A study on best simultaneous approximation through Chebyshev centers and related geometric properties in Banach spaces

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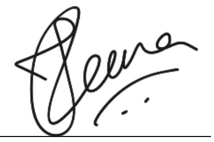

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-Teena Thomas

Abstract

In this work, we study the existence of (restricted) Chebyshev centers and the set-valued generalization of strong proximinality in Banach spaces. We mainly explore the concepts above in L_1-predual spaces and their subspaces.

Although it is well-known that a Chebyshev center exists for compact subsets of an L_1-predual space, we approach this problem differently. Interestingly, this approach leads us to an explicit description of the Chebyshev centers of the compact subsets of the spaces in question. Furthermore, we establish the validity of a geometric identity in terms of the (restricted) Chebyshev radius in L_1-predual spaces and characterize L_1-predual spaces using it. This identity was first established in 2000 by R. Espínola, A. Wiśnicki and J. Wośko for the space of real-valued continuous functions on a compact Hausdorff space S, denoted by $C(S)$, which forms a major subclass of the L_1-predual spaces. We also yield a few geometric characterizations of the ideals in L_1-predual spaces. In particular, we obtain characterizations for a compact convex subset of a locally convex topological space to be a Choquet simplex.

The study of strong (ball) proximinality gained momentum in the recent years and the main motivation to study this property is it results in some "nice" continuity properties of the metric projections. With the same motivation, we extend the study of the set-valued generalization of strong proximinality, which was initiated by J. Mach in the literature. This generalization is termed as property-(P_1) by Mach. For a non-empty closed convex subset V of a Banach space X and a family $\mathscr{F}$ of non-empty closed bounded subsets of X, property-(P_1) is defined for a triplet $(X, V, \mathscr{F})$. We study the interconnnection between property-(P_1) of a subspace and that of its closed unit ball in a Banach space in detail. Expanding on some of the works by C. R. Jayanarayanan and S. Lalithambigai, we establish the equivalence of strong ball proximinality and property-(P_1) of the closed unit ball of the finite co-dimensional subspaces of the L_1-predual spaces. For a general subspace of a Banach space, we prove that property-(P_1) of the closed unit ball of the subspace implies property-(P_1) of the subspace itself. We also establish a similar implication in the case of the Hausdorff metric continuity of the restricted Chebyshev-center map of the subspace and that of its closed unit ball.

We further investigate property-(P_1) and the continuity properties of the restricted Chebyshev-center maps in vector-valued continuous function spaces. We derive that if Y is a proximinal finite co-dimensional closed linear subspace of c_0 then the closed unit ball of Y satisfies property-(P_1) for the non-empty closed bounded subsets of ℓ_∞ and the restricted Chebyshev-center map of the closed unit ball of Y is Hausdorff metric continuous on the class of non-empty closed bounded subsets of ℓ_∞ with equi-bounded restricted Chebyshev radii. We also prove a few stability results of property-(P_1) and the continuity of the restricted Chebyshev-center maps in ℓ_∞-direct sum of Banach spaces. Finally, we discuss a few positive results on the existence of restricted Chebyshev centers and property-(P_1) of an ideal in an L_1-predual space and in particular, of an L_1-predual space in its bidual.

Contents

Chapter 0

List of Notations

$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
X	a Banach space over the field $\mathbb{R}$
X^*	the dual space of X
X^{**}	the bidual of X or in other words, the set $(X^*)^*$
$B_X[x,r]$	the closed ball in X centered at $x \in X$ and radius $r > 0$
$B_X(x,r)$	the open ball in X centered at $x \in X$ and radius $r > 0$
B_X	the unit ball of X, that is, the set $B_X[0,1]$
S_X	the unit sphere of X, that is, the set $\{x \in X : \|x\| = 1\}$
$ker(f)$	the kernel of a linear function $f \colon X \to Z$, for Banach spaces X and Z.
$Y^\perp$	the annihilator of a closed linear subspace Y of X, that is, the set $\{x^* \in X^* : x^*(y) = 0$, for each $y \in Y\}$
$span\ (A)$	the linear span of a set A
$X \cong Z$	X is isometric and linearly isomorphic to Z, for Banach spaces X and Z
$x + A$	the set $\{x + a \colon a \in A\}$, for an element $x \in X$ and subset A of X
λA	the set $\{\lambda a : a \in A\}$, for a subset A of X and $\lambda > 0$
$conv(K)$	the convex hull of a set K
$ext(K)$	the set of all extreme points of a set K
$S(\mu)$	the support of a regular Borel measure μ on a compact Hausdorff space
$\mathcal{CV}(X)$	the set $\{V \subseteq X : V \neq \emptyset$, V is closed and convex.$\}$
$\mathcal{CB}(V)$	the set $\{B \subseteq V : B \neq \emptyset$, B is closed and bounded.$\}$, for each $V \in \mathcal{CV}(X)$
$\mathcal{K}(V)$	the set $\{F \subseteq V : F \neq \emptyset$ and F is compact.$\}$, for each $V \in \mathcal{CV}(X)$
$\mathcal{F}(V)$	the set $\{F \subseteq V : F \neq \emptyset$ and F is finite.$\}$, for each $V \in \mathcal{CV}(X)$

$\mathcal{F}_4(V)$	the set $\{F \subseteq V : F \neq \emptyset,\ F \text{ is a four-point set.}\}$, for each $V \in \mathcal{CV}(X)$
$\mathcal{S}(V)$	the set $\{\{x\} : x \in V\}$, for each $V \in \mathcal{CV}(X)$
$d(x, A)$	the distance of a point $x \in X$ and a non-empty bounded subset A of X, defined as the number $\inf\{\|x - a\| : a \in A\}$
$d(A, B)$	the distance between two non-empty bounded subsets A and B of X, defined as the number $\inf\{\|a - b\| : a \in A,\ b \in B\}$
$C_b(T, X)$	the Banach space of X-valued bounded continuous functions on a topological space T, equipped with the supremum norm
$C(S)$	the Banach space of real-valued continuous functions on a compact Hausdorff space S, equipped with the supremum norm
$C_0(T)$	the Banach space of real-valued continuous functions on a locally compact Hausdorff space T vanishing at infinity, equipped with the supremum norm
$A(K)$	the Banach space of real-valued affine continuous functions on a compact convex subset K of a locally convex topological vector space (in short, lctvs), equipped with the supremum norm

Chapter 1

Introduction

In approximation theory, the concept of best simultaneous approximation, also called restricted Chebyshev center, in normed linear spaces is of great interest and significance. This concept is the root of one of the classical problems in this field of study: the restricted Chebyshev center problem. From a geometric standpoint, this problem deals with the existence of a ball of minimal radius, among those centered at the points in a given closed convex subset of a normed linear space, to cover another given bounded set of data points in the space. The recent article [3] is a survey dedicated to understanding the current state of this problem.

In this thesis, we study the restricted Chebyshev center problem and a stronger notion, namely property-(P_1), related to restricted Chebyshev centers for the class of all Banach spaces and its specific subclasses. In the following sections, we define some basic notions and notations apart from those defined in Chapter 0 relevant to this work. Moreover, we provide an overview of the existing literature, this work's motivation and objectives and the thesis's structural outline.

1.1 Basic definitions and notations

In this thesis, we consider Banach spaces over $\mathbb{R}$ and all the subspaces considered are assumed to be linear and norm-closed.

Let X be a Banach space. We consider X as a subspace of X^{**} under the canonical embedding. If Y is a subspace of X then the unit ball of Y, defined as $B_X \cap Y$, is denoted by B_Y. For each element $x \in X$, $V \in \mathcal{CV}(X)$ and non-empty bounded subset A of X, we define $r(x, A) = \sup\{\|x-a\| \colon a \in A\}$. The quantity $rad_V(A) := \inf_{v\in V} r(v, A)$ is called the *restricted Chebyshev radius* of A in V. The elements in the set $cent_V(A) := \{v \in V \colon A \subseteq B_X[v, rad_V(A)]\}$ are called the *restricted Chebyshev centers* of A in V. If $V = X$, then $rad_X(A)$ is called the *Chebyshev radius* of A in X and the elements in $cent_X(A)$ are called the *Chebyshev centers* of A in X.

Let $V \in \mathcal{CV}(X)$ and A be a non-empty bounded subset of X. Let $\overline{A}$ denote the norm-closure of A. It is easy to observe that $rad_V(A) = rad_V(\overline{A})$. Indeed, $rad_V(A) \leq rad_V(\overline{A})$ and for each $\varepsilon > 0$, $rad_V(\overline{A}) \leq rad_V(A) + \varepsilon$.

Definition 1.1.1 ([48])**.** Let X be a Banach space, $V \in \mathcal{CV}(X)$ and $\mathscr{F} \subseteq \mathcal{CB}(X)$.

(i) We say X *admits centers* for $\mathscr{F}$ if for each $F \in \mathscr{F}$, $cent_X(F) \neq \emptyset$.

(ii) The pair $(V, \mathscr{F})$ is said to satisfy the *restricted center property* (in short, r.c.p.) if for each $F \in \mathscr{F}$, $cent_V(F) \neq \emptyset$.

(iii) Let $(V, \mathscr{F})$ have r.c.p.. The set-valued function on $\mathscr{F}$, denoted by $cent_V(.)$, which maps each $F \in \mathscr{F}$ to the set $cent_V(F)$ is called the *restricted Chebyshev-center map* on $\mathscr{F}$.

In particular, if $V = X$, then the map $cent_X(.)$ is called the *Chebyshev-center map* on $\mathscr{F}$.

Remark 1.1.2 ([31]). Consider the notations as in Definition 1.1.1.

(i) If $(V, \mathcal{S}(X))$ satisfies r.c.p., then V is said to be *proximinal* in X. In this case, it is easy to see that for each $x \in X$, the number $rad_V(\{x\}) = d(x, V)$. For each $x \in X$, we denote the set $cent_V(\{x\})$ by $P_V(x)$.

(ii) Let V be proximinal in X. In this case, the map $cent_V(.)$ is called the *metric projection from X onto V* and is denoted by P_V.

Given a set $V \in \mathcal{CV}(X)$, the continuity properties of the map $cent_V(.)$ is discussed with respect to the Hausdorff metric. Given a Banach space X, the *Hausdorff metric*, denoted by d_H, is defined as follows: For each $B_1, B_2 \in \mathcal{CB}(X)$,

$$d_H(B_1, B_2) = \inf\{a > 0 \colon B_1 \subseteq B_2 + aB_X, B_2 \subseteq B_1 + aB_X\}.$$

Let T be a topological space. Let Φ be a set-valued map from T into $\mathcal{CB}(X)$. We say Φ is *lower Hausdorff semi-continuous* (l.H.s.c.) at $t \in T$ if for each $\varepsilon > 0$, there exists a neighbourhood $\mathscr{N}(t, \varepsilon)$ of t such that for each $s \in \mathscr{N}(t, \varepsilon)$ and $z \in \Phi(t)$,

$$\Phi(s) \cap B_X(z, \varepsilon) \neq \emptyset.$$

The map Φ is *upper Hausdorff semi-continuous* (u.H.s.c.) at $t \in T$ if for each $\varepsilon > 0$, there exists a neighbourhood $\mathscr{N}(t, \varepsilon)$ of t such that for each $s \in \mathscr{N}(t, \varepsilon)$,

$$\Phi(s) \subseteq \Phi(t) + \varepsilon B_X.$$

We say Φ is l.H.s.c. (or u.H.s.c. respectively) on T if Φ is l.H.s.c. (or u.H.s.c. respectively) at each point $t \in T$. The map Φ is *Hausdorff metric continuous* on T if for each $t \in T$, the single-valued map Φ from T into the metric space $(\mathcal{CB}(X), d_H)$ is continuous at $t \in T$. The map Φ is Hausdorff metric continuous on T if and only if Φ is both l.H.s.c. and u.H.s.c. on T; see [31, Remark 2.8]. We refer to [31] for the terminologies defined above related to the (semi-) continuity properties of the set-valued maps.

We now recall a few concepts which are stronger than proximinality. Before proceeding, we introduce the following notations.

Notation 1.1.3. Let X be a Banach space, $B \in \mathcal{CB}(X)$, $V \in \mathcal{CV}(X)$ and $\lambda > 0$.

(i) $S_\lambda(B) \colon = \{x \in X \colon r(x, B) \leq \lambda\}$.

(ii) $cent_V(B, \lambda) \colon = \{v \in V \colon r(v, B) \leq rad_V(B) + \lambda\} = V \cap S_{rad_V(B)+\lambda}(B)$.

By the definition of restricted Chebyshev centers, we have $cent_V(B) = V \cap S_{rad_V(B)}(B)$. In particular, for each $x \in X$, the set $cent_V(\{x\}, \lambda)$ is denoted by $P_V(x, \lambda)$ (see [23]).

Definition 1.1.4. Let X be a Banach space, Y be a subspace of X and $V \in \mathcal{CV}(X)$.

(i) ([23]) The set V is said to be *strongly proximinal* at $x \in X$ if $P_V(x) \neq \emptyset$ and for each $\varepsilon > 0$, there exists $\delta(\varepsilon, x) > 0$ such that $P_V(x, \delta) \subseteq P_V(x) + \varepsilon B_X$.

We say that V is strongly proximinal in X if it is strongly proximinal at each point in X.

(ii) ([8]) A subspace Y of a Banach space X is said to be *ball proximinal* in X if for each $x \in X$, $(B_Y, \mathcal{S}(X))$ satisfies r.c.p..

A subspace Y of a Banach space X is said to be *strongly ball proximinal* in X if B_Y is strongly proximinal in X.

(iii) ([62]) A subspace Y of a Banach space X is said to have the $1\frac{1}{2}$*-ball property* in X if for each $y \in Y$, $x \in X$ and $r_1, r_2 > 0$, the conditions $\|x - y\| \leq r_1 + r_2$ and $Y \cap B_X[x, r_2] \neq \emptyset$ imply that for each $\varepsilon > 0$, $Y \cap B_X[y, r_1 + \varepsilon] \cap B_X[x, r_2 + \varepsilon] \neq \emptyset$.

An equivalent way of saying that Y has $1\frac{1}{2}$-ball property in X is that for each $x \in X$ and $r > 0$, if $\|x\| \leq r + 1$ and $Y \cap B_X[x, r] \neq \emptyset$, then for each $\varepsilon > 0$, $Y \cap B_X[0, 1 + \varepsilon] \cap B_X[x, r + \varepsilon] \neq \emptyset$.

We next define the set-valued generalization of strong proximinality called as property-(P_1). The following definition of this property is a reformulation of that given by J. Mach in [47]. The equivalence of these two definitions has been justified in [40].

Definition 1.1.5 ([40, Definition 1.1])**.** Let X be a Banach space, $V \in \mathcal{CV}(X)$ and $\mathscr{F} \subseteq \mathcal{CB}(X)$ such that $(V, \mathscr{F})$ has r.c.p.. The triplet $(X, V, \mathscr{F})$ has *property-*(P_1) if for each $\varepsilon > 0$ and $F \in \mathscr{F}$, there exists $\delta(\varepsilon, F) > 0$ such that $\mathit{cent}_V(F, \delta) \subseteq \mathit{cent}_V(F) + \varepsilon B_X$.

Examples of triplets satisfying property-(P_1) can be found in [47].

Remark 1.1.6. The following statements are equivalent to Definition 1.1.5.

(i) The pair $(V, \mathscr{F})$ has r.c.p. and for each $F \in \mathscr{F}$ and a sequence $\{v_n\} \subseteq V$, if $r(v_n, F) \to \mathit{rad}_V(F)$, then $d(v_n, \mathit{cent}_V(F)) \to 0$.

(ii) The pair $(V, \mathscr{F})$ has r.c.p. and for each $\varepsilon > 0$ and $F \in \mathscr{F}$, there exists $\delta(\varepsilon, F) > 0$ such that

$$\mathcal{S}(F, \delta) \coloneqq \sup\{d(v, \mathit{cent}_V(F)) \colon v \in V \text{ and } r(v, F) \leq \mathit{rad}_V(F) + \delta\} < \varepsilon.$$

Some of the main objects of study in this thesis are L_1-predual spaces, M-ideals and ideals. We now define these notions one by one.

Definition 1.1.7 ([43])**.** A Banach space X is said to be an L_1*-predual space* if X^* is isometric to an $L_1(\mu)$ space for some positive measure μ.

Few classical examples of L_1-predual spaces are the space $C(S)$ for each compact Hausdorff space S; the space $C_0(T)$ for each locally compact Hausdorff space T and the space $A(K)$ for each Choquet simplex K. One can refer [43] for more examples of an L_1-predual space. We refer to [1] and [7] for the notions related to Choquet simplex and Choquet theory in general.

Definition 1.1.8. Let X be a Banach space.

(i) ([28]) A linear projection P on X is said to be an L*-projection* (M*-projection*) if for each $x \in X$, $\|x\| = \|Px\| + \|x - Px\|$ ($\|x\| = \max\{\|Px\|, \|x - Px\|\}$).

(ii) ([28]) A subspace J of X is said to be an L*-summand* (M*-summand*) in X if it is the range of an L-projection (M-projection).

(iii) ([28]) A subspace J of X is said to be an *M-ideal* in X if $J^{\perp}$ is an L-summand.

(iv) ([24]) A subspace Y of a Banach space X is an *ideal* in X if there exists a norm one linear projection P on X^* such that $ker(P) = Y^{\perp}$.

Clearly, M-summands are M-ideals (see the discussion in [28, p. 2]) and M-ideals are ideals. Rao proved in [49, Proposition 1] that a subspace Y of an L_1-predual space is an ideal in X if and only if Y itself is an L_1-predual. The concept of M-ideals is stronger than strong proximinality; see [28] and [14, Proposition 3.3]. One can refer to [24] and [28] for a detailed study of ideals and M-ideals respectively.

1.2 Literature review and objectives

We begin this section by stating the restricted Chebyshev center problem.

Question 1.2.1 (Restricted Chebyshev center problem)**.** Let X be a Banach space, $V \in \mathcal{CV}(X)$ and $B \in \mathcal{CB}(X)$. Does there exist $v_0 \in V$ such that $rad_V(B) = r(v_0, B)$?

In Question 1.2.1, if $V = X$ then we call it simply the Chebyshev center problem. For most of this work, we investigate Question 1.2.1 and related questions in L_1-predual spaces. The Chebyshev center problem for the compact subsets of L_1-predual spaces is solved in the literature. It is an easy consequence of the results in [9, Corollary 3.4], [45, Theorem 4.8, p. 38 and Theorem 6.1, p. 62]. We formally state it as follows:

Proposition 1.2.2. *Let X be an L_1-predual space. Then for each $F \in \mathcal{K}(X)$, $cent_X(F) \neq \emptyset$.*

In [6, Corollary 4.8], the authors proved the existence of a Choquet simplex K such that $cent_{A(K)}(B) = \emptyset$ for some bounded set $B \subseteq A(K)$. Nevertheless, for a compact Hausdorff space S and $B \in \mathcal{CB}(C(S))$, J. D. Ward described the elements in $cent_{C(S)}(B)$ along with its non-emptiness as follows:

Theorem 1.2.3 ([61, Theorem I.2.2])**.** *Let S be a compact Hausdorff space. Let $B \in \mathcal{CB}(C(S))$. Consider the following functions and number: for each $t \in S$,*

$$\begin{aligned} m_B(t) &= \inf\{b(t) \colon b \in B\}, & M_B(t) &= \sup\{b(t) \colon b \in B\}, \\ n_B(t) &= \liminf_{s \to t} m_B(s), & N_B(t) &= \limsup_{s \to t} M_B(s) \end{aligned} \tag{1.1}$$

and

$$r_B = \frac{1}{2} \sup\{N_B(t) - n_B(t) \colon t \in S\}. \tag{1.2}$$

Then the set $cent_{C(S)}(B)$ is non-empty, $rad_{C(S)}(B) = r_B$ and

$$cent_{C(S)}(B) = \{f \in C(S) \colon N_B - r_B \leq f \leq n_B + r_B\}. \tag{1.3}$$

P. W. Smith and Ward provided the following necessary and sufficient condition for the existence of a restricted Chebyshev center in a closed convex subset of the space $C(S)$.

Theorem 1.2.4 ([53, Theorem 2.2])**.** *Let S be a compact Hausdorff space, $V \in \mathcal{CV}(C(S))$ and $B \in \mathcal{CB}(C(S))$. Then the set $cent_V(B) \neq \emptyset$ if and only if $d(V, cent_{C(S)}(B)) = \|v_0 - x_0\|$ for some $v_0 \in V$ and $x_0 \in cent_{C(S)}(B)$.*

There are various types of characterizations available in the literature for L_1-predual spaces; for example, see [39, Chapter 7, Section 21] and [49]. The authors in [16] characterized the spaces of the type $C(S)$ and $C_0(T)$ in terms of the restricted Chebyshev radius. A rigorous examination of the proofs in [16] enables us to state the result therein in the following manner.

Theorem 1.2.5 ([16])**.** *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X is isometric to either a $C(S)$ space or a $C_0(T)$ space for some compact Hausdorff space S or locally compact Hausdorff space T.*

(ii) *For each $V \in \mathcal{CV}(X)$ and $B \in \mathcal{CB}(X)$, $rad_V(B) = rad_X(B) + d(V, cent_X(B))$.*

(iii) *For each $V \in \mathcal{CV}(X)$ and $B \in \mathcal{CB}(X)$, $rad_V(B) = rad_X(B) + \lim_{\delta \to 0^+} d(V, cent_X(B, \delta))$.*

An important characterization of L_1-predual spaces, proved by Duan and Lin, is the following result.

Theorem 1.2.6 ([13, Theorem 2.7])**.** *Let X be a Banach space. Then X is isometric to an L_1-predual space if and only if for each $F \in \mathcal{F}_4(X)$ (or $\mathcal{F}(X)$ or $\mathcal{K}(X)$), $rad_X(F) = \frac{1}{2} diam(F)$.*

In [12], the authors proved the following result for an M-ideal in an L_1-predual space.

Theorem 1.2.7 ([12, Theorem 2.6])**.** *Let X be an L_1-predual space and J be an M-ideal in X. Then for each $F \in \mathcal{F}(X)$, $rad_J(F) = rad_X(F) + d(J, cent_X(F))$.*

We mention here in passing that another instance where the identity in Theorem 1.2.5 (ii) holds true can be found in [51, Proposition 13].

Given a compact Hausdorff space S and an M-ideal J in the space $C(S)$, we investigate the possibility of the set $cent_J(B)$ to have a similar description as in Theorem 1.2.3, in other words, the possibility of describing the set $cent_J(B)$ as the set of those functions in J which are interposed between two special functions defined in terms of the elements of B. In light of the results discussed so far, we raise the following questions which form the basis of a part of our investigation.

Question 1.2.8. Let X be an Banach space.

(i) Let X be an L_1 predual space and $F \in \mathcal{K}(X)$.

  (1) Is it possible to explicitly describe the elements of the set $cent_X(F)$?

  (2) For each set $V \in \mathcal{CV}(X)$, is $rad_V(F) = rad_X(F) + d(V, cent_X(F))$?

  (3) Is the Chebyshev-center map $cent_X(.)$ Hausdorff metric continuous on $\mathcal{K}(X)$?

(ii) If for each $F \in \mathcal{K}(X)$ and $V \in \mathcal{CV}(X)$, $rad_V(F) = rad_X(F) + d(V, cent_X(F))$, then is X isometric to an L_1-predual space?

We answer Question 1.2.8 entirely in the affirmative in this thesis. We implement a different approach to answer the traditional question of existence of Chebyshev centers in L_1-predual spaces. It involves a well-known isometric identification of a Banach space and the selection and separation theorems in an L_1-predual space. This approach does not invoke the intersection properties of balls in the L_1-predual spaces, which is a usual technique adopted in the literature.

We now turn our attention to a few stronger notions of proximinality and r.c.p.. It is proved in [8] that strong ball proximinality implies strong proximinality of a subspace in a Banach space. In general, the converse is not true (see [19] and [26]). However, C. R. Jayanarayanan proved the converse for a finite co-dimensional subspace of an L_1-predual space in [33]. Furthermore, it is proved in [14, Proposition 3.3] that $1\frac{1}{2}$-ball property is stronger than strong proximinality.

We recall the following intersection property of closed balls.

Definition 1.2.9 ([45])**.** Let X be a Banach space and $n \in \mathbb{N}$. Then X is said to have the $n.2.I.P.$ if for each family of pairwise intersecting balls $\{B_X[x_i, r_i] \colon i = 1, \ldots, n\}$, $\bigcap_{i=1}^{n} B_X[x_i, r_i] \neq \emptyset$.

It is proved in [45] that X is an L_1-predual space if and only if for each $n \in \mathbb{N}$, X has $n.2.I.P.$. It is known that M-ideals in an L_1-predual space are ball proximinal in the space. In fact, the authors in [35] prove a more generalized result in the following manner.

Theorem 1.2.10 ([35, Corollary 4.8])**.** *Let X be a Banach space and J be an M-ideal in X. If X has the $3.2.I.P.$, then J is ball proximinal in X.*

The following result is a strengthening of Theorem 1.2.10.

Theorem 1.2.11 ([34, Theorem 3.5])**.** *Let X be a Banach space and J be an M-ideal in X. If X has the $3.2.I.P.$, then J is strongly ball proximinal in X. In particular, if X is an L_1-predual space, then J is strongly ball proximinal in X.*

In [18], the authors defined the following notion of strong subdifferentiability which characterizes strongly proximinal hyperplanes.

Definition 1.2.12. The norm of a Banach space X is *strongly subdifferentiable* (in short, SSD) at $x \in X$ if the one-sided limit $\lim_{t \to 0^+} \frac{\|x+ty\| - \|x\|}{t}$ exists uniformly for each $y \in S_X$. We say in this case that x is a SSD-point of X.

In [23], it is proved that given a Banach space X and $x^* \in X^*$, x^* is a SSD-point of X^* if and only if $ker(x^*)$ is strongly proximinal in X. It is also established that if Y is a strongly proximinal finite co-dimensional subspace of a Banach space then $Y^{\perp}$ is contained in the set of all SSD-points of X^*. The converse of the above result is true in the case of an L_1-predual space X (see [35, Proposition 3.20]).

Notation 1.2.13 ([23]). Let X be a Banach space. For each $x \in X$ and $x^* \in X^*$, we denote

$$\begin{aligned} J_{X^*}(x) &= \{x^* \in B_{X^*} \colon x^*(x) = \|x\|\} \text{ and} \\ J_X(x^*) &= \{x \in B_X \colon x^*(x) = \|x^*\|\}. \end{aligned} \tag{1.4}$$

The functionality of the sets defined in (1.4) has been explored in detail in [23]. The authors in [32, Theorem 3.2] derived that for each $x^* \in S_{X^*}$, if $ker(x^*)$ is ball proximinal in X then $J_X(x^*)$ is proximinal at each $x \in X$ with $x^*(x) = 1$. Furthermore, they also established in [32, Theorem 3.3] that if $x^* \in S_{X^*}$ is a SSD-point of X^*, then strong proximinality of $J_{X^{**}}(x^*)$ in X^{**} implies strong proximinality of $J_X(x^*)$ in X.

The various aspects of property-(P_1) have been explored in [11], [40] and [47]. In this work, we aim to study property-(P_1) of triplets which are set-valued analogues of strong proximinality and strong ball proximinality of a subspace of a Banach space and explore the interconnection between them. Therefore, we ask the following meaningful questions which constitute our next set of objectives.

Question 1.2.14. Let X be a Banach space.

(i) Let Y be a subspace of X and $\mathscr{F} \subseteq \mathcal{CB}(X)$.

(1) If $(B_Y, \mathscr{F})$ satisfies r.c.p., then does the pair $(Y, \mathscr{F})$ satisfy r.c.p.?

(2) If $(X, B_Y, \mathscr{F})$ satisfies property-(P_1), does the triplet $(X, Y, \mathscr{F})$ satisfy property-(P_1)?

(ii) Let X be an L_1-predual space.

(1) Let J be an M-ideal in X. Then does $(X, B_J, \mathcal{K}(X))$ satisfy property-(P_1)?

(2) Let Y be a finite co-dimensional subspace of X. Then does $(X, Y, \mathcal{K}(X))$ satisfy property-(P_1) if and only if $(X, B_Y, \mathcal{K}(X))$ satisfies property-(P_1)?

Question 1.2.15. Let X be a Banach space and x^* be a SSD-point of X^*. If the triplet $(X^{**}, J_{X^{**}}(x^*), \mathcal{K}(X^{**}))$ satisfies property-(P_1), then does $(X, J_X(x^*), \mathcal{K}(X))$ satisfy property-(P_1)?

In this work, we give affirmative answers to Question 1.2.14, and Question 1.2.15 is answered positively for a generalized class of triplets. In [35], the authors solved a variant of the transitivity problem for strong proximinality. We pose a similar problem for property-(P_1) and investigate it. The question is as follows:

Question 1.2.16 ([55, Question 1.7])**.** Let X be a Banach space, J be an M-ideal in X and Y be a finite co-dimensional subspace of X such that $Y \subseteq J$. If the triplets $(J, Y, \mathcal{CB}(J))$ (or $(J, Y, \mathcal{K}(J))$ or $(J, Y, \mathcal{F}(J))$ respectively) and $(X, J, \mathcal{CB}(X))$ (or $(X, J, \mathcal{K}(X))$ or $(X, J, \mathcal{F}(X))$ respectively) have property-(P_1), then does $(X, Y, \mathcal{CB}(X))$ (or $(X, Y, \mathcal{K}(X))$ or $(X, Y, \mathcal{F}(X))$ respectively) have property-(P_1)?

We do not know the complete answer to Question 1.2.16. However, in this thesis, we answer it positively for two cases. It should also be noted that in Question 1.2.16, we impose the additional assumption that the triplet involving M-ideals satisfy property-(P_1) because the example provided by Vesely in [60] shows that in general, an M-ideal of a Banach space may not admit Chebyshev centers for its closed bounded subsets.

The main motivation to study the notion of property-(P_1) is that it is one of the means to explore the continuity properties of the restricted Chebyshev-center maps. In this context, we recall the following result by Mach.

Theorem 1.2.17 ([47, Theorem 5])**.** *Let X be a Banach space, $V \in \mathcal{CV}(X)$ and $\mathscr{F} \subseteq \mathcal{CB}(X)$. If the triplet $(X, V, \mathscr{F})$ has property-(P_1) then the map $cent_V(.)$ is u.H.s.c. on $\mathscr{F}$.*

In relation to Question 1.2.14 (ii), we raise the following questions.

Question 1.2.18. Let Y be a finite co-dimensional subspace of an L_1-predual space X.

(i) If $(X, Y, \mathcal{K}(X))$ satisfies property-(P_1) then is the map $cent_Y(.)$ continuous on $\mathcal{K}(C(S))$?

(ii) If $(X, B_Y, \mathcal{K}(X))$ satisfies property-(P_1) then is the map $cent_{B_Y}(.)$ continuous on $\mathcal{K}(C(S))$?

We do not know the answer to Question 1.2.18. Nevertheless, we answer this question for a particular case in Chapter 5. Jayanarayanan and Lalithambigai positively answered Question 1.2.18 for the triplet $(X, Y, \mathcal{S}(X))$ in the following result.

Theorem 1.2.19 ([34, Theorem 6.3])**.** *Let Y be a strongly proximinal finite co-dimensional subspace of an L_1-predual space X. Then the metric projection from X onto Y, P_Y, is Hausdorff continuous.*

In [22, Theorem 3], Godefroy and Indumathi proved that for each subspace X of c_0 and a finite co-dimensional subspace Y of X, if every hyperplane of X containing Y is proximinal in X then Y is proximinal in X. Further, Indumathi proved the following result.

Theorem 1.2.20 ([31, Theorem 4.1])**.** *Let Y be a proximinal finite co-dimensional subspace of c_0. Then Y is proximinal in ℓ_∞ and the metric projection P_Y from ℓ_∞ onto Y is Hausdorff metric continuous.*

Jayanarayanan and Lalithambigai improved Theorem 1.2.20 and proved the following result.

Proposition 1.2.21 ([34, Corollary 3.7])**.** *If Y is a proximinal finite co-dimensional subspace of c_0, then B_Y is strongly proximinal in ℓ_∞.*

They also proved in [34, Theorem 6.3] that if Y is a strongly proximinal finite co-dimensional subspace of an L_1-predual space X, then the metric projection from X onto Y, P_Y, is Hausdorff metric continuous on X. The present work investigates the approximation properties such as r.c.p. and property-(P_1) in finite co-dimensional subspaces of c_0. With the notations as in Theorem 1.2.21, we aim to generalize this result and investigate property-(P_1) of the triplets $(\ell_\infty, Y, \mathcal{CB}(\ell_\infty))$ and $(\ell_\infty, B_Y, \mathcal{CB}(\ell_\infty))$. We also aim to investigate the interconnection between the continuity properties of the restricted Chebyshev-center maps of a subspace and that of its unit ball in a Banach space.

In light of the discussion above, we also briefly discuss the following questions related to the generalization of the so-called proxbid spaces (a Banach space which is proximinal in its bidual). We refer to [29], [42], [50] and the references therein for the study related to proxbid spaces.

Question 1.2.22. Let X be an L_1-predual space and Y be an ideal in X.

(i) Do the pairs $(Y, \mathcal{K}(X))$ or $(Y, \mathcal{CB}(X))$ have r.c.p.?

(ii) Do the triplets $(X, Y, \mathcal{K}(X))$ or $(X, Y, \mathcal{CB}(X))$ have property-(P_1)?

A well-known characterization of an L_1-predual space is as follows: A Banach space X is an L_1-predual space if and only if $X^{**} \cong C(S)$ for some compact Hausdorff space S. This characterization follows from [39, Theorem 6, pg. 92 and Theorem 6, pg. 212]. Therefore, a particular case of Question 1.2.22 is the following

Question 1.2.23. Let X be an L_1-predual space.

(i) Do the pairs $(X, \mathcal{K}(X^{**}))$ or $(X, \mathcal{CB}(X^{**}))$ have r.c.p.?

(ii) Do the triplets $(X^{**}, X, \mathcal{K}(X^{**}))$ or $(X^{**}, X, \mathcal{CB}(X^{**}))$ have property-(P_1)?

We do not know the answer to Questions 1.2.22 and 1.2.23 in its entirety. However, we consider a few triplets for which Questions 1.2.22 and 1.2.23 can be positively answered.

1.3 Major outcomes and organization of the thesis

This thesis is organized into four main chapters following the introductory Chapters 0 and 1. We summarize and present the major results obtained in each chapter.

Let X be an L_1-predual space and B_{X^*} be endowed with the weak* topology. In Chapter 2, by using an isometric identification of X to a subspace of $A(B_{X^*})$ (see Proposition 2.2.5), we not only prove the existence but also describe the Chebyshev centers of a compact subset of X.

Coupled with the separation and selection theorems in L_1-predual spaces by A. J. Lazar and J. Lindenstrauss in [43], this description leads to a few interesting consequences. One such consequence is a geometrical characterization of an L_1-predual space X as a Banach space satisfying the equality $rad_V(F) = rad_X(F) + d(V, cent_X(F))$, for each $V \in \mathcal{CV}(X)$ and $F \in \mathcal{F}_4(X)$. Another consequence is that the Chebyshev-center map of an L_1-predual space is 2-Lipschitz Hausdorff metric continuous on $\mathcal{K}(X)$.

Furthermore, in this chapter, we obtain some new characterizations of ideals in an L_1-predual space in Theorem 2.5.3 and Corollary 2.5.4. In particular, for a compact Hausdorff space S and a subspace $\mathcal{A}$ of $C(S)$ which contains the constant function 1 and separates the points of S, we prove that the state space of $\mathcal{A}$ is a Choquet simplex if and only if $d(\mathcal{A}, cent_{C(S)}(F)) = 0$, for every $F \in \mathcal{F}_4(\mathcal{A})$. We also derive characterizations for a compact convex subset of a lctvs to be a Choquet simplex. In the above discussions, equivalently, one may replace the class $\mathcal{F}_4(X)$ by $\mathcal{K}(X)$ (or $\mathcal{F}(X)$). Sections 2.2, 2.3, 2.4 and 2.5 have appeared in [58].

In Chapter 3, we positively answer Question 1.2.14 (i) as follows: for a subspace Y of a Banach space X and $\mathscr{F} = \mathcal{CB}(X)$, $\mathcal{K}(X)$ or $\mathcal{F}(X)$, if $(B_Y, \mathscr{F})$ has r.c.p. then so does $(Y, \mathscr{F})$ (Proposition 3.2.2) and if $(X, B_Y, \mathscr{F})$ has property-(P_1) then so does $(X, Y, \mathscr{F})$ (Proposition 3.2.3). Further, we establish in Theorem 3.3.3 that for an M-ideal J in an L_1-predual space X, $(X, B_J, \mathcal{K}(X))$ satisfies property-(P_1). In Theorem 3.4.7, we expand the characterizations of a strongly ball proximinal finite co-dimensional subspace of an L_1-predual space provided by Jayanarayanan in [33]. In this result, we establish that for a strongly ball proximinal finite co-dimensional subspace Y of an L_1-predual space X, $(X, B_Y, \mathcal{K}(X))$ has property-(P_1), and also prove that if $(X, Y, \mathcal{K}(X))$ has property-(P_1), then so does the triplet $(X, B_Y, \mathcal{K}(X))$. This answers Question 1.2.14 (ii) in the affirmative. Sections 3.2, 3.3 and 3.4 have appeared in [56].

In this chapter, we also prove that for a Banach space X, if x^* is a SSD-point of X^* and the triplet $(X^{**}, J_{X^{**}}(x^*), \mathcal{K}(X^{**}))$ satisfies property-(P_1), then so does $(X, J_X(x^*), \mathcal{K}(X))$. In fact, a much more generalized version of the result above is proved in Theorem 3.5.8. We conclude this chapter by demonstrating through an example that $1\frac{1}{2}$-ball property is not a sufficient condition for r.c.p.. Section 3.6 of this chapter has appeared in [56].

In Chapter 4, we deviate slightly from our main subject to understand a well-known representation of the closed linear sublattices of the space $C(S)$, whenever S is a compact Hausdorff space, given in [36]. This representation is determined by a set of algebraic relations. We provide a simple alternative proof for the same in Theorem 4.2.1. As a consequence, in Corollary 4.3.2, we algebraically represent the sublattices of the space $C_0(T)$, whenever T is a locally compact Hausdorff space. The representations above come in useful in the subsequent chapter. Sections 4.2 and 4.3 have appeared in [57].

In Chapter 5, for a topological space T and a uniformly convex Banach space X, we first prove that the triplet $(C_b(T, X), B_{C_b(T,X)}, \mathcal{CB}(C_b(T, X)))$ satisfies property-(P_1) and the map $cent_{B_{C_b(T,X)}}(.)$ is uniformly Hausdorff metric continuous on subfamilies of sets in $\mathcal{CB}(C_b(T, X))$ with equi-bounded restricted Chebyshev radii. As a consequence, we establish that if Y is a proximinal finite co-dimensional closed linear subspace of c_0 then the triplet $(\ell_\infty, B_Y, \mathcal{CB}(\ell_\infty))$ satisfies property-(P_1) and the map $cent_{B_Y}(.)$ is uniformly Hausdorff metric continuous on subfamilies of sets in $\mathcal{CB}(\ell_\infty)$ with equi-bounded restricted Chebyshev radii. To this end, we derive some stability results of property-(P_1) and the semi-continuity properties of restricted Chebyshev-center maps in the

ℓ_∞-direct sum of two Banach spaces. Finally, we prove in Theorem 5.5.1 that for an M-summand Y of a Banach space X and a subspace Z of Y, if $(Y, Z, \mathcal{CB}(Y))$ has property-(P_1), then $(X, Z, \mathcal{CB}(X))$ has property-(P_1). We also provide a positive answer to Question 1.2.16 in the case of an L_1-predual space in Proposition 5.5.2.

In this chapter, we also positively answer Questions 1.2.22 and 1.2.23 for a few particular cases. Let X be an L_1-predual space. Then we prove that $(X^{**}, B_X, \mathcal{CB}(X^{**}))$ has property-(P_1) if X is a closed subalgebra of a $C(S)$ or $C_0(T)$ space, whenever S is a compact Hausdorff space and T is a locally compact Hausdorff space. Sections 5.2, 5.3, 5.4, 5.5 and a few parts of Section 5.6 have appeared in [55].

Chapter 2

Various geometric properties of L_1-predual spaces determined by Chebyshev centers

2.1 Summary of results

In this chapter, we study several geometric properties of L_1-predual spaces which are determined by Chebyshev centers and in fact, establish that some of these properties characterize L_1-predual spaces.

In Section 2.2, we lay the groundwork for the results in the subsequent sections. In this section, for a compact Hausdorff space S, a set $V \in \mathcal{CV}(C(S))$ and $B \in \mathcal{CB}(C(S))$, we mainly observe that the set $cent_V(B)$ is exactly the set of functions in V which are interposed between two special functions defined in terms of the elements of B; see Proposition 2.2.3. This result serves as a starting point for the study carried out in this chapter.

In Section 2.3, we prove in Theorem 2.3.6 that for a compact Hausdorff space S, an M-summand J in $C(S)$ and $B \in \mathcal{CB}(J)$, the set $cent_J(B) \neq \emptyset$ and is precisely the set $cent_{C(S)}(B) \cap J$. We illustrate with examples that the description above is not necessarily true in the case of M-ideals which are not M-summands in $C(S)$.

Let X be an L_1-predual space. In Section 2.4, we precisely describe the elements in $cent_X(F)$ for each $F \in \mathcal{K}(X)$ via a well-known identification of a Banach space to a function space; see Lemma 2.2.5 and Theorem 2.4.2. This description leads us to the following consequences. In Theorem 2.4.4, we prove that the map $cent_X(.)$ is 2-Lipschitz Hausdorff metric continuous on $\mathcal{K}(X)$ and that the constant 2 is an optimal choice. Furthermore, for $V \in \mathcal{CV}(X)$ and $F \in \mathcal{K}(X)$, we prove in Theorem 2.4.8 that the set $cent_V(F) \neq \emptyset$ if and only if the infimum defining $d(V, cent_X(F))$ is attained.

In the last Section 2.5, we provide a few more appplications of Theorem 2.4.2. We first characterize an L_1-predual space to be a Banach space which satisfies for each $F \in \mathcal{K}(X)$ and $V \in \mathcal{CV}(X)$, the equality, $rad_V(F) = rad_X(F) + d(V, cent_X(F))$. We prove in Theorem 2.5.3 that a subspace Y of an L_1-predual space is an ideal in X if and only if for each $F \in \mathcal{F}_4(Y)$, $d(Y, cent_X(F)) = 0$.

2.2 Preliminaries

Let S be a compact Hausdorff space. Let $B \in \mathcal{CB}(C(S))$. We define the following functions in a similar way as in Theorem 1.2.3. For each $t \in S$,

$$\begin{aligned} m_B(t) &= \inf\{b(t) \colon b \in B\}, \\ n_B(t) &= \liminf_{s \to t} m_B(s), \\ M_B(t) &= \sup\{b(t) \colon b \in B\}, \\ \text{and } N_B(t) &= \limsup_{s \to t} M_B(s). \end{aligned} \tag{2.1}$$

Further, we define

$$r_B = \frac{1}{2} \sup\{N_B(t) - n_B(t) \colon t \in S\}. \tag{2.2}$$

Remark 2.2.1 ([58, Remark 2.1]). The following properties of the functions and number defined in (2.1) and (2.2) respectively can be easily verified.

(i) If $B \in \mathcal{CB}(C(S))$ then the functions N_B, m_B are upper semi-continuous and n_B, M_B are lower semi-continuous functions on S.

(ii) If $F \in \mathcal{K}(C(S))$ then the functions M_F and m_F are continuous on S and consequently, $N_F = M_F$ and $n_F = m_F$. It follows from the fact that a compact set $F \subseteq C(S)$ is an equicontinuous family of functions in $C(S)$.

(iii) For each $B \in \mathcal{CB}(C(S))$ and $V \in \mathcal{CV}(C(S))$, $r_B \leq rad_{C(S)}(B) \leq rad_V(B)$. One can find a proof of the former inequality in [61, Lemma I.2.1] and the latter inequality follows from the definition of (restricted) Chebyshev radius.

Lemma 2.2.2 ([58, Lemma 2.2])**.** *Let S be a compact Hausdorff space. If $B \in \mathcal{CB}(C(S))$ then $\frac{1}{2} diam(B) \leq r_B$. Moreover, if $F \in \mathcal{K}(C(S))$ then $\frac{1}{2} diam(F) = r_F$.*

Proof. Let $B \in \mathcal{CB}(C(S))$. Let $L = \sup\{N_B(t) - n_B(t) \colon t \in S\} = 2r_B$. The upper semi-continuity of m_B and $-M_B$ results in $b(t) \leq M_B(t) \leq N_B(t)$ and $n_B(t) \leq m_B(t) \leq b(t)$, for each $t \in S$ and $b \in B$. Therefore, for each $b_1, b_2 \in B$ and $t \in S$,

$$\pm(b_1(t) - b_2(t)) \leq N_B(t) - n_B(t) \leq L. \tag{2.3}$$

It follows that $diam(B) \leq L$.

Now, let $F \in \mathcal{K}(C(S))$. By Remark 2.2.1 (ii), $n_F = m_F$ and $N_F = M_F$. Now, for each $t \in S$, the evaluation functional δ_t, defined as $\delta_t(f) = f(t)$ for each $f \in C(S)$, is norm continuous on $C(S)$. Therefore, due to the compactness of F, for each $t \in S$, there exists $z_1, z_2 \in F$ such that

$$M_F(t) - m_F(t) = \max_{z' \in F} \delta_t(z') - \min_{z' \in F} \delta_t(z') = z_1(t) - z_2(t) \leq diam(F). \tag{2.4}$$

It follows that $2r_F = \sup\{M_F(t) - m_F(t) \colon t \in S\} \leq diam(F)$. ☐

For a set $V \in \mathcal{CV}(C(S))$ and $B \in \mathcal{CB}(C(S))$, the following result gives us a description of $cent_V(B)$.

Proposition 2.2.3 ([58, Proposition 2.3])**.** *Let S be a compact Hausdorff space. Let $V \in \mathcal{CV}(C(S))$ and $B \in \mathcal{CB}(C(S))$. Then*

$$cent_V(B) = \{f \in V \colon N_B - rad_V(B) \leq f \leq n_B + rad_V(B)\}.$$

Proof. Let $B \in \mathcal{CB}(C(S))$. Without loss of generality, we assume that $cent_V(B) \neq \emptyset$. Suppose $f \in cent_V(B)$. Then $rad_V(B) = r(f, B)$. It follows that for each $t \in S$ and $b \in B$,

$$b(t) - rad_V(B) \leq f(t) \leq b(t) + rad_V(B). \tag{2.5}$$

Therefore, from the definitions in (2.1), for each $t \in S$,

$$N_B(t) - rad_V(B) \leq f(t) \leq n_B(t) + rad_V(B). \tag{2.6}$$

Now, if $f \in V$ such that the inequalities in (2.6) hold, then using Remark 2.2.1 (i), it is easy to deduce that $rad_V(B) = r(f, B)$. □

We refer to [6, Section 4] for examples of subspaces of $C(S)$ which admit (restricted) Chebyshev centers for certain subfamilies of closed bounded subsets of $C(S)$.

Let X be a Banach space. If $V \in \mathcal{CV}(X)$ then the following result shows that the map $rad_V(.)$, defined as $B \mapsto rad_V(B)$ for each $B \in \mathcal{CB}(X)$, is Hausdorff metric continuous on $\mathcal{CB}(X)$. A proof of this result can be found in [11, Theorem 2.5]. We include it here for the sake of completeness.

Lemma 2.2.4 ([56, Lemma 4.1])**.** *Let V be a non-empty closed convex subset of a Banach space X and $B_1, B_2 \in \mathcal{CB}(X)$. Then for each $v \in V$, $|r(v, B_1) - r(v, B_2)| \leq d_H(B_1, B_2)$ and $|rad_V(B_1) - rad_V(B_2)| \leq d_H(B_1, B_2)$.*

Proof. Let $v \in V$. Now, let $b_1 \in B_1$ and $\varepsilon > 0$. Choose $b_2 \in B_2$ such that $\|b_1 - b_2\| < d_H(B_1, B_2) + \varepsilon$. Then

$$\|v - b_1\| \leq \|v - b_2\| + \|b_2 - b_1\| < r(v, B_2) + d_H(B_1, B_2) + \varepsilon. \tag{2.7}$$

It follows that

$$r(v, B_1) \leq r(v, B_2) + d_H(B_1, B_2). \tag{2.8}$$

Further, by swapping B_1 with B_2 in the argument above, we obtain the following inequality.

$$r(v, B_2) \leq r(v, B_1) + d_H(B_1, B_2). \tag{2.9}$$

The first conclusion of the result follows from the inequalities in (2.8) and (2.9).

The inequalities in (2.8) and (2.9) hold true for every $v \in V$. Hence, the final conclusion of the result follows. □

Let X be a Banach space. By Banach-Alaoglu's theorem, B_{X^*}, when equipped with the weak* topology, is a weak*-compact Hausdorff space. There exists a natural affine homeomorphism from B_{X^*} onto B_{X^*} given by the map $\sigma(x^*) = -x^*$ for each $x^* \in B_{X^*}$. We define

$$A_\sigma(B_{X^*}) = \{a \in A(B_{X^*}) \colon a = -a \circ \sigma\}.$$

Clearly $A_\sigma(B_{X^*})$ is a Banach space. The following result shows us that every Banach space can be viewed as a function space.

Lemma 2.2.5 ([39, Lemma 8, p. 213])**.** *Let X be a Banach space. Then $X \cong A_\sigma(B_{X^*})$ under the mapping $x \mapsto \overline{x}$, defined as $\overline{x}(x^*) = x^*(x)$, for each $x^* \in B_{X^*}$ and $x \in X$.*

Remark 2.2.6 ([58, Remark 2.5]). Let X be a Banach space and $F \in \mathcal{K}(X)$. By Lemma 2.2.5, we view F as a compact subset of $A_\sigma(B_{X^*}) \subseteq C(B_{X^*})$. Hence for each $x^* \in B_{X^*}$, the norm continuity of x^* on X and the compactness of F implies that

$$\begin{aligned} M_F(x^*) &= \max\{x^*(x) \colon x \in F\}, \\ m_F(x^*) &= \min\{x^*(x) \colon x \in F\} \\ \text{and } r_F &= \frac{1}{2}\max\{M_F(x^*) - m_F(x^*) \colon x^* \in B_{X^*}\}. \end{aligned} \tag{2.10}$$

2.3 Restricted center property of M-ideals in $C(S)$

Let S be a compact Hausdorff space. In this thesis, we use the following notation:

Notation 2.3.1. For a subset D of a compact Hausdorff space S, we define

$$J_D = \{h \in C(S) \colon h(t) = 0, \text{ for each } t \in D\}.$$

We recall that J is an M-ideal in $C(S)$ if and only if there exists a closed subset D of S such that $J = J_D$ and J is an M-summand in $C(S)$ if and only if there exists a clopen subset D of S such that $J = J_D$ (see [28, Example 1.4 (a), p. 3]).

We now present the following easy observation.

Proposition 2.3.2 ([58, Proposition 3.1])**.** *Let S be a compact Hausdorff space. If D is a closed subset of S then the subspace J_D of $C(S)$ admits centers for $\mathcal{K}(J_D)$ and for each $F \in \mathcal{K}(J_D)$, $rad_{J_D}(F) = r_F$ and*

$$cent_{J_D}(F) = \{h \in J_D \colon M_F - r_F \le h \le m_F + r_F\}. \tag{2.11}$$

Proof. Let $F \in \mathcal{K}(J_D)$. From Remark 2.2.1 (ii), M_F and m_F are continuous functions on S. Clearly, for each $t \in D$, $M_F(t) = 0 = m_F(t)$. Therefore

$$\frac{M_F + m_F}{2} \in \{h \in J_D \colon M_F - r_F \le h \le m_F + r_F\}. \tag{2.12}$$

It follows that $rad_{J_D}(F) \le r_F$. Thus from Remark 2.2.1 (iii) and Proposition 2.2.3, we obtain $rad_{J_D}(F) = r_F$ and the desired description of the set $cent_{J_D}(F)$ as in (2.11). □

Let T be a locally compact Hausdorff space. It is known that $C_0(T)$ admits centers for $\mathcal{CB}(C_0(T))$; see [16]. We denote T_∞ to be the one-point compactification of T and t_∞ to be the "point at infinity". Consider the restriction map $\Phi \colon J_{\{t_\infty\}} \to C_0(T)$ defined as $\Phi(f) = f|_T$ for each $f \in J_{\{t_\infty\}}$. Then the map Φ is an isometric lattice and algebra isomorphism. Hence the following result follows immediately from Proposition 2.3.2.

Corollary 2.3.3 ([58, Corollary 3.2]). *Let T be a locally compact Hausdorff space. Let $F \in \mathcal{K}(C_0(T))$. Consider the following functions and number: for each $t \in T$,*

$$\begin{aligned} M_F(t) &= \sup\{f(t)\colon f \in F\}, \\ m_F(t) &= \inf\{f(t)\colon f \in F\} \\ \text{and } r_F &= \frac{1}{2}\sup\{M_F(t) - m_F(t)\colon t \in T\}. \end{aligned} \tag{2.13}$$

Then the set $cent_{C_0(T)}(F) \neq \emptyset$, $rad_{C_0(T)}(F) = r_F$ and

$$cent_{C_0(T)}(F) = \{h \in C_0(T)\colon M_F - r_F \leq h \leq m_F + r_F \text{ on } T\}. \tag{2.14}$$

We now recall and state the well-known insertion theorem by Katětov according to our purpose.

Theorem 2.3.4 ([37] and [38]). *Let S be a compact Hausdorff space. If $g, -f$ are upper semi-continuous functions on S such that $g \leq f$ on S then there exists $h \in C(S)$ such that $g \leq h \leq f$ on S.*

The following result is a variant of Theorem 2.3.4. We need it to prove the main result of this section.

Lemma 2.3.5 ([58, Lemma 3.4]). *Let D be a clopen subset of a compact Hausdorff space S. If $g, -f$ are upper semi-continuous functions on S such that $g \leq f$ on S and for each $t \in D$, $g(t) \leq \alpha \leq f(t)$, then there exists $h \in C(S)$ such that $g \leq h \leq f$ on S and for each $t \in D$, $h(t) = \alpha$.*

Proof. Without loss of generality, we assume that D is a non-empty proper subset of S. By our assumption, both D and $S\backslash D$ are clopen in S. Thus by Theorem 2.3.4, there exists $k \in C(S\backslash D)$ such that $g \leq k \leq f$ on $S\backslash D$. We now define $h\colon S \to \mathbb{R}$ as $h = k$ on $S\backslash D$ and $h(t) = \alpha$, for each $t \in D$. It is now easy to see that $h \in C(S)$. Moreover, $g \leq h \leq f$ on S. □

Theorem 2.3.6 ([58, Theorem 3.5]). *Let S be a compact Hausdorff space and J be an M-summand in $C(S)$. Then J admits centers for $\mathcal{CB}(J)$. Moreover, for each $B \in \mathcal{CB}(J)$, $rad_J(B) = r_B$ and*

$$cent_J(B) = \{h \in J\colon N_B - r_B \leq h \leq n_B + r_B \text{ on } S\}. \tag{2.15}$$

Proof. Let J be an M-summand in $C(S)$. Thus there exists a clopen subset D of S such that $J = J_D$. Let $B \in \mathcal{CB}(J_D)$. Clearly, for each $t \in D$, $M_B(t) = 0 = m_B(t)$. Using the assumption that D is clopen in S, by the definitions of the functions N_B and $-n_B$ and Remark 2.2.1 (i), it follows that for each $t \in D$, $N_B(t) = 0 = n_B(t)$. Hence for each $t \in D$,

$$N_B(t) - r_B \leq 0 \leq n_B(t) + r_B. \tag{2.16}$$

By the definition of r_B, Remark 2.2.1 (i) and Lemma 2.3.5, there exists $h \in J_D$ such that $N_B - r_B \leq h \leq n_B + r_B$. This shows that the set on the right-hand side in (2.15) is non-empty. It follows that $rad_{J_D}(B) \leq r_B$. Therefore, by Remark 2.2.1 (iii) and Proposition 2.2.3, we obtain $rad_{J_D}(B) = r_B$ and the description of the set $cent_{J_D}(B)$ as in (2.15). □

By [54, Theorem 1], for a compact Hausdorff space S, an M-ideal in $C(S)$ admits centers for $\mathcal{CB}(C(S))$. In Theorem 2.3.6, if $B \in \mathcal{CB}(C(S))$ then it is not necessary that the set $cent_{J_D}(B)$ has a description as given in (2.15). We provide an example below to illustrate this fact.

Example 2.3.7 ([58, Example 3.6])**.** Consider the subspace $J = \{h \in C(\{0,1\}) \colon h(0) = 0\}$ of $C(\{0,1\})$. Define the functions $f, g \colon \{0,1\} \to \mathbb{R}$ as $f(0) = 2$, $f(1) = 0$, $g(0) = 3$ and $g(1) = 1$. Let $B = \{f, g\} \subseteq C(\{0,1\}) - J$. It is easy to see that if $h \in C(\{0,1\})$ such that $M_B - r_B \leq h \leq m_B + r_B$, then $h(0) = \frac{5}{2}$ and $h(1) = \frac{1}{2}$ and hence $h \notin J$.

In Theorem 2.3.6, if the set D is not clopen then the set $cent_{J_D}(B)$ need not have the description as given in (2.15) for each $B \in \mathcal{CB}(J_D)$. The following example illustrates this fact.

Example 2.3.8 ([58, Example 3.7])**.** Let $0 < a < b < 1$. Consider the space $C([0,1])$ and $D = [a, b]$. Let

$$B = \{f \in J_D \colon 0 \leq f(t) \leq 1, \text{ for each } t \in [0,1]\} \in \mathcal{CB}(J_D) - \mathcal{K}(J_D). \tag{2.17}$$

It is easy to see that for each $t \in [0,1] - (a,b)$, $N_B(t) = 1$ and $n_B(t) = 0$ and for each $t \in (a,b)$, $N_B(t) = 0 = n_B(t)$. Hence $r_B = \frac{1}{2}$. If $h \in C([0,1])$ such that $N_B - r_B \leq h \leq n_B + r_B$, then for each $t \in [0,1] - (a,b)$, $h(t) = \frac{1}{2}$ and thus $h \notin J_D$.

2.4 Restricted center property in L_1-predual spaces

In this section, we first prove the existence and provide a description of a Chebyshev center of a compact subset of an L_1-predual space using the isometric identification in Lemma 2.2.5. To this end, we recall the following separation theorem in L_1-predual spaces.

Theorem 2.4.1 ([41, Theorem 2.3])**.** *Let X be an L_1-predual space. If f is a real-valued weak*-lower semi-continuous concave function on B_{X^*} such that for each $x^* \in B_{X^*}$, $f(x^*) + f(-x^*) \geq 0$, then there exists $a \in A_\sigma(B_{X^*})$ such that $a \leq f$.*

Theorem 2.4.2 ([58, Theorem 4.2])**.** *Let X be an L_1-predual space. If $F \in \mathcal{K}(A_\sigma(B_{X^*}))$, then $cent_{A_\sigma(B_{X^*})}(F) \neq \emptyset$, $rad_{A_\sigma(B_{X^*})}(F) = r_F$ and*

$$cent_{A_\sigma(B_{X^*})}(F) = \{a \in A_\sigma(B_{X^*}) \colon M_F - r_F \leq a \leq m_F + r_F\}. \tag{2.18}$$

Proof. Let $F \in \mathcal{K}(A_\sigma(B_{X^*}))$. Define $f = m_F + r_F$ on B_{X^*}. It follows from Remark 2.2.6 that f is a weak*-continuous concave function on B_{X^*}. Let $x^* \in B_{X^*}$. Then,

$$\begin{aligned} & f(x^*) + f(-x^*) \\ & = m_F(x^*) + r_F + m_F(-x^*) + r_F \\ & = (m_F(x^*) + r_F) - (M_F(x^*) - r_F) \geq 0. \end{aligned} \tag{2.19}$$

Therefore, by Theorem 2.4.1, there exists $a \in A_\sigma(B_{X^*})$ such that $a \leq m_F + r_F$. Now, for each $x^* \in B_{X^*}$, $a(-x^*) \leq m_F(-x^*) + r_F$ and hence, $M_F(x^*) - r_F \leq a(x^*)$. Therefore, $M_F - r_F \leq a \leq m_F + r_F$ on B_{X^*}. It follows that $rad_{A_\sigma(B_{X^*})}(F) = r_F$ and from Proposition 2.2.3, we can deduce that $cent_{A_\sigma(B_{X^*})}(F) \neq \emptyset$ and

$$cent_{A_\sigma(B_{X^*})}(F) = \{a \in A_\sigma(B_{X^*}) \colon M_F - r_F \leq a \leq m_F + r_F \text{ on } B_{X^*}\}. \tag{2.20}$$

□

Remark 2.4.3 ([58, Remark 4.3]). Let X be an L_1-predual space and $F \in \mathcal{K}(X)$. Using Lemma 2.2.5, we view F as a compact subset of $A_\sigma(B_{X^*})$. Therefore, applying Theorem 2.4.2, it is easy to see that $cent_X(F) \neq \emptyset$ and $rad_X(F) = r_F$. Moreover,

$$cent_X(F) = \{x \in X \colon M_F - r_F \leq \overline{x} \leq m_F + r_F \text{ on } B_{X^*}\}. \tag{2.21}$$

We now look at two applications of Theorem 2.4.2. For an L_1-predual space X, we prove the continuity of the map $cent_X(.)$ on $\mathcal{K}(X)$ in the Hausdorff metric in the following result.

Theorem 2.4.4 ([58, Theorem 4.5])**.** *Let X be an L_1-predual space. Then the map $cent_X(.)$ is 2-Lipschitz Hausdorff metric continuous on $\mathcal{K}(X)$.*

Proof. From Lemma 2.2.5, it suffices to prove that the map $cent_{A_\sigma(B_{X^*})}(.)$ is 2-Lipschitz Hausdorff metric continuous on $\mathcal{K}(A_\sigma(B_{X^*}))$.

Let $\varepsilon > 0$ and $F_1, F_2 \in \mathcal{K}(A_\sigma(B_{X^*}))$ be such that $d_H(F_1, F_2) < \frac{\varepsilon}{2}$. Thus there exists $0 < \delta < \frac{\varepsilon}{2}$ such that

$$F_1 \subseteq F_2 + \delta B_X \text{ and } F_2 \subseteq F_1 + \delta B_X. \tag{2.22}$$

Then by Theorem 2.4.2, for each $i = 1, 2$,

$$cent_{A_\sigma(B_{X^*})}(F_i) = \{a \in A_\sigma(B_{X^*}) \colon M_{F_i} - r_{F_i} \leq a \leq m_{F_i} + r_{F_i} \text{ on } B_{X^*}\}. \tag{2.23}$$

and $r_{F_i} = rad_{A_\sigma(B_{X^*})}(F_i)$. By Lemma 2.2.4, $|r_{F_1} - r_{F_2}| \leq d_H(F_1, F_2)$. Hence we can choose δ such that $r_{F_1} - \delta < r_{F_2} < r_{F_1} + \delta$ and (2.22) hold true.

Let $a \in cent_{A_\sigma(B_{X^*})}(F_1)$ and $z_2 \in F_2$. It follows from (2.22) that there exists $z_1 \in F_1$ and $z_0 \in B_X$ such that $z_2 = z_1 + \delta z_0$. Thus using (2.23), for each $x^* \in B_{X^*}$,

$$\begin{aligned} z_2(x^*) - r_{F_2} &= z_1(x^*) + \delta z_0(x^*) - r_{F_2} \\ &\leq z_1(x^*) + \delta - r_{F_1} + \delta \\ &\leq a(x^*) + 2\delta. \end{aligned} \tag{2.24}$$

$$\begin{aligned} z_2(x^*) + r_{F_2} &= z_1(x^*) + \delta z_0(x^*) + r_{F_2} \\ &\geq z_1(x^*) - \delta + r_{F_1} - \delta \\ &\geq a(x^*) - 2\delta. \end{aligned} \tag{2.25}$$

It follows from (2.24), (2.25) and the definitions of M_{F_2} and m_{F_2} that $M_{F_2} - r_{F_2} \leq a + 2\delta$ and $a - 2\delta \leq m_{F_2} + r_{F_2}$ on B_{X^*}. We define $g = \max\{M_{F_2} - r_{F_2}, a - 2\delta\}$ and $f = \min\{a + 2\delta, m_{F_2} + r_{F_2}\}$ on B_{X^*}. Clearly, $g \leq f$. Moroever, the functions $-g$ and f are weak*-continuous concave functions on B_{X^*}. For each $x^* \in B_{X^*}$, by using the fact that $M_{F_2}(-x^*) = -m_{F_2}(x^*)$, it is easy to see that

$$f(x^*) + f(-x^*) = f(x^*) - g(x^*) \geq 0. \tag{2.26}$$

Therefore, by Theorem 2.4.1, there exists $a' \in A_\sigma(B_{X^*})$ such that $a' \leq f$. Now, for each $x^* \in B_{X^*}$, since $f(x^*) = -g(-x^*)$, it follows that $g(-x^*) \leq -a'(x^*) = a'(-x^*)$. Thus $g \leq a' \leq f$ on B_{X^*}. It follows that $a' \in cent_{A_\sigma(B_{X^*})}(F_2)$ such that $\|a' - a\| \leq 2\delta < \varepsilon$. Therefore, this proves that $cent_{A_\sigma(B_{X^*})}(F_1) \subseteq cent_{A_\sigma(B_{X^*})}(F_2) + \varepsilon B_X$. Similarly, we can prove that $cent_{A_\sigma(B_{X^*})}(F_2) \subseteq cent_{A_\sigma(B_{X^*})}(F_1) + \varepsilon B_X$. Hence $d_H(cent_{A_\sigma(B_{X^*})}(F_1), cent_{A_\sigma(B_{X^*})}(F_2)) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $d_H(cent_{A_\sigma(B_{X^*})}(F_1), cent_{A_\sigma(B_{X^*})}(F_2)) \leq 2d_H(F_1, F_2)$. □

Remark 2.4.5 ([58, Remark 4.6]). In Theorem 2.4.4, the constant 2 is the optimal choice. For example, consider $\mathbb{R}^2$ equipped with the supremum norm. Let $F = \{(-1,0),(1,0)\}$ and $G = \{(0,1)\}$. Then $cent_{\mathbb{R}^2}(F) = \{(0,\lambda)\colon -1 \leq \lambda \leq 1\}$ and $cent_{\mathbb{R}^2}(G) = \{(0,1)\}$. Moreover, $d_H(F,G) = 1$ and $d_H(cent_{\mathbb{R}^2}(F), cent_{\mathbb{R}^2}(G)) = 2$.

We next discuss a necessary and sufficient condition for the existence of restricted Chebyshev centers in L_1-predual spaces. Before proceeding, we need the following definition.

Definition 2.4.6 ([43])**.** Let C be a convex subset of a lctvs and E be another lctvs. Let Φ be a set-valued map from C to the family of non-empty convex subsets of E. The map Φ is said to be

(i) a *convex function* on C if for each $0 \leq \alpha \leq 1$ and $c_1, c_2 \in C$,

$$\alpha\Phi(c_1) + (1-\alpha)\Phi(c_2) \subseteq \Phi(\alpha c_1 + (1-\alpha)c_2);$$

(ii) a *lower semi-continuous function* on C if for each open subset U of E, the set

$$\{c \in C\colon \Phi(c) \cap U \neq \emptyset\}$$

is a relatively open subset of C and

(iii) a *symmetric function* on C if $\Phi(-c) = -\Phi(c)$ whenever $c, -c \in C$.

We also need the following selection theorem by Lazar and Lindenstrauss, stated according to our purpose.

Theorem 2.4.7 ([43, Theorem 2.2])**.** *Let X be an L_1-predual space. If $\Phi\colon B_{X^*} \to \mathcal{CV}(\mathbb{R})$ is a convex symmetric weak*-lower semi-continuous set-valued function on B_{X^*} then Φ admits a selection from $A_\sigma(B_{X^*})$, that is, there exists $a \in A_\sigma(B_{X^*})$ such that for each $x^* \in B_{X^*}$, $a(x^*) \in \Phi(x^*)$.*

We are now ready to prove the following necessary and sufficient condition in L_1-predual spaces.

Theorem 2.4.8 ([58, Theorem 4.9])**.** *Let X be an L_1-predual space. Let $V \in \mathcal{CV}(X)$ and $F \in \mathcal{K}(X)$. Then the following statements are true.*

(*i*) *The equality*

$$rad_V(F) = rad_X(F) + d(V, cent_X(F)) \tag{2.27}$$

holds true.

(*ii*) *A necessary and sufficient condition for the set $cent_V(F) \neq \emptyset$ is that there exists $v_0 \in V$ and $x_0 \in cent_X(F)$ such that $d(V, cent_X(F)) = \|v_0 - x_0\|$. Furthermore, the set*

$$cent_V(F) = \{v \in V\colon \text{ there exists } x \in cent_X(F) \text{ such that } d(V, cent_X(F)) = \|v - x\|\}.$$

Proof. Using Lemma 2.2.5, it suffices to prove the following claims. Let $F \in \mathcal{K}(A_\sigma(B_{X^*}))$ and $V \in \mathcal{CV}(A_\sigma(B_{X^*}))$.

Claim 1: The following formula holds true.

$$rad_V(F) = rad_{A_\sigma(B_{X^*})}(F) + d(V, cent_{A_\sigma(B_{X^*})}(F)). \tag{2.28}$$

Claim 2: The set $cent_V(F) \neq \emptyset$ if and only if there exists $v_0 \in V$ and $a_0 \in cent_{A_\sigma(B_{X^*})}(F)$ such that $d(V, cent_{A_\sigma(B_{X^*})}(F)) = \|v_0 - a_0\|$.

From Theorem 2.4.2, $cent_{A_\sigma(B_{X^*})}(F) \neq \emptyset$ and $rad_{A_\sigma(B_{X^*})}(F) = r_F$. Let $R = d(V, cent_{A_\sigma(B_{X^*})}(F))$. In order to prove the formula in (2.28), it suffices to show that $rad_V(F) \geq r_F + R$ because the reverse inequality is a simple consequence of triangle inequality. We implement the proof techniques used in [53, Theorem 2.2].

Let $S_n = rad_V(F) - r_F + \frac{1}{n}$, for $n = 1, 2, \ldots$. There exists $v_n \in V$ such that

$$r(v_n, F) < rad_V(F) + \frac{1}{n}. \tag{2.29}$$

Hence for each $z \in F$ and $x^* \in B_{X^*}$,

$$z(x^*) - rad_V(F) - \frac{1}{n} < v_n(x^*) < z(x^*) + rad_V(F) + \frac{1}{n}. \tag{2.30}$$

Therefore, from the inequalities in (2.30) and (2.10), for each $x^* \in B_{X^*}$,

$$M_F(x^*) - rad_V(F) - \frac{1}{n} \leq v_n(x^*) \leq m_F(x^*) + rad_V(F) + \frac{1}{n}. \tag{2.31}$$

This implies

$$M_F(x^*) - r_F - S_n \leq v_n(x^*) \leq m_F(x^*) + r_F + S_n. \tag{2.32}$$

We define the following set-valued function. For each $x^* \in B_{X^*}$,

$$\Phi_n(x^*) = [v_n(x^*) - S_n,\, v_n(x^*) + S_n] \cap [M_F(x^*) - r_F, m_F(x^*) + r_F]. \tag{2.33}$$

Clearly, $\Phi_n(x^*)$ is closed, convex and bounded for each $x^* \in B_{X^*}$. Moreover, the inequalities in (2.32) guarantee that $\Phi_n(x^*)$ is non-empty for each $x^* \in B_{X^*}$. The map Φ_n can be proved to be a weak*-lower semi-continuous function on B_{X^*} using the same argument as in the proof of [53, Theorem 2.2].

Now, let $0 \leq \alpha \leq 1$ and $x_1^*, x_2^* \in B_{X^*}$. From the facts that $v_n \in A_\sigma(B_{X^*})$ and the functions M_F and $-m_F$ are convex on B_{X^*}, it is easy to verify that

$$\alpha\Phi(x_1^*) + (1-\alpha)\Phi(x_2^*) \subseteq \Phi(\alpha x_1^* + (1-\alpha)x_2^*). \tag{2.34}$$

Moreover, if $x^* \in B_{X^*}$, then from the facts that $v_n \in A_\sigma(B_{X^*})$ and $M_F(-x^*) = -m_F(x^*)$, it follows that $-\Phi_n(x^*) = \Phi_n(-x^*)$. Therefore, Φ_n is a convex symmetric weak*-lower semi-continuous function on B_{X^*} and hence, by Theorem 2.4.7, there exists $a_n \in A_\sigma(B_{X^*})$ such that $a_n(x^*) \in \Phi_n(x^*)$, for each $x^* \in B_{X^*}$. Therefore, for each $x^* \in B_{X^*}$,

$$\begin{aligned} v_n(x^*) - S_n \leq a_n(x^*) \leq v_n(x^*) + S_n \\ \text{and } M_F(x^*) - r_F \leq a_n(x^*) \leq m_F(x^*) + r_F. \end{aligned} \tag{2.35}$$

It follows that $a_n \in cent_{A_\sigma(B_{X^*})}(F)$ such that $\|v_n - a_n\| \leq S_n$. Hence for each $n = 1, 2, \ldots$, $R \leq S_n$. This proves the formula in (2.28).

We now prove Claim 2. Suppose $R = \|v_0 - a_0\|$, for some $v_0 \in V$ and $a_0 \in cent_{A_\sigma(B_{X^*})}(F)$. This implies

$$r(v_0, F) \leq \sup_{z \in F}\{\|v_0 - a_0\| + \|a_0 - z\|\} \leq R + r_F = rad_V(F). \tag{2.36}$$

Therefore, $rad_V(F) = r(v_0, F)$ and hence $v_0 \in cent_V(F)$.

Conversely, if $v_0 \in cent_V(F)$, then an argument similar to the one above proves that the set-valued map defined as $\Phi(x^*) = [v_0(x^*) - R, v_0(x^*) + R] \cap [M_F(x^*) - r_F, m_F(x^*) + r_F]$, for each $x^* \in B_{X^*}$, is convex symmetric weak*-lower semi-continuous function on B_{X^*}. Hence, by Theorem 2.4.7, there exists a selection $a_0 \in cent_{A_\sigma(B_{X^*})}(F)$ such that $R = \|v_0 - a_0\|$. □

2.5 Some geometrical characterizations of L_1-predual spaces

We begin this section by providing a geometrical characterization of L_1-predual spaces in terms of the identity in (2.27) in the following result.

Theorem 2.5.1 ([58, Theorem 4.10]). *Let X be a Banach space. Then the following statements are equivalent.*

(*i*) *X is an L_1-predual space.*

(*ii*) *For each $V \in \mathcal{CV}(X)$ and $F \in \mathcal{F}_4(X)$,*

$$rad_V(F) = rad_X(F) + d(V, cent_X(F)).$$

(*iii*) *For each $V \in \mathcal{CV}(X)$ and $F \in \mathcal{F}_4(X)$,*

$$rad_V(F) = rad_X(F) + \lim_{\delta \to 0^+} d(V, cent_X(F, \delta)).$$

Proof. *(i) ⇒ (ii)* follows from Theorem 2.4.8 and *(ii) ⇒ (iii)* follows from the following chain of inequalities. For each $F \in \mathcal{K}(X)$,

$$rad_V(F) \leq rad_X(F) + \lim_{\delta \to 0^+} d(V, cent_X(F, \delta)) \leq rad_X(F) + d(V, cent_X(F)). \tag{2.37}$$

In order to prove *(iii) ⇒ (i)*, by Theorem 1.2.6, it suffices to show that for each $F \in \mathcal{F}_4(X)$, $rad_X(F) = \frac{1}{2} diam(F)$. The proof idea is similar to that in [16, Theorem 3.4].

Let $F = \{x_1, x_2, x_3, x_4\} \subseteq X$. Without loss of generality, let $diam(F) = \|x_1 - x_2\|$. Define $R = \frac{1}{2} diam(F)$. Now, let $F' = \{x_1, x_2\}$ and $V = \{x_3\}$. Then $rad_X(F') = R$. By our assumption,

$$rad_{\{x_3\}}(F') = rad_X(F') + \lim_{\delta \to 0^+} d(\{x_3\}, cent_X(F', \delta)). \tag{2.38}$$

Therefore, $2R \geq R + \lim_{\delta\to 0^+} d(\{x_3\}, cent_X(F', \delta))$ and hence, for each $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that $r(x_\varepsilon, \{x_1, x_2, x_3\}) \leq R + \varepsilon$. It follows that $rad_X(\{x_1, x_2, x_3\}) = R$. We next consider $F' = \{x_1, x_2, x_3\}$ and $V = \{x_4\}$ and follow the arguments above to obtain $rad_X(F) = R$. $\square$

The following result follows from Theorems 2.4.8 and 2.5.1.

Corollary 2.5.2 ([58, Corollary 4.11]). *Let X be a Banach space. Then the following statements are equivalent.*

(*i*) *X is an L_1-predual space.*

(*ii*) *For each $V \in \mathcal{CV}(X)$ and $F \in \mathcal{F}(X)$,*

$$rad_V(F) = rad_X(F) + d(V, cent_X(F)).$$

(*iii*) *For each $V \in \mathcal{CV}(X)$ and $F \in \mathcal{F}(X)$,*

$$rad_V(F) = rad_X(F) + \lim_{\delta\to 0^+} d(V, cent_X(F, \delta)).$$

(*iv*) *For each $V \in \mathcal{CV}(X)$ and $F \in \mathcal{K}(X)$,*

$$rad_V(F) = rad_X(F) + d(V, cent_X(F)).$$

(*v*) *For each $V \in \mathcal{CV}(X)$ and $F \in \mathcal{K}(X)$,*

$$rad_V(F) = rad_X(F) + \lim_{\delta\to 0^+} d(V, cent_X(F, \delta)).$$

We conclude this chapter with one more application of Theorem 2.4.2: we derive a few characterizations for the ideals of an L_1-predual.

Theorem 2.5.3 ([58, Theorem 4.12]). *Let Y be a subspace of an L_1-predual space X. Then the following statements are equivalent.*

(*i*) *Y is an ideal in X.*

(*ii*) *For each $F \in \mathcal{F}_4(Y)$, $cent_Y(F) \neq \emptyset$ and*

$$cent_Y(F) = cent_X(F) \cap Y.$$

(*iii*) *For each $F \in \mathcal{F}_4(Y)$, $d(Y, cent_X(F)) = 0$.*

Proof. We first prove (*i*) $\Rightarrow$ (*ii*). By [49, Proposition 1], Y is an L_1-predual space. Thus by Lemma 2.2.5, $Y \cong A_\sigma(B_{Y^*})$ and $X \cong A_\sigma(B_{X^*})$.

Let $F = \{y_1, y_2, y_3, y_4\} \subseteq Y$. Clearly, $cent_X(F) \cap Y \subseteq cent_Y(F)$. Under the mapping given in Lemma 2.2.5, let $\overline{F} = \{\overline{y_1}, \overline{y_2}, \overline{y_3}, \overline{y_4}\} \in \mathcal{F}(A_\sigma(B_{Y^*}))$. We define the following functions and number. For each $y^* \in B_{Y^*}$,

$$\begin{aligned} \tilde{m}_{\overline{F}}(y^*) &= \min\{y^*(y_i) \colon 1 \leq i \leq 4\}, \\ \tilde{M}_{\overline{F}}(y^*) &= \max\{y^*(y_i) \colon 1 \leq i \leq 4\} \\ \text{and } \tilde{r}_{\overline{F}} &= \frac{1}{2}\max\{M_{\overline{F}}(y^*) - m_{\overline{F}}(y^*) \colon y^* \in B_{Y^*}\}. \end{aligned} \tag{2.39}$$

Then by Theorems 1.2.6 and 2.4.2,

$$cent_{A_\sigma(B_{Y^*})}(\overline{F}) = \{a \in A_\sigma(B_{Y^*}) \colon \tilde{M}_{\overline{F}} - \tilde{r}_{\overline{F}} \le a \le \tilde{m}_{\overline{F}} + \tilde{r}_{\overline{F}}\} \neq \emptyset \tag{2.40}$$

and $rad_{A_\sigma(B_{Y^*})}(\overline{F}) = \tilde{r}_{\overline{F}} = \frac{1}{2}diam(\overline{F})$.

Since $\overline{F} \in \mathcal{F}(A_\sigma(B_{X^*}))$, we define the following functions and number. For each $x^* \in B_{X^*}$,

$$\begin{aligned} m_{\overline{F}}(x^*) &= \min\{x^*(y_i) \colon 1 \le i \le 4\}, \\ M_{\overline{F}}(x^*) &= \max\{x^*(y_i) \colon 1 \le i \le 4\} \\ \text{and } r_{\overline{F}} &= \frac{1}{2}\max\{M_{\overline{F}}(x^*) - m_{\overline{F}}(x^*) \colon x^* \in B_{X^*}\} \end{aligned} \tag{2.41}$$

Then we apply Theorems 1.2.6 and 2.4.2 and get

$$cent_{A_\sigma(B_{X^*})}(\overline{F}) = \{a \in A_\sigma(B_{X^*}) \colon M_{\overline{F}} - r_{\overline{F}} \le a \le m_{\overline{F}} + r_{\overline{F}}\} \neq \emptyset \tag{2.42}$$

and $rad_{A_\sigma(B_{X^*})}(\overline{F}) = r_{\overline{F}} = \frac{1}{2}diam(\overline{F})$.

Let $y_0 \in Y$ such that $\overline{y_0} \in cent_{A_\sigma(B_{Y^*})}(\overline{F})$. Then for each $x^* \in B_{X^*}$,

$$\begin{aligned} M_{\overline{F}}(x^*) - r_{\overline{F}} = \tilde{M}_{\overline{F}}(x^*|_Y) - \tilde{r}_{\overline{F}} \le x^*|_Y(y_0) &= x^*(y_0) \\ &\le \tilde{m}_{\overline{F}}(x^*|_Y) + \tilde{r}_{\overline{F}} \\ &= m_{\overline{F}}(x^*) + r_{\overline{F}}. \end{aligned} \tag{2.43}$$

Therefore, $\overline{y_0} \in cent_{A_\sigma(B_{X^*})}(\overline{F})$. It follows from Remark 2.4.3 that $cent_Y(F) \subseteq cent_X(F) \cap Y$.

$(ii) \Rightarrow (iii)$ is easy to observe.

Finally, we prove $(iii) \Rightarrow (i)$. Let $F \in \mathcal{F}_4(Y)$. By our assumption, $d(Y, cent_X(F)) = 0$. Using the identity in Theorem 2.5.1 and by Theorem 1.2.6, $rad_Y(F) = rad_X(F) = \frac{1}{2}diam(F)$. Therefore, (i) follows from Theorem 1.2.6 and [49, Proposition 1]. □

We use Theorem 1.2.6 and an argument similar to that in Theorem 2.5.3 to deduce the following result.

Corollary 2.5.4 ([58, Corollary 4.13]). *Let Y be a subspace of an L_1-predual space X. Then the following statements are equivalent.*

(*i*) *Y is an ideal in X.*

(*ii*) *For each $F \in \mathcal{F}(Y)$, $cent_Y(F) \neq \emptyset$ and*

$$cent_Y(F) = cent_X(F) \cap Y.$$

(*iii*) *For each $F \in \mathcal{F}(Y)$, $d(Y, cent_X(F)) = 0$.*

(*iv*) *For each $F \in \mathcal{K}(Y)$, $cent_Y(F) \neq \emptyset$ and*

$$cent_Y(F) = cent_X(F) \cap Y.$$

(*v*) *For each $F \in \mathcal{K}(Y)$, $d(Y, cent_X(F)) = 0$.*

We now recall the notion of state spaces.

Definition 2.5.5 ([7])**.** Let S be a compact Hausdorff space and $\mathcal{A}$ be a subspace of $C(S)$ that contains the constant function 1 and separates the points of S. The *state space* of a subspace $\mathcal{A}$, denoted by $\mathcal{S}_{\mathcal{A}}$, is defined as

$$\mathcal{S}_{\mathcal{A}} = \{L \in \mathcal{A}^* \colon L(1) = 1 = \|L\|\}.$$

It is easily seen that $\mathcal{S}_{\mathcal{A}}$ is a weak*-compact convex subset of $\mathcal{A}^*$. We recall the following result in [20].

Theorem 2.5.6 ([20, Theorem 1.4])**.** *Let S be a compact Hausdorff space and $\{s_1, \ldots, s_n\} \subseteq S$ such that $S \backslash \{s_1, \ldots, s_n\} \neq \emptyset$. Let $\mu_1, \ldots, \mu_n$ be postive measures on S such that for each $i = 1, \ldots, n$, $\|\mu_i\| \leq 1$. Then the subspace of $C(S)$*

$$\mathcal{A} \colon = \{f \in C(S) \colon f(s_i) = \mu_i(f), \textit{ for } i = 1, \ldots, n\}$$

equipped with the relative norm and order is an L_1-predual space.

We also recall that by [7, Theorem 4.9, p. 14], a subspace $\mathcal{A}$ of $C(S)$ which contains the constant function 1 and separates points of S is order isometric to the space $A(\mathcal{S}_{\mathcal{A}})$. Theorem 2.5.6 provides plenty of examples of such subspaces $\mathcal{A}$ of $C(S)$ such that $\mathcal{S}_{\mathcal{A}}$ is a Choquet simplex. For instance, the following subspaces $\mathcal{A}$ of $C([0,1])$ fall under the aforementioned category.

Example 2.5.7. (*a*) Let λ be the Lebesgue measure on $[0,1]$. Then $\mathcal{A} = \{f \in C([0,1]) \colon f(0) = \int_0^1 f(t) d\lambda(t)\}$.

(*b*) $\mathcal{A} = \left\{f \in C([0,1]) \colon f(\frac{1}{2}) = \frac{f(0)+f(1)}{2}\right\}$.

A special case of Theorem 2.5.3 is

Proposition 2.5.8 ([58, Proposition 4.14])**.** *Let S be a compact Hausdorff space and $\mathcal{A}$ be a subspace of $C(S)$ that contains the constant function 1 and separates the points of S. Then the following statements are equivalent.*

(*i*) *$\mathcal{S}_{\mathcal{A}}$ is a Choquet simplex.*

(*ii*) *For each $F \in \mathcal{F}_4(\mathcal{A})$, $cent_{\mathcal{A}}(F) \neq \emptyset$ and*

$$cent_{\mathcal{A}}(F) = cent_{C(S)}(F) \cap \mathcal{A}.$$

(*iii*) *For each $F \in \mathcal{F}_4(\mathcal{A})$, $d(\mathcal{A}, cent_{C(S)}(F)) = 0$.*

(*iv*) *For each $F \in \mathcal{F}(\mathcal{A})$, $cent_{\mathcal{A}}(F) \neq \emptyset$ and*

$$cent_{\mathcal{A}}(F) = cent_{C(S)}(F) \cap \mathcal{A}.$$

(*v*) *For each $F \in \mathcal{F}(\mathcal{A})$, $d(\mathcal{A}, cent_{C(S)}(F)) = 0$.*

(*vi*) *For each $F \in \mathcal{K}(\mathcal{A})$, $cent_{\mathcal{A}}(F) \neq \emptyset$ and*

$$cent_{\mathcal{A}}(F) = cent_{C(S)}(F) \cap \mathcal{A}.$$

(*vii*) *For each* $F \in \mathcal{K}(\mathcal{A})$, $d(\mathcal{A}, cent_{C(S)}(F)) = 0$.

Proof. By [7, Theorem 4.9, p. 14], $\mathcal{A}$ is order isometric to the space $A(\mathcal{S}_{\mathcal{A}})$. Therefore, $\mathcal{S}_{\mathcal{A}}$ is a Choquet simplex if and only if $A(\mathcal{S}_{\mathcal{A}})$ is an L_1-predual space if and only if $\mathcal{A}$ is an L_1-predual space if and only if $\mathcal{A}$ is an ideal in $C(S)$. Now by applying Theorem 2.5.3 and Corollary 2.5.4, we get the desired equivalence. □

If K is a compact convex subset of a lctvs then K is affinely homeomorphic to $\mathcal{S}_{A(K)}$; [7, Theorem 4.7, p. 14]. Hence a direct consequence of Proposition 2.5.8 is

Corollary 2.5.9 ([58, Corollary 4.15]). *Let* K *be a compact convex subset of a lctvs. Then the following statements are equivalent*

(*i*) K *is a Choquet simplex.*

(*ii*) *For each* $F \in \mathcal{F}_4(A(K))$ *(or* $\mathcal{F}(A(K))$ *or* $\mathcal{K}(A(K))$*),* $cent_{A(K)}(F) \neq \emptyset$ *and* $cent_{A(K)}(F) = cent_{C(K)}(F) \cap A(K)$.

(*iii*) *For each* $F \in \mathcal{F}_4(A(K))$ *(or* $\mathcal{F}(A(K))$ *or* $\mathcal{K}(A(K))$*),* $d(A(K), cent_{C(K)}(F)) = 0$.

Chapter 3

Property-(P_1) in Banach spaces

3.1 Summary of results

Let X be a Banach space, Y be a subspace of X and $\mathscr{F} \subseteq \mathcal{CB}(X)$. This chapter is mainly dedicated to understanding the interconnection between property-(P_1) of the triplets $(X, Y, \mathscr{F})$ and $(X, B_Y, \mathscr{F})$ and the interrelation between the four concepts, namely strong proximinality, strong ball proximinality and property-(P_1) of the triplets $(X, Y, \mathcal{K}(X))$ and $(X, B_Y, \mathcal{K}(X))$, whenever Y is a finite co-dimensional subspace of an L_1-predual space X.

We prove in Section 3.2 that if $(X, B_Y, \mathscr{F})$ has property-(P_1), then so does $(X, Y, \mathscr{F})$, whenever $\mathscr{F} = \mathcal{CB}(X)$, $\mathcal{K}(X)$ or $\mathcal{F}(X)$. The converse of the result above is proved to be true for the triplet $(X, Y, \mathcal{K}(X))$, whenever X is an L_1-predual space and Y is a finite co-dimensional subspace Y of X (Theorem 3.4.7).

In Section 3.3, we derive that the following classes of triplets satisfy property-(P_1):

(i) $(X, B_J, \mathcal{K}(X))$, for an L_1-predual space X and an M-ideal J in X.

(ii) $(C(S), B_Y, \mathcal{K}(C(S)))$, for a compact Hausdorff space S and a finite co-dimensional subspace Y of $C(S)$ such that the measures determining Y are finitely supported.

(iii) $(A(K), B_Y, \mathcal{K}(A(K)))$, for a Choquet simplex K and a finite co-dimensional subspace Y of $A(K)$ such that the measures determining Y are finitely supported and their supports are contained in $ext(K)$.

In Section 3.4, we characterize a strongly proximinal finite co-dimensional subspace Y of an L_1-predual space X in terms of property-(P_1) of the triplets $(X, Y, \mathcal{K}(X))$ and $(X, B_Y, \mathcal{K}(X))$. Furthermore, in Section 3.5 of this chapter, we prove that for a Banach space X, if x^* is a SSD-point of X^* and $(X^{**}, J_{X^{**}}(x^*), \mathcal{K}(X^{**}))$ has property-(P_1), then $(X, J_X(x^*), \mathcal{K}(X))$ satisfies property-(P_1). In fact, we prove a much more generalized version of the result above in Theorem 3.5.8.

In the last Section 3.6, we illustrate that $1\frac{1}{2}$-ball property is not a sufficient condition for r.c.p..

3.2 Property-(P_1) of a subspace in relation to that of its closed unit ball

In this section, for a subspace Y of a Banach space X, we discuss the interconnection between property-(P_1) of the triplets $(X, Y, \mathscr{F})$ and $(X, B_Y, \mathscr{F})$ whenever $\mathscr{F} = \mathcal{CB}(X)$, $\mathcal{K}(X)$ or $\mathcal{F}(X)$. We use ideas similar to that in [8].

Lemma 3.2.1 ([56, Lemma 2.1])**.** *Let Y be a subspace of a Banach space X and $B \in \mathcal{CB}(X)$.*

(i) For each $\lambda > 0$, $\lambda cent_{B_Y}(\frac{1}{\lambda}B) = cent_{\lambda B_Y}(B)$.

(ii) For each $\lambda \geq \sup_{b \in B} \|b\| + rad_Y(B)$, $cent_Y(B) \subseteq cent_{\lambda B_Y}(B)$.

(iii) For each $\lambda > \sup_{b \in B} \|b\| + rad_Y(B)$, $cent_Y(B) = cent_{\lambda B_Y}(B)$.

Proof. (i). Let $\lambda > 0$ and $y_0 \in B_Y$. $\lambda y_0 \in \lambda cent_{B_Y}(\frac{1}{\lambda}B) \Leftrightarrow$ for each $y \in B_Y$, $r(y_0, \frac{1}{\lambda}B) \leq r(y, \frac{1}{\lambda}B)$ $\Leftrightarrow$ for each $y \in B_Y$, $r(\lambda y_0, B) \leq r(\lambda y, B) \Leftrightarrow \lambda y_0 \in cent_{\lambda B_Y}(B)$.

(ii). Let $\lambda \geq \sup_{b \in B} \|b\| + rad_Y(B)$ and $y_0 \in cent_Y(B)$. Then for each $b \in B$, $\|y_0\| \leq \|b\| + \|y_0 - b\| \leq \sup_{b \in B} \|b\| + r(y_0, B) = \sup_{b \in B} \|b\| + rad_Y(B) \leq \lambda$. Hence, $y_0 \in \lambda B_Y$ and it follows that $y_0 \in cent_{\lambda B_Y}(B)$.

(iii). Let $\lambda > \sup_{b \in B} \|b\| + rad_Y(B)$ and $y_0 \in cent_{\lambda B_Y}(B)$. Let $R = rad_Y(B)$. It is easy to see that for each $\delta > 0$, $R = \inf\{r(y, B) \colon y \in S_{R+\delta}(B) \cap Y\}$. In particular, let $\delta = \lambda - (\sup_{b \in B} \|b\| + R)$. If $y \in S_{R+\delta}(B) \cap Y$, then $y \in \lambda B_Y$. Hence, $r(y_0, B) \leq r(y, B)$. It follows that $y_0 \in cent_Y(B)$. □

Proposition 3.2.2 ([56, Proposition 2.2])**.** *Let Y be a subspace of a Banach space X and $\mathscr{F} = \mathcal{CB}(X)$, $\mathcal{K}(X)$ or $\mathcal{F}(X)$. If $(B_Y, \mathscr{F})$ has r.c.p. then $(Y, \mathscr{F})$ has r.c.p..*

Proof. We prove the result only for $\mathcal{CB}(X)$ because the same proof works for $\mathcal{K}(X)$ and $\mathcal{F}(X)$. Let $B \in \mathcal{CB}(X)$ and $\lambda > \sup_{b \in B} \|b\| + rad_Y(B)$. Since $(B_Y, \mathcal{CB}(X))$ has r.c.p., $(B_Y, \mathcal{CB}(B_X))$ has r.c.p.. Therefore, $(\lambda B_Y, \mathcal{CB}(\lambda B_X))$ has r.c.p.. Clearly, for each $b \in B$, $b \in \lambda B_X$. Therefore, from Lemma 3.2.1 (iii), $cent_Y(B) = cent_{\lambda B_Y}(B) \neq \emptyset$. □

Proposition 3.2.3 ([56, Proposition 2.3])**.** *Let Y be a subspace of a Banach space X and $B \in \mathcal{CB}(X)$. Then*

(i) For each $\lambda > 0$ and $\delta > 0$, $cent_{\lambda B_Y}(B, \delta) = \lambda cent_{B_Y}(\frac{1}{\lambda}B, \frac{\delta}{\lambda})$.

(ii) For each $\lambda > 0$, $(X, \lambda B_Y, \{B\})$ has property-(P_1) if and only if $(X, B_Y, \{\frac{1}{\lambda}B\})$ has property-(P_1).

(iii) Let $\mathscr{F} = \mathcal{CB}(X)$, $\mathcal{K}(X)$ or $\mathcal{F}(X)$. If $(X, B_Y, \mathscr{F})$ has property-(P_1) then $(X, Y, \mathscr{F})$ has property-(P_1).

Proof. (i) follows from a similar argument as in Lemma 3.2.1 (i).

(ii) easily follows from (i) and Lemma 3.2.1 (i).

(iii). We prove the result only for $\mathcal{CB}(X)$ because the same proof works for $\mathcal{K}(X)$ and $\mathcal{F}(X)$. Assume $(X, B_Y, \mathcal{CB}(X))$ has property-(P_1). Obviously, $(X, B_Y, \mathcal{CB}(B_X))$ has property-(P_1) and from Proposition 3.2.2, it follows that $(Y, \mathcal{CB}(X))$ has r.c.p.. Let $B \in \mathcal{CB}(X)$ and $\lambda > \sup_{b \in B} \|b\| + rad_Y(B)$. Since $\frac{1}{\lambda}B \in \mathcal{CB}(B_X)$, $(X, B_Y, \{\frac{1}{\lambda}B\})$ has property-(P_1). Hence, from (ii), $(X, \lambda B_Y, \{B\})$ has property-(P_1). Now, using the same argument as in Lemma 3.2.1 (iii), for $0 < \delta < \lambda - (\sup_{b \in B} \|b\| + rad_Y(B))$, $cent_Y(B, \delta) \subseteq \lambda B_Y$ and hence, $cent_Y(B, \delta) = cent_{\lambda B_Y}(B, \delta)$. Moreover,

$cent_Y(B) = cent_{\lambda B_Y}(B)$. It follows that $(X, Y, \{B\})$ has property-(P_1). Therefore, $(X, Y, \mathcal{CB}(X))$ has property-(P_1). □

3.3 Property-(P_1) in some L_1-predual spaces

In this section, we study property-(P_1) in the M-ideals and finite co-dimensional subspaces of L_1-predual spaces.

We first recall and prove a few intersection properties of closed balls of M-ideals in an L_1-predual space. The following result follows from [44, Theorem 2.17 and Proposition 6.5] and [2, Theorem 5.8].

Lemma 3.3.1. *Let J be an M-ideal in an L_1-predual space X. Then for every $n \in \mathbb{N}$, $\{x_1, \ldots, x_n\} \subseteq X$ and $r_1, r_2, \ldots, r_n > 0$, if for each $i = 1, 2, \ldots, n$, $B_X[x_i, r_i] \cap J \neq \emptyset$ and $\bigcap_{i=1}^n B_X[x_i, r_i] \neq \emptyset$ then $\bigcap_{i=1}^n B_X[x_i, r_i] \cap J \neq \emptyset$.*

The following intersection property is obtained by minor modifications to the proof of [46, Lemma 2.1]. We include the proof here for the sake of completeness.

Lemma 3.3.2 ([56, Lemma 3.1])**.** *Let X be an L_1-predual space and J be an M-ideal in X. Let $F \in \mathcal{K}(X)$, $\{x_1, \ldots, x_n\} \subseteq X$ and $r, r_1, \ldots, r_n > 0$. If for each $x \in F$, $B_X[x, r] \cap J \neq \emptyset$; for each $i = 1, \ldots, n$, $B_X[x_i, r_i] \cap J \neq \emptyset$ and $S_r(F) \cap \bigcap_{i=1}^n B_X[x_i, r_i] \neq \emptyset$, then $S_r(F) \cap \bigcap_{i=1}^n B_X[x_i, r_i] \cap J \neq \emptyset$.*

Proof. For each $m \in \mathbb{N}$, let $F_m \subseteq F$ be a finite $\frac{1}{m}$-net such that $F_m \subseteq F_{m+1}$. Then by Lemma 3.3.1, there exists an element $y_1 \in \bigcap_{y \in F_1} B_X[y, r] \cap \bigcap_{i=1}^n B_X[x_i, r_i] \cap J$. We assume that for each $m \in \mathbb{N}$ and $i = 1, 2, \ldots, m$, the elements

$$y_i \in \bigcap_{y \in F_i} B_X[y, r] \cap \bigcap_{i=1}^{n} B_X[x_i, r_i] \cap \bigcap_{j=1}^{i-1} B_X\left[y_j, \frac{1}{j}\right] \cap J \tag{3.1}$$

have been constructed (we use the convention that $\bigcap_{j=1}^0 B_X\left[y_j, \frac{1}{j}\right] = X$). Then it is easy to see that for $i = 1, 2, \ldots, m$ and each $y \in F$, $y_i \in B_X\left[y, r + \frac{1}{i}\right]$. Therefore, the closed balls in the collection

$$\{B_X[y, r] \colon y \in F\} \cup \{B_X[x_i, r_i]\}_{i=1}^n \cup \left\{B_X\left[y_j, \frac{1}{j}\right]\right\}_{j=1}^m \tag{3.2}$$

intersect pairwise. Then by [45, Theorem 4.5, pg. 38] and [39, Theorem 6, pg. 212],

$$\bigcap_{y \in F_{m+1}} B_X[y, r] \cap \bigcap_{i=1}^{n} B_X[x_i, r_i] \cap \bigcap_{i=1}^{m} B_X\left[y_j, \frac{1}{j}\right] \neq \emptyset. \tag{3.3}$$

We apply Lemma 3.3.1 again to obtain an element

$$y_{m+1} \in \bigcap_{y \in F_{m+1}} B_X[y, r] \cap \bigcap_{i=1}^{n} B_X[x_i, r_i] \cap \bigcap_{j=1}^{m} B_X\left[y_j, \frac{1}{j}\right] \cap J. \tag{3.4}$$

Then for each $y \in F$, $y_{m+1} \in B_X[y, r + \frac{1}{m+1}]$. Proceeding inductively, we construct a sequence $\{y_m\}$ which satisfies the following properties:

(i) for each $m \in \mathbb{N}$ and $y \in F$, $y_m \in B_X[y, r + \frac{1}{m}]$,

(ii) for each $m, k \in \mathbb{N}$ with $m > k$, $y_m \in B_X[y_k, \frac{1}{k}]$ and

(iii) for each $m \in \mathbb{N}$, $y_m \in \bigcap_{i=1}^{n} B_X[x_i, r_i]$.

It is easy to see that $\{y_m\}$ is a Cauchy sequence. In fact, for an $\varepsilon > 0$, we choose $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0} < \frac{\varepsilon}{2}$. Then for each $m, k > m_0$, $\|y_m - y_k\| \leq \|y_m - y_{m_0}\| + \|y_{m_0} - y_k\| \leq \frac{2}{m_0} < \varepsilon$. Hence let $y_0 = \lim_{m\to\infty} y_m$. Then $y_0 \in S_r(F) \cap \bigcap_{i=1}^{n} B_X[x_i, r_i] \cap J$. $\square$

Theorem 3.3.3 ([56, Theorem 3.2]). *Let X be an L_1-predual space and J be an M-ideal in X. Then $(X, B_J, \mathcal{K}(X))$ has property-(P_1).*

Proof. Let $\varepsilon > 0$ and $F \in \mathcal{K}(X)$. Let $x \in cent_{B_J}(F, \varepsilon) = S_{rad_{B_J}(F)+\varepsilon}(F) \cap B_J$. Obviously, $B_X[x, \varepsilon] \cap B_X \cap J \neq \emptyset$ and for each $y \in F$, $B_X[x, \varepsilon] \cap B_X[y, rad_{B_J}(F)] \neq \emptyset$. By [35, Corollary 4.8], J is ball proximinal in X. Hence for each $y \in F$, $B_X[y, d(y, B_J)] \cap B_J \neq \emptyset$. Moreover, it is easy to see that for each $y \in F$, $d(y, B_J) \leq rad_{B_J}(F)$. It follows that for each $y \in F$, $B_X[y, rad_{B_J}(F)] \cap B_X \neq \emptyset$. By [46, Theorem 2.2], $cent_J(F) = S_{rad_J(F)}(F) \cap J \neq \emptyset$. Since $rad_J(F) \leq rad_{B_J}(F)$, $S_{rad_{B_J}(F)}(F) \cap J \neq \emptyset$. Now, $\{B_X[y, rad_{B_J}(F)] \colon y \in F\} \cup \{B_X[x, \varepsilon], B_X\}$ is a collection of closed balls which intersect pairwise. Therefore, by [45, Theorem 4.5, pg. 38] and [39, Theorem 6, pg. 212],

$$S_{rad_{B_J}(F)}(F) \cap B_X[x, \varepsilon] \cap B_X \neq \emptyset. \tag{3.5}$$

It is easily observed that each of the closed balls above intersects J. Therefore, by Lemma 3.3.2,

$$S_{rad_{B_J}(F)}(F) \cap B_X[x, \varepsilon] \cap B_X \cap J \neq \emptyset. \tag{3.6}$$

$\square$

Next, for a compact Hausdorff space S and a finite co-dimensional subspace Y of $C(S)$, we provide a a sufficient condition for the triplet $(C(S), B_Y, \mathcal{K}(C(S)))$ to satisfy property-(P_1). Before we proceed, we need the following technical result.

Lemma 3.3.4 ([56, Lemma 3.3]). *Let X be a Banach space, $V \in \mathcal{CV}(X)$ and $B \in \mathcal{CB}(X)$. Then for each $\varepsilon > 0$ and $\gamma > 0$, there exists $\delta > 0$ such that*

$$cent_V(B, \gamma + \delta) \subseteq cent_V(B, \gamma) + \varepsilon B_X.$$

Proof. Let $\varepsilon > 0$, $\gamma > 0$ and without loss of generality, assume that $R := rad_V(B) > 0$. We choose $\delta > 0$ such that $\delta < \min\left\{R, \frac{\varepsilon\gamma}{6R+4\gamma}\right\}$. Let $v \in cent_V(B, \gamma + \delta)$. Then $r(v, B) \leq R + \gamma + \delta$. Further, let $v' \in cent_V\left(B, \frac{\gamma}{2}\right)$. We define $\lambda = \frac{2\delta}{2\delta+\gamma}$ and $\tilde{v} = (1-\lambda)v + \lambda v'$. Now, for each $b \in B$,

$$\begin{aligned}
\|\tilde{v} - b\| &= \|(1-\lambda)v + \lambda v' - b\| \\
&\leq (1-\lambda)\|v - b\| + \lambda\|v' - b\| \\
&< (1-\lambda)(R+\gamma+\delta) + \lambda\left(R + \frac{\gamma}{2}\right) \\
&= R + (1-\lambda)(\gamma+\delta) + \lambda\frac{\gamma}{2} \\
&= R + (1-\lambda)\gamma + \lambda\gamma - \lambda\frac{\gamma}{2} + (1-\lambda)\delta \\
&= R + \gamma + \delta - \left(\frac{2\delta}{2\delta+\gamma}\right)\left(\delta + \frac{\gamma}{2}\right) \\
&= R + \gamma.
\end{aligned} \tag{3.7}$$

Hence, it follows that $r(\tilde{v}, B) \leq R + \gamma$. Moreover, for each $b \in B$,

$$\|v - \tilde{v}\| \leq \lambda(\|v - b\| + \|v' - b\|) < \frac{2\delta}{2\delta+\gamma}(3R + 2\gamma) < \varepsilon. \tag{3.8}$$

□

Theorem 3.3.5 ([56, Theorem 3.5]). *Let S be a compact Hausdorff space and $\{\mu_1, \ldots, \mu_n\} \subseteq S_{C(S)^*}$. If for each $i = 1, \ldots, n$, $S(\mu_i)$ is finite and $Y = \bigcap_{i=1}^{n} ker(\mu_i)$, then $(C(S), B_Y, \mathcal{K}(C(S)))$ has property-(P_1).*

Proof. We apply techniques similar to those used in the proof of [34, Proposition 4.2]. We prove the result only for $n = 2$ because the same ideas work to prove the result for $n \neq 2$. Let $\{k_1, \ldots, k_m, t_1, \ldots, t_r\} \subseteq S$ and $\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_r\} \subseteq \mathbb{R}$ be such that $\mu_1 = \sum_{i=1}^{m} \alpha_i \delta_{k_i}$, $\mu_2 = \sum_{j=1}^{r} \beta_j \delta_{t_j}$, $Y = ker(\mu_1) \cap ker(\mu_2)$ and $F \in \mathcal{K}(C(S))$.

Case 1: $S(\mu_1) \cap S(\mu_2) = \emptyset$.

We define

$$A = \left\{(\gamma_1, \ldots, \gamma_m, \gamma'_1, \ldots, \gamma'_r) \in [-1,1]^{m+r} \colon \sum_{i=1}^{m} \alpha_i \gamma_i = 0 \text{ and } \sum_{j=1}^{r} \beta_j \gamma'_j = 0\right\} \tag{3.9}$$

and

$$\alpha = \inf\left\{\sup_{f \in F} \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq r}} \{|\gamma_i - f(k_i)|, |\gamma'_j - f(t_j)|\} \colon (\gamma_1, \ldots, \gamma_m, \gamma'_1, \ldots, \gamma'_r) \in A\right\}. \tag{3.10}$$

For each $f \in F$, the continuity of the map

$$(\gamma_1, \ldots, \gamma_m, \gamma'_1, \ldots, \gamma'_r) \mapsto \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq r}} \{|\gamma_i - f(k_i)|, |\gamma'_j - f(t_j)|\} \tag{3.11}$$

on $\mathbb{R}^{m+r}$ implies the lower semi-continuity of the map

$$(\gamma_1, \ldots, \gamma_m, \gamma'_1, \ldots, \gamma'_r) \mapsto \sup_{f \in F} \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq r}} \{|\gamma_i - f(k_i)|, |\gamma'_j - f(t_j)|\} \tag{3.12}$$

on $\mathbb{R}^{m+r}$. The set $A \subseteq \mathbb{R}^{m+r}$ is non-empty and compact and hence, the infimum in (3.10) is attained. Let $(\eta_1, \dots, \eta_m, \eta'_1, \dots, \eta'_r) \in A$ be such that

$$\alpha = \sup_{f \in F} \max_{\substack{1 \le i \le m \\ 1 \le j \le r}} \{|\eta_i - f(k_i)|, |\eta'_j - f(t_j)|\}. \tag{3.13}$$

Therefore, for each $f \in F$,

$$\begin{aligned} &-\alpha + \eta_i \le f(k_i) \le \alpha + \eta_i \text{ for } i = 1, \dots, m \text{ and} \\ &-\alpha + \eta'_j \le f(t_j) \le \alpha + \eta'_j \text{ for } j = 1, \dots, r. \end{aligned} \tag{3.14}$$

Let $R = rad_{B_Y}(F)$. It follows from the definition of α that $R \ge \alpha$. Therefore, from the inequalities in (3.14), it follows that for each $f \in F$,

$$\begin{aligned} &-R + \eta_i \le f(k_i) \le R + \eta_i \text{ for } i = 1, \dots, m \text{ and} \\ &-R + \eta'_j \le f(t_j) \le R + \eta'_j \text{ for } j = 1, \dots, r. \end{aligned} \tag{3.15}$$

Now, by [61, Theorem I.2.2], $cent_{C(S)}(F) = S_{rad_{C(S)}(F)}(F) \neq \emptyset$. Since $rad_{C(S)}(F) \le R$, $S_R(F) \neq \emptyset$. By [34, Proposition 4.2], Y is ball proximinal in $C(S)$. Therefore, for each $f \in F$, $B_{C(S)}[f, d(f, B_Y)] \cap B_Y \neq \emptyset$. It follows that for each $f \in F$, $B_{C(S)}[f, R] \cap B_{C(S)} \neq \emptyset$. Since $C(S)$ is an L_1-predual space and F is compact, by [45, Theorem 4.5, pg. 38], $S_R(F) \cap B_{C(S)} \neq \emptyset$. Let $g_0 \in S_R(F) \cap B_{C(S)}$. Then for each $f \in F$ and $t \in S$,

$$f(t) - R \le g_0(t) \le f(t) + R. \tag{3.16}$$

It follows that for each $t \in S$,

$$\sup_{f \in F} f(t) - R \le \inf_{f \in F} f(t) + R. \tag{3.17}$$

It also follows from (3.16) that for each $f \in F$ and $t \in S$,

$$-1 - R \le f(t) \le R + 1. \tag{3.18}$$

Now, choose $g \in B_{C(S)}$ such that $g(k_i) = \eta_i$, for $i = 1, \dots, m$ and $g(t_j) = \eta'_j$, for $j = 1, \dots, r$. Let $h_0 \colon S \to \mathbb{R}$ be defined as $h_0 = \min\{g, \inf_{f \in F} f + R\}$. The compactness of F ensures $h_0 \in C(S)$. Further, define $h \colon S \to \mathbb{R}$ as $h = \max\{h_0, \sup_{f \in F} f - R\}$. Then from the inequalities in (3.15), (3.17) and (3.18), it follows that $h \in B_{C(S)}$; $h(k_i) = \eta_i$, for $i = 1, \dots, m$; $h(t_j) = \eta'_j$, for $j = 1, \dots, r$ and for each $t \in S$, $\sup_{f \in F} f(t) - R \le h(t) \le \inf_{f \in F} f(t) + R$. Therefore, $h \in cent_{B_Y}(F)$.

Now, we prove that $(C(S), B_Y, \{F\})$ satisfies property-(P_1). Let $\varepsilon > 0$. Let $X = \mathbb{R}^{m+r}$, equipped with the supremum norm and

$$\tilde{F} = \{x_f = (f(k_1), \dots, f(k_m), f(t_1), \dots, f(t_r)) \in X \colon f \in F\} \in \mathcal{K}(X). \tag{3.19}$$

Subcase 1: $R = \alpha$.

Due to the compactness of the set A, $(X, A, \mathcal{CB}(X))$ has property-(P_1). Hence, there exists $0 < \delta < \varepsilon$ such that $cent_A(\tilde{F}, \delta) \subseteq cent_A(\tilde{F}) + \varepsilon B_X$.

Let $g \in cent_{B_Y}(F, \delta)$. Then $x_g = (g(k_1), \dots, g(k_m), g(t_1), \dots, g(t_r)) \in cent_A(\tilde{F}, \delta)$. Therefore, there exists $z = (z_1, \dots, z_m, z'_1, \dots, z'_r) \in cent_A(\tilde{F})$ such that $\|x_g - z\| \le \varepsilon$. Now, choose $g' \in B_{C(S)}$

such that $g'(k_i) = z_i$, for $i = 1, \ldots, m$ and $g'(t_j) = z'_j$, for $j = 1, \ldots, r$. Let $f_1 = \max\{\sup_{f\in F} f - R, g-\varepsilon, -1\}$ and $f_2 = \min\{\inf_{f\in F} f + R, g+\varepsilon, 1\}$. The compactness of F implies that $f_1, f_2 \in C(S)$. Moreover, $f_1 \leq g' \leq f_2$ on $\{k_1, \ldots, k_m, t_1, \ldots, t_r\}$. Let $h_1 = \max\{f_1, g'\}$ and $h_2 = \min\{h_1, f_2\}$. Clearly, $h_1, h_2 \in B_{C(S)}$. Since $r(g, F) \leq R + \delta < R + \varepsilon$. It follows that $\sup_{f\in F} f - R \leq g + \varepsilon$ and $g - \varepsilon \leq \inf_{f\in F} f + R$. From the inequalities in (3.18), it follows that $\sup_{f\in F} f - R \leq 1$ and $-1 \leq \inf_{f\in F} f + R$. Further, since $g \in B_Y$, $-1 \leq g \leq 1$ and hence, $g - \varepsilon \leq 1$. Therefore, $f_1 \leq f_2$ and $f_1 \leq h_1$. We can then conclude that $h_2 = g'$ on $\{k_1, \ldots, k_m, t_1, \ldots, t_r\}$ and $f_1 \leq h_2 \leq f_2$ on S. Therefore, $h_2 \in B_Y$, $\sup_{f\in F} f - R \leq h_2 \leq \inf_{f\in F} f + R$ and $g - \varepsilon \leq h_2 \leq g + \varepsilon$. This implies $h_2 \in cent_{B_Y}(F)$ and $\|g - h_2\| \leq \varepsilon$. Hence, $(C(S), B_Y, \{F\})$ satisfies property-(P_1).

SUBCASE 2: $R > \alpha$.

Let $\beta = R - \alpha$. By Lemma 3.3.4, there exists $0 < \delta < \varepsilon$ such that $cent_A(\tilde{F}, \beta+\delta) \subseteq cent_A(\tilde{F}, \beta) + \varepsilon B_X$.

Let $g \in cent_{B_Y}(F, \delta)$. Then $x_g = (g(k_1), \ldots, g(k_m), g(t_1), \ldots, g(t_r)) \in cent_A(\tilde{F}, \beta+\delta)$. Therefore, there exists $z = (z_1, \ldots, z_m, z'_1, \ldots, z'_r) \in cent_A(\tilde{F}, \beta)$ such that $\|x_g - z\| \leq \varepsilon$. Therefore, $r(z, \tilde{F}) \leq \alpha + \beta = R$. Now, choose $g' \in B_{C(S)}$ such that $g'(k_i) = z_i$ and $g'(t_j) = z'_j$, for $i = 1, \ldots, m$ and $j = 1, \ldots, r$. Then by following the same steps as in the last paragraph of SUBCASE 1, we can prove that $(C(S), B_Y, \{F\})$ satisfies property-(P_1).

CASE 2: $S(\mu_1) \cap S(\mu_2) \neq \emptyset$.

Without loss of generality and for simplicity, we assume that for each $i \in \mathbb{N}$ such that $1 \leq i \leq s \leq \min\{m, r\}$, $k_i = t_i$ and hence $S(\mu_1) \cap S(\mu_2) = \{k_1, \ldots, k_s\}$. We define

$$B = \{(\gamma_1, \ldots, \gamma_m, \gamma'_1, \ldots, \gamma'_r) \in [-1,1]^{m+r} : \gamma_i = \gamma'_i \text{ for } 1 \leq i \leq s; \sum_{i=1}^{m} \alpha_i \gamma_i = 0 \text{ and } \sum_{j=1}^{r} \beta_j \gamma'_j = 0\} \tag{3.20}$$

and

$$\alpha' = \inf \left\{ \sup_{f\in F} \max_{\substack{1\leq i\leq m \\ 1\leq j\leq r}} \{|\gamma_i - f(k_i)|, |\gamma'_j - f(t_j)|\} : (\gamma_1, \ldots, \gamma_m, \gamma'_1, \ldots, \gamma'_r) \in B \right\}. \tag{3.21}$$

Applying the same argument as in CASE 1, we can show that the infimum in (3.21) is attained say at $(\eta_1, \ldots, \eta_m, \eta'_1, \ldots, \eta'_r) \in B$. We further proceed the same way as in CASE 1 to first prove that $cent_{B_Y}(F) \neq \emptyset$ and then that $(C(S), B_Y, \{F\})$ satisfies property-(P_1). □

The following result provides a characterization for strongly proximinal finite co-dimensional subspaces of a $C(S)$ space. This result is important to characterize the finite co-dimensional strongly proximinal subspace of an L_1-predual space. This is due to the fact that the bidual of an L_1-predual space is of the type $C(S)$ for some compact Hausdorff space S (see [39, Chapter 7]).

Theorem 3.3.6 ([56, Theorem 3.6]). *Let S be a compact Hausdorff space and Y be a finite co-dimensional linear subspace of $C(S)$. Then the following statements are equivalent:*

(*i*) *Y is strongly proximinal in $C(S)$.*

(*ii*) *Y is strongly ball proximinal in $C(S)$.*

(*iii*) *$(C(S), Y, \mathcal{K}(C(S)))$ has property-(P_1).*

(iv) $(C(S), B_Y, \mathcal{K}(C(S)))$ *has property-*(P_1).

(v) $Y^\perp \subseteq \{\mu \in C(S)^* \colon \mu$ *is a SSD-point of* $C(S)^*\}$.

Proof. By [34, Theorem 4.3], $(i) \Leftrightarrow (ii) \Leftrightarrow (v)$. The implication $(v) \Rightarrow (iv)$ follows from [15, Theorem 2.1] and Theorem 3.3.5. Clearly, $(iii) \Rightarrow (i)$. Moreover, from Proposition 3.2.3, $(iv) \Rightarrow (iii)$. □

For a Choquet simplex K and a finite co-dimensional subspace Y of $A(K)$, the following result provides a sufficient condition for the triplet $(A(K), B_Y, \mathcal{K}(A(K)))$ to satisfy property-(P_1). We recall that for a Choquet simplex K, if $\mu \in A(K)^*$ then it means $\mu \in C(K)^*$ is a restriction map on $A(K)$. We also recall that a subset G of K is called a face of K if $\lambda x + (1-\lambda)y \in G$ whenever $x, y \in K$ and $0 < \lambda < 1$, then $x, y \in G$.

Theorem 3.3.7 ([56, Theorem 3.7])**.** *Let K be a Choquet simplex and $\{\mu_1, \ldots, \mu_n\} \subseteq S_{A(K)^*}$. If for each $i = 1, \ldots, n$, $S(\mu_i)$ is finite, $S(\mu_i) \subseteq ext(K)$ and $Y = \bigcap_{i=1}^n ker(\mu_i)$, then $(A(K), B_Y, \mathcal{K}(A(K)))$ has property-(P_1).*

Proof. We apply methods similar to those used in the proof of [34, Theorem 5.4]. We prove the result only for $n = 2$ because the same ideas work to prove the result for $n \neq 2$. Let $\{k_1, \ldots, k_m, t_1, \ldots, t_r\} \subseteq ext(K)$ and $\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_r\} \subseteq \mathbb{R}$ be such that $\mu_1 = \sum_{i=1}^m \alpha_i \delta_{k_i}$, $\mu_2 = \sum_{j=1}^r \beta_j \delta_{t_j}$ and $Y = ker(\mu_1) \cap ker(\mu_2)$. Let $F \in \mathcal{K}(A(K))$.

Case 1: $S(\mu_1) \cap S(\mu_2) = \emptyset$.

Let α, A be as defined in the proof of Case 1 of Theorem 3.3.5 and following the same argument as in that proof, let $(\eta_1, \ldots, \eta_m, \eta'_1, \ldots, \eta'_r) \in A$ be such that

$$\alpha = \sup_{f \in F} \max_{\substack{1 \le i \le m \\ 1 \le j \le r}} \{|\eta_i - f(k_i)|, |\eta'_j - f(t_j)|\}. \tag{3.22}$$

Let $R = rad_{B_Y}(F)$. Then from the definition of α it follows that $R \geq \alpha$ and hence for each $f \in F$,

$$\begin{aligned} -R + \eta_i &\le f(k_i) \le R + \eta_i \text{ for } i = 1, \ldots, m \text{ and} \\ -R + \eta'_j &\le f(t_j) \le R + \eta'_j \text{ for } j = 1, \ldots, r. \end{aligned} \tag{3.23}$$

Now, by Proposition 1.2.2, $cent_{A(K)}(F) = S_{rad_{A(K)}(F)}(F) \neq \emptyset$. Since $rad_{A(K)}(F) \le R$, $S_R(F) \neq \emptyset$. By [34, Theorem 5.4], Y is ball proximinal in $A(K)$. Therefore, for each $f \in F$, $B_{A(K)}[f, d(f, B_Y)] \cap B_Y \neq \emptyset$. For each $f \in F$, since $d(f, B_Y) \le R$, it follows that $B_{A(K)} \cap B_{A(K)}[f, R] \neq \emptyset$. Hence, by [45, Theorem 4.5, pg. 38], $B_{A(K)} \cap S_R(F) \neq \emptyset$. Let $g_0 \in B_{A(K)} \cap S_R(F)$. Then for each $f \in F$ and $t \in K$,

$$f(t) - R \le g_0(t) \le f(t) + R. \tag{3.24}$$

It follows that for each $t \in K$,

$$\sup_{f \in F} f(t) - R \le \inf_{f \in F} f(t) + R. \tag{3.25}$$

It also follows from (3.24) that for each $f \in F$ and $t \in K$,

$$-1 - R \le f(t) \le R + 1. \tag{3.26}$$

We choose $g \in B_{C(K)}$ such that $g(k_i) = \eta_i$, for $i = 1, \ldots, m$ and $g(t_j) = \eta'_j$, for $j = 1, \ldots, r$. Define $h_0 \colon K \to \mathbb{R}$ as follows: for each $t \in K$,

$$h_0(t) = \begin{cases} \inf_{f \in F} f(t) + R & \text{, if } g(t) \geq \inf_{f \in F} f(t) + R \\ g(t) & \text{, if } \sup_{f \in F} f(t) - R \leq g(t) \leq \inf_{f \in F} f(t) + R \\ \sup_{f \in F} f(t) - R & \text{, if } g(t) \leq \sup_{f \in F} f(t) - R. \end{cases} \tag{3.27}$$

The compactness of F and the inequalities in (3.26) ensure $h_0 \in B_{C(K)}$. By the definition of h_0, $\sup_{f \in F} f - R \leq h_0 \leq \inf_{f \in F} f + R$ on K. From the inequalities in (3.23), it follows that for $i = 1, \ldots, m$, $h_0(k_i) = \eta_i$ and for $j = 1, \ldots, r$, $h_0(t_j) = \eta'_j$. Hence, $\sum_{i=1}^{m} \alpha_i h_0(k_i) = 0 = \sum_{j=1}^{r} \beta_j h_0(t_j)$.

Now, by [1, Theorem II.3.12], there exists $h \in B_{A(K)}$ such that for each $i = 1, \ldots, m$ and $j = 1, \ldots, r$, $h(k_i) = h_0(k_i)$ and $h(t_j) = h_0(t_j)$. Let $G = conv(\{k_1, \ldots, k_m, t_1, \ldots, t_r\})$. Then G is a closed face of K. Further, for each $f \in F$, $f - R \leq h \leq f + R$ on G and hence, $\sup_{f \in F} f - R \leq h \leq \inf_{f \in F} f + R$ on G. Also, $-1 \leq h \leq 1$ on G. Therefore, from the inequalities in (3.26), it follows that

$$\max\left\{-1, \sup_{f \in F} f - R\right\} \leq h \leq \min\left\{1, \inf_{f \in F} f + R\right\} \text{ on } G \tag{3.28}$$

and

$$\max\left\{-1, \sup_{f \in F} f - R\right\} \leq \min\left\{1, \inf_{f \in F} f + R\right\} \text{ on } K. \tag{3.29}$$

Note that $\max\left\{-1, \sup_{f \in F} f - R\right\}$ and $-\min\left\{1, \inf_{f \in F} f + R\right\}$ are convex continuous functions on K. Therefore, by [7, Corollary 7.7, p. 73], there exists $\tilde{h} \in A(K)$ such that $\tilde{h} = h$ on G and $\max\left\{-1, \sup_{f \in F} f - R\right\} \leq \tilde{h} \leq \min\left\{1, \inf_{f \in F} f + R\right\}$ on K. It follows that $\tilde{h} \in cent_{B_Y}(F)$.

Now, we prove that $(A(K), B_Y, \{F\})$ satisfies property-(P_1). Let $\varepsilon > 0$. Let $X = \mathbb{R}^{m+r}$, equipped with the supremum norm and

$$\tilde{F} = \{x_f = (f(k_1), \ldots, f(k_m), f(t_1), \ldots, f(t_r)) \in X \colon f \in F\} \in \mathcal{K}(X). \tag{3.30}$$

Subcase 1: $R = \alpha$.

The set $A \subseteq X$ is compact and hence, $(X, A, \mathcal{CB}(X))$ has property-(P_1). Therefore, there exists $0 < \delta < \varepsilon$ such that $cent_A(\tilde{F}, \delta) \subseteq cent_A(\tilde{F}) + \varepsilon B_X$.

Let $g \in cent_{B_Y}(F, \delta)$. Then $x_g = (g(k_1), \ldots, g(k_m), g(t_1), \ldots, g(t_r)) \in cent_A(\tilde{F}, \delta)$. Therefore, there exists $z = (z_1, \ldots, z_m, z'_1, \ldots, z'_r) \in cent_A(\tilde{F})$ such that $\|x_g - z\| \leq \varepsilon$. Now, choose $g' \in B_{C(K)}$ such that $g'(k_i) = z_i$ and $g'(t_j) = z'_j$, for $i = 1, \ldots, m$ and $j = 1, \ldots, r$. Then by [1, Theorem II.3.12], there exists $h' \in B_{A(K)}$ such that $h'(k_i) = g'(k_i) = z_i$, for $i = 1, \ldots, m$ and $h'(t_j) = g'(t_j) = z'_j$, for $j = 1, \ldots, r$. Therefore, $\sum_{i=1}^{m} \alpha_i h'(k_i) = 0 = \sum_{j=1}^{r} \beta_j h'(t_j)$.

Let $G = conv(\{k_1, \ldots, k_m, t_1, \ldots, t_r\})$. Then G is a closed face of K. Clearly, for each $t \in G$, $\sup_{f \in F} f(t) - R \leq h'(t) \leq \inf_{f \in F} f(t) + R$, $g(t) - \varepsilon \leq h'(t) \leq g(t) + \varepsilon$ and $-1 \leq h'(t) \leq 1$. Since $r(g, F) \leq R + \delta < R + \varepsilon$, it follows that $\sup_{f \in F} f - R \leq g + \varepsilon$ on K and $g - \varepsilon \leq \inf_{f \in F} f + R$ on K. Since $g \in B_Y$, $-1 \leq g \leq 1$ and hence $g - \varepsilon \leq 1$ on K. Therefore,

$$\max\left\{\sup_{f \in F} f - R, g - \varepsilon, -1\right\} \leq h' \leq \min\left\{\inf_{f \in F} f + R, g + \varepsilon, 1\right\} \text{ on } G \tag{3.31}$$

and

$$\max\left\{\sup_{f\in F} f-R, g-\varepsilon, -1\right\} \leq \min\left\{\inf_{f\in F} f+R, g+\varepsilon, 1\right\} \text{ on } K. \tag{3.32}$$

Moreover, we note that $\max\{\sup_{f\in F} f-R, g-\varepsilon,-1\}$ and $-\min\{\inf_{f\in F} f+R, g+\varepsilon, 1\}$ are convex continuous functions on K. Therefore, by [7, Corollary 7.7, p. 73], there exists $h\in A(K)$ such that $h=h'$ on G and

$$\max\{\sup_{f\in F} f-R, g-\varepsilon,-1\} \leq h \leq \min\{\inf_{f\in F} f+R, g+\varepsilon, 1\} \text{ on } K. \tag{3.33}$$

It follows that $h\in cent_{B_Y}(F)$ such that $\|g-h\|\leq\varepsilon$. Hence, $(A(K), B_Y, \{F\})$ satisfies property-(P_1).

Subcase 2: $R>\alpha$.

Let $\beta=R-\alpha$. By Lemma 3.3.4, there exists $0<\delta<\varepsilon$ such that $cent_A(\tilde{F},\beta+\delta)\subseteq cent_A(\tilde{F},\beta)+\varepsilon B_X$.

Let $g\in cent_{B_Y}(F,\delta)$. Then $x_g=(g(k_1),\ldots,g(k_m),g(t_1),\ldots,g(t_r))\in cent_A(\tilde{F},\beta+\delta)$. Therefore, there exists $z=(z_1,\ldots,z_m,z'_1,\ldots,z'_r)\in cent_A(\tilde{F},\beta)$ such that $\|x_g-z\|\leq\varepsilon$. Therefore, $r(z,\tilde{F})\leq\alpha+\beta=R$. Now, choose $g'\in B_{C(K)}$ such that $g'(k_i)=z_i$ and $g'(t_j)=z'_j$, for $i=1,\ldots,m$ and $j=1,\ldots,r$. Therefore, by [1, Theorem II.3.12], there exists $h'\in B_{A(K)}$ such that $h'(k_i)=g'(k_i)=z_i$, for $i=1,\ldots,m$ and $h'(t_j)=g'(t_j)=z'_j$, for $j=1,\ldots,r$. Then by following the same steps as in the last paragraph of Subcase 1, we can prove that $(A(K),B_Y,\{F\})$ satisfies property-(P_1).

Case 2: $S(\mu_1)\cap S(\mu_2)\neq\emptyset$.

For simplicity and without loss of generality, we assume that for each $i\in\mathbb{N}$ such that $1\leq i\leq s\leq\min\{m,r\}$, $k_i=t_i$ and hence $S(\mu_1)\cap S(\mu_2)=\{k_1,\ldots,k_s\}$. Let B and α' be defined as in the proof of Case 2 of Theorem 3.3.5. We further proceed the same way as in Case 1 to prove that $(A(K),B_Y,\{F\})$ satisfies property-(P_1). □

The following characterization is an easy consequence of [34, Theorem 5.3], Theorem 3.3.7, Proposition 3.2.3 and [33, Theorem 2.6].

Theorem 3.3.8 ([56, Theorem 3.8]). *Let K be a Choquet simplex; $\{\mu_1,\ldots,\mu_n\}\subseteq A(K)^*$ be such that for each $i=1,\ldots,n$, $S(\mu_i)\subseteq ext(K)$ and $Y=\bigcap_{i=1}^n ker(\mu_i)$. Then the following statements are equivalent:*

(*i*) *Y is strongly proximinal in $A(K)$.*

(*ii*) *Y is strongly ball proximinal in $A(K)$.*

(*iii*) *$(A(K),Y,\mathcal{K}(A(K)))$ has property-(P_1).*

(*iv*) *$(A(K),B_Y,\mathcal{K}(A(K)))$ has property-(P_1).*

(*v*) *$Y^\perp\subseteq\{\mu\in A(K)^*\colon \mu$ is a SSD-point of $A(K)^*\}$.*

3.4 Characterizations of strongly proximinal finite co-dimensional subspaces of an L_1-predual space

In this section, we generalize the characterization in Theorem 3.3.6 for the strongly proximinal finite co-dimensional subspaces of an L_1-predual space. To this end, we need a few biduality results.

We first recall the extended version of principle of local reflexivity given by Behrends. We state it according to our purpose.

Theorem 3.4.1 ([10, Theorem 3.2])**.** *Let Y be a subspace of a Banach space X. Then for each finite dimensional subspaces $E \subseteq X^{**}$ and $F \subseteq X^*$ and $\varepsilon > 0$, there exists a bounded linear map $T\colon E \to X$ such that*

(i) $\|T\|, \|T^{-1}\| \leq 1 + \varepsilon$;

(ii) for each $x^{**} \in E$ *and* $x^* \in F$, $x^*(T(x^{**})) = x^{**}(x^*)$;

(iii) for each $x^{**} \in E \cap X$, $T(x^{**}) = x^{**}$;

(iv) for each $y^{**} \in E \cap Y^{\perp\perp}$, $T(y^{**}) \in Y$ *and*

(v) for each $y^* \in Y^\perp$ *and* $x^{**} \in E$ *such that* x^{**} *is weak*-continuous on* $Y^\perp$, $(T(x^{**}))(y^*) = x^{**}(y^*)$.

Lemma 3.4.2 ([56, Lemma 4.2])**.** *Let Y be a subspace of a Banach space X. Then for each $F \in \mathcal{K}(X)$, $rad_{B_{Y^{\perp\perp}}}(F) = rad_{B_Y}(F)$.*

Proof. First we prove the result for each set in $\mathcal{F}(X)$. Let $F = \{x_1, \ldots, x_n\} \in \mathcal{F}(X)$. Clearly, $rad_{B_{Y^{\perp\perp}}}(F) \leq rad_{B_Y}(F)$. Suppose $rad_{B_{Y^{\perp\perp}}}(F) < rad_{B_Y}(F)$. We choose $\varepsilon > 0$ and $\Phi \in B_{Y^{\perp\perp}}$ such that $r(\Phi, F) < rad_{B_Y}(F) - \varepsilon$. Now, choose $0 < \varepsilon' < \frac{\varepsilon}{1+r(\Phi,F)}$ and define $E = span(\{x_1, \ldots, x_n, \Phi\}) \subseteq X^{**}$. Then by Theorem 3.4.1, there exists a bounded linear map $T\colon E \to X$ such that $T(x_i) = x_i$, for each $i = 1, \ldots, n$; $T(\Phi) \in Y$ and $\|T\| \leq 1 + \varepsilon'$. Let $y = \frac{T(\Phi)}{1+\varepsilon'} \in B_Y$. Then for each $i = 1, \ldots, n$,

$$\begin{aligned} \|x_i - y\| &\leq \|T(x_i) - T(\Phi)\| + \left\| T(\Phi) - \frac{T(\Phi)}{1+\varepsilon'} \right\| \\ &\leq (1+\varepsilon')\|x_i - \Phi\| + \varepsilon' \\ &\leq r(\Phi, F) + \varepsilon'(1 + r(\Phi, F)) \\ &< r(\Phi, F) + \varepsilon. \end{aligned} \tag{3.34}$$

It follows that $r(y, F) \leq r(\Phi, F) + \varepsilon$. Now, from the inequalities $rad_{B_Y}(F) \leq r(y, F)$ and $r(\Phi, F) < rad_{B_Y}(F) - \varepsilon$, it follows $rad_{B_Y}(F) < rad_{B_Y}(F)$, which is a contradiction. Therefore, $rad_{B_{Y^{\perp\perp}}}(F) = rad_{B_Y}(F)$.

Now, for a set $F \in \mathcal{K}(X)$, it follows from Lemma 2.2.4; the fact that for each $\varepsilon > 0$, there exists a finite ε-net F_ε such that $d_H(F_\varepsilon, F) < \varepsilon$ and the first part of the proof that $rad_{B_{Y^{\perp\perp}}}(F) = rad_{B_Y}(F)$. □

Lemma 3.4.3 ([56, Lemma 4.3])**.** *Let Y be a subspace of a Banach space X. If $(X^{**}, B_{Y^{\perp\perp}}, \mathcal{K}(X))$ has property-(P_1) then for each $F \in \mathcal{K}(X)$ and $y \in Y$, $d(y, cent_{B_{Y^{\perp\perp}}}(F)) = d(y, cent_{B_Y}(F))$.*

Proof. Let $F \in \mathcal{K}(X)$ and $y \in Y$. We define $r = d(y, cent_{B_{Y^{\perp\perp}}}(F))$ and $r' = rad_{B_Y}(F)$. By Lemma 3.4.2, $r' = rad_{B_{Y^{\perp\perp}}}(F)$. Hence $cent_{B_Y}(F) \subseteq cent_{B_{Y^{\perp\perp}}}(F)$ and for each $\delta > 0$, $cent_{B_Y}(F, \delta) \subseteq cent_{B_{Y^{\perp\perp}}}(F, \delta)$. Therefore, $d(y, cent_{B_{Y^{\perp\perp}}}(F)) \leq d(y, cent_{B_Y}(F))$.

We now prove the reverse inequality. By our assumption, for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that whenever $v \in cent_{B_Y}(F, \delta_\varepsilon)$, we have $d(v, cent_{B_{Y^{\perp\perp}}}(F)) < \varepsilon$.

Now, let $\varepsilon > 0$ be fixed.

We choose $0 < \beta < \frac{\varepsilon}{3}$ and define $\delta = \delta_{\frac{\varepsilon}{2^2}}$. For each $m \in \mathbb{N}$, let $F_m \subseteq F$ be a finite $\frac{\delta}{2^{m+2}}$-net such that $F_m \subseteq F_{m+1}$ and define $r'_m = rad_{B_Y}(F_m)$. By Lemma 3.4.2, $r'_m = rad_{B_{Y^{\perp\perp}}}(F_m)$. Clearly, for each $m \in \mathbb{N}$, $r'_m \leq r'_{m+1}$. Further, by Lemma 3.3.4, for each $m \in \mathbb{N}$ and $\varepsilon' > 0$, there exists $0 < \gamma^m_{\varepsilon'} < \frac{\delta}{2}$ such that $d(v, cent_{B_{Y^{\perp\perp}}}(F_m, \sum_{k=1}^{m} \frac{\delta}{2^{k+1}})) < \varepsilon'$, whenever $v \in cent_{B_Y}(F_m, \sum_{k=1}^{m} \frac{\delta}{2^{k+1}} + \gamma^m_{\varepsilon'})$.

Now, since $cent_{B_{Y^{\perp\perp}}}(F_1, \frac{\delta}{2^2})$ is weak*-compact, it is proximinal and hence there exists $\Phi_0 \in cent_{B_{Y^{\perp\perp}}}(F_1, \frac{\delta}{2^2})$ such that $d(y, cent_{B_{Y^{\perp\perp}}}(F_1, \frac{\delta}{2^2})) = \|y - \Phi_0\|$. Define $r_0 = d(y, cent_{B_{Y^{\perp\perp}}}(F_1, \frac{\delta}{2^2}))$. It is easy to see that $cent_{B_{Y^{\perp\perp}}}(F) \subseteq cent_{B_{Y^{\perp\perp}}}(F_1, \frac{\delta}{2^2})$. Indeed, it follows from Lemma 2.2.4 that $r' \leq r'_1 + \frac{\delta}{2^3}$ and hence, for $\Phi' \in cent_{B_{Y^{\perp\perp}}}(F)$, $r(\Phi', F_1) \leq r(\Phi', F) = r' \leq r'_1 + \frac{\delta}{2^3} < r'_1 + \frac{\delta}{2^2}$. Therefore, it follows that $r_0 \leq r$.

We choose $0 < \varepsilon_1 < \min\left\{\frac{3\beta}{2^2(r_0+1)}, \frac{\gamma^1_{\frac{\beta}{2^2}}}{1+r'_1+\frac{\delta}{2^2}}\right\}$. Let $E_1 = span(F_1 \cup \{y, \Phi_0\}) \subseteq X^{**}$. Then by Theorem 3.4.1, there exists a bounded linear map $T_1 \colon E_1 \to X$ such that $T_1(x) = x$, for each $x \in F_1$; $T_1(y) = y$; $T_1(\Phi_0) \in Y$ and $\|T_1\| \leq 1 + \varepsilon_1$. Now, let $y_1 = \frac{T_1(\Phi_0)}{1+\varepsilon_1} \in B_Y$. Then

$$
\begin{aligned}
\|y - y_1\| &\leq \|T_1(y) - T_1(\Phi_0)\| + \left\|T_1(\Phi_0) - \frac{T_1(\Phi_0)}{1+\varepsilon_1}\right\| \\
&\leq (1+\varepsilon_1)r_0 + \varepsilon_1 \\
&\leq r + \varepsilon_1(1 + r_0) \\
&< r + \frac{3\beta}{2^2}.
\end{aligned}
\tag{3.35}
$$

Moreover, for each $x \in F_1$,

$$
\begin{aligned}
\|x - y_1\| &\leq \|T_1(x) - T_1(\Phi_0)\| + \left\|T_1(\Phi_0) - \frac{T_1(\Phi_0)}{1+\varepsilon_1}\right\| \\
&\leq (1+\varepsilon_1)r(\Phi_0, F_1) + \varepsilon_1 \\
&\leq r'_1 + \frac{\delta}{2^2} + \varepsilon_1\left(1 + r'_1 + \frac{\delta}{2^2}\right) \\
&< r'_1 + \frac{\delta}{2^2} + \gamma^1_{\frac{\beta}{2^2}}.
\end{aligned}
\tag{3.36}
$$

It follows that $r(y_1, F_1) \leq r'_1 + \frac{\delta}{2^2} + \gamma^1_{\frac{\beta}{2^2}}$. Thus, $y_1 \in cent_{B_Y}(F_1, \frac{\delta}{2^2} + \gamma^1_{\frac{\beta}{2^2}})$. This implies $d(y_1, cent_{B_{Y^{\perp\perp}}}(F_1, \frac{\delta}{2^2})) < \frac{\beta}{2^2}$. Now, let $\Phi_1 \in cent_{B_{Y^{\perp\perp}}}(F_1, \frac{\delta}{2^2})$ such that $\|y_1 - \Phi_1\| < \frac{\beta}{2^2}$.

We also make the following observation: Let $x \in F_2$. Then there exists $x_1 \in F_1$ such that $\|x - x_1\| < \frac{\delta}{2^3}$ and hence,

$$
\begin{aligned}
\|x - \Phi_1\| &\leq \|x - x_1\| + \|x_1 - \Phi_1\| \\
&< \frac{\delta}{2^3} + r(\Phi_1, F_1) \\
&\leq \frac{\delta}{2^3} + r'_1 + \frac{\delta}{2^2} \\
&\leq r'_2 + \frac{\delta}{2^2} + \frac{\delta}{2^3}.
\end{aligned}
\tag{3.37}
$$

It follows that $r(\Phi_1, F_2) \leq r'_2 + \frac{\delta}{2^2} + \frac{\delta}{2^3}$.

We choose $0 < \varepsilon_2 < \min\left\{\frac{\beta}{2^3(1+\frac{\beta}{2^2})}, \frac{\gamma^2_{\frac{\beta}{2^3}}}{1+r_2'+\frac{\delta}{2^2}+\frac{\delta}{2^3}}\right\}$. Let $E_2 = span(F_2 \cup \{\Phi_1, y_1\}) \subseteq X^{**}$. Then, applying Theorem 3.4.1 again, there exists a bounded linear map $T_2 \colon E_2 \to X$ such that $T_2(x) = x$, for each $x \in F_2$; $T_2(y_1) = y_1$; $T_2(\Phi_1) \in Y$ and $\|T_2\| \leq 1 + \varepsilon_2$. Now, let $y_2 = \frac{T_2(\Phi_1)}{1+\varepsilon_2} \in B_Y$. Then

$$\begin{aligned}\|y_1 - y_2\| &\leq \|T_2(y_1) - T_2(\Phi_1)\| + \left\|T_2(\Phi_1) - \frac{T_2(\Phi_1)}{1+\varepsilon_2}\right\| \\ &< (1+\varepsilon_2)\frac{\beta}{2^2} + \varepsilon_2 \\ &= \frac{\beta}{2^2} + \varepsilon_2\left(1 + \frac{\beta}{2^2}\right) \\ &< \frac{\beta}{2^2} + \frac{\beta}{2^3} = \frac{3\beta}{2^3}.\end{aligned} \tag{3.38}$$

Moreover, for each $x \in F_2$,

$$\begin{aligned}\|x - y_2\| &\leq \|T_2(x) - T_2(\Phi_1)\| + \left\|T_2(\Phi_1) - \frac{T_2(\Phi_1)}{1+\varepsilon_2}\right\| \\ &\leq (1+\varepsilon_2) r(\Phi_1, F_2) + \varepsilon_2 \\ &\leq r_2' + \frac{\delta}{2^2} + \frac{\delta}{2^3} + \varepsilon_2\left(1 + r_2' + \frac{\delta}{2^2} + \frac{\delta}{2^3}\right) \\ &< r_2' + \frac{\delta}{2^2} + \frac{\delta}{2^3} + \gamma^2_{\frac{\beta}{2^3}}.\end{aligned} \tag{3.39}$$

It follows that $r(y_2, F_2) \leq r_2' + \frac{\delta}{2^2} + \frac{\delta}{2^3} + \gamma^2_{\frac{\beta}{2^3}}$. Thus, $y_2 \in cent_{B_Y}(F_2, \frac{\delta}{2^2} + \frac{\delta}{2^3} + \gamma^2_{\frac{\beta}{2^3}})$. This implies $d(y_2, cent_{B_{Y^{\perp\perp}}}(F_2, \frac{\delta}{2^2} + \frac{\delta}{2^3})) < \frac{\beta}{2^3}$. Now, let $\Phi_2 \in cent_{B_{Y^{\perp\perp}}}(F_2, \frac{\delta}{2^2} + \frac{\delta}{2^3})$ such that $\|y_2 - \Phi_2\| < \frac{\beta}{2^3}$. Similar to the earlier observation, we can conclude that $r(\Phi_2, F_3) \leq r_3' + \frac{\delta}{2^2} + \frac{\delta}{2^3} + \frac{\delta}{2^4}$.

Proceeding inductively, we get a sequence $\{y_n\} \subseteq B_Y$ such that $\|y_n - y_{n+1}\| < \frac{3\beta}{2^{n+2}}$ and $r(y_n, F_n) \leq r_n' + \sum_{k=1}^{n} \frac{\delta}{2^{k+1}} + \gamma^n_{\frac{\beta}{2^{n+1}}} < r' + \sum_{k=1}^{n} \frac{\delta}{2^{k+1}} + \frac{\delta}{2}$. Clearly, $\{y_n\}$ is Cauchy in B_Y and hence, let $z_1 \in B_Y$ with $z_1 = \lim_{n\to\infty} y_n$. Then we have

$$\|y - z_1\| \leq r + \sum_{n=1}^{\infty} \frac{3\beta}{2^{n+1}} = r + \frac{3\beta}{2} < r + \frac{\varepsilon}{2}. \tag{3.40}$$

Now, let $\varepsilon' > 0$ and $x \in F$. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{\delta}{2^{n_0+2}} < \frac{\varepsilon'}{3}$, $\|y_{n_0} - z_1\| < \frac{\varepsilon'}{3}$ and $\sum_{k=1}^{n_0} \frac{\delta}{2^{k+1}} < \frac{\delta}{2} + \frac{\varepsilon'}{3}$ and $x_{n_0} \in F_{n_0}$ such that $\|x - x_{n_0}\| < \frac{\delta}{2^{n_0+2}}$. Therefore,

$$\begin{aligned}\|x - z_1\| &\leq \|x - x_{n_0}\| + \|x_{n_0} - y_{n_0}\| + \|y_{n_0} - z_1\| \\ &< \frac{\delta}{2^{n_0+2}} + r(y_{n_0}, F_{n_0}) + \frac{\varepsilon'}{3} \\ &< \frac{\varepsilon'}{3} + r' + \sum_{k=1}^{n_0} \frac{\delta}{2^{k+1}} + \frac{\delta}{2} + \frac{\varepsilon'}{3} \\ &< \frac{\varepsilon'}{3} + r' + \frac{\delta}{2} + \frac{\varepsilon'}{3} + \frac{\delta}{2} + \frac{\varepsilon'}{3} = r' + \delta + \varepsilon'.\end{aligned} \tag{3.41}$$

It follows that $r(z_1, F) \leq r' + \delta + \varepsilon'$. Since ε' is arbitrary, $r(z_1, F) \leq r' + \delta = r' + \delta_{\frac{\varepsilon}{2^2}}$.

Thus, $z_1 \in cent_{B_Y}(F, \delta_{\frac{\varepsilon}{2^2}})$ and hence, $d(z_1, cent_{B_{Y^{\perp\perp}}}(F)) < \frac{\varepsilon}{2^2}$. Now, for each $m \in \mathbb{N}$, choose a finite $\frac{\delta_{\varepsilon/2^3}}{2^{m+2}}$-net $G_m \subseteq F$ such that $G_m \subseteq G_{m+1}$. Therefore, there exists $\psi \in cent_{B_{Y^{\perp\perp}}}(G_1, \frac{\delta_{\varepsilon/2^3}}{2^2})$ such that $\|z_1 - \psi\| < \frac{\varepsilon}{2^2}$. Then by applying similar arguments as above, there exists an element $z_2 \in B_Y$ such that $\|z_1 - z_2\| < \frac{\varepsilon}{2^2}$ and $r(z_2, F) \leq r' + \delta_{\frac{\varepsilon}{2^3}}$.

We now proceed inductively and obtain a sequence $\{z_n\} \subseteq B_Y$ such that $\|z_n - z_{n+1}\| < \frac{\varepsilon}{2^{n+1}}$ and $r(z_n, F) \leq r' + \delta_{\frac{\varepsilon}{2^{n+1}}}$. Without loss of generality, we assume $\delta_{\frac{\varepsilon}{2^{n+1}}} \to 0$. Clearly, $\{z_n\}$ is Cauchy in B_Y and hence, let $z_0 \in B_Y$ such that $z_0 = \lim_{n\to\infty} z_n$. Let $x \in F$. Then $\|x - z_0\| = \lim_{n\to\infty} \|x - z_n\| \leq \lim_{n\to\infty} r(z_n, F) \leq r'$. It follows that $r(z_0, F) \leq r'$ and hence, $z_0 \in cent_{B_Y}(F)$. Also, $\|y - z_0\| \leq r + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = r + \varepsilon$. Therefore, $d(y, cent_{B_Y}(F)) \leq \|y - z_0\| \leq d(y, cent_{B_{Y^{\perp\perp}}}(F)) + \varepsilon$. Since ε is arbitrary, $d(y, cent_{B_Y}(F)) \leq d(y, cent_{B_{Y^{\perp\perp}}}(F))$. This proves the result. □

The next result connects property-(P_1) of the closed unit ball of a subspace of a Banach space with that of its bidual.

Proposition 3.4.4 ([56, Proposition 4.4])**.** *Let Y be a subspace of a Banach space X. If $(X^{**}, B_{Y^{\perp\perp}}, \mathcal{K}(X))$ has property-(P_1) then $(X, B_Y, \mathcal{K}(X))$ has property-(P_1).*

Proof. Let $F \in \mathcal{K}(X)$. It follows from the proof of Lemma 3.4.3 that $cent_{B_Y}(F) \neq \emptyset$. Now, let $\{y_n\}$ be a sequence in B_Y such that $r(y_n, F) \to rad_{B_Y}(F)$. By Lemma 3.4.2, $rad_{B_Y}(F) = rad_{B_{Y^{\perp\perp}}}(F)$. Therefore, $d(y_n, cent_{B_{Y^{\perp\perp}}}(F)) \to 0$. Hence, by Lemma 3.4.3, $d(y_n, cent_{B_Y}(F)) \to 0$. Therefore, $(X, B_Y, \{F\})$ satisfies property-(P_1). □

We now provide an instance where the converse of Proposition 3.4.4 holds true.

Proposition 3.4.5 ([56, Proposition 4.5])**.** *Let Y be a finite co-dimensional subspace of an L_1-predual space X. Then $(X^{**}, B_{Y^{\perp\perp}}, \mathcal{K}(X^{**}))$ has property-(P_1) if and only if $(X, B_Y, \mathcal{K}(X))$ has property-(P_1).*

Proof. Assume $(X, B_Y, \mathcal{K}(X))$ has property-(P_1). Then, in particular, Y is strongly ball proximinal in X. Therefore, by [35, Theorem 3.10], $Y^{\perp\perp}$ is a strongly proximinal finite co-dimensional subspace of X^{**}. By our assumption, X^{**} is isometric to a $C(S)$ space, for some compact Hausdorff space S. Therefore, by Theorem 3.3.6, it follows that $(X^{**}, B_{Y^{\perp\perp}}, \mathcal{K}(X^{**}))$ has property-(P_1).

The converse of the result follows from Proposition 3.4.4. □

Next, we demonstrate that property-(P_1) is stable through the weak*-dense subset X in X^{**} and hence, we generalize [33, Corollary 2.5].

Corollary 3.4.6 ([56, Corollary 4.6])**.** *Let X be an L_1-predual space and Z be a finite co-dimensional weak*-closed linear subspace of X^{**}. If $(X^{**}, B_Z, \mathcal{K}(X))$ has property-(P_1), then so does $(X^{**}, B_Z, \mathcal{K}(X^{**}))$.*

Proof. Since Z is a finite co-dimensional weak*-closed subspace of X^{**}, there exists a basis $\{x_1^*, \ldots, x_n^*\} \subseteq X^*$ for $Z^\perp$. Now, let $Y = \bigcap_{i=1}^n ker(x_i^*)$. Then $Y^{\perp\perp} = Z$. Hence, by Proposition 3.4.4, $(X, B_Y, \mathcal{K}(X))$ has property-(P_1). Therefore, the result follows from Proposition 3.4.5. □

We now prove the main result of this section.

Theorem 3.4.7 ([56, Theorem 4.7])**.** *Let Y be a finite co-dimensional subspace of an L_1-predual space X. Then the following statements are equivalent:*

(*i*) *Y is strongly proximinal in X.*

(ii) *Y is strongly ball proximinal in X.*

(iii) *$(X, Y, \mathcal{K}(X))$ has property-(P_1).*

(iv) *$(X, B_Y, \mathcal{K}(X))$ has property-(P_1).*

(v) *$Y^{\perp} \subseteq \{x^* \in X^* \colon x^*$ is a SSD-point of $X^*\}$.*

Proof. By [33, Theorem 2.6], $(i) \Leftrightarrow (ii) \Leftrightarrow (v)$. Obviously, $(iii) \Rightarrow (i)$ and from Proposition 3.2.3, $(iv) \Rightarrow (iii)$.

Now, we prove that $(ii) \Rightarrow (iv)$. Assume Y is strongly ball proximinal in X. Since $(ii) \Rightarrow (i)$, by [35, Theorem 3.10], $Y^{\perp\perp}$ is a strongly proximinal finite co-dimensional subspace of X^{**}. Now, by [45, Theorem 6.1], X^{**} is isometric to $C(S)$, for some compact Hausdorff space K. It follows from Theorem 3.3.6 that $(X^{**}, B_{Y^{\perp\perp}}, \mathcal{K}(X^{**}))$ has property-(P_1). Then, by Proposition 3.4.5, $(X, B_Y, \mathcal{K}(X))$ has property-(P_1). □

We conclude this section by providing characterizations for a strongly proximinal finite co-dimensional subspace of an L_1-predual space which are similar and in addition to those stated in [33, Corollary 2.7].

Corollary 3.4.8 ([56, Corollary 4.8]). *Let Y be a finite co-dimensional subspace of an L_1-predual space X. Then the following statements are equivalent*:

(i) *$(X, Y, \mathcal{K}(X))$ has property-(P_1).*

(ii) *$(X, B_Y, \mathcal{K}(X))$ has property-(P_1).*

(iii) *Y is the intersection of finitely many hyperplanes $Y_1, \ldots, Y_n$ such that for each $i = 1, \ldots, n$, $(X, Y_i, \mathcal{K}(X))$ has property-(P_1).*

(iv) *Y is the intersection of finitely many hyperplanes $Y_1, \ldots, Y_n$ such that for each $i = 1, \ldots, n$, $(X, B_{Y_i}, \mathcal{K}(X))$ has property-(P_1).*

Proof. It follows from Theorem 3.4.7 and [35, Corollary 3.21] that $(i) \Leftrightarrow (iii)$ and $(ii) \Leftrightarrow (iv)$. Clearly, $(i) \Leftrightarrow (ii)$ follows from Theorem 3.4.7. □

3.5 Some results on the biduality of the approximation properties

In this section, we prove a few biduality properties of property-(P_1) of triplets concerning the special sets defined in (1.4). We first recall some notations from [35].

Notation 3.5.1 ([35]). Let X be a Banach space and $\{x_1^*, \ldots, x_s^*\}$ be a set of linearly independent functionals in X^*. Let $M_1 = M_1^* = \|x_1^*\|$. Furthermore, by Hahn-Banach theorem, $J_{X^{**}}(x_1^*) = \{x^{**} \in B_{X^{**}} \colon x^{**}(x_1^*) = \|x_1^*\|\}$ is non-empty. Now, suppose for each $i \in \{2, \ldots, s\}$, $J_X(x_1^*, \ldots, x_{i-1}^*)$ is defined and is non-empty. Then we define

$$\begin{aligned} M_i &= \sup\{x_i^*(x) \colon x \in J_X(x_1^*, \ldots, x_{i-1}^*)\}, \\ M_i^* &= \sup\{x^{**}(x_i^*) \colon x^{**} \in J_{X^{**}}(x_1^*, \ldots, x_{i-1}^*)\}, \\ J_X(x_1^*, \ldots, x_i^*) &= \{x \in J_X(x_1^*, \ldots, x_{i-1}^*) \colon x_i^*(x) = M_i\}, \\ J_{X^{**}}(x_1^*, \ldots, x_i^*) &= \{x^{**} \in J_{X^{**}}(x_1^*, \ldots, x_{i-1}^*) \colon x^{**}(x_i^*) = M_i^*\}. \end{aligned} \tag{3.42}$$

For each $\varepsilon > 0$, we define $J_X(x_1^*, \varepsilon) = \{x \in B_X : x_1^*(x) > \|x_1^*\| - \varepsilon\}$. Now, for each $i = 2, \ldots, s$, we define

$$J_X(x_1^*, \ldots, x_i^*, \varepsilon) = \{x \in J_X(x_1^*, \ldots, x_{i-1}^*, \varepsilon) \colon x_i^*(x) > M_i - \varepsilon\}. \tag{3.43}$$

Remark 3.5.2. We recall and observe the following facts.

(i) For each $i = 1, \ldots, s$, $J_{X^{**}}(x_1^*, \ldots, x_i^*)$ is a non-empty weak*-compact subset of X^{**}. Indeed, it is true for $i = 1$ by Hahn-Banach theorem. Moroever, it follows from that $J_{X^{**}}(x_1^*)$ is a weak*-compact subset of X^{**}. Now consider $i = 2$. The evaluation functional $\delta_{x_2^*} \colon X^{**} \to \mathbb{R}$, defined as $\delta_{x_2^*}(x^{**}) = x^{**}(x_2^*)$ for each $x^{**} \in X^{**}$, is weak*-continuous on X^{**}. Since $J_{X^{**}}(x_1^*)$ is weak*-compact, we have that $J_{X^{**}}(x_1^*, x_2^*)$ is a weak*-compact subset of X^{**}. Therefore, $M_2^* = \sup\{\delta_{x_2^*}(x^{**}) \colon x^{**} \in J_{X^{**}}(x_1^*)\}$ is attained at some $x^{**} \in J_{X^{**}}(x_1^*)$, hence proving that $J_{X^{**}}(x_1^*, x_2^*) \neq \emptyset$. By using an induction argument, we can prove that for each $i > 2$, $J_{X^{**}}(x_1^*, \ldots, x_i^*)$ is a weak*-compact subset of X^{**} and $J_{X^{**}}(x_1^*, \ldots, x_i^*) \neq \emptyset$.

(ii) By [30, Theorem 1], if Y is a proximinal finite co-dimensional subspace of X, then for each basis $\{x_1^*, \ldots, x_s^*\}$ of $Y^\perp$ and for each $i = 1, \ldots, s$, $J_X(x_1^*, \ldots, x_i^*) \neq \emptyset$.

The following result provides a few relations satisfied by the notations defined in Notation 3.5.1.

Proposition 3.5.3 ([35, Proposition 3.4])**.** *Let Y be a strongly proximinal finite co-dimensional subspace of a Banach space X and let $\{x_1^*, \ldots, x_s^*\} \subseteq S_{Y^\perp}$ be a basis of $Y^\perp$. For each $i = 1, \ldots, s$, let M_i, M_i^*, $J_X(x_1^*, \ldots, x_i^*)$ and $J_{X^{**}}(x_1^*, \ldots, x_i^*)$ be defined as in Notation 3.5.1. Then for each $i = 1, \ldots, s$,*

(i) $M_i = M_i^*$ and

(ii) $J_{X^{**}}(x_1^*, \ldots, x_i^*)$ is the weak-closure of $J_X(x_1^*, \ldots, x_i^*)$.*

In this section, the following characterization of a strongly proximinal finite co-dimensional subspace is used to prove the subsequent results.

Theorem 3.5.4 ([23])**.** *Let Y be a proximinal finite co-dimensional subspace of a Banach space X. Then Y is strongly proximinal in X if and only if for each basis $\{x_1^*, \ldots, x_s^*\}$ of $Y^\perp$ and $i = 1, \ldots, s$,*

$$\lim_{\varepsilon \to 0}[\sup\{d(x, J_X(x_1^*, \ldots, x_i^*)) \colon x \in J_X(x_1^*, \ldots, x_i^*, \varepsilon)\}] = 0. \tag{3.44}$$

The following result generalizes the equality in [23, Remark 1.2 (1)].

Lemma 3.5.5. *Let X be a Banach space. Let $F \in \mathcal{K}(X)$ and $x^* \in S_{X^*}$. If x^* is a SSD-point of X^*, then $rad_{J_{X^{**}}(x^*)}(F) = rad_{J_X(x^*)}(F)$.*

Proof. We first prove the desired equality for each $F \in \mathcal{F}(X)$. Let $F = \{x_1, \ldots, x_n\} \subseteq X$. By Proposition 3.5.3, the weak* closure of $J_X(x^*)$ is $J_{X^{**}}(x^*)$. Therefore, clearly, $rad_{J_{X^{**}}(x^*)}(F) \leq rad_{J_X(x^*)}(F)$.

Suppose $rad_{J_{X^{**}}(x^*)}(F) < rad_{J_X(x^*)}(F)$. Then there exists $x^{**} \in J_{X^{**}}(x^*)$ and $\varepsilon > 0$ such that $r(x^{**}, F) < rad_{J_X(x^*)}(F) - \varepsilon$. Since x^* is a SSD-point of X^*, $ker(x^*)$ is strongly proximinal in X. Therefore, by Theorem 3.5.4, there exists $\delta > 0$ such that $d(x, J_X(x^*)) < \varepsilon/2$, whenever $x \in B_X$ and $x^*(x) > 1 - \delta$.

Let $0 < \varepsilon' < \min\left\{\frac{1}{1-\delta} - 1, \frac{\varepsilon}{2(r(x^{**},F)+1)}\right\}$. Moreover, we define $E = span(\{x_1, \ldots, x_n, x^{**}\}) \subseteq X^{**}$ and $Y = span(\{x^*\}) \subseteq X^*$. Then by Theorem 3.4.1, there exists a bounded linear map

$T\colon E \to X$ such that $T(x_i) = x_i$, for $i = 1, \ldots, n$; $\|T\| \le 1 + \varepsilon'$; $T(x^{**}) \in X$ and for each $y^* \in Y$, $y^*(T(x^{**})) = x^{**}(y^*)$. We define $x = \frac{T(x^{**})}{1+\varepsilon'} \in B_X$. Therefore

$$x^*(x) = x^*\left(\frac{T(x^{**})}{1+\varepsilon'}\right) = \frac{1}{1+\varepsilon'} x^{**}(x^*) = \frac{1}{1+\varepsilon'} > 1 - \delta. \tag{3.45}$$

Therefore, there exists $x_0 \in J_X(x^*)$ such that $\|x - x_0\| < \varepsilon/2$. Now, for each $i = 1, \ldots, n$,

$$\begin{aligned} \|x_0 - x_i\| &\le \|x_0 - x\| + \|x - x_i\| \\ &< \frac{\varepsilon}{2} + \|T(x^{**}) - T(x_i)\| + \left\| T(x^{**}) - \frac{T(x^{**})}{1+\varepsilon'} \right\| \\ &\le \frac{\varepsilon}{2} + \|T\|\|x^{**} - x_i\| + \varepsilon' \\ &\le \frac{\varepsilon}{2} + (1+\varepsilon')r(x^{**}, F) + \varepsilon' = r(x^{**}, F) + \frac{\varepsilon}{2} + \varepsilon'(r(x^{**}, F) + 1) \\ &< r(x^{**}, F) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r(x^{**}, F) + \varepsilon. \end{aligned} \tag{3.46}$$

It follows that

$$rad_{J_X(x^*)}(F) \le r(x_0, F) \le r(x^{**}, F) + \varepsilon < rad_{J_X(x^*)}(F). \tag{3.47}$$

This is a contradiction. Therefore, $rad_{J_X(x^*)}(F) = rad_{J_{X^{**}}(x^*)}(F)$.

Now, for a set $F \in \mathcal{K}(X)$, we can easily prove that $rad_{J_{X^{**}}(x^*)}(F) = rad_{J_X(x^*)}(F)$ by using Lemma 2.2.4; the fact that for each $\varepsilon > 0$, there exists a finite ε-net F_ε such that $d_H(F_\varepsilon, F) < \varepsilon$ and the argument in the first part of the proof. □

Lemma 3.5.6. *Let Y be a strongly proximinal finite co-dimensional subspace of a Banach space X. Let $\{x_1^*, \ldots, x_s^*\} \subseteq S_{Y^\perp}$ be a basis of $Y^\perp$. Then for each $F \in \mathcal{K}(X)$ and $i = 1, \ldots, s$, $rad_{J_{X^{**}}(x_1^*, \ldots, x_i^*)}(F) = rad_{J_X(x_1^*, \ldots, x_i^*)}(F)$.*

Proof. Let $i = 1$. Since $Y = ker(x_1^*)$ is strongly proximinal, x_1^* is an SSD-point of X^*. Hence, from Lemma 3.5.5, for each $F \in \mathcal{K}(X)$, $rad_{J_X(x_1^*)}(F) = rad_{J_{X^{**}}(x_1^*)}(F)$.

We now only prove for $i = 2$ since the same ideas work for $i > 2$. We consider first $F = \{x_1, \ldots, x_m\} \subseteq X$. By Proposition 3.5.3, $rad_{J_{X^{**}}(x_1^*, \ldots, x_i^*)}(F) \le rad_{J_X(x_1^*, \ldots, x_i^*)}(F)$. To prove the reverse inequality, we employ same proof techniques as in [35, Lemma 3.7].

Let $R = rad_{J_{X^{**}}(x_1^*, x_2^*)}(F)$. Since $J_{X^{**}}(x_1^*, x_2^*)$ is a weak*-compact set, $cent_{J_{X^{**}}(x_1^*, x_2^*)}(F) \ne \emptyset$. Let $\phi \in J_{X^{**}}(x_1^*, x_2^*)$ such that $r(\phi, F) = R$.

Since Y is strongly proximinal in X, by Theorem 3.5.4, for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $d(x, J_X(x_1^*, x_2^*)) < \varepsilon$, whenever $x \in J_X(x_1^*, x_2^*, \delta_\varepsilon)$.

Now let $\varepsilon' > 0$ such that $0 < \varepsilon' < \min\left\{\delta_{\varepsilon/2^2}, \frac{\varepsilon}{2(R+1)}\right\}$. Let $E = span(\{x_1, \ldots, x_m, \phi\}) \subseteq X^{**}$ and $F = span(\{x_1^*, x_2^*\}) \subseteq X^*$. Then by Theorem 3.4.1, there exists a bounded linear map $T\colon E \to X$ such that for $i = 1, \ldots, m$, $T(x_i) = x_i$; $\|T\| \le 1 + \varepsilon'$; $T(\phi) \in X$ and for $i = 1, 2$, $x_i^*(T(\phi)) = \phi(x_i^*)$.

Now, let $z_1 = \frac{T(\phi)}{1+\varepsilon'} \in B_X$. Then, for each $i = 1, \ldots, m$,

$$\begin{aligned} \|x_i - z_1\| &\le \|T(x_i) - T(\phi)\| + \left\| T(\phi) - \frac{T(\phi)}{1+\varepsilon'} \right\| \\ &\le (1+\varepsilon')r(\phi, F) + \varepsilon' \\ &= R + \varepsilon'(R+1) < R + \frac{\varepsilon}{2}. \end{aligned} \tag{3.48}$$

This implies that $r(z_1, F) \leq R + \frac{\varepsilon}{2}$. Further, for $i = 1, 2$, by Proposition 3.5.3, we have

$$x_i^*(z_1) = x_i^*\left(\frac{T(\phi)}{1+\varepsilon'}\right) = \frac{\phi(x_i^*)}{1+\varepsilon'} = \frac{M_i^*}{1+\varepsilon'} = \frac{M_i}{1+\varepsilon'} = M_i - \frac{M_i\varepsilon'}{1+\varepsilon'} > M_i - \varepsilon' > M_i - \delta_{\varepsilon/2^2}. \quad (3.49)$$

Thus, $z_1 \in J_X(x_1^*, x_2^*, \delta_{\varepsilon/2^2})$ and $d(z_1, J_{X^{**}}(x_1^*, x_2^*)) \leq d(z_1, J_X(x_1^*, x_2^*)) < \varepsilon/2^2$. Let $\phi_1 \in J_{X^{**}}(x_1^*, x_2^*)$ such that $\|z_1 - \phi_1\| < \frac{\varepsilon}{2^2}$. Then by Theorem 3.4.1, there exists $z_2 \in B_X$ such that $\|z_1 - z_2\| < \frac{\varepsilon}{2^2}$ and for $i = 1, 2$, $x_i^*(z_2) > M_i - \delta_{\varepsilon/2^3}$.

Proceeding inductively, we obtain a sequence $\{z_n\} \subseteq B_X$ such that $\|z_n - z_{n+1}\| < \frac{\varepsilon}{2^{n+1}}$ and for each $n = 1, 2, \ldots$ and $i = 1, 2$, $x_i^*(z_n) > M_i - \delta_{\frac{\varepsilon}{2^{n+1}}}$. Without loss of generality, assume that $\delta_{\frac{\varepsilon}{2^n}} \to 0$. Clearly, $\{z_n\}$ is a Cauchy sequence and there exists $z_\varepsilon \in B_X$ such that $z_\varepsilon = \lim_{n\to\infty} z_n$. Now, for $i = 1, 2$, $x_i^*(z_\varepsilon) = M_i$ and hence $z_\varepsilon \in J_X(x_1^*, x_2^*)$. Moreover, for each $i = 1, \ldots, m$ and $n \in \mathbb{N}$, $\|x_i - z_n\| \leq R + \frac{\varepsilon}{2} + \ldots + \frac{\varepsilon}{2^n}$. By taking limit as $n \to \infty$, it follows that for $i = 1, \ldots, m$, $\|x_i - z_\varepsilon\| \leq R + \varepsilon$. This implies $r(z_\varepsilon, F) \leq R + \varepsilon$. Since $\varepsilon > 0$ is arbitrary and $z_\varepsilon \in J_X(x_1^*, x_2^*)$,

$$rad_{J_X(x_1^*,x_2^*)}(F) \leq R = rad_{J_{X^{**}}(x_1^*,x_2^*)}(F). \quad (3.50)$$

Therefore, for each $F \in \mathcal{F}(X)$, $rad_{J_X(x_1^*,x_2^*)}(F) = rad_{J_{X^{**}}(x_1^*,x_2^*)}(F)$.

Further, for a set $F \in \mathcal{K}(X)$, by Lemma 2.2.4; the fact that for each $\varepsilon > 0$, there exists a finite ε-net F_ε such that $d_H(F_\varepsilon, F) < \varepsilon$ and the argument in the first part of the proof, we can conclude that $rad_{J_X(x_1^*,x_2^*)}(F) = rad_{J_{X^{**}}(x_1^*,x_2^*)}(F)$. □

Lemma 3.5.7. *Let Y be a strongly proximinal finite co-dimensional subspace of a Banach space X. Let $\{x_1^*, \ldots, x_s^*\} \subseteq S_{Y^\perp}$ be a basis of $Y^\perp$. Then for each $i = 1, \ldots, s$, $F \in \mathcal{K}(X)$ and $x \in X$, $d(x, cent_{J_{X^{**}}(x_1^*,\ldots,x_s^*)}(F)) = d(x, cent_{J_X(x_1^*,\ldots,x_s^*)}(F))$.*

Proof. By Remark 3.5.2 (ii), for each $i = 1, \ldots, s$, $J_X(x_1^*, \ldots, x_i^*) \neq \emptyset$. We prove the result only for $i = 2$ since the same ideas work for $i \neq 2$.

Let $F \in \mathcal{K}(X)$ and $x \in X$. We define $r = d(x, cent_{J_{X^{**}}(x_1^*,x_2^*)}(F))$. Now, by Lemma 3.5.6, $r' := rad_{J_{X^{**}}(x_1^*,x_2^*)}(F) = rad_{J_X(x_1^*,x_2^*)}(F)$. Therefore, $cent_{J_X(x_1^*,x_2^*)}(F) \subseteq cent_{J_{X^{**}}(x_1^*,x_2^*)}(F)$ and for each $\delta > 0$,

$$cent_{J_X(x_1^*,x_2^*)}(F, \delta) \subseteq cent_{J_{X^{**}}(x_1^*,x_2^*)}(F, \delta). \quad (3.51)$$

Hence,

$$d(x, cent_{J_{X^{**}}(x_1^*,x_2^*)}(F)) \leq d(x, cent_{J_X(x_1^*,x_2^*)}(F)). \quad (3.52)$$

We next prove the reverse inequality. By our assumption and (3.51), for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that whenever $z \in cent_{J_X(x_1^*,x_2^*)}(F, \delta_\varepsilon)$,

$$d(z, cent_{J_{X^{**}}(x_1^*,x_2^*)}(F)) < \varepsilon. \quad (3.53)$$

Now, let $\varepsilon > 0$ be fixed.

We choose $\beta > 0$ such that $\beta < \frac{\varepsilon}{3}$. We define $\delta = \delta_{\frac{\varepsilon}{2^2}}$. We also choose $\beta' > 0$ such that $\frac{3\beta}{2} + 2\beta' < \frac{\varepsilon}{2}$ and $2\beta' < \frac{\delta}{4}$. For each $m \in \mathbb{N}$, let $F_m \subseteq F$ be a finite $\frac{\delta}{2^{m+2}}$-net such that $F_m \subseteq F_{m+1}$. By Lemma 3.5.6, $r'_m := rad_{J_X(x_1^*,x_2^*)}(F_m) = rad_{J_{X^{**}}(x_1^*,x_2^*)}(F_m)$. Clearly, for each $m \in \mathbb{N}$, $r'_m \leq r'_{m+1}$.

Further, by Lemma 3.3.4, for each $m \in \mathbb{N}$ and $\varepsilon' > 0$, there exists $0 < \gamma^m_{\varepsilon'} < \frac{\delta}{4}$ such that whenever

$$z \in cent_{J_X(x_1^*, x_2^*)}\left(F_m, \sum_{k=1}^{m}\left(\frac{\delta}{2^{k+1}} + \frac{\beta'}{2^{k-1}}\right) + \gamma^m_{\varepsilon'}\right), \tag{3.54}$$

we have

$$d\left(z, cent_{J_{X^{**}}(x_1^*, x_2^*)}\left(F_m, \sum_{k=1}^{m}\left(\frac{\delta}{2^{k+1}} + \frac{\beta'}{2^{k-1}}\right)\right)\right) < \varepsilon'. \tag{3.55}$$

Since Y is strongly proximinal in X, by Theorem 3.5.4, for each $\varepsilon' > 0$, there exists $\theta_{\varepsilon'} > 0$ such that whenever $z \in J_X(x_1^*, x_2^*, \theta_{\varepsilon'})$, we have $d(z, J_X(x_1^*, x_2^*)) < \varepsilon'$.

Now, since $cent_{J_{X^{**}}(x_1^*, x_2^*)}(F_1, \frac{\delta}{2^2})$ is weak*-compact, it is proximinal and hence there exists $\Phi_0 \in cent_{J_{X^{**}}(x_1^*, x_2^*)}(F_1, \frac{\delta}{2^2})$ such that $d(x, cent_{J_{X^{**}}(x_1^*, x_2^*)}(F_1, \frac{\delta}{2^2})) = \|x - \Phi_0\|$. We define $r_0 = d(x, cent_{J_{X^{**}}(x_1^*, x_2^*)}(F_1, \frac{\delta}{2^2}))$. It is easy to see that $cent_{J_{X^{**}}(x_1^*, x_2^*)}(F) \subseteq cent_{J_{X^{**}}(x_1^*, x_2^*)}(F_1, \frac{\delta}{2^2})$. Indeed, it follows from Lemma 2.2.4 that $r' \leq r_1' + \frac{\delta}{2^3}$ and hence, for $\Phi' \in cent_{J_{X^{**}}(x_1^*, x_2^*)}(F)$, $r(\Phi', F_1) \leq r(\Phi', F) = r' \leq r_1' + \frac{\delta}{2^3} < r_1' + \frac{\delta}{2^2}$. Therefore, it follows that $r_0 \leq r$.

We choose $0 < \varepsilon_1 < \min\left\{\frac{3\beta}{2^2(r_0+1)}, \frac{\gamma^1_{\frac{\beta}{2^2}}}{1 + r_1' + \frac{\delta}{2^2}}, \theta_{\beta'}\right\}$. Let $E_1 = span(F_1 \cup \{x, \Phi_0\}) \subseteq X^{**}$ and $G = span(\{x_1^*, x_2^*\}) \subseteq X^*$. Then by Theorem 3.4.1, there exists a bounded linear map $T_1 \colon E_1 \to X$ such that for each $f \in F_1$, $T_1(f) = f$; $T_1(x) = x$; $T_1(\Phi_0) \in X$ and $\|T_1\| \leq 1 + \varepsilon_1$. Now, let $z_1 = \frac{T_1(\Phi_0)}{1+\varepsilon_1} \in B_X$. Now, for each $i = 1, 2$, by Proposition 3.5.3, we have

$$x_i^*\left(\frac{T(\Phi_0)}{1+\varepsilon_1}\right) = \frac{\Phi_0(x_i^*)}{1+\varepsilon_1} = \frac{M_i^*}{1+\varepsilon_1} = \frac{M_i}{1+\varepsilon_1} = M_i - \frac{M_i\varepsilon_1}{1+\varepsilon_1} > M_i - \varepsilon_1 > M_i - \theta_{\beta'}. \tag{3.56}$$

This implies that $z_1 \in J_X(x_1^*, x_2^*, \theta_{\beta'})$. Therefore there exists $y_1 \in J_X(x_1^*, x_2^*)$ such that $\|y_1 - z_1\| < \beta'$. Then

$$\begin{aligned}
\|x - y_1\| &\leq \|T_1(x) - T_1(\Phi_0)\| + \left\|T_1(\Phi_0) - \frac{T_1(\Phi_0)}{1+\varepsilon_1}\right\| + \|z_1 - y_1\| \\
&< (1+\varepsilon_1)r_0 + \varepsilon_1 + \beta' \\
&\leq r + \varepsilon_1(1 + r_0) + \beta' \\
&< r + \frac{3\beta}{2^2} + \beta'.
\end{aligned} \tag{3.57}$$

Moreover, for each $f \in F_1$,

$$\begin{aligned}
\|f - y_1\| &\leq \|T_1(f) - T_1(\Phi_0)\| + \left\|T_1(\Phi_0) - \frac{T_1(\Phi_0)}{1+\varepsilon_1}\right\| + \|z_1 - y_1\| \\
&< (1+\varepsilon_1)r(\Phi_0, F_1) + \varepsilon_1 + \beta' \\
&\leq r_1' + \frac{\delta}{2^2} + \varepsilon_1\left(1 + r_1' + \frac{\delta}{2^2}\right) + \beta' \\
&< r_1' + \frac{\delta}{2^2} + \beta' + \gamma^1_{\frac{\beta}{2^2}}.
\end{aligned} \tag{3.58}$$

It follows that $r(y_1, F_1) \leq r_1' + \frac{\delta}{2^2} + \beta' + \gamma^1_{\frac{\beta}{2^2}}$. Thus, $y_1 \in cent_{J_X(x_1^*, x_2^*)}(F_1, \frac{\delta}{2^2} + \beta' + \gamma^1_{\frac{\beta}{2^2}})$. This implies $d(y_1, cent_{J_{X^{**}}(x_1^*, x_2^*)}(F_1, \frac{\delta}{2^2} + \beta')) < \frac{\beta}{2^2}$. Now, let $\Phi_1 \in cent_{J_{X^{**}}(x_1^*, x_2^*)}(F_1, \frac{\delta}{2^2} + \beta')$ such that

$\|y_1 - \Phi_1\| < \frac{\beta}{2^2}$.

We also make the following observation: Let $f \in F_2$. Then there exists $f_1 \in F_1$ such that $\|f - f_1\| < \frac{\delta}{2^3}$ and hence

$$\begin{aligned} \|f - \Phi_1\| &\leq \|f - f_1\| + \|f_1 - \Phi_1\| \\ &< \frac{\delta}{2^3} + r(\Phi_1, F_1) \\ &\leq \frac{\delta}{2^3} + r_1' + \frac{\delta}{2^2} + \beta' \\ &\leq r_2' + \frac{\delta}{2^2} + \frac{\delta}{2^3} + \beta'. \end{aligned} \tag{3.59}$$

It follows that $r(\Phi_1, F_2) \leq r_2' + \frac{\delta}{2^2} + \frac{\delta}{2^3} + \beta'$.

We choose $0 < \varepsilon_2 < \min\left\{ \frac{\beta}{2^3(1+\frac{\beta}{2^2})}, \frac{\gamma^2_{\frac{\beta}{2^3}}}{1+r_2'+\frac{\delta}{2^2}+\frac{\delta}{2^3}}, \theta_{\frac{\beta'}{2}} \right\}$. Let $E_2 = span(F_2 \cup \{y_1, \Phi_1\}) \subseteq X^{**}$ and $G = span(\{x_1^*, x_2^*\}) \subseteq X^*$. Then, applying Theorem 3.4.1 again, there exists a bounded linear map $T_2 \colon E_2 \to X$ such that for each $f \in F_2$, $T_2(f) = f$; $T_2(y_1) = y_1$; $T_2(\Phi_1) \in X$ and $\|T_2\| \leq 1 + \varepsilon_2$. Now, let $z_2 = \frac{T_2(\Phi_1)}{1+\varepsilon_2} \in B_X$. Using the earlier argument, it is easy to conclude that $z_2 \in J_X(x_1^*, x_2^*, \theta_{\frac{\beta'}{2}})$. Therefore, there exists $y_2 \in J_X(x_1^*, x_2^*)$ such that $\|z_2 - y_2\| < \frac{\beta'}{2}$. Then

$$\begin{aligned} \|y_1 - y_2\| &\leq \|T_2(y_1) - T_2(\Phi_1)\| + \left\| T_2(\Phi_1) - \frac{T_2(\Phi_1)}{1+\varepsilon_2} \right\| + \|z_2 - y_2\| \\ &< (1+\varepsilon_2)\frac{\beta}{2^2} + \varepsilon_2 + \frac{\beta'}{2} \\ &= \frac{\beta}{2^2} + \varepsilon_2\left(1 + \frac{\beta}{2^2}\right) + \frac{\beta'}{2} \\ &< \frac{\beta}{2^2} + \frac{\beta}{2^3} + \frac{\beta'}{2} = \frac{3\beta}{2^3} + \frac{\beta'}{2}. \end{aligned} \tag{3.60}$$

Moreover, for each $f \in F_2$,

$$\begin{aligned} \|f - y_2\| &\leq \|T_2(f) - T_2(\Phi_1)\| + \left\| T_2(\Phi_1) - \frac{T_2(\Phi_1)}{1+\varepsilon_2} \right\| + \|z_2 - y_2\| \\ &< (1+\varepsilon_2) r(\Phi_1, F_2) + \varepsilon_2 + \frac{\beta'}{2} \\ &\leq r_2' + \frac{\delta}{2^2} + \frac{\delta}{2^3} + \beta' + \varepsilon_2\left(1 + r_2' + \frac{\delta}{2^2} + \frac{\delta}{2^3}\right) + \frac{\beta'}{2} \\ &< r_2' + \frac{\delta}{2^2} + \frac{\delta}{2^3} + \beta' + \frac{\beta'}{2} + \gamma^2_{\frac{\beta}{2^3}}. \end{aligned} \tag{3.61}$$

It follows that $r(y_2, F_2) \leq r_2' + \frac{\delta}{2^2} + \frac{\delta}{2^3} + \beta' + \frac{\beta'}{2} + \gamma^2_{\frac{\beta}{2^3}}$. Thus,

$$y_2 \in cent_{J_X(x_1^*, x_2^*)}\left(F_2, \frac{\delta}{2^2} + \frac{\delta}{2^3} + \beta' + \frac{\beta'}{2} + \gamma^2_{\frac{\beta}{2^3}}\right). \tag{3.62}$$

This implies

$$d\left(y_2, cent_{J_{X^{**}}(x_1^*, x_2^*)}\left(F_2, \frac{\delta}{2^2} + \frac{\delta}{2^3} + \beta' + \frac{\beta'}{2}\right)\right) < \frac{\beta}{2^3}. \tag{3.63}$$

Now, let $\Phi_2 \in cent_{J_{X^{**}}(x_1^*,x_2^*)}(F_2, \frac{\delta}{2^2} + \frac{\delta}{2^3} + \beta' + \frac{\beta'}{2})$ such that $\|y_2 - \Phi_2\| < \frac{\beta}{2^3}$. Similar to the earlier observation, we can conclude that $r(\Phi_2, F_3) \leq r'_3 + \frac{\delta}{2^2} + \frac{\delta}{2^3} + \frac{\delta}{2^4} + \beta' + \frac{\beta'}{2}$.

Proceeding inductively, we get a sequence $\{y_n\} \subseteq J_X(x_1^*, x_2^*)$ such that $\|y_n - y_{n+1}\| < \frac{3\beta}{2^{n+2}} + \frac{\beta'}{2^n}$ and $r(y_n, F_n) \leq r'_n + \sum_{k=1}^{n} \left(\frac{\delta}{2^{k+1}} + \frac{\beta'}{2^{k-1}}\right) + \gamma^n_{\frac{\beta}{2^{n+1}}} < r' + \sum_{k=1}^{n} \left(\frac{\delta}{2^{k+1}} + \frac{\beta'}{2^{k-1}}\right) + \frac{\delta}{4}$. Clearly, $\{y_n\}$ is Cauchy in $J_X(x_1^*, x_2^*)$ and hence, let $y_0^1 \in B_X$ with $y_0^1 = \lim_{n\to\infty} y_n$. Moreover, for each $i = 1, 2$, $x_i^*(y_0^1) = \lim_{n\to\infty} x_i^*(y_n) = \lim_{n\to\infty} M_i = M_i$ and hence $y_0^1 \in J_X(x_1^*, x_2^*)$. Then we have

$$\|x - y_0^1\| \leq r + \sum_{n=1}^{\infty} \frac{3\beta}{2^{n+1}} + \sum_{n=0}^{\infty} \frac{\beta'}{2^n} = r + \frac{3\beta}{2} + 2\beta' < r + \frac{\varepsilon}{2}. \tag{3.64}$$

Now, let $\varepsilon' > 0$ and $f \in F$. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{\delta}{2^{n_0+2}} < \frac{\varepsilon'}{3}$, $\|y_{n_0} - y_0^1\| < \frac{\varepsilon'}{3}$ and $\sum_{k=1}^{n_0} \left(\frac{\delta}{2^{k+1}} + \frac{\beta'}{2^{k-1}}\right) < \frac{\delta}{2} + 2\beta' + \frac{\varepsilon'}{3}$ and $f_{n_0} \in F_{n_0}$ such that $\|f - f_{n_0}\| < \frac{\delta}{2^{n_0+2}}$. Therefore,

$$\begin{aligned}
\|f - y_0^1\| &\leq \|f - f_{n_0}\| + \|f_{n_0} - y_{n_0}\| + \|y_{n_0} - y_0^1\| \\
&< \frac{\delta}{2^{n_0+2}} + r(y_{n_0}, F_{n_0}) + \frac{\varepsilon'}{3} \\
&< \frac{\varepsilon'}{3} + r' + \sum_{k=1}^{n_0} \left(\frac{\delta}{2^{k+1}} + \frac{\beta'}{2^{k-1}}\right) + \frac{\delta}{4} + \frac{\varepsilon'}{3} \\
&< \frac{\varepsilon'}{3} + r' + \frac{\delta}{2} + 2\beta' + \frac{\varepsilon'}{3} + \frac{\delta}{4} + \frac{\varepsilon'}{3} \\
&< \frac{\varepsilon'}{3} + r' + \frac{\delta}{2} + \frac{\delta}{4} + \frac{\varepsilon'}{3} + \frac{\delta}{4} + \frac{\varepsilon'}{3} = r' + \delta + \varepsilon'.
\end{aligned} \tag{3.65}$$

It follows that $r(y_0^1, F) \leq r' + \delta + \varepsilon'$. Since ε' is arbitrary, $r(y_0^1, F) \leq r' + \delta = r' + \delta_{\frac{\varepsilon}{2^2}}$.

Thus, $y_0^1 \in cent_{J_X(x_1^*,x_2^*)}(F, \delta_{\frac{\varepsilon}{2^2}})$ and hence, $d(y_0^1, cent_{J_{X^{**}}(x_1^*,x_2^*)}(F)) < \frac{\varepsilon}{2^2}$. Now, for each $m \in \mathbb{N}$, choose a finite $\frac{\delta_{\varepsilon/2^3}}{2^{m+2}}$-net $H_m \subseteq F$ such that $H_m \subseteq H_{m+1}$. Therefore, there exists $\psi \in cent_{J_{X^{**}}(x_1^*,x_2^*)}(H_1, \frac{\delta_{\varepsilon/2^3}}{2^2})$ such that $\|y_0^1 - \psi\| < \frac{\varepsilon}{2^2}$. Then by applying similar arguments as above, there exists an element $y_0^2 \in J_X(x_1^*, x_2^*)$ such that $\|y_0^1 - y_0^2\| < \frac{\varepsilon}{2^2}$ and $r(y_0^2, F) \leq r' + \delta_{\frac{\varepsilon}{2^3}}$.

We now proceed inductively and obtain a sequence $\{y_0^n\} \subseteq J_X(x_1^*, x_2^*)$ such that $\|y_0^n - y_0^{n+1}\| < \frac{\varepsilon}{2^{n+1}}$ and $r(y_0^n, F) \leq r' + \delta_{\frac{\varepsilon}{2^{n+1}}}$. Without loss of generality, we assume $\delta_{\frac{\varepsilon}{2^{n+1}}} \to 0$. Clearly, $\{y_0^n\}$ is Cauchy in $J_X(x_1^*, x_2^*)$ and hence, let $y_0 \in B_X$ such that $y_0 = \lim_{n\to\infty} y_0^n$. Moreover, for each $i = 1, 2$, $x_i^*(y_0) = \lim_{n\to\infty} x_i^*(y_0^n) = \lim_{n\to\infty} M_i = M_i$ and hence $y_0 \in J_X(x_1^*, x_2^*)$. Let $f \in F$. Then $\|f - y_0\| = \lim_{n\to\infty} \|f - y_0^n\| \leq \lim_{n\to\infty} r(y_0^n, F) \leq r'$. It follows that $r(y_0, F) \leq r'$ and hence, $y_0 \in cent_{J_X(x_1^*,x_2^*)}(F)$. Moreover, $\|x - y_0\| \leq r + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = r + \varepsilon$. Therefore, $d(x, cent_{J_X(x_1^*,x_2^*)}(F)) \leq \|x - y_0\| \leq d(x, cent_{J_{X^{**}}(x_1^*,x_2^*)}(F)) + \varepsilon$. Since ε is arbitrary, $d(x, cent_{J_X(x_1^*,x_2^*)}(F)) \leq d(x, cent_{J_{X^{**}}(x_1^*,x_2^*)}(F))$. This proves the result. □

We now prove the main result of this section.

Theorem 3.5.8. *Let Y be a strongly proximinal finite co-dimensional subspace of a Banach space X. Let $\{x_1^*, \ldots, x_s^*\} \subseteq S_{Y^\perp}$ be a basis of $Y^\perp$. Then for each $i = 1, \ldots, s$, if the triplet $(X^{**}, J_{X^{**}}(x_1^*, \ldots, x_i^*), \mathcal{K}(X))$ has property-(P_1) then $(X, J_X(x_1^*, \ldots, x_i^*), \mathcal{K}(X))$ has property-(P_1).*

Proof. Let $F \in \mathcal{K}(X)$ and $i \in \{1, \ldots, s\}$. It follows from the proof of Lemma 3.5.7 that $cent_{J_X(x_1^*,\ldots,x_i^*)}(F) \neq \emptyset$. Now, let $\{x_n\}$ be a sequence in $J_X(x_1^*, \ldots, x_i^*)$ such that $r(x_n, F) \to rad_{J_X(x_1^*,\ldots,x_i^*)}(F)$. By Lemma 3.5.6, $rad_{J_X(x_1^*,\ldots,x_i^*)}(F) = rad_{J_{X^{**}}(x_1^*,\ldots,x_i^*)}(F)$. Therefore,

$d(x_n, cent_{J_{X^{**}}(x_1^*,\ldots,x_i^*)}(F)) \to 0$. Hence, by Lemma 3.5.7, $d(x_n, cent_{J_X(x_1^*,\ldots,x_i^*)}(F)) \to 0$. Therefore, $(X, J_X(x_1^*, \ldots, x_i^*), \{F\})$ satisfies property-(P_1). $\square$

3.6 A subspace of a Banach space which satisfies $1\frac{1}{2}$-ball property and does not have r.c.p.

A. L. Garkavi provided an example in [19] of a hyperplane in a non-reflexive Banach space which is proximinal but does not admit restricted Chebyshev center for a two-point set, after a (equivalent) renorming, in the resulting renormed space. We can observe that this hyperplane satisfies $1\frac{1}{2}$-ball property in the renormed Banach space. This shows that $1\frac{1}{2}$-ball property and hence strong proximinality is not a sufficient condition for r.c.p.. We now briefly explain Garkavi's example and further, we prove that it satisfies $1\frac{1}{2}$-ball property for the sake of completeness.

Example 3.6.1 ([56, Example 5.1])**.** Let X be a non-reflexive Banach space. Let $x^* \in X^*\backslash\{0\}$. We define $Y = ker(x^*)$. Then Y is also non-reflexive and by James' theorem, there exists a linear functional $\Phi \in Y^*$ such that $\|\Phi\| = 1$ and Φ does not attain its norm on B_Y. We define $D = \{y \in B_Y : \Phi(y) \geq \frac{3}{4}\}$ and then choose a $\gamma > 0$ and $y_0 \in D$ such that $B_X[y_0, \gamma] \cap Y$ is contained in the interior of the set D, with respect to Y. Let $\alpha = \inf\{\Phi(y) : y \in B_X[y_0, \gamma] \cap Y\}$. Then $\frac{3}{4} \leq \alpha < 1$. Further, we define $U = \{y \in B_Y : |\Phi(y)| \leq \alpha\}$. Now, $U \cap B_X[y_0, \gamma] \cap Y = \emptyset$ because the infimum defining α is not attained on $B_X[y_0, \gamma] \cap Y$.

We fix $x_0 \in X\backslash Y$ such that $x^*(x_0) = 1$. We define $B_\gamma = B_X[0, \gamma] \cap Y$ and $V = x_0 + B_\gamma$. Let B denote the closure of the set $conv(U \cup V \cup -V)$. Then B is a closed bounded symmetric subset of X. Let X' denote the Banach space X, renormed to have B as the closed unit ball. Let the renorming be denoted by $\|.\|_B$. Then the new norm $\|.\|_B$ on X' is equivalent to the old one on X. It is proved in [20] that Y is proximinal in X' and $cent_Y(\{0, x_0 + y_0\}) = \emptyset$ in X'.

Let Y be a subspace of a Banach space X. For an element $x \in X$ and $\varepsilon = 0$, we define $P_Y(x, \varepsilon) = P_Y(x)$. We recall the following characterisation of $1\frac{1}{2}$-ball property, which is obtained as a consequence of [27, Remark 6, p. 50 and Corollary 4, p. 52].

Proposition 3.6.2 ([56, Proposition 5.2])**.** *Let Y be a subspace of a Banach space X. Then Y has $1\frac{1}{2}$-ball property in X if and only if Y is proximinal in X and for each $x \in X$ and $\varepsilon \geq 0$, $P_Y(x, \varepsilon) = \{y \in Y : d(y, P_Y(x)) \leq \varepsilon\}$.*

The next result is proved using ideas similar to that used in [8, Example 3.3].

Proposition 3.6.3 ([56, Proposition 5.3])**.** *Let Y be a closed hyperplane in a non-reflexive Banach space X and X' be the Banach space X with the renorming $\|.\|_B$ as defined in Example 3.6.1. Then Y satisfies $1\frac{1}{2}$-ball property in X'.*

Proof. Clearly, if $x \in X'$, then there exists $\lambda \in \mathbb{R}$ and $y \in Y$ such that $x = y + \lambda x_0$. Also, clearly, $P_Y(y + \lambda x_0) = y + \lambda P_Y(x_0)$ and $P_Y(y + \lambda x_0, \delta) = y + \lambda P_Y(x_0, \frac{\delta}{|\lambda|})$, for $\delta > 0$ and $\lambda \neq 0$. Therefore, applying Proposition 3.6.2 and by translation, it suffices to prove that for each $\varepsilon \geq 0$, $P_Y(x_0, \varepsilon) = \{y \in Y : d(y, P_Y(x_0)) \leq \varepsilon\}$. Now, $d(x_0, Y) = 1$ and $P_Y(x_0) = B_\gamma$. Let $\varepsilon \geq 0$. By [27, Remark 5, p. 50], we have $\{y \in Y : d(y, P_Y(x_0)) \leq \varepsilon\} \subseteq P_Y(x_0, \varepsilon)$. For $\varepsilon = 0$, it is trivial to see that $P_Y(x_0) \subseteq \{y \in Y : d(y, P_Y(x_0)) = 0\}$. Thus, it remains to show that for each $\varepsilon > 0$,

$P_Y(x_0,\varepsilon) \subseteq \{y \in Y : d(y, P_Y(x_0)) \leq \varepsilon\}$, or in other words, we prove that if $\varepsilon > 0$ and $y \in Y$ is such that $\|y - x_0\|_B \leq 1+\varepsilon$ then we have $d(y, B_\gamma) \leq \varepsilon$.

Let $y \in Y$ such that $\eta = \|y - x_0\|_B \leq 1+\varepsilon$. Without loss of generality, assume $\eta > 1$. Therefore, $\frac{y-x_0}{\eta} \in B$. Thus, there exists sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\nu_n\} \subseteq [0,1]$ such that for each n, $\alpha_n + \beta_n + \nu_n = 1$ and sequences $\{u_n\}$, $\{u'_n\} \subseteq B_\gamma$; $\{y_n\} \subseteq U$ such that

$$\frac{y-x_0}{\eta} = \lim_{n\to\infty} [\alpha_n u_n + \beta_n u'_n + \nu_n y_n + (\alpha_n - \beta_n)x_0]. \tag{3.66}$$

Without loss of generality, there exists $\alpha, \beta, \nu \in [0,1]$ such that $\alpha+\beta+\nu = 1$, $\alpha_n \to \alpha$, $\beta_n \to \beta$ and $\nu_n \to \nu$. Therefore, it follows that $\beta - \alpha = \frac{1}{\eta}$ and $y = \lim_{n\to\infty} \eta[\alpha u_n + \beta u'_n + \nu y_n]$. Now, $\frac{1}{\eta} \leq \alpha + \frac{1}{\eta} = \beta \leq 1$ and for each n, $\|u_n\|_B$, $\|u'_n\|_B$, $\|y_n\|_B \leq 1$. Therefore,

$$\begin{aligned} d(y, B_\gamma) &\leq \inf_n \|y - u'_n\|_B \\ &\leq \liminf_n \|\eta(\alpha u_n + \beta u'_n + \nu y_n) - u'_n\|_B \\ &= \liminf_n \|\eta\alpha u_n + (\eta\beta - 1)u'_n + \eta\nu y_n\|_B \\ &\leq \eta\alpha + (\eta\beta - 1) + \eta\nu = \eta - 1 \leq \varepsilon. \end{aligned} \tag{3.67}$$

□

Chapter 4

An algebraic characterization of closed linear sublattices of $C(S)$

4.1 Summary of results

In this chapter, we study the algebraic representation of the closed linear sublattices of the space $C(S)$, whenever S is a compact Hausdorff space, given by S. Kakutani [36].

In Section 4.2, we recall a few known facts and identify precisely the closed linear sublattices of the two-dimensional vector space, $\mathbb{R}^2$. Further, in Section 4.3, we present an elementary proof of Kakutani's representation of a closed linear sublattice and subalgebra of a $C(S)$ space, without using any deep results of lattice theory or functional analysis. As a consequence, in Corollary 4.3.2, we also provide an algebraic representation of the sublattices and subalgebras of the space $C_0(T)$, whenever T is a locally compact Hausdorff space.

4.2 Preliminaries

Let S be a compact Hausdorff space. It is well-known that $C(S)$ is a lattice under the operation of pointwise maximum or minimum of a pair of functions in $C(S)$. Moreover, the Banach space $C(S)$ is an algebra under the operation of pointwise multiplication of a pair of functions in $C(S)$. Kakutani presented an algebraic characterization of the closed linear sublattices of $C(S)$ as follows:

Theorem 4.2.1 ([36, Theorem 3, pg. 1005])**.** *Let S be a compact Hausdorff space. Let $\mathcal{A}$ be a closed linear subspace of $C(S)$. Then $\mathcal{A}$ is a sublattice of $C(S)$ if and only if there exists an index set I and co-ordinates $(t_i, s_i, \lambda_i) \in S \times S \times [0,1]$ for each $i \in I$ such that*

$$\mathcal{A} = \{f \in C(S) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}. \tag{4.1}$$

The following result gives us an explicit description of the sublattices in $\mathbb{R}^2$ under the lattice operation of co-ordinate-wise maximum or minimum of a pair of vectors in $\mathbb{R}^2$.

Lemma 4.2.2 ([57, Lemma 1.2])**.** *Consider $\mathbb{R}^2$ as a lattice under the operation of co-ordinate-wise maximum or minimum of a pair of vectors in $\mathbb{R}^2$. Then the only closed linear sublattices of $\mathbb{R}^2$ are*

$\{(0,0)\}$, $\mathbb{R}^2$ and span$(\{(a,b)\})$, for those $(a,b) \in \mathbb{R}^2$ satisfying $0 \le a, b \le 1$.

Proof. Let $\mathcal{A}$ be a closed linear subspace of $\mathbb{R}^2$. Now, the possible dimensions of $\mathcal{A}$ are $0, 1$ or 2. If the dimension of $\mathcal{A}$ is either 0 or 2, then $\mathcal{A} = \{(0,0)\}$ or $\mathcal{A} = \mathbb{R}^2$ respectively; obviously, these subspaces are sublattices of $\mathbb{R}^2$. Assume that $\mathcal{A}$ is a one-dimensional subspace of $\mathbb{R}^2$. We know that if $\mathcal{A} = span(\{(a,b)\})$ for some $(a,b) \in \mathbb{R}^2$, then $\mathcal{A} = span(\{(\lambda a, \lambda b)\})$ for each $\lambda \in \mathbb{R}\backslash\{0\}$. Therefore, without loss of generality, there exists $a, b \in [-1, 1]$ such that $\mathcal{A} = span(\{(a,b)\})$. It is easy to see that if $-1 \le a, b \le 0$ or $0 \le a, b \le 1$ then $\mathcal{A}$ is a sublattice of $\mathbb{R}^2$. Furthermore, if $-1 \le a < 0 < b \le 1$ then the minimum of (a,b) and $(2a, 2b)$ is $(2a, b) \notin \mathcal{A}$; hence $\mathcal{A}$ is not a sublattice of $\mathbb{R}^2$. Using a similar argument, if $-1 \le b < 0 < a \le 1$ then $\mathcal{A}$ is not a sublattice of $\mathbb{R}^2$. Therefore, without loss of generality, we get the desired conclusion. □

Similarly, the possible closed linear subalgebras of $\mathbb{R}^2$ are as follows:

Lemma 4.2.3 ([17, Lemma 4.46])**.** *Consider $\mathbb{R}^2$ as an algebra under co-ordinate-wise addition and multiplication of a pair of vectors in $\mathbb{R}^2$. Then the only subalgebras of $\mathbb{R}^2$ are $\{(0,0)\}$, $\mathbb{R}^2$, span$(\{(1,0)\})$, span$(\{(0,1)\})$ and span$(\{(1,1)\})$.*

We now recall the following interconnection between a subalgebra and a sublattice of $C(S)$.

Lemma 4.2.4 ([17, Lemma 4.48])**.** *Let S be a compact Hausdorff space. If $\mathcal{A}$ is a closed linear subalgebra of $C(S)$, then $\mathcal{A}$ is a sublattice of $C(S)$.*

For a closed linear sublattice $\mathcal{A}$ of $C(S)$, we also recall the following sufficient condition for a function in $C(S)$ to be in $\mathcal{A}$.

Lemma 4.2.5 ([17, Lemma 4.49])**.** *Let S be a compact Hausdorff space. Let $\mathcal{A}$ be a closed linear sublattice of $C(S)$ and $f \in C(S)$. If for every $x, y \in S$ there exists $g_{xy} \in \mathcal{A}$ such that $g_{xy}(x) = f(x)$ and $g_{xy}(y) = f(y)$ then $f \in \mathcal{A}$.*

4.3 Main results

We now prove our main result. This proof has appeared in [57, Section 2].

Proof of Theorem 4.2.1. We assume that S contains at least two points because if S is a singleton set then $C(S)$ is simply $\mathbb{R}$ and the only closed linear sublattices or subalgebras are $\{0\}$ and $\mathbb{R}$. If $\mathcal{A}$ has the description as in (4.1) then clearly $\mathcal{A}$ is a sublattice of $C(S)$.

Now, assume that $\mathcal{A}$ is a sublattice of $C(S)$. For every two distinct points $x, y \in S$, we define

$$\mathcal{A}_{xy} = \{(g(x), g(y)) \in \mathbb{R}^2 \colon g \in \mathcal{A}\}. \tag{4.2}$$

Since $\mathcal{A}$ is a lattice, $\mathcal{A}_{xy}$ is a sublattice of $\mathbb{R}^2$ (under the operation of co-ordinate-wise maximum of a pair of vectors in $\mathbb{R}^2$).

Case 1: Assume $\mathcal{A}$ separates the points of S.

If for every $x, y \in S$, $\mathcal{A}_{xy} = \mathbb{R}^2$ then by Lemma 4.2.5, $\mathcal{A} = C(S)$. Hence $\mathcal{A}$ has the description as in (4.1). Otherwise, there exists two distinct points $x_0, y_0 \in S$ such that $\mathcal{A}_{x_0 y_0}$ is a proper sublattice of $\mathbb{R}^2$. Consider the following collection:

$$I = \{(t, s, \lambda) \in S \times S \times [0,1] \colon f(t) = \lambda f(s), \text{ for each } f \in \mathcal{A}\}. \tag{4.3}$$

We now show that $I \neq \emptyset$. Since $\mathcal{A}$ separates the points of S, $\mathcal{A}_{x_0y_0}$ cannot be $\{(0,0)\}$ or $span(\{(1,1)\})$. Thus $\mathcal{A}_{x_0y_0} = span(\{(a,b)\})$ for some $0 \leq a, b \leq 1$ and $a \neq b$. If $a = 0$ and $b > 0$ then for each $g \in \mathcal{A}$, $g(x_0) = 0$ and hence $(x_0, y_0, 0) \in I$. If $a > 0$ and $b = 0$ then for each $g \in \mathcal{A}$, $g(y_0) = 0$ and hence $(y_0, x_0, 0) \in I$. If without loss of generality $0 < a < b$ then for each $g \in \mathcal{A}$, there exists $r_g \in \mathbb{R}$ such that $(g(x_0), g(y_0)) = (r_g a, r_g b)$. It follows that $g(x_0) = \frac{a}{b} g(y_0)$. Thus $(x_0, y_0, \frac{a}{b}) \in I$.

We index I by I itself, that is, each element of I is indexed by itself. Further, we define

$$\mathcal{A}' = \{f \in C(S) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}. \tag{4.4}$$

By the definition of I, it is clear that $\mathcal{A} \subseteq \mathcal{A}'$.

We next show that $\mathcal{A}' \subseteq \mathcal{A}$. Let $f \in \mathcal{A}'$. In order to show $f \in \mathcal{A}$, by Lemma 4.2.5, it suffices to show that for each $x, y \in S$, there exists $g_{xy} \in \mathcal{A}$ such that $g_{xy}(x) = f(x)$ and $g_{xy}(y) = f(y)$. Therefore, it suffices to show that for each $x, y \in S$, $(f(x), f(y)) \in \mathcal{A}_{xy}$.

Let $x, y \in S$. Since $\mathcal{A}$ separates the points of S, $\mathcal{A}_{xy}$ cannot be $\{(0,0)\}$ or $span(\{(1,1)\})$. We remark here that in this case, we use the assumption that $\mathcal{A}$ separates the points of S only to prove that $I \neq \emptyset$ and to rule out the above two possibilities of $\mathcal{A}_{xy}$.

If $\mathcal{A}_{xy} = \mathbb{R}^2$ then clearly $(f(x), f(y)) \in \mathbb{R}^2 = \mathcal{A}_{xy}$. If $\mathcal{A}_{xy} = span(\{(0,1)\})$ then $(0, f(y)) \in \mathcal{A}_{xy}$ and for each $g \in \mathcal{A}$, $g(x) = 0$. Thus $(x, y, 0) \in I$. Since $f \in \mathcal{A}'$, $f(x) = 0$. Hence $(f(x), f(y)) \in \mathcal{A}_{xy}$. Similar arguments hold if $\mathcal{A}_{xy} = span(\{(1,0)\})$.

Without loss of generality, let $0 < a < b < 1$. Consider $\mathcal{A}_{xy} = span(\{(a,b)\})$ then for each $g \in \mathcal{A}$, $g(x) = \frac{a}{b} g(y)$. Thus $(x, y, \frac{a}{b}) \in I$. Since $f \in \mathcal{A}'$, let $\frac{f(x)}{a} = \frac{f(y)}{b} = r$ (say). It follows that $(f(x), f(y)) \in span(\{(a,b)\}) = \mathcal{A}_{xy}$.

Case 2: Assume that $\mathcal{A}$ does not separate the points of S. Thus there exists two distinct points $x_0, y_0 \in S$ such that $f(x_0) = f(y_0)$, for each $f \in \mathcal{A}$. Consider the following collection:

$$I = \{(t, s, \lambda) \in S \times S \times [0,1] \colon f(t) = \lambda f(s), \text{ for each } f \in \mathcal{A}\}. \tag{4.5}$$

Since $(x_0, y_0, 1), (y_0, x_0, 1) \in I$, clearly $I \neq \emptyset$. We index I by I itself, that is, each element of I is indexed by itself. We define

$$\mathcal{A}' = \{f \in C(S) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}. \tag{4.6}$$

Clearly $\mathcal{A} \subseteq \mathcal{A}'$. In order to show $\mathcal{A}' \subseteq \mathcal{A}$, by Lemma 4.2.5 it suffices to show that for each $f \in \mathcal{A}'$ and each $x, y \in S$, $(f(x), f(y)) \in \mathcal{A}_{xy}$.

Let $x, y \in S$. We first consider the following possibilities of $\mathcal{A}_{xy}$: $\mathbb{R}^2$, $span(\{(0,1)\})$, $span(\{(1,0)\})$ and $span(\{(a,b)\})$ for those $a, b \in \mathbb{R}$ satisfying $0 < a < b < 1$ (without loss of generality). For each of the above possibilities, we apply arguments similar to that used in Case 1 to show that $(f(x), f(y)) \in \mathcal{A}_{xy}$.

In this case, due to our assumption that $\mathcal{A}$ does not separate the points of S, we need to consider the two possibilities which we rule out in Case 1. They are as follows: If $\mathcal{A}_{xy} = \{(0,0)\}$ then for each $g \in \mathcal{A}$, $g(x) = 0 = g(y)$. Hence, $(x, y, 0), (y, x, 0) \in I$. Since $f \in \mathcal{A}'$, $(f(x), f(y)) = (0,0) \in \mathcal{A}_{xy}$. If $\mathcal{A}_{xy} = span(\{(1,1)\})$ then for each $g \in \mathcal{A}$, $g(x) = g(y)$. Thus $(x, y, 1) \in I$. Since $f \in \mathcal{A}'$, $(f(x), f(y)) = (f(x), f(x)) \in \mathcal{A}_{xy}$. $\square$

Along similar lines, we obtain a representation of the closed linear subalgebras of the $C(S)$ space with the value of the coefficients λ_i being either 0 or 1 in (4.1). With the help of Lemmas 4.2.3, 4.2.4 and 4.2.5, we use a similar argument as in Theorem 4.2.1 to prove the following result. Hence we omit it.

Theorem 4.3.1 ([57, Theorem 2.1]). *Let S be a compact Hausdorff space. Let $\mathcal{A}$ be a closed linear subspace of $C(S)$. Then $\mathcal{A}$ is a subalgebra of $C(S)$ if and only if there exists an index set I and co-ordinates $(t_i, s_i, \lambda_i) \in S \times S \times \{0, 1\}$ for each $i \in I$ such that*

$$\mathcal{A} = \{f \in C(S)\colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}. \tag{4.7}$$

For a locally compact Hausdorff space T, we obtain similar characterizations for the sublattices and subalgebras of $C_0(T)$ by combining Theorems 4.2.1 and 4.3.1 and the fact that $C_0(T)$ is isometrically lattice and algebra isomorphic to a subalgebra of a $C(S)$ space for some compact Hausdorff space S. We state it formally as follows:

Corollary 4.3.2 ([57, Corollary 2.2]). *Let T be a locally compact Hausdorff space. Let $\mathcal{A}$ be a closed linear subspace of $C_0(T)$. Then*

(*i*) *$\mathcal{A}$ is a sublattice of $C_0(T)$ if and only if there exists an index set I and co-ordinates $(t_i, s_i, \lambda_i) \in T \times T \times [0, 1]$, for each $i \in I$ such that*

$$\mathcal{A} = \{f \in C_0(T)\colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}. \tag{4.8}$$

(*ii*) *$\mathcal{A}$ is a subalgebra of $C_0(T)$ if and only if there exists an index set I and co-ordinates $(t_i, s_i, \lambda_i) \in T \times T \times \{0, 1\}$, for each $i \in I$ such that*

$$\mathcal{A} = \{f \in C_0(T)\colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}. \tag{4.9}$$

Remark 4.3.3 ([57, Remark]). Kakutani provided a precise identification of a much general class of Banach spaces, namely abstract (M)-spaces (see [36, pg. 994] for the definition), with a subspace of $C(S)$ for some compact Hausdorff space S. Let S be a compact Hausdorff space and $\mathcal{A}$ be a closed subspace of $C(S)$. If $\mathcal{A}$ is an abstract (M)-space, then by [36, Theorem 1, pg. 998], there exists a compact Hausdorff space Ω such that $\mathcal{A}$ is isometric and lattice isomorphic to the subspace, $\{f \in C(\Omega)\colon f(t_i) = \lambda_i f(s_i), \text{for each } i \in I\}$, of $C(\Omega)$ for some index set I and co-ordinates $(t_i, s_i, \lambda_i) \in \Omega \times \Omega \times [0, 1]$ for each $i \in I$. However, it is not necessary that $\mathcal{A}$, being a subspace of $C(S)$, has a description in $C(S)$ as given in (4.1). For example, consider the Banach space $A([0, 1])$, which is a closed subspace of $C([0, 1])$ but not a sublattice of $C([0, 1])$. Therefore, by Theorem 4.2.1, $A([0, 1])$ does not have a description as in (4.1), for any given subfamily of co-ordinates in $[0, 1] \times [0, 1] \times [0, 1]$. Nevertheless, it is easy to see that $A([0, 1])$ is isometric and lattice isomorphic to $C(\{0, 1\})$ and hence is an abstract (M)-space.

Chapter 5

Semi-continuity properties of restricted Chebyshev-center maps of Banach spaces

5.1 Summary of results

As the title of this chapter suggests, we investigate the semi-continuity properties of the restricted Chebyshev-center maps of Banach spaces which stem from the concept of property-(P_1) in Banach spaces. We also discuss property-(P_1) in the Banach space c_0 (as a subspace of ℓ_∞) and its subspaces.

In Section 5.2, we prove that the triplet $(\ell_\infty, B_{c_0}, \mathcal{CB}(\ell_\infty))$ has property-(P_1). In fact, we improve a result by Amir [5] in Theorem 5.2.2. It is observed that for a topological space T and a uniformly convex Banach space X, the triplet $(C_b(T,X), B_{C_b(T,X)}, \mathcal{CB}(C_b(T,X)))$ satisfies property-(P_1).

In Section 5.3, we establish the stability of r.c.p. and property-(P_1) in the ℓ_∞-direct sum of two Banach spaces. As a consquence of the results in Sections 5.2 and 5.3, we prove that for a proximinal finite co-dimensional subspace Y of c_0, the triplet $(\ell_\infty, B_Y, \mathcal{CB}(\ell_\infty))$ satisfies property-(P_1).

In Section 5.4, we derive various stability results of the semi-continuity properties of the restricted Chebyshev-center maps of the ℓ_∞-direct sum of two Banach spaces. These results lead us to conclude that for a proximinal finite co-dimensional subspace Y of c_0, the map $cent_{B_Y}(.)$ is uniformly Hausdorff continuous on subfamilies of sets in $\mathcal{CB}(\ell_\infty)$ with equi-bounded restricted Chebyshev radii. We also establish that for a subspace Y of a Banach space X, if $(B_Y, \mathcal{CB}(X))$ has r.c.p. then the Hausdorff metric continuity of the map $cent_{B_Y}(.)$ on $\mathcal{CB}(X)$ implies that of the map $cent_Y(.)$ on $\mathcal{CB}(X)$.

In Section 5.5, we prove in Theorem 5.5.1 that for a Banach space X, an M-summand Y in X and a subspace Z of Y, if $(Y, Z, \mathcal{CB}(Y))$ has property-(P_1) then so does $(X, Z, \mathcal{CB}(X))$. We further positively answer Question 1.2.16 for an L_1-predual space. In fact, we establish in Proposition 5.5.2 that for an L_1-predual space X, a finite co-dimensional subspace Y of X and an M-ideal J in X such that $Y \subseteq J$, if Y is strongly proximinal in J then the triplet $(X, Y, \mathcal{K}(X))$ satisfies property-(P_1).

In the last Section 5.6, we mainly present the following observations: Let Y be an ideal in an L_1-predual space X. Then $(X, B_Y, \mathcal{CB}(X))$ has property-(P_1) if (i) for a compact Hausdorff space S, Y is a closed linear subalgebra of $C(S)$ and $X = C(S)^{**}$ and (ii) for a locally compact Hausdorff space T, Y is a closed linear subalgebra of $C_0(T)$ and $X = C_0(T)^{**}$.

5.2 Property-(P_1) in vector-valued continuous function spaces

In this section, we establish that for a compact Hausdorff space S, if $\mathcal{A}$ is a closed linear subalgebra of $C(S)$ then the triplet $(C(S)^{**}, B_{\mathcal{A}}, \mathcal{CB}(C(S)^{**}))$ satisfies property-(P_1). We first recall the following characterization of a uniformly convex Banach space.

Lemma 5.2.1 ([5, Lemma 1]). *Let X be a Banach space. Then X is uniformly convex if and only if for each $\varepsilon > 0$, there exists $\delta'(\varepsilon) > 0$ such that if $x, y \in X$ and $\Phi \in X^*$ such that $\|x\| = \|y\| = 1 = \|\Phi\| = \Phi(y)$ and $\Phi(x) > 1 - \delta'(\varepsilon)$, then $\|x - y\| < \varepsilon$. We can choose $\delta'(\varepsilon) \leq \frac{\varepsilon}{2}$.*

The following result is obtained through a few modifications in the proof of [5, Theorem 2]. For the sake of thoroughness, we present the modifications in its proof here.

Theorem 5.2.2 ([55, Theorem 2.3]). *Let T be a topological space and X be a uniformly convex Banach space. Then the triplet $(C_b(T,X), B_{C_b(T,X)}, \mathcal{CB}(C_b(T,X)))$ satisfies property-(P_1) and the map $cent_{B_{C_b(T,X)}}(.)$ is uniformly Hausdorff metric continuous on $\{B \in \mathcal{CB}(C_b(T,X))\colon rad_{B_{C_b(T,X)}}(B) \leq R\}$, for each $R > 0$.*

Proof. Let $B \in \mathcal{CB}(C_b(T,X))$. We define $R = rad_{B_{C_b(T,X)}}(B)$. We assume $R = 1$ and fix $\varepsilon > 0$. Then we obtain $\delta'(\varepsilon) > 0$ satisfying the condition in Lemma 5.2.1. Now there exists $f_0 \in B_{C_b(T,X)}$ such that $r(f_0, B) \leq 1 + \delta'(\varepsilon)$. We claim the following:

Claim: There exists $f_1 \in B_{C_b(T,X)}$ such that $r(f_1, B) \leq 1 + \delta'(\varepsilon/2)$ and $\|f_0 - f_1\| \leq 2\varepsilon$.

Indeed, there exists $g \in B_{C_b(T,X)}$ such that $r(g, B) \leq 1 + \delta'(\varepsilon/2)$. We now define $\alpha\colon T \to [0,1]$ and $f_1\colon T \to X$ as follows: for each $t \in T$,

$$\alpha(t) = \begin{cases} 1, & \text{if } \|g(t) - f_0(t)\| \leq 2\varepsilon; \\ \dfrac{2\varepsilon}{\|g(t) - f_0(t)\|}, & \text{if } \|g(t) - f_0(t)\| > 2\varepsilon. \end{cases} \tag{5.1}$$

and

$$f_1(t) = f_0(t) + \alpha(t)(g(t) - f_0(t)). \tag{5.2}$$

Clearly, $f_1 \in B_{C_b(T,X)}$ and $\|f_1 - f_0\| \leq 2\varepsilon$. For each $b \in B$, we now claim that for each $t \in T$,

$$\|f_1(t) - b(t)\| \leq 1 + \delta'(\varepsilon/2) \tag{5.3}$$

and hence $r(f_1, B) \leq 1 + \delta'(\varepsilon/2)$.

In order to prove the claim above, we apply the same arguments as in the proof of [5, Theorem 2]. Let $t \in T$. The inequality in (5.3) is true if $\alpha(t) = 1$ (since in this case $f_1(t) = g(t)$) or $\alpha(t) < 1$ and $\|g(t) - b(t)\| \geq \|f_0(t) - b(t)\|$ (since in this case $f_1(t)$ lies in the line segment joining $f_0(t)$ and $g(t)$). Therefore, we assume that $\alpha(t) < 1$ and $\|g(t) - b(t)\| < \|f_0(t) - b(t)\| \leq 1 + \delta'(\varepsilon)$. We denote $u = f_0(t) - b(t)$ and $v = g(t) - b(t)$. Thus $\|v\| \leq 1 + \delta'(\frac{\varepsilon}{2})$ and $\|v\| < \|u\| \leq 1 + \delta'(\varepsilon)$. We want to show that by moving a distance of 2ε from u towards v, we enter the ball $B_X[0, 1 + \delta'(\frac{\varepsilon}{2})]$. Since this is true if $\|v\| = 0$, it suffices to show for the case when $\|v\| = 1 + \delta'(\frac{\varepsilon}{2})$.

Consider the 2-dimensional space spanned by u and v. Let z be a point with $\|z\| = \|v\|$ on the same side of the line through 0 and u as v is such that the line passing through u and z supports the sphere $\|v\|S_X$. Extend this line to a hyperplane $H\colon = \psi^{-1}(1)$ supporting $B_X[0, \|v\|]$ in X. Then it is clear that $\|\psi\| = \frac{1}{\|v\|}$. Let $\phi = \|v\|\psi$, $x = \frac{u}{\|u\|}$ and $y = \frac{z}{\|z\|}$. Then $\|\phi\| = 1 = \phi(y) = \|y\| = \|x\|$ and $\phi(x) = \frac{\|v\|}{\|u\|} \geq \frac{1}{\|u\|} \geq \frac{1}{1+\delta'(\varepsilon)} > 1 - \delta'(\varepsilon)$. Hence, by Lemma 5.2.1, $\|x - y\| < \varepsilon$ and $\|u - z\| < \varepsilon + \|u - x\| + \|z - y\| \leq \varepsilon + \delta'(\varepsilon) + \delta'(\frac{\varepsilon}{2}) < 2\varepsilon$. This proves our claim since the distance from u to $B_X[0, \|v\|]$ in the direction of v is less than the maximum of the distances in the directions x (which is less than or equal to $\delta'(\varepsilon)$) and z (which is less than 2ε).

We now proceed inductively to find a sequence $\{f_n\} \subseteq B_{C_b(T,X)}$ such that for each $n = 1, 2, \ldots$, $\|f_{n+1} - f_n\| \leq 2\frac{\varepsilon}{2^n}$ and $r(f_{n+1}, B) \leq 1 + \delta'(\varepsilon/2^{n+1})$. Since $\{f_n\}$ is Cauchy, there exists $f \in B_{C_b(T,X)}$ such that $\lim_{n\to\infty} f_n = f$. Hence $\|f - f_0\| \leq 4\varepsilon$ and $r(f, B) \leq \lim_{n\to\infty} r(f_n, B) \leq 1$. Thus $f \in cent_{B_{C_b(T,X)}}(B)$. It also follows that $cent_{B_{C_b(T,X)}}(B, \delta'(\varepsilon)) \subseteq cent_{B_{C_b(T,X)}}(B) + 4\varepsilon B_X$. Hence $(C_b(T,X), B_{C_b(T,X)}, \{B\})$ has property-(P_1).

Now assume $0 < R \neq 1$. Then $\inf_{f\in B_{C_b(T,X)}} r\left(\frac{f}{R}, \frac{1}{R}B\right) = 1$. In the argument above, if f_0, g are chosen in $\frac{1}{R}B_{C_b(T,X)}$ then $f_1 \in \frac{1}{R}B_{C_b(T,X)}$. Hence replacing $B_{C_b(T,X)}$ and B by $\frac{1}{R}B_{C_b(T,X)}$ and $\frac{1}{R}B$ respectively in the argument above, we can conclude that $(C_b(T,X), \frac{1}{R}B_{C_b(T,X)}, \{\frac{1}{R}B\})$ has property-(P_1). Thus it follows from Proposition 3.2.3 (ii) that $(C_b(T,X), B_{C_b(T,X)}, \{B\})$ has property-(P_1).

In order to show that the map $cent_{B_{C_b(T,X)}}(.)$ is uniformly Hausdorff continuous on subfamilies of sets in $\mathcal{CB}(C_b(T,X))$ with equi-bounded restricted Chebyshev radii, we fix $\varepsilon, R > 0$. We now obtain a $\delta'(\varepsilon) > 0$ satisfying the condition in Lemma 5.2.1. Choose $0 < \delta < \frac{R\delta'(\varepsilon)}{2}$. Let $A, B \in \mathcal{CB}(C_b(T,X))$ such that $rad_{B_{C_b(T,X)}}(B), rad_{B_{C_b(T,X)}}(A) < R$ and $d_H(B,A) < \delta$. Then by Lemma 2.2.4, $|rad_{B_{C_b(T,X)}}(B) - rad_{B_{C_b(T,X)}}(A)| < \delta$. Let $f \in cent_{B_{C_b(T,X)}}(A)$. Thus

$$r(f, B) < rad_{B_{C_b(T,X)}}(A) + \delta < rad_{B_{C_b(T,X)}}(B) + 2\delta < (1 + \delta'(\varepsilon))R. \tag{5.4}$$

Now using the arguments above, we obtain $f_0 \in cent_{B_{C_b(T,X)}}(B)$ such that $\|f - f_0\| \leq 4\varepsilon R$. It follows that

$$cent_{B_{C_b(T,X)}}(A) \subseteq cent_{B_{C_b(T,X)}}(B) + 4\varepsilon R B_{C_b(T,X)}. \tag{5.5}$$

Similarly, we prove that

$$cent_{B_{C_b(T,X)}}(B) \subseteq cent_{B_{C_b(T,X)}}(A) + 4\varepsilon R B_{C_b(T,X)}. \tag{5.6}$$

Hence $d_H(cent_{B_{C_b(T,X)}}(B)), cent_{B_{C_b(T,X)}}(A)) \leq 4\varepsilon R$. ☐

Corollary 5.2.3 ([55, Corollary 2.4])**.** *Let S be a compact Hausdorff space and $\mathcal{A}$ be a closed linear subspace of $C(S)$ described as follows*:

$$\mathcal{A} = \{f \in C(S)\colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}, \tag{5.7}$$

for some index I and co-ordinates $(t_i, s_i, \lambda_i) \in S \times S \times \{-1, 0, 1\}$ for each $i \in I$. Then the triplet $(C(S), B_{\mathcal{A}}, \mathcal{CB}(C(S)))$ satisfies property-(P_1) and the map $cent_{B_{\mathcal{A}}}(.)$ is uniformly Hausdorff metric continuous on $\{B \in \mathcal{CB}(C(S))\colon rad_{B_{\mathcal{A}}}(B) \leq R\}$, for each $R > 0$.

Proof. In the proof of Theorem 5.2.2, if we choose f_0 and g in $B_{\mathcal{A}}$ then clearly $\|f_1\| \leq 1$ and from the description of $\mathcal{A}$, $f_1 \in B_{\mathcal{A}}$. Hence the result follows. $\square$

By using the representation given in Theorem 4.3.1 of closed linear subalgebras of $C(S)$ and applying Corollary 5.2.3, we obtain the following result.

Corollary 5.2.4 ([55, Corollary 2.5]). *Let S be a compact Hausdorff space and $\mathcal{A}$ be a closed linear subalgebra of $C(S)$. Then the triplet $(C(S), B_{\mathcal{A}}, \mathcal{CB}(C(S)))$ satisfies property-(P_1) and the map $cent_{B_{\mathcal{A}}}(.)$ is uniformly Hausdorff metric continuous on $\{B \in \mathcal{CB}(C(S)) \colon rad_{B_{\mathcal{A}}}(B) \leq R\}$, for each $R > 0$.*

The fact that c_0 is strongly proximinal in ℓ_∞ follows from the well-known fact that c_0 is an M-ideal in ℓ_∞. The next result follows from the fact that c_0 is a subalgebra in $\ell_\infty \cong C(\beta\mathbb{N})$ (here $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$) and Corollary 5.2.4.

Corollary 5.2.5 ([55, Corollary 2.7]). *The triplet $(\ell_\infty, B_{c_0}, \mathcal{CB}(\ell_\infty))$ has property-(P_1) and the map $cent_{B_{c_0}}(.)$ is uniformly Hausdorff metric continuous on $\{B \in \mathcal{CB}(\ell_\infty) \colon rad_{B_{c_0}}(B) \leq R\}$, for each $R > 0$.*

5.3 Stability of property-(P_1) in ℓ_∞-direct sums

We first establish some notations which are used in the present and subsequent sections.

Notation 5.3.1 ([55, Section 3]). Let X_1 and X_2 be two Banach spaces. Then the ℓ_∞-direct sum of X_1 and X_2, given as

$$X \coloneqq X_1 \oplus_\infty X_2 = \{(x_1, x_2) \in X_1 \times X_2 \colon x_1 \in X_1 \text{ and } x_2 \in X_2\}, \tag{5.8}$$

is again a Banach space equipped with the maximum norm defined as follows: for each $x = (x_1, x_2) \in X$, $\|x\| = \max\{\|x_1\|, \|x_2\|\}$.

For each $B \in \mathcal{CB}(X)$, we denote

$$\begin{aligned} B(1) &= \{b_1 \in X_1 \colon \text{there exists } b \in B \text{ and } b_2 \in X_2 \text{ such that } b = (b_1, b_2)\} \\ \text{and } B(2) &= \{b_2 \in X_2 \colon \text{there exists } b \in B \text{ and } b_1 \in X_1 \text{ such that } b = (b_1, b_2)\}. \end{aligned} \tag{5.9}$$

For each $B \in \mathcal{CB}(X)$ and $i \in \{1, 2\}$, we also denote

$$r_i(B) = rad_{V_i}(B(i)). \tag{5.10}$$

Remark 5.3.2 ([55, Section 3]). Let X_1 and X_2 be two Banach spaces. Let $X = X_1 \oplus_\infty X_2$ and $B \in \mathcal{CB}(X)$.

(i) For each $i \in \{1, 2\}$, if $V_i \in \mathcal{CV}(X_i)$ then $V_1 \times V_2 \in \mathcal{CV}(X)$.

(ii) For each $i \in \{1, 2\}$, $B(i) \in \mathcal{CB}(X_i)$ and $B \subseteq B(1) \times B(2)$.

The following result provides a formula for the restricted Chebyshev radius of a closed bounded subset of an ℓ_∞-direct sum.

Proposition 5.3.3 ([55, Proposition 3.1]). *For each $i \in \{1, 2\}$, let X_i be a Banach space and $V_i \in \mathcal{CV}(X_i)$. Let $X = X_1 \oplus_\infty X_2$, $V = V_1 \times V_2$ and $B \in \mathcal{CB}(X)$. Then $rad_V(B) = \max\{r_1(B), r_2(B)\}$ and for each $v = (v_1, v_2) \in V$, $r(v, B) = \max\{r(v_1, B(1)), r(v_2, B(2))\}$.*

Proof. Let $v = (v_1, v_2) \in V$ and for each $i \in \{1, 2\}$, $b_i \in B(i)$. Then there exists $b, b' \in B$ and for each $i \in \{1, 2\}$, $b'_i \in X_i$ such that $b = (b_1, b'_2)$ and $b' = (b'_1, b_2)$. Thus

$$\|b_1 - v_1\| \leq \|b - v\| \leq r(v, B) \text{ and } \|b_2 - v_2\| \leq \|b' - v\| \leq r(v, B). \tag{5.11}$$

It follows that for each $i \in \{1, 2\}$, $r(v_i, B(i)) \leq r(v, B)$. From here, it is easy to conclude that $\max\{r_1(B), r_2(B)\} \leq rad_V(B)$.

Conversely, for each $\varepsilon > 0$ and $i \in \{1, 2\}$, there exists $v_i \in V_i$ such that $r(v_i, B(i)) < r_i(B) + \varepsilon$. Let $v = (v_1, v_2) \in V$ and $b = (b_1, b_2) \in B$. Then for each $i \in \{1, 2\}$, $b_i \in B(i)$ and hence

$$\|v - b\| = \max\{\|v_1 - b_1\|, \|v_2 - b_2\|\} \leq \max\{r_1(B), r_2(B)\} + \varepsilon. \tag{5.12}$$

It follows that $rad_V(B) \leq \max\{r_1(B), r_2(B)\} + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain the desired equality.

It follows from the arguments above that for each $v = (v_1, v_2) \in V$, $r(v, B) = \max\{r(v_1, B(1)), r(v_2, B(2))\}$. □

We now prove the stability of r.c.p. under the ℓ_∞-direct sum in the following result.

Proposition 5.3.4 ([55, Proposition 3.2])**.** *For each $i \in \{1, 2\}$, let X_i be a Banach space and $V_i \in \mathcal{CV}(X_i)$. Let $X = X_1 \oplus_\infty X_2$ and $V = V_1 \times V_2$. If for each $i \in \{1, 2\}$, $(V_i, \mathcal{CB}(X_i))$ has r.c.p. then $(V, \mathcal{CB}(X))$ has r.c.p..*

Proof. Assume that for each $i \in \{1, 2\}$, $(V_i, \mathcal{CB}(X_i))$ has r.c.p.. Let $B \in \mathcal{CB}(X)$. For each $i \in \{1, 2\}$, since $B(i) \in \mathcal{CB}(X_i)$, let $v_i \in cent_{V_i}(B(i))$. We define $v = (v_1, v_2) \in V$. Let $b = (b_1, b_2) \in B$. Then for each $i \in \{1, 2\}$, $b_i \in B(i)$ and by Proposition 5.3.3,

$$\begin{aligned} \|v - b\| &= \max\{\|v_1 - b_1\|, \|v_2 - b_2\|\} \\ &\leq \max\{r(v_1, B(1)), r(v_2, B(2))\} \\ &= \max\{r_1(B), r_2(B)\} \\ &= rad_V(B). \end{aligned} \tag{5.13}$$

It follows that $r(v, B) \leq rad_V(B)$ and hence $v \in cent_V(B)$. □

Remark 5.3.5 ([55, Remark 3.3]). For each $i \in \{1, 2\}$, let X_i be a Banach space and $V_i \in \mathcal{CV}(X_i)$. For each $i \in \{1, 2\}$, suppose $(V_i, \mathcal{CB}(X_i))$ has r.c.p.. Let $X = X_1 \oplus_\infty X_2$ and $V = V_1 \times V_2$. It is easy to verify the following facts.

(i) For each $B \in \mathcal{CB}(X)$,

$$cent_V(B) = \begin{cases} cent_{V_1}(B(1)) \times cent_{V_2}(B(2)), & \text{if } r_1(B) = r_2(B); \\ \bigcap\limits_{b_1 \in B(1)} B_{X_1}[b_1, r_2(B)] \cap V_1 \times cent_{V_2}(B(2)), & \text{if } r_1(B) < r_2(B); \\ cent_{V_1}(B(1)) \times \bigcap\limits_{b_2 \in B(2)} B_{X_2}[b_2, r_1(B)] \cap V_2, & \text{if } r_2(B) < r_1(B). \end{cases} \tag{5.14}$$

We note here that in each case above,

$$cent_V(B) \supseteq cent_{V_1}(B(1)) \times cent_{V_2}(B(2)). \tag{5.15}$$

(ii) For each $B, A \in \mathcal{CB}(X)$,

$$\max\{d_H(B(1), A(1)), d_H(B(2), A(2))\} \leq d_H(B, A). \tag{5.16}$$

We need the following important auxiliary result, which is a generalization of [31, Fact 3.2].

Lemma 5.3.6 ([55, Lemma 3.4])**.** *Let X be a Banach space, $V \in \mathcal{CV}(X)$ and $\mathscr{F} \subseteq \mathcal{CB}(X)$ such that $(V, \mathscr{F})$ has r.c.p.. Let $F \in \mathscr{F}$ and $\alpha > rad_V(F)$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $F' \in \mathscr{F}$ with $d_H(F, F') < \delta$ and scalar β such that $|\alpha - \beta| < \delta$, we have*

$$d_H\left(\bigcap_{z\in F} B_X[z, \alpha] \cap V, \bigcap_{z'\in F'} B_X[z', \beta] \cap V\right) < \varepsilon.$$

Proof. We define $R_F = rad_V(F)$. We fix $\varepsilon > 0$. Further, we define $2\gamma = \alpha - R_F$, $L = \alpha + R_F + 2$ and $\delta = \min\{1, \frac{\gamma}{2}, \frac{\gamma\varepsilon}{2L}\}$.

Let $F' \in \mathscr{F}$ be such that $d_H(F, F') < \delta$ and β be a scalar such that $|\alpha - \beta| < \delta$. For simplicity, let $R_{F'} = rad_V(F')$. Then, from Lemma 2.2.4, we have $|R_F - R_{F'}| < \delta$. Moreover,

$$\beta - R_{F'} = \beta - \alpha + R_F - R_{F'} + \alpha - R_F > 2\gamma - 2\delta \geq 2\gamma - \gamma = \gamma. \tag{5.17}$$

Now, let $v \in \bigcap_{z\in F} B_X[z, \alpha] \cap V$ and $z' \in F'$. Then there exists $z \in F$ such that $\|z - z'\| < \delta$. Therefore, we have

$$\|z' - v\| \leq \|z' - z\| + \|z - v\| < \delta + \alpha < \beta + 2\delta. \tag{5.18}$$

Thus for each $z' \in F'$, $\|z' - v\| < \beta + 2\delta$. Let $v_0 \in cent_V(F')$ and $\lambda = \frac{\beta - R_{F'}}{\beta - R_{F'} + 2\delta}$. We define $v' = \lambda v + (1-\lambda)v_0$. Then $v' \in V$ and for each $z' \in F'$, we have

$$\begin{aligned}\|z' - v'\| &\leq \lambda\|z' - v\| + (1-\lambda)\|z' - v_0\| \\ &< \lambda(\beta + 2\delta) + (1-\lambda)R_{F'} = \lambda(\beta - R_{F'} + 2\delta) + R_{F'} = \beta.\end{aligned} \tag{5.19}$$

It follows that $v' \in \bigcap_{z'\in F'} B_X[z', \beta] \cap V$. Further, let $z' \in F'$. Since $d_H(F, F') < \delta$, there exists $z \in F$ such that $\|z - z'\| < \delta$. Now,

$$\begin{aligned}\|v - v'\| = (1-\lambda)\|v - v_0\| &= \frac{2\delta}{\beta - R_{F'} + 2\delta}\|v - v_0\| \\ &< \frac{2\delta}{\gamma}(\|v - z\| + \|z - z'\| + \|z' - v_0\|) \\ &< \frac{2\delta}{\gamma}(\alpha + \delta + r(v_0, F')) = \frac{2\delta}{\gamma}(\alpha + \delta + R_{F'}) \\ &< \frac{2\delta}{\gamma}(\alpha + R_F + 2\delta) < \frac{2\delta L}{\gamma} \leq \varepsilon.\end{aligned} \tag{5.20}$$

Therefore, $v' \in \bigcap_{z'\in F'} B_X[z', \beta] \cap V$ such that $\|v - v'\| < \varepsilon$.

Similarly, for each $w' \in \bigcap_{z' \in F'} B_X[z', \beta] \cap V$, we obtain $w \in \bigcap_{z \in F} B_X[z, \alpha] \cap V$ such that $\|w - w'\| < \varepsilon$. This completes the proof. $\square$

We next prove the stability of property-(P_1) under ℓ_∞-direct sums in the following result.

Theorem 5.3.7 ([55, Theorem 3.5])**.** *For each $i \in \{1,2\}$, let X_i be a Banach space and $V_i \in \mathcal{CV}(X_i)$. Let $X = X_1 \oplus_\infty X_2$ and $V = V_1 \times V_2$. If for each $i \in \{1,2\}$, $(X_i, V_i, \mathcal{CB}(X_i))$ has property-(P_1) then $(X, V, \mathcal{CB}(X))$ has property-(P_1).*

Proof. Firstly we observe that by Proposition 5.3.4, $(V, \mathcal{CB}(X))$ has r.c.p.. Now let $B \in \mathcal{CB}(X)$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that for each $i \in \{1,2\}$,

$$\mathit{cent}_{V_i}(B(i), \delta) \subseteq \mathit{cent}_{V_i}(B(i)) + \varepsilon B_{X_i}. \tag{5.21}$$

Case 1: $r_1(B) = r_2(B)$

By Remark 5.3.5 (i), $\mathit{cent}_V(B) = \mathit{cent}_{V_1}(B(1)) \times \mathit{cent}_{V_2}(B(2))$. Similarly, for each $\gamma > 0$, we also have $\mathit{cent}_V(B, \gamma) = \mathit{cent}_{V_1}(B(1), \gamma) \times \mathit{cent}_{V_2}(B(2), \gamma)$. Therefore, it follows from (5.21) that

$$\mathit{cent}_V(B, \delta) \subseteq \mathit{cent}_V(B) + \varepsilon B_X. \tag{5.22}$$

Case 2: $r_1(B) \neq r_2(B)$

Without loss of generality, assume $r_1(B) < r_2(B)$, since the same arguments work for the reverse inequality. By Remark 5.3.5 (i), $\mathit{cent}_V(B) = \bigcap_{b_1 \in B(1)} B_{X_1}[b_1, r_2(B)] \cap V_1 \times \mathit{cent}_{V_2}(B(2))$. Similarly, for each $\gamma > 0$, we also have

$$\mathit{cent}_V(B, \gamma) = \bigcap_{b_1 \in B(1)} B_{X_1}[b_1, r_2(B) + \gamma] \cap V_1 \times \mathit{cent}_{V_2}(B(2), \gamma). \tag{5.23}$$

Let $\varepsilon > 0$. Now replacing X, V and B by X_1, V_1 and $B(1)$ respectively in Lemma 5.3.6, we obtain $\delta' > 0$ such that for each scalar β with $|\alpha - \beta| < 2\delta'$, we have

$$d_H\left(\bigcap_{b_1 \in B(1)} B_{X_1}[b_1, \alpha] \cap V_1, \bigcap_{b_1 \in B(1)} B_{X_1}[b_1, \beta] \cap V_1\right) < \varepsilon. \tag{5.24}$$

Choose $\delta_0 = \min\{\delta, \delta'\}$. We take $\alpha = r_2(B)$ and $\beta = r_2(B) + \delta_0$. Then we have

$$d_H\left(\bigcap_{b_1 \in B(1)} B_{X_1}[b_1, r_2(B)] \cap V_1, \bigcap_{b_1 \in B(1)} B_{X_1}[b_1, r_2(B) + \delta_0] \cap V_1\right) < \varepsilon. \tag{5.25}$$

Let $v = (v_1, v_2) \in \mathit{cent}_V(B, \delta_0)$ such that $v_1 \in \bigcap_{b_1 \in B(1)} B_{X_1}[b_1, r_2(B) + \delta_0] \cap V_1$ and $v_2 \in \mathit{cent}_{V_2}(B(2), \delta_0)$. Then by (5.25) and (5.21), there exists $w_1 \in \bigcap_{b_1 \in B(1)} B_{X_1}[b_1, r_2(B)] \cap V_1$ and $w_2 \in \mathit{cent}_{V_2}(B(2))$ such that for each $i \in \{1,2\}$, $\|w_i - v_i\| < \varepsilon$. Hence $(w_1, w_2) \in \mathit{cent}_V(B)$ and $\|v - (w_1, w_2)\| < \varepsilon$. $\square$

One of the instances where the converses of Proposition 5.3.4 and Theorem 5.3.7 hold true is as follows:

Proposition 5.3.8 ([55, Proposition 3.6]). *For each $i \in \{1,2\}$, let X_i be a Banach space and Y_i be a non-trivial subspace of X_i. Let $X = X_1 \oplus_\infty X_2$ and $Y = Y_1 \oplus_\infty Y_2$. Then*

(*i*) *If $(Y, \mathcal{CB}(X))$ has r.c.p. then for each $i \in \{1,2\}$, $(Y_i, \mathcal{CB}(X_i))$ has r.c.p..*

(*ii*) *If $(X, Y, \mathcal{CB}(X))$ has property-(P_1) then for each $i \in \{1,2\}$, $(X_i, Y_i, \mathcal{CB}(X_i))$ has property-(P_1)*

Proof. (i): Without loss of generality, we only prove that $(Y_1, (CB)(X_1))$ has r.c.p.. Let $B_1 \in \mathcal{CB}(X_1)$. We can choose $B_2 \in \mathcal{CB}(X_2)$ such that $rad_{Y_2}(B_2) = rad_{Y_1}(B_1)$. Let $B = B_1 \times B_2 \in \mathcal{CB}(X)$. Then $B(1) = B_1$ and $B(2) = B_2$ and hence $r_1(B) = r_2(B)$. By Remark 5.3.5 (i), $cent_Y(B) = cent_{Y_1}(B_1) \times cent_{Y_2}(B_2)$. Thus by our assumption, it follows that $cent_{Y_1}(B_1) \neq \emptyset$.

(ii): Without loss of generality, we only prove that $(X_1, Y_1, (CB)(X_1))$ has property-(P_1). By (i), $(Y_1, \mathcal{CB}(X_1))$ has r.c.p.. Let $B_1 \in \mathcal{CB}(X_1)$. We choose $B_2 \in \mathcal{CB}(X_2)$ such that $rad_{Y_2}(B_2) = rad_{Y_1}(B_1)$. Define $B = B_1 \times B_2 \in \mathcal{CB}(X)$. Then $B(1) = B_1$ and $B(2) = B_2$ and hence $r_1(B) = r_2(B)$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$cent_Y(B, \delta) \subseteq cent_Y(B) + \varepsilon B_X. \tag{5.26}$$

By Remark 5.3.5 (i), $cent_Y(B) = cent_{Y_1}(B_1) \times cent_{Y_2}(B_2)$. Similarly, $cent_Y(B, \delta) = cent_{Y_1}(B_1, \delta) \times cent_{Y_2}(B_2, \delta)$. Let $y_1 \in cent_{Y_1}(B_1, \delta)$ and $y_2 \in cent_{Y_2}(B_2, \delta)$. Hence $(y_1, y_2) \in cent_Y(B, \delta)$. Then there exists $y_1' \in cent_{Y_1}(B_1)$ and $y_2' \in cent_{Y_2}(B_2)$ such that $\|(y_1, y_2) - (y_1', y_2')\| \leq \varepsilon$. Thus $\|y_1 - y_1'\| \leq \varepsilon$. □

We next provide an application of the above stability results.

Proposition 5.3.9 ([55, Proposition 3.7]). *Let Y be a proximinal finite co-dimensionsal subspace of c_0. Then the triplet $(\ell_\infty, B_Y, \mathcal{CB}(\ell_\infty))$ has property-(P_1).*

Proof. Let Y be a proximinal finite co-dimensional subspace of c_0. Let $NA(c_0)$ denote the set of all norm attaining functionals on c_0. Then the set $NA(c_0)$ is precisely the set of all finite sequences in ℓ_1. Now, $Y^\perp$ is a finite dimensional subspace of ${c_0}^*$. Further, $Y^\perp \subseteq NA(c_0)$ (see [25, Proposition III.5]). Therefore, there exists an integer n_0 such that for each $y = (y_n) \in Y^\perp$, we have $y_n = 0$, for each $n \geq n_0$.

Let us now consider the decompositions of ℓ_∞ and Y as done in [31, Section 4]. Let $\{e_n : n = 1, 2, \ldots\}$ be the canonical basis of c_0. For each sequence of real scalars $x = (x_n)$, we define $x' = \sum_{i=1}^{n_0} x_i e_i$. We also define the following spaces:

$$\begin{aligned} X_1 &= span(\{e_1, \ldots, e_{n_0}\}), \\ X_2 &= \{(x_n) \in \ell_\infty : x_n = 0 \text{ for } 1 \leq i \leq n_0\}, \\ Y_1 &= \{x' : x \in Y\} \\ \text{and } Y_2 &= \{(x_n) \in c_0 : x_n = 0 \text{ for } 1 \leq i \leq n_0\}. \end{aligned}$$

Thus $\ell_\infty = X_1 \oplus_\infty X_2$ and $Y = Y_1 \oplus_\infty Y_2$. Clearly, $B_Y = B_{Y_1} \times B_{Y_2}$.

It is easily seen that $X_2 \cong \ell_\infty$ and $Y_2 \cong c_0$. By Corollary 5.2.5, $(X_2, B_{Y_2}, \mathcal{CB}(X_2))$ has property-(P_1). Further, since X_1 is finite dimensional space and $Y_1 \subseteq X_1$, by using a compactness argument we observe that $(X_1, B_{Y_1}, \mathcal{CB}(X_1))$ has property-(P_1). Therefore by Theorem 5.3.7, we conclude that $(\ell_\infty, B_Y, \mathcal{CB}(\ell_\infty))$ has property-(P_1). □

5.4 Semi-continuity of restricted Chebyshev-center maps of ℓ_∞-direct sums

In this section, we derive a few stability results concerning the semi-continuity properties of restricted Chebyshev-center maps of ℓ_∞-direct sums.

Proposition 5.4.1 ([55, Proposition 4.1]). *For each $i \in \{1,2\}$, let X_i be a Banach space and $V_i \in \mathcal{CV}(X_i)$ such that $(V_i, \mathcal{CB}(X_i))$ has r.c.p.. Let $X = X_1 \oplus_\infty X_2$ and $V = V_1 \times V_2$. If for each $i \in \{1,2\}$, $cent_{V_i}(.)$ is l.H.s.c. on $\mathcal{CB}(X_i)$ then the map $cent_V(.)$ is l.H.s.c. on $\mathcal{CB}(X)$.*

Proof. By Proposition 5.3.4, $(V, \mathcal{CB}(X))$ has r.c.p.. Let $B \in \mathcal{CB}(X)$ and $\varepsilon > 0$. Using lower Hausdorff semi-continuity of the maps $cent_{V_1}(.)$ and $cent_{V_2}(.)$ at $B(1)$ and $B(2)$ respectively, there exists $\delta > 0$ such that for each $i \in \{1,2\}$, whenever

$$A \in \mathcal{CB}(X) \text{ with } d_H(B,A) < \delta \text{ and } \rho_i \in cent_{V_i}(B(i)) \Rightarrow B_{X_i}(\rho_i, \varepsilon) \cap cent_{V_i}(A(i)) \neq \emptyset. \quad (5.27)$$

CASE 1: $r_1(B) = r_2(B)$

In this case, by Remark 5.3.5 (i), $cent_V(B) = cent_{V_1}(B(1)) \times cent_{V_2}(B(2))$. Let $i \in \{1,2\}$. Let $\rho_i \in cent_{V_i}(B(i))$ and $A \in \mathcal{CB}(X)$ such that $d_H(B,A) < \delta$. Thus by Remark 5.3.5 (ii), $d_H(B(i), A(i)) < \delta$ and hence by (5.27), there exists $v_i \in B_{X_i}(\rho_i, \varepsilon) \cap cent_{V_i}(A(i))$. By Remark 5.3.5 (i), $(\rho_1, \rho_2) \in cent_V(B)$ and $(v_1, v_2) \in cent_V(A)$. Moreover, $\|(\rho_1,\rho_2) - (v_1,v_2)\| < \varepsilon$. Hence $B_X((\rho_1,\rho_2),\varepsilon) \cap cent_V(A) \neq \emptyset$. Thus $cent_V(.)$ is l.H.s.c. at B.

CASE 2: $r_1(B) \neq r_2(B)$

Without loss of generality, assume $r_1(B) < r_2(B)$, since the same arguments work for the reverse inequality. Let $2\gamma = r_2(B) - r_1(B)$. Replacing X, V, B and α by X_1, V_1, $B(1)$ and $r_2(B)$ respectively in Lemma 5.3.6, we obtain $0 < \delta < \frac{\gamma}{2}$ such that whenever $A \in \mathcal{CB}(X)$ with $d_H(B,A) < \delta$, we have $r_2(A) - r_1(A) > \gamma$ and

$$d_H\left(\bigcap_{b_1 \in B(1)} B_{X_1}[b_1, r_2(B)] \cap V_1, \bigcap_{a_1 \in A(1)} B_{X_1}[a_1, r_2(A)] \cap V_1\right) < \varepsilon. \quad (5.28)$$

Without loss of generality, assume that δ is so chosen that (5.27) is also satisfied. Let $A \in \mathcal{CB}(X)$ with $d_H(B,A) < \delta$. Then $r_2(A) > r_1(A)$ and hence by Remark 5.3.5 (i), $cent_V(A) = \bigcap_{a_1 \in A(1)} B_{X_1}[a_1, r_2(A)] \cap V_1 \times cent_{V_2}(A(2))$. Let $v_1 \in \bigcap_{b_1 \in B(1)} B_{X_1}[b_1, r_2(B)] \cap V_1$ and $v_2 \in cent_{V_2}(B(2))$ and hence by Remark 5.3.5 (i), $v = (v_1, v_2) \in cent_V(B)$. Therefore, by (5.27) and (5.28), there exists $w_1 \in \bigcap_{a_1 \in A(1)} B_{X_1}[a_1, r_2(A)] \cap V_1$ and $w_2 \in cent_{V_2}(A(2))$ such that $\|v_1 - w_1\| < \varepsilon$ and $\|v_2 - w_2\| < \varepsilon$. Thus $(w_1, w_2) \in B_X(v, \varepsilon) \cap cent_V(A)$. ☐

Proposition 5.4.2 ([55, Proposition 4.2]). *For each $i \in \{1,2\}$, let X_i be a Banach space and $V_i \in \mathcal{CV}(X_i)$. Let $X = X_1 \oplus_\infty X_2$ and $V = V_1 \times V_2$. If for each $i \in \{1,2\}$, $(X_i, V_i, \mathcal{CB}(X_i))$ has property-(P_1), then the map $cent_V(.)$ is u.H.s.c. on $\mathcal{CB}(X)$.*

Proof. By Proposition 5.3.4, $(V, \mathcal{CB}(X))$ has r.c.p.. By Theorem 1.2.17, for each $i \in \{1,2\}$, $cent_{V_i}(.)$ is u.H.s.c. on $\mathcal{CB}(X_i)$. Let $B \in \mathcal{CB}(X)$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that for each

$i \in \{1,2\}$, if

$$A \in \mathcal{CB}(X) \text{ with } d_H(B,A) < \delta \Rightarrow cent_{V_i}(A(i)) \subseteq cent_{V_i}(B(i)) + \varepsilon B_{X_i}. \tag{5.29}$$

Case 1: $r_1(B) = r_2(B)$

By Remark 5.3.5 (i), $cent_V(B) = cent_{V_1}(B(1)) \times cent_{V_2}(B(2))$. By our assumption and Remark 1.1.6, we can choose $\theta > 0$ such that for each $i \in \{1,2\}$, $\mathcal{S}(B(i),\theta) < \varepsilon$. We further choose $0 < \delta < \frac{\theta}{4}$ such that (5.29) holds valid. Let $A \in \mathcal{CB}(X)$ such that $d_H(B,A) < \delta$. If $r_1(A) = r_2(A)$, then $cent_V(A) = cent_{V_1}(A(1)) \times cent_{V_2}(A(2))$. Thus by (5.29), we obtain $cent_V(A) \subseteq cent_V(B) + \varepsilon B_X$.

Now without loss of generality, we assume that $r_1(A) < r_2(A)$. For each $i \in \{1,2\}$, since

$$|r_i(B) - r_i(A)| \leq d_H(B(i),A(i)) \leq d_H(B,A) < \frac{\theta}{4}, \tag{5.30}$$

we have

$$|r_1(B) - r_2(A)| \leq |r_1(B) - r_2(B)| + |r_2(B) - r_2(A)| < \frac{\theta}{4}. \tag{5.31}$$

Now, by Remark 5.3.5 (i),

$$cent_V(A) = \bigcap_{a_1 \in A(1)} B_{X_1}[a_1, r_2(A)] \cap V_1 \times cent_{V_2}(A(2)). \tag{5.32}$$

We select $v_1 \in \bigcap_{a_1 \in A(1)} B_{X_1}[a_1, r_2(A)] \cap V_1$. Then using (5.31) and the assumption that $d_H(B,A) < \delta$, for each $b_1 \in B(1)$, there exists $a_1 \in A(1)$ such that $\|a_1 - b_1\| < \frac{\theta}{4}$ and hence

$$\|v_1 - b_1\| \leq \|v_1 - a_1\| + \|a_1 - b_1\| < r_2(A) + \frac{\theta}{4} < r_1(B) + \frac{\theta}{2}. \tag{5.33}$$

It follows that $r(v_1, B(1)) \leq r_1(B) + \frac{\theta}{2}$. Now, since $\mathcal{S}(B(1),\theta) < \varepsilon$, $d(v_1, cent_{V_1}(B(1))) < \varepsilon$. Thus there exists $w_1 \in cent_{V_1}(B(1))$ satisfying $\|v_1 - w_1\| < \varepsilon$. It follows that

$$\bigcap_{a_1 \in A(1)} B_{X_1}[a_1, r_2(A)] \cap V_1 \subseteq cent_{V_1}(B(1)) + \varepsilon B_{X_1}. \tag{5.34}$$

Further, we also have $cent_{V_2}(A(2)) \subseteq cent_{V_2}(B(2)) + \varepsilon B_{X_2}$. Hence, $cent_V(A) \subseteq cent_V(B) + \varepsilon B_X$.

Case 2: $r_1(B) \neq r_2(B)$

Without loss of generality, we assume that $r_1(B) < r_2(B)$ since the same arguments work for the reverse inequality. Let $2\gamma = r_2(B) - r_1(B)$. Replacing X, V, B and α by X_1, V_1, $B(1)$ and $r_2(B)$ respectively in Lemma 5.3.6, we obtain $0 < \delta < \frac{\gamma}{2}$ such that whenever $A \in \mathcal{CB}(X)$ with $d_H(B,A) < \delta$, we have $r_2(A) - r_1(A) > \gamma$ and

$$d_H\left(\bigcap_{b_1 \in B(1)} B_{X_1}[b_1, r_2(B)] \cap V_1, \bigcap_{a_1 \in A(1)} B_{X_1}[a_1, r_2(A)] \cap V_1\right) < \varepsilon. \tag{5.35}$$

We choose $\delta > 0$ such that (5.29) is also satisfied. If $A \in \mathcal{CB}(X)$ such that $d_H(B,A) < \delta$, then

$r_2(A) > r_1(A)$ and hence by Remark 5.3.5 (i),

$$cent_V(A) = \bigcap_{a_1 \in A(1)} B_{X_1}[a_1, r_2(A)] \cap V_1 \times cent_{V_2}(A(2)). \tag{5.36}$$

Let $A \in \mathcal{CB}(X)$ such that $d_H(B, A) < \delta$. If $v_1 \in \bigcap_{a_1 \in A(1)} B_{X_1}[a_1, r_2(A)] \cap V_1$ and $v_2 \in cent_{V_2}(A(2))$, then by (5.29) and (5.35), we choose $w_1 \in \bigcap_{b_1 \in B(1)} B_{X_1}[b_1, r_2(B)] \cap V_1$ and $w_2 \in cent_{V_2}(B(2))$ satisfying $\|v_i - w_i\| < \varepsilon$, for each $i \in \{1, 2\}$. Thus $(w_1, w_2) \in cent_V(B)$ and $\|(v_1, v_2) - (w_1, w_2)\| < \varepsilon$. It follows that $cent_V(A) \subseteq cent_V(B) + \varepsilon B_X$. □

We note that the assumptions in above Proposition 5.4.2 cannot be weakened; see [31, Remark 3.5]. The following result follows from Propositions 5.4.1 and 5.4.2 and [31, Remark 2.8].

Proposition 5.4.3 ([55, Proposition 4.3])**.** *For each $i \in \{1, 2\}$, let X_i be a Banach space and $V_i \in \mathcal{CV}(X_i)$ such that $(X_i, V_i, \mathcal{CB}(X_i))$ has property-(P_1). Let $X = X_1 \oplus_\infty X_2$ and $V = V_1 \times V_2$. If for each $i \in \{1, 2\}$, $cent_{V_i}(.)$ is Hausdorff metric continuous on $\mathcal{CB}(X_i)$, then $cent_V(.)$ is Hausdorff metric continuous on $\mathcal{CB}(X)$.*

The stability results proved in Sections 5.3 and 5.4 are true if we replace the class of all non-empty closed bounded subsets by that of all non-empty compact or finite subsets of the respective spaces.

We now recall the definition of a polyhedral space. A finite dimensional Banach space X is called *polyhedral* if B_X has only finitely many extreme points. An infinite dimensional Banach space X is called polyhedral if each of the finite dimensional subspace of X is polyhedral. The sequence space c_0 is a well-known example of an infinite dimensional polyhedral space. We refer to [21] and the references therein for a study on polyhedral spaces. We recall the following recent result due to Tsar'kov.

Theorem 5.4.4 ([59, p. 243] and [3, Theorem 6.7, pg. 801])**.** *Let V be a non-empty polyhedral subset of a finite dimensional polyhedral Banach space X. Then the map $cent_V(.)$ is globally Lipschitz Hausdorff metric continuous on $\mathcal{CB}(X)$ and admits a Lipschitz selection.*

We now prove our main result.

Proposition 5.4.5 ([55, Proposition 4.5])**.** *Let Y be a proximinal finite co-dimensional subspace of c_0. Then the map $cent_{B_Y}(.)$ is Hausdorff metric continuous on $\{B \in \mathcal{CB}(\ell_\infty) \colon rad_{B_Y}(B) \leq R\}$, for each $R > 0$.*

Proof. For each $i \in \{1, 2\}$, let X_i and Y_i be defined as in the proof of Theorem 5.3.9. Then $\ell_\infty = X_1 \oplus_\infty X_2$ and $Y = Y_1 \oplus_\infty Y_2$. Clearly, $B_Y = B_{Y_1} \times B_{Y_2}$. Moreover, $X_2 \cong \ell_\infty$, $Y_2 \cong c_0$ and $Y_1 \subseteq X_1$. Hence from Corollary 5.2.5, $cent_{B_{Y_2}}(.)$ is Hausdorff metric continuous on $\mathcal{CB}(X_2)$. Further, X_1 is a finite dimensional subspace of c_0 and hence is a polyhedral space. Thus by Theorem 5.4.4, $cent_{B_{Y_1}}(.)$ is Hausdorff metric continuous on $\mathcal{CB}(X_1)$. Therefore, we conclude from Theorems 5.3.7 and 5.4.3 that $cent_{B_Y}(.)$ is Hausdorff metric continuous on $\{B \in \mathcal{CB}(\ell_\infty) \colon rad_{B_Y}(B) \leq R\}$, for each $R > 0$. □

For a subspace Y of a Banach space X, we conclude this section by discussing the interconnection between the semi-continuity properties of the maps $cent_{B_Y}(.)$ and $cent_Y(.)$ in the following result.

Proposition 5.4.6 ([55, Proposition 4.6]). *Let X be a Banach space and Y be a subspace of X. Let the pair $(B_Y, \mathcal{CB}(X))$ have r.c.p..*

(*i*) *For each $\lambda > 0$ and $B \in \mathcal{CB}(X)$, $cent_{\lambda B_Y}(.)$ is l.H.s.c. at B if and only if $cent_{B_Y}(.)$ is l.H.s.c. at $\frac{1}{\lambda}B$.*

(*ii*) *If $cent_{B_Y}(.)$ is l.H.s.c. on $\mathcal{CB}(X)$, then $cent_Y(.)$ is l.H.s.c. on $\mathcal{CB}(X)$.*

(*iii*) *For each $\lambda > 0$ and $B \in \mathcal{CB}(X)$, $cent_{\lambda B_Y}(.)$ is u.H.s.c. at B if and only if $cent_{B_Y}(.)$ is u.H.s.c. at $\frac{1}{\lambda}B$.*

(*iv*) *If $cent_{B_Y}(.)$ is u.H.s.c. on $\mathcal{CB}(X)$, then $cent_Y(.)$ is u.H.s.c. on $\mathcal{CB}(X)$.*

(*v*) *For each $\lambda > 0$ and $B \in \mathcal{CB}(X)$, $cent_{\lambda B_Y}(.)$ is Hausdorff metric continuous at B if and only if $cent_{B_Y}(.)$ is Hausdorff metric continuous at $\frac{1}{\lambda}B$.*

(*vi*) *If $cent_{B_Y}(.)$ is Hausdorff metric continuous on $\mathcal{CB}(X)$, then $cent_Y(.)$ is Hausdorff metric continuous on $\mathcal{CB}(X)$.*

Proof. By Proposition 3.2.2, $(Y, \mathcal{CB}(X))$ has r.c.p..

(*i*) We first assume that $cent_{\lambda B_Y}(.)$ is l.H.s.c. at B. We fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that if

$$A \in \mathcal{CB}(X) \text{ with } d_H(B, A) < \delta \text{ and } y \in cent_{\lambda B_Y}(B) \Rightarrow B_X(y, \lambda\varepsilon) \cap cent_{\lambda B_Y}(A) \neq \emptyset. \tag{5.37}$$

We now set $\gamma = \frac{\delta}{\lambda}$. Let $A \in \mathcal{CB}(X)$ such that $d_H(\frac{1}{\lambda}B, A) < \gamma$ and $y \in cent_{B_Y}(\frac{1}{\lambda}B)$. This implies $d_H(B, \lambda A) < \gamma\lambda = \delta$ and from Lemma 3.2.1, $\lambda y \in cent_{\lambda B_Y}(B)$. Therefore, by (5.37), let $z \in B_X(\lambda y, \lambda\varepsilon) \cap cent_{\lambda B_Y}(\lambda A)$. Thus by Lemma 3.2.1, $\frac{z}{\lambda} \in cent_{B_Y}(A)$. It follows that $\|\frac{z}{\lambda} - y\| < \varepsilon$ and hence $B_X(y, \varepsilon) \cap cent_{B_Y}(A) \neq \emptyset$.

Conversely, let $\varepsilon > 0$. Then there exists $\delta > 0$ such that if

$$A \in \mathcal{CB}(X) \text{ with } d_H\left(\frac{1}{\lambda}B, A\right) < \delta \text{ and } y \in cent_{B_Y}\left(\frac{1}{\lambda}B\right) \Rightarrow B_X\left(y, \frac{\varepsilon}{\lambda}\right) \cap cent_{B_Y}(A) \neq \emptyset. \tag{5.38}$$

We now set $\gamma = \lambda\delta$. Let $A \in \mathcal{CB}(X)$ such that $d_H(B, A) < \gamma$ and $y \in cent_{\lambda B_Y}(B)$. This implies $d_H(\frac{1}{\lambda}B, \frac{1}{\lambda}A) < \frac{\gamma}{\lambda} = \delta$ and from Lemma 3.2.1, $\frac{y}{\lambda} \in cent_{B_Y}(\frac{1}{\lambda}B)$. Therefore, by (5.38), let $z \in B_X(\frac{y}{\lambda}, \frac{\varepsilon}{\lambda}) \cap cent_{B_Y}(\frac{1}{\lambda}A)$. Thus by Lemma 3.2.1, $\lambda z \in cent_{\lambda B_Y}(A)$. It follows that $\|\lambda z - y\| < \varepsilon$ and hence $B_X(y, \varepsilon) \cap cent_{B_Y}(A) \neq \emptyset$.

(*ii*) Let $B \in \mathcal{CB}(X)$ and $\lambda > \sup_{b \in B} \|b\| + rad_Y(B)$. From our assumption, $cent_{B_Y}(.)$ is l.H.s.c. at $\frac{1}{\lambda}B$. Therefore from (*i*), $cent_{\lambda B_Y}(.)$ is l.H.s.c. at B. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that if

$$A \in \mathcal{CB}(X) \text{ with } d_H(B, A) < \delta \text{ and } y \in cent_{\lambda B_Y}(B) \Rightarrow B_X(y, \varepsilon) \cap cent_{\lambda B_Y}(A) \neq \emptyset. \tag{5.39}$$

We choose $0 < \gamma < \min\{2\delta, \lambda - (\sup_{b \in B} \|b\| + rad_Y(B))\}$. Let $A \in \mathcal{CB}(X)$ such that $d_H(B, A) < \frac{\gamma}{2}$ and $y \in cent_Y(B)$. Now by Lemma 2.2.4,

$$rad_Y(A) + \sup_{a \in A} \|a\| \leq rad_Y(B) + \frac{\gamma}{2} + \frac{\gamma}{2} + \sup_{b \in B} \|b\| < \lambda. \tag{5.40}$$

Hence by Lemma 3.2.1 (*iii*), $cent_Y(B) = cent_{\lambda B_Y}(B)$ and $cent_Y(A) = cent_{\lambda B_Y}(A)$. Therefore, it follows from (5.39) that

$$B_X(y, \varepsilon) \cap cent_{B_Y}(A) \neq \emptyset. \tag{5.41}$$

(iii) We first assume that $cent_{\lambda B_Y}(.)$ is u.H.s.c. at B. We fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that if

$$A \in \mathcal{CB}(X) \text{ such that } d_H(B, A) < \delta \Rightarrow cent_{\lambda B_Y}(A) \subseteq cent_{\lambda B_Y}(B) + \lambda\varepsilon B_X. \tag{5.42}$$

We set $\gamma = \frac{\delta}{\lambda}$. Let $A \in \mathcal{CB}(X)$ such that $d_H(\frac{1}{\lambda}B, A) < \delta$. Hence, $d_H(B, \lambda A) < \gamma\lambda = \delta$. Therefore, from (5.42),

$$cent_{\lambda B_Y}(\lambda A) \subseteq cent_{\lambda B_Y}(B) + \lambda\varepsilon B_X. \tag{5.43}$$

From Lemma 3.2.1, $\lambda cent_{B_Y}(A) \subseteq \lambda cent_{B_Y}(\frac{1}{\lambda}B) + \lambda\varepsilon B_X$. Hence, $cent_{B_Y}(A) \subseteq cent_{B_Y}\left(\frac{1}{\lambda}B\right) + \varepsilon B_X$.

Conversely, let $\varepsilon > 0$. Then there exists $\delta > 0$ such that if

$$A \in \mathcal{CB}(X) \text{ such that } d_H\left(\frac{1}{\lambda}B, A\right) < \delta \Rightarrow cent_{B_Y}(A) \subseteq cent_{B_Y}\left(\frac{1}{\lambda}B\right) + \frac{\varepsilon}{\lambda}B_X. \tag{5.44}$$

We now set $\gamma = \lambda\delta$. Let $A \in \mathcal{CB}(X)$ such that $d_H(B, A) < \gamma$. Thus $d_H(\frac{1}{\lambda}B, \frac{1}{\lambda}A) < \frac{\gamma}{\lambda} = \delta$. Therefore, by (5.44), $cent_{B_Y}(\frac{1}{\lambda}A) \subseteq cent_{B_Y}\left(\frac{1}{\lambda}B\right) + \frac{\varepsilon}{\lambda}B_X$. Now, by Lemma 3.2.1, $\frac{1}{\lambda}cent_{\lambda B_Y}(A) \subseteq \frac{1}{\lambda}cent_{\lambda B_Y}(B) + \frac{\varepsilon}{\lambda}B_X$ and hence

$$cent_{\lambda B_Y}(A) \subseteq cent_{\lambda B_Y}(B) + \varepsilon B_X. \tag{5.45}$$

(iv) Let $B \in \mathcal{CB}(X)$ and $\lambda > \sup_{b\in B}\|b\| + rad_Y(B)$. From our assumption, $cent_{B_Y}(.)$ is u.H.s.c. at $\frac{1}{\lambda}B$. Therefore, from (iii), $cent_{\lambda B_Y}(.)$ is u.H.s.c. at B. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that if

$$A \in \mathcal{CB}(X) \text{ such that } d_H(B, A) < \delta \Rightarrow cent_{\lambda B_Y}(A) \subseteq cent_{\lambda B_Y}(B) + \varepsilon B_X. \tag{5.46}$$

We choose $0 < \gamma < \min\{2\delta, \lambda - (\sup_{b\in B}\|b\| + rad_Y(B))\}$. Let $A \in \mathcal{CB}(X)$ such that $d_H(B, A) < \delta$. Applying the same argument as in (ii), we obtain that $\lambda > rad_Y(A) + \sup_{a\in A}\|a\|$. Thus by Lemma 3.2.1 (iii), $cent_Y(B) = cent_{\lambda B_Y}(B)$ and $cent_Y(A) = cent_{\lambda B_Y}(A)$. Therefore, it follows from (5.46) that

$$cent_{B_Y}(A) \subseteq cent_{B_Y}(B) + \varepsilon B_X. \tag{5.47}$$

The result in (v) follows from (i), (iii) and [31, Remark 2.8] and the result in (vi) follows from (ii), (iv) and [31, Remark 2.8]. ☐

5.5 A variant of transitivity property of property-(P_1)

In this section, we provide answers to the transitivity type problem for property-(P_1), stated in Question 1.2.16, for certain cases. On considering an M-summand, in particular, in Question 1.2.16, the following result answers this question positively and generalizes [35, Proposition 3.2].

Proposition 5.5.1 ([55, Proposition 5.1])**.** *Let X be a Banach space, Y be an M-summand in X and Z be a subspace of Y. If $(Y, Z, \mathcal{CB}(Y))$ satisfies property-(P_1), then $(X, Z, \mathcal{CB}(X))$ satisfies property-(P_1).*

Proof. Since Y is an M-summand in X, let $X = Y \oplus_\infty W$ for some subspace W of X. Then the subspace $Z \subseteq Y$, when considered as a subspace of X, is isometrically isomorphic to $Z' = Z \times \{0\}$. We know that $(Y, Z, \mathcal{CB}(Y))$ has property-(P_1) and, trivially, $(W, \{0\}, \mathcal{CB}(W))$ has property-(P_1) as well. Therefore, by Theorem 5.3.7, $(X, Z', \mathcal{CB}(X))$ has property-(P_1). ☐

Another instance where Question 1.2.16 is positively answered is as follows:

Proposition 5.5.2 ([55, Proposition 5.2]). *Let X be an L_1-predual space. Let Y be a finite co-dimensional subspace of X and J be an M-ideal in X such that $Y \subseteq J$. If Y is strongly proximinal in J, then the triplet $(X, Y, \mathcal{K}(X))$ satisfies property-(P_1).*

Proof. By [35, Theorem 3.13], Y is strongly proximinal in X. Therefore, by Theorem 3.4.7, $(X, Y, \mathcal{K}(X))$ satisfies property-(P_1). ☐

5.6 Restricted center property of X in X^{**}

This section is a brief discourse on Questions 1.2.22 and 1.2.23. In this section, we present a few applications to the results in Section 5.2. We provide our first observation as follows:

Proposition 5.6.1 ([55, Corollary 2.6]). *Let S be a compact Hausdorff space and $\mathcal{A}$ be a closed linear subalgebra of $C(S)$. Then the triplet $(C(S)^{**}, B_{\mathcal{A}}, \mathcal{CB}(C(S)^{**}))$ satisfies property-(P_1) and the map $cent_{B_{\mathcal{A}}}(.)$ is uniformly Hausdorff metric continuous on $\{B \in \mathcal{CB}(C(S)^{**})\colon rad_{B_{\mathcal{A}}}(B) \leq R\}$, for each $R > 0$.*

Proof. We know that $C(S)^{**} \cong C(\Omega)$ for some compact Hausdorff space Ω. Moreover, $C(S)$ is a closed linear subalgebra of $C(\Omega)$ under the canonical embedding; see [52]. Since $\mathcal{A}$ is a subalgebra of $C(S)$, $\mathcal{A}$ is a subalgebra of $C(\Omega)$. Now, it follows directly from Corollary 5.2.4 that $(C(S)^{**}, B_{\mathcal{A}}, \mathcal{CB}(C(S)^{**}))$ satisfies property-(P_1) and the map $cent_{B_{\mathcal{A}}}(.)$ is uniformly Hausdorff metric continuous on $\{B \in \mathcal{CB}(C(S)^{**})\colon rad_{B_{\mathcal{A}}}(B) \leq R\}$, for each $R > 0$. ☐

For a subspace $\mathcal{A}$ of $C(S)$, since $\mathcal{A}^{**} \cong \mathcal{A}^{\perp\perp}$ is a subspace of $C(S)^{**}$, an immediate consequence of Proposition 5.6.1 is

Corollary 5.6.2. *Let S be a compact Hausdorff space and $\mathcal{A}$ be a closed linear subalgebra of $C(S)$. Then the triplet $(\mathcal{A}^{**}, B_{\mathcal{A}}, \mathcal{CB}(\mathcal{A}^{**}))$ satisfies property-(P_1) and the map $cent_{B_{\mathcal{A}}}(.)$ is uniformly Hausdorff metric continuous on $\{B \in \mathcal{CB}(\mathcal{A}^{**})\colon rad_{B_{\mathcal{A}}}(B) \leq R\}$, for each $R > 0$.*

We know that for a locally compact Hausdorff space T, $C_0(T)$ is isometrically isomorphic to a closed linear subalgebra of some $C(S)$ space. Therefore

Corollary 5.6.3. *Let T be a locally compact Hausdorff space and $\mathcal{A}$ be a closed linear subalgebra of $C_0(T)$. Then the triplet $(C_0(T)^{**}, B_{\mathcal{A}}, \mathcal{CB}(C_0(T)^{**}))$ satisfies property-(P_1) and the map $cent_{\mathcal{A}}(.)$ is uniformly Hausdorff metric continuous on $\{B \in \mathcal{CB}(C_0(T)^{**})\colon rad_{B_{\mathcal{A}}}(B) \leq R\}$, for each $R > 0$. Consequently, the triplet $(\mathcal{A}^{**}, B_{\mathcal{A}}, \mathcal{CB}(\mathcal{A}^{**}))$ satisfies property-(P_1) and the map $cent_{B_{\mathcal{A}}}(.)$ is uniformly Hausdorff metric continuous on $\{B \in \mathcal{CB}(\mathcal{A}^{**})\colon rad_{B_{\mathcal{A}}}(B) \leq R\}$, for each $R > 0$.*

As per the classification diagram in [43], we investigate if the answer is positive or not for Question 1.2.22 in the case of sublattices of $C(S)$. There exists a sublattice $\mathcal{A}$ of $C([0, 1])$, which does not contain constant functions and a bounded subset $B \subseteq \mathcal{CB}(\mathcal{A})$ such that $cent_{\mathcal{A}}(B) = \emptyset$; see [6, Example 4.7]. However, the authors in [6] proved the following result for a specific type of sublattices of $C(S)$.

Proposition 5.6.4 ([6, Corollary 4.5])**.** *Let S be a compact Hausdorff space and $\mathcal{A}$ be a closed linear sublattice of $C(S)$ described as follows:*

$$\mathcal{A} = \{f \in C(S) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}, \tag{5.48}$$

for some index set I and co-ordinates $(t_i, s_i, \lambda_i) \in S \times S \times [0,1]$ for each $i \in I$ such that $\inf\{\lambda_i \colon i \in I \text{ and } \lambda_i > 0\} > 0$*. Then the pair $(\mathcal{A}, \mathcal{CB}(C(S)))$ has r.c.p..*

Proposition 5.6.4 leads us to the following result.

Proposition 5.6.5. *Let S be a compact Hausdorff space and $\mathcal{A}$ be a closed linear sublattice of $C(S)$ described as follows:*

$$\mathcal{A} = \{f \in C(S) \colon f(t_i) = \lambda_i f(s_i), \text{ for each } i \in I\}, \tag{5.49}$$

for some index set I and co-ordinates $(t_i, s_i, \lambda_i) \in S \times S \times [0,1]$ for each $i \in I$ such that $\inf\{\lambda_i \colon i \in I \text{ and } \lambda_i > 0\} > 0$*. Then the pair $(\mathcal{A}, \mathcal{CB}(C(S)^{**}))$ has r.c.p.. Consequently, the pair $(\mathcal{A}, \mathcal{CB}(\mathcal{A}^{**}))$ has r.c.p..*

Proof. We know that there exists a compact Hausdorff space Ω such that $C(S)^{**} \cong C(\Omega)$. Furthermore, $C(S)$ is a closed linear sublattice of $C(\Omega)$ under the canonical embedding; see [4, Theorem 1.69]. Since $\mathcal{A}$ is a sublattice of $C(S)$, $\mathcal{A}$ is a sublattice of $C(\Omega)$. Therefore, it follows directly from Proposition 5.6.4 that $(\mathcal{A}, \mathcal{CB}(C(S)^{**}))$ has r.c.p.. □

We now make the following easy observation.

Proposition 5.6.6. *Let S be a compact Hausdorff space. Let $\mathcal{A}$ be a closed linear sublattice of $C(S)$ containing constant functions. Then $\mathcal{A}$ is a subalgebra of $C(S)$.*

Proof. By Theorem 4.2.1, let I be the index set and $(s_i, t_i, \lambda_i) \in S \times S \times [0,1]$ for each $i \in I$ such that

$$\mathcal{A} = \{f \in C(S) \colon f(s_i) = \lambda_i f(t_i), \text{ for each } i \in I\}. \tag{5.50}$$

Since $\mathcal{A}$ contains constant functions, it is easy to see that for each $i \in I$, $\lambda_i \notin [0,1)$. Hence, for each $i \in I$, $\lambda_i = 1$. It follows from Theorem 4.3.1 that $\mathcal{A}$ is a subalgebra of $C(S)$. □

Therefore, Proposition 5.6.1 holds true for closed linear sublattices of $C(S)$ containing constant functions.

Further, according to the classification diagram in [43], we look at the class of G-spaces. It is described as follows: A Banach space X is a *G-space* if X is isometric to the Banach space described as follows:

$$\{f \in C(S) \colon f(s_i) = \lambda_i f(t_i) \text{ for each } i \in I\}, \tag{5.51}$$

for some compact Hausdorff space S, index set I and co-ordinates $(s_i, t_i, \lambda_i) \in S \times S \times \mathbb{R}$ for each $i \in I$. The result in Corollary 5.2.3 gives a positive answer to Question 1.2.22 for certain types of G-subspaces in a $C(S)$ space. The discussion above motivates us to ask the following general question to which we do not know the answer.

Question 5.6.7. Let S be a compact Hausdorff space and $\mathcal{A}$ be a closed linear subspace of $C(S)$ described as follows:

$$\mathcal{A} = \{f \in C(S) \colon f(s_i) = \lambda_i f(t_i) \text{ for each } i \in I\}, \tag{5.52}$$

for some index set I and co-ordinates $(s_i, t_i, \lambda_i) \in S \times S \times \mathbb{R}$ for each $i \in I$. Then is $\mathcal{A}$ proximinal in $C(S)$?

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