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# Order from non-associative operations

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**Abstract**

Algebraic structures are often converted to ordered structures to gain information about the algebra using the properties of partially ordered sets. Such studies have been predominantly undertaken for semigroups, using various proposed relations. This has led to a spate of works dealing with associative fuzzy logic connectives (FLCs) and the orders that they generate. One such relation, proposed by Clifford, is employed both for its generality as well as utility. In a recent work, Gupta and Jayaram classified the semigroups that yield a partial order through the relation. In this work, we characterise groupoids that would give a partial order by introducing a property called the Generalised Quasi-Projectivity. Further, for the groupoids that lead to an ordered set, we explore the monotonicity of the underlying groupoid operation on the obtained poset. Finally, in light of the above results, we explore the major non-associative fuzzy logic connectives along these lines, thus complementing and augmenting, already existing works in the literature. Our work also shows when an FLC from a given class of operations remains one even w.r.to the order generated from it.


*Keywords:* Non-associative operations; Ordered sets; Posets; Lattices; Semi-copulas; Fuzzy implications

## 1. Introduction

Order-theoretic exploration of semigroups has been undertaken in the classical works of Mitsch, Nambooripad, and Clifford [18–20,7], to name a few. The impact of such studies is twofold. Firstly, we can associate a partially ordered set to an algebra and secondly, depending on the type of poset obtained, we can gain insights into the algebra itself. Associating a partially-ordered set to an algebra can help us intuitively interpret an algebra, much like a graph of the function can help us understand the algebraic expression better. Also, it can help us gather valuable insights into the nature of the algebra. A point in case is the seminal result of Clifford, which states that a commutative semigroup can be written as an ordinal sum of subsemigroups if Clifford's relation yields a total order.

Inspired by these classical studies on semigroups, researchers proposed orders based on associative fuzzy logic connectives (FLCs). Karaçal and Kesicioğlu suggested an order in [13], based on a $t$-norm $T$ defined on a bounded lattice $\mathbb{L}$ by conveniently modifying the relation given by Clifford in [7]. The relation $\preceq_T$ was defined as follows:

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$$x \preceq_T y \iff \text{there exists an } \ell \in \mathbb{L} \text{ such that } T(\ell, y) = x \;. \tag{1}$$

Note the relation in (1) is a modified form of the dual of the relation in [7], as reflexivity is assumed in the latter.

We shall use this relation and its dual depending on the considered fuzzy logic connective. We shall call the relations multiplicative and additive variants of Clifford's relation and denote them by $\preceq_F$ and $\sqsubseteq_F$, respectively. The respective relations, (MCR) and (ACR), are given in Definition 3. In the sequel, we may simply use the phrase *Clifford's relations* to refer to (MCR) and (ACR), when no confusion can arise in the considered context, and *Clifford posets* to refer to the corresponding posets.

### *1.1. Motivation for and main contributions of this work*

While Clifford's seminal results were under the assumption of the relation (ACR) giving rise to total order, there were no available results on the conditions under which (ACR) itself would lead to a partial order on the underlying set.

In [12], the authors characterise semigroups that lead to an order using (MCR). Note that the duality of the orders implies the same conditions hold for (ACR), and in [11], the same authors discuss the conditions under which implicative type operations, which are not typically associative, lead to an order using the relation (ACR). However, a complete characterisation of groupoids[1] leading to posets through the above relations is unknown, which forms our first motivation for this work as captured below:

(**M1**) Let $(\mathbb{P}, F)$ be a groupoid. Characterise the binary operations $F$ that yield a partial order through (ACR) and (MCR), i.e., characterise operations $F$ that ensure $(\mathbb{P}, \sqsubseteq_F)$ and $(\mathbb{P}, \preceq_F)$ are partially ordered sets.

In fact, as a groupoid operation, the considered $F$ is not expected to possess any further algebraic properties, though it very well may. However, if we do obtain a Clifford poset, it would be worthwhile and interesting to study if the obtained order infuses any type of monotonicity onto $F$. This gives us the second of our motivations, viz.,

(**M2**) Let $(\mathbb{P}, F)$ be a groupoid. Given $(\mathbb{P}, \preceq_F)$ is a partially ordered set, characterise the operations $F$ that are monotonic or antitonic w.r.to $\preceq_F$ in each variable.

In fact, in the context of fuzzy logic operations, one already has a poset underneath and the considered operations thus possess some form of monotonicity. However, neither the original poset's properties nor the operation's monotonicity may be preserved by and on the obtained Clifford poset.

Consider for example the Gödel implication given in Example 6. Clearly, $[0, 1]$ w.r.to the usual order is bounded. However, the poset obtained by (ACR), given in Fig. 6(i), is not bounded below.

We see from the example mentioned above that even though we start with a bounded set, the obtained Clifford poset may not be bounded. Similarly, given a monotonic operation $F$ on the original order, even if it leads to a Clifford poset, it need not be monotonic on the obtained Clifford poset. Example 3 shows such a semi-copula.

The above observations prompt us to explore the behaviour of non-associative FLCs on the obtained Clifford poset. Since every FLC satisfies some boundary conditions, in this context, it is also important to discuss the boundedness of the obtained Clifford poset. These form the last of our motivations for the work contained in this submission:

(**M3**) Let $([0, 1], F)$ be the considered groupoid where $F$ is a non-associative fuzzy logic connective. Given $([0, 1], \preceq_F)$ is a partially ordered set, determine the conditions under which
- $([0, 1], \preceq_F)$ is bounded, and
- $F$ is monotonic and/or antitonic on the obtained poset.

**Remark 1.** Note that some preliminary explorations relating to the above motivations were presented in [21] and [22]. This submission contains an in-depth study of the above with related explorations. For instance, we explore the conditions under which the obtained Clifford poset coincides with the underlying order on the set. We also present

[1] A groupoid is a non-empty set endowed with a closed binary operation.

results showing that every poset is, in fact, a Clifford poset and characterise groupoids with differing monotonicities w.r.to the Clifford order. The contents of this submission augment the earlier works by providing examples showcasing the variety of Clifford posets obtainable from semi-copulas and fuzzy implications and enlarging the admissible families of fuzzy implications that can yield a Clifford poset.

The proofs, if reproduced from [21], are either for completeness or due to their augmentation, and these are clearly indicated through a footnote in the appropriate context.

### 1.2. *Outline of this submission:*

In Section 2, we begin by presenting some definitions from order theory. We then define the relations (ACR) and (MCR), and propose necessary and sufficient conditions for them to become a partial order. We also show that any partially ordered set can be seen as a Clifford poset. In Section 3, we introduce the properties (namely (LA), (LEP), (LLC), and (LRC)) that characterise monotonic and antitonic operations in both the variables. In Section 4, we discuss some non-associative operations on [0, 1] and explore the conditions under which they yield a poset. We also examine if and when they are monotonic and-/or antitonic in each variable w.r.to the new order. Finally, we present our conclusions in Section 5.

## 2. Order from non-associative operations

In this section, we begin by presenting some definitions from order theory. We then characterise groupoids yielding a partially ordered set through (ACR) and (MCR). After providing a few examples of such groupoids and the posets they generate, we also show that the characterising properties are mutually independent. We then show that every poset is a Clifford poset obtained from an appropriate binary operation.

### 2.1. *Some order-theoretic concepts*

We begin by presenting some notions from order theory, and for further details, refer the readers to the book of Davey and Priestly [8].

**Definition 1.** [cf. [8]]

(i) Let $\mathbb{P} \neq \emptyset$. A partial order on $\mathbb{P}$ is a binary relation $\leq$ on $\mathbb{P}$ such that, for all $a, b, c \in \mathbb{P}$, the following properties hold:
- Reflexivity: $a \leq a$,
- Antisymmetry: If $a \leq b$ and $b \leq a$, then $a = b$,
- Transitivity: If $a \leq b$ and $b \leq c$, then $a \leq c$.

**Definition 2.** [cf. [8]] Let $(\mathbb{P}, \leq)$ be a poset.

(i) An element $a$ in $\mathbb{P}$ is said to be
- a **maximal element** if there does not exist any $b \in \mathbb{P}$ such that $a \leq b$.
- a **minimal element** if there does not exist any $b \in \mathbb{P}$ such that $b \leq a$.
- the **greatest element** (maximum element) if for every element $b$ in $\mathbb{P}$ we have that $a \geq b$.
- the **least element** (minimum element) if for every element $b$ in $\mathbb{P}$ we have that $a \leq b$.

(ii) A pair of elements $a, b \in \mathbb{P}$ is said to be **comparable**, denoted $a \sim b$, if either $a \leq b$ or $b \leq a$. If not, we denote it by $a \nsim b$.

(iii) Let $y \in \mathbb{P}$. Then
- $y \downarrow = \{a \in \mathbb{P} \mid a \leq y\}$ is called the **downset** of $y$.
- $y \uparrow = \{a \in \mathbb{P} \mid y \leq a\}$ is called the **upset** of $y$.

(iv) Let $Y \subseteq \mathbb{P}$.
- An element $a \in \mathbb{P}$ is an **upper bound** of $Y$ if $y \leq a$ for all $y \in Y$.
- An element $a \in \mathbb{P}$ is a **lower bound** of $Y$ if $y \geq a$ for all $y \in Y$.

(v) $\mathbb{P}$ is said to be
- a **chain** or **totally ordered** if for any $a, b \in \mathbb{P}$ either $a \leq b$ or $b \leq a$, i.e., $a \sim b$ for every $a, b \in \mathbb{P}$.
- an **anti-chain** if no two distinct elements are comparable.
- **bounded** if there exist elements $\alpha, \beta \in \mathbb{P}$ such that, for all $x \in \mathbb{P}$, $\alpha \leq x \leq \beta$. In such a case, for emphasis, the poset will be denoted as $(\mathbb{P}, \leq, \alpha, \beta)$.
- a **join-semilattice** if every pair of elements $x, y \in \mathbb{P}$ has a least upper bound. It will be denoted by $x \vee y$, where the operation $\vee$ is also known as the **join**.
- a **meet-semilattice** if every pair of elements $x, y \in \mathbb{P}$ has a greatest lower bound. It will be denoted by $x \wedge y$, where the operation $\wedge$ is also known as the **meet**.
- a **lattice** if it is both a meet-semilattice and a join-semilattice. In other words, for any pair of elements in $\mathbb{P}$ both the join and meet exist.
- a **complete lattice** if it is a lattice and for all $Y \subseteq \mathbb{P}$ both the least upper bound of $Y$ - denoted sup $Y$ or $\bigvee Y$, and the greatest lower bound of $Y$ - denoted inf $Y$ or $\bigwedge Y$, exist.
- a **distributive lattice** if it is a lattice and if the following identity holds for all $x, y, z \in \mathbb{P}$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

- a **modular lattice** if it is a lattice and if for any $a, b \in \mathbb{P}$ such that $a \leq b$, modular law is satisfied for every $x \in \mathbb{P}$, i.e.,

$$a \vee (x \wedge b) = (a \vee x) \wedge b.$$

**Remark 2.** We make the following observations on the notations used in the sequel.

(i) In the sequel, $\mathbb{P}$ will always denote a non-empty set with no further structure assumed on it.
(ii) $\mathbb{L}$ will denote a bounded poset $(\mathbb{L}, \leq, 0, 1)$.
(iii) By $[a, b] \subseteq \mathbb{L}$ we denote an interval w.r.to the order $\leq$ defined on $\mathbb{L}$, i.e., $[a, b] = \{x \in \mathbb{L} \mid a \leq x \leq b\}$.
(iv) $F$ will denote a binary operation on $\mathbb{P}$ without any further assumption except closure, i.e., $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$.
(v) By $F^x$ we mean the first partial function of $F$, i.e., $F^x(m) = F(m, x)$.
(vi) Given a function $F$, we shall denote its range by $\mathcal{R}an(F)$.

### 2.2. *Clifford posets and some functional equations*

We shall now define the multiplicative and additive variants of Clifford's relation that we shall employ in the sequel.

**Definition 3.** Let $\mathbb{P} \neq \emptyset$ and $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$. The multiplicative and additive Clifford's relation, denoted by $\preceq_F$ and $\sqsubseteq_F$, respectively, are defined as follows:

$$x \preceq_F y \iff \text{there exists } \ell \in \mathbb{P} \text{ s.t. } F(\ell, y) = x \ , \tag{MCR}$$

$$x \sqsubseteq_F y \iff \text{there exists } \ell \in \mathbb{P} \text{ s.t. } F(\ell, x) = y \ . \tag{ACR}$$

Note that $\preceq_F$ is nothing but the dual of $\sqsubseteq_F$, i.e., $x \preceq_F y$ if and only if $y \sqsubseteq_F x$.

In [12], the authors characterise semigroups that yield a partial order through Clifford's relation. For this purpose, they introduce the properties of (**LLI**) and (**QP**) and show that associativity is not necessary to obtain a partial order. Here, we provide a complete characterisation of groupoids that yield a poset.

In the following definition, we introduce the property of (**GQP**) and define relevant functional equations that help us characterise the groupoids that yield a partial order.

**Definition 4.** Let $\mathbb{P} \neq \emptyset$ and $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$. $F$ is said to satisfy the

- **Local Left Identity** property [12], if for every $x \in \mathbb{P}$, there exists an $\ell \in \mathbb{P}$, possibly depending on $x$, such that

$$F(\ell, x) = x, \tag{LLI}$$

i.e., every element has a *local left identity* w.r.to $F$.

Table 1
Functions $F_i$ for $i = 1, 2, 3, 4, 5$ in Example 1.

| $F_1$ | $x$ | $y$ | $z$ | 1 |
|---|---|---|---|---|
| $x$ | $x$ | $y$ | $z$ | 1 |
| $y$ | $x$ | $y$ | $z$ | 1 |
| $z$ | $x$ | $y$ | $z$ | 1 |
| 1 | $x$ | $y$ | $z$ | 1 |

| $F_2$ | $x$ | $y$ | $z$ | 1 |
|---|---|---|---|---|
| $x$ | $x$ | 1 | 1 | 1 |
| $y$ | 1 | $y$ | 1 | 1 |
| $z$ | 1 | 1 | $z$ | 1 |
| 1 | 1 | 1 | 1 | 1 |

| $F_3$ | $x$ | $y$ | $z$ | 1 |
|---|---|---|---|---|
| $x$ | 1 | 1 | 1 | 1 |
| $y$ | $z$ | 1 | 1 | 1 |
| $z$ | $y$ | 1 | 1 | 1 |
| 1 | $x$ | $y$ | $z$ | 1 |

| $F_4$ | $x$ | $y$ | $z$ | 1 |
|---|---|---|---|---|
| $x$ | $x$ | $y$ | $x$ | $x$ |
| $y$ | $x$ | $y$ | $z$ | $y$ |
| $z$ | $x$ | $y$ | $z$ | 1 |
| 1 | $x$ | $y$ | $z$ | $z$ |

| $F_5$ | $x$ | $y$ | $z$ | 1 |
|---|---|---|---|---|
| $x$ | $x$ | $y$ | $x$ | $x$ |
| $y$ | $x$ | $x$ | $z$ | $y$ |
| $z$ | $x$ | $x$ | $y$ | $z$ |
| 1 | $x$ | $x$ | $x$ | 1 |

- **Quasi-Projection** property [12], if for any $x, y, z \in \mathbb{P}$,

$$F(x, F(y, z)) = z \Longrightarrow F(y, z) = z \, . \tag{QP}$$

- **Generalised Quasi-Projection** property, if for any $x, y, z, w \in \mathbb{P}$ such that

$$F(x, F(y, z)) = w, \text{ there exists } \ell \in \mathbb{P} \text{ such that } F(\ell, z) = w \, . \tag{GQP}$$

**Theorem 1** *(cf. Theorem 1, [21]). Let $\mathbb{P} \neq \emptyset$ and $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$.*
*The following are equivalent[2]:*

(i) *$(\mathbb{P}, \sqsubseteq_F)$ is a poset.*
(ii) *$(\mathbb{P}, \preceq_F)$ is a poset.*
(iii) *$F$ satisfies* (**LLI**), (**QP**), *and* (**GQP**).

**Proof.** Since $\preceq_F$ and $\sqsubseteq_F$ are the dual of each other, (i) and (ii) are equivalent.

We thus complete the proof by showing that (ii) is equivalent to (iii).

Suppose, $(\mathbb{P}, \preceq_F)$ is a poset. That the reflexivity of $\preceq_F$ is equivalent to (**LLI**) and its antisymmetry is equivalent to (**QP**) follows from Theorem 2.4 in [12].

Thus it suffices to show the equivalence between the transitivity of $\preceq_F$ and (**GQP**).

Let us suppose that $\preceq_F$ is transitive and that $F(x, F(y, z)) = w$ for some $x, y, z, w \in \mathbb{P}$. Thus, from (MCR), we have $w \preceq_F F(y, z)$. Since $F(y, z) = F(y, z)$, we have that $F(y, z) \preceq_F z$ and by transitivity of $\preceq_F$, we have $w \preceq_F z$. Clearly, from (MCR), there exists an $\ell \in \mathbb{P}$ such that $F(\ell, z) = w$. Thus, $F$ satisfies (**GQP**).

Suppose $F$ satisfies (**GQP**) and that $x \preceq_F y$, $y \preceq_F z$. Thus, there exist $\ell, m \in \mathbb{P}$ such that $F(m, z) = y$ and

$$F(\ell, y) = x \Longrightarrow F(\ell, F(m, z)) = x \, .$$

Now, by (**GQP**) we have an $n \in \mathbb{P}$ such that $F(n, z) = x$ and hence, $x \preceq_F z$, i.e., $\preceq_F$ is transitive. □

We present in the following examples some binary functions on a finite set, satisfying the properties mentioned above and the corresponding Hasse diagrams obtained through (MCR).

**Example 1.** Let $\mathbb{P} = \{x, y, z, 1\}$. The functions given in Table 1 satisfy (**LLI**), (**QP**), and (**GQP**) and hence, yield a poset.

- Note that the poset given in Fig. 1(i) obtained from $F_1$ is an anti-chain, i.e., every element is both maximal and minimal and there are no maximum and minimum elements.
- The poset obtained from $F_2$, given in Fig. 1(ii), is bounded below by the element 1 and has *three* maximal elements.

[2] The proof is reproduced from [21] for the sake of completeness.

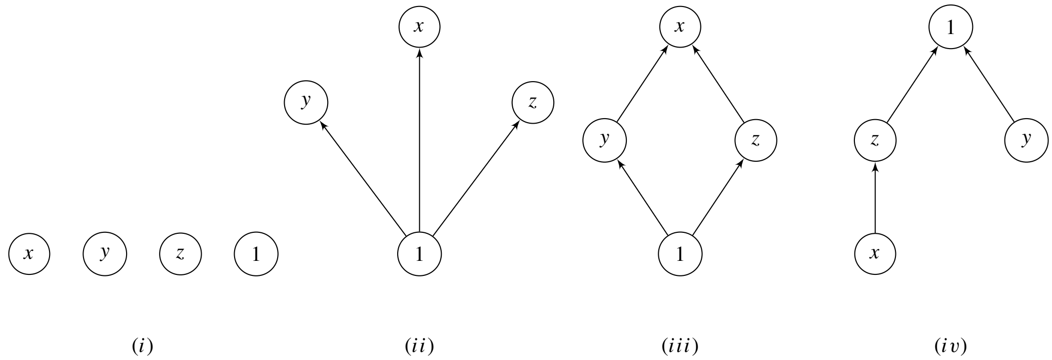


Fig. 1. Hasse Diagrams of the Clifford posets obtained from functions $F_i$, $i = 1, 2, 3, 4$ presented in Table 1.

Table 2
Functions $F_i'$ for $i = 1, 2, 3$ in Remark 4.

| $F_1'$ | 0 | $x$ | $y$ | $z$ | 1 |
|---|---|---|---|---|---|
| 0 | $x$ | $x$ | $x$ | $x$ | $x$ |
| $x$ | $x$ | $x$ | $y$ | $x$ | $x$ |
| $y$ | $x$ | $x$ | $x$ | $x$ | $x$ |
| $z$ | $x$ | $x$ | $x$ | $x$ | $x$ |
| 1 | $x$ | $x$ | $x$ | $x$ | $x$ |

| $F_2'$ | 0 | $x$ | $y$ | $z$ | 1 |
|---|---|---|---|---|---|
| 0 | $z$ | 1 | $x$ | $y$ | 0 |
| $x$ | 1 | $y$ | $z$ | 0 | $x$ |
| $y$ | $x$ | $z$ | 0 | 1 | $y$ |
| $z$ | $y$ | 0 | 1 | $x$ | $z$ |
| 1 | 0 | $x$ | $y$ | $z$ | 1 |

| $F_3'$ | 0 | $x$ | $y$ | $z$ | 1 |
|---|---|---|---|---|---|
| 0 | 0 | $x$ | $y$ | $z$ | 1 |
| $x$ | 0 | $x$ | 1 | $z$ | 1 |
| $y$ | 0 | $x$ | $y$ | $y$ | 1 |
| $z$ | 0 | $x$ | $y$ | $z$ | 1 |
| 1 | 0 | $x$ | $y$ | $z$ | 1 |

- The poset obtained from $F_3$, given in Fig. 1(iii), is a bounded lattice.
- The poset obtained by $F_4$, given in Fig. 1(iv), is bounded above by the element 1 and has *two* minimal elements.
- The poset obtained from $F_5$ is a chain, i.e., totally ordered with $x \preceq_F y \preceq_F z \preceq_F 1$.

**Remark 3.** If $F$ is associative, it satisfies (**GQP**). However, associativity is only sufficient and not necessary for (**GQP**). Note that while the functions $F_1$ and $F_2$ are associative, $F_3$, $F_4$, and $F_5$ are not, since

$$F_3(x, F_3(y, z)) = 1 \neq z = F_3(F_3(x, y), z) ,$$
$$F_4(y, F_4(z, 1)) = y \neq 1 = F_4(F_4(y, z), 1) ,$$
$$F_5(x, F_4(y, z)) = x \neq z = F_5(F_5(x, y), z) .$$

**Remark 4.** Let $\mathbb{P} = \{0, x, y, z, 1\}$. In the functions $F_i' : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ given in Table 2, exactly one of (**LLI**), (**QP**), and (**GQP**) is not satisfied, showing that the properties are mutually independent.

(i) Clearly, $F_1'$ does not satisfy (**LLI**), since there exists no $\ell \in \mathbb{P}$ such that $F_1'(\ell, 0) = 0$. However, $F_1'$ satisfies (**QP**) and (**GQP**).
(ii) Similarly, $F_2'$ satisfies (**LLI**) and (**GQP**). However $F_2'(y, F_2'(z, x)) = x \neq 0 = F_2'(z, x)$, i.e., $F_2'$ does not satisfy (**QP**).
(iii) Note that $F_3'$ satisfies (**LLI**) and (**QP**). However, $F_3'(x, F_3'(y, z)) = 1 \neq F_3'(\ell, z)$ for any $\ell \in \mathbb{P}$, i.e., $F_3'$ does not satisfy (**GQP**).

### *2.3. Relation between the underlying and Clifford relation*

For some groupoids $(\mathbb{P}, F)$, the underlying set $\mathbb{P}$ may have a preexisting order $\leq$. In that case, if the Clifford poset exists, it is natural to ask when the Clifford relation coincides with the preexisting relation, i.e., when will $(\mathbb{P}, \leq) = (\mathbb{P}, \preceq_F)$? In the following results, we characterise such groupoids. Recall the definition of downset and upset of an element given in Definition 2. Given $y \in \mathbb{P}$. Then

- $y \downarrow = \{a \in \mathbb{P} \mid a \leq y\}$ is called the **downset** of $y$.

- $y\uparrow = \{a \in \mathbb{P} \mid y \leq a\}$ is called the **upset** of $y$.

**Theorem 2.** *Given an ordered groupoid* $(\mathbb{P}, \leq, F)$ *such that* $\preceq_F$ *is a partial order, then* $(\mathbb{P}, \leq) = (\mathbb{P}, \preceq_F)$ *if and only if* $\mathcal{R}an(F^y) = y\downarrow$ *for all* $y \in \mathbb{P}$.

**Proof.** Let $x, y \in \mathbb{P}$ be arbitrary.

Suppose $(\mathbb{P}, \leq) = (\mathbb{P}, \preceq_F)$. Then

$$\begin{aligned} x \in \mathcal{R}an(F^y) &\iff \exists \ell \in \mathbb{P} \text{ such that } F(\ell, y) = x \\ &\iff x \preceq_F y \\ &\iff x \leq y \\ &\iff x \in y\downarrow . \end{aligned}$$

Thus, $\mathcal{R}an(F^y) = y\downarrow$.

Conversely, suppose $\mathcal{R}an(F^y) = y\downarrow$. Then

$$\begin{aligned} x \leq y &\iff x \in y\downarrow \\ &\iff x \in \mathcal{R}an(F^y) \\ &\iff \exists \ell \in \mathbb{P} \text{ such that } F(\ell, y) = x \\ &\iff x \preceq_F y. \end{aligned}$$

Thus, $(\mathbb{P}, \leq) = (\mathbb{P}, \preceq_F)$. □

The proof of the following theorem for the additive Clifford relation follows along similar lines.

**Theorem 3.** *Given an ordered groupoid* $(\mathbb{P}, \leq, F)$ *such that* $\sqsubseteq_F$ *is a partial order, then* $(\mathbb{P}, \leq) = (\mathbb{P}, \sqsubseteq_F)$ *if and only if* $\mathcal{R}an(F^y) = y\uparrow$ *for all* $y \in \mathbb{P}$.

### 2.4. Every poset is a Clifford poset

While in the previous section, we discussed the coincidence of the underlying and the Clifford's order, in this section, we discuss if every poset can be considered a Clifford poset for a suitable operation.

Similar to our exploration, Neggers and Kim [14] have discussed obtaining an order from a groupoid. Given a groupoid $(\mathbb{P}, F)$, Neggers [23] defines it to be a *pogroupoid*, if for all $x, y, z \in \mathbb{P}$, the following properties or functional equations are valid:

$$F(x, y) \in \{x, y\}, \tag{2}$$

$$F(x, F(y, x)) = F(y, x), \tag{3}$$

$$F(F(x, y), z) = F(F(x, y), F(y, z)). \tag{4}$$

Further, for a given pogroupoid $(\mathbb{P}, F)$, they associate a partial order $\leq_*$ with $\mathbb{P}$ as follows:

$$x \leq_* y \iff F(y, x) = y. \tag{5}$$

Conversely, given a poset $(\mathbb{P}, \leq)$, they define a pogroupoid $(\mathbb{P}, F')$, where $F' : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ is given as follows:

$$F'(x, y) = \begin{cases} x, & \text{if } y \leq x, \\ y, & \text{otherwise.} \end{cases} \tag{6}$$

Note that (5) can be seen as a special case of (ACR) with $\ell = y$. Further, every pogroupoid $(\mathbb{P}, F)$ also satisfies (**LLI**), (**QP**), and (**GQP**) as shown below.

**Theorem 4.** *If* $(\mathbb{P}, F)$ *is a pogroupoid, then* $F$ *satisfies* (**LLI**)*,* (**QP**)*, and* (**GQP**).

**Proof.** Since $F$ satisfies (2), it is idempotent. Thus it satisfies (**LLI**).

Now suppose $F(x, F(y, z)) = z$. Then by (2), $z = x$ or $z = F(y, z)$.

- If $z = F(y, z)$, (**QP**) is satisfied.
- If $z = x$, we have $F(x, F(y, x)) = x$. By (3), we get $F(y, x) = x$. Thus (**QP**) is satisfied.

Suppose $F(x, F(y, z)) = w$. Then we need to show that there exists an $l \in \mathbb{P}$ such that $F(l, z) = w$. Now, by (2), either $w = x$ or $w = F(y, z)$.

- If $w = F(y, z)$, then $l = y$ and (**GQP**) is satisfied.
- If $w = x$, we have $F(x, F(y, z)) = x$. By (2), we have $F(y, z) = z$ or $F(y, z) = y$.
  - If $F(y, z) = z$, we have $F(x, z) = x = w$ and with $l = x$ we see that (**GQP**) is satisfied.
  - If $F(y, z) = y$, we have $F(x, y) = x$. Thus $F(x, F(y, z)) = F(F(x, y), F(y, z)) = x$. By (4), we get $F(F(x, y), z) = x$, i.e., $F(x, z) = x = w$ and with $l = x$ we see that (**GQP**) is satisfied. □

**Remark 5.** Note however that the converse of the above theorem is not true. A function satisfying (**LLI**), (**QP**), and (**GQP**), need not satisfy the axioms of a pogroupoid. The product $t$-norm $T_{\mathbf{P}}(x, y) = xy$ defined on $[0, 1]$, satisfies (**LLI**), (**QP**), and (**GQP**), but is not locally internal, i.e., $T_{\mathbf{P}}(x, y) \notin \{x, y\}$ for any $x, y \in (0, 1)$, and hence $([0, 1], T_{\mathbf{P}})$ is not a pogroupoid.

From Theorem 4, we see that every pogroupoid, in fact, gives rise to a Clifford poset. Taking cue from the construction proffered in (6), we can also show that every poset is a multiplicative Clifford poset obtained from an appropriate groupoid.

**Theorem 5.** *Given a poset $(\mathbb{P}, \leq)$, there exists an operation $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ such that $(\mathbb{P}, \leq) = (\mathbb{P}, \preceq_F)$.*

**Proof.** Let us define $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ as follows:

$$F(x, y) = \begin{cases} x, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases} \tag{7}$$

We claim that given $x, y \in \mathbb{P}$, $x \leq y$ iff $x \preceq_F y$. Suppose $x \leq y$, then $F(x, y) = x$, which implies that $x \preceq_F y$.

Conversely, if $x \preceq_F y$, then there exists an $\ell \in \mathbb{P}$ such that $F(\ell, y) = x$. Since $F$ is locally internal, $x = y$ or $x = \ell$. If $x = y$, $x \leq y$ follows trivially, and if $x = \ell$ then $F(x, y) = x$ and $x \leq y$ from (7). Hence, $(\mathbb{P}, \leq) = (\mathbb{P}, \preceq_F)$. □

Note that $F'$ defined in (6) leads to an additive Clifford poset. The following dual result can be obtained easily.

**Theorem 6.** *Given a poset $(\mathbb{P}, \leq)$, there exists an operation $F' : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ such that $(\mathbb{P}, \leq) = (\mathbb{P}, \sqsubseteq'_F)$.*

The main line of exploration in [14] is to characterise semigroups that are also pogroupoids while our intention in the sequel is to investigate the behaviour of $F$ on the obtained Clifford poset, which we take up in Section 3.

## 3. Monotonicity of the groupoid operations on the obtained Clifford posets

In the previous section, we studied the minimal conditions a groupoid should possess to give a partial order. Given an operation $F$ that yields a partial order, we obtain an ordered algebra, i.e., $(\mathbb{P}, \preceq_F, F)$. It is then natural to ask if $F$ is imbued with monotonicity w.r.to $\preceq_F$ in either of the variables. In the sequel, we investigate the conditions required on $F$ that would ensure the desired monotonicity of the induced poset.

Similar to the vital role played by the functional equations in Definition 4, the following conditional functional equations will prove immensely useful in our quest to study the conditions on $F$ to ensure the appropriate type of monotonicity on the Clifford poset.

**Definition 5.** A function $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ is said to satisfy the

- **Localised**[3] **associativity** property, if for any $x, y, z \in \mathbb{P}$, there exists an $\ell \in \mathbb{P}$ such that

$$F(F(x, y), z) = F(\ell, F(y, z)). \tag{LA}$$

- **Localised exchange principle**, if for any $x, y, z \in \mathbb{P}$, there exists an $\ell \in \mathbb{P}$ such that

$$F(x, F(y, z)) = F(\ell, F(x, z)). \tag{LEP}$$

- **Localised Left Contraction property**, if for any $x, y, z \in \mathbb{P}$, there exists an $\ell \in \mathbb{P}$ such that

$$F(\ell, F(F(y, x), z)) = F(x, z). \tag{LLC}$$

- **Localised Right Contraction property**, if for any $x, y, z \in \mathbb{P}$, there exists an $\ell \in \mathbb{P}$ such that

$$F(\ell, F(F(x, y), z)) = F(x, z). \tag{LRC}$$

We now present the results[4] that show the importance of the functional equations introduced in Definition 5.

**Theorem 7.** *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ be such that $(\mathbb{P}, \preceq_F)$ is a poset. Then the following are equivalent:*

(i) *$F$ is increasing in the first variable w.r.to $\preceq_F$.*
(ii) *$F$ satisfies* (LA).

**Proof.** Suppose $F$ is increasing in the first variable. Let $x, y, z \in \mathbb{P}$ be arbitrary. Since $F(x, y) = F(x, y)$, we can say that $F(x, y) \preceq_F y$. Since $F$ is increasing in the first variable, $F(F(x, y), z) \preceq_F F(y, z)$. Hence, there exists an $\ell$ such that $F(F(x, y), z) = F(\ell, F(y, z))$.

Conversely, suppose $F$ satisfies (LA). Let $x, y \in \mathbb{P}$ such that $x \preceq_F y$, and $z \in \mathbb{P}$ be arbitrary. Then there exists an $m \in \mathbb{P}$ such that $F(m, y) = x$. Using (LA), we can say that there exists an $\ell \in \mathbb{P}$ such that, $F(x, z) = F(F(m, y), z) = F(\ell, F(y, z))$. Hence $F(x, z) \preceq_F F(y, z)$. □

**Theorem 8.** *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ be such that $(\mathbb{P}, \preceq_F)$ is a poset. Then the following are equivalent:*

(i) *$F$ is increasing in the second variable w.r.to $\preceq_F$.*
(ii) *$F$ satisfies* (LEP).

**Proof.** Suppose $F$ is increasing in the second variable. Let $x, y, z \in \mathbb{P}$ be arbitrary. Since $F(y, z) = F(y, z)$, we can say that $F(y, z) \preceq_F z$. Since $F$ is increasing in the second variable, $F(x, F(y, z)) \preceq_F F(x, z)$. Hence, there exists an $\ell \in \mathbb{P}$ such that $F(x, F(y, z)) = F(\ell, F(x, z))$.

Conversely, suppose $F$ satisfies (LEP). Let $y, z \in \mathbb{P}$ such that $y \preceq_F z$, and $x \in \mathbb{P}$ be arbitrary. Then there exists an $m \in \mathbb{P}$ such that $F(m, z) = y$. Using (LEP), we can say that there exists an $\ell \in \mathbb{P}$ such that, $F(x, y) = F(x, F(m, z)) = F(\ell, F(x, z))$. Hence $F(x, y) \preceq_F F(x, z)$. □

Once again, by duality, we have the following result:

**Theorem 9.** *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ be such that $(\mathbb{P}, \preceq_F)$ is a poset. Then the following are equivalent:*

(i) *$F$ is increasing (decreasing) in the first (second) variable w.r.to $\preceq_F$.*
(ii) *$F$ is increasing (decreasing) in the first (second) variable w.r.to $\sqsubseteq_F$.*

**Proof.** Suppose $F$ is increasing in the first variable w.r.to $\preceq_F$. Let $z \in \mathbb{P}$ be arbitrary and $x, y \in \mathbb{P}$ such that $x \sqsubseteq_F y$, i.e., there exists an $\ell \in \mathbb{P}$ such that $F(\ell, x) = y$. Hence $y \preceq_F x$. Thus, $F(y, z) \preceq_F F(x, z)$. Hence there exists an

[3] It is worthy to highlight here that the conditional functional equations (CFEs) in Definition 5 are quite different from the usual CFEs in that we allow an argument to be substituted with another, albeit depending on the considered fixed triple and hence the nomenclature of being 'localised'.

[4] Note that the Theorems 7 - 10 appear in [22] without the proofs.

$m \in \mathbb{P}$ such that $F(m, F(x,z)) = F(y,z)$. Hence, $F(x,z) \sqsubseteq_F F(y,z)$. Thus $F$ is increasing in the first variable w.r.to $\sqsubseteq_F$.

The converse of the above can be proven similarly. □

In the following theorems, we characterise the functions that are decreasing on the induced poset, showcasing the importance of (LLC) and (LRC). We shall use (ACR) in the following results, since it is considered to obtain order from fuzzy implications, and it would be interesting to investigate when a fuzzy implication is decreasing in the first variable w.r.to the induced order.[5]

**Theorem 10.** *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ be such that $(\mathbb{P}, \sqsubseteq_F)$ is a poset. Then the following are equivalent:*

(i) *$F$ is decreasing in the first variable w.r.to $\sqsubseteq_F$.*
(ii) *$F$ satisfies* (LLC).

**Proof.** Suppose $F$ is decreasing in the first variable w.r.to $\sqsubseteq_F$. Let $x, y, z \in \mathbb{P}$ be arbitrary. Since $F(x,y) = F(x,y)$, we have $y \sqsubseteq_F F(x,y)$. Thus, $F(F(x,y),z) \sqsubseteq_F F(y,z)$. Hence there exists an $l \in \mathbb{P}$ such that $F(l, F(F(x,y),z)) = F(y,z)$. Thus $F$ satisfies (LLC).

Conversely, suppose $F$ satisfies (LLC). Let $z \in \mathbb{P}$ be arbitrary and $x, l \in \mathbb{P}$ such that $x \sqsubseteq_F l$. Hence there exists a $y \in \mathbb{P}$ such that $F(y,x) = l$. Given $x, y, z$, there exists an $m \in \mathbb{P}$ such that we have $F(x,z) = F(m, F(F(y,x),z)) = F(m, F(l,z))$. Hence, $F(l,z) \sqsubseteq_F F(x,z)$. Since $z$ is arbitrary, $F$ is decreasing in the first variable. □

**Theorem 11.** *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ be such that $(\mathbb{P}, \sqsubseteq_F)$ is a poset. Then the following are equivalent:*

(i) *$F$ is decreasing in the second variable w.r.to $\sqsubseteq_F$.*
(ii) *$F$ satisfies* (LRC).

**Proof.** Suppose $F$ is decreasing in the second variable w.r.to $\sqsubseteq_F$. Let $x, y, z \in \mathbb{P}$ be arbitrary. Since $F(y,z) = F(y,z)$, we have $z \sqsubseteq_F F(y,z)$. Thus, $F(x, F(y,z)) \sqsubseteq_F F(x,z)$. Hence there exists an $\ell \in \mathbb{P}$ such that $F(\ell, F(x, F(y,z))) = F(x,z)$. Thus $F$ satisfies (LRC).

Conversely, suppose $F$ satisfies (LRC). Let $x \in \mathbb{P}$ be arbitrary and $m, z \in \mathbb{P}$ such that $z \sqsubseteq_F m$. Hence there exists a $y \in \mathbb{P}$ such that $F(y,z) = m$. Given $x, y, z$, there exists an $\ell \in \mathbb{P}$ such that we have $F(x,z) = F(\ell, F(x, F(y,z))) = F(\ell, F(x,m))$. Hence, $F(x,m) \sqsubseteq_F F(x,z)$. Since $x$ is arbitrary, $F$ is decreasing in the second variable. □

## 4. Clifford posets induced on [0, 1] by non-associative operations

Among the non-associative fuzzy logic connectives, the well-studied classes are that of semi-copulas - and their variants, viz., quasi-copulas and copulas - fuzzy implications, overlap, and grouping functions. In this section, we deal with each of them and discuss if and when they lead to a Clifford poset, which essentially means studying the satisfaction of the properties (**LLI**), (**QP**), and (**GQP**) by these operations. We follow it up by discussing the order-theoretic behaviour of the induced ordered algebras, i.e., boundedness of the obtained Clifford posets and monotonicity of the operations w.r.to the new order.

We begin by giving a general sufficiency result for an operation $F$ to satisfy (**LLI**), (**QP**), and (**GQP**).

**Lemma 1** *(cf. [21]).* [6] *Let $F : [0,1]^2 \to [0,1]$.*

(i) *If $F$ has a left neutral element $e \in [0,1]$, then it satisfies* (**LLI**).
(ii) *Further, if the neutral element of $F$ is $e = 0$ or $e = 1$, and $F$ is monotonic, then it satisfies* (**QP**).
(iii) *Further, if $F$ is continuous, then it satisfies* (**GQP**).

[5] Also, please see the discussion at the end of Section 4.1.
[6] We reproduce the proof from [21] for the sake of completeness.

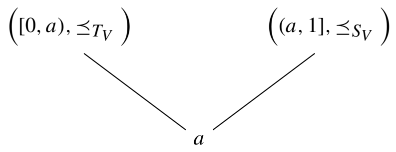


Fig. 2. The general structure of the Clifford poset $([0,1], \preceq_V)$ obtained from a nullnorm $V = \langle S_V, a, T_V \rangle$ on $\mathbb{P} = [0,1]$.

**Proof.** (i) is obvious and (ii) follows from Proposition 4.1 in [12].

To see (iii), let us consider the case $e = 1$ and $F$ is increasing. The other cases can be shown similarly. Suppose $F(x, F(y,z)) = w$. Then $F(x, F(y,z)) = w \le F(y,z) = F(1, F(y,z))$. Since $F(1,0) = 0$, by monotonicity we have

$$0 = F(0,z) \le w \le F(y,z)\ .$$

Now, by the continuity of $F$, there exists $\ell \in [0,1]$ such that $F(\ell, z) = w$, i.e., $F$ satisfies (**GQP**). □

**Remark 6.** Note that the converse of the above statements is not true, as can be seen from the following examples.

- Consider the function
$$F(x,y) = \begin{cases} x, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$
While it does not have a neutral element, it satisfies (**LLI**).
- Consider the uninorm
$$U(x,y) = \begin{cases} \max(x,y), & \text{if } x,y \in [0.5,1], \\ \min(x,y), & \text{otherwise.} \end{cases}$$
While it satisfies (**LLI**) and (**QP**), it does not have $e = 0$ or $1$ as a neutral element. Instead $e = 0.5$.
- Consider the drastic t-norm
$$T_{\mathbf{D}}(x,y) = \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise.} \end{cases}$$
While it satisfies (**LLI**) having 1 as a neutral element, (**QP**), and (**GQP**), it is not continuous.

Note that every fuzzy logic connective is either monotonic or mixed monotonic, thus it is defined on a set with a pre-existing order. Also, each FLC satisfies some boundary conditions, which implies that the underlying set is also bounded. To examine, if an FLC remains an FLC on the induced Clifford poset, we first require the obtained poset to be bounded. Before analysing FLC's in isolation, we discuss the conditions on an algebraic operation that would ensure the boundedness of the obtained poset.

### *4.1. On the boundedness of the obtained poset*

Given that we obtain a Clifford poset, we would now like to peek into the nature of the obtained poset by characterising the conditions under which the underlying operation of the groupoid gives a bounded poset. In Example 6, we observe that while $[0,1]$ w.r.to the usual order is bounded, the poset obtained by (ACR), given in Fig. 6(i), is not bounded below. One might suspect it is the lack of associativity of $I_{\mathbf{GD}}$ that leads to the lack of boundedness of the obtained Clifford poset but this is not the case. Consider a nullnorm $V$ on $\mathbb{P} = [0,1]$, which by definition, is associative. It has been shown in [12] that while all nullnorms do give rise to Clifford posets, such posets are only bounded below, and not bounded above. Note that any nullnorm $V$ with annihilator $a \in ]0,1[$ leads to a poset whose Hasse diagram is given in Fig. 2.

The following, easy to obtain results, show the necessary and sufficient conditions required to ensure that the Clifford posets are bounded below and above. Note that we shall employ the following results in the setting of unit interval but they hold true in a more general setting as presented below.

**Lemma 2** *(cf. [12] Proposition 2.10).* *$(\mathbb{P}, \preceq_F)$ is bounded below iff there exists an $\ell \in \mathbb{P}$ such that $\ell \in \mathcal{R}an(F^x)$ for all $x \in \mathbb{P}$.*

**Proof.**

$$\begin{aligned}(\mathbb{P}, \preceq_F) \text{ is bounded below } &\iff \exists \ell \in \mathbb{P} \text{ such that, for all } x \in \mathbb{P},\ \ell \preceq_F x\\ &\iff \text{for each } x \in \mathbb{P},\ \exists m_x \in \mathbb{P} \text{ such that } F(m_x, x) = \ell\\ &\iff \text{for all } x \in \mathbb{P},\ \ell \in \mathcal{R}an(F^x). \quad \square\end{aligned}$$

**Corollary 1** *(cf. [12] Lemma 6.1).* *Suppose that $\ell$ is a left annihilator of $(\mathbb{P}, F)$. Then $(\mathbb{P}, \preceq_F)$ is bounded below by $\ell$.*

**Lemma 3** *(cf. [12] Proposition 6.2).* *$(\mathbb{P}, \preceq_F)$ is bounded above iff there exists a $t \in \mathbb{P}$ such that $F^t : \mathbb{P} \to \mathbb{P}$ is onto.*

**Proof.**

$$\begin{aligned}(\mathbb{P}, \preceq_F) \text{ is bounded above } &\iff \exists t \in \mathbb{P} \text{ such that, for all } x \in \mathbb{P},\ x \preceq_F t\\ &\iff \text{for each } x \in \mathbb{P},\ \exists m_x \text{ such that } F(m_x, t) = x\\ &\iff F^t : \mathbb{P} \to \mathbb{P} \text{ is onto.} \quad \square\end{aligned}$$

**Corollary 2** *(cf. [12] Lemma 6.1).* *Suppose $t$ is the right identity element of $(\mathbb{P}, F)$, then $(\mathbb{P}, \preceq_F)$ is bounded above by $t$.*

Since (ACR) is the dual of (MCR), the dual of the corresponding results given in Section 4.1 hold, i.e., $(\mathbb{P}, \preceq_F)$ is bounded below (above) iff $(\mathbb{P}, \sqsubseteq_F)$ is bounded above (below). It also follows from here that given the existence of an annihilator or identity element, $(\mathbb{P}, \sqsubseteq_F)$ is bounded above by the annihilator, and below by the identity element.

Note that in the literature, typically, (MCR) is employed to obtain the order from FLCs that are monotonic in both the variables and whose function values do not exceed the meet of their arguments, i.e., $F \le \wedge$, see [13,12]. On the contrary, (ACR) is usually considered to obtain the order from FLCs whose function values always exceed the join of their arguments, i.e., $F \ge \vee$, see [11]. We shall choose the order for the considered operations accordingly. For fuzzy implications, in case we consider (MCR), 1 would be the least element on the Clifford poset as $I(0, x) = 1$ for all $x \in L$, and $I(1, 1) = 1$, which is not the top element. We shall thus consider (ACR) for fuzzy implications.

### 4.2. Posets from semi-copulas

We now explore whether and when the non-associative FLCs yield a Clifford poset, essentially studying the satisfaction of the properties (**LLI**), (**QP**), and (**GQP**), starting with a semi-copula. A semi-copula can be seen as a generalisation of the classical notion of intersection. They generalise $t$-norms as they are not required to be commutative or associative. Quasi-copulas and copulas form a subclass of semi-copulas. We investigate the orders obtained from them, and investigate if they retain their order-theoretic properties on the induced poset.[7]

**Definition 6** *(Definition 2.1, [9]).* A function $S : [0, 1]^2 \to [0, 1]$ is said to be a

- semi-copula, if for all $x \in [0, 1]$,
  - $S(1, x) = S(x, 1) = x$,
  - $S$ is increasing in both variables.
- quasi-copula, if it is a semi-copula and if for all $x_1, x_2, y_1, y_2 \in [0, 1]$, it satisfies the 1-Lipschitz property,
  - $|S(x_1, y_1) - S(x_2, y_2)| \le |x_1 - x_2| + |y_1 - y_2|$.
- copula, if it is a semi-copula and if for all $x, x_1, x_2, y_1, y_2 \in [0, 1]$ such that $x_1 \le x_2$ and $y_1 \le y_2$, it satisfies the 2-increasing property,

[7] For the proofs of the results in this subsection, please refer to [21].

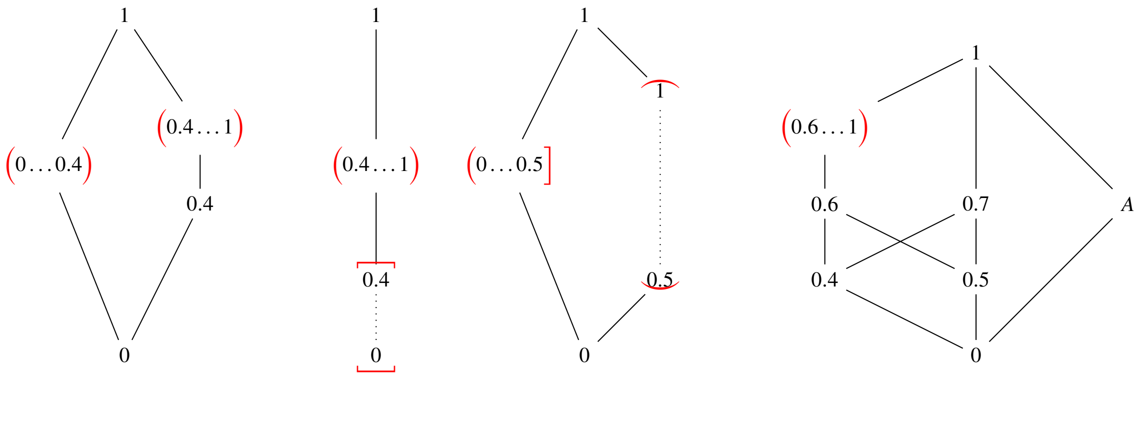


Fig. 3. Hasse Diagrams of the Clifford posets obtained from Semi-Copulas in Example 2.

– $S(x_1, y_1) + S(x_2, y_2) \geq S(x_1, y_2) + S(x_2, y_1)$.

**Remark 7** *([9]).*

- Quasi-copulas and copulas are continuous functions.
- Every copula is a quasi-copula but the converse is not true.
- We will use the notation $Q$ for a quasi-copula and $C$ for a copula.

From the definition of a semi-copula $S$ and Lemma 1, we see that semi-copulas always satisfy (**LLI**) and (**QP**). However, they may not satisfy (**GQP**) and thus may not always give rise to a Clifford poset on $[0, 1]$.

For instance, consider the semi-copula

$$S(x, y) = \begin{cases} \min(x, y), & \text{if } \max(x, y) = 1, \\ 0, & \text{if } (x, y) \in [0, 0.4]^2, \\ \frac{1}{2}\min(x, y), & \text{otherwise.} \end{cases}$$

While $S(0.7, S(0.5, 0.4)) = S(0.7, 0.2) = 0.1$, there does not exist any $\ell \in [0, 1]$ s.t. $S(\ell, 0.4) = 0.1$. Thus $S$ does not satisfy (**GQP**), and the relation $\preceq_S$ does not give rise to a poset on $[0, 1]$. In the following example, we present examples of semi-copulas that give rise to Clifford poset.

**Example 2.** In Table 3, we present some examples of semi-copulas that give rise to posets. The corresponding Hasse diagrams can be seen in Fig. 6. We see that semi-copulas can give rise to a variety of ordered structures. Notice that while the semi-copula is defined on a totally ordered set, the poset obtained through multiplicative Clifford relation may not be a chain.

- The Hasse diagrams of the posets obtained from $S_1$, $S_2$, $S_3$, and $S_4$ are given in Figs. 3(i) – (iv), respectively. As shown, $S_1$, $S_2$ and $S_3$ yield bounded lattices, whereas $S_4$ does not yield a lattice. Note that $A = (0, 0.6) \setminus \{0.4, 0.5\}$ in Fig. 3(iv).
- All the obtained lattices, viz. through $S_1$, $S_2$, and $S_3$ are non-distributive. However, a semi-copula may give rise to a distributive lattice. For example if $S(x, y) = \min(x, y)$, we obtain a total order, which is distributive.
- The posets obtained from $S_1$, and $S_3$ are not modular, while the poset obtained from $S_2$ is.

Note that since associativity implies (**GQP**), every associative semi-copula gives rise to a poset. Also, from Theorem 2 it follows that $([0, 1], \preceq_S)$ is a chain if and only if $S$ is continuous in the first variable. Thus, every continuous semi-copula yields a chain.

Table 3
Some examples of semi-copulas that retain their properties on the Clifford poset.

| Semi-copula | $([0,1], \preceq_S)$ |
|---|---|
| $S_1(x,y) = \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{if } \min(x,y) \leq 0.4 \text{ and } \max(x,y) < 1, \\ 0.4, & \text{otherwise.} \end{cases}$ | Fig. 6(i) |
| $S_2(x,y) = \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1 \text{ or } \min(x,y) \leq 0.4, \\ 0.4, & \text{otherwise.} \end{cases}$ | Fig. 6(ii) |
| $S_3(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,0.5] \times [0,1), \\ \min(x,y), & \text{otherwise.} \end{cases}$ | Fig. 6(iii) |
| $S_4(x,y) = \begin{cases} 0.6, & \text{if } (x,y) \in [0.6,1) \times (0.6,1) \cup (0.4,0.6] \times (0.7,1), \\ 0.4, & \text{if } (x,y) \in [0.4,0.5) \times 0.7 \cup [0.6,0.7] \times 0.6, \\ 0.5, & \text{if } (x,y) \in [0.5,0.6) \times 0.7 \cup [0.7,1) \times 0.6, \\ \min(x,y), & \max(x,y) = 1, \\ 0, & \text{otherwise.} \end{cases}$ | Fig. 6(iv) |

We now move our attention specifically to quasi-copulas and copulas. From Lemma 1, we see that every quasi-copula $Q$ and a copula $C$ give rise to a poset. In fact, due to their continuity, they give rise to a totally ordered complete lattice on $[0, 1]$.

**Lemma 4.** *If* $F : [0,1]^2 \to [0,1]$ *is either a quasi-copula or a copula, it satisfies* (**LLI**), (**QP**) *and* (**GQP**).

**Lemma 5.** *Given a quasi-copula* $Q$*, the poset obtained from* $\preceq_Q$ *on* $[0, 1]$ *is a chain, and is in fact, the usual order on* $[0, 1]$*, i.e.,* $([0,1], \preceq_Q) = ([0,1], \leq)$.

**Corollary 3.** *Given a copula* $C$*, the poset obtained from* $\preceq_C$ *on* $[0, 1]$ *is a chain and is the usual order on* $[0, 1]$.

Thus, for copulas and quasi-copulas, unlike semi-copulas, the multiplicative Clifford relation always yields a totally ordered set.

**Remark 8.**

- In the case of a semi-copula, due to the presence of 0 as its annihilator and 1 as its identity, it is clear from Corollary 1, and Corollary 2 that if it does lead to a Clifford poset, it is bounded below by 0 and bounded above by 1.
- A semi-copula, as is well known, is not associative or commutative, and thus its satisfaction of (LA) and (LEP) does not follow directly. In general, a semi-copula $S$ may or may not be monotonic on the induced poset, and hence may or may not be a semi-copula. Table 3 presents some examples of semi-copulas that satisfy both (LA) and (LEP), and are thus semi-copulas on the obtained Clifford posets.
  Note that the semi-copula $S$ in Example 3 does not satisfy (LA) and hence, is not increasing in the first variable. The semi-copula in Example 4 does not satisfy (LEP) and hence, is not increasing in the second variable.
- Since quasi-copulas, and copulas give the usual order, they retain their properties on the induced poset.

**Example 3.** Consider the semi-copula $S : [0,1]^2 \to [0,1]$ given as follows (see Fig. 4(i)):

$$S(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0.1, & \text{if } x = 0.2,\ y \in [0.7,1), \\ 0.2, & \text{if } (x,y) \in (0.2,0.9) \times [0.7,1), \\ 0.6, & \text{if } (x,y) \in [0.9,1) \times [0.7,1), \\ 0, & \text{otherwise.} \end{cases}$$

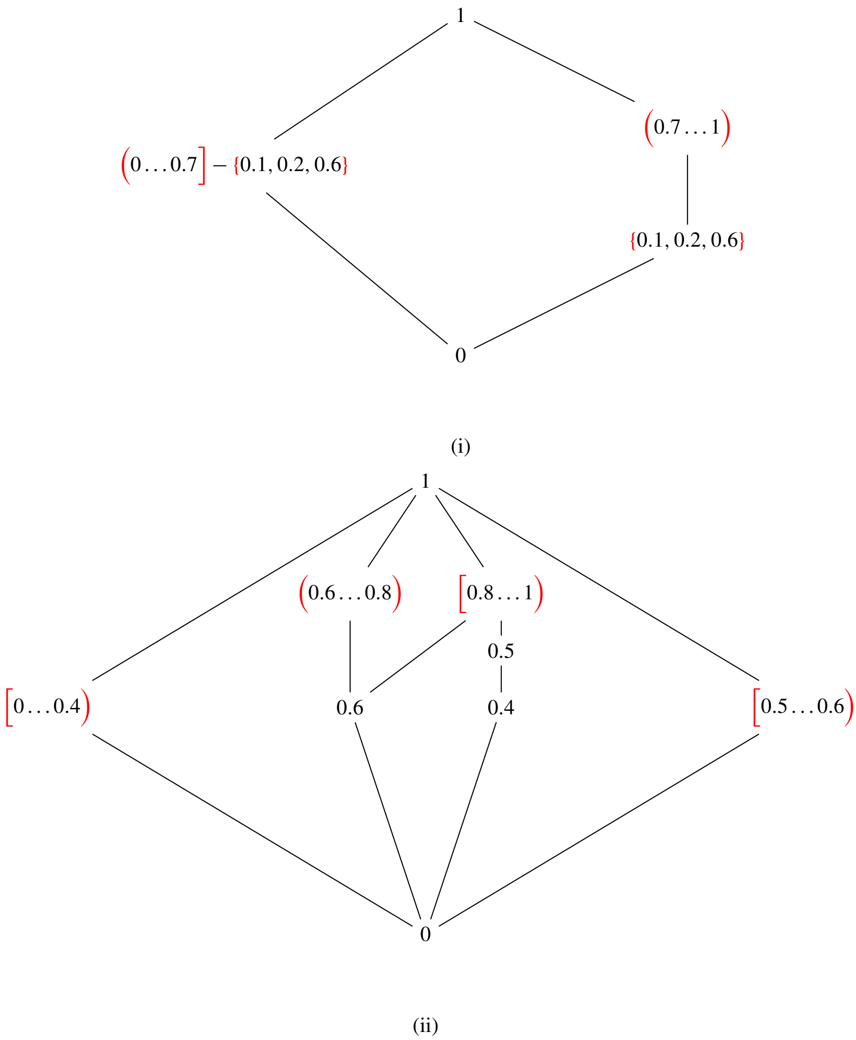


Fig. 4. Hasse Diagrams of the Clifford posets obtained from Semi-Copulas in Examples 3 and 4.

Note that $S$ does not satisfy (LA), since there does not exist any $\ell \in [0, 1]$ such that $0.1 = S(S(0.5, 0.9), 0.7) = S(\ell, S(0.9, 0.7)) = S(\ell, 0.6)$. Thus $S$ is not increasing in the first variable w.r.to $\preceq_S$ as

$$0.2 \preceq_S 0.9 \text{ but } 0.1 = S(0.2, 0.7) \not\preceq_S S(0.9, 0.7) = 0.6.$$

while it is on $([0, 1], \leq)$.

**Example 4.** Consider the semi-copula $S : [0, 1]^2 \to [0, 1]$ given as follows (see Fig. 4(ii)):

$$S(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1, \\ 0.4, & \text{if } (x, y) \in [0.8, 1) \times [0.5, 0.6) \cup [0.5, 0.6) \times [0.8, 1), \\ 0.5, & \text{if } (x, y) \in [0.6, 0.8) \times [0.8, 1), \\ 0.6, & \text{if } (x, y) \in [0.8, 1) \times [0.6, 1), \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \text{Note that } S(0.8, S(0.7, 0.9)) &= S(0.8, 0.5) \\ &= 0.4 \neq S(\ell, 0.6) = S(\ell, S(0.8, 0.9)) \text{ for any } \ell \in [0, 1]. \end{aligned}$$

Thus (LEP) is not satisfied, and $S$ is not increasing in the second variable w.r.to $\preceq_S$, as

$0.5 \preceq_S 0.9$ but $0.4 = S(0.8, 0.5) \not\preceq_S S(0.8, 0.9) = 0.6$.

### 4.3. Posets from overlap and grouping functions

Introduced by Bustince et al. [26,27], overlap and grouping functions also generalise the notion of classical intersection and union, respectively. They play an important role in image processing and have been a subject of theoretical investigation in recent works. Since they are not required to be associative, we investigate the order induced from them and compare it with the results of [24]. We then see if their order-theoretic properties are retained on the induced poset.[8]

**Definition 7.** Let $F : [0, 1]^2 \to [0, 1]$ be a commutative and a continuous function that is non-decreasing in both variables. $F$ is said to be

- an overlap function if, further, for any $x, y \in [0, 1]$
  - $F(x, y) = 0 \iff xy = 0$;
  - $F(x, y) = 1 \iff xy = 1$;
- a grouping function if, further, for any $x, y \in [0, 1]$
  - $F(x, y) = 0 \iff x = y = 0$;
  - $F(x, y) = 1 \iff x = 1$ or $y = 1$;

We denote an overlap function by $O$ and a grouping function by $G$.

From [24], we have the result that an overlap function $O$ yields a poset w.r.to the order defined in (MCR) if and only if $O$ has a neutral element 1, i.e., $O(1, x) = x$ for every $x \in [0, 1]$. We show in the following result that the above condition is equivalent to an overlap function satisfying (**LLI**), (**QP**), and (**GQP**).

**Proposition 12.** *Given an overlap function* $O : [0, 1]^2 \to [0, 1]$*, the following are equivalent:*

(i) $O(1, x) = x$*, for all* $x \in [0, 1]$.
(ii) $O$ *satisfies* (**LLI**)*,* (**QP**)*, and* (**GQP**).

Note that the above result emphasises the value of order-theoretic exploration of algebraic structures. Whether a given overlap function $O$ has a neutral element or not is characterised by whether it leads to a Clifford poset or not. Once again, by the continuity of an overlap function and 1, we have the following result:

**Lemma 6.** *Given an overlap function* $O$ *such that* 1 *is the left neutral element of* $O$*, the poset obtained from* $\preceq_O$ *on* $[0, 1]$ *is a chain, and is in fact, the usual order on* $[0, 1]$*, i.e.,* $([0, 1], \preceq_O) = ([0, 1], \leq)$.

**Remark 9.** Since overlap functions yield the usual order through multiplicative Clifford relation, their order-theoretic properties are retained on the induced poset.

For grouping functions, we consider the relation (ACR). The results follow along similar lines, hence, we present only the statements without their proofs.

**Proposition 13.** *Given a grouping function* $G : [0, 1]^2 \to [0, 1]$*, the following are equivalent:*

(i) $G(0, x) = x$*, for all* $x \in [0, 1]$.
(ii) $G$ *satisfies* (**LLI**)*,* (**QP**)*, and* (**GQP**).

[8] For the proofs of the results in this subsection, please refer to [21].

**Lemma 7.** *Given a grouping function $G$ such that $0$ is the left neutral element of $G$, the poset obtained from $\sqsubseteq_G$ on $[0,1]$ is a chain, and is in fact, the usual order on $[0,1]$, i.e., $([0,1], \sqsubseteq_G) = ([0,1], \leq)$.*

**Remark 10.** Like overlap functions, grouping functions also yield the usual order through additive Clifford relation. Hence their order-theoretic properties are also retained.

Note that while the continuous FLCs such as quasi-copulas, copulas, overlap and grouping functions always yield a total order, not all continuous functions yield a total order. For instance, consider the projection map $F(x, y) = y$. While the function is continuous, it does not yield a total order. In fact, $\preceq_F$ is an anti-chain.

### 4.4. Posets from fuzzy implications

Fuzzy implications generalise the notion of classical implication and are one of the important fuzzy logic connectives. They play a vital role in fuzzy control [15], fuzzy logic systems [25], and approximate reasoning [3,28,16]. Unlike semi-copulas, overlap, and grouping functions, fuzzy implications do not include any associative versions of them. We investigate the posets obtained from them in this section.[9]

**Definition 8.** An $I : [0,1]^2 \to [0,1]$ is said to be a fuzzy implication if:

- $I$ is decreasing in the first variable, i.e.,

$$x_1 \leq x_2 \implies I(x_1, y) \geq I(x_2, y) \tag{I1}$$

- $I$ is increasing in the second variable, i.e.,

$$y_1 \leq y_2 \implies I(x, y_1) \leq I(x, y_2) \tag{I2}$$

- $I$ satisfies the boundary conditions:

$$I(0,0) = I(1,1) = 1, I(1,0) = 0 \tag{I3}$$

**Definition 9.** [4] A fuzzy implication $I$ is said to satisfy the

- **left neutrality** property if

$$I(1, y) = y, \qquad y \in [0,1]\,. \tag{NP}$$

- **law of importation** w.r.to $C$ if there exists a binary operation $C$ such that

$$I(x, I(y,z)) = I(C(x,y), z), \qquad x, y, z \in [0,1]\,. \tag{LI(C)}$$

In [11], the authors introduced the concept of an importation algebra. They showed that (ACR) imposes a poset structure on the underlying set of such an algebra. We present below the definition and the corresponding result.

**Definition 10** *(cf. [11], Definition 2.1).* Let $\mathbb{P} \neq \emptyset$. An algebra $(\mathbb{P}, I, C)$ is an importation algebra, if $I$ satisfies (**NP**), (**QP**), and the pair $(I, C)$ satisfies (**LI**($C$)).

**Theorem 14** *(cf. [11], Proposition 2.7). Let $(\mathbb{P}, I, C)$ be an importation algebra. Then the relation (ACR) is a partial order on $\mathbb{P}$.*

**Remark 11.** In the literature, if an $I$ satisfies (**LI**($C$)) w.r.to a $t$-norm, i.e., $C = T$, it is simply called the law of importation. If $C$ is increasing, conjunctive, and commutative, it is called the weak law of importation [17].

[9] For the proofs of the results presented in the conference in this subsection, please refer to [21]. The other results are either stated for the first time or have been provided in [22] without proofs.

The following lemma shows that any fuzzy implication satisfying (**NP**) and (**LI**($C$)) w.r.to any $C$ yields a partial order through Clifford's relation.

**Lemma 8.** *A fuzzy implication $I$ satisfying* (**NP**) *and* (**LI**($C$))*, satisfies* (**LLI**), (**QP**)*, and* (**GQP**).

Note that the converse of the above lemma is not necessarily true as illustrated in the following example.

**Example 5.** Consider the fuzzy implication $I_{\mathbf{VG}}$[11], defined as

$$I_{\mathbf{VG}}(x, y) = \begin{cases} \max(1-\sqrt{x}, y), & \text{if } y \le 0.5, \\ \max(1-x^2, y), & \text{if } y > 0.5. \end{cases}$$

In [11], it was shown that $I_{\mathbf{VG}}$ does not satisfy (**LI**($C$)) with respect to any $C$. However, it satisfies (**NP**), and thus (**QP**). Despite being discontinuous, it satisfies (**GQP**) and yields the usual order.

There exist many families of fuzzy implications. Among the major ones, it is well-known that the families of $(S, N)$-, $f$- and $g$-implications satisfy both (**NP**) and (**LI**($C$)) w.r.to an appropriate $t$-norm, see for instance, [2,1,6]. It is also known that an $R$-implication generated from a left-continuous $T$, also satisfies (**LI**($C$)), see [3], **Theorem 7.3.5**. Thus when these families are considered, $\sqsubseteq_I$ do give rise to posets. This has been extensively dealt with in [10,11].

The family of $QL$-implications however has not been investigated. Also, there exist $QL$-implications that do not always satisfy (**LI**($C$)). Thus it is interesting to study if fuzzy implications $I$ belonging to this family give rise to a poset w.r.to the relation $\sqsubseteq_I$ as defined in (ACR). In the following, we introduce this family of fuzzy implications and study certain sufficient conditions under which they do yield a poset.

**Definition 11.** [3] A function $I : [0, 1]^2 \to [0, 1]$ is called a QL-operation if there exist a $t$-norm $T$, a $t$-conorm $S$, and a fuzzy negation $N$ such that $I(x, y) = S(N(x), T(x, y))$ for all $x, y \in [0, 1]$. We will denote such an $I$ by $I_{T,S,N}$. If a QL-operation $I_{T,S,N}$ is a fuzzy implication, then we call $I_{T,S,N}$ a QL-implication.

**Remark 12.** It can be easily seen that a QL-implication always satisfies (**NP**). Hence, a QL-implication always satisfies (**LLI**) and (**QP**).

**Definition 12.** Let $S$ be a $t$-conorm and $N$ be a fuzzy negation. The pair $(S, N)$ is said to satisfy the law of excluded middle if:

$$S(N(x), x) = 1, \quad x \in [0, 1]. \tag{LEM}$$

**Lemma 9.** *[5] If $I_{T,S,N}$ is a QL-implication, then the pair $(S, N)$ satisfies* (LEM).

**Proposition 15.** *Consider a QL-operation $I_{T,S,N}$. If the pair $(S, N)$ does not satisfy* (LEM)*, $I_{T,S,N}$ does not satisfy* (**QP**).

**Proof.** Since $(S, N)$ does not satisfy (LEM), there exists a $y \in (0, 1)$ such that $S(N(y), y) < 1$. Now, $I(0, I(y, 1)) = 1$, but $I(y, 1) = S(N(y), T(y, 1)) = S(N(y), y) < 1$. Hence $I_{T,S,N}$ does not satisfy (**QP**). □

**Remark 13.**

(i) We infer from the result above that the satisfaction of (**QP**) implies (LEM). However, the satisfaction of (LEM) may not ensure (**QP**). For instance, consider the product $t$-norm $T_{\mathbf{P}}(x, y) = xy$, the fuzzy negation $N_{\mathbf{C}}(x) = 1 - x$ and the nilpotent maximum $t$-conorm

$$S_{\mathbf{nM}}(x, y) = \begin{cases} \max(x, y), & \text{if } x + y < 1, \\ 1, & \text{otherwise.} \end{cases}$$

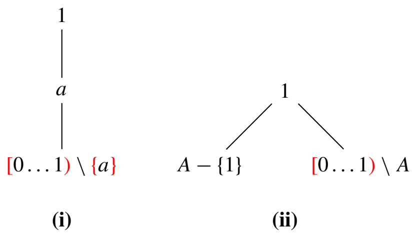


Fig. 5. Hasse diagrams of the Clifford posets obtained from the QL-operation given in Remark 13((ii),(iv)), where $A = \mathcal{R}an(N)$.

It can be seen that $(S_{\mathbf{nM}}, N_{\mathbf{C}})$ satisfies (LEM). However,

$$I_{T_{\mathbf{P}},S_{\mathbf{nM}},N_{\mathbf{C}}}(x,y) = \begin{cases} 1, & \text{if } y = 1\,, \\ \max(1-x, xy), & \text{otherwise.} \end{cases}$$

does not satisfy (**QP**), since

$$I_{T_{\mathbf{P}},S_{\mathbf{nM}},N_{\mathbf{C}}}(0.8, I_{T_{\mathbf{P}},S_{\mathbf{nM}},N_{\mathbf{C}}}(0.5, 0.4)) = 0.4 \neq I_{T_{\mathbf{P}},S_{\mathbf{nM}},N_{\mathbf{C}}}(0.5, 0.4) = 0.5\,.$$

(ii) Also, it is not true that every QL-operation that satisfies (**QP**) or (**GQP**) is a QL-implication.
The following is an example of a QL-operation $I$ which is not a fuzzy implication but satisfies (**LLI**), (**QP**), and (**GQP**). Let $S = S_{\mathbf{D}}$ be the drastic $t$-conorm defined as:

$$S_{\mathbf{D}}(x,y) = \begin{cases} \max(x,y), & \text{if } \min(x,y) = 0\,, \\ 1, & \text{otherwise}\,. \end{cases}$$

$T = T_{\mathbf{D}}$ be the drastic $t$-norm defined as:

$$T_{\mathbf{D}}(x,y) = \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

and $N_a(x) = \begin{cases} 1, & x = 0, \\ 0, & x = 1, \\ a, & \text{otherwise.} \end{cases}$ where $a \in (0,1)$ is arbitrary but fixed. Then

$$I_{S_{\mathbf{D}},T_{\mathbf{D}},N_{\mathbf{a}}}(x,y) = \begin{cases} 1, & x = 0 \text{ or } y = 1, \\ y, & x = 1, \\ a, & \text{otherwise.} \end{cases}$$

where $a \in (0, 1)$. The corresponding Hasse diagram is given in Fig. 5(i).
(iii) [5] For any $t$-conorm $S$, the pair $(S, N_{\mathbf{D_2}})$ always satisfies (LEM) where

$$N_{\mathbf{D_2}}(x) = \begin{cases} 0, & \text{if } x = 1, \\ 1, & \text{otherwise.} \end{cases}$$

In fact, given any $t$-norm $T$, $I_{T,S,N_{\mathbf{D_2}}}$ is the Weber implication

$$I_{\mathbf{WB}}(x,y) = \begin{cases} y, & \text{if } x = 1, \\ 1, & \text{otherwise.} \end{cases}$$

It can be easily verified that the Weber implication satisfies (**QP**) and (**GQP**) and hence leads to a Clifford poset, whose Hasse diagram is the same as given in Fig. 6(i). Notice that we obtain a join-semilattice where 1 is the greatest element, and all the other elements are minimal.

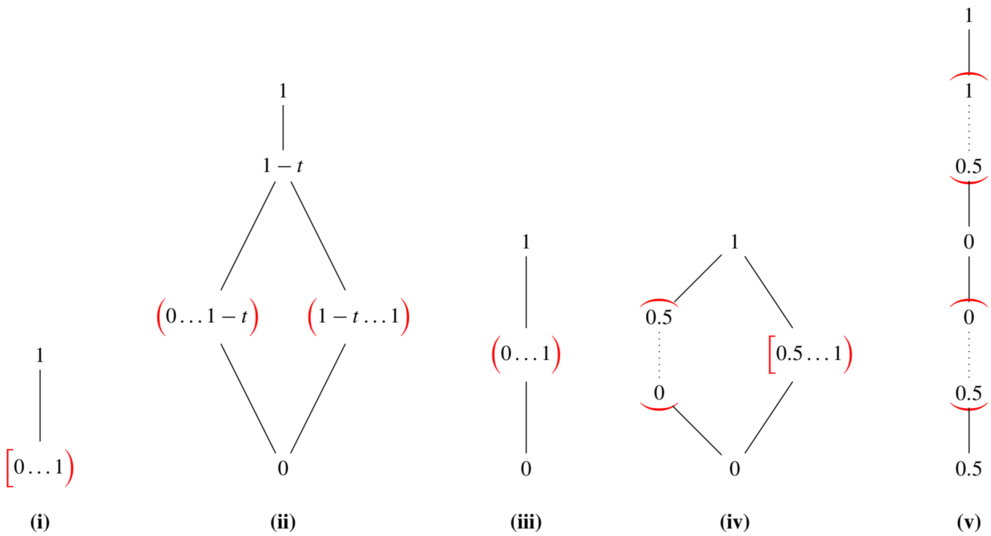

Fig. 6. Hasse diagrams of the Clifford posets obtained from fuzzy implications in Tables 3 and 4, respectively.

(iv) Let $S = S_{\mathbf{D}}$ be the drastic $t$-conorm and $N$ be any non-vanishing negation, i.e., $N(x) = 0 \iff x = 1$. If $T$ is a positive $t$-norm, i.e., $T(x, y) = 0 \iff xy = 0$, then the QL-operation obtained from the triple $(T, S_{\mathbf{D}}, N)$ satisfies (**LLI**), (**QP**), and (**GQP**).
The QL-operation obtained by the above construction is an implication and is of the given form:

$$I(x, y) = \begin{cases} N(x), & \text{if } y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

The corresponding Hasse diagram is given in Fig. 5(ii), where $A = \mathcal{R}an(N)$.

Note that it is well known that if a fuzzy implication $I$ does not satisfy the exchange principle (EP)

$$I(x, I(y, z)) = I(y, I(x, z)), \qquad x, y, z \in [0, 1]\,, \tag{EP}$$

then it does not satisfy (**LI**($C$)) with any commutative $C$ [3].

*4.4.1. Bounded additive Clifford posets from fuzzy implications:*

We saw that all the operations considered above give a bounded poset due to the presence of an annihilator, and an identity element. However, as can be seen in the following example, an implication may not have a right identity element and hence may not lead to a bounded poset.

**Example 6.** Consider the Godel implication

$$I_{\mathbf{GD}}(x, y) = \begin{cases} 1, & \text{if } x \le y\,, \\ y, & \text{if } x > y\,, \end{cases}$$

defined on $[0, 1]^2$. Clearly, $[0, 1]$ w.r.to the usual order is bounded. However, the poset obtained by (ACR), given in Fig. 6(i), is not bounded below.

We thus investigate the conditions under which an implication yields a bounded poset.

**Definition 13.** Let $I : [0, 1]^2 \to [0, 1]$ be a fuzzy implication. The function $N_I(x) = I^0(x) = I(x, 0)$ is a fuzzy negation, called the natural negation of $I$.

From the results and discussion in Sections 4.1, we have the following result:

**Lemma 10.** $([0,1], \sqsubseteq_I)$ *is bounded below by* $\ell$ *iff* $I^\ell$ *is onto.*

Since $I^0$ is onto iff $N_I$ is continuous, we have the following corollary.

**Corollary 4.** *A fuzzy implication* $I$ *on* $[0,1]$ *is bounded below on* $([0,1], \sqsubseteq_I)$ *by* $0$ *iff* $N_I$ *is continuous.*

**Corollary 5.** *If* $I : [0,1]^2 \to [0,1]$ *satisfies the neutrality property and* $\sqsubseteq_I$ *yields a bounded partial order, then* $0$ *is its bottom element.*

**Proof.** Let $l \in [0,1]$ be the bottom element of $([0,1], \sqsubseteq_I)$. Note that the range of $I^l$, when $I$ satisfies (**NP**), is $[I(1,l), I(0,l)] = [l, 1]$. Hence, by Lemma 10, we have $l = 0$. □

Interestingly, as the following result shows, from the bottom element of the obtained poset, one may be able to determine if the original $I$ is an implication on the induced poset.

**Lemma 11.** *Let* $I$ *be an implication on the bounded poset* $([0,1], \leq, 0, 1)$*. If* $([0,1], \sqsubseteq_I)$ *is a poset bounded below by an* $l \neq 0$*, then* $I$ *is not an implication on* $(\mathbb{L}, \sqsubseteq_I)$.

**Proof.** Suppose $I$ is an implication on $([0,1], \sqsubseteq_I, l, 1)$, i.e., $([0,1], \sqsubseteq_I)$ is bounded below by $l$. We can then say that $I(1,l) = l$ and since $x \leq 1$ for all $x \in [0,1]$, we have that $l \leq I(x,l)$. Thus $0 < l$ is not in the range of $I^l$. Clearly then $I^l$ is not onto and $([0,1], \sqsubseteq_I)$ can not be bounded below by $l$. Hence our assumption that $I$ is an implication on $([0,1], \sqsubseteq_I)$ is incorrect. □

*4.4.2. Monotonicity of fuzzy implications on additive Clifford posets:*

Below we examine if and when the mixed monotonicity of an $I$ is preserved on the obtained poset, i.e., the satisfaction of (LLC), and (LEP) by $I$.

**Lemma 12.** *If* $I$ *satisfies* (EP)*, it satisfies* (LEP).

**Lemma 13.** *Every* $I$ *defined on* $[0,1]$ *satisfying the neutrality property such that* $\mathcal{R}an(I^l) = [l, 1]$ *for all* $l \in [0,1]$*, satisfies* (LLC).

**Proof.** Let $x, y, z \in [0,1]$ be arbitrary. We have, by the antitonicity in the first variable and neutrality of $I$, $y = I(1,y) \leq I(x,y)$. Thus, $I(I(x,y), z) \leq I(y,z)$. Hence there will exist an $l \in [0,1]$ such that $I(l, I(I(x,y),z)) = I(y,z)$ since $\mathcal{R}an(I^{I(I(x,y),z)}) = [I(I(x,y),z), 1]$. Thus, $I$ satisfies (LLC). □

**Corollary 6.** *([22]) Every continuous fuzzy implication* $I : [0,1]^2 \to [0,1]$ *with* $1$ *as the left neutral element satisfies* (LLC).

Based on the above results, we now present the main result of this section, that completely characterises functions that give rise to fuzzy implications on the obtained Clifford poset.

**Theorem 16.** *Let* $I : [0,1]^2 \to [0,1]$*. The following statements are equivalent:*

(i) $I$ *is a fuzzy implication on* $([0,1], \sqsubseteq_I, 0, 1)$.
(ii) $I$ *satisfies* (LLC), (LEP)*,* $I(1,0) = 0$*,* $I(1,1) = 1$*,* $I^0$ *is onto,* $I(0,x) = 1$ *for all* $x \in [0,1]$.

From Theorem 16, we see that the function $I$ need not be an implication on $([0,1], \leq)$ to begin with. For a fixed but arbitrary $t \in (0,1)$, the function given by

$$I_t(x,y) = \begin{cases} 1-t, & \text{if } (x,y) \in (0,1)^2, \\ \max(1-x, y), & \text{otherwise.} \end{cases}$$

Table 4
Some examples of Clifford posets from Fuzzy Implications.

| Implication | $([0,1], \sqsubseteq_I)$ |
|---|---|
| $I_{\mathbf{T}^{\#}_{\mathbf{M}},\mathbf{N_C}}(x,y) = \begin{cases} 1, & \text{if } x \in [0,0.5] \\ & \& \ y \in (0,1], \\ \max(1-x,y), & \text{otherwise.} \end{cases}$ | Fig. 6(iv) |
| $I_{\mathbf{LK}}(x,y) = \min(1, 1-x+y)$ | Usual order on [0, 1] |
| $I_{\mathbf{DP}}(x,y) = \begin{cases} y, & \text{if } x = 1, \\ 1-x, & \text{if } y = 0, \\ 1, & \text{if } x < 1 \text{ and } y > 0. \end{cases}$ | Fig. 6(iii) |

is not a fuzzy implication on [0, 1], but is one on the induced poset. The induced poset is shown in Fig. 6(ii).

Now the following result specifying when a fuzzy implication on the original poset will again be one on the Clifford poset follows easily.

**Corollary 7.** *Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication. The following statements are equivalent:*

(i) *$I$ is a fuzzy implication on $([0,1], \sqsubseteq_I)$.*
(ii) *$I$ satisfies (LLC), (LEP), and $N_I$ is continuous.*

**Example 7.** Note that not every fuzzy implication satisfies (LLC). Consider, for example, the following implication:

$$I(x,y) = \begin{cases} \min(1-x,y), & \text{if } \max(1-x,y) \le 0.5, \\ \max(1-x,y), & \text{otherwise.} \end{cases}$$

The Hasse diagram is given in Fig. 6(v). Note that there exists no $\ell \in [0,1]$ such that $I(\ell, I(I(1,0.3),0.5)) = I(0.3,0.5)$, and hence $I$ does not satisfy (LLC).

**Example 8.** In Table 4, we present some examples of fuzzy implications, that satisfy the conditions of Corollary 7, and hence are fuzzy implications on the additive Clifford posets as well. We see that fuzzy implications yield a variety of partially ordered sets. Notice that while the fuzzy implication is defined on a totally ordered set, the poset obtained through additive Clifford relation may not be a chain.

- Since $I_{\mathbf{LK}}$ yields a total order on [0, 1], it is bounded, distributive and modular. In fact, from Theorem 3, it follows that $([0,1], \sqsubseteq_I)$ is a chain if and only if $I$ is continuous in the first variable and satisfies (**NP**).
- $I_{\mathbf{DP}}$ yields a modular but non-distributive lattice on [0, 1]. The corresponding poset is obtained in Fig. 6(iii).
- $I_{\mathbf{T}^{\#}_{\mathbf{M}},\mathbf{N_C}}$ yields neither a distributive nor a modular lattice on [0, 1]. The corresponding poset is obtained in Fig. 6(iv).
- Note that fuzzy implications do not always yield a lattice. $I_{\mathbf{GD}}$ defined in Example 6, does not yield a lattice on [0, 1]. The Hasse diagram is given in Fig. 6(i). For any $x, y \in [0,1)$, the infimum of $x$ and $y$ does not exist.

## 5. Concluding remarks

In this submission, we characterise groupoids that yield a partial order through Clifford's relation. Also, given a groupoid, we study if and when we can obtain an ordered groupoid using Clifford's relation, i.e., we provide the necessary and sufficient conditions under which the original groupoid operation is monotonic or antitonic w.r.to the new order. Further, we employ the obtained results for fuzzy logic connectives and see if and when they retain their properties on the induced poset.

We see that the monotonic (or antitonic) operations on the induced poset are characterised through conditional functional equations. The importance of these results can be understood from Example 3, where the semi-copula $S$ while increasing in the first variable w.r.to the original order, is not increasing in the first variable w.r.to the new order. We can conclude by Theorem 7 that this semi-copula $S$ does not satisfy (LA), and in turn we can conclude that $S$ is not associative. Similarly, if the operation is not increasing in the second variable w.r.to the new order, we can conclude that the operation does not satisfy the exchange principle. Thus, we see that knowing about the order-theoretic behaviour on the induced poset can provide us insights on the algebraic properties of the operation.

In future work, we would like to characterise operations that would yield special Clifford posets, namely lattices, distributive lattices, or modular lattices, and investigate if these ordered structures correspond to specific algebraic properties, thus enriching the pursuit of order-theoretic exploration of algebras.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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## References

[1] M. Baczyński, B. Jayaram, On the characterizations of (S, N)-implications, Fuzzy Sets Syst. 158 (2007) 1713–1727.
[2] M. Baczyński, B. Jayaram, Yager's classes of fuzzy implications: Some properties and intersections, Kybernetika 43 (2007) 157–182.
[3] M. Baczyński, B. Jayaram, Fuzzy Implications, Studies in Fuzziness and Soft Computing, vol. 231, Springer-Verlag, Berlin, Heidelberg, 2008.
[4] M. Baczynski, B. Jayaram, Fuzzy Implications: Some Recently Solved Problems, Springer Berlin Heidelberg, Berlin, Heidelberg, 2013, pp. 177–204.
[5] M. Baczyński, B. Jayaram, QL-implications: some properties and intersections, Fuzzy Sets Syst. 161 (2010) 158–188. Featured issue: Selected Papers from the FSTA 2008 Conference.
[6] J. Balasubramaniam, Yager's new class of implications $J_f$ and some classical tautologies, Inf. Sci. 177 (2007) 930–946.
[7] A.H. Clifford, Naturally totally ordered commutative semigroups, Am. J. Math. 76 (1954) 631–646.
[8] B. Davey, H. Priestley, Introduction to Lattices and Order, Cambridge University Press, Cambridge, 1990.
[9] F. Durante, C. Sempi, Semicopulæ, Kybernetika 41 (2005) 315–328.
[10] V.K. Gupta, B. Jayaram, Importation algebras, in: International Summer School on Aggregation Functions, Springer, 2019, pp. 83–94.
[11] V.K. Gupta, B. Jayaram, Importation lattices, Fuzzy Sets Syst. 405 (2021) 1–17.
[12] V.K. Gupta, B. Jayaram, Order based on associative operations, Inf. Sci. 566 (2021) 326–346.
[13] F. Karaçal, M.N. Kesicioğlu, A T-partial order obtained from t-norms, Kybernetika 47 (2011) 300–314.
[14] J.N.H.S. Kim, Modular posets and semigroups, Semigroup Forum 53 (1996) 57–62.
[15] C. Lee, Fuzzy logic in control systems: fuzzy logic controller. I, IEEE Trans. Syst. Man Cybern. 20 (1990) 404–418.
[16] S. Mandal, B. Jayaram, Monotonicity of SISO fuzzy relational inference with an implicative rule base, IEEE Trans. Fuzzy Syst. 24 (2016) 1475–1487.
[17] S. Massanet, J. Torrens, The law of importation versus the exchange principle on fuzzy implications, Fuzzy Sets Syst. 168 (2011) 47–69.
[18] H. Mitsch, A natural partial order for semigroups, Proc. Am. Math. Soc. 97 (1986) 384–388.
[19] H. Mitsch, Semigroups and their natural order, Math. Slovaca 44 (1994) 445–462.
[20] K.S. Nambooripad, The natural partial order on a regular semigroup, Proc. Edinb. Math. Soc. 23 (1980) 249–260.
[21] K. Nanavati, B. Jayaram, Order based on non-associative operations, in: Joint Proceedings of the 19th World Congress of the International Fuzzy Systems Association (IFSA), the 12th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT), and the 11th International Summer School on Aggregation Operators (AGOP), Atlantis Press, 2021, pp. 675–681.
[22] K. Nanavati, B. Jayaram, On the order-compatibility of fuzzy logic connectives on the generated Clifford poset, in: Information Processing and Management of Uncertainty in Knowledge-Based Systems, Springer International Publishing, Cham, 2022, pp. 714–725.
[23] J. Neggers, Partially ordered sets and groupoids, Kyungpook Math. J. 16 (1976) 7–20.
[24] J. Qiao, On binary relations induced from overlap and grouping functions, Int. J. Approx. Reason. 106 (2019) 155–171.
[25] P. Smets, P. Magrez, Implication in fuzzy logic, Int. J. Approx. Reason. 1 (1987) 327–347.
[26] H.B. Sola, J. Fernández, R. Mesiar, J. Montero, R. Orduna, Overlap index, overlap functions and migrativity, in: J.P. Carvalho, D. Dubois, U. Kaymak, J.M. da Costa Sousa (Eds.), Proceedings of the Joint 2009 International Fuzzy Systems Association World Congress and 2009 European Society of Fuzzy Logic and Technology Conference, Lisbon, Portugal, July 20-24, 2009, 2009, pp. 300–305.

[27] H.B. Sola, M. Pagola, R. Mesiar, E. Hüllermeier, F. Herrera, Grouping, overlap, and generalized bientropic functions for fuzzy modeling of pairwise comparisons, IEEE Trans. Fuzzy Syst. 20 (2012) 405–415.

[28] M. Štěpnička, B. Jayaram, Interpolativity of at-least and at-most models of monotone fuzzy rule bases with multiple antecedent variables, Fuzzy Sets Syst. 297 (2016) 26–45.