Chaotic Dynamical System

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M.Sc. Thesis

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Declaration

I hereby declare that the matter embodied in this report is the result of investigation carried out by me in the Department of Mathematics, Indian Institute of Technology Hyderabad under the supervision of Dr. D, Sukumar.

In keeping with general practice of reporting scientific observations, due acknowledgement has been made wherever the work described is based on the findings of other investigators.

08/05/2015

Signature of the Supervisor
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Acknowledgement

Foremost, I would like to express my sincere gratitude to my guide Dr. D Sukumar for the continuous support he gave me during my project work, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of doing project and writing of this thesis. Last but not the least, I would like to thank my parents and my friends for their continuous support to me.

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1 What is Dynamical System ?

Definition 1. *Let X be a topological space and $F : X \to X$ is a continuous functions. Then the pair (X, F) is called a Dynamical System. Here, we have considered $X = \mathbb{R}$.*

2 Orbits

Definition 2. *Given $x_0 \in \mathbb{R}$. We define* ***Orbit*** *of x_0 under F to be the sequence of points.*

$$x_0, F(x_0), F^2(x_0), \ldots, F^n(x_0), \ldots$$

where $F^n(x_0) = \underbrace{F \circ F \circ \cdots \circ F}_{n \ times}(x_0)$ and $F^0(x_0) = x_0$. The point x_0 is called the seed of the orbit.

Example 1. *Consider $F : \mathbb{R} \to \mathbb{R}$ defined by F(x)=$\sqrt{x}$. with $x_0 = 256$ then the orbit of x_0 under F is*

$$x_0 = 256, F(x_0) = 16, F^2(x_0) = 4, F^3(x_0) = 2, F^4(x_0) = \sqrt{2}, \ldots$$

2.1 Type of Periodic Points

2.1.1 Fixed Point

Definition 3. *: A* ***fixed point*** *of a dynamical system $(\mathbb{R}, f)$ is a point $x_0 \in \mathbb{R}$ s.t. $F(x_0)$= x_0.*

The orbit of a fixed point x_0 is the constant sequence x_0,x_0,$x_0, \ldots$

Example 2. *0,1,-1 are the fixed point for $F(x) = x^3$.*

Example 3. *while 0 is the fixed point for $F(x) = sin(x)$.*

Can we say something graphically?

Fixed Point is the intersection of the graph of given function with the line $y = x$.

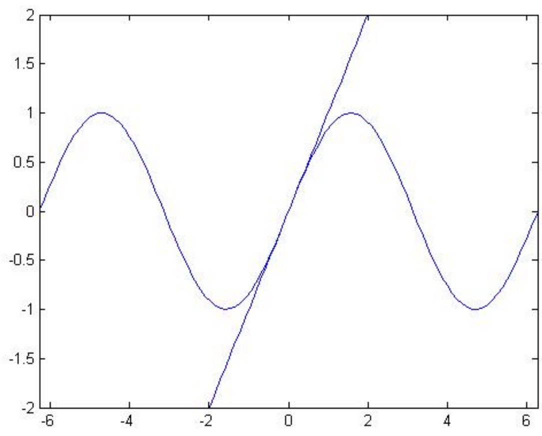


Figure 1: $y = sin(x)$.

2.1.2 Periodic Points

Definition 4. *A point x_0 is said to be a* ***periodic point*** *of the function F of period $n > 0$ if $F^n(x_0) = x_0$.*

The least such n is called the prime period of the orbit. If x_0 is periodic with prime period n then the orbit of x_0 is :

$$x_0, F(x_0), \ldots, F^{n-1}(x_0), x_0, F(x_0), \dot{F}^{n-1}(x_0), \ldots$$

i.e. The first term of the sequence occur again and again.

Example 4. *0 is a periodic point of period 2 for $F(x) = x^2 - 1$.*

2.1.3 Eventually Periodic or Eventually Fixed Point

Definition 5. *A point x_0 is called* ***Eventually periodic or Eventually fixed*** *if x_0 is not a periodic point or fixed point but some point on the orbit of x_0 is fixed or periodic.*

Example 5. *Consider $F(x) = x^2 - 1$.*
The point 1 is the eventually periodic point.

Example 6. *Consider $F(x) = x^2$.*
The point -1 is the eventually fixed point.

3 Graphical Analysis

Method: Suppose, we have the function F and we want to display the orbit of x_0 under F . We begin at the point (x_0, x_0) on the diagonal directly above x_0 on the x-axis. We first draw a vertical line to the graph of F. When this line meets the graph, we have reached the point $(x_0, F(x_0))$. We then draw a horizontal line from this point to the diagonal. We reach the diagonal at the point whose coordinate is $(F(x_0), F(x_0))$. Now we continue this procedure. Draw a vertical line from $(F(x_0), F(x_0))$ on the diagonal to the graph. This yields the point $(F(x_0), F^2(x_0))$ then a horizontal line to the diagonal reaches at $(F^2(x_0), F^2(x_0))$ directly above the next point in the orbit and continue the process. This results a "staircase" diagram.

Example 7. *Take $F(x) = \sqrt{x}$*

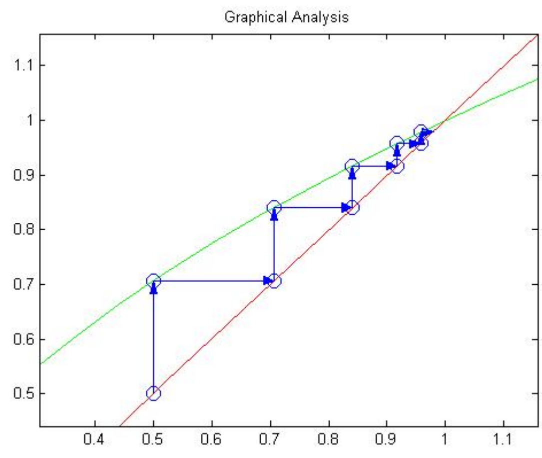


Figure 2: $F(x) = \sqrt{x}$.

3.1 Orbit Analysis

Graphical analysis some time allow us to describe the behavior of all orbits of a dynamical systems.

Example 8. *Take* $F(x) = x^3$.

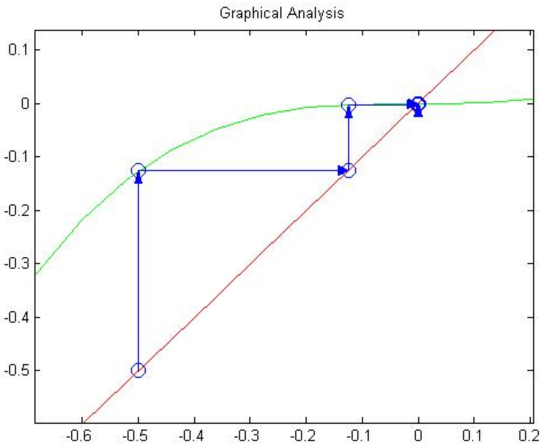


Figure 3: $F(x) = x^3$.

What graph of F tells us?
The Graph of F shows that there are fixed points at 0,1 and -1.It allow us to read off the following behavior;if $|x_0| < 1$ then the orbis of x_0 tends to 0. On the other hand if $|x_0| > 1$ then the orbit of x_0 tends to $\pm\infty$

3.2 The Phase Portrait

This is the picture on the real line of the orbits.
Method: We represent fixed point by solid dots and the dynamics along orbits by arrows.

Example 9. *Take* $F(x) = x^3$.

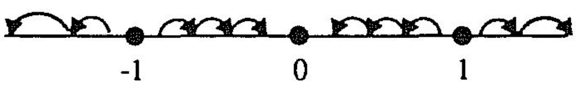


Figure 4: Phase Portrait of $F(x) = x^3$.

Example 10. *Take* $F(x) = x^2$

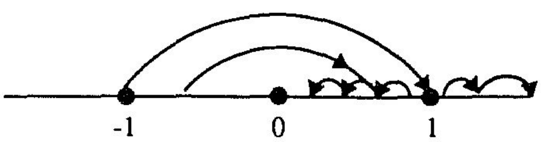


Figure 5: Phase Portrait of $F(x) = x^2$.

Reamrks:

- Graph of a function.

  The graph of a function on the reals provides information about its first iterate, but gives very little information about subsequent iterates.

- Phase Portrait.

  A much more efficient, geometric method for describing the orbits of a dynamical system is the phase portrait.This is a picture, on the real line itself.

4 Fixed and Periodic Points

Theorem 1. ***Intermediate Value Theorem*** *Let $F : [a, b] \to \mathbb{R}$ is continuous. Suppose $F(a) < y_0 < F(b)$. then there exists an $x_0 \in [a, b]$ with $F(x_0) = y_0$.*

Theorem 2. ***Fixed Point Theorem*** *Let $F : [a, b] \to [a, b]$ is continuous then there is a fixed point of F in $[a, b]$ i.e. $F(x) = x$.*

4.1 Attracting and Repulsing Fixed Point

Example 11. *Consider $F : \mathbb{R} \to \mathbb{R}$ defined as $F(x) = x^2$. Here fixed points are 0 and 1. Observation: If we choose any x_0 with $|x_0| < 1$. then the orbit of $x_0 \to 0$.*

$$0.1, 0.01, 0.0001, 0.000000001, \ldots$$

i.e. for any x_0 with $0 \leq x_0 < 1$ leads to an orbit far from 1 and close to 0. Similarly, orbit of 0.9 is

$$0.9, 0.81, 0.6561, \ldots, 0.0017, \ldots$$

We call 0 as attracting and 1 as repelling fixed point for the above function F.

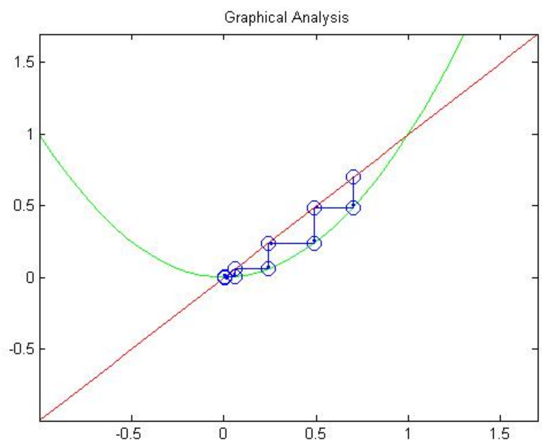


Figure 6: $F(x) = x^2$.

Definition 6. *Let x_0 be a fixed point for a function F. Then x_0 is an* ***attracting fixed point*** *if $|F'(x_0)| < 1$.*

Definition 7. *The above point x_0 is a* ***repelling fixed point****, if $|F'(x_0)| > 1$*

Definition 8. *The above point x_0 is a* ***neutral fixed point****, if $|F'(x_0)| = 1$.*

Example 12. *Take $F(x) = 2x(1 - x)$.*
Here 0 and 1/2 are fixed point for F.
$F'(x) = 2 - 4x, F'(0) = 2, F'(1/2) = 0.$
$\therefore 0$ *is a repelling fixed point. and 1/2 is attracting fixed point.*

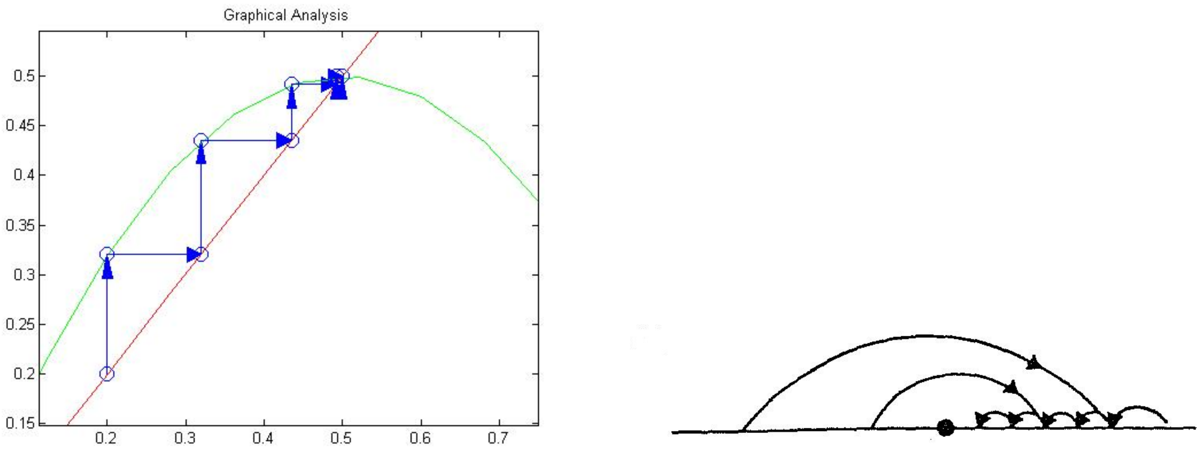


Figure 7: Graphical Analysis and Phase Portrait of $F(x) = 2x(1 - x)$

• One Major difference between attracting and repelling fixed point is that attracting point are visible on the computer where repelling fixed point generally are not.

Theorem 3. ***Attracting Fixed Point Theorem*** *Suppose x_0 is an attracting fixed point for F. Then there exists an interval I that contains x_0 in its interior and in which the following condition is satisfied. If $x \in I$ then $F^n(x) \in I$ for all n and moreover $F^n(x) \to x_0$ as $n \to \infty$*

Theorem 4. ***Repelling Fixed Point Theorem*** *Let x_0 is a repelling fixed point for F.Then there exists an interval I that contains x_0 in its interior and in which the following condition hold if $x \in I$ and $x \neq x_0$ then there exists an integer $n > 0$ s.t. $F^n(x) \notin I$*

4.2 Periodic Points

Definition 9. *A* ***periodic point*** *of $F : \mathbb{R} \to \mathbb{R}$ of period n is said to be attracting (repelling), if it is an attracting (repelling) fixed point for F^n.*

• **How to calculate $(F^n)'$ along a cycle ?**
Suppose $x_0, x_1, \ldots x_{n-1}$ lie on a cycle of period n for F with $x_i = F^i(x_0)$.
Then $(F^n)'(x_0) = F'(x_{n-1}) \ldots F'(x_1)F'(x_0)$

5 Bifurcations

Here onwards we study the dynamics of quadratic family $Q_c(x) = x^2 + c$; c is a constant.

5.1 Dynamics of Quadratics Map

Fixed Points of Q_c :

There are at most two fixed points for every $Q_c(x) = x^2 + c$ given by
$x = \frac{1 \pm \sqrt{(1-4c)}}{2} = p_{\pm}$
where p_+ and p_- are real iff $1 - 4c \geq 0 \Rightarrow c \leq 1/4$
• when $c > 1/4$, Q_c has no fixed point.
• when $c = 1/4$, Q_c has fixed point $p_+ = p_- = 1/2$.
• when $c < 1/4$, $1 - 4c > 0$ so fixed points p_+ and p_- are real and distinct and $p_+ > p_-$

Few Observation:

For the one parameter quadratic family, we have $Q'_c(x) = 2x$
Therefore $Q'_c(p_+) = 1 + \sqrt{(1-4c)}$. and $Q'_c(p_-) = 1 - \sqrt{(1-4c)}$.
So we have, $Q'_c(p_+) = 1$ when $c = 1/4$.
$Q'_c(p_+) > 1$ when $c < 1/4$
$Q'_c(p_-) = 1$ when $c = 1/4$
$Q'_c(p_-) < 1$ when $c < 1/4$
• p_+ is neutral fixed point when $c = 1/4$ and repelling when $c < 1/4$.
• so p_- is neutral fixed point when $c = 1/4$ and attracting when $c < 1/4$.
In particular ,
$|Q'_c(p_-)| < 1$
$\Rightarrow -1 < Q'_c(p_-) < 1$
$\Rightarrow -1 < 1 - \sqrt{(1-4c)} < 1$
$\Rightarrow 4 > 1 - 4c > 0$
$\Rightarrow -3/4 < c < 1/4$
i.e. p_- is an attracting fixed point for Q_c when $-3/4 < c < 1/4$.
$Q'_c(p_-) = 1 - \sqrt{(1+3)} = -1$ when $c = -3/4$.
$\therefore p_-$ is neutral.
$Q'_c(p_-) < -1$ when $c < -3/4$
$\therefore p_-$ is repelling.

Proposition 1. ***The First Bifurcation for the family*** $Q_c(x) = x^2 + c$
1. All orbits tend to infinity if $c > 1/4$.
2. When $c = 1/4$, Q_c *has a single fixed point at* $p_+ = p_- = 1/2$ *i.e. neutral.*
3. For $c < 1/4$, Q_c *has two fixed points at* p_+ *and* p_-. *The fixed point* p_+ *is always repelling and for* p_- *we have three cases:*
a. If $-3/4 < c < 1/4$, p_- *attracting.*
b. If $c = -3/4$, p_- *neutral.*
c. If $c < -3/4$, p_- *repelling.*

- Note that $Q_c(-p_+) = p_+$, So $-p_+$ is an eventually fixed point.

What happens as c decreases below $-3/4$?

Let us check if there exists any cycles of period 2.We obtain the cycles of period 2 by solving the equation : $Q_c^2(x) = x$.
i.e. $x^4 + 2cx^2 - x + c^2 + c = 0$.
We already know two solution namely p_-andp_+
$\therefore (x - p_+)(x - p_-)$ is a factor.
$\Rightarrow \frac{x^4+2cx^2-x+c^2+c}{x^2+c-x} = x^2 + x + c + 1$.
Then the solutions of $x^2 + x + c + 1 = 0$ give fixed points of Q_c^2 that are periodic points of period 2 for Q_c. These roots are : $q_\pm = \frac{-1 \pm \sqrt{(-4c-3)}}{2}$
These are real $\iff -4c - 3 \geq 0$.
$\Rightarrow c \leq -3/4$.

Proposition 2. ***The Second Bifurcation for the family*** $Q_c(x) = x^2 + c$:
1. For $-3/4 < c < 1/4$, Q_c has an attracting fixed point at p_- and no 2-cycles.
2. For $c = -3/4$, Q_c has a neutral fixed point at $p_- = q_\pm = 1/2$and no 2-cycles.
3. For $-5/4 < c < -3/4$, Q_c has repelling fixed points at p_- and an attracting 2-cycle at $q_\pm$.

5.2 The Saddle-Node Bifurcation

Definition 10. *A one-parameter family of functions F_λ undergoes a saddle- node (or tangent) bifurcation at the parameter value λ_0 if there is an open interval I and an $\epsilon > 0$ such that:*
1. For $\lambda_0 - \epsilon < \lambda < \lambda_0$, F_λ has no fixed points in the interval I.
2. For $\lambda = \lambda_0$, F_λ has one fixed point in I and this fixed point is neutral.
3. For $\lambda_0 < \lambda < \lambda_0 + \epsilon$, F_λ has two fixed points in I, one attracting and one repelling.

Remarks: Bifurcation theory is a **local** theory in that we are only concerned about changes in the periodic point structure near the parameter value λ_0. That is the reason for the ϵ in the definition.

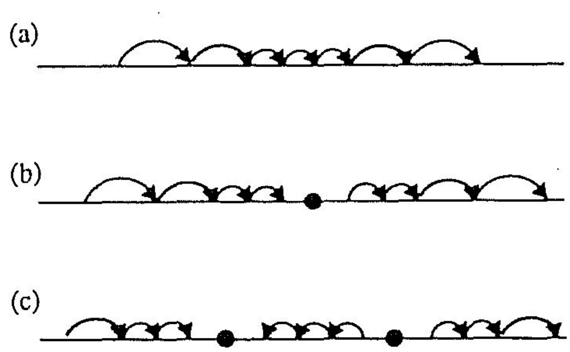


Figure 8: Phase portraits for (a) $\lambda < \lambda_0$, (b) $\lambda = \lambda_0$, and (c) $\lambda > \lambda_0$.

Example 13. *The quadratic family $Q_c(x) = x^2 + c$ has a saddle-node bifurcation at $c = 1/4$, as discussed earlier.*

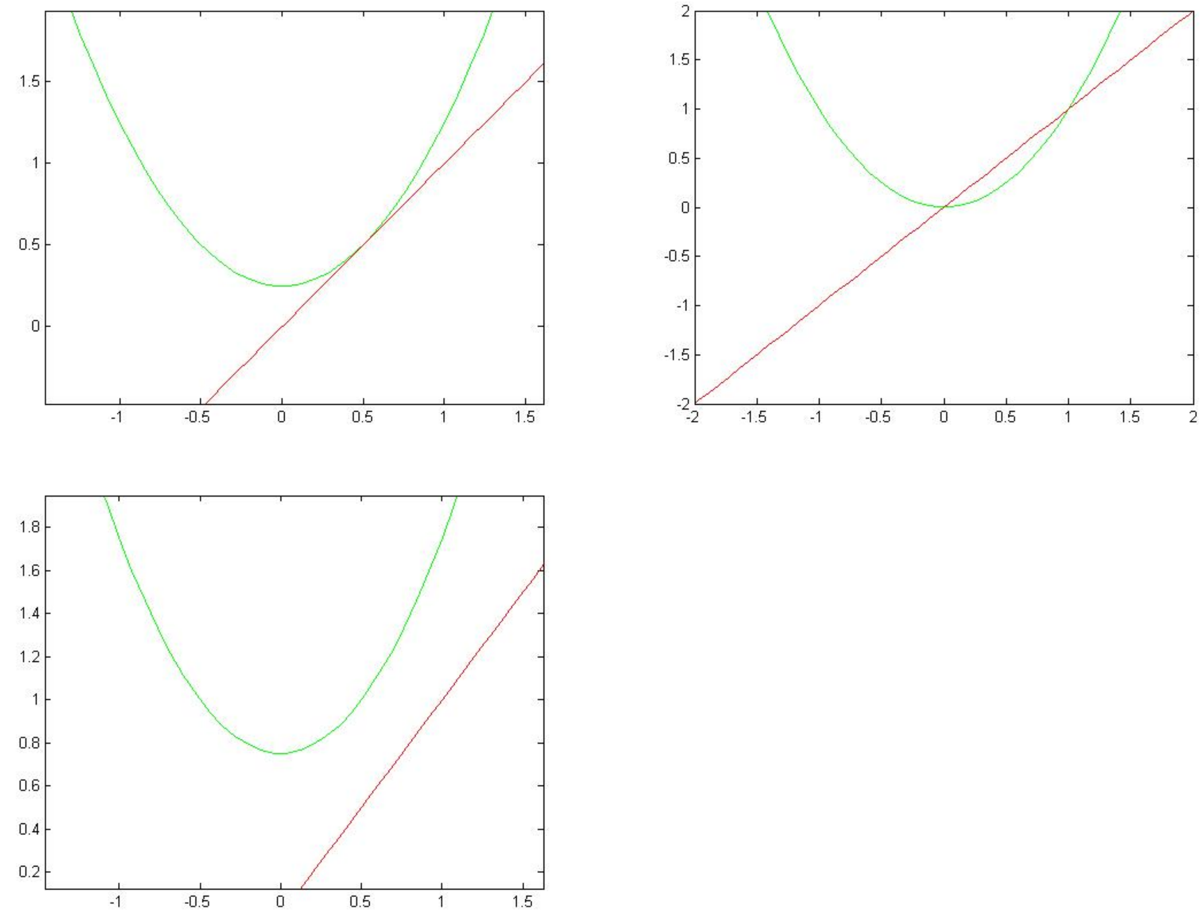
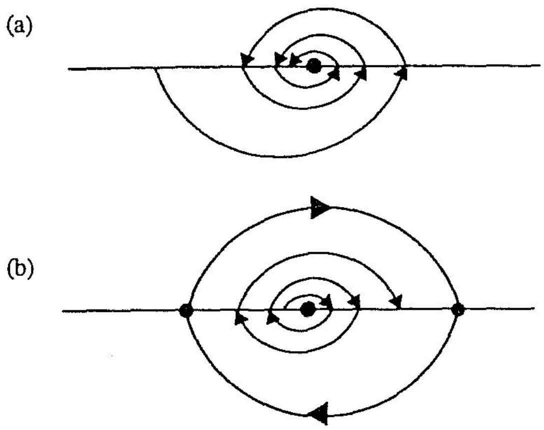


Figure 9: (a) Graph of $Q_{1/4}(x)$, (b) Graph of $Q_0(x)$, and (c) Graph of $Q_{3/4}$.

5.3 The Period-Doubling Bifurcation

Definition 11. *A one-parameter family of functions F_λ undergoes a period- doubling bifurcation at the parameter value $\lambda = \lambda_0$ if there is an open interval I and an $\epsilon > 0$ such that:*
1. For each λ in the interval $[\lambda_0 - \epsilon, \lambda_0 + \epsilon]$, $\exists$ a unique fixed point p_λ for F_λ in I.
2. For $\lambda_0 - \epsilon < \lambda < \lambda_0$, F_λ has no cycles of period 2 in I and p_λ is attracting (resp. repelling).
3. For $\lambda_0 < \lambda < \lambda_0 + \epsilon$, there is a unique 2-cycle q^1_λ,q^2_λ in I with $F_\lambda(q^1_\lambda) = q^2_\lambda$.This 2-cycle is attracting (resp. repelling). Meanwhile,the fixed point p_λ is repelling (resp. attracting).
4. as $\lambda \to \lambda_0 \Rightarrow q^1_\lambda \to p_\lambda$.

Example 14. *For Quadratic family $Q_c(x) = x^2 + c$, we already described analytically the period-doubling bifurcation occurs at $c = -3/4$.*



(a) $c > -3/4$,(b) $c < -3/4$

Figure 10: Phase portraits of the period-doubling bifurcation for Q_c

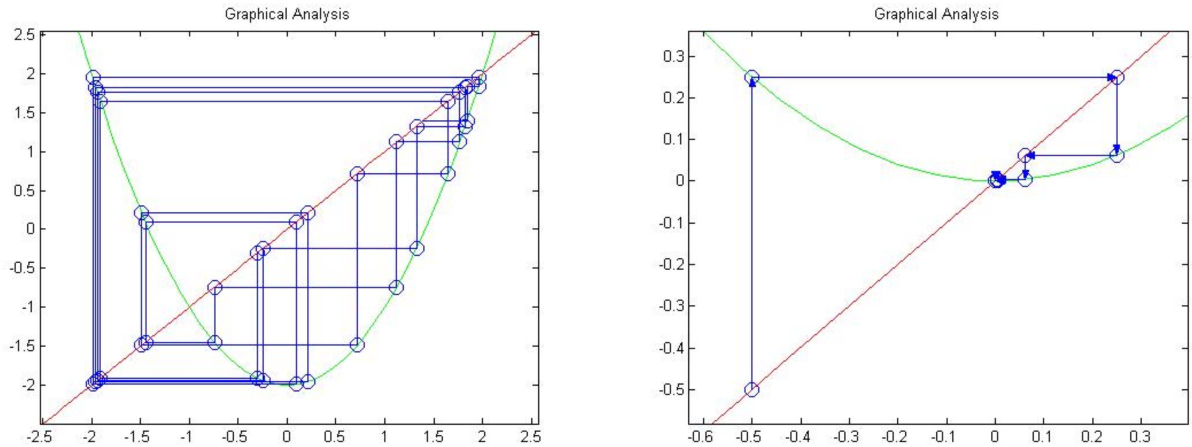


Figure 11: Graph of $Q_c(x)$ for (a) $c < -3/4$,(b) $c > -3/4$

6 The Quadratic Family $Q_c(x) = x^2 + c$

6.1 The Case c= -2

We know all of the interesting dynamics of Q_c take place in the interval $-p_+ < x < p_+$ where p_+ is the repelling fixed point with $p_+ > 0$. All other orbits of Q_c tend to infinity. Here, $c = -2 \Rightarrow p_+ = 2$. So, we concentrate on $[-2, 2] \equiv I$.

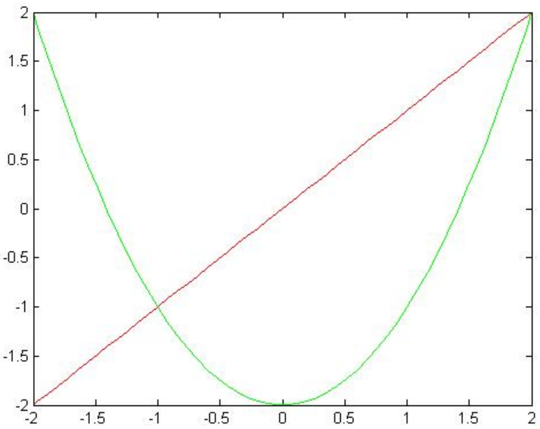


Figure 12: The graph of Q_{-2} on the interval $[-2, 2]$

Q_{-2} is increasing on [0,2] and takes this subinterval onto entire interval I in one to one fashion. Similarly Q_{-2} is decreasing on subinterval [-2,0] and also takes this interval onto I in one-to-one fashion. Thus every point (except -2) in I has exactly two preimages in I.

What happens in next iteration?

Let us approach graphically.

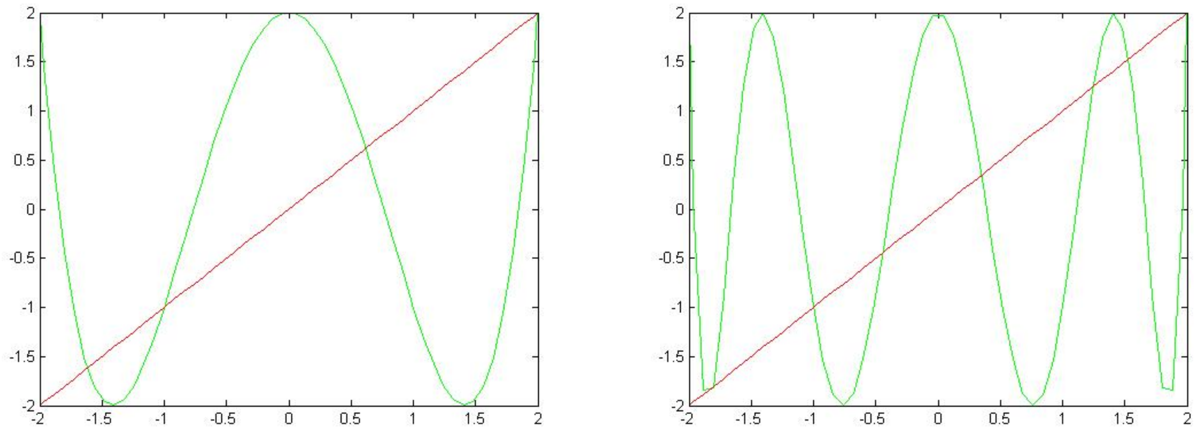


Figure 13: The graphs of higher iterates of Q_{-2} on $[-2, 2]$

Observation:

Continuing in this fashion, we see that the graph of Q^n_{-2} has 2^{n-1} valleys in the interval I. Hence this graph must cross the diagonal at least 2^n times over I.

Theorem 5. *The function Q_{-2} has at least 2^n periodic points of period n in the interval [-2,2].*

6.2 The Case c< -2

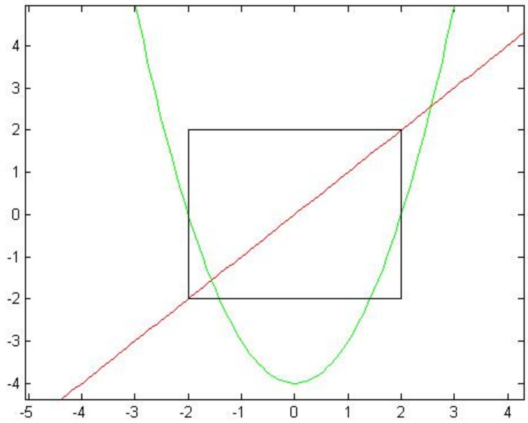


Figure 14: The graph of Q_c for $c = -4$

Observation:

There exists an open interval in I containing 0 that is mapped outside of I by Q_c. This interval is precisely the set of points lying above the portion of the graph outside of the box. Let A_1 is the set of points that escape from I after just one iteration of Q_c. Now any orbit that eventually leaves I must tend to infinity, so we understand the fate of all these orbits. Therefore it remains to understand the fate of orbits that never escape from I.

Define: The set $\Lambda = \{x \in I | Q_c^n(x) \in I \ \forall n\}$.

We see that there are a pair of open intervals that have the property that, if x is in one of these intervals, then $Q_c \in A_1$. Hence the orbit of x escapes from I after two iterations. We call this pair of intervals A_2. So if $x \in A_2 \Rightarrow Q_c^n(x) \to \infty$ and $x \notin \Lambda$.
Let A_n denote the set of points in I whose orbit leaves I after exactly n iterations. The set A_n consists of exactly $2n - 1$ open intervals. If a point has an orbit that eventually escapes from I, then this point must lie in A_n for some n. Hence $\Lambda^{\complement} = \cup_n A_n$

Theorem 6. *Suppose $c < -2$. Then the set of points Λ, whose orbits under Q_c do not tend to infinity is a non empty closed set in I that contains no intervals.*

Proof. We will prove this theorem imposing some condition on c. Let us assume that $c < -\frac{5+2\sqrt{5}}{4} = -2.368$.

Lemma 1. *There exists a constant $\mu > 1$ such that $|Q_c'(x)| \geq \mu > 1 \ \forall x \in I - A_1$.*

Proof. We have $|Q_c(x)| = |2x| > \mu > 1$ if $|x| > 1/2\mu \ \forall x \in I - A_1$.
So we need to arrange that A_1 contains the interval $[-1/2, 1/2]$ in its interior. i.e. we need to be sure if $|Q_c(1/2)| < -p_+$.
On solving, $1/4 + c = -\frac{1+\sqrt{(1-4c)}}{2}$
$c = \frac{5+2\sqrt{5}}{2}$ as the lower root.
Therefore the lemma holds, $Q_c(1/2) < -p_+$ □

Claim 1: Λ is non empty.
The fixed points at $p_\pm \in \Lambda$.

Claim 2: Λ contains no interval.
Now suppose that Λ contains an interval J. Let length of J is $k > 0$.
Now we already have ,$|Q_c'(x)| > \mu \ \forall x \in J$. Take any two points x and y in J, the Mean Value Theorem shows that

$$|Q_c(x) - Q_c(y)| > \mu|x - y| = \mu k$$

$\because J \in \Lambda$ we also have $Q_c(J) \in \Lambda$. Hence we may apply the same argument to the interval $Q_c(J)$ and then $|Q_c^2(J)| > \mu|Q_c(J)| > \mu^2 k$. Continuing this way we have, $|Q_c^n(J)| > \mu^n k$. But $\mu > 1$. So,

$$|Q_c^n(J)| \to \infty \text{ as } n \to \infty$$

the length of $Q_c^n(J)$ becomes arbitrarily large. However, the interval $Q_c^n(J)$ must lie within I, which has finite length. A contradiction arises. Therefore, the result established.
Claim 3: Λ is a closed subset of I.
$\Lambda^{\complement} = \cup A_n \cup (-\infty, -p_+) \cup (p_+, \infty)$ the union of a collection of open intervals is open, it follows that the complement of Λ is an open set and so Λ is closed. □

Remark: This theorem is valid for all $c < -2$, but the cases that are not so difficult to prove are considered.

7 Transition into Chaos

Again we consider the quadratic family $Q_c(x) = x^2 + c$.

7.1 The Orbit Diagram

This is one of the most instructive and complicated images in all the dynamical systems. The orbit diagram is an attempt to capture the dynamics of Q_c for many different c values in one picture.
• **How one draw the orbit diagram?**
We plot the parameter c on the horizontal axis versus the asymptotic orbit of 0 under Q_c on the vertical axis.
•**What is asymptotic orbit ?**
By this we simply mean that we do not plot the first few iterations (usually 100 or so) of 0. This allows the orbit to *settle down* and reach its *eventual behavior*. Thus one can eliminated the "transient behavior" of the orbit by not plotting the first few iterations.
• **Now we construct the Orbit Diagram of $Q_c(x) = x^2 + c$.**
For the orbit diagram of Q_c we initially choose the parameter c in the range $-2 < c < 1/4$. We plot the asymptotic orbit of 0 under Q_c on the vertical line over c. We know, the orbit of 0 remains in the interval $[-2, 2]$ for these c-values, so the coordinates on the vertical axis run from -2 to 2.
• **Why we choose the orbit of 0 to plot?**

Definition 12. *Suppose $F : \mathbb{R} \to \mathbb{R}$. A point x_0 is a critical point of F if $F'(x_0) = 0$. The critical point x_0 is nondegenerate if $F''(x_0) \neq 0$. Otherwise, the critical point is degenerate.*

Here, '0' is the only critical point for Q_c and is non degenerate.

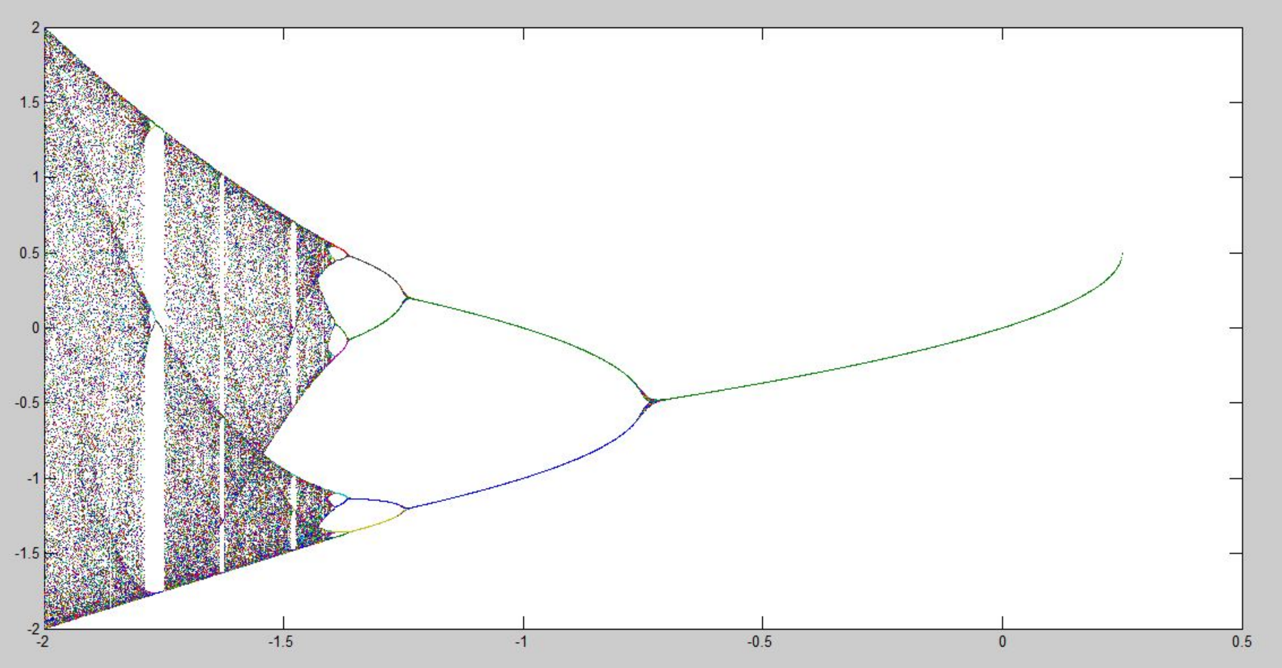


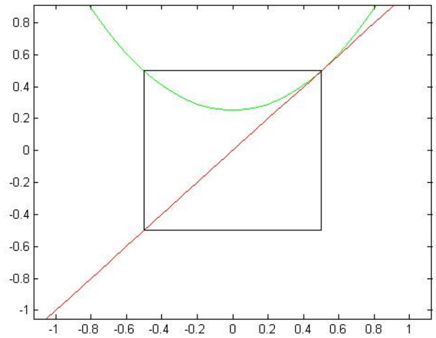
Figure 15: Orbit Diagram of $Q_c(x) = x^2 + c$

Observation 1: *As c decreases, we seem to see a succession of period doubling bifurcations. It seems that periodic points first appear in the order* $1, 2, 4, 8, \ldots, 2^n, \ldots$
Observation 2: There is a region that is not completely white but rather containing three tiny black regions at the top, middle, and bottom. This region is named as *period 3 window*. This window containing an attracting 3-cycle, that undergo a sequence of period-doubling bifurcations. So, in general we can say :
In each period-n window, there exists an attracting n-cycle followed by a succession of period-doubling bifurcations.
Observation 3: *The orbit diagram appears to be self-similar i.e. if one magnify certain portions of the picture, the resulting image resembles to the original figure.*
Observation 4: *There exits is at most one attracting cycle for each* Q_c*.*
Observation 5: *There exists a large set of c-values for which the orbit of* 0 *is not attracted to an attracting cycle.*

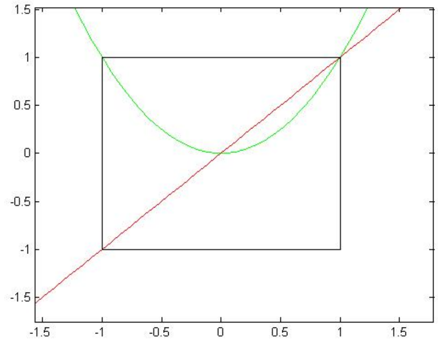
7.2 Graphical Views of Some Dynamical Properties

Now plot the graphs of Q_c for six different c-values. In each case we have superimposed a *square on the graph with vertices at* (p_+, p_+) *and* $(-p_+, -p_+)$; where $p_+ = \frac{1+\sqrt{(1-4c)}}{2}$. Here six different c-values exhibit six dynamical bhaviour of Q_c. They are :

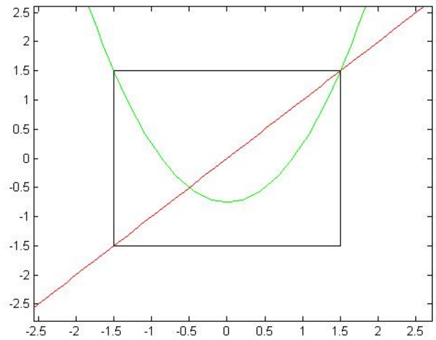
1. Saddle-node bifurcation point.
2. Critical point is fixed.
3. Period-doubling bifurcation point.
4. Critical point has period 2.
5. Chaotic c-value.
6. Cantor set case.



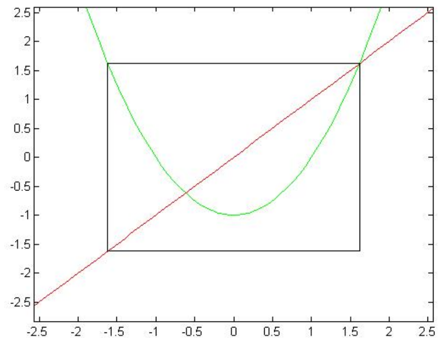
(a) Saddle-node bifurcation when c=1/4



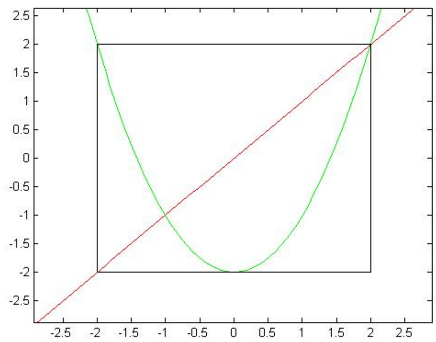
(b) Critical point is fixed when c=0



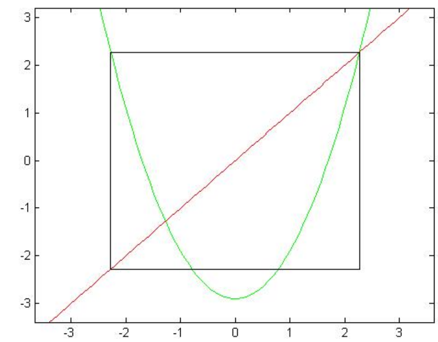
(c) Period-doubling bifurcation when c=−3/4



(d) Critical point has period 2 when c=−1



(e) Chaotic c-value when c=−2



(f) Cantor set case when c=−2.9

8 Symbolic Dynamics for $Q_c(x) = x^2 + c$

It is most powerful tools for understanding the chaotic behavior of dynamical system. Here, we are introducing a space of sequence and a mapping on ths space that will later serve as a model for the quadratic maps.

8.1 Itineraries

Again we consider the quadratic map $Q_c(x) = x^2 + c$.

Definition 13. *Let $x \in \Lambda$. The itinerary of x is the infinite sequence of 0's and 1's given by*

$$S(x) = (s_0, s_1, s_2, \dots)$$

where

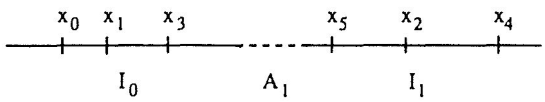
$$s_j = \begin{cases} 0 & \text{if } Q_c^j(x) \in I_0 \\ 1 & \text{if } Q_c^j(x) \in I_1 \end{cases} \tag{1}$$

Example 15. *Take* $p_+ \in \Lambda \Rightarrow S(p_+) = (1, 1, 1, \dots)$

Example 16. *Take* $p_- \in \Lambda \Rightarrow S(p_-) = (0, 1, 1, 1, \dots)$

Example 17.

Suppose $x_0 \in \Lambda$ with the following diagram



Then the itinerary of x_0 is $S(x_0) = (0, 0, 1, 0, 1, 1, \dots)$.

8.2 Sequence Space

Definition 14. *The sequence space on two symbols is the set*

$$\sum = \{(s_0, s_1, s_2, \dots)\ s.t.\ s_j = 0\ or\ 1\}$$

i.e. The set $\sum$ *consists of all possible sequences of* 0*'s and* 1*'s.*

Remark: The elements of the $\sum$ are not numbers, they are sequences .

Example 18. $(0, 0, 0, \dots)$,$(0, 1, 1, \dots)$,$(1, 1, 1, \dots) \in \sum$

Now, we need a way to measure distances between two elements or points in the set. Mathematically, this means we need a metric on the set.

Let's define it:

Definition 15. *Let* $s = (s_0, s_1, s_2, \dots)$ *and* $t = (t_0, t_1, t_2, \dots)$ *be two points in* $\sum$*. The distance between s and t is given by*

$$d[s, t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

Example 19. *Take* $s = (0, 0, 0, \dots)$ *and* $t = (1, 1, 1, \dots)$
$\Rightarrow$

$$\begin{aligned} d[s, t] &= \sum_{i=0}^{\infty} (1/2)^i \\ &= \frac{1}{1 - 1/2} \\ &= 2 \end{aligned}$$

Remark: Here the series defined $d[s, t]$ is always convergent. Indeed since s_i and t_i is either $0\ or\ 1 \Rightarrow |s_i - t_i| = 0\ or\ 1$ So,the series $d[s, t]$ always dominated by 2.

Proposition 3. *The distance d on $\sum$ given by*

$$d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

is a metric on $\sum$.

Proof. Let $s = (s_0, s_1, s_2, \dots), t = (t_0, t_1, t_2, \dots)$ *and* $u = (u_0, u_1, u_2, \dots) \in \sum$.

(a) Clearly, $d[s,t] = 0$.

(b) Clearly, $d[s,t] = 0 \Leftrightarrow s = t$

(c) since,$|s_i - t_i| = |t_i - s_i| \Rightarrow d[s,t] = d[t,s]$.

(d) for any three real numbers s_i, t_i, u_i , We have the triangular inequality

$$|s_i - t_i| + |t_i - u_i| \geq |s_i - u_i|$$

and $d[s,t]$ is always convergent and bounded by $2 \Rightarrow d[s,t] + d[t,u] \geq d[s,u]$.

□

Theorem 7. ***The Proximity Theorem*** *Let $s, t \in \sum$ and suppose $s_i = t_i$ for $i = 0, 1, \dots, n$. then $d[s,t] \leq \frac{1}{2^n}$. Conversely, if $d[s,t] < \frac{1}{2^n}$. then $s_i = t_i$ for $i \leq n$.*

Proof. if $s_i = t_i$ for $i \leq n$.

$$\begin{aligned} d[s,t] &= \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \\ &\leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} \\ &= \frac{1}{2^n}. \end{aligned}$$

Conversely, if $s_i \neq t_j$ for some $j \leq n$.Then

$$d[s,t] \geq \frac{1}{2^j} \geq \frac{1}{2^n}.$$

Hence , if $d[s,t] < \frac{1}{2^n} \Rightarrow s_i = t_i$. for $i \leq n$. □

8.3 The Shift Map 'σ'

The shift map is a map on $\sum$.

Definition 16. *The shift map $\sigma : \sum \to \sum$ is given by*

$$\sigma(s_0, s_1, s_2, s_3 \dots) = (s_1, s_2, s_3, \dots)$$

- **What is the benifit of σ map ?**

It is easy to find the periodic points of σ of period 'n'. Take s=$(\overline{s_0, s_1, \dots, s_{n-1}}) \Rightarrow \sigma^n(s) = s$.
- It is different for usual polynomials and trigonometric, exponential function.

8.4 Some Properties of 'σ' function

8.4.1 Continuity of σ map

Definition 17. ***Continuity*** *Let (X, d) be a metric space. $F : X \to X$. Then F is said to be continuous at $x_0 \in X$ if*

$$\forall \epsilon > 0 \; \exists \, a \, \delta > 0 \; s.t. \; d[x, x_0] < \delta \Rightarrow d[F(x), F(x_0)] < \epsilon$$

F is said to be continuous on X if it is continuous at all $x_0 \in X$

(a) σ **is continous at fixed point** $(0, 0, 0, \dots)$

Proof. Let $\epsilon > 0$ be given.We have to find a $\delta > 0$ s.t

$$d[s, (0, 0, 0, \dots)] < \delta \Rightarrow d[\sigma(s), (0, 0, 0, \dots)] < \epsilon$$

Choose $n > 0$ s.t. $1/2^n < \epsilon$.
Now By Proximity Th. We have if

$$s_i = 0 \, \forall \, i = 0, \dots, n. \Rightarrow d[s, (0, 0, 0, \dots)] \leq 1/2^n$$

Then choose $\delta = 1/2^{n+1}$. Then

$$d[s, (0, 0, 0, \dots)] < \delta \Rightarrow d[\sigma(s), (0, 0, 0, \dots)] = d[(s_1, s_2, s_3, \dots), (0, 0, 0, \dots)] \leq 1/2^n < \epsilon$$

□

(b) σ **is continuous at all points** $\in \sum$

Proof. Let $\epsilon > 0$ and $s = (s_0, s_1, s_2, \dots)$ be given.We have to prove σ is continuous at s.
Choose $n > 0$ s.t. $1/2^n < \epsilon$.Then Choose $\delta = 1/2^{n+1}$.
If $t \in \sum$ s.t. $d[s, t] < \delta \Rightarrow s_i = t_i$ for $i = 0, \dots, n + 1$.
Therefore $d[\sigma(s), \sigma(t)] \leq 1/2^n < \epsilon$.
Therefore σ is continuous at s. Since, s is choosen arbitraily we have σ is continuous on $\sum$. □

8.4.2 Conjugacy

Definition 18. *Let $F : X \to X$ and $G : Y \to Y$ be two functions . We call F and G are conjugate if $\exists$ a homeomorpism $h : X \to Y$ s.t.*

$$h \circ F = G \circ h$$

The map 'h' is called ***Conjugacy****.*

Theorem 8. *If $x \in \Lambda$, then $S \circ Q_c(x) = \sigma \circ S(x)$.*

Proof. if $x \in \Lambda$ has itinerary $S(x) = (s_0, s_1, s_2, \dots)$ then

$x \in I_{s_0}$, $Q_c(x) \in I_{s_1}$, $Q_c^2(x) \in I_{s_2}$ and so on. Where I_{s_j}= either I_1 or I_0.

Therefore $S(Q_c(x)) = (s_1, s_2, s_3, \dots) = \sigma(S(x))$. □

Therefore We have a commutative diagram:

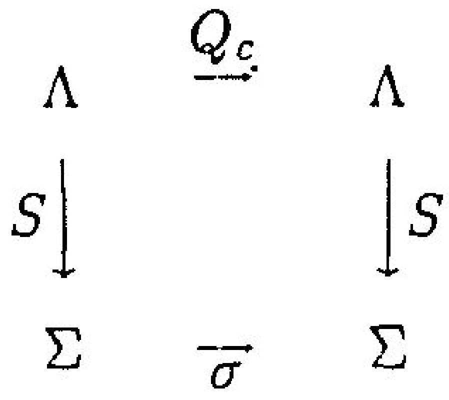


Figure 17: Commutative diagram

•What we got from above Theorem?
S converts orbits of Q_c to orbits of σ.

Theorem 9. ***The Conjugacy Theorem*** *The shift map σ on $\sum$ is conjugate to Q_c on Λ when $c < \frac{-(5+2\sqrt{5})}{4} = -2.368$.*

Proof. We prove $S : \Lambda \to \sum$ homeomorphism. To prove this we need to show that S is one-to-one and onto and that both S and S^{-1} are continuous.

(a) **S is One-to-One**

Proof. Let $x, y \in \Lambda$ with $x \neq y$. s.t. $S(x) = S(y)$.
$\Rightarrow Q_c^n(x)$ and $Q_c^n(y)$ always lie in the same interval I_0 or I_1.Now Q_c is one-to-one on each of these interval and continuous. Now consider the interval $[x, y]$. Then for each n we have the following bijective correspondence

$$[x, y] \xrightarrow{Q_c^n} [Q_c^n(x), Q_c^n(y)]$$

Moreover, $|Q_c'(x)| > \mu > 1 \,\forall\, x \in I_0 \cup I_1$ and some μ.The by MVT we have

$$\text{length } [Q_c^n(x), Q_c^n(y)] \geq \mu^n \text{length } [x, y]$$

But $\mu > 1$ So as $n \to \infty \Rightarrow$ length $[Q_c^n(x), Q_c^n(y)] \to \infty$.
A contradiction occurs. Since $x, y \in \Lambda$ with $x \neq y$. Hence $x = y$. □

(b) **S is onto**

Proof. Let $J \in I$ be a closed interval. Define $Q_c^{-n}(J) = \{x \in I | Q_c^n(x) \in J\}$.
For $n = 1$ it denotes the preimage of J. Now,$Q_c^{-1}(J)$consists of two closed subintervals, one in I_0 and other in I_1

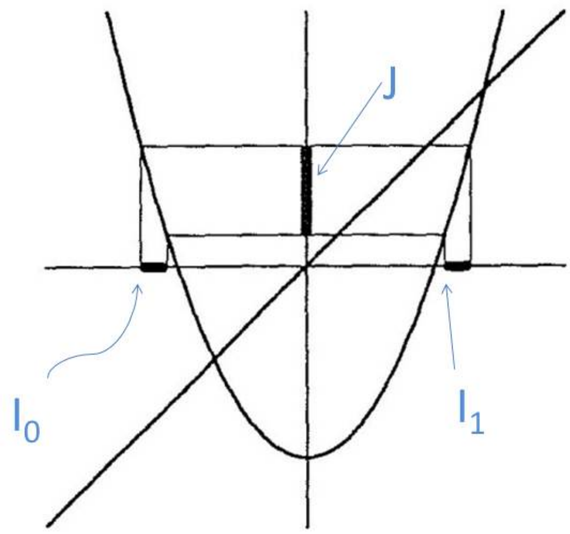


Figure 18: Graphical View Of $Q_c^{-1}(J)$

Claim: Given $s \in \sum$ We have to find $x \in \Lambda$ s.t. $S(x) = s$.
Define

$$\begin{aligned} I_{s_0,s_1,\ldots,s_n} &= \{x \in I | x \in I_{s_0}, Q_c(x) \in I_{s_1}, \ldots, Q_c^n(x) \in I_{s_n}\}; Where\, s_j = 0\, or\, 1. \\ \therefore I_{s_0,s_1,\ldots,s_n} &= I_{s_0} \cap Q_c^{-1}(I_{s_1}) \cap \ldots Q_c^{-n}(I_{s_n}) \\ &= I_{s_0} \cap Q_c^{-1}(I_{s_1,s_2,\ldots,s_n}) \end{aligned}$$

Claim: $I_{s_0,\ldots,s_n}$ are closed intervals that are nested.
Closed Prop. We use induction to prove it.
Clearly , I_{s_0} is a closed Interval.
Induction Hypothesis: $I_{s_1,\ldots,s_n}$ is closed interval. $Q_c^{-1}(I_{s_1,\ldots,s_n})$ consists of pair of closed intervals one in I_0 and one in I_1.
$\therefore I_{s_0} \cap Q_c^{-1}(I_{s_1,\ldots,s_n}) = I_{s_0,\ldots,s_n}$ is a single closed interval.
Nested Prop. These intervals are nested because

$$I_{s_0,\ldots,s_n} = I_{s_0,\ldots,s_{n-1}} \cap Q_c^{-n}(I_{s_n}) \in I_{s_0,\ldots,s_{n-1}}$$

Therefore , We conclude that $\cap_{n\geq 0} I_{s_0,\ldots,s_n}$ is non empty. (By Nested Interval Prop.)
if $x \in \cap_{n\geq 0} I_{s_0,\ldots,s_n}$. then $x \in I_{s_0}, Q_c(x) \in I_{s_1}$ and so forth.
$\Rightarrow S(x) = (s_0, s_1, \ldots)$
$\Rightarrow S$ is onto. □

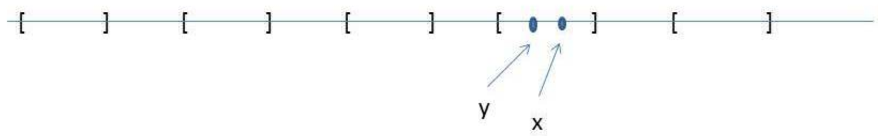
(c) **Continuity**

Proof. We will show continuity from the first principle
Now, $S : \Lambda \to \sum$. Let $x \in \Lambda$ and $S(x) = (s_0, s_1, s_2, \ldots)$.
Claim: S is continuous at x_0.

Let $\epsilon > 0$. Pick n so that $1/2^n < \epsilon$.
We already defined

$$I_{t_0,t_1,\ldots,t_n} = \{x \in I | x \in I_{t_0}, Q_c(x) \in I_{t_1}, \ldots, Q_c^n(x) \in I_{t_n}\}; Where\, t_j = 0\, or\, 1.$$
$$\Lambda = \{x \in I | Q_c^n(x) \in I, \forall n\}$$

Obs: These subintervals are disjoint and $\Lambda \subset \cup_n I_{t_0,t_1,\ldots,t_n}$. There are 2^{n+1} such subintervals and $I_{s_0,s_1,\ldots,s_n}$ is one of them.
Now, We can choose δ s.t. $|x - y| < \delta$ and $y \in \Lambda$
$\Rightarrow y \in I_{s_0,s_1,\ldots,s_n}$.
We choose δ so small such that interval of length 2δ centered at x overlaps only $I_{s_0,s_1,\ldots,s_n}$.
This is possible as shown in picture



So, the first $n + 1$ entries of $S(x)$ and $S(y)$ are same. By Proximity Theorem we have $d[S(x), S(y)] \leq 1/2^n < \epsilon$.
Therefore, S is a continuous at x.

□

Similarly, We can show S^{-1} is also continuous.
Hence, S is homeomorphism. □

9 Chaos in $Q_c(x) = x^2 + c$

Here we will introduce the notion chaos. We will show there are many dynamical systems hat are chaotic but we can study them with the help of mathematical model.

9.1 Notion of Dense Set

Definition 19. *Let X is a set and $Y \subset X$. We say Y is dense in X if $\overline{Y} = X$. i.e. for any $x \in X$ we can find a sequence of points $\{y_n\} \in Y$ that converges to x.*

Example 20. *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Example 21. *(a, b) is dense in $[a, b]$.*

9.2 Chaotic Systems

Definition 20. ***Chaotic System*** *Let X be a set. A continuous map $F : X \to X$ is said to be chaotic on X if*

(a) F is Topologically Transitive.

(b) The Periodic points of F are dense in X.

(c) F has sensitive dependence on initial conditions.

Let us give the notions of above terminologies.

Definition 21. ***Transitivity*** *Let (X, d) be a metric space. A Dynamical system $(X, F)$is said to be transitive if*

$$\forall\ x \text{ and } y \text{ and } \epsilon > 0\ \exists\ z \in B(x, \epsilon) \text{ and } k > 0 \text{ s.t. } F^k(z) \cap B(y, \epsilon) \neq \phi$$

More generally, Let X be a topological space. $F : X \to X$ is topologically transitive if for any pair of open non-empty sets $U, W \subset V\ \exists$ a $k > 0$ s.t. $F^k(U) \cap W \neq \phi$

Definition 22. ***Sensitive Dependence on Initial Conditions*** *Let (X, d) be a metric space. A Dynamical system (X, F) depends sensitively on initial conditions if there exists a $\beta > 0$ s.t. for any x and any $\epsilon > 0$ there exists a $y \in B(x, \epsilon)$ and a $k > 0$ s.t. $d|F^k(x) - F^k(y)| \geq \beta$*

Remarks:

(i) The Definition of sensitivity does not require that the orbit of y remain far from x $\forall$ iterations. We only need one point on the orbit to be far from corresponding iterate of x.

(ii) This is important in study of application of dynamical systems. Small errors in numerical computation may throw us from the intended orbit. As a consequence we may look at an orbit that eventually diverges from the true orbit.

Theorem 10. *The shift map $\sigma : \sum \to \sum$ is a chaotic dynamical system.*

Proof. (a) **Claim:** The subset of $\sum$ that consists of all periodic points of σ is a dense subset in$\sum$. Let $\epsilon > 0$ be given and Take $s = (s_0, s_1, \dots, s_n) \in \sum$ Choose $n \in \mathbb{Z}$ s.t. $1/2^n < \epsilon$ Let $t_n = (s_0, s_1, \dots, s_n, \overline{s_0, s_1, \dots, s_n})$. Now the first $(n + 1)$ entries of s and t_n are same. Then by Proximity Theorem

$$d[s, t_n] \leq 1/2^n < \epsilon$$

But t_n is a repeating sequence and so it is a periodic point of period $n+1$ for σ. Now we have a sequence of periodic points $\{t_n\} \to s$ as $n \to \infty$ So, we got our claim.

(b) **Claim:** σ is transitive.
We will show it by some other way. First we will show

$$\exists \textit{ a point} \in \sum \textit{ whose orbit under } \sigma \textit{ is dense in } \sum.$$

Consider the point $\hat{s} = (0\ 1\ 00\ 01\ 10\ 11\ 000\ 001 \dots)$.
i.e. $\hat{s}$ is the sequence which consists of all possible blocks of 0's and 1's of length 1, followed by all such blocks of length 2, then length 3, and so on.
Choose an arbitrary $s = (s_0, s - 1, \dots)$ and an $\epsilon > 0$.
Choose n s.t. $1/2^n < \epsilon$. Now, Far to the right in the expression for $\hat{s}$, $\exists$ block of length $n+1$ that consists of the digits $s_0, s_1, \dots, s_n$. Suppose the entry s_0 is at the k th place in the sequence. Then by Proximity Theorem

$$d[\sigma^k(\hat{s}), s] \leq 1/2^n < \epsilon$$

Clearly, a dynamical system that has a dense orbit is transitive, because the dense orbit comes arbitrarily close to all points.

(c) **Claim:** σ depends sensitively on initial conditions.
Take $\beta = 1$. for any $s \in \sum$ and $\epsilon > 0$. Choose n s.t. $1/2^n < \epsilon$.
Let $t \in \sum$ satisfy $d[s,t] < 1/2^n$ but since $t \neq s \Rightarrow \exists\ k > n\ s.t. s_k \neq t_k$. So we have $|s_k - t_k| = 1$.
Now consider $\sigma^k(s), \sigma^k(t)$. The initial entries of these sequence are different, So we have

$$d[\sigma^k(s), \sigma^k(t)] \geq \frac{|s_k - t_k|}{2^0} + \sum_{i=1}^{\infty} \frac{0}{2^i} = 1$$

Therefore, we proved the sensitivity for the shift map σ.

□

Remarks On Definition Of Chaos:

(i) In [1] It is proved that in a metric space, a system that has a dense set of periodic points and is transitive also depends sensitively on initial conditions. So, third condition follows from first two.

(ii) In [4] It is proved that if I be a interval, not necessarily finite and $F : I \to I$ is a continuous and topologically transitive map. Then the periodic points of F are dense in I and F has sensitive dependence on initial conditions.

i.e. In an interval topologically transitivity $\Rightarrow$ chaos.

Proposition 4. ***The Density Proposition*** *Let $f : X \to Y$ is a continuous map that is onto and suppose also that $D \subset X$ is a dense subset. Then $f(D)$ is dense in Y.*

Proof. Let O be a non empty open set in Y. Then $f^{-1}(O)$ is a non empty open set in X. Since, D is dense in X. $f^{-1}(O)$ contain some member of D then O contain some member of $f(D)$. So every non empty open set in Y contains at least one memeber of $f(D)$.

$$\overline{f(D)} = Y.$$

□

Theorem 11. *Suppose $c < -\frac{(5+2\sqrt{5})}{4}$. Then the Quadratic map $Q_c(x) = x^2 + c$ is chaotic on the set Λ.*

Proof. Under this condition on c the itinerary map $S : \Lambda \to \sum$ is a conjugacy. Then $S^{-1} : \sum \to \Lambda$ is a homeomorphism.

(a) Now, S^{-1} carries periodic points for σ to periodic point for Q_c.
Therefore The Density Proposition guarantees that the set of periodic points for Q_c is dense in Λ.

(b) if $\hat{s}$ lies on a dense orbit for σ. Then The Density Proposition says that the $S^{-1}(\hat{s})$ lies on a dense orbit for Q_c.

(c) **Claim:** Q_c depends sensitively on initial conditions.
Now $\Lambda \subset I_0 \cup I_1$.
Choose $\beta < d(I_0, I_1)$. Now we will show

Any two Q_c orbits eventually separate by at least β.

Let x and $y \in \Lambda$ and $x \neq y$.
Since S is a homeomorphism $\Rightarrow S(x) \neq S(y)$. Therefore these two sequence differ at some entry say, k^{th} .

$$\therefore Q^k(x) \; and \; Q^k(y) \text{ lie in differnet } I_j.$$
$$\therefore |Q^k(x) - xQ^k(y)| > \beta.$$
$$\Rightarrow Q \text{ is transitive.}$$

Hence, Q_c is chaotic on the set Λ.

□

In a brief We can say chaotic map possesses three ingredients:

(a) Unpredictability.

(b) Indecomposable.

(c) An element Of Regularity.

9.3 Other Chaotic Systems

Definition 23. ***Semi Conjugacy*** *Let $F : X \to X$ and $G : Y \to Y$ are two dynamical systems. A mapping $h : X \to Y$ is called a semi conjugacy if h is continuous, onto, at most n to one and satisfies $h \circ F = G \circ h$.*

Theorem 12. *The function $Q_{-2}(x) = x^2 - 2$ is chaotic on the entire interval[-2,2]*

Proof. Instead of dealing with $x^2 - 2$. We will consider a simple system equivalent to $x^2 - 2$. Let $V(x) = 2|x| - 2$.

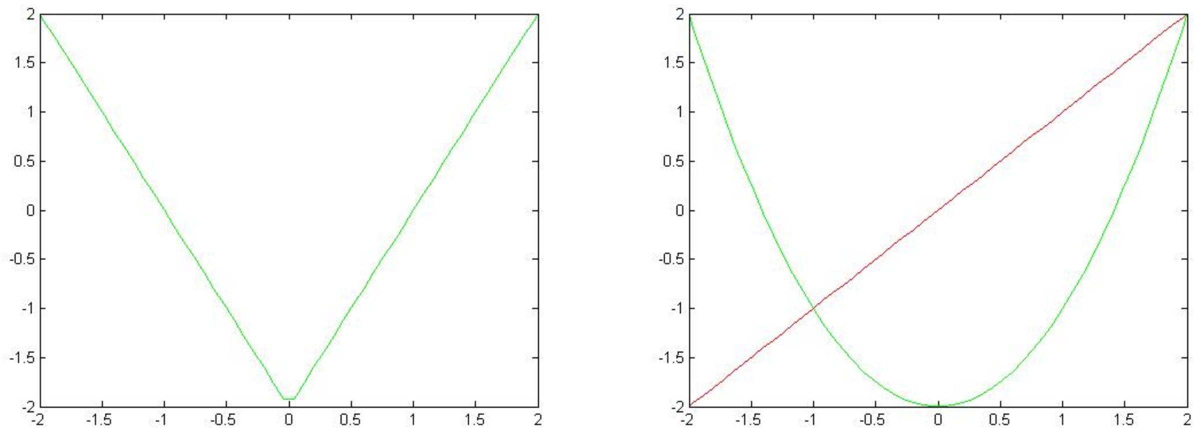


Figure 19: The Graph of $V(x) = 2|x| - 2$ and $Q_{-2}(x) = x^2 - 2$

The graph takes $[-2, 2]$ to itself as exactly $x^2 - 2$ does.
if$|x| > 2$ then the orbit of x under V tends to infinity as same happen for $x^2 - 2$. Now,

$$V^2(x) = \begin{cases} 4x - 6 & \text{if } x \geq 1 \\ -4x - 6 & \text{if } x \leq 1 \\ -4x + 2 & \text{if } 0 \leq x \leq 1 \\ 4x + 2 & \text{if } -1 \leq x \leq 0 \end{cases}$$

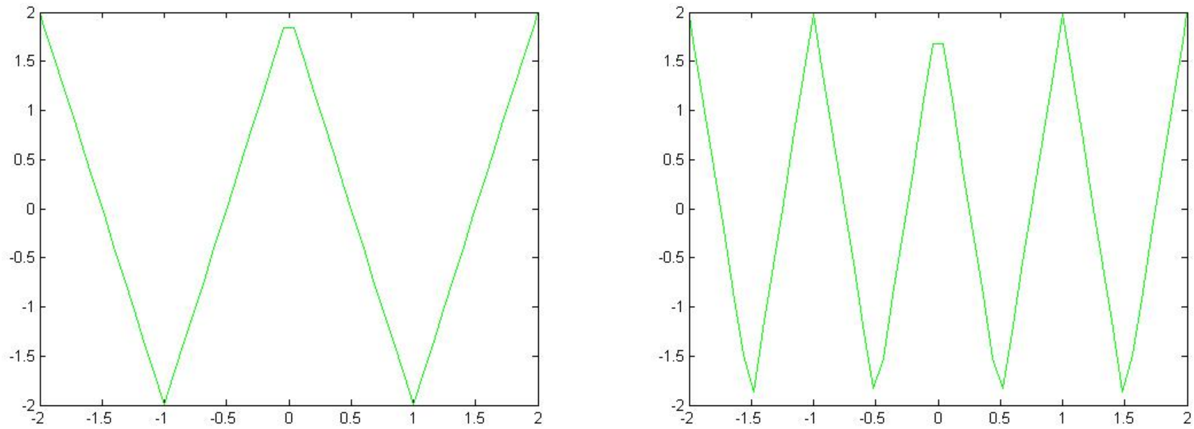


Figure 20: The Graph of V^2 and V^3.

From graph we have the followings
• V^n consists of 2^n pieces, each of which is a straight line with slope $\pm 2^n$.
• each of these linear portions of the graph is defined on an interval of length $1/2^{n-2}$.
Claim: V is chaotic on [-2,2].

(a) Consider an open subinterval J in $[-2, 2]$. Also, we find a subinterval of J of length $1/2^{n-2}$ on which the graph of v^n stretches from -2 to 2 and intersects $y = x$ graph. So it has a fixed point in J. So this implies that periodic points are dense in $[-2, 2]$

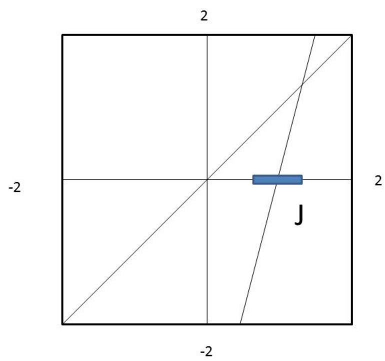


Figure 21: The graph of V^n stretches J over $[-2, 2]$.

(b) Image of J covers the entire interval $[-2, 2]$ so V is transitive.

(c) For any $x \in J$, there exists a $y \in J$ s.t. $|V^n(x) - V^n(y)| \geq 2$. Therefore choose $\beta = 2$. Therefore, we have sensitive dependence on initial conditions.

Therefore, V is chaotic on $[-2, 2]$.
Consider the function $C : [-2, 2] \twoheadrightarrow [-2, 2]$ defined as $C(x) = -2cos(\pi x/2)$. Every point in $[-2, 2]$ has exactly two preimages in $[-2, 2]$ with the exception of -2 which has only one.

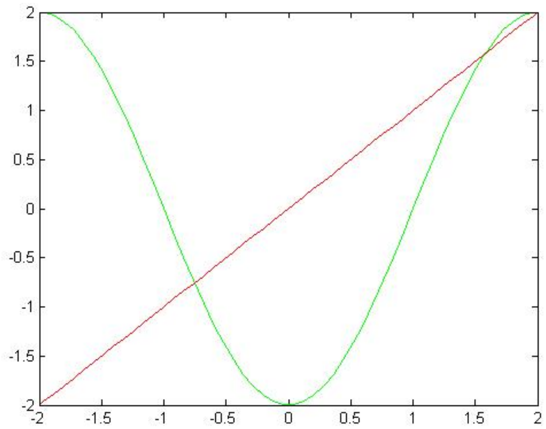


Figure 22: The graph of $C(x) = -2cos(\pi x/2)$.

Now, we will show C is a semi-conjugacy.
i.e. we have the following commutative diagram.

$$\begin{array}{ccc} x & \xrightarrow{V} & 2|x|-2 \\ C\big\downarrow & & \big\downarrow C \\ -2\cos(\pi x/2) & \xrightarrow[?]{} & -2\cos(\pi|x|-\pi) \end{array}$$

Now

$$\begin{aligned} -2cos(\pi|x|-\pi) =& 2cos(\pi x) \\ =& 2cos(\pi x) \\ =& 2cos(2\frac{\pi x}{2}) \\ =& (2cos(\frac{\pi x}{2}))^2 - 2 \\ =& (-2cos(\frac{\pi x}{2}))^2 - 2 \end{aligned}$$

Hence we have the following commutative diagram.

$$\begin{array}{ccc} [-2,2] & \xrightarrow{V} & [-2,2] \\ C\big\downarrow & & \big\downarrow C \\ [-2,2] & \xrightarrow[Q_{-2}]{} & [-2,2] \end{array}$$

Since, C is not one-one. It appears C is a semi conjugacy between V and Q_{-2}. But C is at most two to one. So it carries cycles to cycles.
Claim: Q_{-2} is chaotic.

(a) Since, C is both continuous and onto . Therefore by Density Proposition shows that Q_{-2} has periodic points that are dense as well as dense orbits.

(b) Finally, n may be chosen so that V^n maps arbitrarily small intervals onto all of $[-2, 2]$. The same must be true for Q_{-2}. This proves sensitive dependence of Q_{-2}.

Hence, We proved Q_{-2} is chaotic. □

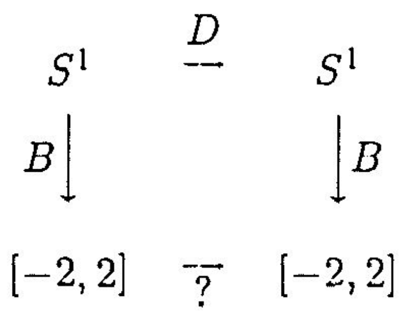
Let S^{-1} denote the unit circle in the plane. i.e.

$$S^1 = \{(x, y) \in \mathbb{R}^2 \ |x^2 + y^2 = 1\}$$

We describe a point on S^1 by giving its polar angle θ.
Define $D : S^1 \to S^1$,$D(\theta) = 2\theta$.
This D is called the doubling map on S^1.

Theorem 13. *Doubling map D is chaotic on the circle S^1.*

Proof. We will prove it by semi-conjugacy. We consider the map $B : S^1 \to [-2, 2]$ defined as $B(\theta) = 2cos\theta$. Consider the diagram



Again we may write

$$\begin{aligned} 2cos2\theta =& 2.2cos^2\theta - 2 \\ =& (2cos\theta)^2 - 2 \\ =& Q_{-2}(2cos\theta). \end{aligned}$$

Therefore B is a semi-conjugacy between D and Q_{-2}. Hence by similar argument as before we have D is chaotic. □

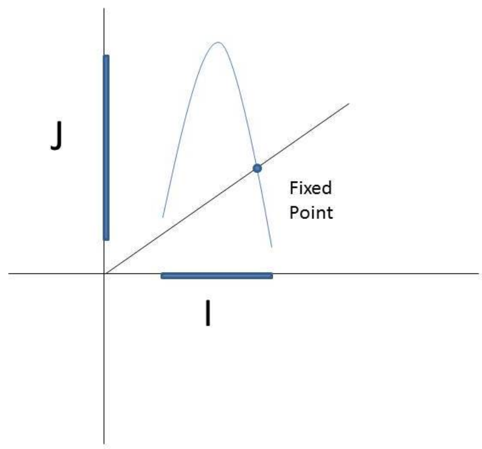
10 Sarkovskii's Theorem

10.1 Period 3 Implies Chaos

Observation 1:

Suppose $I = [a, b]$ and $J = [c, d]$ are closed intervals and $I \subset J$. If $F(I) \supset J$ then F has a fixed point I.

Proof. Since, $F(I) \supset J \supset I$. We can pick $x, y \in I \Rightarrow F(x) \leq a$ and $F(y) \geq b$.
Define, $G(x) = F(x) - x$



Four cases may occur.
Case 1: $F(x) \leq a$ and $F(y) \geq b$.
Here, we already have $a, b \in I$ s.t. $F(a) = a$ & $F(b) = b$.
That give two fixed points in I.
Case 2:$F(x) < a$ and $F(y) \geq b$
Here, we have $b \in I$ s.t. $F(b) = b$.
Case 3:$F(x) \leq a$ and $F(y) > b$
Here,we have $a \in I$ s.t. $F(a) = a$
Case 4:$F(x) < a$ and $F(y) > b$
Now,

$$\begin{aligned} & x > a \\ \Rightarrow & -x < -a \\ \Rightarrow & F(x) - x < F(x) - a < 0 \end{aligned}$$

Similarly , $F(y) - y > 0$.

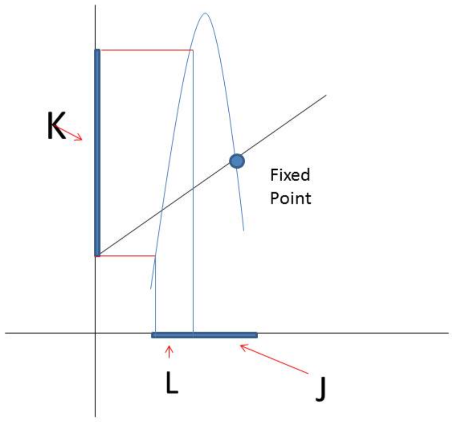
$$\begin{aligned} \text{i.e. } & G(x) < 0 \textit{ and } G(y) > 0 \\ & \exists\ z \in (x, y) \subset I \text{ s.t } G(z) = 0 \\ & \Rightarrow F(z) = z \end{aligned}$$

□

Observation 2:

Suppose J and K are two closed intervals and $f(J) \supset K$ then $\exists$ a closed subinterval $L \in J$ s.t. $f(L) = K$.

Proof. Let $K = [a, b]$; $c = sup\{x \in J | f(x) = a\}$
If $f(x) = b$ for some $x \in J$ with $x > c$.Let d be the largest among them. Then take $L = [c, d]$.



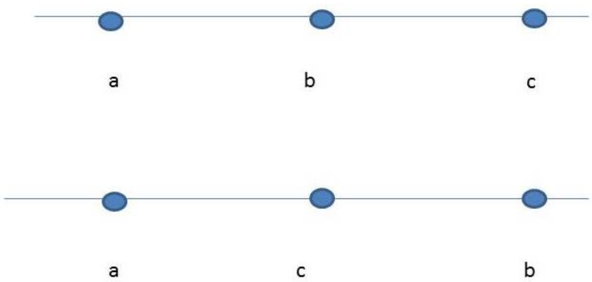
Otherwise , f(x)=b for some $x \in J$ with $x < c$. Let c' be the greatest and let $d' \leq c$ be the least $x \in J$ with $x > c'$ for which $f(x) = a$. Then take $L = [c', d']$. □

Theorem 14. ***The Period*** 3 ***Theorem*** *Let* $F : \mathbb{R} \to \mathbb{R}$ *is continuous. Suppose also that* F *has a periodic point of prime period* 3 *. Then* F *also has periodic points of all other periods.*

Proof. Let F has a 3 cycle given by

$$a \longmapsto b \longmapsto c \longmapsto a \longmapsto \dots$$

If we assume that a is the left most point of the orbit. then there are two possibilities for relative positions of the points on this orbit.



Take the First case :$a < b < c$.
$I_0 = [a, b], I_1 = [b, c]$ and $F(a) = b, F(b) = c \Rightarrow F(I_0) \supset I_1 \ \& F(I_1) \supset I_0 \cup I_1$.
Case 1: First we will produce a cycle of period $n > 3$.
To find a periodic point of period n, we will invoke *Observation 2* n times.
Now, $F(I_1) \supset I_1 \Rightarrow \exists a$ closed interval $A_1 \subset I_1$ s.t. $F(A_1) = I_1$
Since , $A_1 \subset I_1 \ \& F(A_1) = I_1 \supset A_1$
By invoking Observation 2,
$\exists$ closed interval $A_2 \subset A_1$ s.t. $F(A_2) = A_1$.Therefore We got

$$A_2 \subset A_1 \subset I_1 \ \& F^2(A_2) = I_1$$

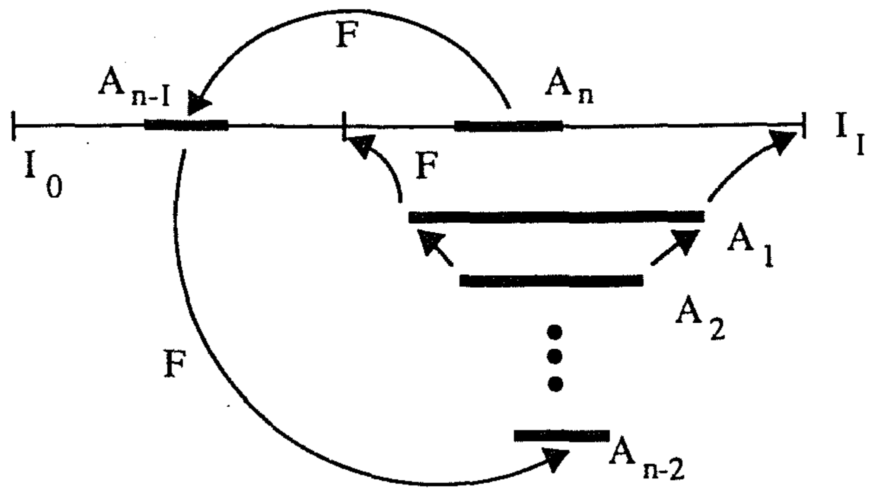
Now continuing these way for $n - 2$ steps we have

$$A_{n-2} \subset A_{n-3} \subset \cdots \subset A_2 \subset A_1 \subset I_1$$
$$\text{s.t. } F^{n-2}(A_{n-2}) = I_1; \ A_{n-2} \subset I_1.$$

Now, $F(I_0) = I_1 \supset A_{n-2} \ \exists$ a closed interval $A_{n-1} \subset I_0$ s.t. $F(A_{n-1}) = A_{n-2}$.
Since, $F(I_1) \supset I_0 \supset A_{n-1} \ \exists$ a closed interval $A_n \subset I_1$ s.t. $F(A_n) = A_{n-1}$.

$$\therefore A_n \xrightarrow{F} A_{n-1} \xrightarrow{F} \dots \xrightarrow{F} A_1 \xrightarrow{F} I_1$$
$$\text{with } \ F^n(A_n) = I_1.$$

We have the following diagrammatic sketch



Now $A_n \subset I_1$. Therefore by observation 1 there exists a point $X_0 \in A_n$ i.e. x_0 is a fixed point for F^n.
$\therefore x_0$ has period n under F.
Claim: x_0 has prime period n.
Now, $F(x_0) \in A_{n-1} \subset I_0$ but $F^i(x_0) \in I_1$ for $i = 2, \dots, n$.
So, First iterate of x_0 lies in I_0 but all other iterates lies in I_1.
$\Rightarrow x_0$ has period $\geq n$. So, x_0 has prime period n.
Case 2: $n = 1$ and $n = 2$
$F(I_1) \supset I_1, \exists$ a fixed point in I_1.
Similarly, $F(I_0) \supset I_1$ and $F(I_1) \supset I_0$
$\therefore$ a fixed point in I_0 for F i.e. $F^2(x) = x$
Therefore, we got a 2-cycle. □

There is an order in the existence of periodic points for a function. The ordering is called *Sarkovskii's Ordering* of natural numbers given by

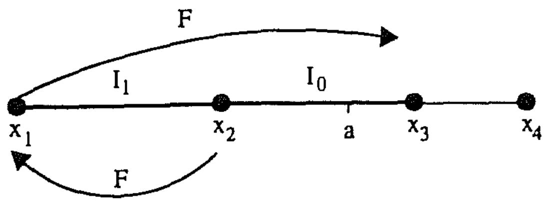
$$3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2.3 \rhd 2.5 \rhd 2.7 \rhd \cdots \rhd 2^2.3 \rhd 2^2.5 \rhd 2^2.7 \rhd \dots$$
$$\rhd 2^3.3 \rhd 2^3.5 \rhd 2^3.7 \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

Theorem 15. ***Sarkovskii's Theorem*** *Let $F : \mathbb{R} \to \mathbb{R}$ is continuous. Suppose that F has a periodic point of period n and that n precedes k in the Sarkovskii's Ordering of then F also has a periodic point of prime period k.*

Proof. We will proof this theorem for some special cases.
Case 1: *Period $k \Rightarrow$ Period* 1.
Let $x_1, x_2, \dots x_k$ lies on the k cycle with

$$x_1 < x_2 < \cdots < x_k$$

Now, $F(x_1) = x_i$ with $i > 1$ and $F(x_k) = x_i$; $i < k$.
Consider the function $G(x) = F(x) - x$.
Then we have $G(x_1) > 0$ and $G(x_k) < 0 \Rightarrow \exists$ an $x \in (x-1, x_k)$
s.t. $G(x) = 0 \Rightarrow F(x) = x$ (By I.V.P)
$\Rightarrow x$ is a fixed point.
Case 2: *Period* $4 \Rightarrow$ *Period* 2.
Let x_1, x_2, x_3, x_4 form the 4 cycle with $x_1 < x_2 < x_3 < x_4$.
Choose a point 'a' between x_2 & x_3.
Then three cases arises.
Sub case 1: Both $F(x_1) > a$ & $F(x_2) > a$
Then We must have $F(x_3) < a$ & $F(x_4) < a$.
Let $I_0 = [x_1, x_2]$ & $I_1 = [x_3, x_4] \Rightarrow F(I_1) \supset I_0$ & $F(I_0) \supset I_1$.
By Observation 1 $\exists$ $a2$ cycle either in I_0 or I_1.
Sub case 2: When one of x_1 or x_2 is mapped to the right of 'a' but others not.
Let $F(x_1) > a$ & $F(x_2) < a \Rightarrow F(x_2) = x_1$.
Let $I_0 = [x_2, x_3]$ & $I_1 = [x_1, x_2]$. i.e. we have following diagram.



Then $F(I_0) \supset I_1$ & $F(I_1) \supset I_0 \cup I_1 \supset I_0$
Again By Observation 2 we have a cycle of period 2.
Case 3: *Period* $2^n \Rightarrow$ *Period* 2^k *; when* $n > k$.
For $n = 1$ or $n = 2$ we are done.
take $n > 3$.
Let $l = 2^{n-2}$ & $G(x) = F^l(x)$.
If F has a cycle of period 2^n.

$$\begin{aligned}
&\Rightarrow F^{2^n}(x) = x \\
&\Rightarrow F^{2^{n-2}.2^2}(x) = x \\
&\Rightarrow G^4(x) = x \\
&\Rightarrow x \text{ is a cycle of period 4 for } G \\
&\Rightarrow x \text{ is a cycle of period 2 for } G \quad (\textit{Since, Period } 4 \Rightarrow \textit{ Period } 2) \\
&\Rightarrow G^2(x) = x \\
&\Rightarrow F^{2^{n-2}.2}(x) = x \\
&\Rightarrow F^{2^{n-1}}(x) = x \\
&\Rightarrow x \text{ is a } 2^{n-1} \text{ cycle for } F
\end{aligned}$$

Hence, we are done. □

Theorem 16. ***Converse of Sarkovskii's Theorem*** *For all $n \in \mathbb{N}$ there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ that has a cycle of period n but no cycles of periods that precedes n in the Sarkovskii's Ordering.*

10.2 Example of Converse of Sarkovskii's Theorem

Example 22. *Draw the function $F(x)$ in $[1,5]$. Defined as*

$$\begin{aligned} F(1) &= 3 \\ F(3) &= 4 \\ F(4) &= 2 \\ F(2) &= 5 \\ F(5) &= 1 \end{aligned}$$

So, that we have a 5 cycle

$$1 \longmapsto 3 \longmapsto 4 \longmapsto 2 \longmapsto 5 \longmapsto 1$$

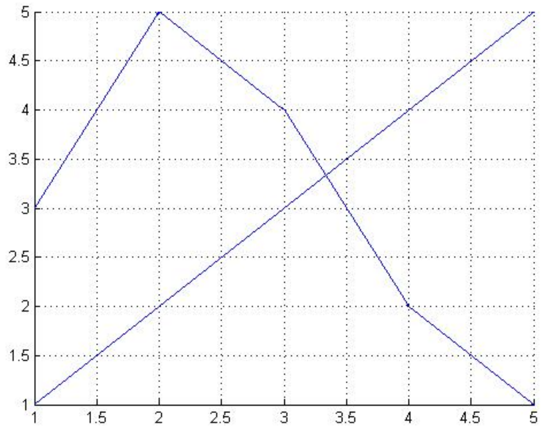


Figure 23: The Graph of $F(x)$

Claim: *F has a no periodic point of period 3.*
Let it be true .

$$\begin{aligned} F([1,2]) =& [3,5] \\ F([3,5]) =& [1,4] \\ F([1,4]) =& [2,5] \end{aligned}$$

Hence $F^3([1,2]) = [2,5]$ so $F^3([1,2]) \cap [1,2] = \{2\}$ which has fixed point in $[1,2]$.
Similar argument shows that there are no 3 cycle in either of interval $[2,3] or [4,5]$.
We can not use the same argument in the interval $[3,4]$. Since, F itself has a fixed point within this interval.

$$F : [3,4] \to [2,4]$$

is a decreasing function. Also

$$\begin{aligned} F &: [2,4] \to [2,5] \\ \text{and } F &: [2,5] \to [1,5] \end{aligned}$$

are also decreasing.Hence

$$F^3 : [3, 4] \to [1, 5]$$

is decreasing. Thus the graph of F^3 *on* $[3, 4]$ *meets the diagonal over* $[3, 4]$ *in one point but it is a fixed point for* F*.*
$\Rightarrow F$ *has no* 3 *cycle in* $[3, 4]$*.*
Consequently, F *has period* 5 *point but no period* 3 *point.*

11 Elementary Definitions

11.1 Types of Orbits

Definition 24. ***Forward Orbit*** *The forward orbit of x under the map F is the set of points $x, F(x), F^2(x)\dots$ and is denoted by $O^+(x)$.*

Definition 25. ***Backward Orbit*** *The backward orbit of x under the map x is the set of points $x, F^{-1}(x), F^{-2}(x)\dots$ and is denoted by $O^-(x)$.*

11.2 Different Type of Periodic Points

Definition 26. ***Fixed Point*** *The point x is a fixed point for a function F if $F(x) = x$.*
The set of fixed points for a function f are denoted by $Fix(f)$.

Definition 27. ***Periodic Point*** *The point x is said to be a Periodic Point for a function F for period n if $F^n(x) = x$.*
The set of periodic points of period n for a function f are denoted by $Per_n(f)$.

Example 23. *Let S^1 denote the unit circle in the plane. Any point in S^1 is denoted by θ measured in radians.*
Let us take the function

$$f(\theta) = 2\theta.$$

Now, on the circle we have $f(\theta + 2\pi) = f(\theta)$.
Since, $f^n(\theta) = 2\theta$ and θ is a periodic point of period n iff

$$2^n\theta = \theta + 2k\pi \text{ for some } k$$
$$\Rightarrow \theta = \frac{2k\pi}{2^n - 1} \text{ Where } \theta \leq k \leq 2^n.$$

Hence the periodic points of the function f is the $(2^n - 1)^{th}$ root of unity.

Definition 28. ***Eventually Periodic*** *A point x is eventually periodic of period n if x is not periodic but $\exists\ m > 0$ s.t. $f^{n+i}(x) = f^i(x)\ \forall i \geq m$.*

Example 24. *Let us consider again the example of S^1 with the function $f(\theta) = 2\theta$ then 0 is a fixed point. If we consider $\theta = \frac{2k\pi}{2^n}$. then $f^n(\theta) = 2k\pi$. Hence θ is a eventually fixed point.*

11.3 Asymptotic Point

Definition 29. ***Forward Asymptotic Point:*** *Let p be periodic point of period n. A point x is forward asymptotic to p if*

$$\lim_{i\to\infty} f^{in}(x) = p$$

.

Definition 30. ***Stable Set:*** *The stable set of p, denoted by $W^s(p)$ consists of all points that are forward asymptotic to p.*

Remark : if p is non periodic we may still define forward asymptotic points by demanding

$$|f^i(x) - f^i(p)| \to 0 \text{ as } i \to \infty.$$

If f is invertible then we have backward asymptotic points by letting $i \to -\infty$.

Definition 31. ***Backward Asymptotic*** *The set of points backward asymptotic to p is called the unstable set of p and is denoted by $W^u(p)$.*

Example 25. *Let us take the following example $f(x) = x^3$. For this map $0, 1, -1$ are fixed points .Then we have*

(a) $W^s(0) = (-1, 1)$.

(b) $W^u(1) =$ *positive real axis.*

(c) $W^u(-1) =$ *negative real axis.*

Proof. From definitions we have ,

$$\begin{aligned}
W^s(0) &= \{x| \lim_{i\to\infty} f^i(x) = 0\} \\
&= \{x| \lim_{i\to\infty} x^{3^i} = 0\} \\
&= \{x| -1 < x < 1\} \\
W^u(1) &= \{x| \lim_{i\to-\infty} f^i(x) = 1\} \\
&= \{x| \lim_{i\to-\infty} x^{3^i} = 1\} \\
&= \{x| \lim_{i\to\infty} x^{3^{-i}} = 1\} \\
&= \{x|x > 0\} \\
W^u(-1) &= \{x| \lim_{i\to-\infty} f^i(x) = -1\} \\
&= \{x| \lim_{i\to-\infty} x^{3^i} = -1\} \\
&= \{x| \lim_{i\to\infty} x^{3^{-i}} = -1\} \\
&= \{x|x < 0\}
\end{aligned}$$

□

Definition 32. ***Critical Point*** *A point x is said to be a critical point of $f \in \mathcal{C}^2$ if $f^{'}(x) = 0$.The critical point is non degenerate if $f^{''}(x) = 0$. The critical point is degenerate if $f^{''}(x) = 0$.*

Example 26. *Let $f(x) = x^2$ then 0 is a non degenerated critical point.*

Goal: The goal of dynamical systems is to understand the nature of all orbits under a particular map, and to identify the set of orbits which are periodic, eventually periodic,asymptotic, etc. i.e. we are more interested on eventual behavior of the orbits.

Geometric techniques to understand the dynamics of a given system Recall from chapter 3 that we have two types of graphical analysis

(i) Orbit Analysis.

(ii) Phase Portrait.

11.4 Translations on the circle

Example 27. *Let $\lambda \in \mathbb{R}$. Define $T_\lambda : S^1 \to S^1$ by $T_\lambda(\theta) = \theta + 2\lambda\pi$. Here we have two cases:*
Case 1: Let $\lambda \in \mathbb{Q}$. Let $\lambda = \frac{p}{q}$. where $p, q \in \mathbb{Z}$ with $q \neq 0$. Then

$$
\begin{aligned}
T_\lambda(\theta) &= \theta + \frac{2p\pi}{q} \\
T_\lambda^2(\theta) &= \theta + 2.\frac{2p\pi}{q} \\
&\cdots \\
T_\lambda^q(\theta) &= \theta + q.\frac{2p\pi}{q} \\
&= \theta + 2p\pi.
\end{aligned}
$$

$\therefore T_\lambda^q$ fixed all points on S^1 .
Case 2: Let $\lambda \in \mathbb{Q}^c$. Here, we have an important result named as Jacobi's Theorem.

Theorem 17. *Each Orbit T_λ is dense in S^1 if λ is irrational.*

Proof. Let $\theta \in S^1$ arbitrarily be choosen.
Claim: The points on the orbit of θ are distinct. If not so then we have $\exists m, n \in \mathbb{Z}$ s.t.

$$
\begin{aligned}
&T_\lambda^n(\theta) = T_\lambda^m(\theta) \\
&\Rightarrow \theta + 2\lambda n\theta = \theta + 2\lambda m\theta + 2k\pi \text{ where } k \in \mathbb{Z}. \\
&\Rightarrow (n-m)\lambda \in \mathbb{Z}.
\end{aligned}
$$

But $\lambda \in \mathbb{Q}^c \;\Rightarrow\; n = m$. i.e. the points on the orbit of θ are distinct.
Now, the orbits of θ under T_λ is a bounded sequence on S^1. Then by Bolzano Weirstrass theorem we have a convergent subsequence, say $(T_\lambda^{nk}(\theta))$. Then by Cauchy Criterion
for all $\epsilon > 0$ $\exists N_0 \in \mathbb{N}$ s.t.
$|T_\lambda^{nk_1}(\theta) - T_\lambda^{nk_2}(\theta)| < \epsilon$ where $k_1, k_2 > N_0$.
Let $k = nk_1 - nk_2$
$\Rightarrow |T_\lambda^k(\theta) - \theta| < \epsilon$.
Now, T_λ preserves the length.
$\because \; |\theta - \theta'| = |T_\lambda(\theta) - T_\lambda(\theta')|$.
$\therefore$ the points $\theta, T_\lambda^k(\theta), T_\lambda^{2k}(\theta), \ldots$ partitions S^1 into arc length of less than ϵ.
So,every open sets on S^1 contains at least one point from the orbit of θ.
Let $\xi \in S^1$ arbitrary.
Let $\eta = |\xi - \theta|$. then then number of ϵ neighborhood to cover ξ and θ is $\lceil \frac{1}{\epsilon} \rceil + 1 = k'$. Then we have $|T^{kk'} - \xi| < \epsilon$.
From this we get one element from orbit of θ within ϵ neighborhood of ξ.
This complete the proof. □

12 Hyperbolicity

Definition 33. ***Hyperbolicity*** *Let p be a periodic point of prime period n.The point p is said to be hyperbolic if*

$$|(f^n)'(p)| \neq 1.$$

The number $(f^n)'(p)$ is called the multiplier of the periodic point.

Example 28. *Consider the diffeomorphism $f(x) = \frac{1}{2}(x^3 + x)$.*
Here, the fixed points are $0, 1$ and -1. Now,$f'(x) = \frac{3x^2+1}{2}$
$f'(0) = \frac{1}{2}$ and $f'(\pm 1) = 2$.
Here, each fixed point is a hyperbolic point.

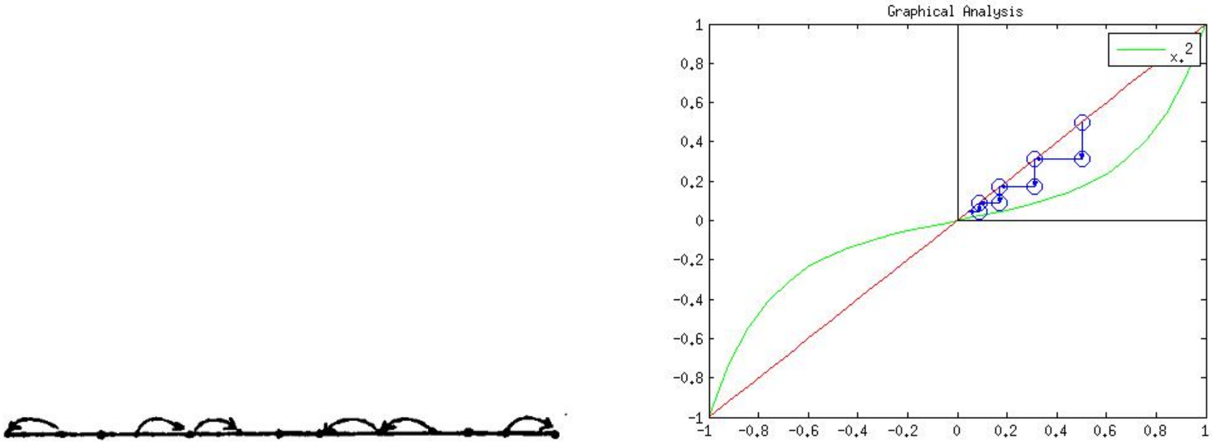


Figure 24: Graphical Analysis and Phase Portrait of $y = \frac{1}{2}(x^3 + x)$

12.1 Properties of Hyperbolic Points

Proposition 5. *Let p be a hyperbolic fixed point with $|f'(p)| < 1$.Then there exist an open interval U about p such that if $x \in U$ then*

$$\lim_{n\to\infty} f^n(x) = p \tag{2}$$

that is there is an open interval about p in which every point is forward asymptotic point to p.

Proof. Given $|f'(x)| < 1$ then there exist $0 < A < 1$ such that $|f'(x)| < A < 1$and $f \in \mathcal{C}'$ then by continuity of f' we have $\epsilon > 0$ s.t. $|f'(x)| < A$ when $x \in (p - \epsilon, p + \epsilon)$. Then for an arbitrary $x \in (p - \epsilon, p + \epsilon)$ by Mean Value Theorem we have,

$$|f(x) - p| = |f(x) - f(p)| \leq A|x - p| < |x - p| \leq \epsilon$$

Hence, $f(x) \in (p - \epsilon, p + \epsilon)$ Using same argument we have $|f^n(x) - p| \leq A^n|x - p|$.
Since $0 < A < 1$ we have $f^n(x) \to p$ as $n \to \infty$. □

Remarks:

i $(p - \epsilon, p + \epsilon) \in W^s(p)$ i.e. in the stable set associated with p

ii Similar result is true for hyperbolic periodic points of period n In that case we have an neighborhood U of p which is mapped inside itself by f^n with the assumption $|f^n(p)| < 1$.

12.2 Types of Hyperbolic Periodic Points

Definition 34. ***Attracting Periodic Point*** *Let p be a hyperbolic periodic point of period n with $|(f^n)^{'}(p)| < 1$. Then p is said to be attracting periodic point or attractor or a sink.*

Remarks: This type of points have the nature that we have (a) and (b) discussed in the previous remark. Such neighborhood U is called local stable set and denoted by W^s_{loc}.
Graph of different types of Attracting Fixed Points:

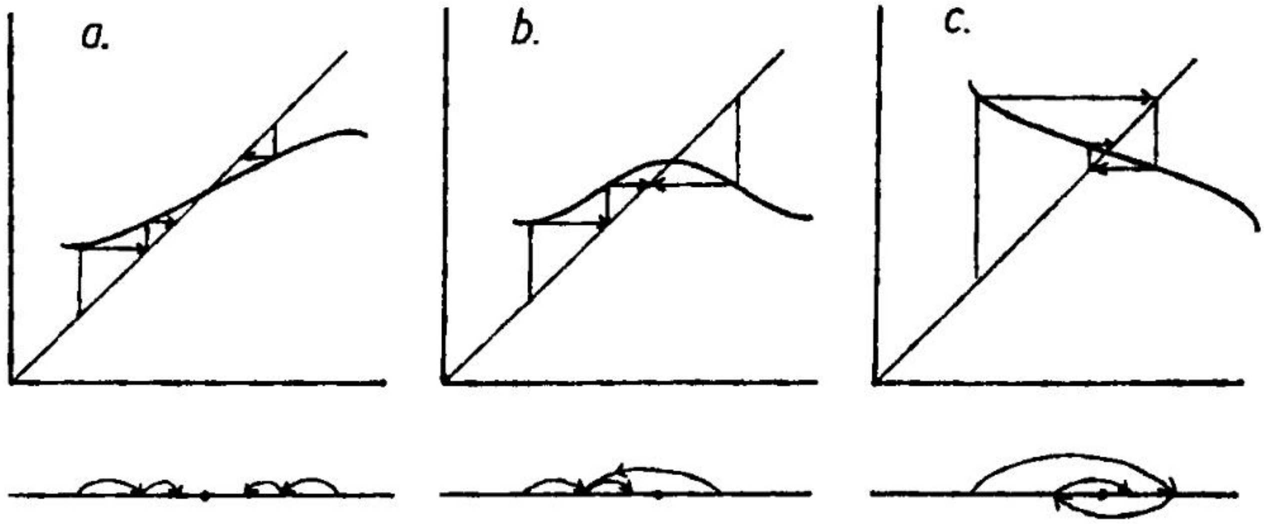


Figure 25: Graph and Phase Portrait when (a) $0 < f^{'}(p) < 1$,(b) $f^{'}(p) = 0$,(c) $-1 < f^{'}(p) < 0$.

Proposition 6. *Let $f \in \mathcal{C}^{'}$ and p be a hyperbolic fixed point with $|f^{'}(p)| > 1$ then there exist an open interval U of p such that $x \in U$ and $x \neq p \;\Rightarrow \exists\; k > 0$such that $f^k(x) \notin U$.*

Proof. Given $|f^{'}(p)| > 1$ then there exist $0 < \lambda < 1$ s.t.$|f^{'}(p)| > \lambda > 1$.
By continuity of $f^{'}$ we have $\epsilon > 0$ s.t. $|f^{'}(x)|\lambda$ when $x \in (p-\epsilon, p+\epsilon)$. Then for an arbitrary $x \in (p-\epsilon, p+\epsilon)$ by Mean Value Theorem we have
$|f(x) - p| = |f(x) - f(p)| > \lambda|x - p|$
As $\lambda > 1$ we have $f(x)$ is further from p than x originally was.
If $f(x) \notin (p-\epsilon, p+\epsilon)$ then we are done.
if not then repeat the process i.e. again using the Mean Value Theorem we have

$$|f^2(x) - p| = |f^2(x) - f^2(p)| > \lambda|f(x) - f(p)| > \lambda^2|x - p|. \tag{3}$$

$\because \lambda^2 > \lambda > 1$ we have $f^2(x)$ is further from p than $f^2(x)$ originally was. Continue the process with the motive to increase the distance between $f^n(x)$ and$f^n(p)$.It follows that $|f^n(x) - p| > \lambda^n|x - p|$ as long as $f^{n-1}(x) \in (p-\epsilon, p+\epsilon)$.
Now $\lambda^n \to \infty$ as $n \to \infty$ there exists $k > 0$ s.t.$\lambda^n > \frac{\epsilon}{|x-p|}$ $\forall n \geq k$. Then we have $|f^n(x) - p| > \lambda^k|x - p| > \epsilon$. It implies that $f^k(x) \notin (p-\epsilon, p+\epsilon)$. $\square$

Definition 35. ***Repelling Fixed Point:*** *A fixed point p with $|f^{'}(p)| > 1$ is called a repelling fixed point or a repellor or source.*
The neighborhood we got here is called the local unstable set of p and is denoted by $W^u(p)$. This type of point has a small neighborhood around them in which every point eventually leave the interval after few iterations.

Example 29. *Consider the family of quadratic functions $Q_c(x) = x^2 + c$,where c is a parameter. The graphs of Q_c assume three different positions depending upon whether $c > 1/4,\ c = 1/4,$ or $c < /4$. Note that Q_c has no fixed points for $c > 1/4$. When $c = 1/4$ It has a unique non-hyperbolic fixed point at $x = 1/2$ And when $c < 1/4$ It has a pair of fixed points, one attracting and one repelling. This change is an example of a bifurcation.That is when parameter change it changes the behavior of the fixed points.*

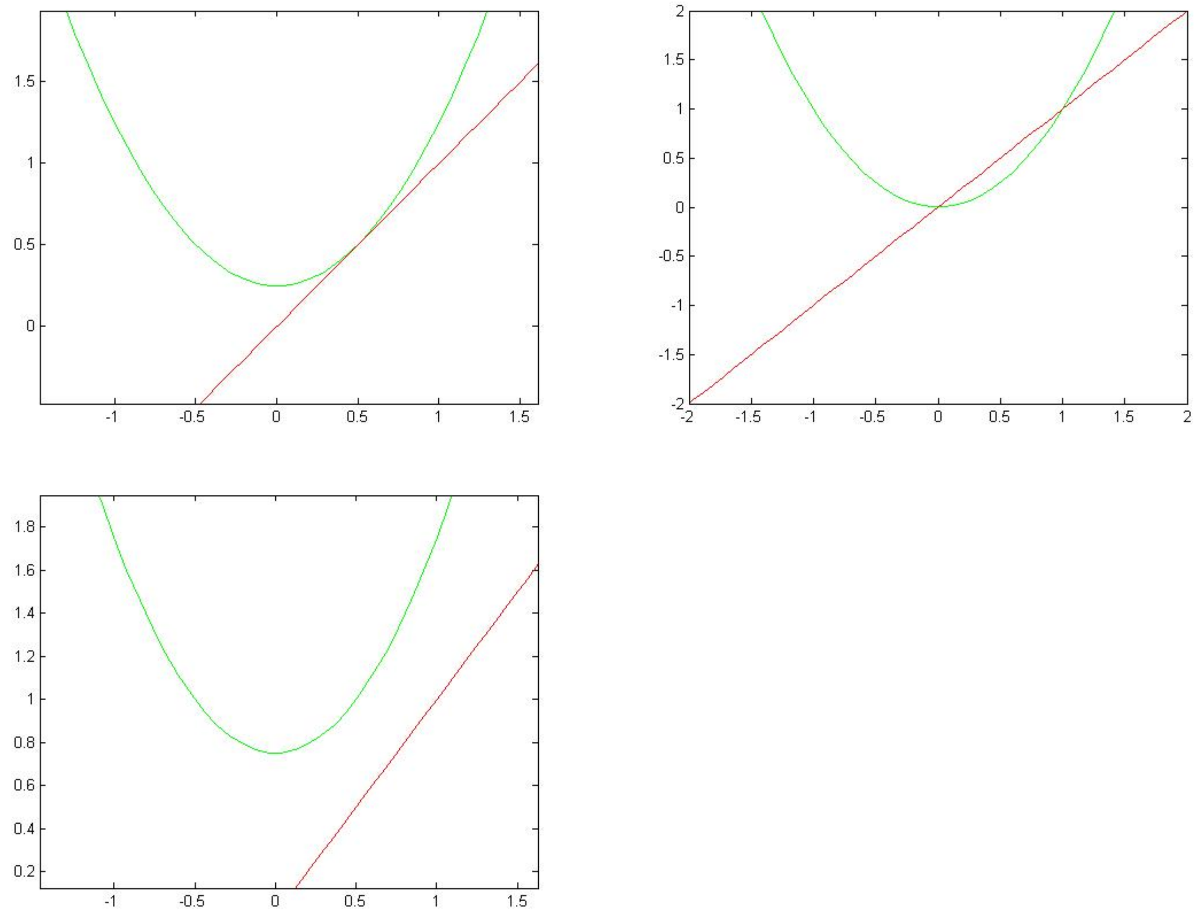


Figure 26: (a) Graph of $Q_{1/4}(x)$, (b) Graph of $Q_0(x)$, and (c) Graph of $Q_{3/4}$.

13 Quadratic Family $F_\mu(x) = \mu x(1-x)$

Consider the function $F_\mu(x) = \mu x(1-x)$ with $\mu > 1$.

Let us try to find out the fixed points

$$F_\mu(x) = x$$
$$\mu x(1-x) = x$$
$$x(\mu - \mu x - 1) = 0$$
$$\Rightarrow \; x = 0 \textit{ and } x = \frac{\mu-1}{\mu} = p_\mu.$$

Now,$F_\mu'(x) = \mu - 2\mu x$.Then $F_\mu'(0) = \mu$ and $F_\mu'(p_\mu) = 2 - \mu$
Hence, 0 is a repelling fixed point for $\mu > 1$ and p_μ is attracting for $1 < \mu < 3$.

Proposition 7. *1. $F_\mu(0) = 0 = F_\mu(1)$ and $F_\mu(p_\mu) = p_\mu$ where $p_\mu = \frac{\mu-1}{\mu}$.*

2. $0 < p_\mu < 1$ when $\mu > 1$.

13.1 The Case $1 < \mu < 3$

Proposition 8. *Suppose $\mu > 1$. If $x < 0$ then $F_\mu^n(x) \to -\infty$ as $n \to \infty$. and If $x > 1$ then $F_\mu^n(x) \to -\infty$ as $n \to \infty$.*

Proof. Given $\mu > 1$. Now assume $x < 0$ then we have,

$$x(1-x) < x$$
$$\mu x(1-x) < \mu x$$
$$\text{but } \mu x < x$$
$$F_\mu(x) < x$$

Similarly,$F_\mu^2(x) < F_\mu(x) < x$. Hence $F_\mu^n(x)$ is a decreasing sequence points. Let us assume $F_\mu^n(x) \to p$. then we would have $F_\mu^{n+1}(x) \to F_\mu(x) < p$
Hence, we have $F_\mu^n(x) \to -\infty$.
If $x > 1$ then $(1-x) < 0$ hence the same argument follows. □

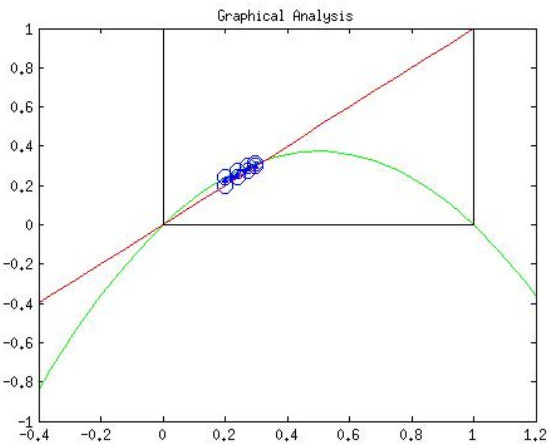


Figure 27: $F_\mu(x) = \mu x(1-x)$ when $\mu > 1$

We are more interested on what happen to $F_\mu(x)$ when $x \in I \equiv \{x|0 \le x \le 1\}$

Proposition 9. *Let* $1 < \mu < 3$

1. *F_μ has an attracting fixed point at $p_\mu = \frac{\mu-1}{\mu}$ and a repelling fixed point at* 0*.*
2. *If $0 < x < 1$ then*

$$\lim_{n\to\infty} F_\mu^n(x) = p_\mu$$

.

Proof. 1. We know that $F_\mu^{'}(x) = \mu - 2\mu x$.Then $F_\mu^{'}(0) = \mu$and $F_\mu^{'}(p_\mu) = 2 - \mu$. Here, 0 is a repelling fixed point.and p_μ is attracting fixed point.

2. **Case: 1** Let $1 < \mu \le 2$. and $F_\mu(1/2) = \mu/4 \le 1/2$.
Using the graph of the function it is clear that $p_\mu \le 1/2$. where $p_\mu = \frac{\mu-1}{\mu}$
Since $F_\mu^{'}(x) = \mu - 2\mu x$. We have $F_\mu^{'}(x) > 0 \; \forall x \in (0, p_\mu)$.
Hence we have F_μ is monotonically increasing and the graph lies above the diagonal when $x \in (0, p_\mu)$. Thus for $x_0 \in (0, p_\mu)$ $F_\mu^j(x_0)$ is monotonically increasing sequence which must converge to fixed point p_μ.

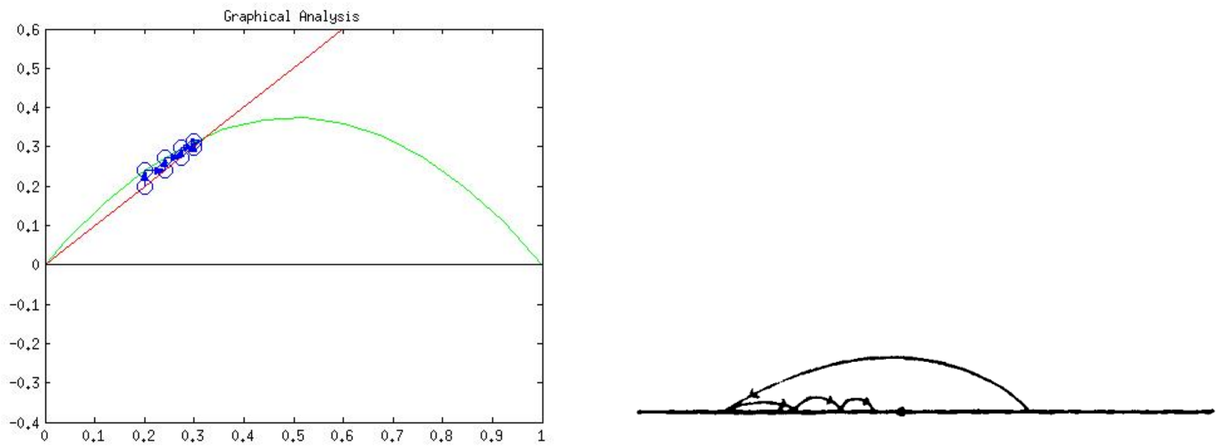


Figure 28: Graph and Phase Portrait of $F_{1.5}(x) = (1.5)x(1-x)$

Similarly,in $(p_\mu, 1/2]$ the function is monotonically increasing and the graph lies below the diagonal. Thus for $y_0 \in (p_\mu, 1/2]$ $F_\mu^j(y_0)$ is monotonically decreasing to p_μ .
Finally, $x_0 \in (1/2, 1) \Rightarrow F_\mu(x_0) \in (0, 1/2)$.So, $F_\mu^j(x_0) \to p_\mu$.
Case 2: Assume $2 < \mu < 3 \Rightarrow 1/2 < p_\mu < 1$.

Let $\hat{p_\mu}$ denote the unique point in the interval $(0, 1/2)$ that is mapped onto p_μ by F_μ.
Then F_μ^2 maps $[\hat{p_\mu}, p_\mu]$ inside $[1/2, p_\mu]$.
It follows that $F_\mu^n(x) \to p_\mu$ as $n \to \infty$ $\forall x \in [\hat{p_\mu}, p_\mu]$.Now let $x < p_\mu$. Again graphical analysis shows that $\exists\ k > 0$ s.t. $F_\mu^k(x) \in [\hat{p_\mu}, p_\mu]$. Thus, $F_\mu^{k+n}(x) \to p_\mu$ as $n \to \infty$.
Therefore, F_μ maps $(p_\mu, 1)$ onto $(0, p_\mu)$. Since, $(0, 1) = (0, \hat{p}_\mu) \cup [\hat{p}_\mu, p_\mu] \cup (p_\mu, 1)$ So, the result follows.
a

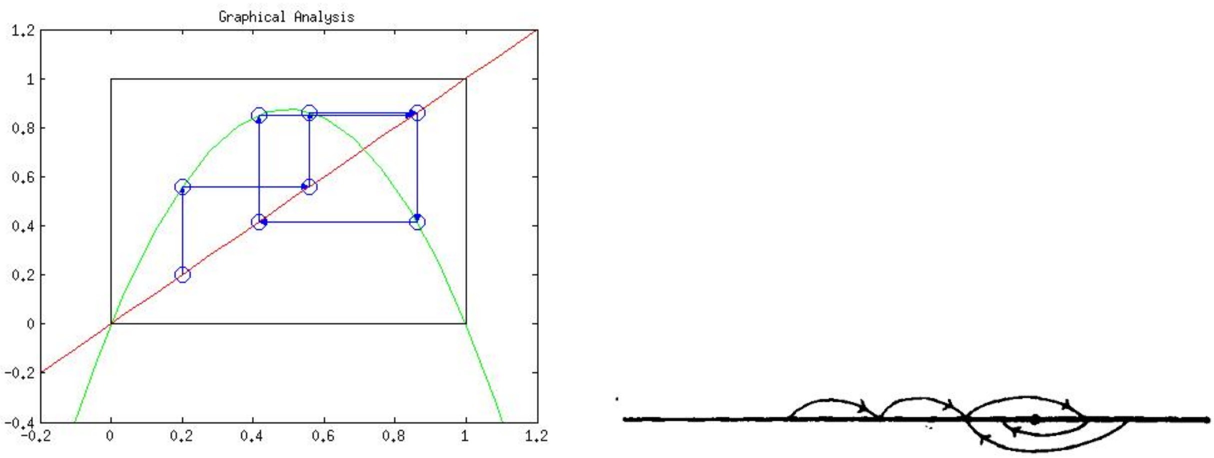


Figure 29: Graph and Phase Portrait of $F_{3.5}(x) = (3.5)x(1-x)$

□

13.2 The Case $\mu > 4$

We will drop μ and instead of F_μ we will write only F.
Here onward we are concentrating more on the interval $I \equiv (0, 1)$. Since, $\mu > 4$, the maximum value of F is $\frac{\mu}{4} > 1$. Hence, certain points leaves the interval I after first iteration. Denote the collection of such points by A_0. Clearly, A_0 is an open interval centered at 1/2 and has the property that if $x \in A_0$,then $F(x) > 1$, so $F^2(x) < 0$ and $F^n(x) \to -\infty$.

$$A_0 = \{x \in I : F(x) > 1\}$$

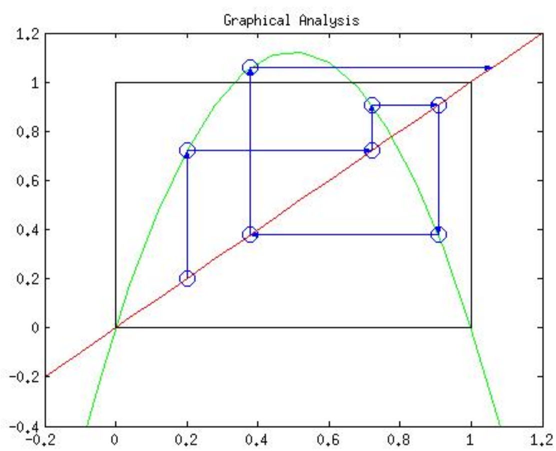


Figure 30: Graph of $F_{4.5}(x) = (4.5)x(1-x)$

Let $A_1 = \{x \in I | F(x) \in A_0\}$ then $F^2(x) > 1$ and $F^3(x) < 0$ then $F^n(x) \to -\infty$. The set of points in I which leaves I after second Iterations. More generally we can say :

$$A_n = \{x \in I | F^n(x) \in A_0\}$$

Let us investigate on the interval $I - (\cup_{n=0}^{\infty} A_n) \equiv \Lambda$
This is set of points which will stay in the interval for ever. Now, A_0 is the set centered at 1/2. Hence, if we remove A_0 from I we are left with two intervals say, I_0 and I_1. i.e.

$$I - A_0 = I_0 \cup I_1.$$

Similarly we can remove both A_0 and A_1 then we left with four intervals. This set is denoted by A_2. and F^2 maps this interval onto A_0.
Observation: If we see the graph of F^2 we have two humps there.

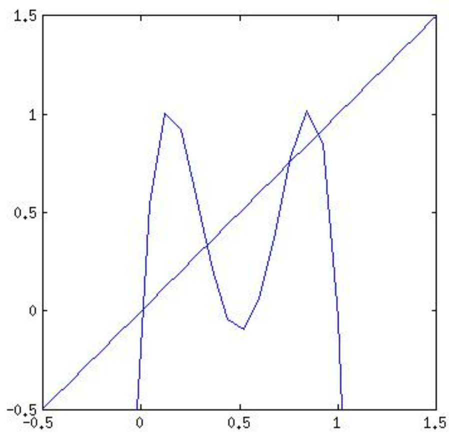


Figure 31: Graph of $F_\mu^2(x)$ when $\mu = 4.1$

Continuing this manner we have A_n consists of 2^n disjoint open intervals. Hence, $I - (A_0 \cup A_1 \cdots \cup A_n)$ consists of $2n + 1$ closed intervals.Secondly, F^{n+1} maps each of these closed intervals monotonically onto I. In fact, the graph of F^{n+1} is alternately increasing and decreasing on these intervals. Thus the graph of F^{n+1} has exactly 2^n humps on I.
The Graph of F^n cross the $y = x$ line exactly 2^n times.
It follows that F^n has at least 2^n fixed points.

13.3 A set Reminiscent of Cantor Middle Third Set

The set Λ is the reminiscent of the Cantor Middle Third Set.
To define Cantor set we need the following definitions

Definition 36. ***Totally disconnected*** *A set is said to be totally disconnected if it does not contains any intervals.*

Definition 37. ***Perfect Set*** *A set is said to be Perfect Set if every point in it is a limit point of other points in the set.*

Definition 38. ***Cantor Set*** *A set A is set to be Cantor Set if it has the following properties*

1. *Closed.*
2. *Totally disconnected.*
3. *Perfect Set.*

Theorem 18. *If $\mu > 2+\sqrt{5}$ then the set Λ is a Cantor Set.*

Proof. 1. By construction $I - (\cup_{n=0}^{\infty} A_n) \equiv \Lambda$
$\Rightarrow \Lambda^c = I - (\cap_{n=0}^{\infty} A_n^c)$
Now each A_n is close set hence $\cap_{n=0}^{\infty} A_n^c$ is also closed.It follows that the Λ^c is open and hence Λ is closed .

2. To prove Λ is totally disconnected we need some hypothesis.

Lemma 2. $|F^{'}(x)| > 1\ \forall x \in I_0 \cup I_1$ *when* $\mu > 2+\sqrt{5}$.

Proof. Now,$F_\mu(x) = \mu x(1-x)$.
$\Rightarrow F_\mu^{'}(x) = \mu - 2\mu x$.

$$|F_\mu^{'}(x)| > 1$$
$$\Rightarrow |\mu - 2\mu x| > 1$$

Case:1

$$\mu - 2\mu x > 1$$
$$\Rightarrow \mu > 2\mu x + 1$$
$$\Rightarrow \frac{\mu-1}{2\mu} > x$$

Case:2

$$\mu - 2\mu x < -1$$
$$\Rightarrow \frac{\mu+1}{2\mu} < x$$

Hence, $\frac{\mu+1}{2\mu} < x < \frac{\mu-1}{2\mu}$
Now, $F_\mu(\frac{\mu+1}{2\mu}) = \mu\frac{\mu-1}{2\mu}(1-\frac{\mu-1}{2\mu})$
$\Rightarrow\ \frac{\mu-1}{2}\frac{\mu+1}{2\mu} = 1$
$\Rightarrow\ \mu^2 - 4\mu - 1 = 0$ and $\Rightarrow \mu = 2+\sqrt{5}$ □

Hence for these μ values there exists a $\lambda > 1$ such that $|F^{'}(x)| > \lambda$ for all $x \in \Lambda \subset\ I_0 \cup I_1$.
Now,

$$(F^n)^{'}(x) = F^{'}(F^{n-1}(x)).F^{'}(F^{n-2}(x))\dots F^{'}(x)$$

Since, $x \in \Lambda,$ it follows $F^n(x) \in \Lambda$ by the property of Λ and then by previous lemma we have $|(F^n)^{'}(x)| > \lambda^n$.

Claim: Λ contains no intervals.
Suppose Λ contains intervals.Then we could choose $x, y \in \Lambda$ with $x \neq y$ such that $[x, y] \subset \Lambda$ choose n such that $\lambda^n|y - x| > 1$ then apply Mean Value Theorem for the interval $[x, y]$ with the function F^n we have,

$$
\begin{aligned}
|F^n(x) - F^n(y)| &= |(F^n)^{'}(\xi)||y - x| \text{ where } \xi \in (x, y) \\
&\geq \lambda^n|y - x| \\
&> 1
\end{aligned}
$$

which implies either $F^n(x)$ or $F^n(y)$ is outside the unit interval I.
This is a contradiction and hence we proved Λ is totally disconnected.

3. **Claim:** Λ is a perfect set.
Let us prove it by contradiction. Now, any endpoint of an A_n is in Λ and these points are eventually mapped to 0.So, they stay in the interval I under any iteration of F.
Let $p \in \Lambda$ is isolated point.So, near by points leave I after some iterations. and such type of points must be in some A_n.Then either there exists a sequence of endpoints of A_n converging to p or all the pints in the deleted neighborhood of p are mapped out of I at some iterations of F.In earlier case the we are done since end points are mapped to 0.and 0 already in I.
In the second case assume F^n maps to 0 and all other in the neighborhood of p to the negative real axis.But in that case F^n has critical point at p.Then $(F^n)^{'}(p) = 0$. Hence we have $F^{'}(F^i(p)) = 0$ for some $i < n$.
i.e. $F^i(p) = \frac{1}{2}$. It implies that $F^{i+1}(p) \ni I$ which follows $F^n(p) \to -\infty$
It is a contradiction to the fact that $F^n(p) = 0$. Hence p is not a isolated point and we are done.

□

Remark: The theorem is also true for $\mu > 4$.

13.4 Orbit Diagram

In the Chapter 7 I explained how to construct the Orbit diagram of quadratic family $Q_c(x) = x^2 + c$. Similarly we can do it for this logistic function also.

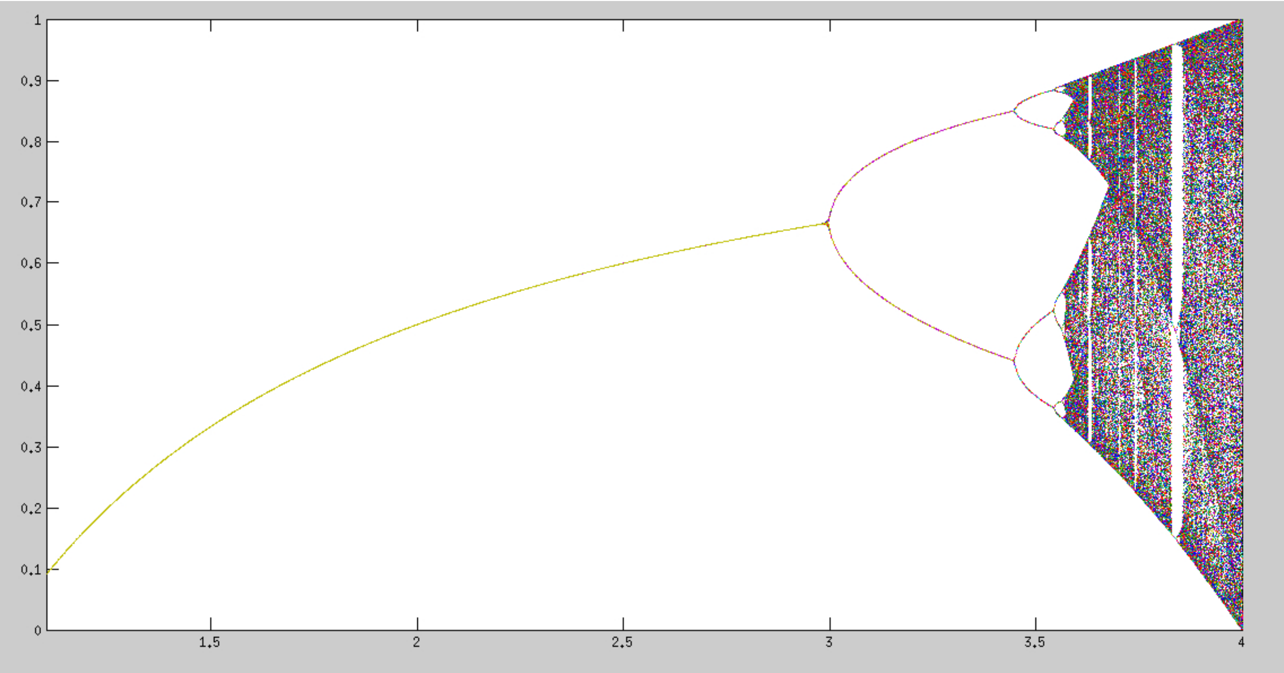


Figure 32: Orbit Diagram of $F_\mu(x) = \mu x(1 - x)$ when $\mu > 1$

14 Symbolic Dynamics for $F_\mu(x) = \mu x(1-x)$

Goal: To give a model of rich dynamical structure of quadratic map on cantor set Λ.
For this we need to set up a space on which our model map will act. The points in this space is the infinite sequence of 0 and 1's.

14.1 Some Definitions

Definition 39. $\Sigma_2 = \{s = (s_0, s_1, \dots) | s_j = \ 0 \ or \ 1\}$.

Σ_2 is called the sequence space on the two symbols 0 and 1
Similarly Σ_n denote the infinite sequence consisting of integers between 0 and $n-1$.
Let us define a metric on the space Σ_2.

$$d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

Clearly, the series is dominated by $\sum_{i=0}^{\infty} \frac{1}{2^i} = 2$.
hence it converges. Recall that proximity theorem which says that

Theorem 19. *Let $s, t \in \Sigma_2$ and Suppose $s_i = t_i \ \forall i = 0, 1, \dots, n$. Then $d[s,t] \leq \frac{1}{2^n}$. Conversely $d[s,t] < \frac{1}{2^n}$ then $s_i = t_i$ for $i \leq n$.*

I already prove this theorem in Chapter 8.I am going to use this theorem to prove the following results.

Definition 40. *The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is given by*
$\sigma(s_0, s_1, s_2, \dots) = (s_1, s_2, s_3, \dots)$.

This shift operator is know as left shift operator.

Proposition 10.
1. *Cardinality of $Per_n(\sigma)$ is 2^n.*
2. *$Per(\sigma)$ is dense in Σ_2.*
3. *There exists a dense orbit for σ in Σ_2.*

Proof.
1. Periodic Points correspond exactly to repeating sequence
   i.e. sequence of the form $(s_0, s_1, \dots, s_{n-1}, s_0, \dots, s_{n-1}, \dots)$.
   Now for each position we have only two choice i.e. either 0 or 1.
   Hence there are 2^n periodic points of period n for σ.
2. To prove denseness , given any point $s = (s_0, s_1, \dots) \in \Lambda$ we have to produce a sequence of periodic points τ_n which converges to s in Σ_2.
   Now define $\tau_n = (s_0, s_1, \dots, s_n, s_0, \dots, s_n, \dots)$.i.e. τ_n is a repeating sequence whose first $n+1$ entries agrees with s.Then by earlier proposition we have
   $d[s, \tau_n] \leq \frac{1}{2^n}$ so we achieved that $\tau_n \to s$.
3. To prove this one we have to show that there exists points in Σ_2 s.t. the closure of the orbit of that points under σ is Σ_2 itself.
   In the other words we can say there are points in Σ_2 whose orbit comes arbitrarily close to any given sequence in Σ_2 .
   Consider $s^* = (0, 1, |0, 0, 0, 1, 1, 0, 1, 1|0, 0, 0, 0, 0, 1 \dots 1, \dots)$.
   i.e. listing all blocks of 0's and 1's of length n, then $n+1$ etc.Then given any point s

if we apply some iterate of σ on s^* yields a sequence which agrees in an arbitrary large number of places with s.
This property of σ is called **Topologically Transitivity**.

$\square$

15 Topological Conjugacy

Here our main aim is to relate the shift map to our quadratic map. So that by studying the shift map we can exhibit the property of quadratic map $F_\mu = \mu x(1-x)$. when μ is large enough.
Observation: All points in $\mathbb{R}$ tends to ∞ under iteration of F_μ with exception to the points belongs to Λ.

Definition 41. ***Itinerary*** *The Itinerary of x is a sequence $S(x)$ defined as*

$$S(x) = (s_0, s_1, \dots)$$
$$\text{Where } s_j = \begin{cases} 0 & \text{if } F_\mu^j(x) \in I_0 \\ 1 & \text{if } F_\mu^j(x) \in I_1 \end{cases}$$

Therefore, $S(x) \in \Sigma_2$.
So, S is a map form Λ to Σ_2.

15.1 Construction of Conjugacy

Theorem 20. *When $\mu > 2 + \sqrt{5}$ then $S : \Lambda \to \Sigma_2$ is a homeomorphism.*

Proof. (i) **Claim:** S is one to one.
Let $x, y \in \Lambda$ and suppose $S(x) = S(y)$. Then for each n, $F_\mu^n(x)$ and $F_\mu^n(y)$ lie in the same side of 1/2. either in I_0 or I_1. This implies that F_μ is monotonic on the interval between $F_\mu^n(x)$ and $F_\mu^n(y)$. Consequently all points in this interval remain in $I_0 \cup I_1$.
It implies Λ contain interval. But it is a contradiction to the fact that Λ is totally disconnected.

(ii) **Claim:** S is onto.
Let $J \subset I$ be a closed interval. Now define

$$F_\mu^{-n}(J) = \{x \in I | F_\mu^n(x) \in J\}$$

i.e. $F_\mu^{-1}(J)$ denote the preimage of J under F. If $J \subset I$ is a closed interval then $F_\mu^{-1}(J)$ consists of two subintervals one in I_0 and one in I_1.

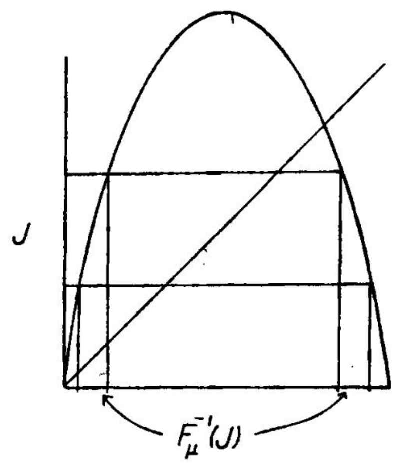


Figure 33: Graphical Argument of above Inference

Now let $s = (s_0, s_1, \dots)$ we have to produce $x \in \Lambda$ such that $S(x) = s$.We define

$$\begin{aligned} I_{s_0,s_1,\dots,s_n} =& \{x \in I | x \in I_{s_0}, F_\mu(x) \in I_{s_1}, \dots, F_\mu^n(x) \in I_{s_n}\} \\ =& I_{s_0} \cap F_\mu^{-1}(I_{s_1}) \cap \dots \cap F_\mu^{-n}(I_{s_n}) \end{aligned}$$

Subclaim: As $n \to \infty$ $I_{s_0,\dots,s_n}$ form a nested sequence of non empty closed intervals.

(I) Now we can write $I_{s_0,\dots,s_n} = I_{s_0} \cap F_\mu^{-1}(I_{s_1,\dots,s_n})$.
By induction $I_{s_1,\dots,s_n}$ is a non empty interval.$F_\mu^{'}(I_{s_1,\dots,s_n})$ is the union of two closed intervals one in I_0 and other in I_1. Now, I_{s_0} is also a closed interval and intersection of closed intervals is closed.
Therefore,$I_{s_0} \cap F_\mu^{'}(I_{s_1,\dots,s_n})$ is a single closed interval.

(II) Now these intervals are nested because

$$\begin{aligned} I_{s_1,\dots,s_n} &= I_{s_1,\dots,s_{n-1}} \cap F_\mu^{-n}(I_{s_n}) \\ &\subset I_{s_1,\dots,s_{n-1}}. \end{aligned}$$

By Cantor Nested Interval Theorem we have

$$\cap_{n \geq 0} I_{s_1,\dots,s_n} \neq \phi$$

Let $x \in \cap_{n\geq 0} I_{s_1,\dots,s_n}$ then $S(x) = (s_0, s_1, \dots, s_n, \dots)$.
This implies S is onto.

(iii) **Claim:** S is continuous.
Given that $S : \Lambda \to \Sigma_2$. Choose $x \in \Lambda$ arbitrarily and $S(x) = (s_0, s_1, \dots)$.
Let $\epsilon > 0$ choose n such that $\frac{1}{2^n} < \epsilon$.
Consider $I_{t_0,\dots,t_n}$ we have such 2^{n+1} intervals and $I_{s_0,\dots,s_n}$ be one of them.
We may choose δ such that $|x - y| < \delta$ and $y \in I$
$\Rightarrow y \in I_{s_0,\dots,s_n}$.Therefore, $d[S(x), S(y)] < \frac{1}{2^n} < \epsilon$. This is because $S(x)$ and $S(y)$ agrees in $(n + 1)$ positions.
It implies that S is continuous at $x \in \Lambda$.

(iv) Similarly we can prove S^{-1} is also continuous.

□

Theorem 21. *$S \circ F_\mu = \sigma \circ S$.*

Proof. Let $x \in \Lambda$. Then $S(x) = (s_0, s_1, \dots)$.
It implies that $x \in I_{s_0},\ F_\mu(x) \in I_{s_1}, \dots$.
Then $S(F_\mu(x)) = (s_1, s_2, \dots) = \sigma \circ S(x)$. □

Definition 42. ***Topological Conjugacy*** *Let $f : A \to A$ and $g : B \to B$ be two maps.then f and g are said to be topologically conjugate if there exists a homeomorphism $h : A \to B$ such that $h \circ f = g \circ h$.*

Example 30. *F_μ on Λ is topologically conjugate to σ on Σ_2.*

Advantage of Topological Conjugacy: Mapping which are topologically conjugate are completely equivalent in terms of their dynamics.Let say f is topologically conjugate to g by via h .

i Let p be a fixed point for f.then $h(p)$ is a fixed point for g.Since, $h(p) = hf(p) = gh(p)$.

ii h gives a one-to-one correspondence between $Per_n(f)$ and $Per_n(g)$.

iii We can also get connection of asymptotic orbits and eventually periodic points of f or g if we have information of other one.

iv f is topologically transitive iff g is so.

Theorem 22. *Let $F_\mu(x) = \mu x(1-x)$ and $\mu > 2+\sqrt{5}$. Then we have*

1. *The cardinality of $Per_n(F_\mu)$ is 2^n.*
2. *$Per(F_\mu)$ is dense in Λ.*
3. *F_μ has dense orbit in Λ.*

Remark: Now, computing the 2^n periodic points of period n for F_μ is a tough task.
But with the help of topological conjugacy we can easily show it. Here, the map $S : \Lambda \to \Sigma_2$ is the required homeomorphism such that $S \circ F_\mu = \sigma \circ S$.

16 Chaos in $F_\mu(x) = \mu x(1-x)$

The quadratic map exhibits a interesting property i.e. the chaotic behavior in the orbits of dynamical system. So before going to define the term *Chaos* let's define some more terminology.

16.1 Some Definitons

Recalling some definitions regarding Chaos from chapter 9.

Definition 43. ***Transitivity*** *Let $f : J \to J$ is said to be topologically transitive if for any pair of open sets U, V and J there exists $k > 0$ such that $f^k(U) \cap V \neq \phi$.*

Roughly speaking if a dynamical system is topologically transitive means there exist points in the space whose orbit can go any where in the space. On other hand it also implies that there exists points which eventually move under iterations from one arbitrarily small neighborhood to any other.

Definition 44. ***Sensitive Dependence*** *Let $f : J \to J$ has sensitive dependence on initial conditions if there exists a $\delta > 0$ such that for any x in J and for neighborhood N of x there exists y in N and $n > 0$ such that*

$$|f^n(x) - f^n(y)| > \delta.$$

16.2 Sensitivity of $F_\mu(x)$

Example 31. *The quadratic map $F_\mu(x) = \mu x(1-x)$ with $\mu > 2 + \sqrt{5}$ posses sensitive dependence on initial conditions on Λ.*

Proof. Choose $\delta < diam(A_0)$. Let $x, y \in \Lambda$ with $x \neq y$.
Then $S(x) \neq S(y)$. So the itinerary of x and y must differ at least one index say, the n th .It implies that $F_\mu^n(x)$ and $F_\mu^n(y)$ must lie on the opposite sides of A_0 so that

$$|F_\mu^n(x) - F_\mu^n(y)| > \delta$$

. □

Definition 45. *Let V be a set. $f : V \to V$ is said to be Chaotic on V if it satisfy the following conditions*

1. *f has sensitive dependence on initial conditions .*
2. *f is topologically transitive.*
3. *Periodic Points are dense in V.*

Remark: In a brief Chaotic map has three ingredients :

a Unpredictability.

b Independence.

c An element of regularity.

Definition 46. ***Expansive*** *$f : J \to J$ is expensive if there exists $\nu > 0$ such that for any $x, y \in J$ with $x \neq y$. there exists $n \in \mathbb{N}$ such that*

$$|f^n(x) - f^n(y)| > \nu$$

.

Example 32. *The quadratic map $F_\mu(x) = \mu x(1-x)$ are chaotic on Λ when $\mu > 2 + \sqrt{5}$.*

17 Structural Stability

A very important notion in the study of dynamical system is the stability or the persistence of the system under small changes or perturbations.
In a brief, no matter how we perturb f or change f slightly we get an equivalent dynamical system. It implies that the dynamical structure of f is stable.

Definition 47. C^0 ***Distance:*** *Let f and g be two maps. Then the C^0 distance between f and g written $d_0(f,g)$ is given by*

$$d_0(f,g) = \sup_{x\in\mathbb{R}} |\, f(x) - g(x)|$$

.

Definition 48. C^r ***Distance:*** *The C^r distance $d_r(f,g)$ is given by*

$$d_r(f,g) = \sup_{x\in\mathbb{R}} \{\, |f(x) - g(x)|, |f^{'}(x) - g^{'}(x)|, \ldots, |f^r(x) - g^r(x)|\}$$

.

We may also consider the C^r distance between two maps on an interval $J \subset \mathbb{R}$ by suitably restricting x and y.

Example 33. *Let $f(x) = 2x$ and $g(x) = (2+\epsilon)x$ have c^0 distance infinity provide I consider the whole $\mathbb{R}$.*

$$d_0(f,g) = \sup_{x\in\mathbb{R}} |\, \epsilon x| = \infty$$

.
But, Let $J = [0, 10]$. then

$$d_0(f,g) = \sup_{x\in J} |\, \epsilon x| = 10\ \epsilon$$

.

Example 34. *$f(x) = 2x$ and $g(x) = 2x + \epsilon$ are $C^r - \epsilon$ distance apart for all r.*

Definition 49. ***Structural Stability:*** *Let $f : J \to J$ then f is said to be structurally stable on J if there exist $\epsilon > 0$ such that whenever $d_r(f,g) < \epsilon$ for $g : J \to J$ it follows that f is topologically conjugate to g.*

17.1 Stability of the map $L(x) = \frac{1}{2}x$

Theorem 23. *Let $L(x) = \frac{1}{2}x$. Then L is C^1 structurally stable on $\mathbb{R}$.*

Proof. We have to show the existence of an $\epsilon > 0$ such that whenever $d_1(L,g) < \epsilon$
then L and$g$ are topologically conjugate.
Claim: any $\epsilon < \frac{1}{2}$ will work.
Now, for if $d_1(L,g) < \epsilon$ then we have

$$\sup_{x\in\mathbb{R}} \{\, |L(x) - g(x)|, |L^{'}(x) - g^{'}(x)|\} < \epsilon$$

. This implies that

$$
\begin{aligned}
&|L^{'}(x) - g^{'}(x)| < \epsilon \\
&\Rightarrow |\frac{1}{2} - g^{'}(x)| < \epsilon \\
&\Rightarrow -\epsilon < g^{'}(x) - \frac{1}{2} < \epsilon \\
&\Rightarrow -\epsilon + \frac{1}{2} < g^{'}(x) < \epsilon + \frac{1}{2}
\end{aligned}
$$

Therefore we have $0 < g^{'}(x) < 1$ for all $x \in \mathbb{R}$. it also implies $g(x)$ is increasing every where.
Claim: $g(x)$ has a unique attracting fixed point p in $\mathbb{R}$. Let say g has two fixed point p and q .Then by Mean value Theorem we have
$|g(p) - g(q)| = |g^{'}(\xi)||p - q| \Rightarrow |g^{'}(\xi)| = 1$ where $\xi \in (p, q)$. This shows that L and g have nearly same dynamics. But to be precise we have to show the existence of topological conjugacy between L and g. To do this we introduce the concept of fundamental domain. By the fundamental domain we mean the orbit of any point (except 0) enter in this set exactly ones under the iteration of the given map.
Let us consider the intervals $5 < |x| < 10$. This is the fundamental domain for the function L. Also for g we can have similar domain .These are $g(10) < x \leq 10$ and $-10 \leq x < g(-10)$.
Construction of conjugacy h.
Define h as following

$$
\begin{aligned}
&h : [5, 10] \to [g(10), 10] \\
&h : [-10, -5] \to [-10, g(-10)]
\end{aligned}
$$

We require h to be increasing so that $h(\pm 10) = \pm 10$.

(a) Let $x \neq 0$. Then by definition of fundamental domain there exist $n \in \mathbb{Z}$ such that $L^n(x)$ belong to fundamental domain of L. Hence $h \circ L^n(x)$ is well defined.
Set $h(x) = g^{-n} \circ h \circ L^n(x)$.
Note, $h(x)$ is well defined since g is homeomorphism. So, g^{-n} make sense.Clearly, we have $g^n \circ h(x) = h \circ L^n(x)$. Then we can extend it to get $g \circ h(x) = h \circ L(x)$.

(b) Finally define $h(0) =$ fixed point of g.

The way we define h is a homeomorphism. Therefore we establish a conjugacy between the map L and g. Hence we are done. □

17.2 Stability of the map $F_\mu(x) = \mu x(1 - x)$

Theorem 24. *Let $F_\mu(x) = \mu x(1 - x)$ where $\mu > 4$. Then we have F_μ is C^1 stable on $[-2, 2]$.*

Proof. Consider the case $\mu > 2 + \sqrt{5}$. Choose δ such that $\frac{\mu}{4} - 1 - \delta > 0$.
Let $I_\delta \equiv [-\delta, 1 + \delta]$ where $\delta > 0$.
Now define $F_\mu^{-1}(I_\delta) = \{x | F_\mu(x) \in I_\delta\}$ for $\delta > 0$ small enough. If $z \in [I_\delta \cap F_\mu^{-1}(I_\delta)] \cup [-2, 0] \cup [1, 2]$ then we have $|F_\mu^{'}(z)| > \lambda > 1$ (This fact we already proved).
Now,

$$F_\mu(x) = \mu x(1 - x) \Rightarrow F_\mu^{'}(x) = \mu - 2\mu x \Rightarrow F_\mu^{'}(\frac{1}{2}) = 0$$

So, to prove the structural stability we have to produce the $\epsilon > 0$ such that when $d_1(f, g) < \epsilon$ g is topologically conjugate to f.
Choose $\epsilon > 0$ such that

1. $g(-\delta) < -\delta$.
2. $g(1+\delta) < -\delta$.
3. $g(\frac{1}{2}) > 1+\delta$.
4. $|g^{'}(z)| > \lambda$.

when $z \in [I_\delta \cap g^{-1}(I_\delta)] \cup [-2,0] \cup [1,2]$
i.e. when $d_1(f,g) < \epsilon \Rightarrow |F(x) - g(x)| < \epsilon$ and $|F^{'}(x) - g^{'}(x)| < \epsilon$.
Choose ϵ_1 such that the following holds true

$$\begin{aligned} g(-\delta) &< -\mu\delta(1+\delta) + \epsilon_1 \\ g(1+\delta) &< -\mu\delta(1+\delta) + \epsilon_1 \end{aligned}$$

Choose ϵ_2 such that

$$\begin{aligned} &g(\frac{1}{2}) > F_\mu(\frac{1}{2} - \epsilon_2) > 1+\delta \\ &\Rightarrow \frac{\mu}{4} - \epsilon_2 > 1+\delta \\ &\Rightarrow \epsilon_2 < \frac{\mu}{4} - 1 - \delta \end{aligned}$$

Choose ϵ_3 such that the following hold true

$$\begin{aligned} &||F_\mu^{'}(z)| - |g^{'}(z)|| < |F^{'}(z) - g^{'}(z)| < \epsilon_3 \\ &|F^{'}(z)| - \epsilon_3 < |g^{'}(z)| < |F^{'}(z)| + \epsilon_3 \\ &|g^{'}(z)| > |F^{'}(z)| - \epsilon_3 > \lambda \end{aligned}$$

Now, choose $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$. Take such a map g with $d_1[F_\mu, g] < \epsilon$. Then from the conditions above we get that $g|_{I_\delta}$ covers I_δ twice.

Lemma 3. *If $g^k(x) \in [-2,2]$ for all $k \geq 0$ then $g^k(x) \in I_\delta$ for all $k \geq 0$.*

Proof. If $g^k(x) \neq I_\delta \equiv [-\delta, 1+\delta]$. Then either $g^k(x) < \delta$ or $g^k(x) > 1+\delta$
Consider the case $g^k(x) < \delta$ and we have the map g is monotone.Then

$$\begin{aligned} &g^{k+1}(x) < g(-\delta) < -\delta \\ &g^{k+2}(x) - (\delta) < g^{k+2}(x) - g(-\delta) < \lambda[g^{k+1}(x) - (-\delta)] \\ &\Rightarrow g^{k+2}(x) < (\delta) \end{aligned}$$

As long as $g^{k+j+1}(x) \in [-2, -\delta]$ we have $g^{k+j+1}(x) < -\delta$.
Now by Mean value Theorem we have

$$g^{k+j}(x) - (-\delta) < \lambda^{j-1}[g^{k+1}(x) - (-\delta)]$$

Since, $\lambda > 1$ after some iterates $g^{k+j}(x) < -2$ for some j.
It is a contradiction to the fact that $g^k(x) \in [-2,2]$. Hence our assumption is wrong.We are done. □

Lemma 4. *Let* $\Lambda_g = \cap_{k=0}^{\infty} g^{-k}(I_\delta)$*. Then* $g|_{\Lambda_g}$ *is topologically conjugate to* σ *on* Σ_2*.*

Proof. $g|_{I_\delta}$ covers I_δ twice. Now, $g^{-1}(I_\delta) \cap I_\delta = I_1^g \cup I_2^g$.

(a) By the Mean Value Theorem $|L(I_1^g)|$ and $|L(I_2^g)| < \lambda^{-1}$ where L denote the length of the set.

(b) g is monotone on each I_j with $|g^{'}(z)| > \lambda$.

Set $I_{j,k}^g = g^{-1}(I_k) \cap I_j$. Now, $S_1^g = \cap g^{-i}(I_\delta)$ is the union of the four intervals $I_{j,k}^g$.
By Mean Value Theorem each of these intervals has bounded length as follows

$$L(I_{j,k}^g) \leq \lambda^{'} L(I_k) \leq \lambda^{-2} L(I_\delta)$$

By induction $S_n^g = \cap_{i=0}^{n} g^{-i}(I_\delta)$ is the union of 2^n intervals length less than $\lambda^{-1} L(I_\delta)$. Therefore, $\Lambda_g = \cap_{k=0}^{\infty} g^{-k}(I_\delta)$ is a Cantor set.
Hence, $g|_{\Lambda_g}$ is topologically conjugate to σ on Σ_2. □

Now, we have $g|_{\Lambda_g}$ is topologically conjugate to σ on Σ_2 which is conjugate to $F_\mu|_{\Lambda_{F_\mu}}$. Therefore, we establish the structural stability of F_μ. Hence, we are done. □

MATLAB Codes

In this thesis, the graph are drawn in *MATLAB*. Here I am giving the *MATLAB* codes that I coded to generate the graph of functions. By just changing the function we can generate the graph of any given function. They are

Graph of any given function with axis limits

```
    x= linspace(-20,20,500);
    c=-3/4;
    y=@(x)x.^2+c;
    p=(1+sqrt(1-4*c))/2;
    %% The only type of function you can define inside a MATLAB script is an
    anonymous function
    figure
    plot(x,y(x),'green')
    hold on
    plot(x,x,'red');
    hold on;
   % rectangle('Position',[-p,-p,2*p,2*p])
    xlim([-p,p])
    ylim([-p,p])
```

Orbit Diagram for the Quadratic family $Q_c(x) = x^2 + c$

```
orbit=zeros(1,100);
j=0;
% parameter range
for r=-2:0.001:0.25
    j=j+1;
    % random nitiation of iteration
    xn1=0;
        for i=1:200
        % calculate logistic map
        xn=xn1;
        xn1=xn.^2+r;
       % wait for transients
       if(i>100)
           % store the orbit points
           orbit(i-100)=xn1;
       end
       end
    plot(r,orbit);
    if(j==1)
        axis([-2 0.5 -2 2]);
    hold;
    end
 end
```

Orbit Diagram for the Logistic family $F_\mu(x) = \mu x(1-x)$

```
orbit=zeros(1,300);
j=0;
% parameter range
for(r=1.1:0.001:4)
    j=j+1;
    % random nitiation of iteration
    xn1=rand(1);
        for(i=1:600)
        % calculate logistic map
        xn=xn1;
        xn1=r*xn*(1-xn);
       % wait for transients
       if(i>300)
           % store the orbit points
           orbit(i-300)=xn1;
       end
    end
    plot(r,orbit);
    if(j==1)
        axis([1.1 4 0 1]);
    hold;
    end
    end
```

Graphical Analysis for given function

```
x= linspace(-20,20,500);
y=@(x)x.^2-2;
%% The only type of function you can define inside a MATLAB script is an anonymous function
figure
plot(x,y(x),'green');
title('Graphical Analysis')
hold on;
plot(x,x,'red');
hold on;
z=-(0.5);
plot(z,z,'o','MarkerSize',10);
m=1;
a=z;
b=z;
while m<20
    hold on ;
    arrowline([z z],[z y(z)]);
    arrowline([z y(z)],[y(z) y(z)]);
     plot(z,z,'o','MarkerSize',10);
     plot(z,y(z),'o','MarkerSize',10);
     z=y(z);
    a=max([a z]);
    b=min([b z]);
    m=m+1;
end
xlim([b-1,a+1])
ylim([b-1,a+1])
```

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