Spectral Approximations for Characteristic Roots of Delay Differential Equations

C.P. Vyasarayani[1], S. Subhash[1] and T. Kalmár-Nagy[2*]

[1]Department of Mechanical Engineering
Indian Institute of Technology Hyderabad,
Ordnance Factory Estate,
Andhra Pradesh, India, 502205

[2]Mitsubishi Electric Research Laboratory
201 Broadway
Cambridge, MA 02139, USA

Abstract

In this paper we develop approximations to the characteristic roots of delay differential equations using the spectral tau and spectral least squares approach. We study the influence of different choices of basis functions in the spectral solution on the numerical convergence of the characteristic roots. We found that the spectral tau method performed better than the spectral least squares method. Legendre and Chebyshev bases provide much better convergence properties than the mixed Fourier basis.

1 Introduction

Delays are inherent in many natural and physical processes, for example our ability to locate the direction of the sound source comes from the capacity of our ears to detect the small time lag (delay) between the sound perceived by our left and right ear [1]. In engineering, DDEs are used as mathematical models in analyzing manufacturing processes [2], real-time sub-structuring [3], and in control theory [4].

Stability analysis of DDEs is important, particularly to find parameters for which the physical process is stable or to estimate the largest delay that a system can tolerate to remain stable. In machine tool vibrations, for example, the tool dynamics are governed by DDEs [5] and with the help of stability analysis we can find machining parameters for decreased surface roughness.

*Address all correspondence to this author. E-mail: kalmar-nagy@merl.com

DDEs are infinite dimensional systems, inasmuch as their characteristic equations have infinitely many roots. The Lambert W function [6, 7, 8], Laplace transforms [9], and D-subdivision methods [10] can be used to study stability of DDEs with single delay. Asymptotics can be used to calculate roots of scalar DDEs with few delays [11]. Lyapunov functions [12, 13, 14] can also be used to determine the stability of DDEs. General methods for obtaining the stability of DDEs with multiple delays first convert the DDE into a partial differential equation (PDE) [15, 16, 17, 18] with a linear boundary condition. Then, ordinary differential equations (ODEs) based approximation for the PDE can be developed by using spatial discretization methods, such as:

- Semi-discretization [19]
- Spectral least squares [20]
- Spectral tau methods [15, 21]
- Pseudo-spectral collocation [22, 23, 24]
- Time finite elements [25, 26]
- Continuous time approximation [27, 28]
- Finite difference methods [17, 29]

As we represent a DDE using a PDE with a linear boundary condition, the way in which we incorporate the boundary condition while developing the ODE approximations will influence the spectrum (roots of the characteris equation of the DDE). For example the boundary condition can be incorporated using spectral-tau method [15, 21], or by spectral-least squares [20] method. In this paper we compare the spectral-tau and spectral-least squares approaches for obtaining the characteristic roots of DDEs. Spectral methods are advantageous due to their exponential convergence rates [30] to the actual solution. We also study the influence of the choice of spectral basis, i.e, shifted Chebyshev, shifted Legendre, and mixed Fourier basis on the convergence of the characteristic roots for some example problems.

2 Recasting and solving a delay equation as an advection equation

We consider the scalar delay equation with m delays

$$\dot{x}(t) + ax(t) + \sum_{q=1}^{m} b_q x(t - \tau_q) = 0, \quad \tau_q > 0. \tag{1}$$

The initial function is specified as

$$x(t) = \theta(t), \; -\tau \le t < 0, \tag{2}$$

where $\tau = \max(\tau_1, \tau_2, ..., \tau_m)$. By introducing the so-called shift of time $y(s,t) = x(t+s)$, $s \in [-\tau, 0)$, the initial value problem (Eqs. (1)-(2)) can be recast into the following initial-boundary value problem for the advection equation [16, 17]

$$\frac{\partial y(s,t)}{\partial t} = \frac{\partial y(s,t)}{\partial s}, \quad s \in [-\tau, 0)\,, \tag{3}$$

$$\left.\frac{\partial y(s,t)}{\partial t}\right|_{s=0} = -ay(0,t) - \sum_{q=1}^{m} b_q y(-\tau_q, t), \tag{4}$$

$$y(s,0) = \theta(s),\ s \in [-\tau, 0]\,. \tag{5}$$

We discuss two methods, the spectral-tau method and spectral-least squares method for the approximate solution of Eqs. (3-5).

2.1 Spectral-tau method

In the spectral-tau method we assume a solution to the PDE (Eq. (3)) of the following form:

$$y(s,t) = \sum_{i=1}^{\infty} \phi_i(s)\eta_i(t), \tag{6}$$

where $\phi_i(s)$ are the basis functions and $\eta_i(t)$ are the time dependent coordinates. For practical reasons the sum is terminated at N terms, i.e.

$$y(s,t) = \boldsymbol{\phi}(s)^T \boldsymbol{\eta}(t), \tag{7}$$

where $\boldsymbol{\phi}(s) = [\phi_1(s), \phi_2(s), ..., \phi_N(s)]^T$ and $\boldsymbol{\eta}(t) = [\eta_1(t), \eta_2(t), ..., \eta_N(t)]^T$. Substituting the series solution Eq. (7) in Eq. (3) we get (the symbol $'$ denotes derivative with respect to s)

$$\boldsymbol{\phi}(s)^T \dot{\boldsymbol{\eta}}(t) = \boldsymbol{\phi}'(s)^T \boldsymbol{\eta}(t). \tag{8}$$

Pre-multipling Eq. (8) with $\boldsymbol{\phi}(s)$ and integrating over the domain we get:

$$\int_{-\tau}^{0} \boldsymbol{\phi}(s)\boldsymbol{\phi}(s)^T ds \dot{\boldsymbol{\eta}}(t) = \int_{-\tau}^{0} \boldsymbol{\phi}(s)\boldsymbol{\phi}'(s)^T ds \boldsymbol{\eta}(t). \tag{9}$$

In matrix form

$$\mathbf{A}\dot{\boldsymbol{\eta}}(t) = \mathbf{B}\boldsymbol{\eta}(t), \tag{10}$$

with

$$\mathbf{A} = \int_{-\tau}^{0} \boldsymbol{\phi}(s)\boldsymbol{\phi}(s)^T ds, \tag{11}$$

$$\mathbf{B} = \int_{-\tau}^{0} \boldsymbol{\phi}(s)\boldsymbol{\phi}'(s)^T ds. \tag{12}$$

Substituting Eq. (7) in Eq. (4) we get the scalar equation

$$\boldsymbol{\phi}(0)^T\dot{\boldsymbol{\eta}}(t) = \left[-a\boldsymbol{\phi}(0)^T - \sum_{q=1}^{m} b_q\boldsymbol{\phi}(-\tau_q)^T\right]\boldsymbol{\eta}(t). \tag{13}$$

Note that (10, 13) provide $N+1$ independent equations. To arrive at a determinate system we truncate the system (10) and augment it with (13) to form

$$\mathbf{M}_{Tau}\dot{\boldsymbol{\eta}}(t) = \mathbf{K}_{Tau}\boldsymbol{\eta}(t), \tag{14}$$

where

$$\mathbf{M}_{Tau} = \begin{bmatrix} \bar{\mathbf{A}} \\ \boldsymbol{\phi}(0)^T \end{bmatrix}, \tag{15}$$

$$\mathbf{K}_{Tau} = \begin{bmatrix} \bar{\mathbf{B}} \\ -a\boldsymbol{\phi}(0)^T - \sum_{q=1}^{m} b_q\phi(-\tau_q)^T \end{bmatrix}, \tag{16}$$

and matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ are obtained by deleting the last row of matrix $\mathbf{A}$ and $\mathbf{B}$, respectively. The initial conditions for Eq. (14) is $\boldsymbol{\eta}(0) = \mathbf{M}^{-1}\int_{-\tau}^{0}\boldsymbol{\phi}(s)\theta(s)ds$ and the solution of the DDE can be obtained as $x(t) = y(0,t) = \boldsymbol{\phi}(0)^T\boldsymbol{\eta}(t)$. The finite dimensional system (14) represents an approximation for Eq. (1).

2.2 Spectral least-squares method

The error in the PDE (3) due to the substitution of the truncated approximate solution $y(s,t) = \sum_{i=1}^{N}\phi_i(s)\eta_i(t)$ is

$$e(s,t) = \boldsymbol{\phi}(s)^T\dot{\boldsymbol{\eta}}(t) - \boldsymbol{\phi}^{'}(s)^T\boldsymbol{\eta}(t). \tag{17}$$

A good approximation is characterized by a "small" error $e(s,t)$ subject to the boundary constraint Eq. (13). To minimize the error, we aim to solve the following constrained optimization problem:

$$\min_{\dot{\boldsymbol{\eta}}(t)}\frac{1}{2}\int_{-\tau}^{0} e(s,t)^2 ds = \min_{\dot{\boldsymbol{\eta}}(t)}\frac{1}{2}\int_{-\tau}^{0}\left[\boldsymbol{\phi}(s)^T\dot{\boldsymbol{\eta}}(t) - \boldsymbol{\phi}^{'}(s)^T\boldsymbol{\eta}(t)\right]^2 ds \tag{18}$$

$$\text{s.t. } \boldsymbol{\phi}(0)^T\dot{\boldsymbol{\eta}}(t) = \left[-a\boldsymbol{\phi}(0)^T - \sum_{q=1}^{m} b_q\boldsymbol{\phi}(-\tau_q)^T\right]\boldsymbol{\eta}(t), \tag{19}$$

i.e. we are interested to find $\dot{\boldsymbol{\eta}}(t)$ such that the integral of the square of the error function over the domain is minimized. We introduce a Lagrange multiplier Λ and construct the following Lagrangian

$$\begin{aligned} L(\dot{\boldsymbol{\eta}}(t),\Lambda) = {} & \frac{1}{2}\int_{-\tau}^{0}\left[\boldsymbol{\phi}(s)^T\dot{\boldsymbol{\eta}}(t) - \boldsymbol{\phi}^{'}(s)^T\boldsymbol{\eta}(t)\right]^2 ds \\ & - \Lambda\left[\boldsymbol{\phi}(0)^T\dot{\boldsymbol{\eta}}(t) + a\boldsymbol{\phi}(0)^T\boldsymbol{\eta}(t) + \sum_{q=1}^{m} b_q\boldsymbol{\phi}(-\tau_q)^T\boldsymbol{\eta}(t)\right]. \end{aligned} \tag{20}$$

We seek to minimize $L(\dot{\boldsymbol{\eta}}(t), \Lambda)$. The first order optimality conditions for the minimization of L are [31]

$$\frac{\partial L}{\partial \dot{\boldsymbol{\eta}}(t)} = 0, \tag{21}$$

$$\frac{\partial L}{\partial \Lambda} = 0. \tag{22}$$

Substituting Eq. (20) in Eq. (21) and Eq. (22) we get:

$$\mathbf{A}\dot{\boldsymbol{\eta}}(t) = \mathbf{B}\boldsymbol{\eta}(t) + \boldsymbol{\phi}(0)\Lambda, \tag{23}$$

$$\boldsymbol{\phi}(0)^T\dot{\boldsymbol{\eta}}(t) = -a\boldsymbol{\phi}(0)^T\boldsymbol{\eta}(t) - \sum_{q=1}^{m} b_q\boldsymbol{\phi}(-\tau_q)^T\boldsymbol{\eta}(t), \tag{24}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the basis-dependent matrices defined in Eq. (11) and Eq. (12) respectively. Solving Eq. (23) and Eq. (24) for the Lagrange multiplier yields

$$\Lambda = -\frac{\boldsymbol{\phi}(0)^T\mathbf{A}^{-1}\mathbf{B}}{\boldsymbol{\phi}(0)^T\mathbf{A}\boldsymbol{\phi}(0)}\boldsymbol{\eta}(t) - \frac{1}{\boldsymbol{\phi}(0)^T\mathbf{A}^{-1}\boldsymbol{\phi}(0)}\left(a\boldsymbol{\phi}(0)^T + \sum_{q=1}^{m} b_q\boldsymbol{\phi}(-\tau_q)^T\right)\boldsymbol{\eta}(t). \tag{25}$$

Substituting the value of Λ in Eq. (23) and simplifying we get:

$$\mathbf{A}\dot{\boldsymbol{\eta}}(t) = \mathbf{K}_{LS}\boldsymbol{\eta}(t), \tag{26}$$

where

$$\mathbf{K}_{LS} = \mathbf{B} - \frac{1}{\boldsymbol{\phi}(0)^T\mathbf{A}^{-1}\boldsymbol{\phi}(0)}\left(\boldsymbol{\phi}(0)\boldsymbol{\phi}(0)^T\mathbf{A}^{-1}\mathbf{B} + a\boldsymbol{\phi}(0)\boldsymbol{\phi}(0)^T + \sum_{q=1}^{m} b_q\boldsymbol{\phi}(0)\boldsymbol{\phi}(-\tau_q)^T\right). \tag{27}$$

Equation (26) represents the ODE approximation of the DDE Eq. (1).

3 Computing Spectra

Substituting $x(t) = ce^{\lambda t}$ in Eq. (1) we obtain the characteristic equation ($\mathcal{C}(\lambda)$ is the characteristic function)

$$\mathcal{C}(\lambda) = \lambda + a + \sum_{q=1}^{m} b_q e^{-\lambda\tau_q} = 0. \tag{28}$$

A complex root λ of the characteristic equation (eigenvalue of Eq. (1)) is written as

$$\lambda = \alpha + i\beta. \tag{29}$$

Using Euler's identity in (28) and separating the real and imaginary parts yields

$$\alpha + a + \sum_{q=1}^{m} b_q e^{-\alpha\tau_q}\cos(\beta\tau_q) = 0, \tag{30}$$

$$\beta - \sum_{q=1}^{m} b_q e^{-\alpha\tau_q}\sin(\beta\tau_q) = 0. \tag{31}$$

Equations (30, 31) are transcendental (exponential quasipolynomials) and have infinitely many roots. We define the spectrum of (1) as the set of roots of the characteristic equation, i.e.

$$S = \{\lambda_i \,|\, \mathcal{C}(\lambda_i) = 0,\, \mathrm{Re}\lambda_1 \geq \mathrm{Re}\lambda_2 \geq ..\}. \tag{32}$$

The approximate spectrum obtained by calculating the eigenvalues of the $N \times N$ system Eq. (14) is defined as

$$\hat{S}_{Tau} = \left\{\hat{\lambda}_i \,|\, \det\left(\mathbf{M}_{Tau}\hat{\lambda}_i - \mathbf{K}_{Tau}\right) = 0,\, \mathrm{Re}\hat{\lambda}_1 \geq \mathrm{Re}\hat{\lambda}_2 \geq ..\right\}, \tag{33}$$

and for Eq. (26) as

$$\hat{S}_{LS} = \left\{\hat{\lambda}_i \,|\, \det\left(\mathbf{A}\hat{\lambda}_i - \mathbf{K}_{LS}\right) = 0,\, \mathrm{Re}\hat{\lambda}_1 \geq \mathrm{Re}\hat{\lambda}_2 \geq ..\right\}. \tag{34}$$

The error in the k-th eigenvalue is defined as

$$\varepsilon_k = \left|\mathcal{C}\left(\hat{\lambda}_k\right)\right|. \tag{35}$$

The accuracy of the rightmost R roots of the approximate spectrum $\hat{S}$ is characterized by the"tolerance"

$$T(R) = \max_{1 \leq k \leq R} \varepsilon_k. \tag{36}$$

We expect the choice of basis functions $\boldsymbol{\phi}(s)$ in Eq. (7) to play an important role in the convergence of the eigenvalues. To test this hypothesis we consider three different basis functions: mixed Fourier basis [21]

$$\boldsymbol{\phi}(s) = [1,\, s,\, \sin(\frac{\pi}{\tau}s), \sin(2\frac{\pi}{\tau}s)...]^T, \tag{37}$$

shifted Legendre polynomials [30]

$$\phi_1(s) = 1, \phi_2(s) = 1 + \frac{2s}{\tau}, \phi_i(s) = \frac{(2i-3)\,\phi_2(s)\,\phi_{i-1}(s) - (i-2)\,\phi_{i-2}(s)}{i-1}, i = 3, ..., \tag{38}$$

and shifted Chebyshev polynomials [30]

$$\phi_1(s) = 1, \phi_2(s) = 1 + \frac{2s}{\tau}, \phi_i(s) = 2\phi_2(s)\,\phi_{i-1}(s) - \phi_{i-2}(s), i = 3, ... \tag{39}$$

4 Results

Here we compare the spectral tau and spectral least squares methods introduced in Sections 2.1 and 2.2 on equations containg one, two, and (for a good measure) thirty delays. We use mixed Fourier (Eq. 37), Legendre (Eq. 38), and Chebyshev (Eq. 39) basis functions to study convergence properties of the two methods.
We first consider the equation with one time delay

$$\dot{x} + b_1 x(t - \tau_1) = 0. \tag{40}$$

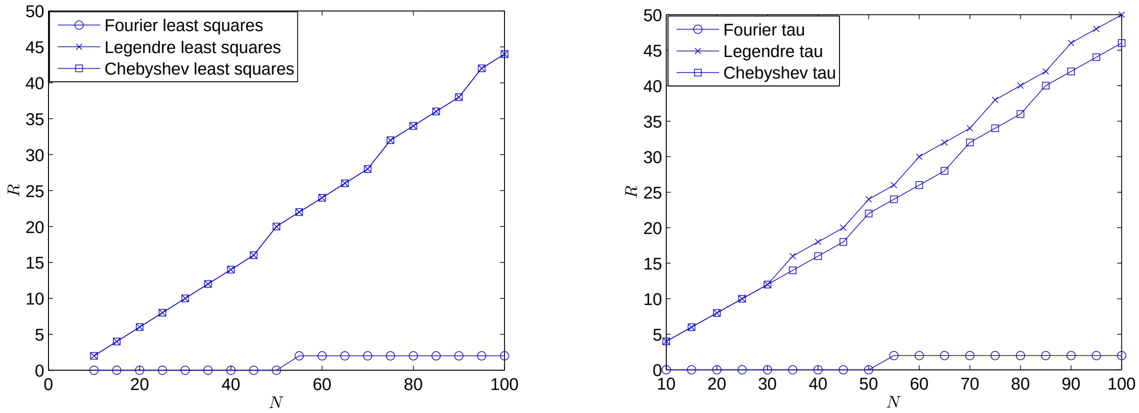
It is an important case, since the eigenvalues of its characteristic equation

$$\lambda + b_1 e^{-\lambda \tau_1} = 0 \tag{41}$$

can be obtained in closed form in terms of Lambert W function [6] as

$$\lambda_r = \frac{1}{\tau_1} W_r(-b_1 \tau_1), \ r = -\infty, ..., -1, 0, 1, ...\infty. \tag{42}$$

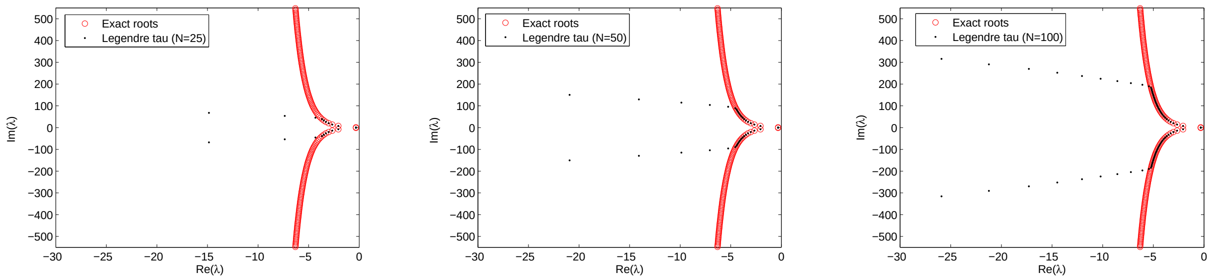
Here W_k corresponds to k^{th} branch of the Lambert W function.
Figure 1 shows the number of converged roots for tolerance $T(R) = 10^{-4}$ for increasing N. As expected, increasing the number of terms in the series solution will yield more and more eigenvalues. It is however surprising to see the poor performance of the mixed Fourier basis for both the tau and the least squares method. Figures 4(a)-(c) show spectra of (41) obtained through



(a) (b)

Figure 1: Number of converged roots for tolerance $T(R) = 10^{-4}$ for increasing N (a) spectral least squares method (b) spectral tau method. The parameters are $\tau_1 = 1$ and $b_1 = 1$.

Eq. (42) for the case $b_1 = \tau_1 = 1$ (other parameter values yield similar results). Superimposed on the graph are the roots obtained by the Legendre tau method for $N = 25$ (Fig. 4(a)), $N = 50$ (Fig. 4(b)), and $N = 100$ (Fig. 4(c)). We again see that for increasing N more and more eigenvalues of the approximate spectrum $\hat{S}$ (Eq. 33) converge to the exact eigenvalues.
After having established the positive influence of increasing N on convergence, we used $N = 25$ to obtain the rest of the results.
Figure 3 shows the average error in each eigenvalue ε_k for 1000 simulations for spectral tau method and for spectral least squares method. The delay τ_1 and b_1 were randomly selected from uniform distributions between $\tau_1 \in [0.1, 1]$ and $b_1 \in [-10, 10]$ respectively. We see that the Legendre tau method has the best performance.
We also studied the total number of roots R that converge to the required tolerance of 10^{-4}. To cover a large number of test cases, 10000 simulations were performed with randomly selected parameters from uniform distributions for a two-delay equation ($a \in [-10, 10]$, $b_1 \in [-10, 30]$, $b_2 \in$



(a) (b) (c)

Figure 2: Comparison of exact roots obtained with Lambert W function with approximate roots obtained from Legendre tau method for (a) $N = 25$ (b) $N = 50$ and (c) $N = 100$. The parameters are $\tau_1 = 1$ and $b_1 = 1$.

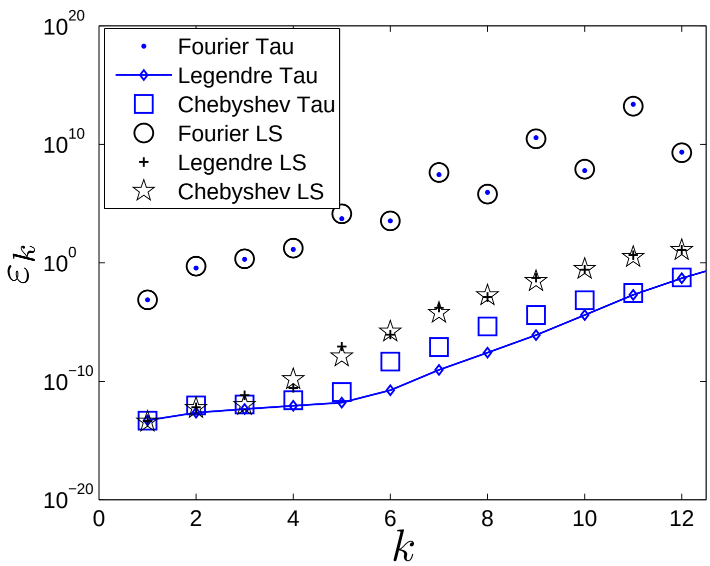


Figure 3: Average error in the roots for spectral least squares and spectral tau methods with $N = 25$. The results are generated using 1000 Monte Carlo simulations with b_1 and τ_1 selected from a uniform distribution on [0.1, 1] and [−10, 10], respectively.

$[-10,\ 50]$, $\tau_1 \in [0.1,\ 5.1]$, and $\tau_2 \in [0.1,\ 10.1]$) and the equation with thirty delays ($a \in [-10,\ 10]$, $\tau_k \in [0.1,\ 10.1]$, $b_k \in [-10,\ 50]$, $k = 1, 2, .., 30$). The results are summarized in Figures 4(a) and (b).

R	0	1	2	3	4	5	6	7	8	9	10	11
Fourier least squares	92	8	0	0	0	0	0	0	0	0	0	0
Legendre least squares	0	0	0	0	0	0	1	7	69	10	13	0
Chebyshev least squares	0	0	0	0	0	0	1	7	70	11	12	0
Fourier tau	18	15	4	1	1	0	0	0	0	0	0	0
Legendre tau	0	0	0	0	0	0	0	3	27	15	54	1
Chebyshev tau	0	0	0	0	0	0	0	3	27	15	53	1

(a)

R	0	1	2	3	4	5	6	7	8	9	10	11
Fourier least squares	100	0	0	0	0	0	0	0	0	0	0	0
Legendre least squares	0	0	0	0	0	0	37	8	48	2	4	0
Chebyshev least square	0	0	0	0	0	0	37	7	48	2	4	0
Fourier tau	100	0	0	0	0	0	0	0	0	0	0	0
Legendre tau	0	0	0	0	1	0	39	8	47	2	3	0
Chebyshev tau	0	0	0	0	0	1	39	8	46	2	3	0

(b)

Figure 4: Percentage of number of converged roots for different methods and for different basis functions. The results are obtained by performing 10000 Monte Carlo simulations with parameters taken from uniform distributions (a) Two-delay equation ($a \in [-10,\ 10]$, $b_1 \in [-10,\ 30]$, $b_2 \in [-10,\ 50]$, $\tau_1 \in [0.1,\ 5.1]$, and $\tau_2 \in [0.1,\ 10.1]$) (b) Thirty-delay equation ($a \in [-10,\ 10]$, $\tau_k \in [0.1,\ 10.1]$, $b_k \in [-10,\ 50]$, $k = 1, 2, .., 30$).

It is clear that the choice of the mixed Fourier basis has an adverse effect on the number of eigenvalues found. We also note that the expected number of "correct" eigenvalues in the thirty-delay case is smaller than that of the two-delay case, but this might be due to the different range of parameters selected for the simulations.

5 Discussion and Conclusions

It is worth investigating why the mixed Fourier basis performs badly in terms of convergence compared to shifted Legendre and shifted Chebyshev basis. In the mixed Fourier basis, we have $\boldsymbol{\phi}(s) = [1,\ s,\ \sin(\pi s/\tau), ..., \sin((N-2)\pi s/\tau)]^T$ and we can see that only the first two terms have non-zero values at $s = -\tau$ and for $s = 0$ only one term remains. The information about the rightmost delay appears in the spectral matrices $\mathbf{M}_{Tau}$, $\mathbf{K}_{Tau}$ and $\mathbf{K}_{LS}$ through a term like $\boldsymbol{\phi}(-\tau)^T\boldsymbol{\eta}(t)$, so the coupling terms for the largest delay are not strong (as most of the terms in $\boldsymbol{\phi}(s)^T$ are zeros). This is not the case with the Legendre and Chebysev bases, where $\boldsymbol{\phi}(s)^T$ is fully populated. We thus conjecture that the sparse nature of $\boldsymbol{\phi}(s)^T$ at $s = 0$ and $s = -\tau$ is a (if not the) reason for the bad convergence behavior for the mixed Fourier basis.

We have studied the spectral-tau method and spectral-least squares method for obtaining characteristic roots of a linear DDE. We found that the spectral tau method performed better than the spectral least squares method. Also the Legendre or Chebyshev basis performed much better in terms of error convergence compared to mixed Fourier basis. The spectral-tau method is easy to code and understand, also it performs better than the spectral-least squares method, so we recommend spectral-tau method for analyzing the stability of linear DDEs.

Acknowledgements

CPV thanks DST for financial support through the fast track scheme for young scientists (ref: SB/FTP/ETA-0462/2012).

References

[1] J Blauert. *Spatial Hearing: The Psychophysics of Human Sound Localization.* MIT press, 1997.

[2] B Balachandran. Nonlinear dynamics of milling processes. *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 359(1781):793–819, 2001.

[3] M I Wallace, J Sieber, S A Neild, D J Wagg, and B Krauskopf. Stability analysis of real-time dynamic substructuring using delay differential equation models. *Earthquake Engineering & Structural Dynamics*, 34(15):1817–1832, 2005.

[4] J P Richard. Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39(10):1667–1694, 2003.

[5] T Insperger and G Stépán. Stability of the milling process. *Mechanical Engineering*, 44(1):47–57, 2000.

[6] F M Asl and A G Ulsoy. Analysis of a system of linear delay differential equations. *Journal of Dynamic Systems, Measurement, and Control*, 125(2):215–223, 2003.

[7] E Jarlebring and T Damm. The lambert w function and the spectrum of some multidimensional time-delay systems. *Automatica*, 43(12):2124–2128, 2007.

[8] S Yi, P W Nelson, and A G Ulsoy. *Time-Delay Systems: Analysis and Control Using the Lambert W Function.* World Scientific, 2010.

[9] T Kalmár-Nagy. Stability analysis of delay-differential equations by the method of steps and inverse laplace transform. *Differential Equations and Dynamical Systems*, 17(1-2):185–200, 2009.

[10] N Olgac and R Sipahi. An exact method for the stability analysis of time-delayed linear time-invariant (LTI) systems. *IEEE Transactions on Automatic Control*, 47(5):793–797, 2002.

[11] P Wahi and A Chatterjee. Asymptotics for the characteristic roots of delayed dynamic systems. *ASME Journal of Applied Mechanics*, 72(4):475–483, 2005.

[12] K Gu, J Chen, and V L Kharitonov. *Stability of Time Delay Systems*. Springer, 2003.

[13] K Gu and S I Niculescu. Stability analysis of time-delay systems: A Lyapunov approach. In *Advanced Topics in Control Systems Theory*, pages 139–170. Springer, 2006.

[14] Emilia Fridman. New Lyapunov–Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & Control Letters*, 43(4):309–319, 2001.

[15] K Ito and R Teglas. Legendre-tau approximations for functional-differential equations. *SIAM Journal on Control and Optimization*, 24(4):737–759, 1986.

[16] A Bellen and S Maset. Numerical solution of constant coefficient linear delay differential equations as abstract cauchy problems. *Numerische Mathematik*, 84(3):351–374, 2000.

[17] T Koto. Method of lines approximations of delay differential equations. *Computers & Mathematics with Applications*, 48(1):45–59, 2004.

[18] D Breda. Solution operator approximations for characteristic roots of delay differential equations. *Applied Numerical Mathematics*, 56(3):305–317, 2006.

[19] T Insperger and G Stépán. *Semi-Discretization for Time-Delay Systems*. Springer New York, 2011.

[20] C P Vyasarayani. Galerkin approximations for higher order delay differential equations. *ASME Journal of Computational and Nonlinear Dynamics*, 7(3):031004, 2012.

[21] P Wahi and A Chatterjee. Galerkin projections for delay differential equations. *Journal of Dynamic Systems, Measurement, and Control*, 127(1):80–87, 2005.

[22] E A Butcher, H Ma, E Bueler, V Averina, and Z Szabo. Stability of linear time-periodic delay-differential equations via chebyshev polynomials. *International Journal for Numerical Methods in Engineering*, 59(7):895–922, 2004.

[23] D Breda, S Maset, and R Vermiglio. Pseudospectral differencing methods for characteristic roots of delay differential equations. *SIAM Journal on Scientific Computing*, 27(2):482–495, 2005.

[24] Z Wu and W Michiels. Reliably computing all characteristic roots of delay differential equations in a given right half plane using a spectral method. *Journal of Computational and Applied Mathematics*, 236(9):2499–2514, 2012.

[25] B P Mann and B R Patel. Stability of delay equations written as state space models. *Journal of Vibration and Control*, 16(7-8):1067–1085, 2010.

[26] F A Khasawneh and B P Mann. A spectral element approach for the stability of delay systems. *International Journal for Numerical Methods in Engineering*, 87(6):566–592, 2011.

[27] J Q Sun. A method of continuous time approximation of delayed dynamical systems. *Communications in Nonlinear Science and Numerical Simulation*, 14(4):998–1007, 2009.

[28] B Song and J Q Sun. Lowpass filter-based continuous-time approximation of delayed dynamical systems. *Journal of Vibration and Control*, 17(8):1173–1183, 2011.

[29] K Engelborghs and D Roose. On stability of LMS methods and characteristic roots of delay differential equations. *SIAM Journal on Numerical Analysis*, 40(2):629–650, 2002.

[30] J P Boyd. *Chebyshev and Fourier Spectral Methods*. Dover Publications, 2001.

[31] D J Luenberger. *Linear and Nonlinear Programming*. Springer, New York, 2003.