Strong proximinality and intersection properties of balls in Banach spaces

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Abstract

We investigate a variation of the transitivity problem for proximinality properties of subspaces and intersection properties of balls in Banach spaces. For instance, we prove that if $Z \subseteq Y \subseteq X$, where Z is a finite co-dimensional subspace of X which is strongly proximinal in Y and Y is an M-ideal in X, then Z is strongly proximinal in X. Towards this, we prove that a finite co-dimensional proximinal subspace Y of X is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly proximinal in X^{**}. We also prove that in an abstract L_1-space, the notions of strongly subdifferentiable points and quasi-polyhedral points coincide. We also give an example to show that M-ideals need not be ball proximinal. Moreover, we prove that in an L_1-predual space, M-ideals are ball proximinal.

Keywords: Proximinality, strong proximinality, ideal, semi M-ideal, M-ideal.

1. Preliminaries

In this article, we consider only Banach spaces over the real field $\mathbb{R}$ and all subspaces we consider are assumed to be closed. For a Banach space X; B_X, S_X and $B[x, r]$ denote the closed unit ball, the unit sphere and the closed ball with centre at x and radius r respectively. We consider every Banach space X, under the canonical embedding, as a subspace of X^{**}.

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Let K be a non-empty closed subset of a Banach space X. For $x \in X$, let $d(x, K) = \inf\{\|x - k\| : k \in K\}$ and $P_K(x) = \{k \in K : d(x, K) = \|x - k\|\}$. The set K is said to be proximinal in X if $P_K(x) \neq \emptyset$ for all $x \in X$. A subspace Y of X is said to be ball proximinal in X if for every $x \in X$, $P_{B_Y}(x) \neq \emptyset$ (see [2, 16] for details).

In [10], Godefroy and Indumathi introduced a stronger version of proximinality called 'strong proximinality'.

Definition 1.1. A proximinal subspace Y of a Banach space X is said to be *strongly proximinal* in X if for every $x \in X$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that $P_Y(x, \delta) \subseteq P_Y(x) + \varepsilon B_Y$, where $P_Y(x, \delta) = \{y \in Y : \|x - y\| < d(x, Y) + \delta\}$.

In [7], Franchetti and Payá introduced the notion of strong subdifferentiability in Banach spaces which in turn characterizes strongly proximinal hyperplanes.

Definition 1.2. The norm of a Banach space X is said to be *strongly subdifferentiable* (in short SSD) at $x \in X$ if the one sided limit

$$d^+(x)(y) := \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for $y \in B_X$. In this case, x is said to be an SSD point of X. If each $x \in S_X$ is an SSD-point of X, then the norm of X is said to be SSD.

The following result by Godefroy and Indumathi connects SSD-points with strongly proximinal subspaces of co-dimension one.

Theorem 1.3 ([10])**.** *Let X be a Banach space. Then, for $f \in X^*$, $\ker(f)$ is strongly proximinal in X if and only if f is an* SSD*-point of X^*.*

In the case of finite co-dimensional strongly proximinal subspaces, we recall the following result.

Theorem 1.4 ([10])**.** *Let Y be a finite co-dimensional subspace of a Banach space X. If Y is strongly proximinal in X, then $Y^\perp$ is contained in the set of all* SSD*-points of X^*.*

The following notion of a quasi-polyhedral point, introduced in [1] by Amir and Deutsch, is stronger than the notion of an SSD-point.

Definition 1.5. A vector x in a Banach space X is said to be a *quasi-polyhedral* (in short QP) point of X if there exists a $\delta > 0$ such that $J_{X^*}(z) \subseteq J_{X^*}(x)$ for $\|z - x\| < \delta$ and $\|z\| = \|x\|$, where $J_{X^*}(x) = \{f \in B_{X^*} : f(x) = \|x\|\}$.

In [10], Godefroy and Indumathi proved that a QP-point is also an SSD-point but the converse need not be true.

The next result follows from the proof of Theorem 3.4 of [10].

Theorem 1.6 ([10])**.** *Let Y be a finite co-dimensional subspace of a Banach space X such that $Y^{\perp}$ is contained in the set of all* QP*-points of X^* . Then Y is strongly proximinal in X.*

We now recall the notion of an M-ideal in a Banach space which is stronger than proximinality (in fact, stronger than strong proximinality).

Definition 1.7 ([12, 23])**.** Let X be a Banach space.

(a) A linear projection P on X is said to be an *M-projection* (L-projection) if $\|x\| = \max\{\|Px\|, \|x - Px\|\}$ ($\|x\| = \|Px\| + \|x - Px\|$) for all $x \in X$. A function $P : X \to X$ is said to be a semi L-projection if $P^2 = P$, $P(\lambda x + P(z)) = \lambda P(x) + P(z)$ for all $\lambda \in \mathbb{R}$, $x, z \in X$ and $\|x\| = \|P(x)\| + \|x - P(x)\|$ for all $x \in X$.

(b) A subspace Y of X is said to be an *M-summand* (*L-summand*) in X if it is the range of an M-projection (L-projection). A subspace Y of X is said to be a semi *L-summand* if it is the range of a semi L-projection.

(c) A subspace Y of X is said to be an *M-ideal* (*semi M-ideal*) in X if $Y^{\perp}$ is an *L-summand* (semi L-summand) in X^*.

(d) A subspace Y of X is said to be an *ideal* in X if $Y^{\perp}$ is the kernel of a norm one projection on X^*.

It is well-known that each Banach space is an ideal in its bidual.

We next recall some of the intersection properties of balls which are closely related to the proximinality properties.

Definition 1.8 ([12])**.** (a) Let $n \in \mathbb{N}$. A subspace Y of a Banach space X is said to have *the* (*strong*) *n-ball property* if, given n closed balls $\{B[a_i, r_i]\}_{i=1}^n$ in X such that $\bigcap_{i=1}^n B[a_i, r_i] \neq \emptyset$ and $Y \bigcap B[a_i, r_i] \neq \emptyset$ for all $i = 1, \ldots, n$, then $Y \bigcap (\bigcap_{i=1}^n B[a_i, r_i + \varepsilon]) \neq \emptyset$ for all ($\varepsilon \geq 0$) $\varepsilon > 0$.

(b) A subspace Y of a Banach space X is said to have *the* (*strong*) $1\frac{1}{2}$-*ball property* if the conditions $x \in X$, $y \in Y$, $Y \cap B[x,r] \neq \emptyset$ and $\|x - y\| \leq r + s$ $(r, s > 0)$ imply that $Y \cap B[x, r+\varepsilon] \bigcap B[y, s+\varepsilon] \neq \emptyset$ for all $(\varepsilon \geq 0)$ $\varepsilon > 0$.

It is well-known that M-ideals are precisely the subspaces having the 3-ball property (see [12]). It is also known that the semi M-ideals are precisely the subspaces having the 2-ball property (see [17, Theorem 6.10]). Proposition 3.3 of [6] shows that a subspace having the weakest of the above intersection properties, namely the $1\frac{1}{2}$-ball property, is already a strongly proximinal subspace. In particular, M-ideals are strongly proximinal.

[12] is a standard reference for any unexplained terminology.

2. Introduction

One of the interesting problems in approximation theory is the transitivity of various degrees of proximinality and intersection properties of balls. Precisely, let (P) be any one of the properties proximinality, strong proximinality, $1\frac{1}{2}$-ball property or 2-ball property and let Y and Z be subspaces of X with $Z \subseteq Y \subseteq X$ such that Z has property (P) in Y and Y has property (P) in X. Then is it necessary that Z has property (P) in X? The motivation for the study of transitivity problem comes from [20] where Pollul established the transitivity of proximinality for finite co-dimensional subspaces of c_0. In [5], Dutta and Narayana proved the transitivity of strong proximinality for finite co-dimensional subspaces of $C(K)$, and in [21], Payá and Yost proved the transitivity of 2-ball property. More results regarding the transitivity problem for the property (P) can be found in [5, 6, 14, 20, 21].

On the other hand, it is also known that most of the properties listed above as (P), in general, are not transitive. Corollary 7 of [14] shows that proximinality need not be transitive. From [21, Example 6], it follows that the $1\frac{1}{2}$-ball property fails to be transitive. Motivated by these, since each M-ideal satisfies property (P), our main theme in this paper is to discuss the following problem, which is a variation of the above mentioned transitivity problem.

Problem 2.1. *Let X, Y, Z be Banach spaces such that $Z \subseteq Y \subseteq X$ and Y be an M-ideal in X. If (P) is a property which is shared by all M-ideals and if Z has property (P) in Y, does it follow that Z has (P) in X?*

The solution to Problem 2.1 is known to be positive when property (P) is the n-ball property (a new and more natural proof is given in Section 4), but the problem is still open when property (P) is strong proximinality.

In Section 3, we give an example to show that the strong proximinality need not be transitive. Moreover, we prove that Problem 2.1 has an affirmative answer when (P) is strong proximinality and Z is of finite co-dimension in X. In order to prove this, we first prove that a finite co-dimensional proximinal subspace Y of a Banach space X is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly proximinal in X^{**}.

In Section 3, we also consider the following problem.

For an SSD-point f of X^*, there always exists a Hahn-Banach extension of f to X^{**} which is an SSD-point of X^{***}, namely the canonical image of f in X^{***}. But it is not known whether each Hahn-Banach extension of f to X^{**} is again an SSD-point of X^{***}. Coming to a more general set up, we consider the following problem.

Problem 2.2. *If Y is a subspace of a Banach space X and $f \in Y^*$ is an* SSD*-point of Y^*, then can we say that all the Hahn-Banach extensions of f are* SSD*-points of X^*?*

We show that the answer to Problem 2.2 is negative in general (see Example 3.15) and is affirmative if the subspace Y is an M-ideal in X.

We now recall that a Banach space X is said to be an L_1*-predual space* if X^* is isometric to $L_1(\mu)$ for some positive measure μ.

In Section 3, we also prove that the converse of Theorem 1.4 and Theorem 1.6 are true for L_1-predual spaces.

In Section 4, we discuss the intersection properties of balls in Banach spaces. We restrict ourselves to the $1\frac{1}{2}$-ball property and semi M-ideals. We give an affirmative answer to Problem 2.1 when (P) is the n-ball property, where $n = 1\frac{1}{2}, 2$.

Corollary 2.5 of [16] claims that M-ideals are ball proximinal subspaces. In Section 4, we disprove this by giving a counterexample and we also prove that in an L_1-predual space, M-ideals are ball proximinal.

In Section 5, we give an example to show that the strong proximinality assumption on a subspace is not sufficient to guarantee that any proximinal subspace of it is also proximinal in the bigger space. We also discuss some examples regarding intersection properties of balls.

3. Strong Proximinality in Banach Spaces

In this section, we discuss Problem 2.1 with property (P) being strong proximinality and then we consider Problem 2.2. Moreover, we characterize finite co-dimensional strongly proximinal subspaces of an L_1-predual space.

3.1. A variation of transitivity problem for strong proximinality

In [22, Remark 2.4], it is observed that there exists a proximinal subspace of c_0, which is not proximinal in ℓ_∞. Since c_0 is an M-ideal in ℓ_∞, this example shows that Problem 2.1 does not have an affirmative answer when (P) is proximinality. But in [15], it is proved that every finite co-dimensional proximinal subspace of c_0 continues to be proximinal in ℓ_∞.

We now prove that for a subspace Y of a Banach space X, strongly proximinal subspace of Y continue to be strongly proximinal in X under a stronger assumption on the subspace Y.

Proposition 3.1. *Let $X = Y \oplus Z$. Let $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a map such that for each $\beta \in \mathbb{R}_+$, $\varphi(\cdot, \beta)$ is an increasing function on $\mathbb{R}_+$ and for any sequence (α_n) in $\mathbb{R}_+$, $\varphi(\alpha_n, \beta) \to \varphi(\alpha, \beta)$ implies $\alpha_n \to \alpha$. Suppose, for $x \in X$, $\|x\| = \varphi(\|y\|, \|z\|)$, where $x = y + z$ with $y \in Y$ and $z \in Z$. If W is a strongly proximinal subspace of Y, then W is a strongly proximinal subspace of X.*

Proof. Let $x \in X$ and let $x = y + z$ with $y \in Y$ and $z \in Z$. If W is proximinal in Y, then the proximinality of W in X follows from the fact that $P_W(y) \subseteq P_W(x)$. We note that the convergence assumption on φ is not used yet.

Now let W be strongly proximinal in Y. Clearly, $d(x, W) = \varphi(d(y, W), \|z\|)$. Let (w_n) be a sequence in W such that $\|x - w_n\| \to d(x, W)$. Then, by the assumption on φ, $\|y - w_n\| \to d(y, W)$ and hence, by the strong proximinality of W in Y, $d(w_n, P_W(y)) \to 0$. Since $P_W(y) \subset P_W(x)$, $d(w_n, P_W(x)) \to 0$ and hence the strong proximinality of W in X follows. $\square$

As an immediate consequence of Proposition 3.1, it follows that if Y is an L-summand in X, then any strongly proximinal subspace W of Y is strongly proximinal in X. When Y is an M-summand in X, the proof of Proposition 3.1 shows that W is proximinal in X if it is so in Y, but this proposition does not give any conclusion regarding the strong proximinality of W in X even if W is strongly proximinal in Y as the convergence assumption on φ need not be satisfied in this case. So we consider this case separately as our next result.

For a Banach space X, let $\mathcal{C}(X)$ denote the class of non-empty, bounded and closed subsets of X. Then the Hausdorff metric on $\mathcal{C}(X)$ is given by

$$h(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{z\in B} d(z, A)\right\} \quad \text{for } A, B \in \mathcal{C}(X).$$

Proposition 3.2. *Let X be a Banach space and Y be an M-summand in X. If W is strongly proximinal in Y, then W is strongly proximinal in X.*

Proof. Let W be strongly proximinal in Y. Clearly, W is proximinal in X. Let $x \in X$ and let $x = y + z$ with $y \in Y$ and $z \in Z$. Then it follows that $d(x, W) = \max\{d(y, W), \|z\|\}$ and $P_W(y) \subseteq P_W(x)$. Let $\varepsilon > 0$.

Suppose $\|z\| > d(y, W)$. Then $P_W(x) = B[y, \|z\|] \cap W$ and $P_W(x, \eta) = B(y, \|z\| + \eta) \cap W$ for all $\eta > 0$. Since $\|z\| > d(y, W)$, by [15, Fact 3.2], there exists a $\delta > 0$ such that for $u \in Y$ with $\|u - y\| < 2\delta$ and for $\beta > 0$ with $|\beta - \|z\|| < 2\delta$, we get

$$h(B(y, \|z\|) \bigcap W,\ B(u, \beta) \bigcap W) < \varepsilon, \tag{1}$$

where h is the Hausdorff metric on $\mathcal{C}(Y)$. Now, by putting $u = y$ and $\beta = \|z\| + \delta$ in (1), we get $h\left(B(y, \|z\|) \bigcap W, B(y, \|z\| + \delta) \bigcap W\right) < \varepsilon$. Thus $B(y, \|z\| + \delta) \bigcap W \subseteq (B(y, \|z\|) \bigcap W) + \varepsilon B_X$ and hence $P_W(x, \delta) \subseteq P_W(x) + \varepsilon B_X$.

Now suppose $\|z\| \leq d(y, W)$. Then $P_W(x) = P_W(y)$ and $P_W(x, \delta) = P_W(y, \delta)$. Since W is strongly proximinal in Y, there exists a $\delta > 0$ such that $P_W(y, \delta) \subseteq P_W(y) + \varepsilon B_Y$. Thus $P_W(x, \delta) \subseteq P_W(x) + \varepsilon B_X$ and hence the result follows. □

We now recall some notation from [13] in order to state a characterization of finite co-dimensional strongly proximinal subspaces in Banach spaces.

Let X be a Banach space and let $\{f_1, \ldots, f_n\}$ be a set of linearly independent functionals in X^*. Let $M_1 = M_1^* = \|f_1\|$, $J_X(f_1) = \{x \in B_X : f_1(x) = \|f_1\|\}$ and $J_{X^{**}}(f_1) = \{x^{**} \in B_{X^{**}} : x^{**}(f_1) = \|f_1\|\}$.

Now suppose, for an $i \in \{2, \ldots, n\}$, $J_X(f_1, \ldots, f_{i-1})$ is defined and is a non-empty set. Then define

$$\begin{aligned}
M_i &= \sup\{f_i(x) : x \in J_X(f_1, \dots, f_{i-1})\},\\
M_i^* &= \sup\{x^{**}(f_i) : x^{**} \in J_{X^{**}}(f_1, \dots, f_{i-1})\},\\
J_X(f_1, \dots, f_i) &= \{x \in J_X(f_1, \dots, f_{i-1}) : f_i(x) = M_i\},\\
J_{X^{**}}(f_1, \dots, f_i) &= \{x^{**} \in J_{X^{**}}(f_1, \dots, f_{i-1}) : x^{**}(f_i) = M_i^*\}.
\end{aligned}$$

For $\varepsilon > 0$, let $J_X(f_1, \varepsilon) = \{x \in B_X : f_1(x) > \|f_1\| - \varepsilon\}$.
For $i = 2, \dots, n$, define

$$J_X(f_1, \dots, f_i, \varepsilon) = \{x \in J_X(f_1, \dots, f_{i-1}, \varepsilon) : f_i(x) > M_i - \varepsilon\}.$$

Using a weak*-compactness argument, one can see that $J_{X^{**}}(f_1, \dots, f_i) \neq \emptyset$ for $i = 1, \dots, n$. In [13, Theorem 1], it is proved that if Y is a finite co-dimensional proximinal subspace of X, then $J_X(f_1, \dots, f_i) \neq \emptyset$ for $i = 1, \dots, n$ and for every basis $\{f_1, \dots, f_n\}$ of $Y^\perp$.

Throughout this section, we use the following characterization of finite co-dimensional strongly proximinal subspace.

Theorem 3.3 ([10])**.** *Let Y be a finite co-dimensional proximinal subspace of a Banach space X. Then Y is strongly proximinal in X if and only if for any basis $\{f_1, \dots, f_n\}$ of $Y^\perp$,*

$$\lim_{\varepsilon \to 0} [\sup\{d(x, J_X(f_1, \dots, f_i)) : x \in J_X(f_1, \dots, f_i, \varepsilon)\}] = 0$$

for $1 \le i \le n$.

In other words, a necessary and sufficient condition for the strong proximinality of a finite co-dimensional subspace Y of X is: if $\{f_1, \dots, f_n\}$ is a basis of $Y^\perp$ and $i \in \{1, \dots, n\}$, then, for every $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that $d(x, J_X(f_1, \dots, f_i)) < \varepsilon$ whenever $x \in J_X(f_1, \dots, f_i, \delta_\varepsilon)$.

We now exhibit some relations between the notations defined above.

Proposition 3.4. *Let Y be a finite co-dimensional strongly proximinal subspace of a Banach space X and let $\{f_1, \dots, f_n\} \subseteq S_{Y^\perp}$ be a basis of $Y^\perp$. For $1 \le i \le n$, let M_i, M_i^*, $J_X(f_1, \dots, f_i)$ and $J_{X^{**}}(f_1, \dots, f_i)$ be defined as above. Then, for $1 \le i \le n$,*

(a) $M_i = M_i^*$,

(b) $J_{X^{**}}(f_1, \ldots, f_i) = \overline{J_X(f_1, \ldots, f_i)}^{w^*}$.

Proof. (a). Clearly, $M_1 = M_1^*$ and $M_2 \le M_2^*$. Let $i \in \{1, \ldots, n\}$. Now suppose that $M_j = M_j^*$ for $1 \le j \le i$. Then $M_{i+1} \le M_{i+1}^*$. Since $J_{X^{**}}(f_1, \ldots, f_i)$ is weak*-compact, f_{i+1} attains its supremum over $J_{X^{**}}(f_1, \ldots, f_i)$ at some element $x_0^{**} \in J_{X^{**}}(f_1, \ldots, f_i)$. Let (x_α) be a net in B_X such that $x_\alpha \to x_0^{**}$ in weak*-sense. Since $x_0^{**} \in J_{X^{**}}(f_1, \ldots, f_i)$, $x_0^{**}(f_j) = M_j^* = M_j$ for $1 \le j \le i$. Hence, for $1 \le j \le i$, $f_j(x_\alpha) \to M_j$. Since Y is strongly proximinal in X, by Theorem 3.3, it follows that $d(x_\alpha, J_X(f_1, \ldots, f_i)) \to 0$. Now let (z_α) be a net in $J_X(f_1, \ldots, f_i)$ such that $\|x_\alpha - z_\alpha\| \to 0$. Then $z_\alpha \to x_0^{**}$ in weak*-sense. Since $f_{i+1}(z_\alpha) \to x_0^{**}(f_{i+1}) = M_{i+1}^*$, we get $M_{i+1}^* = \lim_\alpha f_{i+1}(z_\alpha) \le M_{i+1}$. Now the result follows by induction.

(b). Since f_1 is an SSD-point of X^*, $\overline{J_X(f_1)}^{w^*} = J_{X^{**}}(f_1)$. Clearly, $\overline{J_X(f_1, f_2)}^{w^*} \subseteq J_{X^{**}}(f_1, f_2)$. Let $\phi \in J_{X^{**}}(f_1, f_2)$ and choose a net (x_α) in B_X such that $x_\alpha \to \phi$ in weak*-sense. Since $f_1(x_\alpha) \to \phi(f_1)$, $d(x_\alpha, J_X(f_1)) \to 0$. Choose a net (y_α) in $J_X(f_1)$ such that $\|x_\alpha - y_\alpha\| \to 0$. Hence $y_\alpha \to \phi$ in weak*-sense. Since $f_2(y_\alpha) \to \phi(f_2) = M_2$, $d(y_\alpha, J_X(f_1, f_2)) \to 0$. Hence there exists a net $(z_\alpha) \subseteq J_X(f_1, f_2)$ such that $\|y_\alpha - z_\alpha\| \to 0$, which in turn implies that $z_\alpha \to \phi$ in weak*-sense. i.e., $\overline{J_X(f_1, f_2)}^{w^*} = J_{X^{**}}(f_1, f_2)$. By a similar argument, we can prove (b) for $i > 2$. □

Remark 3.5. Proposition 3.4(b) is a generalization of [10, Remark 1.2(2)].

Remark 3.6. Our next result generalizes a known fact related to the strongly proximinal hyperplanes in Banach spaces (see [10, Remark 1.2(1)]). Precisely, [10, Remark 1.2(1)] can be obtained by putting $n = 1$ in Lemma 3.7. We follow the idea used in the proof of [11, Fact 2] to prove our next result.

Lemma 3.7. *Let Y be a finite co-dimensional strongly proximinal subspace of a Banach space X. Let $\{f_1, \ldots, f_n\} \subset S_{Y^\perp}$ be a basis of $Y^\perp$. Then, for $x \in B_X$ and $1 \le i \le n$, $d(x, J_X(f_1, \ldots, f_i)) = d(x, J_{X^{**}}(f_1, \ldots, f_i))$.*

Proof. If $n = 1$, then the conclusion follows from [10, Remark 1.2(1)].

Since no new ideas are required for $n > 2$, we only prove the case $n = 2$. Hence we have to show that for $x \in B_X$, $d(x, J_X(f_1, f_2)) = d(x, J_{X^{**}}(f_1, f_2))$.

Let $d = d(x, J_{X^{**}}(f_1, f_2))$. Since $J_{X^{**}}(f_1, f_2)$ is weak*-compact, it is proximinal in X^{**}. Choose $\phi \in J_{X^{**}}(f_1, f_2)$ such that $\|x - \phi\| = d$.

Since Y is strongly proximinal in X, for every $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that $d(x, J_X(f_1, f_2)) < \varepsilon$ whenever $x \in J_X(f_1, f_2, \delta_\varepsilon)$.

Now let $\varepsilon > 0$ be arbitrary. Choose an $\varepsilon' > 0$ such that $0 < \varepsilon' < \min\{\delta_{\varepsilon/2^2}, \frac{\varepsilon}{2(d+1)}\}$. Let $E = \text{span}\{x, \phi\} \subseteq X^{**}$ and $F = \text{span}\{f_1, f_2\} \subseteq X^*$. Then, by principle of local reflexivity, there exists a bounded linear map $T : E \to X$ such that $T(x) = x$, $(1-\varepsilon') \leq \|T(z^{**})\| \leq (1+\varepsilon')$ if $z^{**} \in S_E$ and $f_i(T(z^{**})) = z^{**}(f_i)$ for $i = 1, 2$.

Now let $x_1 = \frac{T\phi}{\|T\phi\|}$. Then

$$\begin{aligned} \|x - x_1\| &\leq \|x - T\phi\| + \|T\phi - \frac{T\phi}{\|T\phi\|}\| \\ &= \|T(x-\phi)\| + |1 - \|T\phi\|| \\ &\leq (1+\varepsilon')d + \varepsilon' \\ &= d + \varepsilon'(1+d) < d + \frac{\varepsilon}{2} \end{aligned}$$

and for $i = 1, 2$; by Proposition 3.4(a), we have

$$f_i(x_1) = f_i\left(\frac{T\phi}{\|T\phi\|}\right) \geq \frac{M_i^*}{1+\varepsilon'} = \frac{M_i}{1+\varepsilon'} = M_i - \frac{M_i\varepsilon'}{1+\varepsilon'} > M_i - \varepsilon' > M_i - \delta_{\varepsilon/2^2}.$$

Thus $x_1 \in J_X(f_1, f_2, \delta_{\varepsilon/2^2})$ and $d(x_1, J_{X^{**}}(f_1, f_2)) \leq d(x_1, J_X(f_1, f_2)) < \varepsilon/2^2$. Let $\phi_1 \in J_{X^{**}}(f_1, f_2)$ be such that $\|x_1 - \phi_1\| < \varepsilon/2^2$. Then, again by principle of local reflexivity, there exists an element $x_2 \in B_X$ such that $\|x_1 - x_2\| < \varepsilon/2^2$ and $f_i(x_2) > M_i - \delta_{\varepsilon/2^3}$.

Proceeding inductively, we obtain a sequence (x_n) in B_X such that $\|x_n - x_{n-1}\| < \varepsilon/2^n$ and $f_i(x_n) > M_i - \delta_{\varepsilon/2^{n+1}}$ for all $n \in \mathbb{N}$ and $i = 1, 2$. Without loss of generality, we assume that $\delta_{\varepsilon/2^n} \to 0$.

Clearly, (x_n) is a Cauchy sequence and hence there exists a $z \in B_X$ such that $z = \lim_{n\to\infty} x_n$. Now $f_i(z) = M_i$ for $i = 1, 2$ and hence $z \in J_X(f_1, f_2)$. Also $\|x - x_n\| \leq d + \varepsilon/2 + \ldots + \varepsilon/2^n$ for all $n \in \mathbb{N}$. Now, letting $n \to \infty$, it follows that $\|x - z\| \leq d + \varepsilon$. Since $\varepsilon > 0$ is arbitrary and $z \in J_X(f_1, f_2)$, $d(x, J_X(f_1, f_2)) \leq d = d(x, J_{X^{**}}(f_1, f_2))$ and hence the result follows. $\square$

Combining [7, Theorem 1.2] and [10, Lemma 1.1], we get the following result.

Proposition 3.8. *Let X be a Banach space and $f \in X^*$. Then f is an* SSD*-point of X^* if and only if f is an* SSD*-point of X^{***}.*

Remark 3.9. If Y is a finite co-dimensional subspace of a Banach space X, then $\dim(Y^\perp) = \dim(X^{**}/Y^{\perp\perp})$ and therefore dimension of $Y^\perp$ in X^* equals the dimension of $Y^{\perp\perp\perp}$ in X^{***}.

Now, by combining Theorem 1.3 and Proposition 3.8, it follows that a hyperplane Y in a Banach space X is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly proximinal in X^{**}. Our next result generalizes this to finite co-dimensional subspaces.

Theorem 3.10. *If Y is a finite co-dimensional proximinal subspace of a Banach space X, then Y is strongly proximinal in X if and only if $Y^{\perp\perp}$ is strongly proximinal in X^{**}.*

Proof. Suppose that Y is strongly proximinal in X. Let $\{f_1, \dots, f_n\} \subset S_{Y^{\perp\perp\perp}}$ be a basis of $Y^{\perp\perp\perp}$. As $Y^{\perp}$ is finite dimensional, $Y^{\perp\perp\perp} = Y^{\perp}$. Thus $\{f_1, \dots, f_n\}$ is also a basis of $Y^{\perp}$.

Now let $i \in \{1, \dots, n\}$ and let $\varepsilon > 0$. Since Y is strongly proximinal in X, there exists a $\delta > 0$ such that $d(x, J_X(f_1, \dots, f_i)) < \varepsilon$ whenever $x \in J_X(f_1, \dots, f_i, \delta)$. Then, for $x^{**} \in J_{X^{**}}(f_1, \dots, f_i, \delta)$, $x^{**}(f_j) > M_j - \delta$ for $1 \le j \le i$. Let (x_α) be a net in B_X such that $x_\alpha \to x^{**}$ in weak*-sense. Now, without loss of generality, we assume that $f_j(x_\alpha) > M_j - \delta$ for all α and for $1 \le j \le i$. Hence there exists an element $z_\alpha \in J_X(f_1, \dots, f_i)$ such that $\|x_\alpha - z_\alpha\| < \varepsilon$. Passing to a subnet of (z_α), if necessary, we may assume that $z_\alpha \to \phi$ in weak*-sense for some $\phi \in J_{X^{**}}(f_1, \dots, f_i)$. Thus $(x_\alpha - z_\alpha) \to (x^{**} - \phi)$ in the weak*-sense. Then $\|x^{**} - \phi\| \le \underline{\lim}_\alpha \|x_\alpha - z_\alpha\| \le \varepsilon$. Therefore $d(x^{**}, J_{X^{**}}(f_1, \dots, f_i)) \le \|x^{**} - \phi\| < \varepsilon$. Hence, by Theorem 3.3, $Y^{\perp\perp}$ is strongly proximinal in X^{**}.

Conversely, suppose that $Y^{\perp\perp}$ is a strongly proximinal subspace of X^{**}. Let $\{f_1, \dots, f_n\} \subset S_{Y^{\perp}}$ be a basis of $Y^{\perp}$ and let $\varepsilon > 0$. Since $Y^{\perp\perp\perp} = Y^{\perp}$, $\{f_1, \dots, f_n\}$ is also a basis of $Y^{\perp\perp\perp}$. Let $i \in \{1, \dots, n\}$. Clearly, $J_X(f_1, \dots, f_i, \delta) \subseteq J_{X^{**}}(f_1, \dots, f_i, \delta)$. Since $Y^{\perp\perp}$ is strongly proximinal in X^{**}, there exists a $\delta > 0$ such that $d(x^{**}, J_{X^{**}}(f_1, \dots, f_i)) < \varepsilon$ whenever $x^{**} \in J_{X^{**}}(f_1, \dots, f_i, \delta)$. Then, for $x \in J_X(f_1, \dots, f_i, \delta)$, by Lemma 3.7, $d(x, J_X(f_1, \dots, f_i)) = d(x, J_{X^{**}}(f_1, \dots, f_i)) < \varepsilon$ and this completes the proof. $\square$

We now give an example to show that the strong proximinality need not be transitive. Before going to the proof, we now recall a characterization of SSD-points of ℓ_∞.

Theorem 3.11 ([8, Theorem 5])**.** *An element $x \in \ell_\infty$ is an* SSD*-point of ℓ_∞ if and only if* $\sup\{|x(n)| : |x(n)| \neq \|x\|\} < \|x\|$.

Example 3.12. *There exist two subspaces Z and Y of finite co-dimension in ℓ_1 such that Z is strongly proximinal in Y and Y is strongly proximinal in ℓ_1, but Z is not strongly proximinal in ℓ_1.*

Proof. Let $f = (0, 1, 1, \dots)$ and $g = (1, -\frac{1}{2}, -\frac{1}{3}, \dots)$. Then, by Theorem 3.11, f and g are SSD-points of ℓ_∞ and hence, by [6, Theorem 2.1], f and g are QP-points of ℓ_∞. Let $Z = \ker(f) \cap \ker(g)$ and $Y = \ker(f)$. Since f is a QP-point of ℓ_∞, Y is strongly proximinal in ℓ_1. Also, since g attains its norm on Y and g is a QP-point of ℓ_∞, by the proof of [6, Proposition 4.2], $g|_Y$ is a QP-point of Y^*. Hence $Z = \ker(g|_Y)$ is strongly proximinal in Y. Since, by Theorem 3.11, $f + g \in Z^\perp$ is not an SSD-point of ℓ_∞, it follows from Theorem 1.4 that Z is not strongly proximinal in ℓ_1. □

Our next theorem shows that for an M-ideal Y in a Banach space X, a strongly proximinal subspace of Y having finite co-dimension in X remains to be strongly proximinal in X.

Theorem 3.13. *Let X be a Banach space and Z be a finite co-dimensional proximinal subspace of X. Let Y be an M-ideal in X and $Z \subseteq Y \subseteq X$. If Z is strongly proximinal in Y, then Z is strongly proximinal in X.*

Proof. Let Z be strongly proximinal in Y. Then, by Theorem 3.10, it follows that $Z^{\perp\perp}$ is strongly proximinal in $Y^{\perp\perp}$. Since $Y^{\perp\perp}$ is an M-summand in X^{**}, by Proposition 3.2, $Z^{\perp\perp}$ is strongly proximinal in X^{**}. Then, again by Theorem 3.10, Z is strongly proximinal in X. □

We do not know whether we can replace the M-ideal assumption in Theorem 3.13 by the semi M-ideal assumption. The idea used in the proof of Theorem 3.13 will not be useful in the semi M-ideal case as the bidual of a semi M-ideal is again a semi M-ideal, which we will prove in Lemma 4.2.

Remark 3.14. We do not know whether the finite co-dimensionality assumption on Y in Theorem 3.13 is necessary. The answer is not known even if the strong proximinality in Theorem 3.13 is replaced by proximinality.

3.2. SSD-points and Hahn-Banach extensions

For a subspace Y of a Banach space X, one can ask about the strong subdifferentiability of Hahn-Banach extensions of an SSD-point of Y^*. To begin with, we give an example to show that all the Hahn-Banach extensions of an SSD-point of Y^* need not be SSD-points of X^*.

Example 3.15. *There exist a subspace Y of ℓ_1 and an* SSD*-point of Y^* such that one of its Hahn-Banach extensions is not an* SSD*-point of ℓ_∞.*

Proof. Let f, g, Z and Y be as in Example 3.12. Since $g|_Y$ is an SSD-point of Y^* and $f+g$ is a Hahn-Banach extension of $g|_Y$, the conclusion follows from Example 3.12. □

We now prove that for an M-ideal Y in a Banach space X, the Hahn-Banach extension of an SSD-point of Y^* to X is an SSD-point of X^*.

Our next result is a particular case of [7, Proposition 2.1], but for the sake of completeness, we outline the proof below.

Proposition 3.16. *Let Y be a semi L-summand in a Banach space X and let $y \in Y$ be an* SSD*-point of Y. Then y is also an* SSD*-point of X.*

Proof. Let $P : X \to X$ be a semi L-projection with range Y. Then

$$d^+(y)(x) = d^+(y)(Px) + \|x - Px\|.$$

Now the conclusion follows from the following equation.

$$\frac{\|y+tx\|-1}{t} - d^+(y)(x) = \|Px\| \left(\frac{\|y + t\|Px\|\frac{Px}{\|Px\|}\| - 1}{\|Px\|t} - d^+(y)(\frac{Px}{\|Px\|}) \right).$$

□

Since, by [12, Chapter I, Remark 1.13], for an M-ideal Y in X, $X^* = Y^* \bigoplus_1 Y^\perp$, the following corollary is immediate from Proposition 3.16.

Corollary 3.17. *If Y is an M-ideal in a Banach space X and $f \in Y^*$ is an* SSD*-point of Y^*, then the unique Hahn-Banach extension of f to X is also an* SSD*-point of X^*.*

3.3. Strong proximinality in L_1-predual spaces

Since a QP-point is an SSD-point and also since the converse need not be true, it is natural to ask about the class of Banach spaces where the notions of an SSD-point and a QP-point coincide. We now show that for a positive measure μ, these two notions coincide in $L_1(\mu)$.

Proposition 3.18. *For a positive measure μ, an* SSD*-point of $L_1(\mu)$ is also a* QP*-point of $L_1(\mu)$.*

Proof. Let $f \in L_1(\mu)$ be an SSD-point. Since $L_1(\mu)$ is an L-summand in its bidual, by Proposition 3.16, f is an SSD-point of $L_1(\mu)^{**} = C(K)^*$ (up to an isometry) for some compact Hausdorff space K. Then, by [5, Theorem 2.1], f is a QP-point of $L_1(\mu)^{**}$ and hence f is a QP-point of $L_1(\mu)$. □

Now it follows from the proof of Example 3.12 that the sum of two SSD-points in a Banach space need not be an SSD-point. But in our next result, we prove that the sum of two SSD-points of $L_1(\mu)$ is an SSD-point of $L_1(\mu)$.

Corollary 3.19. *For a positive measure μ, sum of two SSD-points of $L_1(\mu)$ is an SSD-point of $L_1(\mu)$.*

Proof. Let f and g be two SSD-points of $L_1(\mu)$. Since $L_1(\mu)$ is an L-summand in its bidual, by Proposition 3.16, f and g are SSD-points of $L_1(\mu)^{**} = C(K)^*$ (up to an isometry) for some compact Hausdorff space K. Since, by [5, Theorem 2.1], SSD-points of $C(K)^*$ are precisely the finitely supported measures, $f + g$ is an SSD-points of $C(K)^* = L_1(\mu)^{**}$. Hence $f + g$ is an SSD-point of $L_1(\mu)$. □

Our next result characterizes finite co-dimensional strongly proximinal subspaces of L_1-predual spaces. The following result also shows that the converse of Theorem 1.4 and Theorem 1.6 are true in L_1-predual spaces.

Proposition 3.20. *Let X be an L_1-predual space and Y be a finite co-dimensional proximinal subspace of X. Then the following are equivalent*:

(i) *Y is strongly proximinal in X.*

(ii) $Y^\perp \subseteq \{x^* \in X^* : x^*$ *is an* SSD-*point of* $X^*\}$.

(iii) $Y^\perp \subseteq \{x^* \in X^* : x^*$ *is a* QP-*point of* $X^*\}$.

Proof. The implication (i) ⇒ (ii) follows from Theorem 1.3 and the implication (ii) ⇒ (i) follows from Proposition 3.18 and Theorem 1.6. Finally, (ii) ⟺ (iii) follows from Proposition 3.18. □

If Y is a finite co-dimensional strongly proximinal subspace of a Banach space X, then, by Theorem 1.4, Y is the intersection of finitely many strongly proximinal hyperplanes. Our next result shows that the converse of this is true in L_1-predual spaces.

Corollary 3.21. *Let X be an L_1-predual space and let $Y_1, \dots, Y_n$ be strongly proximinal subspaces of finite co-dimension in X. Then $\bigcap_{i=1}^n Y_i$ is strongly proximinal in X.*

Proof. Let $Y = \bigcap_{i=1}^m Y_i$. For $1 \leq i \leq m$, let $f_{i,1}, \dots, f_{i,n_i}$ be SSD-points of X^* such that $Y_i = \bigcap_{k=1}^{n_i} \ker(f_{i,k})$. Thus $Y = \bigcap_{i,k} \ker(f_{i,k})$ and hence, by Corollary 3.19, $Y^\perp = \operatorname{span}\{f_{i,k} : 1 \leq i \leq m, 1 \leq k \leq n_i\} \subseteq \{f \in X^* : f \text{ is an SSD-point of } X^*\}$. Hence, by Proposition 3.20, Y is strongly proximinal in X. □

4. Intersection Properties of Balls in Banach spaces

In this section, we consider Problem 2.1 with property (P) being $1\frac{1}{2}$-ball property or 2-ball property. We also disprove Corollary 2.5 of [16] which states that M-ideals are ball proximinal. Moreover, we prove that in an L_1-predual space, M-ideals are ball proximinal.

4.1. A variation of transitivity problem for n-ball property with $n = 1\frac{1}{2}, 2$

We now prove a variation of transitivity problem for n-ball property with $n \in \mathbb{N}$.

Lemma 4.1. *Let Y be an M-summand in a Banach space X and Z be a subspace of Y. Let $n \in \mathbb{N}$.*

(*a*) *If Z has the (strong) n-ball property in Y, then Z has the (strong) n-ball property in X.*

(*b*) *If Z has the (strong) $1\frac{1}{2}$-ball property in Y, then Z has the (strong) $1\frac{1}{2}$-ball property in X.*

Proof. (*a*) Let Z has the n-ball property in Y. Let $\varepsilon > 0$ and $\{B[x_i, r_i]\}_{1 \leq i \leq n}$ be a family of n balls in X such that

$$B[x_i, r_i] \cap Z \neq \emptyset \text{ for all } i = 1, \dots, n \text{ and } \bigcap_{i=1}^n B[x_i, r_i] \neq \emptyset.$$

Let $x \in \bigcap_{i=1}^n B[x_i, r_i]$ and $P : X \to X$ be an M-projection with range Y. Then $Px \in \bigcap_{i=1}^n B[Px_i, r_i]$ and $B[Px_i, r_i] \cap Z \neq \emptyset$. Then, by the n-ball property of Z in Y, there exists an element $z \in Z \bigcap (\cap_{i=1}^n B[Px_i, r_i + \varepsilon])$. Hence $\|z - x_i\| \leq \max\{\|z - Px_i\|, \|x_i - Px_i\|\} \leq r_i + \varepsilon$ for $1 \leq i \leq n$. Thus Z has the n-ball property in X.

If Z has the strong n-ball property in Y, then the strong n-ball property of Z in X follows by taking $\varepsilon = 0$ in the above proof.

A similar proof also works for (b). □

Our next result is an analogue of Theorem 3.10 in the context of n-ball property with $n = 1\frac{1}{2}, 2$.

Lemma 4.2. *Let Y be a subspace of a Banach space X and let $n = 1\frac{1}{2}, 2$. Then Y has the n-ball property in X if and only if $Y^{\perp\perp}$ has the n-ball property in X^{**}.*

Proof. Suppose that Y has the 2-ball property in X. Then Y is a semi M-ideal in X and hence $Y^{\perp}$ is a semi L-summand in X^*. Then, by [17, Theorem 6.14], $Y^{\perp\perp}$ is a semi M-ideal in X^{**} and hence $Y^{\perp\perp}$ has 2-ball property in X^{**}.

Conversely, suppose that $Y^{\perp\perp}$ has the 2-ball property in X^{**}. Let $\varepsilon > 0$ and let $\{B[x_i, r_i]\}_{i=1,2}$ be two balls in X such that $B[x_i, r_i] \cap Y \neq \emptyset$ for $i = 1, 2$ and $B[x_1, r_1] \cap B[x_2, r_2] \neq \emptyset$.

Since $Y^{\perp\perp}$ is a weak*-closed subspace of X^{**}, $Y^{\perp\perp}$ has the strong 2-ball property in X^{**}. Hence there exists an element $x^{**} \in Y^{\perp\perp}$ such that $\|x^{**} - x_i\| \leq r_i$ for $i = 1, 2$.

Let $E = \text{span}\{x_1, x_2, x^{**}\}$ and $r = \max\{r_1, r_2\}$. Then, by an extended version of principle of local reflexivity (see [4, Theorem 3.2]), there exists a bounded linear map $T_\varepsilon \colon E \to X$ such that $T_\varepsilon(z) = z$ for $z \in E \cap X$, $T_\varepsilon(E \cap Y^{\perp\perp}) \subset Y$ and $\|T_\varepsilon\| \leq 1 + \frac{\varepsilon}{r}$. Now take $z = T_\varepsilon(x^{**})$. Then $z \in Y$ and $\|z - x_i\| \leq r_i + \varepsilon$ for $i = 1, 2$. Hence Y has the 2-ball property in X.

The case $n = 1\frac{1}{2}$ is the (ii)$\Leftrightarrow$ (iv) of [24, Theorem 3]. □

Corollary 4.3. *Let Y be a semi M-ideal in a Banach space X. Then Y is a semi M-ideal in X^{**} if and only if Y is an M-ideal in Y^{**}.*

Proof. Suppose Y is a semi M-ideal in X^{**}. Then Y is a semi M-ideal in $Y^{\perp\perp} = Y^{**}$ and hence, by [18, Corollary 3.4], Y is an M-ideal in Y^{**}.
Conversely, suppose that Y is an M-ideal in Y^{**}. Since Y is a semi M-ideal in X, by Lemma 4.2, $Y^{\perp\perp}$ is a semi M-ideal in X^{**}. Then, by [21, Theorem 5], Y is a semi M-ideal in X^{**}. □

Our next theorem is a particular case of [21, Theorem 5] but our arguments are completely different and should be of interest.

Theorem 4.4. *Let Z and Y be subspaces of a Banach space X such that $Z \subseteq Y \subseteq X$ and Y is an M-ideal in X. Let $n = 1\frac{1}{2}, 2$. If Z has the n-ball property in Y, then Z has the n-ball property in X.*

Proof. **Case 1:** $n = 1\frac{1}{2}$.

Since $Z \subset Y \subset X$, $Z^{\perp\perp} \subset Y^{\perp\perp} \subset X^{**}$. Then, by Lemma 4.2, $Z^{\perp\perp}$ has the $1\frac{1}{2}$-ball property in $Y^{\perp\perp}$ and by Lemma 4.1, $Z^{\perp\perp}$ has the $1\frac{1}{2}$-ball property in X^{**}. Then, by Lemma 4.2, Z has the $1\frac{1}{2}$-ball property in X.

Case 2: $n = 2$.

Since $Z \subset Y \subset X$, $Z^{\perp\perp} \subset Y^{\perp\perp} \subset X^{**}$. Then, by Lemma 4.2, $Z^{\perp\perp}$ is a semi M-ideal in $Y^{\perp\perp}$ and by Lemma 4.1, $Z^{\perp\perp}$ is a semi M-ideal in X^{**}. Then, by Lemma 4.2, Z is a semi M-ideal in X. □

Remark 4.5. We do not know the analogue of Theorem 4.4 in the context of the strong $1\frac{1}{2}$-ball property and the strong 2-ball property.

4.2. M-ideals and Ball Proximinality

In [16], it is proved that a subspace has the strong $1\frac{1}{2}$-ball property if and only if it is ball proximinal and has $1\frac{1}{2}$-ball property. In Corollary 2.5 of [16], it is incorrectly assumed that M-ideals have the strong $1\frac{1}{2}$-ball property, which is not the case, as shown by Example 13 of [24]. Hence Corollary 2.5 of [16] which states that the M-ideals are ball proximinal is incorrect. However, it is well-known that M-ideals have the $1\frac{1}{2}$-ball property and therefore it follows from the results of [16] that an M-ideal is ball proximinal if and only if it has the strong $1\frac{1}{2}$-ball property.

We now give a class of Banach spaces where M-ideals are ball proximinal.

Definition 4.6 ([19])**.** Let X be a Banach space and $n \in \mathbb{N}$. Then X has the $n.2.I.P.$ if any pairwise intersecting family of n balls in X actually intersect.

It is well-known that a Banach space is an L_1-predual space if and only if it has the $4.2.I.P.$ (see [19] for details).

It follows from [17, Proposition 6.5] that an M-ideal in an L_1-predual space has the strong 3-ball property. Our next result generalizes this to any Banach space having the $3.2.I.P.$

Theorem 4.7. *If X has the $3.2.I.P.$, then every M-ideal in X satisfies the strong 3-ball property. In particular, an M-ideal in an L_1-predual space has the strong 3-ball property.*

Proof. Let Y be an M-ideal in X and let $\{B[x_i, r_i]\}_{i=1}^3$ be a family of 3 closed balls satisfying $B[x_i, r_i] \cap Y \neq \emptyset$ for $1 \leq i \leq 3$ and $\bigcap_{i=1}^3 B[x_i, r_i] \neq \emptyset$.

Also, let $\varepsilon > 0$. Since Y is an M-ideal in X, there exists an element $y_0 \in Y$ such that $y_0 \in \bigcap_{i=1}^3 B[x_i, r_i + \varepsilon]$. Now fix an $i \in \{1, 2, 3\}$. Then $\{B[x_j, r_j] : 1 \leq j \leq 3, j \neq i\} \cup \{B[y_0, \varepsilon]\}$ is a pairwise intersecting family of 3 closed balls in X. Since X has the $3.2.I.P.$, the intersection of these three balls is non-empty. Since Y is an M-ideal in X, there exists an element $y_i \in Y$ such that

$$\begin{aligned} \|y_i - x_j\| \leq & r_j + \frac{\varepsilon}{6} \quad \text{for } 1 \leq j \leq 3 \text{ and } j \neq i \quad \text{and} \\ \|y_i - y_0\| \leq & \varepsilon + \frac{\varepsilon}{6}. \end{aligned}$$

We now follow the technique used in [19, Lemma 4.2] for the rest of the proof.

Let $y = \frac{1}{3}\sum_{i=1}^3 y_i$. Then, for $1 \leq j \leq 3$, we get

$$\begin{aligned} \|y - y_0\| \leq & 2\varepsilon \quad \text{and} \\ \|y - x_j\| \leq & \frac{1}{3}\left(\sum_{\substack{1 \leq i \leq 3 \\ i \neq j}} \|y_i - x_j\| + \|y_j - x_j\|\right) \\ \leq & \frac{1}{3}\left(2(r_j + \frac{\varepsilon}{6}) + \|y_j - y_0\| + \|y_0 - x_j\|\right) \\ \leq & r_j + \frac{5}{6}\varepsilon. \end{aligned}$$

Now let $z_0 = y_0$ and $z_1 = y$. Suppose we have constructed $z_1, \ldots, z_m$ such that

$$\begin{aligned} \|z_k - z_{k-1}\| \leq & 2\left(\frac{5}{6}\right)^{k-1} \varepsilon \quad \text{for } 1 \leq k \leq m \quad \text{and} \\ \|z_k - x_j\| \leq & r_j + \left(\frac{5}{6}\right)^k \varepsilon \quad \text{for } 1 \leq k \leq m \text{ and } 1 \leq j \leq 3. \end{aligned}$$

Now fix an $i \in \{1, 2, 3\}$. Then $\{B[x_j, r_j] : 1 \leq j \leq 3, j \neq i\} \cup \{B[z_m, (\frac{5}{6})^m \varepsilon]\}$ is a pairwise intersecting family of 3 closed balls in X. Then, by arguing as above, there exists a $z_{m,i} \in Y$ such that

$$\|z_{m,i} - x_j\| \leq r_j + \frac{1}{6}\left(\frac{5}{6}\right)^m \varepsilon \text{ for } 1 \leq j \leq 3 \text{ and } j \neq i \text{ and}$$
$$\|z_{m,i} - z_m\| \leq \left(\frac{5}{6}\right)^m \varepsilon + \frac{1}{6}\left(\frac{5}{6}\right)^m \varepsilon.$$

Now let $z_m = \frac{1}{3}\sum_{i=1}^{3} z_{m,i}$. Then, for $1 \leq j \leq 3$, we get

$$\|z_{m+1} - z_m\| \leq 2\left(\frac{5}{6}\right)^m \varepsilon \text{ and } \|z_{m+1} - x_j\| \leq r_j + \left(\frac{5}{6}\right)^{m+1} \varepsilon.$$

Thus, by induction, there exists a Cauchy sequence (z_m) in Y such that

$$\|z_m - x_j\| \leq r_j + \left(\frac{5}{6}\right)^m \varepsilon \text{ for } 1 \leq j \leq 3.$$

Now let $z = \lim_{m\to\infty} z_m$. Then $z \in \bigcap_{j=1}^{3} B[x_j, r_j] \cap Y$ and hence the theorem follows. □

Combining Theorem 4.7 and [16, Theorem 2.4], we get the following corollary.

Corollary 4.8. *If X has the 3.2.I.P., then every M-ideal in X is ball proximinal. In particular, M-ideals in L_1-predual spaces are ball proximinal.*

5. Some Examples

Our first example shows that the strong proximinality assumption on a subspace is not sufficient to guarantee that any proximinal subspace of it is also proximinal in the bigger space.

Example 5.1. *There exist two subspaces Z and Y of finite co-dimension in $C[0,1]$ such that Z is proximinal in Y and Y is strongly proximinal in $C[0,1]$, but Z is not proximinal in $C[0,1]$.*

Proof. Let $k \in [0,1] \setminus \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Let $\mu, \nu \in C[0,1]^*$ be defined as $\mu = \sum_{n=1}^{\infty} \frac{1}{2^n}\delta_{\frac{1}{n}}$ and $\nu = \frac{1}{2}(\delta_0 - \delta_k)$. Then $\|\mu\| = \|\nu\| = 1$. Now take $Z = \ker(\mu) \cap \ker(\nu)$ and $Y = \ker(\nu)$. Since $\text{supp}(\nu)$ is finite, by [5, Theorem 2.1], $\ker(\nu)$ is strongly proximinal in $C[0,1]$. Since $1 \in \ker(\nu)$ and $\mu(1) = 1$, $\mu|_{\ker(\nu)}$ is a norm-attaining functional on $ker(\nu)$. Hence $\ker(\mu)\cap\ker(\nu) = \ker(\mu|_{\ker(\nu)})$ is a proximinal subspace of $\ker(\nu)$. Since ν is not absolutely continuous with respect to μ on $\text{supp}(\mu)$, by [9], $\ker(\mu)\cap\ker(\nu)$ is not proximinal in $C[0,1]$. □

Our next example is a variant of Example 5.1. In fact, it shows that the notion of strong proximinality need not pass through ideals.

Example 5.2. *There exist two subspaces Z and Y of finite co-dimension in $C[0,1]$ such that Z is strongly proximinal in Y and Y is an ideal in $C[0,1]$, but Z is not proximinal in $C[0,1]$.*

Proof. Let μ, ν and k be as in the proof of Example 5.1. Take $Z = \ker(\mu) \cap \ker(\nu)$ and $Y = \ker(\mu)$. Choose a continuous function $g\colon [0,1] \to [-1,1]$ such that $g(\frac{1}{n}) = g(0) = 1$ for $n \geq 2$ and $g(1) = g(k) = -1$. Then $g \in \ker(\mu)$ and $\nu(g) = 1$. Since $\nu|_{\ker(\mu)}$ attains its norm over $\ker(\mu)$, $\ker(\mu) \cap \ker(\nu) = \ker(\nu|_{\ker(\mu)})$ is proximinal in $\ker(\mu)$. Let $\lambda = -\sum_{n=2}^{\infty} \frac{1}{2^n} \delta_{\frac{1}{n}}$. Then $\ker(\mu) = \ker(\lambda - \delta_1)$ and $\|\lambda\| \leq 1$ and hence, by [3], $\ker(\mu)$ is an L_1-predual space. Then, by [23, Proposition 1], $\ker(\mu)$ is an ideal in $C[0,1]$. Since ν is not absolutely continuous with respect to μ on $\mathrm{supp}(\mu)$, by [9], $\ker(\mu) \cap \ker(\nu)$ is not proximinal in $C[0,1]$. □

Our next example shows that the semi M-ideals may not pass through L-summands.

Example 5.3. *There exist a Banach space X which is an L-summand in X^{**} and a semi M-ideal Y in X such that Y is not a semi M-ideal in X^{**}.*

Proof. Take $X = \ell_1$. Then X is an L-summand in its bidual. For the constant sequence $1 \in \ell_\infty$, $Y = \ker(1)$ is a semi M-ideal in ℓ_1 (see [12, Chapter I, Remark 2.3]). But $\ker(1)$ is not a semi M-ideal in $(\ell_\infty)^*$. For, if $\ker(1)$ is a semi M-ideal in $(\ell_\infty)^*$, then $\ker(1)$ is a semi M-ideal in $\ker(1)^{\perp\perp}$. Then, by [18, Corollary 3.4], $\ker(1)$ is an M-ideal in $\ker(1)^{\perp\perp}$. Since, by [12, Chapter III, Corollary 3.3.C and Theorem 3.4], a non-reflexive subspace which is an M-ideal in its bidual contains a subspace isomorphic to c_0, $\ker(1)$ is reflexive . But this is a contradiction as ℓ_1 cannot have an infinite dimensional reflexive space. Hence $\ker(1)$ is not a semi M-ideal in $(\ell_\infty)^*$. □

Remark 5.4. Since each Banach space is an ideal in its bidual, Example 5.3 also shows that semi M-ideals may not pass through ideals.

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References

[1] D. Amir and F. Deutsch, *Suns, moons, and quasi-polyhedra*, J. Approx. Theory 6 (1972), 176–201.

[2] P. Bandyopadhyay, Bor-Luh Lin, T. S. S. R. K. Rao, *Ball proximinality in Banach spaces*, Banach Spaces and Their Applications in Analysis (2007), 251–264.

[3] J. B. Bednar and H. E. Lacey, *Concerning Banach spaces whose duals are abstract L-spaces*, Pacific J. Math. 41 (1972), 13–24.

[4] E. Behrends, *On the principle of local reflexivity*, Studia Math. 100 (1991), no. 2, 109–128.

[5] S. Dutta and D. Narayana, *Strongly proximinal subspaces of finite co-dimension in $C(K)$*, Colloq. Math. 109 (2007), no. 1, 119–128.

[6] S. Dutta and D. Narayana, *Strongly proximinal subspaces in Banach spaces*, Contemp. Math. 435, Amer. Math. Soc., Providence, RI, 2007, 143–152.

[7] C. Franchetti and R. Payá, *Banach spaces with strongly subdifferentiable norm*, Boll. Un. Mat. Ital. B (7)7 (1993), no. 1, 45–70.

[8] C. Franchetti, *Lipschitz maps and the geometry of the unit ball in normed spaces*, Arch. Math. (Basel) 46 (1986), no. 1, 76–84.

[9] A. L. Garkavi, *The problem of Helly and best approximation in the space of continuous functions*, Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967), 641–656.

[10] G. Godefroy and V. Indumathi, *Strong proximinality and polyhedral spaces*, Rev. Mat. Complut. 14 (2001), no. 1, 105–125.

[11] G. Godefroy and V. Indumathi, *Norm-to-Weak Upper Semi-Continuity of the Duality and Pre-Duality Mappings*, Set Valued Analysis 10 (2002), 317–330.

[12] P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Math., **1547**, Springer-Verlag, Berlin, 1993.

[13] V. Indumathi, *Proximinal subspaces of finite co-dimension in general normed linear spaces*, Proc. London Math. Soc. 45 (1982), 3, 435-455.

[14] V. Indumathi, *On transitivity of proximinality*, J. Approx. Theory 49 (1987), 2, 130-143.

[15] V. Indumathi, *Semi-continuity of metric projections in ℓ_∞-direct sums*, Proc. Amer. Math. Soc. 133 (2005), no. 5, 1441-1449.

[16] V. Indumathi and S. Lalithambigai, *Ball proximinal spaces*, J. Convex Anal. 18 (2011), no. 2, 353–366.

[17] A. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc. 227 (1977), 1–62.

[18] A. Lima, *Uniqueness of Hahn-Banach extensions and liftings of linear dependences*, Math. Scand. 53 (1983), no. 1, 97–113.

[19] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. **48**, 1964.

[20] W. Pollul, *Reflexivität und Existenz-Teilräume in der linearen Approximationstheorie*, Gesellschaft für Mathematik und Datenverarbeitung, Bonn, Ber. No. 53, 1972.

[21] R. Payá and D. Yost, *The two-ball property: transitivity and examples*, Mathematika 35 (1988), no. 2, 190–197.

[22] T. S. S. R. K. Rao, *Some remarks on proximinality in higher dual spaces*, J. Math. Anal. Appl. 328 (2007), no. 2, 1173-1177.

[23] T. S. S. R. K. Rao, *On ideals in Banach spaces*, Rocky Mountain J. Math. 31 (2001), no. 2, 595–609.

[24] D. Yost, *The n-ball properties in real and complex Banach spaces*, Math. Scand. 50 (1982), no. 1, 100–110.