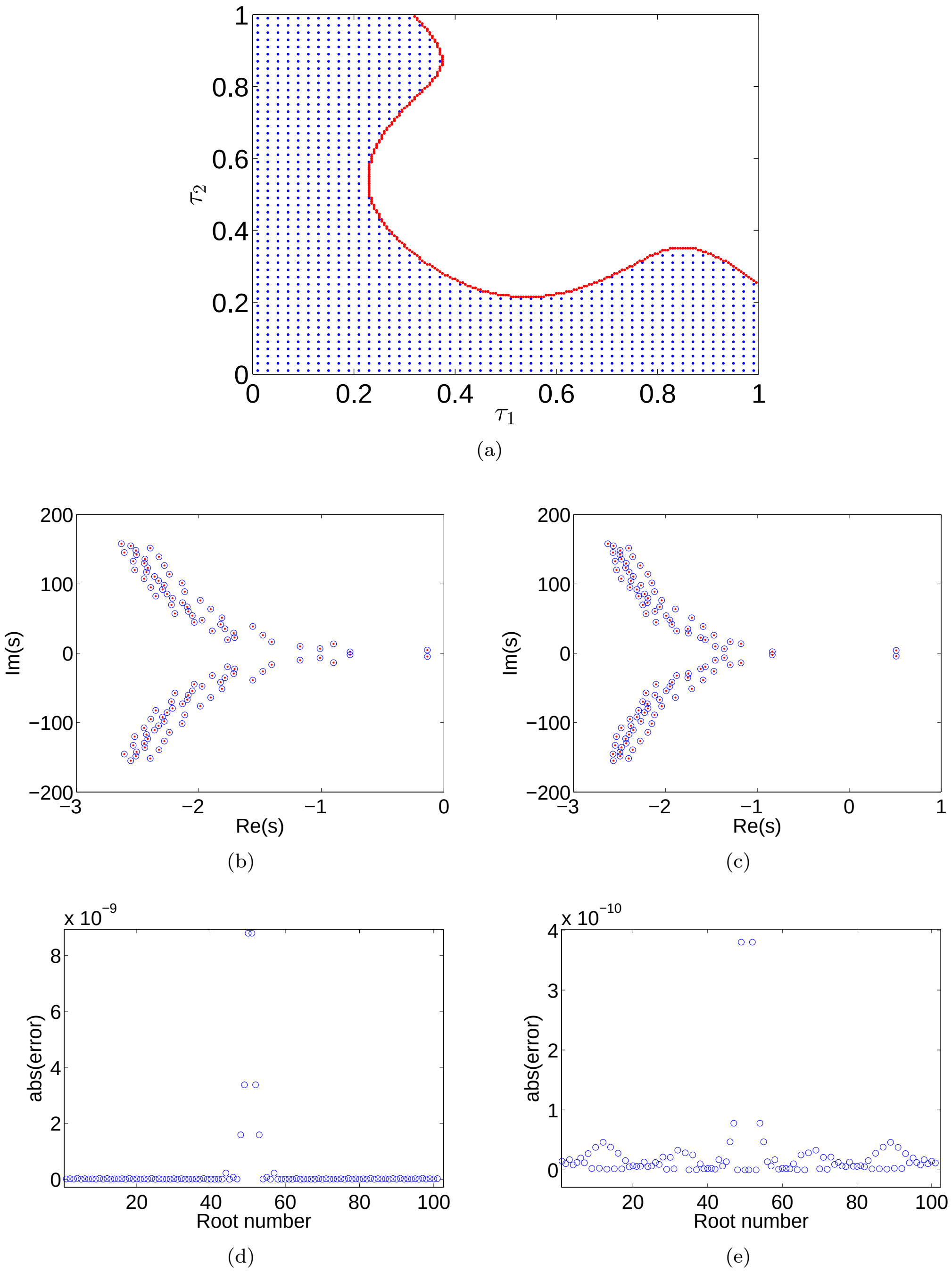


Figure 5.3: (a) Stability diagram of the first-order DDE with five delays. (b,c) Comparison between the roots obtained from homotopy and spectral-tau methods for stable and unstable points respectively. (d,e) Error for homotopy roots

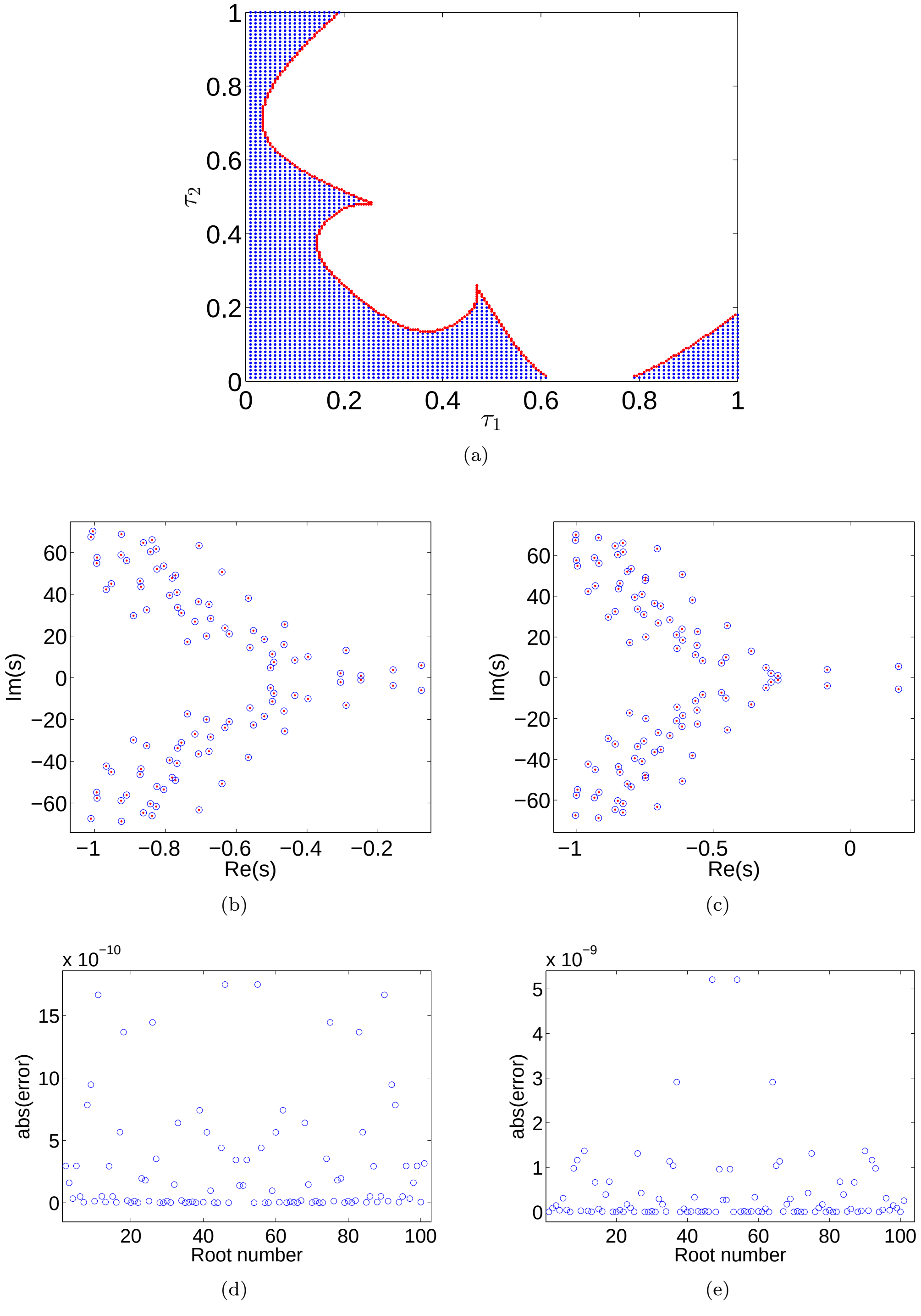


Figure 5.4: (a) Stability diagram for the first-order DDE with ten delays having different coefficients. (b,c) Comparison between the roots obtained from homotopy and spectral-tau methods for stable and unstable points respectively. (d,e) Error for homotopy roots

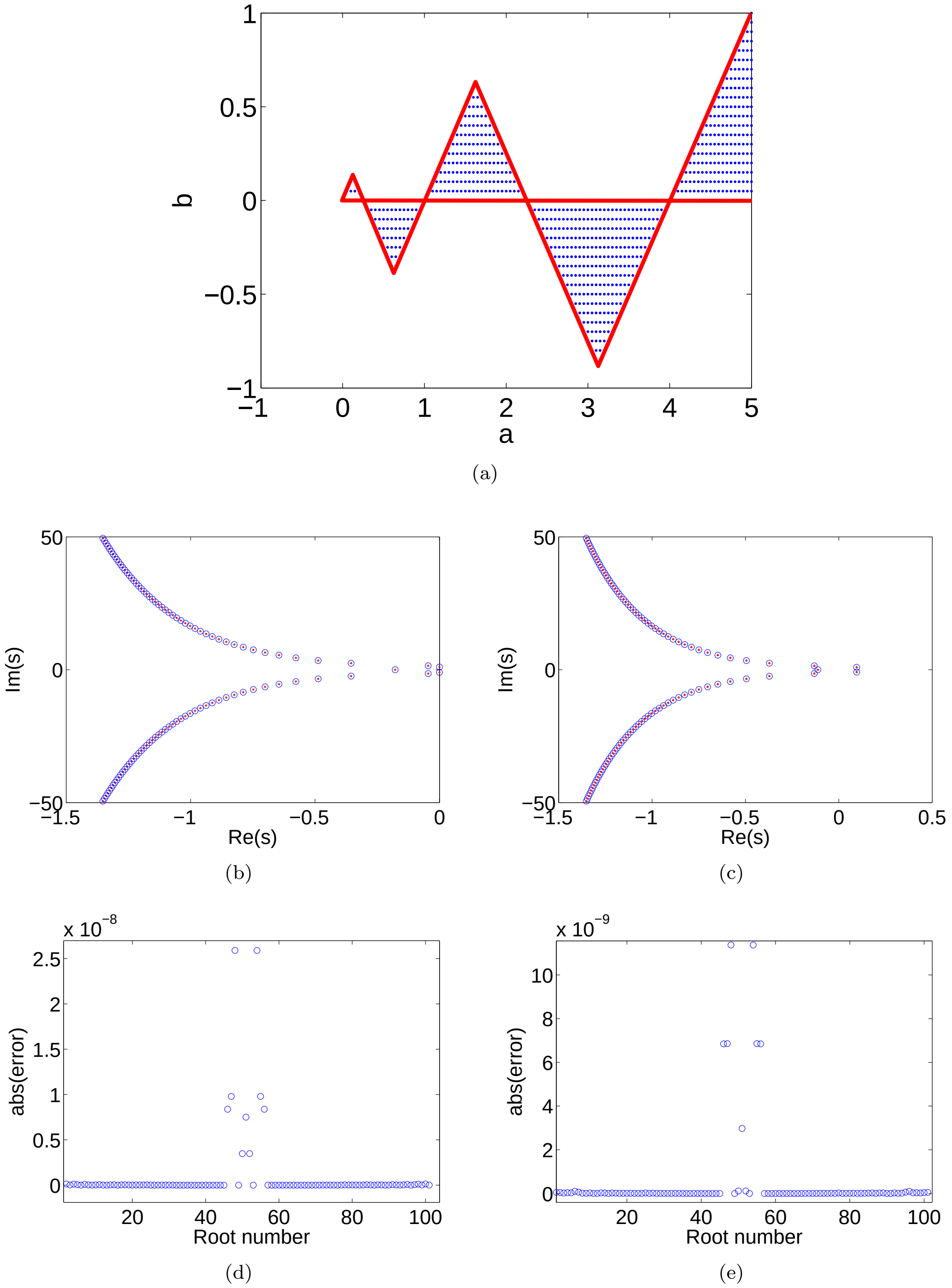


Figure 5.5: (a) Stability diagram for the second-order DDE with single delays. (b,c) Comparison between the roots obtained from homotopy and spectral-tau methods for stable and unstable points respectively. (d,e) Error for homotopy roots

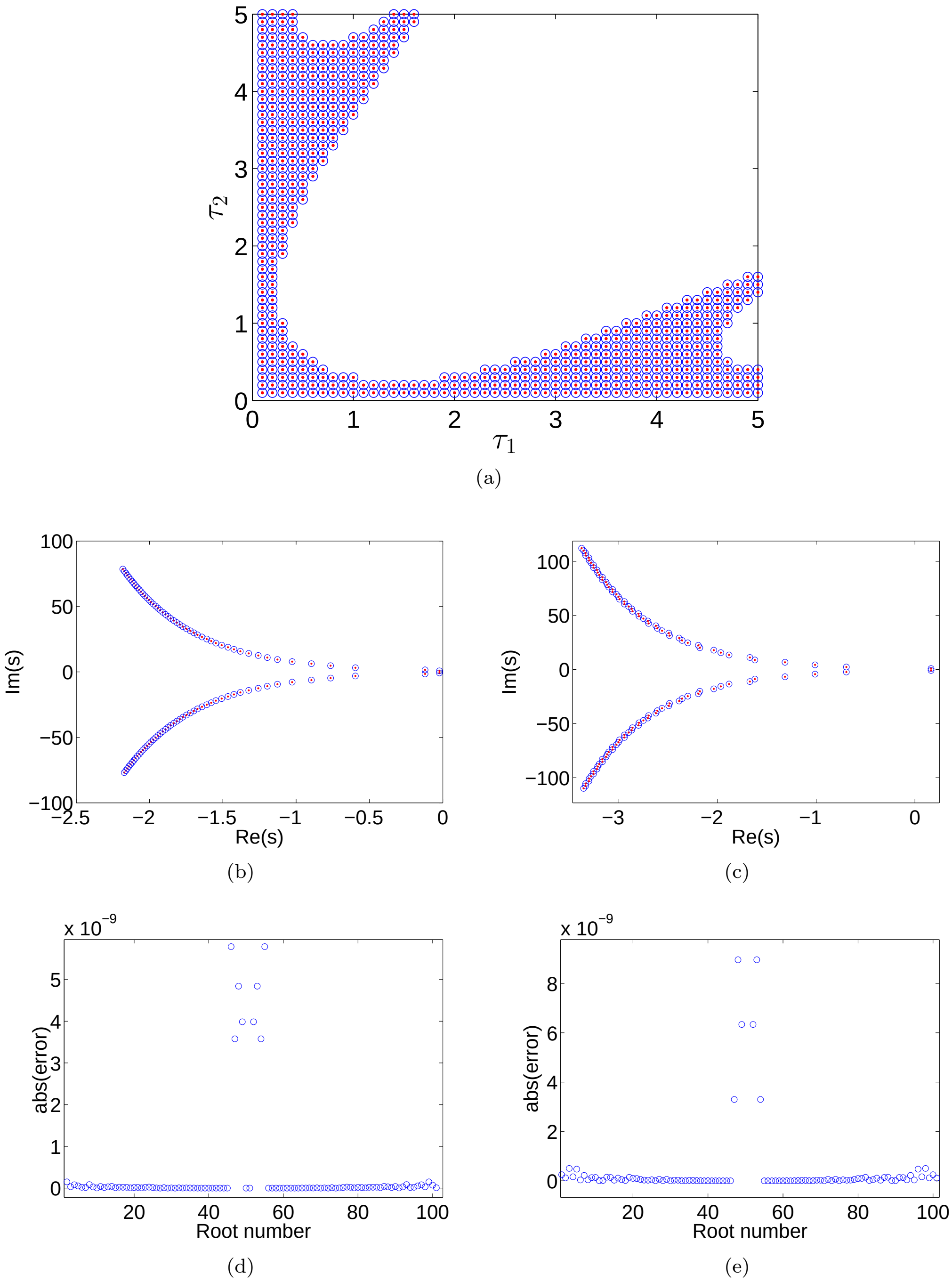
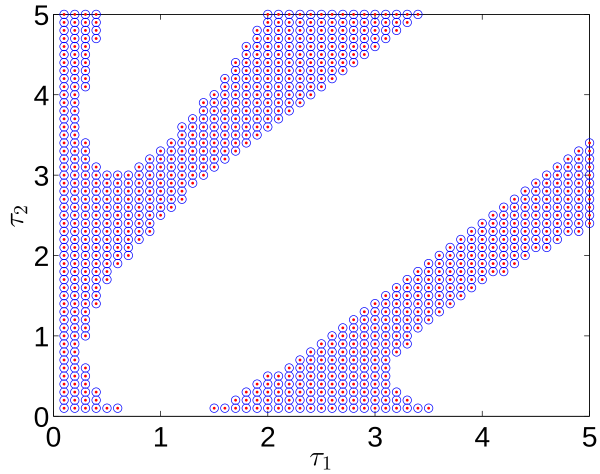
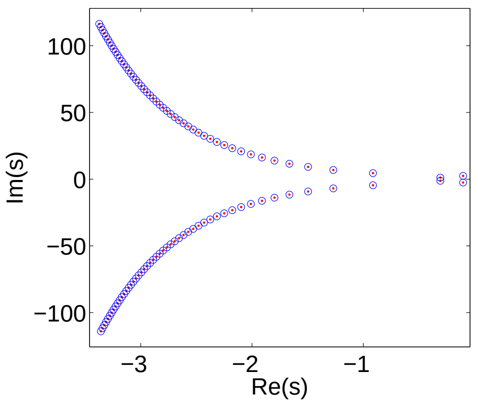


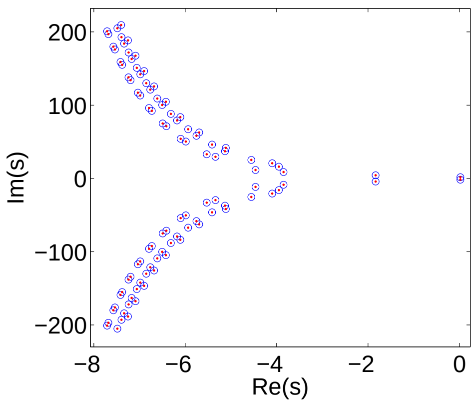
Figure 5.6: (a) Stability diagram for the second-order DDE with two delays. (b,c) Comparison between the roots obtained from homotopy and spectral-tau methods for stable and unstable points respectively. (d,e) Error for homotopy roots



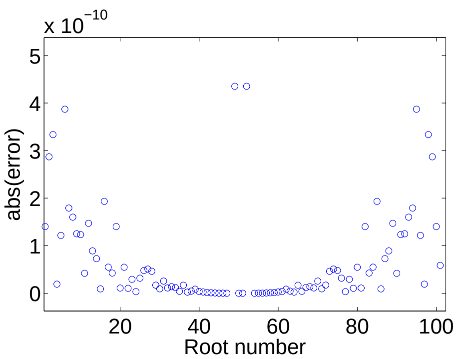
(a)



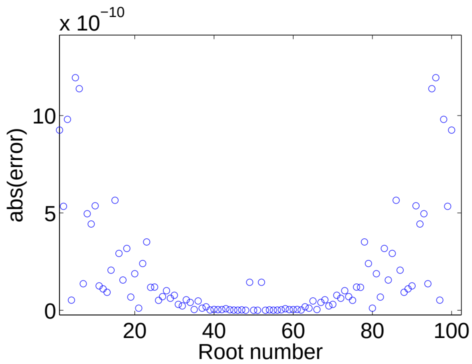
(b)



(c)



(d)



(e)

Figure 5.7: (a) Stability diagram for the second-order DDE with two delays and different coefficients. (b,c) Comparison between the roots obtained from homotopy and spectral-tau methods for stable and unstable points respectively. (d,e) Error for homotopy roots

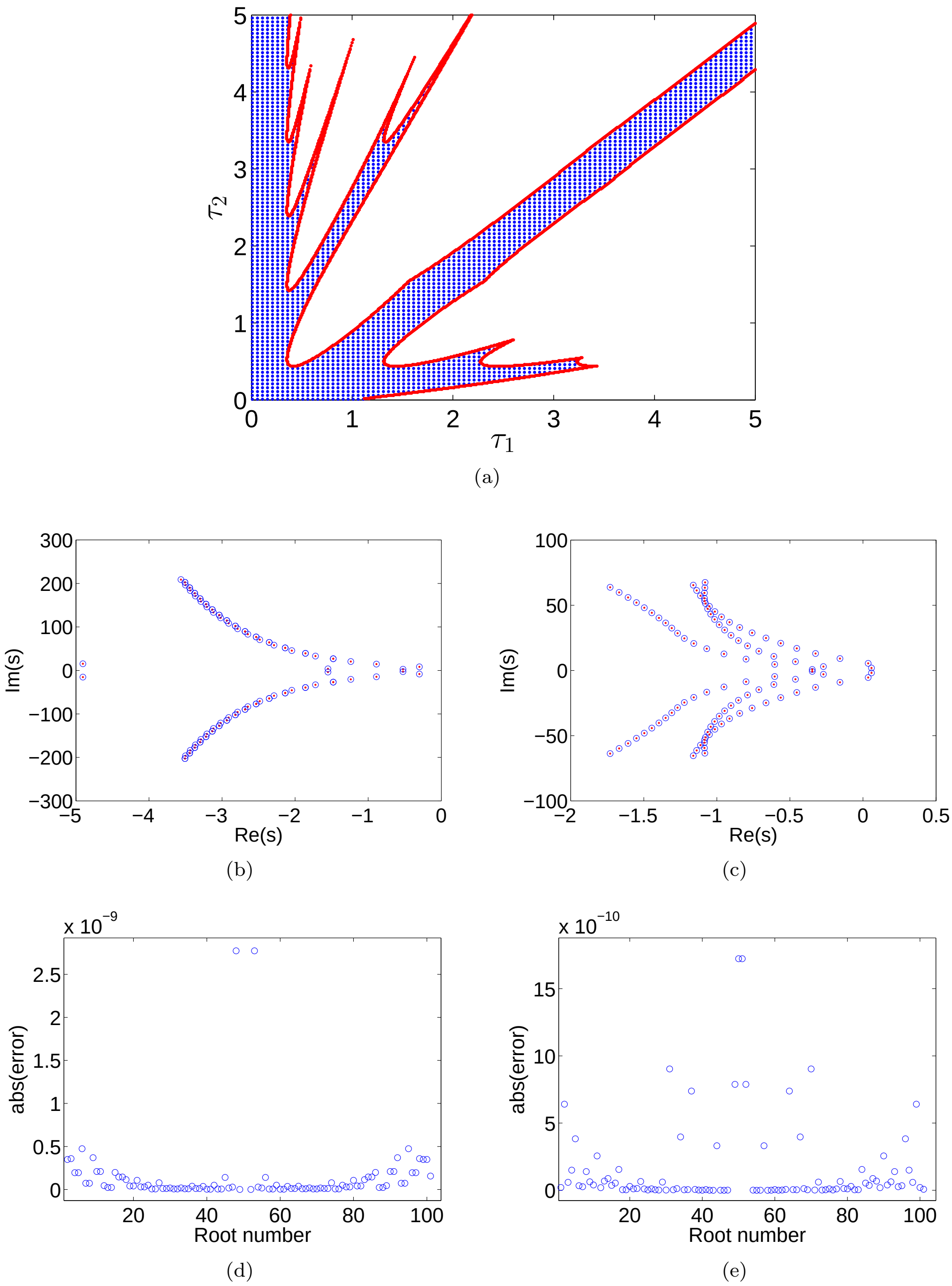
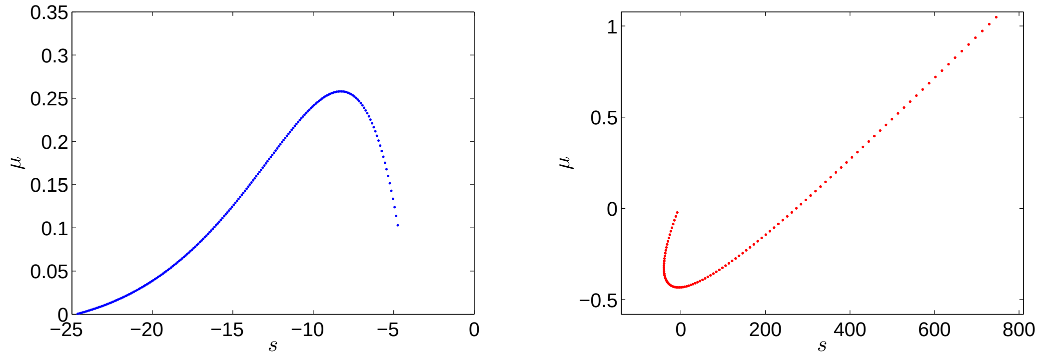


Figure 5.8: (a) Stability diagram for the second-order DDE with three delays. (b,c) Comparison between the roots obtained from homotopy and spectral-tau methods for stable and unstable points respectively. (d,e) Error for homotopy roots

Chapter 6

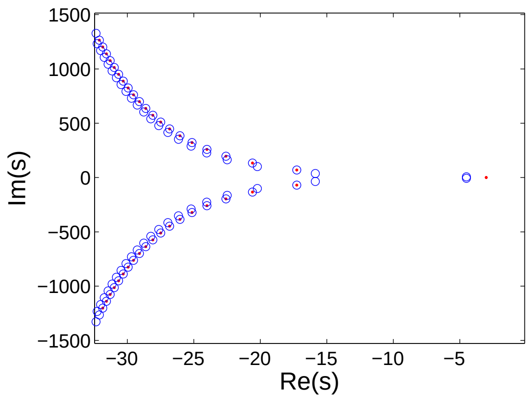
Cases when the method fails

In the case of not considering the dominant term in $P(s)$ (see Eq. (4.2)) some roots of the equation $\mathcal{H}(s,\mu) = 0$ will be missed and some other diverges to a random value that does not satisfy the Eq. (1.18). Therefore, dominant term in the Eq. (1.18) i.e., the term with highest delay must be included in $P(s)$. Figure. 6.1 shows the difference between the s Vs μ plots acquired with and without considering the dominant term in the $P(s)$. Here s Vs μ plots are generated for the non satisfying root of the Eq. (5.2) when $\tau_1 = 0.1$ and $\tau_2 = 0.2$ with a step length $\zeta = 10^{-2}$. Figure. 6.1(a) represents the path traced by the root when the dominant term is not included in $P(s)$ and Fig. 6.1(b) represents the case when dominant term is included. In the case of not considering the dominant term the root turns back to the situation when $\mu = 0$ and this is similar to case that is shown in the Fig. 4.3 with yellow colored paths. Roots acquired with and without considering the dominant term in $P(s)$ is also shown in Fig. 6.2, where the red dots are the roots obtained when the dominant term not considered in $P(s)$ and blue circles represents the roots obtained with the dominant term considered in $P(s)$. Error in finding the roots with the two different approaches is presented in Fig. 6.3. From Fig. 6.2 and Fig. 6.3 it is seen that the roots obtained from the two different approaches on solving the Eq. (5.2) does not match with each other. In the solution obtained by not considering the dominant term, some of the roots are missing and some other does not satisfies the Eq. (5.2). Therefore in order to find the characteristic roots of the DDE using the homotopy method, dominant term must be included in $P(s)$.



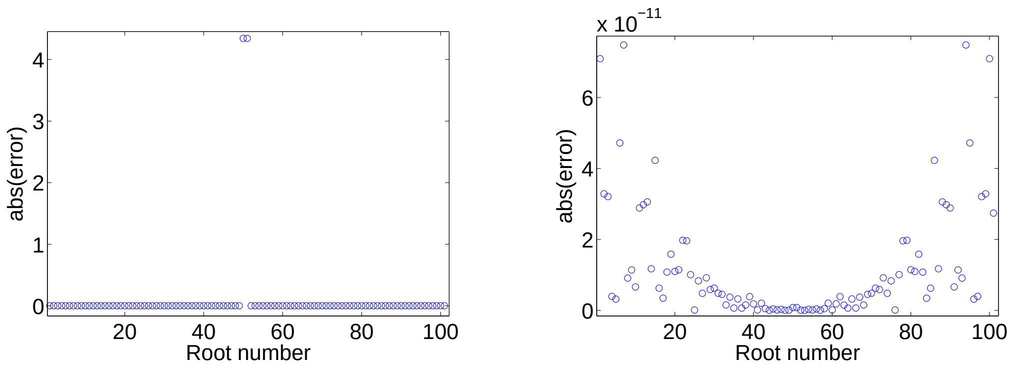
(a) (b)

Figure 6.1: (a) s Vs μ when dominant term is not included in $P(s)$. (b)s Vs μ when dominant term is included in $P(s)$.



(a)

Figure 6.2: (a)Comparison between the roots acquired with and without dominant term in $P(s)$.



(a) (b)

Figure 6.3: (a) Error in roots with out dominant term in $P(s)$. (b) Error in roots with dominant term in $P(s)$.

Chapter 7

Conclusion

In this work we have developed a homotopy method to find the roots of a characteristic equation of DDEs with multiple time delays. We split the characteristic equation of the DDE into two parts among which the roots of first part are known. We introduce a homotopy parameter μ into the characteristic equation such that for $\mu = 0$, the roots are exactly known and for $\mu = 1$ the original characteristic equation is recovered. By using the roots of the characteristic equation for $\mu = 0$ as initial guess, we trace the roots as a function of μ as it is varied from 0 to 1. When μ becomes 1 we obtain the roots of the original characteristic equation. The developed method can be used to obtain the characteristic roots of first and higher order DDEs. The developed method also allows us to determine the stability of a given DDE for different parameter values. We have successfully demonstrated the applicability of our method on seven different DDEs.

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